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Abstract

The asynchronous automaton associated with a Boolean network $f : \{0,1\}^n \to \{0,1\}^n$ is considered in many applications. It is the finite deterministic automaton with set of states $\{0,1\}^n$, alphabet $\{1,\dots,n\}$, where the action of letter i on a state x consists in either switching the ith component if $f_i(x) \neq x_i$ or doing nothing otherwise. This action is extended to words in the natural way. We then say that a word w *fixes* f if, for all states x, the result of the action of w on x is a fixed point of f. In this paper, we ask for the existence of fixing words, and their minimal length. Firstly, our main results concern the minimal length of words that fix *monotone* networks. We prove that, for n sufficiently large, there exists a monotone network f with n components such that any word fixing f has length $\Omega(n^2)$. For this first result we prove, using Baranyai's theorem, a property about shortest supersequences that could be of independent interest: there exists a set of permutations of $\{1,\dots,n\}$ of size $2^{o(n)}$, such that any sequence containing all these permutations as subsequences is of length $\Omega(n^2)$. Conversely, we construct a word of length $O(n^3)$ that fixes all monotone networks with n components. Secondly, we refine and extend our results to different classes of fixable networks, including networks with an acyclic interaction graph, increasing networks, conjunctive networks, monotone networks whose interaction graphs are contained in a given graph, and balanced networks.

1 Introduction

1.1 Asynchronous graph and fixable Boolean networks

A **Boolean network** (**network** for short) is a finite dynamical system usually defined by a function

$$f : \{0,1\}^n \to \{0,1\}^n, \qquad x = (x_1,\dots,x_n) \mapsto f(x) = (f_1(x),\dots,f_n(x)).$$

Boolean networks have many applications. In particular, since the seminal papers of McCulloch and Pitts [21], Hopfield [16], Kauffman [17, 18] and Thomas [28, 29], they are omnipresent in the modeling of neural and gene networks (see [8, 20] for reviews). They are also essential tools in computer science, for the network coding problem in information theory [3, 14] and memoryless computation [9, 10, 15].

The "network" terminology comes from the fact that the **interaction graph** of f is often considered as the main parameter of f: it is the directed graph $G(f)$ with vertex set $[n] := \{1,\dots,n\}$ and an arc from j to i if f_i depends on x_j, that is, if there exist $x, y \in \{0,1\}^n$ that only differ in the component j such that $f_i(x) \neq f_i(y)$.

In many applications, for modelling gene networks in particular, the dynamics derived from f is not the synchronous dynamics, which is simply described by the successive iterations of

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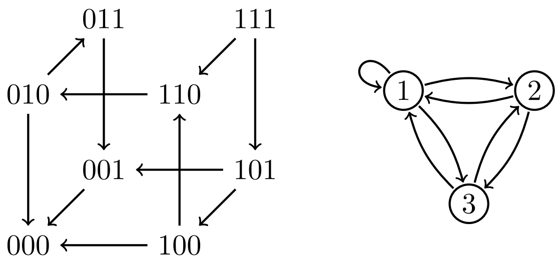
x	$f(x)$
000	000
001	000
010	001
011	001
100	010
101	000
110	010
111	100

$f_1(x) = x_1 \wedge x_2 \wedge x_3$
$f_2(x) = x_1 \wedge \neg x_3$
$f_3(x) = x_2 \wedge \neg x_1.$



3-component network f $\Gamma(f)$ $G(f)$

Figure 1: A network f given under two different forms (its table and a definition of its components by logical formulas) with its asynchronous graph $\Gamma(f)$ and its interaction graph $G(f)$.

f, but the asynchronous dynamics [1]. The latter is usually represented by the directed graph $\Gamma(f)$, called **asynchronous graph** of f and defined as follows. The vertex set of $\Gamma(f)$ is $\{0,1\}^n$, the set of all the possible **states**, and there is an arc from x to y if and only if x and y differs in exactly one component, say i, and $f_i(x) \neq x_i$. An example of a network with its interaction graph and its asynchronous is given in Figure 1.

Before going on, let us review a few basic properties of the asynchronous graph. First, $\Gamma(f)$ completely determines f and vice versa. In particular, x is a fixed point of f (i.e. $f(x) = x$) if and only if it is a sink of $\Gamma(f)$ (i.e. it has no outgoing arcs). Second, since arcs of $\Gamma(f)$ link states that differs in exactly one coordinate, $\Gamma(f)$ is a directed subgraph of the hypercube Q_n (where cycles of length two are allowed). The converse obviously holds: any directed subgraph of the hypercube is the asynchronous graph of a network.

$\Gamma(f)$ can be naturally regarded as a deterministic finite automaton where the set of states is $\{0,1\}^n$, the alphabet is $[n]$, and where

$$f^i(x) := (x_1, \dots, f_i(x), \dots, x_n)$$

is the result of the action of a letter i on a state x. This action is extended to words on the alphabet $[n]$ in the natural way: the result of the action of a word $w = i_1 i_2 \dots i_k$ on a state x is inductively defined by $f^w(x) := f^{i_2 \dots i_k}(f^{i_1}(x))$ or, equivalently,

$$f^w(x) := (f^{i_k} \circ f^{i_{k-1}} \circ \dots \circ f^{i_1})(x).$$

This interpretation is illustrated in Figure 2.

In this paper, we are interested in words w that **fix** f, that is, such that $f^w(x)$ is a fixed point of f for every x. This corresponds to the situation where the start state is undetermined, and the accepting states are exactly the fixed points. As such, there is an obvious connection between fixing word and **synchronizing words**: if f has a unique fixed point, then w is a synchronizing word for $\Gamma(f)$ if and only if w fixes f. If f admits a fixing word we say that f is **fixable**. For instance, the network in Figure 1 is fixed by $w = 1231$, and is hence fixable. It is rather easy to see that f is fixable if and only if there is a path in $\Gamma(f)$ from any initial state to a fixed point of f.

Trivially, if f is fixable, then it has at least one fixed point. Interestingly, Bollobás, Gotsman and Shamir [7] showed that, considering the uniform distribution on the set of n-component networks, the probability for f to be fixable when f has at least one fixed point tends to 1 when $n \to \infty$. Thus almost all networks with a fixed point are fixable. In turn, this shows that for n large, a positive fraction $(1 - 1/e)$ of all n-component networks are fixable.

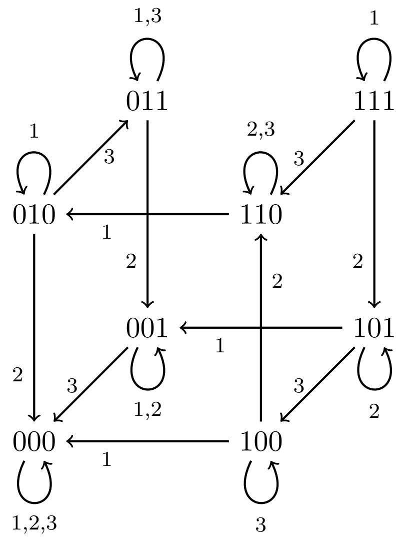
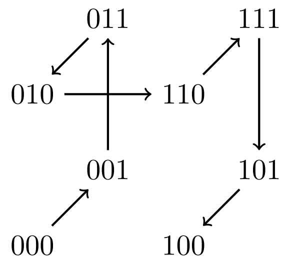


Figure 2: The asynchronous graph of Figure 1 regarded as a deterministic finite automata.

For any fixable network f, the **fixing length** of f is the length of a shortest word fixing f and is denoted as $\lambda(f)$. For instance, we have seen that $w = 1231$ fixes the network in Figure 1, thus this network has fixing length at most 4, and it is easy to see that no word of length three fixes this network, and thus it has fixing length exactly 4. It is easy to construct a fixable n-component network f such that $\lambda(f)$ is exponential in n, as follows. Let $x^1, \dots, x^{2^n}$ be a Gray code ordering of $\{0,1\}^n$, i.e. x^k and x^{k+1} only differ by one coordinate for all k. Then let $f(x^k) := x^{k+1}$ for $1 \le k < 2^n$ and $f(x^k) := x^k$ for $k = 2^n$. It is clear that the asynchronous graph of such a f, illustrated below, only contains a directed hamiltonian path of Q_n, and hence $\lambda(f) = 2^n - 1$.



We are then interested in networks f which can be fixed in polynomial time, i.e. $\lambda(f)$ is bounded by a polynomial in n. More strongly, we extend our concepts to entire families $\mathcal{F}$ of n-component networks. We say that $\mathcal{F}$ is fixable if there is a word w such that w fixes f for all $f \in \mathcal{F}$, which is clearly equivalent to: all the members of $\mathcal{F}$ are fixable. The fixing length $\lambda(\mathcal{F})$ is defined naturally as the length of a shortest word fixing $\mathcal{F}$. We are then interested in families $\mathcal{F}$ which can be fixed in polynomial time. For each family, we aim to derive an upper bound on $\lambda(\mathcal{F})$ and a lower bound on the maximum $\lambda(f)$ amongst all $f \in \mathcal{F}$. Up to our knowledge, the only result of this kind was given in [13], where it is shown that the word $12\dots n$ repeated $n(3n-1)$ times fixes any n-component symmetric threshold network with weights in $\{-1,0,1\}$. This family of threshold networks has thus a cubic fixing length. We could also mention somewhat less connected work concerning the minimal, maximal and average convergence time toward fixed points in the asynchronous graph $\Gamma(f)$ for some specific fixable networks f [4, 22, 23, 12, 11].

1.2 Results

1.2.1 Acyclic networks

We first consider the family $F_A(n)$ of n-component networks with an acyclic interaction graph. Let f be a member of this family, and consider a topological sort $i_1i_2\dots i_n$ of the interaction graph $G(f)$ (no arc from i_p to i_q for $p \geq q$). A first observation is that the word $i_1i_2\dots i_n$ fixes f. As a consequence, the fixing length of f is exactly n. We will prove, more generally, that a word w fixes f if at least one topological sort of $G(f)$ is a subsequence of w. As a consequence, w fixes the whole family $F_A(n)$ if w contains all the permutations of $[n]$ as subsequences, and this sufficient condition is actually also necessary. Interestingly, such a word w appears naturally in many combinatorial contexts. It is called an ***n*-complete word**. The minimal length $\lambda(n)$ of an n-complete word is not exactly known, but it is quadratic: Adleman [2] proved that $\lambda(n) \leq n^2 - 2n + 4$ (see [24] and the references therein for improvements) and Kleitman and Kwiatkowski [19] proved that, for all $\varepsilon > 0$, there exists a positive constant C_ε such that $\lambda(n) \geq n^2 - C_\varepsilon n^{7/4+\varepsilon}$. Thus, to sum up, for the family $F_A(n)$, the picture is very clear:

Theorem 1. *For every $n \geq 1$, the fixing length of any member of $F_A(n)$ is n, and the fixing length of the family $F_A(n)$ is $\lambda(n)$, and $\lambda(n) \to n^2$ as $n \to \infty$.*

1.2.2 Increasing networks

A network f is **increasing** if $x \leq f(x)$ for every x, where $\leq$ is applied componentwise. We denote by $F_I(n)$ the set of n-component increasing networks. A simple exercise shows that the number of increasing networks is doubly exponential: $|F_I(n)| = 2^{n2^{n-1}}$. It is then remarkable that, even though some increasing networks require quadratic time to be fixed, they can all be fixed together in quadratic time still:

Theorem 2. *For any $\varepsilon > 0$ and n sufficiently large, there exists $f \in F_I(n)$ of fixing length at least $(\frac{1}{e} - \varepsilon)n^2$, and, for $n \geq 2$, the fixing length of $F_I(n)$ is $\lambda(n) \leq n^2$.*

The dual $\tilde{f}$ of a network f is defined as $\tilde{f}(x) = \neg f(\neg x)$. It is easily checked that a word fixes f if and only if it fixes $\tilde{f}$. Since a network is increasing if and only if its dual is decreasing ($x \geq f(x)$ for all x), Theorem 2 also holds for decreasing networks.

1.2.3 Monotone networks

The main topic of this paper is the family of **monotone** networks, that is, the n-component networks f such that $x \leq y \Rightarrow f(x) \leq f(y)$ for all $x, y \in \{0,1\}^n$. We denote by $F_M(n)$ the set of n-component monotone networks. The fact that monotone networks are fixable is not obvious and is proved in [25, 22]. Our first contribution concerning monotone network is that some monotone networks have a quadratic fixing length.

Theorem 3. *For any $\varepsilon > 0$ and n sufficiently large, there exists an n-component monotone network with fixing length at least $(\frac{1}{e} - \varepsilon)n^2$.*

The main step in the proof consists in proving that there exists a sub-exponential collection of permutations of $[n]$ such that any word containing all these permutations as subsequences is of quadratic length. The proof uses Baranyai's theorem (if a divides n, then there exist $\binom{n}{a}\frac{a}{n}$ partitions of $[n]$ in a-sets such that each a-set of $[n]$ appears in exactly one of these partitions, see [30]).

Theorem 4. *For any $\varepsilon > 0$ and n sufficiently large, there exists a set of at most $n^{n^{\frac{1}{2}+\varepsilon}}$ permutations of $[n]$ such that any word containing all these permutations as subsequences is of length at least $(\frac{1}{e} - \varepsilon)n^2$.*

We then show that there is a word of cubic length, based on the concatenation of m-complete words for $m = 1, \dots, n-1$, that fixes all monotone networks.

Theorem 5. *For all $n \geq 5$, there is a word of length $\frac{1}{3}n^3$ that fixes $F_M(n)$.*

The quadratic lower-bound in Theorem 3 and the cubic upper-bound in Theorem 5 raise the following natural question: *is it as difficult to fix* one *member of $F_M(n)$ as* all *members of $F_M(n)$?*

1.2.4 Refinements and extensions

We also refine and extend our results for monotone networks in three different fashions.

Firstly, some accurate results can be obtained for subfamilies of monotone networks. We say that the network $f \in F_M(n)$ is **conjunctive** if for all $i \in [n]$ there exists $J_i \subseteq [n]$ such that $f_i(x) = \bigwedge_{j \in J_i} x_j$. We denote by $F_C(n)$ the set of n-component conjunctive networks.

Theorem 6. *For $n \geq 3$, the maximum fixing length of a member of $F_C(n)$ is $2n-2$.*

Secondly, we refine the cubic upper bound for the word fixing all monotone networks by considering the interaction graph. We denote the set of monotone networks whose interaction graph is contained in a directed graph G as $F_M(G)$. Recall that a feedback vertex set of G is a set of vertices intersecting every cycle; the minimum size of a feedback vertex set is usually called the **transversal number** of G and denoted τ. We then prove that the family $F_M(G)$ can be fixed by a word of length $2\tau^2 n + n$. Thus, when we bound the transversal number, the fixing length becomes linear.

Theorem 7. *Let G be a directed graph on $[n]$ with transversal number τ. There is a word of length $2\tau^2 n + n$ that fixes $F_M(G)$.*

On one hand, if G is the complete directed graph on $[n]$, with n^2 arcs, then $\tau = n$ and we obtain a cubic bound, as in Theorem 5. On the other hand, if G is acyclic, then $\tau = 0$, and we obtain an upper-bound of n, which is tight according to Theorem 1.

Thirdly, we consider **balanced networks**. The signed interaction graph is a refined description of the interaction graph, which not only indicates that x_j has an influence on f_i, but also whether that influence is positive (f_i is a non-decreasing function of x_j), negative (f_i is a non-increasing function of x_j) or null (other cases). Monotone networks are exactly the networks such that all the arcs in the signed interaction graph are signed positively (all the interactions are positive). A network f is called balanced if every cycle of its signed interaction graph is positive. Here, the sign of a cycle is defined as the product of the signs of its arcs. Thus monotone networks are a special case of balanced networks. We denote the set of all n-component balanced networks as $F_B(n)$. We once again show that the family of balanced networks can be fixed in polynomial time.

Theorem 8. *For all $n \geq 7$, there is a word of length $\frac{1}{9}n^4$ that fixes $F_B(n)$.*

Our results are summarised in Table 1. A dash means that we did not find any nontrivial result for the given entry: the tightest upper-bound on $\lambda(F_C(n))$ is that of $\lambda(F_M(n))$, the tightest lower bound on $\max_{f \in F_B(n)} \lambda(f)$ is that of $\max_{f \in F_M(n)} \lambda(f)$.

2 Preliminaries

2.1 Notation

Let $w = w_1 \dots w_p$ be a word. Then length p of w is denoted $|w|$. If $S = \{i_1, i_2, \dots i_q\} \subseteq [p]$ with $i_1 < i_2 \dots < i_q$, then we shall sometimes use the notation $w_S = w_{i_1} w_{i_2} \dots w_{i_q}$; if $S = \emptyset$, then $w_S := \epsilon$, where ϵ is the empty word. Any such w_S is a **subsequence** of w. Moreover, for

Networks	$\mathcal{F}$	$\max_{f\in\mathcal{F}} \lambda(f)$	$\lambda(\mathcal{F})$
Acyclic	$F_A(n)$	$= n$	$= \lambda(n)$
Increasing	$F_I(n)$	$\geq (\frac{1}{e} - \varepsilon)n^2$	$= \lambda(n)$
Monotone	$F_M(n)$	$\geq (\frac{1}{e} - \varepsilon)n^2$	$\leq \frac{1}{3}n^3$
Conjunctive	$F_C(n)$	$= 2n - 2$	–
G-monotone	$F_M(G)$	–	$\leq 2\tau^2 n + n$
Balanced	$F_B(n)$	–	$\leq \frac{1}{9}n^4$

Table 1: Summary of results

any integers $a, b \in [p]$ we set $[a, b] = \{a, a + 1, \ldots, b\}$ and hence $w_{[a,b]} := w_a, \ldots, w_b$ if $a \leq b$ and $w_{[a,b]} := \epsilon$ if $a > b$. Any such $w_{[a,b]}$ is a **factor** of w. For any word w and any $k \geq 1$, the word $k \cdot w$ is obtained by repeating w exactly k times; $0 \cdot w$ is the empty word.

Graphs are always directed and may contain loops (arcs from a vertex to itself). Paths and cycles are always directed and without repeated vertices. We denote by $G[I]$ the subgraph of G induced by a set of vertices I. If V is a set, a graph on V is a graph with vertex set V. We refer the reader to the authoritative book on graphs by Bang-Jensen and Gutin [5] for some basic concepts, notation and terminology.

We denote by $F_M(n)$ the family of n-component monotone networks and by $\lambda_M(n)$ the fixing length of $F_M(n)$. More generally, if $F_X(n)$ is any family of n-component fixable networks, then $\lambda_X(n)$ is the fixing length of $F_X(n)$. If G is a graph on $[n]$, then $F(G)$ denotes the set of n-component networks f such that the interaction graph of f is a subgraph of G. Then, $F_X(G) := F_X(n) \cap F(G)$ and $\lambda_X(G)$ is the fixing length of $F_X(G)$.

Let f be an n-component network. We set $f^\epsilon := f$ and, for any integer i and $x \in \{0, 1\}^n$, we define $f^i(x)$ as in the introduction if $i \in [n]$, and $f^i(x) := x$ if $i \notin [n]$. This extends the action of letters in $[n]$ to letters in $\mathbb{N}$, and by extension, this also defines the action of a word over the alphabet $\mathbb{N}$. Let G be a graph on $[n]$. The **conjunctive network on G** is the unique conjunctive network whose interaction graph is G. Namely, it is the n-component network f defined as follows: for all $i \in [n]$,

$$f_i(x) = \bigwedge_{j \in N^-(i)} x_j$$

where $N^-(i)$ is the set of in-neighbors of i in G, and $f_i(x) = 1$ if $N^-(i)$ is empty.

For all $i \in [n]$, we denote as e_i the i-th unit vector, i.e. $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in position i. Given two states x and y, $x + y$ is applied componentwise and computed modulo two. For instance x and $x + e_i$ only differ in the ith position. The state containing only 1s is denoted $\mathbf{1}$, and the state containing only 0s is denoted $\mathbf{0}$. Hence, if G is a strongly connected graph on $[n]$, then $\mathbf{0}$ and $\mathbf{1}$ are two fixed points of the conjunctive network on G. The **Hamming weight** of a state x, denote $w_{\mathrm{H}}(x)$, is the number of 1s in x.

2.2 Acyclic networks

Recall that f is acyclic if its interaction graph is acyclic, and that $F_A(n)$ denotes the set of n-component acyclic networks. An important property of acyclic networks is that they have a unique fixed point [26] and that they have an acyclic asynchronous graph [27]. This obviously implies that $F_A(n)$ is fixable. We show here that the fixing length of acyclic networks are rather easy to understand. The techniques used will be useful later, for analyzing the fixing length of monotone networks.

Lemma 1. *Let G be an acyclic graph on $[n]$ and $f \in F(G)$. If a word w contains, as subsequence, a topological sort of G, then w fixes f. Furthermore, $\lambda(f) = n$.*

Proof. Let y be the unique fixed point of f. Let $u = i_1 i_2 \dots i_n$ be topological sort of G, and let $w = w_1 w_2 \dots w_p$ be any word containing u as subsequence. Hence, there is a increasing sequence of indices $j_1 j_2 \dots j_n$ such that $u = w_{j_1} w_{j_2} \dots w_{j_n}$. Let x^0 be any initial state, and for all $q \in [p]$, let x^q be obtained from x^{q-1} by updating w_q, that is, $x^q := f^{w_q}(x^{q-1})$. Equivalently, $x^q := f^{w_{[1,q]}}(x^0)$. Let us prove, by induction on $k \in [n]$, that $x^q_{i_k} = y_{i_k}$ for all $j_k \leq q \leq p$. Since i_1 is a source of the interaction graph, f_{i_1} is a constant. Thus $f_{i_1}(x^q) = f_{i_1}(y) = y_{i_1}$ for all $q \in [p]$, and since $w_{j_1} = i_1$, we deduce that $x^q_{i_1} = y_{i_1}$ for all $j_1 \leq q \leq p$. Let $1 < k \leq n$. Since f_{i_k} only depends on components i_l with $1 \leq l < k$, and since, by induction, $x^q_{i_l} = y_{i_l}$ for all $1 \leq l < k$ and $j_{k-1} \leq q \leq p$, we have $f_{i_k}(x^q) = f_{i_k}(y) = y_{i_k}$ for all $j_{k-1} \leq q \leq p$. Since $w_{j_k} = i_k$ we deduce that $x^q_{i_k} = y_{i_k}$ for all $j_k \leq q \leq p$, completing the induction step. Hence, $f^w(x^0) = x^q = y$ for any initial state x^0, thus w fixes f.

We deduce that, in particular, any topological sort u of G fixes f, thus $\lambda(f) \leq n$. Conversely, if a word w fixes f, then $f^w(\neg y) = y$, and hence at least n asynchronous updates are required, that is, the length of w is at least n. Thus $\lambda(f) = n$. $\square$

The converse of the previous proposition is false in general (for instance if f is the 3-component network defined by $f_1(x) = 0$, $f_2(x) = x_1$ and $f_3(x) = x_1 \wedge x_2$, then 132 fixes f while 123 is the unique topological sort of the interaction graph of f) but it holds for conjunctive networks.

Lemma 2. *Let G be an acyclic graph on $[n]$ and let f be the conjunctive network on G. A word w fixes f if and only if it contains, as subsequence, a topological sort of G.*

Proof. According to Lemma 1, it is sufficient to prove that if $w = w_1 w_2 \dots w_p$ fixes f then w contains, as subsequence, a topological sort of G. Let $x^0 := \mathbf{0}$ and $x^q = f^{w_q}(x^{q-1})$ for all $q \in [p]$. Since w fixes f and since $\mathbf{1}$ is the unique fixed point of f, we have $f^w(x^0) = x^p = \mathbf{1}$. Thus for each $i \in [n]$, there exists t_i such that $x^{t_i}_i = 1$ and $x^q_i = 0$ for all $0 \leq q < t_i$. We have, obviously, $w_{t_i} = i$. Let $i_1 i_2 \dots i_n$ be the enumeration of the vertices of G such that $t_{i_1} t_{i_2} \dots t_{i_n}$ is increasing. In this way $i_1 i_2 \dots i_n$ is a subsequence of w, and it follows the topological order. Indeed, suppose that G has an arc from i_k to i_l. Since $f_{i_l}(x^{t_{i_l}-1}) = x^{t_{i_l}}_{i_l} = 1$, we have $x^{t_{i_l}-1}_{i_k} = 1$, and thus $t_{i_k} < t_{i_l}$, that is, i_k is before i_l in the enumeration. $\square$

As an immediate application we get the following characterization.

Proposition 1. *Let G be an acyclic graph on $[n]$. A word w fixes $F(G)$ if and only if it contains, as subsequence, a topological sort of G.*

A word w is **complete** for a set S (or S-complete) if it contains, as subsequence, all the permutations of S. An ***n*-complete word** is a $[n]$-complete word. Let $\lambda(n)$ be the length of a shortest n-complete word. Interestingly, $\lambda(n)$ is unknown. Let $w^1, \dots, w^n$ be n permutations of $[n]$ (not necessarily distinct). Then the concatenation $w^1 w^2 \dots w^n$ clearly contains all the permutations of $[n]$. Thus $\lambda(n) \leq n^2$. Conversely, if w contains all the permutations of n, then $\binom{|w|}{n}$ is at least $n!$ and we deduce that $|w| \geq n^2/e^2$ (this simple counting argument will be reused later). This shows that the magnitude of $\lambda(n)$ is quadratic. We have however the following tighter bounds:

$$\begin{array}{lll}
\lambda(n) \leq n^2 - 2n + 4 & \text{for all } n \geq 1 & [2] \\
\lambda(n) \leq n^2 - 2n + 3 & \text{for all } n \geq 10 & [31] \\
\lambda(n) \leq \lceil n^2 - \frac{7}{3}n + \frac{19}{3} \rceil & \text{for all } n \geq 7 & [24] \\
\lambda(n) \geq n^2 - C_\varepsilon n^{7/4+\varepsilon} & & [19]
\end{array}$$

where $\varepsilon > 0$ and where C_ε is a positive constant that only depends on ε. Thus $\lambda(n) = n^2 - o(n^2)$.

Let $F_P(n)$ be the set of conjunctive networks on the $n!$ paths of length $n-1$ with vertex set $[n]$. Thus, for each permutation $i_1i_2\dots i_n$ of $[n]$, there exists exactly one $f\in F_P(n)$ such that $f_{i_1}(x)=1$ and $f_{i_k}(x)=x_{i_{k-1}}$ for all $x\in\{0,1\}^n$ and $1<k\leq n$. We show below that the family $F_P(n)$ has a quadratic fixing length.

Lemma 3. *A word w fixes $F_P(n)$ if and only if it is n-complete. Hence $\lambda_P(n)=\lambda(n)$.*

Proof. By Lemma 1, any n-complete word fixes $F_A(n)$ and thus $F_P(n)$ in particular. Conversely, suppose that w fixes $F_P(n)$. Since each permutation of n is the unique topological sort of the interaction graph of exactly one conjunctive network in $F_P(n)$, by Lemma 2, w contains, as subsequence, the $n!$ permutations of n. Thus w is n-complete. □

As an immediate consequence, we get the following proposition, which implies, with Lemma 1, the Theorem 1 stated in the introduction.

Proposition 2. *A word w fixes $F_A(n)$ if and only if it is n-complete. Hence $\lambda_A(n)=\lambda(n)$.*

Therefore, it is as hard to fixe $F_P(n)$ as to fixe $F_A(n)$: these two families have the same quadratic fixing length, while $F_P(n)$ is much more smaller than $F_A(n)$ (the former has $n!$ members while the latter has $2^{\Theta(2^n)}$ members). We shall use this to our advantage when designing a monotone network with quadratic fixing length in Section 3.1.

2.3 Increasing networks

Recall that a network f is increasing if $x\leq f(x)$ for all x. Those networks are also relatively easy to fix collectively, as seen below. We shall use this fact when constructing a cubic word fixing all monotone networks in Section 3.2.

Lemma 4. *Let f be an n-component network and $x\in\{0,1\}^n$. If $f^u(x)\leq f^{uv}(x)$ for any words u and v, then $f^w(x)$ is a fixed point of f for any word w containing all the permutations of $\{i:x_i=0\}$. Similarly, if $f^u(x)\geq f^{uv}(x)$ for any words u and v, then $f^w(x)$ is a fixed point of f for any word w containing all the permutations of $\{i:x_i=1\}$.*

Proof. Suppose that $f^u(x)\leq f^{uv}(x)$ for any words u and v, and that $w=w_1w_2\dots w_p$ be S-complete, with $S:=\{i:x_i=0\}$. Let $x^0:=x$ and $x^q:=f^{w_q}(x^{q-1})$ for all $q\in[p]$. By hypothesis, $x^0\leq x^1\leq\dots\leq x^q$. Suppose for the sake of contradiction that $x^p=f^w(x)$ is not a fixed point, i.e. there is j such that $f_j(x^p)>x^p_j$. Let $t_1,\dots,t_m\in[p]$ be the set of positions such that $x^{t_k-1}<x^{t_k}$, and let $i_k:=w_{t_k}$ for all $k\in[m]$. Clearly, $j\neq i_k$ for every $k\in[m]$. Setting $t_0:=0$, we have $x^{t_{k-1}}=x^{t_k-1}$, thus $f^{i_k}(x^q)=f^{i_k}(x^{t_k-1})=x^{t_k}>x^{t_k-1}=x^q$ for all $t_{k-1}\leq q<t_k$. We deduce that i_k does not appear in $w_{[t_{k-1},t_k-1]}$ or, equivalently, t_k is the first position of i_k in $w_{[t_{k-1},p]}$. Similarly, we have $x^{t_m}=x^p$, thus $f_j(x^q)=f_j(x^p)>x^p_j=x^q_j$ for all $t_m\leq q\leq p$ and we deduce that j does not appear in $w_{[t_m,p]}$. Since w is S-complete, the sequence $i_1i_2\dots i_mj$ appears in w, say at positions $w_{s_1}w_{s_2}\dots w_{s_m}w_{s_{m+1}}$. Since t_k is the first position of i_k in $w_{[t_{k-1},p]}$, we have $s_k\geq t_k$ for all $k\in[m]$. In particular, $s_m\geq t_m$, thus j appears in $w_{[t_m,p]}$ which is the desired contradiction. If $f^u(x)\geq f^{uv}(x)$ for any words u and v the proof is similar. □

Proposition 3. *A word w fixes $F_I(n)$ if and only if it is n-complete. Hence $\lambda_I(n)=\lambda(n)$.*

Proof. If w is n-complete, then w fixes f by Lemma 4. Conversely, suppose that w fixes all n-component increasing networks and let $i_1i_2\dots i_n$ be any permutation of $[n]$. Let $y^0:=\mathbf{0}$ and $y^k:=y^{k-1}+e_{i_k}$ for all $k\in[n]$. Then $y^0y^1\dots y^n$ is chain from $\mathbf{0}$ to $\mathbf{1}$ in the hypercube Q_n. Let f be the n-component increasing network defined by

$$f(x):=\begin{cases}y^{k+1} & \text{if } x=y^k \text{ and } 1\leq k<n,\\ x & \text{otherwise.}\end{cases}$$

Then $\mathbf{1}$ is the unique fixed point of f reachable from $\mathbf{0}$ in the asynchronous graph, and it is easy to check that $f^w(\mathbf{0})=\mathbf{1}$ if and only if $i_1i_2\dots i_n$ is a subsequence of w. Thus w is n-complete. □

On the other hand, there exists an increasing network requiring quadratic time to be fixed (this and the Proposition 3 above give the Theorem 2 stated in the introduction).

Theorem 9. *For any $\varepsilon > 0$ and n sufficiently large, there exists $f \in F_I(n)$ such that $\lambda(f) \geq (\frac{1}{e} - \varepsilon)n^2$.*

The proof of Theorem 9 requires the machinery developed for monotone networks, and as such we delay its proof until Section 3.1.

3 Monotone networks

3.1 A monotone network with quadratic fixing length

The aim of this section is to exhibit a monotone network with quadratic fixing length. As we saw in Section 2.2, the family of conjunctive networks $F_P(n)$ on the $n!$ paths on $[n]$ has quadratic fixing length. Therefore, our strategy is to "pack" many of these conjunctive networks in the same network f. As an illustration of this strategy, we first describe a monotone network with fixing length of order $(n/\log n)^2$.

Let $n = m + r$ where $m! \leq \binom{r}{r/2}$, and let us write $F_P(m) = \{h^1, \dots, h^{m!}\}$. There is then a surjection $\phi : X \to [m!]$ where X is the set of states in $\{0,1\}^r$ with Hamming weight $r/2$. The n-component network f then views the r last components as controls, that decide, through ϕ, which conjunctive network in $F_P(m)$ to choose on the first m components. More precisely, by identifying $\{0,1\}^n$ with $\{0,1\}^m \times \{0,1\}^r$, we define f as follows:

$$f(x,y) := \begin{cases} (\mathbf{1}, y) & \text{if } w_{\mathrm{H}}(y) > r/2, \\ (h^{\phi(y)}(x), y) & \text{if } w_{\mathrm{H}}(y) = r/2, \\ (\mathbf{0}, y) & \text{if } w_{\mathrm{H}}(y) < r/2. \end{cases}$$

The first and third cases are there to guarantee that f is indeed monotone. Since any network in $F_P(m)$ can appear, a word fixing f must fix $F_P(m)$. Thus a word fixing f is m-complete, and hence has length $\Omega(m^2)$. Choosing $m = \Omega(n/\log n)$ then yields $\Omega(n^2/\log^2 n)$.

The network above reached a fixing length of $\Omega(m^2)$ because it packed all possible networks in $F_P(m)$. However, it did not reach quadratic fixing length because m had to be $o(n)$ in order to embed all $m!$ networks of $F_P(m)$ in X. Thus, we show below that only a subexponential subset of $F_P(m)$ is required to guarantee $\Omega(m^2)$. This is equivalent to prove that there exists a subexponential set of permutations of $[m]$ such that any word containing these permutations as subsequences is of length $\Omega(m^2)$. In that case, we can use $m = (1 - o(1))n$, and hence reach a fixing length of $\Omega(n^2)$.

The main tool is Baranyai's theorem, see [30].

Theorem 10 (Baranyai). *If a divides n, then there exists a collection of $\binom{n}{a}\frac{a}{n}$ partitions of $[n]$ into $\frac{n}{a}$ sets of size a such that each a-subset of $[n]$ appears in exactly one partition.*

Lemma 5. *Let a and b be positive integers, and $n = ab$. There exists a set of $a!\binom{n}{a} \leq n^a$ permutations of $[n]$ such any word containing all these permutations as subsequences is of length at least*

$$\left(n^{-\frac{2b}{a}}\right)\frac{n(n-a)}{e}.$$

Proof. According to Baranyai's theorem, there exists a collection of $r := b^{-1}\binom{n}{a}$ partitions of $[n]$ into b sets of size a, such that each a-subset of $[n]$ appears in exactly one partition. Let $A^0, \dots, A^{r-1}$ be these partitions. For each $0 \leq i < r$, we set

$$A^i = \{A^i_0, \dots, A^i_{b-1}\}.$$

Then, for all $0 \le i < r$ and $0 \le j, k < b$ we set $S_k^{i,j} := A_{j+k}^i$ and

$$S^{i,j} := S_0^{i,j} S_1^{i,j} \dots S_{b-1}^{i,j} = A_{j+0}^i A_{j+1}^i \dots A_{j+b-1}^i$$

where addition is modulo b. So, the $S^{i,j}$ form a set of $\binom{n}{a}$ *ordered* partitions of $[n]$ in b sets of size a. The interesting point is that, for all fixed i and fixed ℓ, the sequence $S_\ell^{i,0} S_\ell^{i,1} \dots S_\ell^{i,b-1}$ is a permutation of A^i (namely $S^{i,\ell}$). Since each a-subset of $[n]$ appears in exactly one A^i, we deduce that, for any fixed ℓ, the set of $S_\ell^{i,j}$ is *exactly* the set of a-subsets of $[n]$.

Given an a-subset X of $[n]$ and a permutation σ of $[a]$, we set $\sigma(X) = i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(a)}$, where $i_1, i_2, \dots, i_a$ is an enumeration of the elements of X in the increasing order. Let $\sigma^0, \dots, \sigma^{a!-1}$ be an enumeration of the permutations of $[a]$. For all $0 \le i < r$, $0 \le j < b$, $0 \le k < a!$, we set

$$\pi^{i,j,k} := \sigma^k(S_0^{i,j}) \dots \sigma^k(S_{b-1}^{i,j}).$$

The $\pi^{i,j,k}$ form a collection of $a!\binom{n}{a}$ permutations of $[n]$. The interesting property is that, for ℓ fixed, the set of $\sigma^k(S_\ell^{i,j})$ is *exactly* the set of words in $[n]^a$ without repetition, simply because, for ℓ fixed, the set of $S_\ell^{i,j}$ is exactly the set of a-subsets of $[n]$, as mentioned above. In particular, for ℓ fixed, the $\sigma^k(S_\ell^{i,j})$ are pairwise distinct.

Let $w = w_1 w_2 \dots w_p$ be a shortest word containing all the permutations $\pi^{i,j,k}$ as subsequences. We know that $|w| \le \lambda(n) \le n^2$. Let

$$\gamma^{i,j,k} := \gamma_0^{i,j,k} \gamma_1^{i,j,k} \dots \gamma_b^{i,j,k}$$

be the *profile* of $\pi^{i,j,k}$, defined recursively as follows: $\gamma_0^{i,j,k} := 0$ and, for all $0 \le \ell < b$, $\gamma_{\ell+1}^{i,j,k}$ is the smallest integer such that $\sigma^k(S_\ell^{i,j})$ is a subsequence of the factor

$$w_{[\gamma_\ell^{i,j,k}+1, \gamma_{\ell+1}^{i,j,k}]}.$$

Since $\gamma_0^{i,j,k} = 0$ and $1 \le \gamma_\ell^{i,j,k} \le n^2$ for all $1 \le \ell \le b$, there are at most n^{2b} possible profiles. Thus there exist at least

$$s \ge \frac{a!\binom{n}{a}}{n^{2b}}$$

permutations $\pi^{i,j,k}$ with the same profile. Let $\pi^{i_1,j_1,k_1}, \dots, \pi^{i_s,j_s,k_s}$ be these permutations, and let $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_b)$ be their profile. For all $0 \le \ell < b$, let

$$w^\ell := w_{[\gamma_\ell+1, \gamma_{\ell+1}]}.$$

By construction, w^ℓ contains, as subsequences, each of $\sigma^{k_1}(S_\ell^{i_1,j_1}), \dots, \sigma^{k_s}(S_\ell^{i_s,j_s})$. Since these s elements of $[n]^a$ are pairwise distinct (because, for fixed ℓ, all the $\sigma^k(S_\ell^{i,j})$ are pairwise distinct), this means that w^ℓ contains at least s distinct subsequences of length a, and thus

$$\binom{|w^\ell|}{a} \ge s.$$

We deduce

$$\frac{|w^\ell|^a}{a!} \ge \binom{|w^\ell|}{a} \ge s \ge \frac{a!\binom{n}{a}}{n^{2b}} \ge \frac{(n-a)^a}{n^{2b}}$$

and thus

$$|w^\ell|^a \ge a!\frac{(n-a)^a}{n^{2b}} \ge \left(\frac{a}{e}\right)^a \frac{(n-a)^a}{n^{2b}} \ge \left[\frac{a(n-a)}{en^{\frac{2b}{a}}}\right]^a.$$

Consequently,

$$|w| \ge \sum_{0 \le \ell < b} |w^\ell| \ge b \cdot \frac{a(n-a)}{en^{\frac{2b}{a}}} = \left(n^{-\frac{2b}{a}}\right) \frac{n(n-a)}{e}.$$

□

We are now in position to prove that there is a subexponential set of permutations that requires a quadratic length to be represented in a supersequence. This is Theorem 4 that we restate below from the introduction.

Theorem 11. *For any $\varepsilon > 0$ and n sufficiently large, there is a set of at most $n^{n^{\frac{1}{2}+\varepsilon}}$ permutations of $[n]$ such that any word containing all these permutations as subsequences is of length at least $(\frac{1}{e} - \varepsilon)n^2$.*

Proof. Let $\varepsilon > 0$ be arbitrarily small. Let n be a positive integer, $a := \lfloor n^{\frac{1}{2}+\varepsilon} \rfloor$, $b := \lfloor n^{\frac{1}{2}-\varepsilon} \rfloor$ and $m := ab$. By the preceding lemma, there exist $s := a!\binom{m}{a} \leq m^a \leq n^{n^{\frac{1}{2}+\varepsilon}}$ permutations $\pi^1, \ldots, \pi^s$ of $[m]$ such that if w is any word containing all the π^i as subsequences then

$$|w| \geq \left(m^{-\frac{2b}{a}}\right) \frac{m(m-a)}{e} \geq \left(n^{-\frac{2b}{a}}\right) \frac{m(m-a)}{e}.$$

First, $n^{-2ba^{-1}} \to 1$ as $n \to \infty$. Second, since $m = n + o(n)$, we have $m(m-a) = n^2 - o(n^2)$. From these two observations we deduce that if n is at least some constant n_0 that only depends on ε then

$$|w| \geq (1-\varepsilon)\frac{n^2}{e} \geq (\frac{1}{e} - \varepsilon)n^2.$$

For each $i \in [\ell]$, let $\tilde{\pi}^i$ be a permutation of $[n]$ that contains π^i as a subsequence. Then any word $\tilde{w}$ containing all the $\tilde{\pi}^i$ also contains all the π^i, so that $|\tilde{w}| \geq (\frac{1}{e} - \varepsilon)n^2$ if $n \geq n_0$. □

Implementing the "packing" strategy described above, we obtain a monotone network with quadratic fixing length. This is Theorem 3 that we restate below from the introduction.

Theorem 12. *For any $\varepsilon > 0$ and n sufficiently large, there exists an n-component monotone network with fixing length at least $(\frac{1}{e} - \varepsilon)n^2$.*

Proof. Let $\varepsilon > 0$, let n be a positive integer, and let m be the largest integer such that

$$m^{m^{\frac{1}{2}+\delta}} \leq \binom{n-m}{\lfloor \frac{n-m}{2} \rfloor}, \tag{1}$$

with $\delta = \varepsilon/2$. Then $m = (1 - o(1))n$ and thus if n is large enough, then

$$m^2 > n^2(1-\delta)$$

and, according to Theorem 11, there exists a collection $\pi^1, \ldots, \pi^p$ of $p \leq m^{m^{\frac{1}{2}+\delta}}$ permutations of $[m]$ such that any word containing all these permutations as subsequences is of length at least $(\frac{1}{e} - \delta)m^2$. Let regard these p permutations as m-vertex paths, and let $h^1, \ldots, h^p$ be the corresponding conjunctive networks. In other words, writing $\pi^k = \pi^k_1 \pi^k_2 \ldots \pi^k_n$, we have, for all $x \in \{0,1\}^m$,

$$h^k_{\pi^k_1}(x) = 1 \text{ and } h^k_{\pi^k_l}(x) = x_{\pi^k_{l-1}} \text{ for all } 1 < l \leq m.$$

According to Lemma 2, if a word w fixes all the networks h^k then it contains all the permutations π^k as subsequences, and thus $|w| \geq (\frac{1}{e} - \delta)m^2$.

Let $r := n - m$. Then according to (1) there is a surjection ϕ from the set of states in $\{0,1\}^r$ with Hamming weight $\lfloor \frac{r}{2} \rfloor$ to $[p]$. By identifying $\{0,1\}^n$ with $\{0,1\}^m \times \{0,1\}^r$, we then define the n-component network f as follows:

$$f(x,y) = \begin{cases} (\mathbf{1}, y) & \text{if } w_{\mathrm{H}}(y) > \lfloor r/2 \rfloor, \\ (h^{\phi(y)}(x), y) & \text{if } w_{\mathrm{H}}(y) = \lfloor r/2 \rfloor, \\ (\mathbf{0}, y) & \text{if } w_{\mathrm{H}}(y) < \lfloor r/2 \rfloor. \end{cases}$$

Let us check that f is monotone. Suppose $(x,y) \le (x',y')$. If $w_{\rm H}(y) < w_{\rm H}(y')$ we easily check that $f(x,y) \le f(x',y')$. Otherwise, we have $y = y'$ and thus $\phi(y) = \phi(y') = k$ for some $k \in [p]$, and, since h^k is monotone, we obtain $f(x,y) = (h^k(x), y) \le (h^k(x'), y') = f(x',y')$.

Let w be any shortest word fixing f. Then it is clear that fixes h^k for all $k \in [p]$. Thus

$$|w| \ge (\frac{1}{e} - \delta)m^2 > (\frac{1}{e} - \delta)(1-\delta)n^2 > \left(\frac{1}{e} - \varepsilon\right)n^2.$$

□

A similar argument works for increasing networks as well.

Proof of Theorem 9. We use the same setup as the proof of Theorem 12, excepted that the m-component networks h^k and the n-component network f are defined as follows. Let $k \in [p]$. We set $y^{k,0} := \mathbf{0}$ and $y^{k,l} := y^{k,l-1} + e_{\pi^k_l}$ for all $l \in [m]$. We then define the m-component increasing network h^k by

$$h^k(x) := \begin{cases} y^{k,l+1} & \text{if } x = y^{k,l} \text{ and } 1 \le l < m, \\ x & \text{otherwise.} \end{cases}$$

Then, as already said in the proof of Proposition 3, a word fixes h^k if and only if it contains π^k as subsequence. Thus if a word w fixes all the networks h^k then it contains all the permutations π^k as subsequences, and thus $|w| \ge (\frac{1}{e} - \delta)m^2$.

Next, we define the n-component network f as follows:

$$f(x,y) = \begin{cases} (\mathbf{1}, y) & \text{if } w_{\rm H}(y) > \lfloor r/2 \rfloor, \\ (h^{\phi(y)}(x), y) & \text{if } w_{\rm H}(y) = \lfloor r/2 \rfloor, \\ (\mathbf{1}, y) & \text{if } w_{\rm H}(y) < \lfloor r/2 \rfloor. \end{cases}$$

We easily check that f is increasing and that w fixes f if and only if it fixes h^k for all $k \in [p]$. We then deduce as above that any word fixing f is of length at least $\left(\frac{1}{e} - \varepsilon\right)n^2$. □

3.2 Cubic word fixing all monotone networks

What about the fixing length $\lambda_M(n)$ of the whole family $F_M(n)$ of n-component monotone networks? We have shown that some members have quadratic fixing length, namely $(\frac{1}{e} - \varepsilon)n^2$, and thus, obviously, $\lambda_M(n) \ge (\frac{1}{e} - \varepsilon)n^2$. But we can say something slightly better: we have shown that the family of conjunctive networks $F_P(n)$ on the $n!$ paths on $[n]$ has fixing length $\lambda(n)$, and since $F_P(n) \subseteq F_M(n)$ we obtain:

$$\lambda_M(n) \ge \lambda(n).$$

We have no better lower-bound. Maybe the family of n-component conjunctive networks whose interactions graphs are disjoint union of cycles has fixing length greater than $\lambda(n)$ (this family can be equivalently defined as the set of monotone isometries of Q_n).

Concerning upper-bounds, we show below that $\lambda_M(n)$ is at most cubic, and this is the best upper-bound we have on the maximum fixing length of a member of $F_M(n)$. For that we construct inductively a word W^n of cubic length that fixes $F_M(n)$. First, let ω^n be a shortest n-complete word (of length $\lambda(n)$). Then let

$$W^1 := 1 \quad \text{and} \quad W^n := W^{n-1}, n, \omega^{n-1}.$$

In particular, we have

$$\begin{aligned} W^2 &= 1,2,1 \\ W^3 &= 121, 3, 121 \\ W^4 &= 1213121, 4, 1213121 \\ W^5 &= 121312141213121, 5, 123412314213. \end{aligned}$$

Theorem 13. *The word W^n fixes $F_M(n)$ for every $n \geq 1$. Therefore,*

$$\lambda_M(n) \leq n + \sum_{i=1}^{n-1} \lambda(i) \leq \frac{n^3}{3} - \frac{3n^2}{2} + \frac{37n}{6}.$$

The main idea is that, once the components 1 to $n-1$ have been fixed, then a monotone network behaves just like an increasing (or decreasing) network. Therefore, the network can be fixed in quadratic time from that point.

Lemma 6. *Let f be an n-component monotone network. If $x \leq f(x)$ then $f^u(x) \leq f^{uv}(x)$ for any words u and v. Similarly, if $x \geq f(x)$ then $f^u(x) \geq f^{uv}(x)$ for any words u and v.*

Proof. Suppose that $x \leq f(x)$ and let $i \in [n]$. Then $x \leq f^i(x)$ so $f^i_i(x) = f_i(x) \leq f_i(f^i(x))$ and $f^i_j(x) = x_j \leq f_j(x) \leq f_j(f^i(x))$ for all $j \neq i$. Thus $x \leq f(x)$ implies $f^i(x) \leq f(f^i(x))$ for every $i \in [n]$. We deduce that, for any word $w = w_1 w_2 \dots w_k$, $x \leq f^{w_1}(x) \leq f^{w_1 w_2}(x) \leq \dots \leq f^{w_1 w_2 \dots w_k}(x)$, and this clearly implies the lemma. □

Proof of Theorem 13. The proof is by induction on n. This is clear for $n = 1$, so suppose it holds for $n-1$. Fix an initial state $x \in \{0,1\}^n$ and, for every $z \in \{0,1\}^n$, let $z_{-n} := (z_1, \dots, z_{n-1})$ and $h(z_{-n}) := f(z_{-n}, x_n)_{-n}$. Then h is a monotone network with $n-1$ components. Let $y := f^{W^{n-1}}(x)$. Since the letter n does not appear in W^{n-1}, we have $y_{-n} = h^{W^{n-1}}(x_{-n})$ and thus, by induction hypothesis, y_{-n} is a fixed point of h. Hence,

$$f(y) = f(y_{-n}, y_n) = (h(y_{-n}), f_n(y)) = (y_{-n}, f_n(y)).$$

We deduce that either y is a fixed point of f, and in that case $y = f^{n,\omega^{n-1}}(y) = f^{W^n}(x)$ so we are done, or $f(y) = y + e_n$. Suppose that $f(y) = y + e_n$ with $y_n = 0$, and remark that $f(y) = f^n(y)$. Setting $y' := f(y)$ we have $y \leq y'$, thus $y' \leq f(y')$, and since $y'_n = 1$, we deduce that ω^{n-1} contains all the permutations of $\{i : y'_i = 0\}$. Hence, according to Lemma 6 and Lemma 4,

$$f^{\omega^{n-1}}(y') = f^{\omega^{n-1}}(f^n(y)) = f^{n,\omega^{n-1}}(y) = f^{W^n}(x)$$

is a fixed point of f. If $f(y) = y + e_n$ with $y_n = 1$ the proof is similar. □

4 Refinements and extensions

4.1 Conjunctive networks

We now determine the maximum fixing length over all n-component conjunctive networks. Clearly the maximum is equal to one if $n = 1$ and to two if $n = 2$. To settle the case $n \geq 3$ and characterize the extremal networks, we need additional definitions. Let C_n denote the n-vertex cycle (there is an arc from i to $i+1$ for all $1 \leq i < n$, and an arc from n to 1). We denote by C_n° the graph obtained from C_n by adding an arc (i,i) for all $i \in [n]$; these additional arcs are called **loops**. A strongly connected component (strong component for short) in a graph G is **initial** if there is no arc from a vertex outside the component to a vertex inside the component.

Theorem 14. *For all $n \geq 3$ and $f \in F_C(n)$,*

$$\lambda(f) \leq 2n - 2,$$

with equality if and only if the interaction graph of f is isomorphic to C_n°.

Proof. Suppose that $n \geq 2$. Let G be a graph on $[n]$, and let f be the conjunctive network on G. A **spanning in-tree** S in G rooted at i is a spanning connected subgraph of G such that all vertices $j \neq i$ have out-degree one in S, and i has out-degree zero in S. A **spanning out-tree** is defined similarly. It is clear that if G is strongly connected (strong for short), then for any vertex i there exists a spanning in-tree of G rooted at i (and similarly for out-trees). A vertex l with in-degree zero in a spanning in-tree S is referred to as a leaf of S. We denote the maximum number of leaves of a spanning in-tree of G as $\phi(G)$.

We first prove the theorem when G is strong. In that case, f has exactly two fixed points: $\mathbf{0}$ and $\mathbf{1}$.

Claim 1. *If G is strong, then $\lambda(f) \leq 2n - \phi(G) - 1$.*

Proof of Claim 1. Let S be a spanning in-tree of G with $\phi := \phi(G)$ leaves. Let $i_1 i_2 \dots i_n$ be a topological sort of S. The root of S is thus i_n, and its leaves are $i_1 \dots i_\phi$. Let T be a spanning out-tree with the same root as S. Let $j_1 j_2 \dots j_n$ be a topological sort of T, so that its root is $j_1 = i_n$. We claim that the word $w := i_{\phi+1} \dots i_n j_2 \dots j_n$ of length $2n - \phi - 1$ fixes f. Let $u := i_{\phi+1} \dots i_n$ and $x \in \{0,1\}^n$. We set $x^\phi := x$ and $x^k := f^{i_k}(x^{k-1})$ for $\phi < k \leq n$. We claim that if $x^n_{i_n} = 1$, then $f^u(x) = x^n = \mathbf{1}$. For otherwise, suppose $x^n_{i_n} = 1$ and $x^n_{i_k} = 0$ for some $\phi \leq k < n$. Let $i_{r_1} i_{r_2} \dots i_{r_p}$ be the path from $i_k = i_{r_1}$ to $i_{r_p} = i_n$ in S. This path follows the topological order, that is, $r_1 < r_2 < \cdots < r_p$. Clearly, for all $1 \leq q < p$ we have

$$x^n_{i_{r_q}} = 0 \;\Rightarrow\; x^{r_q}_{i_{r_q}} = 0 \;\Rightarrow\; x^{r_{q+1}-1}_{i_{r_q}} = 0 \;\Rightarrow\; x^{r_{q+1}}_{i_{r_{q+1}}} = 0 \;\Rightarrow\; x^n_{i_{r_{q+1}}} = 0.$$

Since $x^n_{i_k} = 0$ we deduce that $x^n_{i_n} = 0$, which is the desired contradiction. Hence, if $x^n_{i_n} = 1$ then $f^u(x) = \mathbf{1}$ and thus $f^w(x) = \mathbf{1}$. Otherwise $x^n_{i_n} = 0$ and it is easily shown by induction on $2 \leq k \leq n$ that $f^{j_2 \dots j_k}_{j_k}(x^n) = 0$, thus $f^{j_2 \dots j_n}(x^n) = f^w(x) = \mathbf{0}$. □

We say that G is a **cycle with loops** if G is isomorphic to a graph obtained from C_n by adding some loops.

Claim 2. *If G is strong and not a cycle with loops, then $\phi(G) \geq 2$ and hence $\lambda(f) \leq 2n - 3$.*

Proof of Claim 2. Since adding loops to a graph maintains the value of ϕ, without loss, suppose that G has no loops. Since G is strong but not a cycle, there exists a vertex i in G with in-degree $d \geq 2$. We can then construct a spanning in-tree rooted at i with at least d leaves as follows. For all $0 \leq k < n$, let U_k be the set of vertices j such that $d_G(j,i) = k$ (i.e. k is the minimum length of a path from j to i in G). Then U_0 only contains i, U_1 is the set of in-neighbors of i, and $U_0 \cup U_1 \cup \cdots \cup U_{n-1} = [n]$. For any $j \in U_k$ with $1 \leq k < n$, let j' be any out-neighbor of j in U_{k-1}. Then the arcs (j, j') for all $j \neq i$ form a spanning in-tree rooted at i with at least $|U_1| = d \geq 2$ leaves. □

Suppose that G is a cycle with loops, and let L be the set of vertices with a loop. Given $l, l' \in L$, we say that l' is the successor of l if none of the internal vertices on the path from l to l' belong to L. The maximum distance in G from a vertex in L to its successor is denoted as $d(G)$. By convention, we let $d(G) := n$ if $|L| = 0$ or $|L| = 1$.

Claim 3. *If G is a cycle with loops, then $\lambda(f) \leq 2n - d(G) - 1$. Therefore, if G is not isomorphic to C°_n, then $\lambda(f) \leq 2n - 3$.*

Proof of Claim 3. Without loss, we assume that G is obtained from C_n by adding some loops. Let us first settle the case where $|L| \leq 1$. If L is empty, then it is easy to see that $1, 2, \dots, n-1$ fixes f. If L is a singleton we may assume, without loss, that n is the only vertex with a loop, and then the same strategy works: $1, 2, \dots, n-1$ fixes f. Henceforth, we assume $|L| \geq 2$. Without loss, suppose that n and $d := d(G)$ both belong to L and that d is the successor of n. Then we claim that the word $w := d+1, d+2, \dots, n, 1, \dots, d, d+1, \dots, n-1$ fixes f. Let $x \in \{0,1\}^n$. Firstly, suppose $x_l = 0$ for some $l \in L$; we note that $d \leq l \leq n$. First of all, the value of x_l will remain zero: $f^{d+1,\dots,l}_l(x) = 0$. Afterwards, the 0 will propagate through the cycle: $f^{d+1,\dots,l,\dots,l-1}(x) = \mathbf{0}$. Secondly, if $x_l = 1$ for all $l \in L$, then it is easy to show that $f^w(x) = \mathbf{1}$. □

The two previous claims show that if G is strong then $\lambda(f) \leq 2n-2$, with a strict inequality when G is not isomorphic to C_n°. The lower bound below thus settles the strong case.

Claim 4. *If G is isomorphic to C_n° then $\lambda(f) \geq 2n-2$.*

Proof of Claim 4. For all $1 \leq i \leq n$ and $0 \leq k < n$, we denote by i_k the vertex at distance k from i in G, and we denote by $x^{i,k}$ the state such that $x^{i,k}_j = 0$ if and only if the distance between i and j is at most k. Thus $x^{i,0} = \mathbf{1} + e_i$ and $x^{i,n-1} = \mathbf{0}$. Furthermore, for all $0 \leq k < n-1$,

$$f(x^{i,k}) = x^{i,k} + e_{i_{k+1}} = x^{i,k+1}.$$

We deduce that if $w = w_1 w_2 \dots w_p$ fixes f, then $f^w(x^{i,0}) = \mathbf{0}$ and, necessarily, $i_1 i_2 \dots i_{n-1}$ is a subsequence of w for all i. Let i be the last index to appear in w, then $i = w_q$ for some $q \geq n$; then the word $i_1 i_2 \dots i_{n-1}$ begins in position q of w and does not end before position $q + n - 2 \geq 2n-2$. Hence $\lambda(f) \geq 2n-2$. □

It remains to settle the non-strong case. We first establish an upper-bound on $\lambda(f)$ that depends on the decomposition of G in strong components. Let $\psi_1(G)$ be the number of initial strong components containing a single vertex without a loop, let $\psi_2(G)$ be the number of initial strong components containing a single vertex with a loop, let $\psi_3(G)$ be the number of initial strong components with at least two vertices, and let $\psi_4(G)$ be the number of non-initial strong components.

Claim 5. *$\lambda(f) \leq 2n - \psi_1(G) - 2\psi_2(G) - 2\psi_3(G) - \psi_4(G)$.*

Proof of Claim 5. Let $I_1, \dots, I_k$ denote the strong components of G in the topological order, and let $n^l := |I_l|$. We then consider a word w^l that fixes the conjunctive network on $G[I_l]$.

1. If I_l is an initial strong component containing a single vertex without a loop, then $w^l := i$; w^l has length $2n^l - 1$.

2. If I_l is an initial strong component containing a single vertex with a loop, then w^l is the empty word; w^l has length $2n^l - 2$.

3. If I_l is an initial strong component with at least two vertices we consider two cases. If $G[I_l]$ is not a cycle with loops, then w^l is the word described in the proof of Claim 1. If $G[I_l]$ is a cycle with loops, then w^l is the word described in the proof of Claim 3. In both case, w^l has length at most $2n^l - 2$.

4. Otherwise, I_l is a non-initial strong component. Let S be a spanning in-tree of $G[I_l]$ and let $i_1 i_2 \dots i_n$ be a topological sort of S. The root of S is thus i_n. Let T be a spanning out-tree with the same root, and let $j_1 j_2 \dots j_n$ be a topological sort of T, so that $j_1 = i_n$. Then $w^l := i_1 \dots i_n j_2 \dots j_n$; w^l has length $2n^l - 1$.

Then, by induction on l, it is easily proved that $f^{w^1 w^2 \dots w^l}$ fixes the conjunctive network on the subgraph of G induced by $I_1 \cup I_2 \cup \dots \cup I_l$. Thus $w := w^1 w^2 \dots w^k$ fixes f and has length at most $2n - 2\psi_2(G) - 2\psi_3(G) - \psi_1(G) - \psi_4(G)$. □

We can finally prove that $\lambda(f) \leq 2n-3$ if G is not strong.

Claim 6. *If G is not strong and $n \geq 3$, then $\lambda(f) \leq 2n-3$.*

Proof of Claim 6. Suppose first that $\psi_4(G) = 0$ (then G is the disjoint union of strong graphs). If $\psi_2(G) + \psi_3(G) \geq 2$ then $\lambda(f) \leq 2n-4$, and if $\psi_2(G) + \psi_3(G) = 1$ then $\psi_1(G) \geq 1$, since G is not strong, and thus $\lambda(f) \leq n-3$. Finally, if $\psi_2(G) + \psi_3(G) = 0$ then $\psi_1(G) \geq 3$ since $n \geq 3$, and thus $\lambda(f) \leq n-3$. Suppose now that $\psi_4(G) \geq 1$. If $\psi_1(G) \geq 2$ or $\psi_2(G) \geq 1$ or $\psi_3(G) \geq 1$ or $\psi_4(G) \geq 2$ then $\lambda(f) \leq 2n-3$. So assume that $\psi_1(G) = \psi_4(G) = 1$ and $\psi_2(G) = \psi_3(G) = 0$. This means that G is connected, has a unique initial strong component containing a single vertex without a loop, and has a unique non-initial strong component, with at least two vertices, since $n \geq 3$. Suppose, without loss, that n is the vertex of the initial strong component, and let $x \in \{0,1\}^n$. Since f_n is the empty conjunction, we have $f_n(x) = 1$. Let h be the conjunctive network on the (strong) graph H obtained from G by removing vertex n. Then for any word u we have $f^{n,u}(x) = (h^u(x_1, \dots, x_{n-1}), 1)$. Thus, let u be the word of length at most $2(n-1)-2$ fixing h from the proof of Claim 1 (if H is not a cycle with loops) or Claim 3 (otherwise). Then $w = n, u$ is a word of length at most $2n-3$ fixing f. □

□

We remark that we can strengthen the upper bound for specific graphs. In particular, if G is undirected and connected, then there are lower bounds on the maximum number of leaves of a spanning tree for G (see [6] for instance).

4.2 Monotone networks with a given interaction graph

We now refine Theorem 13 for $F_M(G)$, the family of monotone networks whose interaction graph is contained in a graph G. The main result is that, for fixed transversal number, the fixing length of $F_M(G)$ is linear in the number of vertices. The statement needs additional definitions.

For $i \geq 0$ and $\alpha \geq 0$, a word on $[\alpha + i]$ is **(i, α)-complete** if it contains, as subsequences, all the permutations $j_1, \dots, j_{\alpha+i}$ of $[\alpha + i]$ such that, for all $1 \leq \ell < \alpha + i$, if $j_\ell, j_{\ell+1} \in [\alpha]$ then $j_\ell < j_{\ell+1}$. We denote by $\lambda(i, \alpha)$ the length of a shortest (i, α)-complete word. Thus $\lambda(0, \alpha) = \alpha$. Furthermore, for $i > 0$, we have $\lambda(i, \alpha) \leq \lambda(\alpha + i)$, with equality if and only if $\alpha \in \{0, 1\}$. In a graph G, a **1-feedback vertex set** is a set of vertices I such that all the cycles of $G \setminus I$ are loops (i.e. cycles of length one). The **1-transversal number** τ_1 of G is the minimum size of a 1-feedback vertex set. Clearly, $\tau_1 \leq \min(\tau, n-1)$.

Theorem 15. *For any graph G with n vertices and 1-transversal number τ_1,*

$$\lambda_M(G) \leq n + \sum_{i=1}^{\tau_1} \lambda(i-1, n-\tau_1) \leq \left(\frac{\tau_1^2}{2} + \frac{3\tau_1}{2} + 1 \right) n.$$

Let K_n be the complete directed graph on n vertices (with n^2 arcs). Since $\tau_1(K_n) = n-1$,

$$\lambda_M(n) = \lambda_M(K_n) \leq n + \sum_{i=1}^{n-1} \lambda(i-1, 1) = n + \sum_{i=1}^{n-1} \lambda(i).$$

Thus Theorem 15 indeed contains Theorem 13. Furthermore, since $\tau_1 \leq \tau$, it also contains Theorem 7 stated in the introduction.

Proof of Theorem 15. Let G be a graph on $[n]$ with 1-transversal number τ_1 and let $\alpha = n - \tau_1$. Let $]\alpha, n] = \{\alpha+1, \dots, n\}$. Without loss, we assume that $]\alpha, n]$ is a 1-feedback vertex set. We also assume that $12 \dots \alpha$ is the topological order of $G[\{1, \dots, \alpha\}]$; this order exists, since all the cycles of $G[\{1, \dots, \alpha\}]$ have length one.

For all $1 \leq i \leq n$, let R_i be the set of vertices reachable from i in $G[\{1, \dots, i\}]$. Thus $R_i = \{i\}$ if $i \leq \alpha$, and $R_i \subseteq [i]$ otherwise. Let P_i be the set of enumerations $j_1 j_2 \dots j_k$ of $R_i \setminus \{i\}$ such that, for all $1 \leq \ell < k$, if $j_\ell, j_{\ell+1} \in [\alpha]$ then $j_\ell < j_{\ell+1}$. Let ω^i be a shortest word containing, as subsequences, all the enumerations contained in P_i. Let $w^i := i, \omega^i$ and

$$W := w^1, \dots, w^n.$$

If $i \in [\alpha]$, then $R_i = \{i\}$, thus $\omega^i = \epsilon$. Furthermore, if $i \in]\alpha, n]$, then $R_i \subseteq [i]$ and we deduce that $|\omega^i| \leq \lambda(i - \alpha - 1, \alpha)$. Thus

$$|W| \leq n + \sum_{i=1}^{\tau_1} \lambda(i-1, n-\tau_1).$$

Let us now prove that W fixes $F_M(G)$. For all $i \in [n]$, let

$$W^i := w^1, \ldots, w^i \quad \text{and} \quad G_i := G[\{1, \ldots, i\}].$$

We prove, by induction on i, that W^i fixes $F_M(G_i)$. This is obvious for $i = 1$. Assume that $i \geq 2$. Let $f \in F_M(G_i)$ and $x \in \{0,1\}^i$. We write $x = (x_{-i}, x_i)$ and set

$$f'(x_{-i}) := f(x_{-i}, x_i)_{-i}.$$

In this way, $f' \in F_M(G_{i-1})$. Let

$$y := f^{W^i}(x).$$

Since $y_{-i} = f'^{W^{i-1}}(x_{-i})$, by induction hypothesis, y_{-i} is a fixed point of f'. We deduce that either y is a fixed point of f, and in that case

$$f^{W^i}(x) = f^{w^i}(f^{W^{i-1}}(x)) = f^{w^i}(y) = y$$

is a fixed point of f, and we are done, or $f(y) = y + e_i$.

So it remains to suppose that $f(y) = y + e_i$ and to prove that $f^{w^i}(y)$ is a fixed point. We consider the case where $y \leq f(y)$, the other case being similar. Let

$$y^0 := y \quad \text{and} \quad y^k := f^{w^i_1 w^i_2 \ldots w^i_k}(y)$$

for all $1 \leq k \leq p$, with $p = |w^i|$. According to Lemma 6 we have

$$y^0 \leq y^1 \leq \cdots \leq y^k \leq f(y^k).$$

Let us prove that $y^p = f^{w^i}(y)$ is a fixed point of f. Let $j_1 j_2 \ldots j_d$ be the ordered sequence of coordinates that turned from 0 to 1 during the sequence $y^0, y^1, \ldots, y^p$. In this way, d is the Hamming distance between y^0 and y^p, and $j_1 = i = w^i_1$. Furthermore,

$$f^{j_1 j_2 \ldots j_d}(y) = y^p.$$

Suppose, for the sake of contradiction, that $f_j(y^p) \neq y^p_j$ for some $1 \leq j \leq i$. Since $y^p \leq f(y^p)$, we must have

$$y^p_j < f_j(y^p).$$

Thus $y^k_j = 0$ for all $0 \leq k \leq p$. Hence, j does not appear in the sequence $j_1 j_2 \ldots j_d$. Let

$$j_{d+1} := j,$$

and let us prove that

$$\{j_1, \ldots, j_d, j_{d+1}\} \subseteq R_i. \tag{2}$$

Since $j_1 = i$ we have $j_1 \in R_i$. We now prove $j_k \in R_i$ with $k \neq 1$. Let y^q be the smallest index $0 \leq q \leq p$ such that $y^q_{j_k} < f_{j_k}(y^q)$. Since $k \neq 1$, $j_k \neq i$, and since $f(y) = y + e_i$, we deduce that $q > 1$. Then, by the choice of q, we have $y^{q-1}_{j_k} = f_{j_k}(y^{q-1})$ and thus $y^{q-1}_{j_k} = y^q_{j_k}$. Hence, $f_{j_k}(y^{q-1}) < f_{j_k}(y^q)$. Thus G has an arc from w_{q-1} to j_k, since w_{q-1} is the unique component that differs between y^{q-1} and y^q. Clearly, $w_{q-1} = j_\ell$ for some $1 \leq \ell < k$. Thus, we have proved that for all j_k with $1 < k \leq d+1$, there exists $1 \leq \ell < k$ such that $j_\ell j_k$ is an arc of G. We deduce that all the j_k with $1 < k \leq d+1$ are reachable from $j_1 = i$. This proves (2).

Furthermore, for all $1 \leq \ell \leq d$, if $j_\ell, j_{\ell+1} \in [\alpha]$ and $j_\ell > j_{\ell+1}$, then

$$f^{j_1 j_2 \ldots j_\ell j_{\ell+1} \ldots j_{d+1}}(y) \leq f^{j_1 j_2 \ldots j_{\ell+1} j_\ell \ldots j_{d+1}}(y)$$

since G has no arc from j_ℓ to $j_{\ell+1}$. Thus, by applying such switches several times, we can reorder the sequence $j_1 j_2 \ldots j_{d+1}$ into a sequence $s_1 s_2 \ldots s_{d+1}$ such that

$$f^{s_1 s_2 \ldots s_d s_{d+1}}(y) \geq f^{j_1 j_2 \ldots j_d j_{d+1}}(y)$$

and such that, for all $1 \leq \ell \leq d$, if $s_\ell, s_{\ell+1} \in [\alpha]$ then $s_\ell < s_{\ell+1}$. In this way, $s_1 = j_1 = i$, and $s_2, \dots, s_{d+1}$ is in P_i. Hence, by definition, $s_2 \dots s_{d+1}$ is a subsequence of ω^i, and thus $s_1 s_2 \dots s_{d+1}$ is a subsequence of w^i. Therefore,

$$y_j^p = y_{j_{d+1}}^p = f_{j_{d+1}}^{\omega^i}(y) \geq f_{j_{d+1}}^{s_1 s_2 \dots s_d s_{d+1}}(y) \geq f_{j_{d+1}}^{j_1 j_2 \dots j_d j_{d+1}}(y) = f_{j_{d+1}}(y^p) = f_j(x^p),$$

a contradiction. Thus W^i fixes f, and thus the whole family $F_M(G_i)$.

Therefore, W fixes $F_M(G)$ and it remains to prove that $|W| \leq (\frac{\tau_1^2}{2} + \frac{3\tau_1}{2} + 1)n$. This follows from the proposition below and an easy computation. □

Proposition 4. *For all $i \geq 0$ and $\alpha \geq 0$ we have $\lambda(i, \alpha) \leq i^2 + i\alpha + \alpha$.*

Proof. Let $\beta := \alpha + i$ and consider the word $w := i \cdot (12 \dots \beta), 12 \dots \alpha$, resulting from the concatenation of i copies of $12 \dots \beta$ and the addition of the suffix $12 \dots \alpha$. Let $u = j_1 j_2 \dots j_\beta$ be a permutation of $[\beta]$ such that, for all $1 \leq \ell < \beta$, if $j_\ell, j_{\ell+1} \in [\alpha]$ then $j_\ell < j_{\ell+1}$. We will prove that w is (i, α)-complete and, for that, it is sufficient to prove that u is contained in w. Let $j_{k_1} \dots j_{k_i}$ be the longest subsequence of u with letters in $[\beta] \setminus [\alpha]$. Then,

$$\begin{array}{rll} j_1 \dots j_{k_1} & \text{is a subsequence of} & (1 \dots \beta) = w_1 \dots w_\beta \\ j_{k_1+1} \dots j_{k_2} & \text{is a subsequence of} & (1 \dots \beta) = w_{\beta+1} \dots w_{2\beta}, \\ & \vdots & \\ j_{k_{i-1}+1} \dots j_{k_i} & \text{is a subsequence of} & (1 \dots \beta) = w_{(i-1)\beta+1} \dots w_{i\beta}, \text{ and} \\ j_{k_i+1} \dots j_\beta & \text{is a subsequence of} & (1 \dots \alpha) = w_{i\beta+1} \dots w_{i\beta+\alpha}. \end{array}$$

Thus u is a subsequence of w. Since $|w| = i^2 + i\alpha + \alpha$, this proves the proposition. □

4.3 Balanced networks

We now consider a class of networks (namely, balanced networks) which is more general than monotone networks. Those are defined by their signed interaction graph, hence we review basic definitions and properties of signed graphs first.

A **signed graph** is a couple (G, σ) where G is a graph, and $\sigma : E \to \{-1, 0, 1\}$ is an arc labelling function, that gives a (positive, negative or null) sign to each arc of G. The **sign** of a cycle in (G, σ) is the product of the signs of its arcs, and (G, σ) is **balanced** if all the cycles are positive. The **signed interaction graph** of an n-component network f is the signed graph (G, σ) where G is the interaction graph of f and where σ is defined for each arc of G from j to i as follows:

$$\sigma(ji) := \begin{cases} 1 & \text{if } f_i(x) \leq f_i(x + e_j) \text{ for all } x \in \{0,1\}^n \text{ with } x_j = 0, \\ -1 & \text{if } f_i(x) \geq f_i(x + e_j) \text{ for all } x \in \{0,1\}^n \text{ with } x_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We see that a network f is monotone if and only if all the arcs are positive. By extension, we say f is **balanced** if its signed interaction graph is balanced. A balanced network can be "decomposed" into monotone networks by considering the decomposition of its interaction graph into strong components, as formally described below.

Given $z \in \{0,1\}^n$, the ***z*-switch** of f is the n-component network f' defined by

$$f'(x) = f(x + z) + z$$

for all $x \in \{0,1\}^n$. For instance, the **1**-switch of f is the dual of f. If f' is the z-switch of f, then f and f' have the same interaction graph G, but their signed interaction graph (G, σ) and (G, σ') may differ, since $\sigma'(ji) = \sigma(ji)$ for all arc ji with $z_j = z_i$ but $\sigma'(ji) = -\sigma(ji)$ for all arc ji with $z_j \neq z_i$. Clearly, if f' is the z-switch of f, then f is the z-switch of f', and we then say that f and f' are **switch-equivalent**.

Proposition 5 ([22]). *Let f be a network with a strongly connected interaction graph. Then f is balanced if and only if f is switch-equivalent to a monotone network.*

The proposition above have immediate consequences on the existence of short words fixing the family $F_B(n)$ of n-component balanced networks. Clearly, if f and f' are switch-equivalent, then any word fixing f fixes f' as well. Therefore, let W^n be a word fixing $F_M(n)$ and consider $n \cdot W^n$ (the word W^n repeated n times). Let $f \in F_B(n)$ and denote the strong components of its interaction graph as $I_1, \dots, I_k$ ($k \leq n$). Since f restricted to each strong component is switch-equivalent to a monotone network, W^n fixes each strong component individually, and thus $l \cdot W^n$ fixes the first l strong components. In particular, $n \cdot W^n$ fixes f. Thus, by Theorem 13, there exists a word of length at most $n^4/3$ fixing $F_B(n)$.

The following theorem refines (and gives a formal proof of) the result above and Theorem 8 stated in the introduction. More precisely, let $n = 3q + r$ with $0 \leq r < 3$, let s be the word $s := 12 \dots n$ and let W^n be any word fixing $F_M(n)$ of minimal length. Then define the word

$$\tilde{W}^n := q \cdot (ssW^n), r \cdot s$$

of length $q(2n + \lambda_M(n)) + rn$.

Theorem 16. *The word $\tilde{W}^n$ fixes $F_B(n)$ for every $n \geq 1$. Therefore,*

$$\lambda_M(n) \leq \lambda_B(n) \leq \frac{n}{3}\lambda_M(n) + n^2.$$

Proof. Let $n = 3q + r$, with $0 \leq r < 3$, and let X^n be a word fixing $F_M(n)$. We prove, more generally, that $\tilde{X}^n := (q \cdot (ssX^n), r \cdot s)$ fixes $F_B(n)$. Let $f \in F_B(n)$ and let G be the interaction graph of f.

The main idea of the proof is that each factor $w := ssX^n$ of $\tilde{X}^n$ fixes at least three new vertices of G. Therefore, $q \cdot w$ fixes at least $3q = n - r$ vertices, and finally $r \cdot s$ fixes the last r vertices if need be.

We formally proceed by induction on n. If $n = 1$ then $s = 1$ fixes f, and if $n = 2$, it is easy to check that $ss = 1212$ fixes f. So we assume that $n \geq 3$. We say that a prefix u of $\tilde{W}^n$ fixes a set of vertices $I \subseteq [n]$ if, for any other prefix v longer than u, we have $f_i^u(x) = f_i^v(x)$ for all $i \in I$. We consider three cases, and in each case, we select a subset I of vertices of size at least three fixed by w.

1. *G has an initial strong component I with at least three vertices.* Then let I be this initial strong component, and let g be the restriction of f on I. Since g is switch-equivalent to a monotone network, X^n fixes g, and thus w fixes I.

2. *G has an initial strong component with two vertices, say $I_1 = \{i, j\}$ with $i < j$.* Again, let g be the restriction of f on I_1. The occurrences of i and j in $ss = 12 \dots n12 \dots, n$ are $ijij$, in that order; this contains iji, which fixes g. Therefore, ss fixes I_1. Suppose, without loss, that $i = n - 1$ and $j = n$, and let h be the $(n - 2)$-component network defined by

$$h(y) := (f_1(y, z)), \dots, f_{n-2}(y, z)) \quad \text{with} \quad z := (f_{n-1}^{ss}(x), f_n^{ss}(x))$$

for all $y \in \{0, 1\}^{n-2}$. Then h is balanced and, by a reasoning similar to the first case, X^n fixes an initial strong component I_2 of the interaction graph of h. Thus, w fixes $I := I_1 \cup I_2$.

3. *All the initial strong components of G have one vertex each.* Note that s fixes all the initial strong components. Therefore, if there are three initial strong components $\{i_1\}, \{i_2\}, \{i_3\}$, then s fixes $I := \{i_1, i_2, i_3\}$ and we are done. If there are two initial strong components $\{i_1\}, \{i_2\}$ then s fixes $I_1 := \{i_1, i_2\}$ and again X^n fixes a non-empty subset I_2 of vertices, as shown in the second case. Thus w fixes $I := I_1 \cup I_2$. There is only one case left: $I_1 = \{i_1\}$ is the only initial strong component. We then consider an initial strong component I_2 of $G \setminus I_1$. If $|I_2| \geq 2$, then s fixes I_1 and X^n fixes I_2. Thus w fixes $I := I_1 \cup I_2$ and we are done. If $I_2 = \{i_2\}$, then ss fixes $I_1 \cup I_2$, and again X^n fixes a non-empty subset of vertices I_3. Thus w fixes $I := I_1 \cup I_2 \cup I_3$ and we are done.

Thus, in any case, there exists a subset I of three vertices fixed by w. Suppose, without loss, that $I = \{n-2, n-1, n\}$. Then, let h be the $(n-3)$-component networks defined by

$$h(y) := (f_1(y,z)), \dots, f_{n-3}(y,z)) \quad \text{with} \quad z := (f^w_{n-2}(x), f^w_{n-1}(x), f^w_n(x))$$

for all $y \in \{0,1\}^{n-3}$. Then h is balanced, and thus, by induction, $\tilde{X}^{n-3}$ fixes h. Consequently, w fixes I, and then $\tilde{X}^{n-3}$ fixes $[n] \setminus I$. Since $\tilde{X}^n = w, \tilde{X}^{n-3}$, we deduce that $\tilde{X}^n$ fixes f. □

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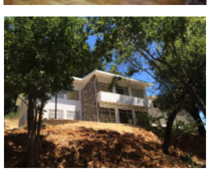
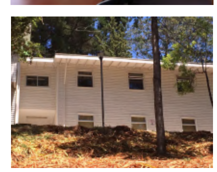

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