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Abstract. We study the dynamics of the solution of a non-linear quantum master equation describing a simple laser under the mean field approximation. The quantum system is formed by a single mode optical cavity and a set of two level atoms that are coupled to two reservoirs. First, we establish the existence of a unique regular stationary state for the non-linear evolution equation under consideration. Second, we examine the free interaction solutions, i.e., the solutions to the non-linear quantum master equation that coincide with unitary evolutions generated by the Hamiltonian resulting from neglecting the interactions between the laser mode, atoms and the bath. We obtain that a family of non-constant free interaction solutions borns at the regular stationary state as a relevant parameter, which is denoted by $C_{\mathfrak{b}}$, passes through the critical value 1. These free interaction solutions yield the periodic solutions of the Maxwell Bloch equations modeling our physical system in the framework of the semiclassical laser theory. Third, in case $C_{\mathfrak{b}} < 1$ we deduce that the system converges exponentially fast to the equilibrium, and so the regular stationary state is a globally asymptotically stable equilibrium solution. Thus, the quantum system has a Hopf bifurcation at $C_{\mathfrak{b}} = 1$.

Keywords: open quantum system, mean-field quantum master equation, laser dynamics, Hopf bifurcation, global attractor, periodic solutions, exponential convergence.

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1. Introduction

This paper develops a physical example of bifurcation in open quantum systems. Namely, we rigorously analyze the qualitative changes in the dynamics of the solution

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to the mean field laser equation

$$
\begin{aligned}
\frac{d}{dt}\rho_t = & -\mathrm{i}\frac{\omega}{2}\left[2\,a^\dagger a+\sigma^3,\rho_t\right]+2\kappa\left(a\,\rho_t a^\dagger-\frac{1}{2}a^\dagger a\rho_t-\frac{1}{2}\rho_t a^\dagger a\right)\\
& +\kappa_-\left(\sigma^-\rho_t\,\sigma^+-\frac{1}{2}\sigma^+\sigma^-\rho_t-\frac{1}{2}\rho_t\,\sigma^+\sigma^-\right)\\
& +\kappa_+\left(\sigma^+\rho_t\,\sigma^--\frac{1}{2}\sigma^-\sigma^+\rho_t-\frac{1}{2}\,\rho_t\,\sigma^-\sigma^+\right)\\
& +g\left[\left(\mathrm{Tr}\left(\sigma^-\rho_t\right)a^\dagger-\mathrm{Tr}\left(\sigma^+\rho_t\right)a\right)+\left(\mathrm{Tr}\left(a^\dagger\rho_t\right)\sigma^--\mathrm{Tr}\left(a\,\rho_t\right)\sigma^+\right),\rho_t\right]
\end{aligned}
\tag{1}
$$

as the parameter $C_{\mathfrak{b}} := 2g^2\left(\kappa_+-\kappa_-\right)/\left(\kappa\left(\kappa_++\kappa_-\right)^2\right)$ varies from $]-\infty,1]$ to $]1,+\infty[$. Here, $\omega\in\mathbb{R}$, g is a non-zero real number, $\kappa,\kappa_+,\kappa_->0$, ρ_t is a non-negative trace-class operator on $\ell^2\left(\mathbb{Z}_+\right)\otimes\mathbb{C}^2$,

$$
\sigma^+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad \sigma^-=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},
$$

and the closed operators $a^\dagger$, a on $\ell^2\left(\mathbb{Z}_+\right)$ are defined by $a^\dagger e_n=\sqrt{n+1}\,e_{n+1}$ for all $n\in\mathbb{Z}_+$ and

$$
ae_n=\begin{cases}\sqrt{n}\,e_{n-1} & \text{if } n\in\mathbb{N}\\ 0 & \text{if } n=0\end{cases},
$$

where $(e_n)_{n\geq 0}$ denotes the standard basis of $\ell^2(\mathbb{Z}_+)$.

The non-linear quantum master equation (1) reproduces the Dicke-Haken-Lax model of the laser (see, e.g., Section 3.7.3 of [1], Section V.E of [2] and [3]). Under the mean field approximation, (1) governs the evolution of a radiation field (with resonance frequency ω) coupled to a set of identical non-interacting two-level systems with transition frequency ω (active medium). The field and the atoms interact weakly with independent reservoirs, causing the photons to leave the resonant mode of the radiation field at rate 2κ and producing the atoms to spontaneously make downward and upward transitions at rates κ_- and κ_+, respectively. The constant g characterizes the coupling between atoms and the field mode. Using (1) we obtain, formally, the closed set of first-order differential equations:

$$
\begin{cases}
\dfrac{d}{dt}\,\mathrm{Tr}\left(a\,\rho_t\right) = -\left(\kappa+\mathrm{i}\omega\right)\mathrm{Tr}\left(a\,\rho_t\right)+g\;\mathrm{Tr}\left(\sigma^-\rho_t\right)\\
\dfrac{d}{dt}\,\mathrm{Tr}\left(\sigma^-\rho_t\right) = -\left(\gamma+\mathrm{i}\omega\right)\mathrm{Tr}\left(\sigma^-\rho_t\right)+g\;\mathrm{Tr}\left(a\,\rho_t\right)\mathrm{Tr}\left(\sigma^3\rho_t\right)\\
\dfrac{d}{dt}\,\mathrm{Tr}\left(\sigma^3\rho_t\right) = -4g\,\Re\left(\mathrm{Tr}\left(a\,\rho_t\right)\,\overline{\mathrm{Tr}\left(\sigma^-\rho_t\right)}\right)-2\gamma\left(\mathrm{Tr}\left(\sigma^3\rho_t\right)-d\right)
\end{cases}
\qquad \forall t\geq 0,
\tag{2}
$$

where $\gamma=\left(\kappa_++\kappa_-\right)/2$ and $d=\left(\kappa_+-\kappa_-\right)/\left(\kappa_++\kappa_-\right)$. In the semiclassical laser theory, the Maxwell-Bloch equations (2) describe the evolution of the field (i.e., $\mathrm{Tr}\left(a\,\rho_t\right)$), the polarization (i.e., $\mathrm{Tr}\left(\sigma^-\rho_t\right)$) and the population inversion (i.e., $\mathrm{Tr}\left(\sigma^3\rho_t\right)$) of ring lasers like far-infrared NH_3 lasers (see, e.g., [4, 5, 6]).

Since the invention of the laser, many experimental and theoretical studies have been undertaken to investigate qualitative properties of the laser dynamics (see, e.g., [5, 7, 8, 9]). Depending on the operating conditions, lasers can show stable or unstable behaviors (see, e.g., [8, 10, 11, 12]). Each laser regime has enabled the development of remarkable applications; for instance, chaotic lasers have been used in secure communications (see, e.g., [8, 13, 14]) and random number generation (see, e.g., [8, 14, 15, 16]). Threshold conditions for the instability of the semiclassical laser equations had been investigated already in the 1960s (see, e.g., [17] and Section 3.4.1 of [7]). In 1975, Haken [18] found an analogy between the Maxwell-Bloch equations for single-mode lasers and the Lorenz equations. From then on, the qualitative behavior of semiclassical laser models has been examined in a number of physical papers by using, for instance, linear stability analysis (see, e.g., [4, 5, 6, 7, 19]). As we recall in Section 2, the Maxwell-Bloch equations (2) develop periodic solutions from the stable fixed point $(0, 0, d)$ as $C_{\mathfrak{b}}$ crosses 1 (see also, e.g., Section 3.7.3 of [1] and [20]). Therefore, (2) undergoes a Hopf bifurcation at $C_{\mathfrak{b}} = 1$.

In this paper we determine how the full quantum dynamics described by (1) yields the bifurcation scenario of (2). Indeed, we prove that (1) has a Hopf bifurcation at $C_{\mathfrak{b}} = 1$. To the best of our knowledge this is the first time that Hopf bifurcation is rigorously established at the level of (infinite dimensional) density matrices in the study of a nonlinear evolution of an open quantum system. In contrast to the semiclassical approach, full quantum models capture very well quantum effects like coherence, spontaneous emissions and photon-number statistics (see, e.g., [4]). This motivates the investigation of the changes in the qualitative behavior of quantum master equations describing open quantum systems, as well as their mean-field approximations. Another important motivation comes from the study of the connections between quantum mechanics and classical chaotic systems, a subject treated in depth by the quantum chaos theory (see, e.g., [21, 22, 23]).

First, we establish the existence and uniqueness of the regular solution to (1), as well as we prove the validity of (2) whenever the initial state is regular enough. Previously, Arnold and Sparber [24] proved the existence and uniqueness of global solutions to a non-linear quantum master equation involving the Hartree potential by means of semigroup techniques.

Second, we study the changes in the invariant sets of the mean field laser equation (1) (namely, stationary and free interaction solutions) as the parameter $C_{\mathfrak{b}}$ varies. We show that

$$\varrho_{\infty} := |e_0\rangle \langle e_0| \otimes \left(\frac{d+1}{2} |e_+\rangle \langle e_+| + \frac{1-d}{2} |e_-\rangle \langle e_-| \right) \tag{3}$$

is the unique regular stationary state for (1) with $\omega \neq 0$, the physical situation we are interested in. This invariant solution yields the unique stationary solution of (2). Moreover, we consider the free interaction solutions to (1) with $\omega \neq 0$, that is, the solutions of (1) that also satisfy

$$\frac{d}{dt}\rho_t = -\mathrm{i}\frac{\omega}{2} \left[2a^{\dagger}a + \sigma^3, \rho_t \right]; \tag{4}$$

the von Neumann equation (4) describes the evolution of the physical system in absence of interactions between the laser mode, atoms and the bath. If $C_{\mathfrak{b}} \leq 1$, then we prove that ϱ_∞ is the unique regular free interaction solution to (1). In case $C_{\mathfrak{b}} > 1$, we obtain that ϱ_∞ splits into ϱ_∞ and a family of non-constant free interaction solutions that yield the periodic solutions of (2). In earlier works, numerical studies of the bifurcation structure of the steady state density operators of quantum master equations in Gorini-Kossakowski-Sudarshan-Lindblad form have been carried out, e.g., by [19, 25, 26]. Moreover, the stationary states for infinite-dimensional quantum Markov semigroups have been treated, e.g., in [27, 28, 29].

Third, we deal with the long time behavior of ρ_t. We show that ρ_t evolves toward ϱ_∞ whenever $\operatorname{Tr}(a\,\rho_0) = \operatorname{Tr}(\sigma^- \rho_0) = 0$. Furthermore, in case $C_{\mathfrak{b}} < 1$, we deduce that ρ_t converges exponentially fast to ϱ_∞ in the trace norm, and so ϱ_∞ is a global attractor for (1). In previous articles, the exponential convergence to the equilibrium state of quantum Markov semigroup has been examined in, e.g., [30, 31, 32].

This paper is organized as follows. Section 2 presents relevant properties of (2). Section 3 is devoted to the main results. Due to its important role in studying (1), we address in Section 4 the linear quantum master equation resulting from replacing in (1) the unknown values of $\operatorname{Tr}(\sigma^- \rho_t)$ and $\operatorname{Tr}(a\,\rho_t)$ by known functions $\alpha(t)$ and $\beta(t)$. To this end, we develop basic properties of general linear master equations by using probabilistic techniques. All proofs are deferred to Section 5.

1.1. Notation

In this article, $(\mathfrak{h}, \langle \cdot, \cdot \rangle)$ is a separable complex Hilbert space whose scalar product $\langle \cdot, \cdot \rangle$ is linear in the second variable and anti-linear in the first one. The canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$ is denoted by $(e_n)_{n\geq 0}$, as well as $e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the standard basis of $\mathbb{C}^2$. We write $\mathcal{D}(A)$ for the domain of A, whenever A is a linear operator in $\mathfrak{h}$. As usual, we set $[A, B] = AB - BA$ in case A, B are linear operators in $\mathfrak{h}$, and $N = a^\dagger a$. If $\mathfrak{X}$, $\mathfrak{Z}$ are normed spaces, then we denote by $\mathfrak{L}(\mathfrak{X}, \mathfrak{Z})$ the set of all bounded operators from $\mathfrak{X}$ to $\mathfrak{Z}$ and we define $\mathfrak{L}(\mathfrak{X}) = \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$. By $\mathfrak{L}_1(\mathfrak{h})$ we mean the set of all trace-class operators on $\mathfrak{h}$ equipped with the trace norm.

Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$. For any $x, y \in \mathcal{D}(C)$ we define the graph scalar product $\langle x, y \rangle_C = \langle x, y \rangle + \langle Cx, Cy \rangle$ and the graph norm $\|x\|_C = \sqrt{\langle x, x \rangle_C}$. We use the symbol $L^2(\mathbb{P}, \mathfrak{h})$ to denote the set of all square integrable random variables from $(\Omega, \mathfrak{F}, \mathbb{P})$ to $(\mathfrak{h}, \mathfrak{B}(\mathfrak{h}))$, where $\mathcal{B}(\mathfrak{Y})$ denotes for the set of all Borel set of the topological space $\mathfrak{Y}$. Moreover, $L^2_C(\mathbb{P}, \mathfrak{h})$ stands for the set of all $\xi \in L^2(\mathbb{P}, \mathfrak{h})$ such that $\xi \in \mathcal{D}(C)$ a.s. and $\mathbb{E}\left(\|\xi\|_C^2\right) < \infty$. We define $\pi_C : \mathfrak{h} \to \mathfrak{h}$ by $\pi_C(x) = x$ if $x \in \mathcal{D}(C)$ and $\pi_C(x) = 0$ if $x \notin \mathcal{D}(C)$.

Recall that $\omega \in \mathbb{R}$, $g \in \mathbb{R} \setminus \{0\}$, and $\kappa, \kappa_+, \kappa_- > 0$. To shorten notation, we take $\gamma = (\kappa_+ + \kappa_-)/2$ and $d = (\kappa_+ - \kappa_-)/(\kappa_+ + \kappa_-)$. Then $\kappa_- = \gamma(1-d)$, $\kappa_+ = \gamma(1+d)$

and

$$C_{\mathfrak{b}} = \frac{g^2\, d}{\kappa\, \gamma}.$$

Using $\kappa_-, \kappa_+ > 0$ we deduce that $\gamma > 0$ and $d \in \left]-1, 1\right[$. In what follows, the letter K denotes generic no-negative constants. We will write $K(\cdot)$ for different non-decreasing non-negative functions on the interval $[0, \infty[$ when no confusion is possible.

2. Complex Lorenz equations

Taking $A(t) = \mathrm{Tr}(a\, \rho_t)$, $S(t) = \mathrm{Tr}(\sigma^- \rho_t)$ and $D(t) = \mathrm{Tr}(\sigma^3 \rho_t)$ we rewrite (2) as

$$\begin{cases} \frac{d}{dt} A(t) = -(\kappa + \mathrm{i}\omega) A(t) + g\, S(t) \\ \frac{d}{dt} S(t) = -(\gamma + \mathrm{i}\omega) S(t) + g\, A(t) D(t) \\ \frac{d}{dt} D(t) = -4g\, \Re\left(\overline{A(t)} S(t)\right) - 2\gamma (D(t) - d) \end{cases}, \tag{5}$$

where $t \geq 0$, $D(t) \in \mathbb{R}$ and $A(t), Y(t) \in \mathbb{C}$. The complex Lorenz equation (5) has received much attention in the physical literature (see, e.g., [20, 33]) due to its important role in the description of laser dynamics. For completeness, we next present relevant properties of (5), together with their mathematical proofs.

Theorem 2.1. *Suppose that $d \in \left]-1, 1\right[$, $\omega \in \mathbb{R}$, $g \in \mathbb{R} \setminus \{0\}$ and $\kappa, \gamma > 0$. Then, for every initial condition $A(0) \in \mathbb{C}$, $S(0) \in \mathbb{C}$, $D(0) \in \mathbb{R}$ there exists a unique solution defined on $[0, +\infty[$ to the system (5). Moreover, we have:*

- *If $d < 0$, then for all $t \geq 0$,*

$$\begin{aligned} &4|d|\,|A(t)|^2 + 4|S(t)|^2 + (D(t) - d)^2 \\ &\leq \mathrm{e}^{-2t \min\{\kappa,\gamma\}} \left(4|d|\,|A(0)|^2 + 4|S(0)|^2 + (D(0) - d)^2\right). \end{aligned} \tag{6}$$

- *If $d \geq 0$, then for any $t \geq 0$,*

$$\begin{aligned} &|A(t)|^2 + \frac{g^2}{\gamma\kappa} |S(t)|^2 + \frac{g^2}{4\gamma\kappa} (D(t) - d)^2 \\ &\leq \mathrm{e}^{-t \min\left\{\kappa - \frac{g^2 d}{\gamma}, \gamma - \frac{g^2 d}{\kappa}\right\}} \left(|A(0)|^2 + \frac{g^2}{\gamma\kappa} |S(0)|^2 + \frac{g^2}{4\gamma\kappa} (D(0) - d)^2\right). \end{aligned} \tag{7}$$

- *If $d \geq 0$ and $C_{\mathfrak{b}} < 1$, then for any $t \geq 0$,*

$$\begin{aligned} &|S(t)|^2 + (D(t) - d)^2/4 \\ &\leq \mathrm{e}^{-(1 - C_{\mathfrak{b}}) \min\{\kappa,\gamma\} t} \left(\frac{4\kappa d}{\gamma} |A(0)|^2 + \left(\frac{4\kappa}{\gamma} + 1\right) |S(0)|^2 + \left(\frac{\kappa}{\gamma} + \frac{1}{4}\right) (D(0) - d)^2\right). \end{aligned} \tag{8}$$

- *If $\omega \neq 0$, then $(A(t), S(t), D(t)) = (0, 0, d)$ is the unique constant solution of (5).*

Proof. Deferred to Section 5.1.1. □

In case $g^2 d < \kappa\gamma$, $(0,0,d)$ is an asymptotically stable equilibrium point of (5); in fact, from (6) and (7) it follows that $A(t)$, $S(t)$ and $D(t) - d$ converge exponentially fast to 0 as t goes to $+\infty$. In order to describe periodic solutions of (5) (whenever $g^2 d \geq \kappa\gamma$), it is usual to set $X(t) = \exp(i\omega t) A(t)$, $Y(t) = \exp(i\omega t) S(t)$ and $Z(t) = D(t) - d$ for all $t \geq 0$. Thus, (5) is transformed into

$$\begin{cases} X'(t) = -\kappa X(t) + g Y(t) \\ Y'(t) = dg X(t) - \gamma Y(t) + g X(t) Z(t) \\ Z'(t) = -4g \Re\left(\overline{X(t)}\, Y(t)\right) - 2\gamma Z(t) \end{cases} . \tag{9}$$

Using simple algebraic manipulations we now find the equilibrium points of (9).

Theorem 2.2. *Let the hypotheses of Theorem 2.1 hold. In case $C_{\mathfrak{b}} \leq 1$, $(0,0,0)$ is the unique equilibrium point of (9). If $C_{\mathfrak{b}} > 1$, then the unique constant solutions of (9) are $(X(t), Y(t), Z(t)) = (0,0,0)$ and the family*

$$\left\{ X(t) = z\sqrt{\frac{\gamma}{2\kappa g^2}(dg^2 - \gamma\kappa)},\, Y(t) = z\sqrt{\frac{\gamma\kappa}{2g^4}(dg^2 - \gamma\kappa)},\, Z(t) = \frac{\gamma\kappa}{g^2} - d : |z| = 1 \right\}.$$

Proof. Deferred to Section 5.1.2. □

According to Theorems 2.1 and 2.2 we have that (5) has a Hopf bifurcation at $C_{\mathfrak{b}} = 1$, because periodic solutions of (5) arise from the stable fixed point $(0,0,d)$ as $C_{\mathfrak{b}}$ crosses 1.

3. Quantum Hopf bifurcation

This section presents the main results of the paper. We start by recalling that a density operator ϱ is C-regular if, roughly speaking, $C\varrho C$ is a trace-class operator, where C is a suitable reference operator (see, e.g., [34, 35]).

Definition 3.1. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$. An operator $\varrho \in \mathfrak{L}_1(\mathfrak{h})$ is called density operator iff ϱ is a non-negative operator with unit trace. The non-negative operator $\varrho \in \mathfrak{L}(\mathfrak{h})$ is said to be C-regular iff $\varrho = \sum_{n\in\mathfrak{I}} \lambda_n |u_n\rangle\langle u_n|$ for some countable set $\mathfrak{I}$, summable non-negative real numbers $(\lambda_n)_{n\in\mathfrak{I}}$ and collection $(u_n)_{n\in\mathfrak{I}}$ of elements of $\mathcal{D}(C)$, which together satisfy: $\sum_{n\in\mathfrak{I}} \lambda_n \|Cu_n\|^2 < \infty$. Let $\mathfrak{L}^+_{1,C}(\mathfrak{h})$ denote the set of all C-regular density operators in $\mathfrak{h}$.*

Next, we establish the existence and uniqueness of the regular solution to (1). We also obtain a Ehrenfest-type theorem describing the evolution of the mean values of the observables $a + a^\dagger$, $\sigma^- + \sigma^+$ and σ^3.

Definition 3.2. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. A family $(\rho_t)_{t\geq 0}$ of operators belonging to $\mathfrak{L}^+_{1,C}(\mathfrak{h})$ is called C-weak solution to (1) iff the function $t \mapsto \mathrm{Tr}(a\rho_t)$ is continuous and for all $t \geq 0$ we have*

$$\frac{d}{dt}\mathrm{Tr}(A\rho_t) = \mathrm{Tr}(A\mathcal{L}_\star(\rho_t)\rho_t) \qquad \forall A \in \mathfrak{L}(\mathfrak{h}),$$

where

$$\mathcal{L}_{\star}\left(\widetilde{\varrho}\right)\varrho = -\frac{\mathrm{i}\omega}{2}\left[2a^{\dagger}a+\sigma^{3},\varrho\right] + 2\kappa\left(a\,\varrho a^{\dagger} - \frac{1}{2}a^{\dagger}a\varrho - \frac{1}{2}\varrho a^{\dagger}a\right)$$
$$+\kappa_{-}\left(\sigma^{-}\varrho\,\sigma^{+} - \frac{1}{2}\sigma^{+}\sigma^{-}\varrho - \frac{1}{2}\varrho\,\sigma^{+}\sigma^{-}\right) + \kappa_{+}\left(\sigma^{+}\varrho\,\sigma^{-} - \frac{1}{2}\sigma^{-}\sigma^{+}\varrho - \frac{1}{2}\varrho\,\sigma^{-}\sigma^{+}\right)$$
$$+g\left[\mathrm{Tr}\left(\sigma^{-}\widetilde{\varrho}\right)a^{\dagger} - \mathrm{Tr}\left(\sigma^{+}\widetilde{\varrho}\right)a,\varrho\right] + g\left[\mathrm{Tr}\left(a^{\dagger}\widetilde{\varrho}\right)\sigma^{-} - \mathrm{Tr}\left(a\,\widetilde{\varrho}\right)\sigma^{+},\varrho\right].$$

Theorem 3.1. *Suppose that $\varrho \in \mathfrak{L}^{+}_{1,N^p}\left(\ell^2\left(\mathbb{Z}_+\right)\otimes\mathbb{C}^2\right)$, with $p\in\mathbb{N}$. Then, there exists a unique N^p-weak solution $\left(\rho_t\right)_{t\geq 0}$ to (1) with initial datum ϱ. Moreover, the Maxwell-Bloch equations (2) hold.*

Proof. Deferred to Section 5.3. □

We now ensure the existence and uniqueness of the regular invariant state for (1) provided that $\omega \neq 0$. This stationary state yields the stationary solution of (2), which is $\mathrm{Tr}\left(a\rho_t\right) = \mathrm{Tr}\left(\sigma^{-}\rho_t\right) = 0$ and $\mathrm{Tr}\left(\sigma^{3}\rho_t\right) = d$.

Definition 3.3. *Consider a C-regular density operator ϱ. We say that ϱ is a stationary state for (1) iff $t \mapsto \varrho$ is a constant C-weak solution to (1).*

Theorem 3.2. *Let the density operator ϱ_∞ be defined by (3). Then ϱ_∞ is a stationary state for (1). Moreover, in case $\omega \neq 0$, ϱ_∞ is the unique N-regular density operator which is a stationary state for (1).*

Proof. Deferred to Section 5.4. □

We turn our attention to the regular solutions of (1) that are also unitary evolutions generated by the Hamiltonian $\frac{\omega}{2}\left(2N+\sigma^3\right)$, which arises from neglecting the interactions between the laser mode, atoms and the bath.

Definition 3.4. *Assume that $\left(\rho_t\right)_{t\geq 0}$ is a C-weak solution to (1). We call $\left(\rho_t\right)_{t\geq 0}$ free interaction solution to (1) if and only if*

$$\rho_t = \exp\left(-\mathrm{i}\frac{\omega}{2}\left(2N+\sigma^3\right)t\right)\varrho_0\exp\left(\mathrm{i}\frac{\omega}{2}\left(2N+\sigma^3\right)t\right) \qquad \forall t\geq 0.$$

Remark 3.1. *If $\left(\rho_t\right)_{t\geq 0}$ is a N-regular free interaction solution to (1), then $\left(\rho_t\right)_{t\geq 0}$ also satisfies the quantum master equation*

$$\frac{d}{dt}\rho_t = -\mathrm{i}\frac{\omega}{2}\left[2a^{\dagger}a+\sigma^3,\rho_t\right].$$

Consider (1) with $\omega \neq 0$. The following theorem asserts that a family of non-constant free interaction solutions borns at the regular stationary state as $C_{\mathfrak{b}}$ passes through the bifurcation value 1. These free interaction solutions yield the periodic solutions of (2) whenever $C_{\mathfrak{b}} > 1$.

Notation 3.1. *The coherent vector associated with $\zeta\in\mathbb{C}$ is defined by*

$$e\left(\zeta\right) = \sum_{n\geq 0}\zeta^n e_n/\sqrt{n!},$$

where $\left(e_n\right)_{n\geq 0}$ denotes the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$.

Theorem 3.3. *Let $\omega \neq 0$. Take $C_{\mathfrak{b}} = dg^2/(\gamma\kappa)$. If $C_{\mathfrak{b}} \leq 1$, then (1) does not have non-constant N-regular free interaction solution. In case $C_{\mathfrak{b}} > 1$ all non-constant N-regular free interaction solutions to (1) are:*

$$e^{-\frac{\gamma^2(C_{\mathfrak{b}}-1)}{2g^2}} \left| e\left(\frac{\gamma\sqrt{C_{\mathfrak{b}}-1}}{\sqrt{2}g} e^{-i(\omega t-\theta)}\right)\right\rangle \left\langle e\left(\frac{\gamma\sqrt{C_{\mathfrak{b}}-1}}{\sqrt{2}g} e^{-i(\omega t-\theta)}\right)\right| \otimes \begin{pmatrix} \frac{1}{2}\left(1+\frac{d}{C_{\mathfrak{b}}}\right) & e^{-i(\omega t-\theta)} \frac{\kappa\gamma}{\sqrt{2}g^2}\sqrt{C_{\mathfrak{b}}-1} \\ e^{i(\omega t-\theta)} \frac{\kappa\gamma}{\sqrt{2}g^2}\sqrt{C_{\mathfrak{b}}-1} & \frac{1}{2}\left(1-\frac{d}{C_{\mathfrak{b}}}\right) \end{pmatrix},$$

where θ is any real number belonging to $[0, 2\pi[$.

Proof. Deferred to Section 5.5. □

Remark 3.2. *Suppose that $\omega \neq 0$. According to the proof of Theorem 3.3 we have that the unique constant N-regular free solution to (1) is described by (3).*

Remark 3.3. *The non-constant N-regular free interaction solutions of (1) are periodic.*

Consider the complex Lorenz equation (5) with initial condition $A(0) = S(0) = 0$ and $D(0) \in \mathbb{R}$. Using $A(0) = S(0) = 0$ we obtain $A(t) = S(t) = 0$ for all $t \geq 0$, and so $\frac{d}{dt}(D(t) - d) = -2\gamma(D(t) - d)$. In the language of the Maxwell-Bloch equations (2), $\operatorname{Tr}(a\,\rho_t) = \operatorname{Tr}(\sigma^-\rho_t) = 0$ and $\operatorname{Tr}(\sigma^3\rho_t)$ evolves toward d in case $\operatorname{Tr}(a\,\rho_0) = \operatorname{Tr}(\sigma^-\rho_0) = 0$. Theorem 3.4 below provides a full quantum explanation for this long time behavior.

Theorem 3.4. *Let ϱ be a N-regular density operator in $\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$ such that $\operatorname{Tr}(a\,\varrho) = \operatorname{Tr}(\sigma^-\varrho) = 0$. Suppose that $(\rho_t)_{t\geq 0}$ is the N-weak solution to (1) with initial state ϱ. Then $\operatorname{Tr}(a\,\rho_t) = \operatorname{Tr}(\sigma^-\rho_t) = 0$ for all $t \geq 0$, and*

$$\operatorname{Tr}(|\rho_t - \varrho_\infty|) \leq 12\exp(-\gamma t)(1+|d|) + 4\exp(-\kappa t)\sqrt{\operatorname{Tr}(\varrho N)} \qquad \forall t \geq 0, \qquad (10)$$

with ϱ_∞ defined by (3).

Proof. Deferred to Section 5.6. □

Let $C_{\mathfrak{b}} < 1$. From Section 2 it follows that $(0,0,d)$ is an asymptotically stable equilibrium solution of the the Maxwell-Bloch equations (2). Indeed, $\lim_{t\to+\infty}\operatorname{Tr}(a\rho_t) = 0$, $\lim_{t\to+\infty}\operatorname{Tr}(\sigma^-\rho_t) = 0$, and $\lim_{t\to+\infty}\operatorname{Tr}(\sigma^3\rho_t) = d$. Next, we show that ρ_t converges to the stationary state (3) with exponential rate, and so ϱ_∞ attracts all N-regular density operators. Moreover, we get the limiting behavior of the mean values of N-bounded operators like $N \otimes (\sigma^+ + \sigma^-)$.

Theorem 3.5. *Let $(\rho_t)_{t\geq 0}$ be the N-weak solution to (1) with $\omega \neq 0$ and initial datum $\rho_0 \in \mathfrak{L}^+_{1,N}(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$. Suppose that $C_{\mathfrak{b}} = d\,g^2/(\gamma\,\kappa) < 1$. Then*

$$\operatorname{Tr}(|\rho_t - \varrho_\infty|) \leq K_{sys}(|g|)\exp(-\delta_{sys}\,t) \qquad \forall t \geq 0, \qquad (11)$$

where ϱ_∞ is given by (3),

$$\delta_{sys} = \begin{cases} \min\{\kappa,\gamma\}/2 & \text{if } d < 0 \\ (1 - C_{\mathfrak{b}})\min\{\kappa,\gamma\}/3 & \text{if } d \geq 0 \end{cases},$$

and $K_{sys}(\cdot)$ *is a non-decreasing non-negative function depending on* $\mathrm{Tr}(\rho_0 N)$ *and the parameters* d, κ *and* γ. *If* $\max\left\{\|A\,x\|^2, \|A^\star x\|^2\right\} \leq K\,\|x\|_N^2$ *for all* $x \in \mathcal{D}(N)$, *then for all* $t \geq 0$ *we have*

$$\left|\mathrm{Tr}(\rho_t A) - \frac{d+1}{2}\langle e_0 \otimes e_+, A\,e_0 \otimes e_+\rangle - \frac{1-d}{2}\langle e_0 \otimes e_-, A\,e_0 \otimes e_-\rangle\right| \leq \widetilde{K}_{sys}(|g|)\exp(-\delta_{sys}\,t), \tag{12}$$

where $\widetilde{K}_{sys}(\cdot)$ *is a non-decreasing non-negative function depending on* A, $\mathrm{Tr}(\rho_0 N)$, d, κ *and* γ.

Proof. Deferred to Section 5.7. □

4. Linear quantum master equations

4.1. General linear quantum master equations

This subsection addresses the well-posedness of the non-autonomous linear quantum master equation

$$\frac{d}{dt}\rho_t = G(t)\,\rho_t + \rho_t G(t)^* + \sum_{k=1}^{\infty} L_k(t)\,\rho_t L_k(t)^* \qquad t \geq 0, \tag{13}$$

where ρ_t is a density operator in $\mathfrak{h}$ and $G(t), L_1(t), L_2(t), \ldots$ are linear operators in $\mathfrak{h}$ satisfying (on appropriate domain) $G(t) = -\mathrm{i}H(t) - \frac{1}{2}\sum_{\ell=1}^{\infty} L_\ell(t)^* L_\ell(t)$ with $H(t)$ self-adjoint operator in $\mathfrak{h}$. Similar to [35], we will obtain basic properties of (13) with the help of the linear stochastic evolution equation in $\mathfrak{h}$:

$$X_t(\xi) = \xi + \int_0^t G(s)\,X_s(\xi)\,ds + \sum_{\ell=1}^{\infty}\int_0^t L_\ell(s)\,X_s(\xi)\,dW_s^\ell, \tag{14}$$

where $W^1, W^2, \ldots$ are real valued independent Wiener processes on a filtered complete probability space $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P}\right)$. Suppose that the density operator ρ_0 is C-regular. According to Theorem 3.1 of [35] we have $\rho_0 = \mathbb{E}\,|\xi\rangle\langle\xi|$ for certain $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$. We set

$$\rho_t := \mathbb{E}\,|X_t(\xi)\rangle\,\langle X_t(\xi)|\,, \tag{15}$$

where we use Dirac notation and $X_t(\xi)$ is the unique strong C-solution of (14) (see Definition 4.1). Then ρ_t is a C-regular density operator (see [35] for details).

Hypothesis 1. *There exists a self-adjoint positive operator* C *in* $\mathfrak{h}$ *such that* $\mathcal{D}(C) \subset \mathcal{D}(G(t))$ *and* $\mathcal{D}(C) \subset \mathcal{D}(L_\ell(t))$ *for all* $t \geq 0$, *and* $G(\cdot)\circ\pi_C$ *and* $L_\ell(\cdot)\circ\pi_C$ *are measurable as functions from* $([0,\infty[\,\times\mathfrak{h}, \mathcal{B}([0,\infty[\,\times\mathfrak{h}))$ *to* $(\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$.

Definition 4.1. *Assume Hypothesis 1. Let* $\mathbb{I}$ *be either* $[0,\infty[$ *or* $[0,T]$, *with* $T \in \mathbb{R}_+$. *By strong* C*-solution of (14) with initial condition* ξ, *on the interval* $\mathbb{I}$, *we mean an* $\mathfrak{h}$*-valued adapted process* $(X_t(\xi))_{t\in\mathbb{I}}$ *with continuous sample paths such that for all* $t \in \mathbb{I}$: $\mathbb{E}\,\|X_t(\xi)\|^2 \leq K(t)\,\mathbb{E}\,\|\xi\|^2$, $X_t(\xi) \in \mathcal{D}(C)$ *a.s.*, $\sup_{s\in[0,t]}\mathbb{E}\,\|CX_s(\xi)\|^2 < \infty$, *and*

$$X_t(\xi) = \xi + \int_0^t G(s)\,\pi_C(X_s(\xi))\,ds + \sum_{\ell=1}^{\infty}\int_0^t L_\ell(s)\,\pi_C(X_s(\xi))\,dW_s^\ell \quad a.s.$$

The following theorem, which extends Theorem 4.4 of [35] to the non-autonomous context, asserts that ρ_t given by (15) is a regular solution to (13).

Definition 4.2. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. A family $(\rho_t)_{t\geq 0}$ of C-regular density operators is called C-weak solution to (13) if and only if*

$$\frac{d}{dt}\operatorname{Tr}(A\rho_t) = \operatorname{Tr}\left(A\left(G(t)\rho_t + \rho_t G(t)^* + \sum_{\ell=1}^{\infty} L_\ell(t)\rho_t L_\ell(t)^*\right)\right) \tag{16}$$

for all $A \in \mathfrak{L}(\mathfrak{h})$ and $t \geq 0$.

Hypothesis 2. *Suppose that C satisfies Hypothesis 1, together with:*

(H2.1) *For any $t \geq 0$ and $x \in \mathcal{D}(C)$, $\|G(t)x\|^2 \leq K(t)\|x\|_C^2$.*

(H2.2) *For any $t \geq 0$ and $x \in \mathcal{D}(C)$, $2\Re\langle x, G(t)x\rangle + \sum_{\ell=1}^{\infty}\|L_\ell(t)x\|^2 = 0$.*

(H2.3) *For any initial datum $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$, (14) has a unique strong C-solution on any bounded interval.*

(H2.4) *There exist functions $f_0, f_1, \ldots : [0,\infty[\times [0,\infty[\to [0,\infty[$ such that: (i) f_k is bounded on bounded subintervals of $[0,\infty[\times [0,\infty[$; (ii) $\lim_{s\to t} f_k(s,t) = 0$; and (iii) for all $s,t \geq 0$ and $x \in \mathcal{D}(C)$ we have $\|G(s)x - G(t)x\|^2 \leq f_0(s,t)\|x\|_C^2$ and $\|L_\ell(s)x - L_\ell(t)x\|^2 \leq f_\ell(s,t)\|x\|_C^2$.*

Theorem 4.1. *Let Hypotheses 1 and 2 hold. Assume that ϱ_0 be C-regular, and that $G(t), L_1(t), L_2(t), \ldots$ are closable for all $t \geq 0$. Then ρ_t given by (15) is a C-weak solution to (13). Moreover, for all $t \geq 0$ we have*

$$\rho_t = \rho_0 + \int_0^t \left(G(s)\rho_s + \rho_s G(s)^* + \sum_{\ell=1}^{\infty} L_\ell(s)\rho_s L_\ell(s)^*\right) ds, \tag{17}$$

where we understand the above integral in the sense of Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$.

Proof. Deferred to Section 5.2.1. □

Remark 4.1. *Sufficient conditions for the regularity of the solution to the linear stochastic Schödinger equation (14) (i.e., Hypothesis 2.3) are given, for instance, in [36, 37, 38].*

4.2. Adjoint quantum master equations

The next theorem introduces the operator $\mathcal{T}_t(A)$ that describes the evolution of the observable A at time t in the Heisenberg picture. Roughly speaking, the maps $A \mapsto \mathcal{T}_t(A)$ is the adjoint operator of the application $\varrho \mapsto \rho_t$, where ρ_t is defined by (15).

Hypothesis 3. *Let Hypothesis 1 hold, together with Conditions H2.1 and H2.3. Suppose that*

(H3.1) *For all* $t \geq 0$ *and* $x \in \mathcal{D}(C)$,

$$2\Re \langle x, G(t) x \rangle + \sum_{\ell=1}^{\infty} \|L_\ell(t) x\|^2 \leq K(t) \|x\|^2 .$$

Theorem 4.2. *Assume that Hypothesis 1 and Conditions H2.1 and H2.3 holds. Consider* $A \in \mathfrak{L}(\mathfrak{h})$. *Then, for every* $t \geq 0$ *there exists a unique* $\mathcal{T}_t(A) \in \mathfrak{L}(\mathfrak{h})$ *for which:*

$$\langle x, \mathcal{T}_t(A) y \rangle = \mathbb{E} \langle X_t(x), A X_t(y) \rangle \qquad \forall x, y \in \mathcal{D}(C) . \tag{18}$$

Moreover, $\sup_{t\in[0,T]} \|\mathcal{T}_t(A)\| < \infty$ *for all* $T \geq 0$.

Proof. Deferred to Section 5.2.2. □

Theorem 4.3 below shows that $\mathcal{T}_t(A)$ is the unique possible solution of the adjoint quantum master equation

$$\frac{d}{dt}\mathcal{T}_t(A) = \mathcal{T}_t(A) G(t) + G(t)^* \mathcal{T}_t(A) + \sum_{k=1}^{\infty} L_k(t)^* \mathcal{T}_t(A) L_k(t) . \tag{19}$$

Thus, we generalize Theorem 2.2 of [35] to the non-autonomous framework.

Theorem 4.3. *Let Hypothesis 3 hold, and let* $\mathcal{T}_t(A)$ *be as in Theorem 4.2 with* $A \in \mathfrak{L}(\mathfrak{h})$. *Assume that* $(\mathcal{A}_t)_{t\geq 0}$ *is a family of operators belonging to* $\mathfrak{L}(\mathfrak{h})$ *such that* $\mathcal{A}_0 = A$, $\sup_{s\in[0,t]} \|\mathcal{A}_s\|_{\mathfrak{L}(\mathfrak{h})} < \infty$, *and*

$$\frac{d}{dt} \langle x, \mathcal{A}_t y \rangle = \langle x, \mathcal{A}_t G(t) y \rangle + \langle G(t) x, \mathcal{A}_t y \rangle + \sum_{\ell=1}^{\infty} \langle L_\ell(t) x, \mathcal{A}_t L_\ell(t) y \rangle \tag{20}$$

for all $x, y \in \mathcal{D}(C)$. *Then* $\mathcal{A}_t = \mathcal{T}_t(A)$ *for all* $t \geq 0$.

Proof. Our assertion can be be proved in much the same way as Theorem 2.2 of [35] (see Section B). □

Remark 4.2. *In the autonomous case, [35, 39] obtain sufficient conditions for* $\mathcal{T}_t(A)$ *defined by (18) to be solution of (19). Using semigroup methods, [34, 40, 41, 42] show the existence and uniqueness of solutions to (13) and (19), in the semigroup sense.*

In order to check Condition H2.3 we establish the following extension of Theorem 2.4 of [36].

Hypothesis 4. *Suppose that* C *satisfies Hypothesis 1, together with:*

(H4.1) *For any* $t \geq 0$ *and* $x \in \mathcal{D}(C)$, $\|G(t) x\|^2 \leq K(t) \|x\|_C^2$.

(H4.2) *For every* $\ell \in \mathbb{N}$ *there exists a non-decreasing function* $K_\ell : [0,\infty[\to [0,\infty[$ *satisfying* $\|L_\ell(t) x\|^2 \leq K_\ell(t) \|x\|_C^2$ *for all* $x \in \mathcal{D}(C)$ *and* $t \geq 0$.

(H4.3) *There exists a non-decreasing function* $\alpha : [0,\infty[\to [0,\infty[$ *and a core* $\mathfrak{D}_1$ *of* C^2 *such that for any* $x \in \mathfrak{D}_1$ *we have*

$$2\Re \left\langle C^2 x, G(t) x \right\rangle + \sum_{\ell=1}^{\infty} \|C L_\ell(t) x\|^2 \leq \alpha(t) \|x\|_C^2 \qquad \forall t \geq 0.$$

(H4.4) *There exists a non-decreasing function $\beta : [0,\infty[\to [0,\infty[$ and a core $\mathfrak{D}_2$ of C such that*

$$2\Re \langle x, G(t)\, x\rangle + \sum_{\ell=1}^{\infty} \|L_\ell (t)\, x\|^2 \le \beta(t)\, \|x\|^2 \qquad \forall t \ge 0 \text{ and } \forall x \in \mathfrak{D}_2.$$

Theorem 4.4. *In addition to Hypothesis 4, we assume that $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$ is $\mathfrak{F}_0$-measurable. Then (14) has a unique strong C-solution $(X_t(\xi))_{t\ge 0}$ with initial condition ξ. Moreover,*

$$\mathbb{E}\, \|C X_t(\xi)\|^2 \le K(t) \left(\mathbb{E}\, \|C\xi\|^2 + \mathbb{E}\, \|\xi\|^2 \right).$$

Proof. The same proof of Theorem 2.4 of [36] works for this theorem. □

Remark 4.3. *We will apply Theorem 4.3 to the case: $L_1 = \sqrt{2\kappa} a^\dagger$, $L_2 = \sqrt{\gamma(1-d)}\sigma^+$, $L_3 = \sqrt{\gamma(1+d)}\sigma^-$ and $G(t) = \mathrm{i} H(t) - \frac{1}{2}\sum_{\ell=1}^3 L_\ell L_\ell^*$ with*

$$H(t) = \frac{\omega}{2}\left(2a^\dagger a + \sigma^3\right) + \mathrm{i} g \left(\alpha(t)\, a^\dagger - \overline{\alpha(t)} a\right) + \mathrm{i} g \left(\overline{\beta(t)}\sigma^- - \beta(t)\, \sigma^+\right).$$

Since $G(t) + G(t)^* + \sum_{\ell=1}^3 L_\ell^* L_\ell = 4\kappa^2 I + 2\gamma^2 (1+d^2)\, \sigma_3$, Condition H2.4 of Theorem 2.4 of [36] does not apply to this case. Theorem 4.4 given above asserts that Theorem 2.4 of [36] still hods if we replace the assumption H2.4 of [36] by Hypothesis H4.4.

4.3. Auxiliary linear quantum master equation

This subsection is devoted to the linear evolution equation obtained by replacing in (1) the unknown functions $t \mapsto g\ \mathrm{Tr}(\sigma^- \rho_t)$ and $t \mapsto g\ \mathrm{Tr}(a\, \rho_t)$ by general functions $\alpha, \beta : [0,\infty[\to \mathbb{C}$. More precisely, we study the linear quantum master equation

$$\frac{d}{dt}\rho_t = \mathcal{L}^h_\star\, \rho_t + \left[\alpha(t)\, a^\dagger - \overline{\alpha(t)} a + \overline{\beta(t)}\sigma^- - \beta(t)\, \sigma^+, \rho_t\right], \tag{21}$$

where $\rho_t \in \mathfrak{L}_1^+(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$,

$$\begin{aligned}\mathcal{L}^h_\star\, \varrho = & \left[-\frac{\mathrm{i}\omega}{2}\left(2a^\dagger a + \sigma^3\right), \varrho\right] + 2\kappa \left(a\, \varrho a^\dagger - \frac{1}{2} a^\dagger a \varrho - \frac{1}{2}\varrho a^\dagger a\right) \\ & + \gamma(1-d)\left(\sigma^- \varrho\, \sigma^+ - \frac{1}{2}\sigma^+\sigma^- \varrho - \frac{1}{2}\varrho\, \sigma^+\sigma^-\right) \\ & + \gamma(1+d)\left(\sigma^+ \varrho\, \sigma^- - \frac{1}{2}\sigma^-\sigma^+ \varrho - \frac{1}{2}\varrho\, \sigma^-\sigma^+\right),\end{aligned} \tag{22}$$

$d \in\,]-1,1[$, $\omega \in \mathbb{R}$ and $\kappa, \gamma > 0$. Though the open quantum system (21) deserves attention in its own right, our main objective is to develop key tools for proving the results of Section 3. First, combining Theorems 4.1, 4.3 and 4.4 we obtain the existence and uniqueness of the regular solution to (21).

Theorem 4.5. *Consider (21) with* $\alpha, \beta : [0, \infty[\to \mathbb{C}$ *continuous. Let* ϱ *be* N^p*-regular, where* $p \in \mathbb{N}$*. Then, there exists a unique* N^p*-weak solution* $(\rho_t)_{t\geq 0}$ *to (21) with initial datum* $\rho_0 = \varrho$*. Moreover, for any* $t \geq 0$ *we have*

$$\rho_t = \rho_0 + \int_0^t \left(\mathcal{L}_\star^h \rho_s + \left[\alpha(s)\, a^\dagger - \overline{\alpha(s)} a + \overline{\beta(s)} \sigma^- - \beta(s)\, \sigma^+, \rho_s \right] \right) ds, \tag{23}$$

where the integral of (23) is understood in the sense of Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$*.*

Proof. Deferred to Section 5.2.3. □

Using the Ehrenfest-type theorem given in [36] we describe the evolution of $\operatorname{Tr}(\rho_t\, a)$, $\operatorname{Tr}(\rho_t\, \sigma^-)$, $\operatorname{Tr}(\rho_t\, \sigma^3)$ and $\operatorname{Tr}(\rho_t\, N)$ by means of a system of ordinary differential equations.

Theorem 4.6. *Under the assumptions of Theorem 4.5,*

$$\frac{d}{dt} \operatorname{Tr}(\rho_t\, a) = -(\kappa + \mathrm{i}\omega) \operatorname{Tr}(\rho_t\, a) + \alpha(t), \tag{24a}$$

$$\frac{d}{dt} \operatorname{Tr}\left(\rho_t\, \sigma^-\right) = -(\gamma + \mathrm{i}\omega) \operatorname{Tr}\left(\rho_t\, \sigma^-\right) + \beta(t) \operatorname{Tr}\left(\rho_t\, \sigma^3\right), \tag{24b}$$

$$\frac{d}{dt} \operatorname{Tr}\left(\rho_t\, \sigma^3\right) = -2 \left(\overline{\beta(t)} \operatorname{Tr}\left(\rho_t \sigma^-\right) + \beta(t) \overline{\operatorname{Tr}\left(\rho_t \sigma^-\right)} \right) - 2\gamma \left(\operatorname{Tr}\left(\rho_t \sigma^3\right) - d \right), \tag{24c}$$

and

$$\frac{d}{dt} \operatorname{Tr}(\rho_t N) = -2\kappa \operatorname{Tr}(\rho_t\, N) + 2\Re \left(\overline{\alpha(t)} \operatorname{Tr}(\rho_t\, a) \right). \tag{25}$$

Proof. Deferred to Section 5.2.4. □

We now deal with the long time behavior of (21) in case $\alpha(t)$ and $\beta(t)$ are constant functions.

Theorem 4.7. *Let* $(\rho_t)_{t\geq 0}$ *be the* N*-weak solution of (21) with* $\alpha(t) \equiv \alpha \in \mathbb{C}$*,* $\beta(t) \equiv \beta \in \mathbb{C}$ *and a* N*-regular density operator as initial datum. Then*

$$\operatorname{Tr}\left(\left| \rho_t - \varrho_\infty^f \otimes \varrho_\infty^a \right| \right) \leq 12\, \mathrm{e}^{-\gamma t} (1 + |d|) + \mathrm{e}^{-\kappa t} \left(\frac{2|\alpha|}{\sqrt{\kappa^2 + \omega^2}} + 4\sqrt{\operatorname{Tr}(\varrho_0 N)} \right), \tag{26}$$

where

$$\varrho_\infty^f = \exp\left(-\frac{|\alpha|^2}{\kappa^2 + \omega^2} \right) \left| e\left(\frac{\alpha}{\kappa + \mathrm{i}\omega} \right) \right\rangle \left\langle e\left(\frac{\alpha}{\kappa + \mathrm{i}\omega} \right) \right|$$

with $e(z)$ *given by Notation 3.1, and*

$$\varrho_\infty^a = \begin{pmatrix} \frac{1}{2} + \frac{d\left(\gamma^2 + \omega^2\right)}{2\left(\gamma^2 + \omega^2 + 2|\beta|^2\right)} & \frac{d\beta(\gamma - \mathrm{i}\omega)}{\gamma^2 + \omega^2 + 2|\beta|^2} \\ \frac{d\bar{\beta}(\gamma + \mathrm{i}\omega)}{\gamma^2 + \omega^2 + 2|\beta|^2} & \frac{1}{2} - \frac{d\left(\gamma^2 + \omega^2\right)}{2\left(\gamma^2 + \omega^2 + 2|\beta|^2\right)} \end{pmatrix}.$$

Proof. Deferred to Section 5.2.5. □

According to Theorem 4.5 we have that

$$\begin{cases} \dfrac{d}{dt}\rho_t^h(\varrho) = \mathcal{L}_\star^h \rho_t^h(\varrho) \\ \rho_0^h(\varrho) \quad = \varrho \end{cases} \tag{27}$$

has a unique N^p-weak solution whenever $\varrho \in \mathfrak{L}_{1,N^p}^+(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$, where $p \in \mathbb{N}$. From Theorems 4.1 and 4.3 of [35] it follows that the family of bounded linear operators $\left(\rho_t^h : \mathfrak{L}_{1,N}^+(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2) \to \mathfrak{L}_1(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)\right)_{t\geq 0}$ can be extended uniquely to a one-parameter semigroup of contractions $\left(\rho_t^h(\cdot)\right)_{t\geq 0}$ on $\mathfrak{L}_1(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$, which indeed is a C_0-semigroup as the next theorem shows.

Theorem 4.8. *The family $\left(\rho_t^h\right)_{t\geq 0}$ is a strongly continuous semigroup on bounded linear operators on $\mathfrak{L}_1(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$. Moreover, for any N-regular density operator ϱ we have*

$$\operatorname{Tr}\left(\left|\rho_t^h(\varrho) - \varrho_\infty\right|\right) \leq 12\, e^{-\gamma t}(1 + |d|) + 4\, e^{-\kappa t}\sqrt{\operatorname{Tr}(\varrho N)}, \tag{28}$$

where ϱ_∞ is defined by (3).

Proof. Deferred to Section 5.2.6. □

We also establish a variation of constants formula for (21), which plays a key role in studying the long time behavior of (21) in case $C_\mathfrak{b} < 1$.

Theorem 4.9. *Let $\alpha, \beta : [0, \infty[\to \mathbb{C}$ be continuous functions. Assume that $(\rho_t)_{t\geq 0}$ is a N-weak solution of (21). Then, for all $t \geq s \geq 0$ we have*

$$\rho_t = \rho_{t-s}^h(\rho_s) + \int_s^t \rho_{t-u}^h\left(\left[\left(\alpha(u)\, a^\dagger - \overline{\alpha(u)} a\right) + \left(\overline{\beta(u)}\sigma^- - \beta(u)\,\sigma^+\right), \rho_u\right]\right) du, \tag{29}$$

where $\rho^h(\cdot)$ is given by (27).

Proof. Deferred to Section 5.2.7. □

5. Proofs

5.1. Proofs of theorems from Section 2

5.1.1. Proof of Theorem 2.1

Proof of Theorem 2.1. Fix $A(0) \in \mathbb{C}$, $S(0) \in \mathbb{C}$ and $D(0) \in \mathbb{R}$. Since (5) is an ordinary differential equation with locally Lipschitz coefficients, (5) has a unique solution defined on a maximal interval $[0, T[$ (see, e.g., [43]).

For all $t \in [0, T[$, we set $X(t) = \exp(i\omega t) A(t)$, $Y(t) = \exp(i\omega t) S(t)$ and $Z(t) = D(t) - d$. Thus, (5) becomes (9), and so

$$\frac{d}{dt}|X(t)|^2 = 2\,\Re\left(X'(t)\,\overline{X(t)}\right) = -2\kappa\,|X(t)|^2 + 2g\,\Re\left(Y(t)\,\overline{X(t)}\right)$$

and

$$\begin{cases} \frac{d}{dt}\left|Y(t)\right|^2 = 2dg\,\Re\left(X(t)\,\overline{Y(t)}\right) - 2\gamma\left|Y(t)\right|^2 + 2g\,Z(t)\,\Re\left(X(t)\,\overline{Y(t)}\right) \\ \frac{d}{dt}Z(t)^2 \quad = -4\gamma Z(t)^2 - 8g\,Z(t)\,\Re\left(\overline{X(t)}Y(t)\right) \end{cases}.$$

Hence,

$$4\,\frac{d}{dt}\left|Y(t)\right|^2 + \frac{d}{dt}Z(t)^2 = 8dg\,\Re\left(X(t)\,\overline{Y(t)}\right) - 8\gamma\left|Y(t)\right|^2 - 4\gamma Z(t)^2. \tag{30}$$

Suppose, for a moment, that $d < 0$. Then

$$-4d\,\frac{d}{dt}\left|X(t)\right|^2 + 4\,\frac{d}{dt}\left|Y(t)\right|^2 + \frac{d}{dt}Z(t)^2 = 8d\kappa\,\left|X(t)\right|^2 - 8\gamma\left|Y(t)\right|^2 - 4\gamma Z(t)^2.$$

This gives

$$\begin{aligned} &\frac{d}{dt}\left(-4d\,\left|X(t)\right|^2 + 4\,\left|Y(t)\right|^2 + (Z(t))^2\right) \\ &\leq -\min\{2\kappa, 2\gamma\}\left(-4d\,\left|X(t)\right|^2 + 4\,\left|Y(t)\right|^2 + Z(t)^2\right), \end{aligned}$$

which implies

$$\begin{aligned} &4\left|d\right|\,\left|X(t)\right|^2 + 4\,\left|Y(t)\right|^2 + Z(t)^2 \\ &\leq \exp\left(-2t\,\min\{\kappa,\gamma\}\right)\left(4\left|d\right|\,\left|X(0)\right|^2 + 4\,\left|Y(0)\right|^2 + Z(0)^2\right) \end{aligned} \tag{31}$$

for any $t \in [0, T[$.

On the other hand, assume that $d \geq 0$. Combining

$$\begin{aligned} &\frac{d}{dt}\left|X(t)\right|^2 + \frac{g^2}{4\gamma\kappa}\left(4\,\frac{d}{dt}\left|Y(t)\right|^2 + \frac{d}{dt}Z(t)^2\right) \\ &= 2g\left(1 + \frac{g^2 d}{\gamma\kappa}\right)\Re\left(X(t)\,\overline{Y(t)}\right) - 2\kappa\left|X(t)\right|^2 - 2\frac{g^2}{\kappa}\left|Y(t)\right|^2 - \frac{g^2}{\kappa}Z(t)^2 \end{aligned}$$

with $2\Re\left(X(t)\,\overline{\frac{g}{\kappa}Y(t)}\right) \leq \left|X(t)\right|^2 + \frac{g^2}{\kappa^2}\left|Y(t)\right|^2$ we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\left|X(t)\right|^2 + \frac{g^2}{\gamma\kappa}\left|Y(t)\right|^2 + \frac{g^2}{4\gamma\kappa}Z(t)^2\right) \\ &\leq \left(-\kappa + \frac{g^2 d}{\gamma}\right)\left|X(t)\right|^2 + \left(-\gamma + \frac{g^2 d}{\kappa}\right)\frac{g^2}{\gamma\kappa}\left|Y(t)\right|^2 - 4\gamma\frac{g^2}{4\gamma\kappa}Z(t)^2. \end{aligned}$$

Therefore, for all $t \in [0, T[$ we have

$$\begin{aligned} &\frac{d}{dt}\left(\left|X(t)\right|^2 + \frac{g^2}{\gamma\kappa}\left|Y(t)\right|^2 + \frac{g^2}{4\gamma\kappa}Z(t)^2\right) \\ &\leq -\min\left\{\kappa - \frac{g^2 d}{\gamma}, \gamma - \frac{g^2 d}{\kappa}\right\}\left(\left|X(t)\right|^2 + \frac{g^2}{\gamma\kappa}\left|Y(t)\right|^2 + \frac{g^2}{4\gamma\kappa}Z(t)^2\right). \end{aligned}$$

This yields

$$
\begin{aligned}
&|X(t)|^2 + \frac{g^2}{\gamma\kappa}|Y(t)|^2 + \frac{g^2}{4\gamma\kappa}Z(t)^2 \\
&\le \mathrm{e}^{-t\min\left\{\kappa-\frac{g^2 d}{\gamma},\gamma-\frac{g^2 d}{\kappa}\right\}}\left(|X(0)|^2 + \frac{g^2}{\gamma\kappa}|Y(0)|^2 + \frac{g^2}{4\gamma\kappa}Z(0)^2\right).
\end{aligned}
\tag{32}
$$

Suppose that $T < +\infty$. According to (31) and (32) we have that $\|(A(t), S(t), D(t))\| < K$, where $K > 0$ and $t \in [0, T[$. This contradicts the property $\lim_{t\to T}\|(A(t), S(t), D(t))\| = \infty$. Therefore, $T = +\infty$. Moreover, (31) and (32) lead to (6) and (7), respectively.

We now assume that $d \ge 0$ and $C_{\mathfrak{b}} := g^2 d/(\kappa\gamma) < 1$. Since

$$
2\,\Re\left(X(t)\overline{Y(t)}\right) = \frac{4\,d\,g}{\gamma}\,\Re\left(X(t)\overline{\frac{\gamma}{2\,d\,g}Y(t)}\right) \le \frac{2\,d\,g}{\gamma}|X(t)|^2 + \frac{\gamma}{2\,d\,g}|Y(t)|^2,
$$

from (30) it follows

$$
\frac{d}{dt}\left(4|Y(t)|^2 + Z(t)^2\right) = \frac{8d^2g^2}{\gamma}|X(t)|^2 - \frac{3}{2}\gamma\left(4|Y(t)|^2 + Z(t)^2\right).
$$

Hence,

$$
4|Y(t)|^2 + Z(t)^2 \le \mathrm{e}^{-\frac{3}{2}\gamma t}\left(4|Y(0)|^2 + Z(0)^2\right) + \frac{8d^2g^2}{\gamma}\mathrm{e}^{-\frac{3}{2}\gamma t}\int_0^t \mathrm{e}^{\frac{3}{2}\gamma s}|X(s)|^2\,ds.
$$

As $g^2 d/(\kappa\gamma) < 1$, according to (7) we have

$$
d|X(t)|^2 \le \mathrm{e}^{-(1-C_{\mathfrak{b}})\min\{\kappa,\gamma\}t}\left(d|A(0)|^2 + |S(0)|^2 + \frac{1}{4}(D(0)-d)^2\right),
$$

and so

$$
\begin{aligned}
4|Y(t)|^2 + Z(t)^2 &\le \mathrm{e}^{-\frac{3}{2}\gamma t}\left(4|Y(0)|^2 + Z(0)^2\right) + \frac{8d^2g^2}{\gamma}\mathrm{e}^{-\frac{3}{2}\gamma t}\int_0^t \mathrm{e}^{\frac{3}{2}\gamma s}|X(s)|^2\,ds \\
&\le \mathrm{e}^{-\frac{3}{2}\gamma t}\left(4|Y(0)|^2 + Z(0)^2\right) + 8\kappa\,\mathrm{e}^{-\frac{3}{2}\gamma t}\int_0^t \mathrm{e}^{\frac{3}{2}\gamma s}d|X(s)|^2\,ds.
\end{aligned}
$$

Using that $3\gamma/2 - (1 - C_{\mathfrak{b}})\min\{\kappa,\gamma\} > \gamma/2$ yields

$$
\begin{aligned}
&4|Y(t)|^2 + Z(t)^2 \le \mathrm{e}^{-\frac{3}{2}\gamma t}\left(4|Y(0)|^2 + Z(0)^2\right) \\
&\quad + \frac{16\kappa}{\gamma}\left(d|A(0)|^2 + |S(0)|^2 + \frac{1}{4}(D(0)-d)^2\right)\left(\mathrm{e}^{-(1-C_{\mathfrak{b}})\min\{\kappa,\gamma\}t} - \mathrm{e}^{-\frac{3}{2}\gamma t}\right).
\end{aligned}
$$

This gives (8).

Let $(A(t), S(t), D(t)) = (A, S, D)$ be a constant solution of (5). Then

$$
-(\kappa + \mathrm{i}\omega)A + g\,S = 0, \tag{33a}
$$

$$
-(\gamma + \mathrm{i}\omega)\,S + g\,A\,D = 0, \tag{33b}
$$

$$
-4g\,\Re\left(A\,\overline{S}\right) - 2\gamma(D-d) = 0. \tag{33c}
$$

Combining ($33a$) with ($33b$) we deduce that

$$A\Big(-(\gamma+\mathrm{i}\omega)(\kappa+\mathrm{i}\omega)+g^2\ D\Big)=0. \tag{34}$$

Using $\omega\neq 0$ and $\kappa,\gamma>0$ we get $(\gamma+\mathrm{i}\omega)(\kappa+\mathrm{i}\omega)\notin\mathbb{R}$. Since $D\in\mathbb{R}$,

$$-(\gamma+\mathrm{i}\omega)(\kappa+\mathrm{i}\omega)+g^2\ D\neq 0,$$

and so (34) yields $A=0$. From ($33a$)-($33c$) we obtain $S=0$ and $D=d$. □

5.1.2. Proof of Theorem 2.2

Proof of Theorem 2.2. Suppose that $(X(t),Y(t),Z(t))=(X,Y,Z)\in\mathbb{C}\times\mathbb{C}\times\mathbb{R}$ satisfies (9). Then

$$\begin{cases}-\kappa\, X+g\, Y=0\\ dg\, X-\gamma\, Y+g\, X\, Z=0\\ -4g\,\Re\left(\overline{X}\, Y\right)-2\gamma\, Z=0\end{cases},$$

which is equivalent to the system

$$Y=\frac{\kappa}{g}X, \tag{35a}$$

$$X\left(dg-\frac{\kappa\gamma}{g}+g\, Z\right)=0, \tag{35b}$$

$$|X|^2=-\frac{\gamma}{2\kappa}Z \tag{35c}$$

From ($35b$) we get $X=0$ or $Z=\kappa\gamma/(g^2)-d$. In the former case, ($35a$) and ($35c$) yields $Y=Z=0$. In the latter case, $dg^2-\gamma\kappa\geq 0$ and $|X|^2=\gamma\left(dg^2-\kappa\gamma\right)/\left(2\kappa g^2\right)$. □

5.2. Proofs of theorems from Section 4

5.2.1. Proof of Theorem 4.1 The proof of Theorem 4.1 follows from combining Lemma 5.2, given below, with the arguments used in the proof of Theorem 4.4 of [35] (see Section A). First, we get the weak continuity of the map $t\mapsto AX_t(\xi)$ in case A is relatively bounded by C.

Lemma 5.1. *Let Condition H2.3 of Hypothesis 2 hold. Suppose that* $\xi\in L^2_C(\mathbb{P},\mathfrak{h})$ *and* $A\in\mathfrak{L}\left(\left(\mathcal{D}(C),\|\cdot\|_C\right),\mathfrak{h}\right)$. *Then, for any* $\psi\in L^2(\mathbb{P},\mathfrak{h})$ *and* $t\geq 0$ *we have*

$$\lim_{s\to t}\mathbb{E}\left\langle\psi,AX_s(\xi)\right\rangle=\mathbb{E}\left\langle\psi,AX_t(\xi)\right\rangle. \tag{36}$$

Proof. Consider a sequence of non-negative real numbers $(s_n)_n$ satisfying $s_n\to t$ as $n\to+\infty$. Since $\left(\left(X_{s_n}(\xi),AX_{s_n}(\xi),CX_{s_n}(\xi)\right)\right)_n$ is a bounded sequence in $L^2(\mathbb{P},\mathfrak{h}^3)$, where $\mathfrak{h}^3=\mathfrak{h}\times\mathfrak{h}\times\mathfrak{h}$, there exists a subsequence $\left(s_{n(k)}\right)_k$ such that

$$\left(X_{s_{n(k)}}(\xi),AX_{s_{n(k)}}(\xi),CX_{s_{n(k)}}(\xi)\right)\longrightarrow_{k\to\infty}(Y,U,V) \tag{37}$$

weakly in $L^2(\mathbb{P},\mathfrak{h}^3)$. Define $\mathfrak{M} = \{(\eta, A\eta, C\eta) : \eta \in L^2_C(\mathbb{P},\mathfrak{h})\}$. Thus,

$$\left(X_{s_{n(k)}}(\xi), AX_{s_{n(k)}}(\xi), CX_{s_{n(k)}}(\xi)\right) \in \mathfrak{M} \qquad \forall k \in \mathbb{N}.$$

Since $\mathfrak{M}$ is a linear manifold of $L^2(\mathbb{P},\mathfrak{h}^3)$ closed with respect to the strong topology (see, e.g., proof of Lemma 7.15 of [35]), (37) implies $(Y, U, V) \in \mathfrak{M}$ (see, e.g., Section III.1.6 of [44]). Using $\mathbb{E}\left(\sup_{s\in[0,t+1]} \|X_s(\xi)\|^2\right) < \infty$, together with the dominated convergence theorem we obtain that $\mathbb{E}\left\|X_{s_{n(k)}}(\xi) - X_t(\xi)\right\|^2 \to 0$ as $k \to +\infty$. Hence $Y = X_t(\xi)$, and so $U = AX_t(\xi)$. Therefore, $AX_{s_{n(k)}}(\xi)$ converges to $AX_t(\xi)$ weakly in $L^2(\mathbb{P},\mathfrak{h})$. □

Lemma 5.2. *Assume Hypothesis 2, together with $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$ and $A \in \mathfrak{L}(\mathfrak{h})$. Then, $t \mapsto L_k(t) X_t(\xi)$ is continuous as a map from $[0,+\infty[$ to $L^2(\mathbb{P},\mathfrak{h})$. Moreover,*

$$\begin{aligned} t \mapsto\ & \mathbb{E}\langle G(t) X_t(\xi), AX_t(\xi)\rangle + \mathbb{E}\langle X_t(\xi), AG(t) X_t(\xi)\rangle \\ & + \sum_{\ell=1}^{\infty} \mathbb{E}\langle L_\ell(t) X_t(\xi), AL_\ell(t) X_t(\xi)\rangle \end{aligned}$$

is a continuous function.

Proof. Suppose that $(t_n)_n$ is a sequence of non-negative real numbers satisfying $t_n \to t$ as $n \to +\infty$. By $\mathbb{E}\left(\sup_{s\in[0,t+1]} \|X_s(\xi)\|^2\right) < \infty$ (see, e.g., Th. 4.2.5 of [45]), using the dominated convergence theorem gives $\mathbb{E}\|X_{t_n}(\xi) - X_t(\xi)\|^2 \longrightarrow_{n\to+\infty} 0$, and hence $AX_{t_n}(\xi) \longrightarrow_{n\to\infty} AX_t(\xi)$ in $L^2(\mathbb{P},\mathfrak{h})$. For any $\psi \in L^2(\mathbb{P},\mathfrak{h})$,

$$\begin{aligned} & |\mathbb{E}\langle \psi, G(s) X_s(\xi)\rangle - \mathbb{E}\langle \psi, G(t) X_t(\xi)\rangle| \\ & \le \mathbb{E}\|\psi\| \|G(s) X_s(\xi) - G(t) X_s(\xi)\| + |\mathbb{E}\langle \psi, G(t) X_s(\xi)\rangle - \mathbb{E}\langle \psi, G(t) X_t(\xi)\rangle|, \end{aligned}$$

and so combining Lemma 5.1 with $\mathbb{E}\|G(s) X_s(\xi) - G(t) X_s(\xi)\|^2 \le f_0(s,t)\,\mathbb{E}\|X_s(\xi)\|_C^2$ yields

$$\lim_{s\to t} \mathbb{E}\langle \psi, G(s) X_s(\xi)\rangle = \mathbb{E}\langle \psi, G(t) X_t(\xi)\rangle. \tag{38}$$

Therefore

$$\lim_{n\to\infty} \mathbb{E}\langle G(t_n) X_{t_n}(\xi), AX_{t_n}(\xi)\rangle = \mathbb{E}\langle G(t) X_t(\xi), AX_t(\xi)\rangle \tag{39}$$

(see, e.g., Section III.1.7 of [44]). Analysis similar to that in (38) shows

$$\lim_{s\to t} \mathbb{E}\langle \psi, L_\ell(s) X_s(\xi)\rangle = \mathbb{E}\langle \psi, L_\ell(t) X_t(\xi)\rangle,$$

and hence

$$L_\ell(t_n) X_{t_n}(\xi) \longrightarrow_{n\to\infty} L_\ell(t) X_t(\xi) \qquad \text{weakly in } L^2(\mathbb{P},\mathfrak{h}). \tag{40}$$

According to (39) with A replaced by A^* we have that $t \mapsto \mathbb{E}\langle A^* X_t(\xi), G(t) X_t(\xi)\rangle$ is a continuous function, then so is $t \mapsto \mathbb{E}\langle X_t(\xi), AG(t) X_t(\xi)\rangle$. Moreover, taking $A = I$ in (39) we deduce that $\mathbb{E}\Re\langle X_{t_n}(\xi), G(t_n) X_{t_n}(\xi)\rangle \to_{n\to\infty} \mathbb{E}\Re\langle X_t(\xi), G(t) X_t(\xi)\rangle$. Applying Condition H2.2 we now get

$$\sum_{\ell=1}^{\infty} \mathbb{E}\|L_\ell(t_n) X_{t_n}(\xi)\|^2 \longrightarrow_{n\to\infty} \sum_{\ell=1}^{\infty} \mathbb{E}\|L_\ell(t) X_t(\xi)\|^2. \tag{41}$$

Combining (40) and (41) yields

$$\limsup_{n\to\infty} \mathbb{E} \left\| L_\ell (t_n) X_{t_n} (\xi) \right\|^2 \leq \mathbb{E} \left\| L_\ell (t) X_t (\xi) \right\|^2$$

(see, e.g., proof of Lemma 7.16 of [35] for details) which, together with (40), implies that $L_\ell (t_n) X_{t_n} (\xi)$ converges strongly in $L^2 (\mathbb{P}, \mathfrak{h})$ to $L_\ell (t) X_t (\xi)$ as $n \to \infty$. Therefore, $t \mapsto L_\ell (t) X_t (\xi)$ is continuous as a function from $[0, +\infty[$ to $L^2 (\mathbb{P}, \mathfrak{h})$.

Using Condition H2.2 we obtain that $\sum_{\ell=1}^{n} \mathbb{E} \langle L_\ell (t) X_t (\xi), AL_\ell (t) X_t (\xi) \rangle$ converges to $\sum_{\ell=1}^{\infty} \mathbb{E} \langle L_\ell (t) X_t (\xi), AL_\ell (t) X_t (\xi) \rangle$ as $n \to \infty$ uniformly on any finite interval. Since

$$\mathbb{E} \langle L_\ell (t_n) X_{t_n} (\xi), AL_\ell (t_n) X_{t_n} (\xi) \rangle \longrightarrow_{n\to\infty} \mathbb{E} \langle L_\ell (t) X_t (\xi), AL_\ell (t) X_t (\xi) \rangle ,$$

the map $t \mapsto \sum_{\ell=1}^{\infty} \mathbb{E} \langle L_\ell (t) X_t (\xi), AL_\ell (t) X_t (\xi) \rangle$ is continuous. □

5.2.2. Proof of Theorem 4.2

Proof of Theorem 4.2. For any $x, y \in \mathcal{D}(C)$ we set $[x, y] = \mathbb{E} \langle X_t (x), AX_t (y) \rangle$. According to Definition 4.1 we have

$$|[x, y]| = |\mathbb{E} \langle X_t (x), AX_t (y) \rangle| \leq K (t) \|A\| \|x\| \|y\| \qquad \forall x, y \in \mathcal{D}(C).$$

Since $\mathcal{D}(C)$ is dense in $\mathfrak{h}$, $[\cdot, \cdot]$ can be extended uniquely to a sesquilinear form $[\cdot, \cdot]$ over $\mathfrak{h} \times \mathfrak{h}$ satisfying $|[x, y]| \leq K (t) \|A\| \|x\| \|y\|$ for any $x, y \in \mathfrak{h}$. Hence, there exists a unique bounded operator $\mathcal{T}_t (A)$ on $\mathfrak{h}$ such that $|[x, y]| = \langle x, \mathcal{T}_t (A) y \rangle$ for all x, y in $\mathfrak{h}$. Moreover, $\|\mathcal{T}_t (A)\| \leq K (t) \|A\|$. □

5.2.3. Proof of Theorem 4.5

Proof of Theorem 4.5. Throughout the proof, we consider the following operators in $\mathfrak{h} = \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$:

$$\begin{cases} H (t) = \dfrac{\omega}{2} \left(2a^\dagger a + \sigma^3 \right) + \mathrm{i} \left(\alpha (t) a^\dagger - \overline{\alpha (t)} a \right) + \mathrm{i} \left(\overline{\beta (t)} \sigma^- - \beta (t) \sigma^+ \right) \\ L_1 = \sqrt{2\kappa}\, a, \qquad L_2 = \sqrt{\gamma (1 - d)}\, \sigma^-, \qquad L_3 = \sqrt{\gamma (1 + d)}\, \sigma^+ \\ G (t) = -\mathrm{i} H (t) - \dfrac{1}{2} \displaystyle\sum_{\ell=1}^{3} L_\ell^* L_\ell \end{cases} . \tag{42}$$

First, we will find a N^p-weak solution to (21). To this end, we will verify that $C = N^p$ satisfies Hypothesis 2. Since $L_2, L_3 \in \mathfrak{L}(\mathfrak{h})$, $L_1, L_1^* L_1$ are relatively bounded with respect to N, and $\|H (t) x\|^2 \leq K \max (|\alpha (t)|, |\beta (t)|) \|x\|_N$ for all $x \in \mathcal{D}(N)$, C fulfills Condition H2.1 of Hypothesis 2. According to (42) we have

$$2\Re \langle x, G (t) x \rangle + \sum_{\ell=1}^{\infty} \|L_\ell (t) x\|^2 = 0 \qquad \forall x \in \mathcal{D}(N),$$

and hence Condition H2.2 holds. Condition H2.4 follows from the continuity of α, β.

In order to check Condition H2.3, we denote by $\mathfrak{D}$ the set of all $x \in \mathfrak{h}$ such that $x(n,\eta) := \langle e_n \otimes e_\eta, x\rangle$ is equal to 0 for all combinations of $n \in \mathbb{Z}_+$ and $\eta = \pm$ except a finite number. Consider $x \in \mathfrak{D}$. A careful computation yields

$$\begin{aligned} & 2\Re\left\langle N^{2p}x, G(t)x\right\rangle + \sum_{\ell=1}^{3} \left\|N^p L_\ell x\right\|^2 \\ & = \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\,\Re\left(\alpha(t)\,x(k,\eta)\,\overline{x(k+1,\eta)}\right)\sqrt{k+1}\left((k+1)^{2p} - k^{2p}\right) \\ & \qquad + \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\kappa\left|x(k,\eta)\right|^2 k\left((k-1)^{2p} - k^{2p}\right). \end{aligned} \tag{43}$$

Since

$$\begin{aligned} & \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\,\Re\left(\alpha(t)\,x(k,\eta)\,\overline{x(k+1,\eta)}\right)\sqrt{k+1}\left((k+1)^{2p} - k^{2p}\right) \\ & \quad \le 2\left|\alpha(t)\right| \sum_{k\in\mathbb{Z}_+,\eta=\pm} \left|x(k,\eta)\right|\left|x(k+1,\eta)\right|\phi(k) \\ & \quad \le 2\left|\alpha(t)\right| \sum_{k\in\mathbb{Z}_+,\eta=\pm} \left|x(k,\eta)\right|^2 \phi(k) \end{aligned}$$

with $\phi(k) = \sqrt{k+1}\left((k+1)^{2p} - k^{2p}\right) = \sqrt{k+1}\sum_{j=0}^{2p-1}\binom{2p}{j} k^j$,

$$\begin{aligned} & \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\,\Re\left(\alpha(t)\,x(k,\eta)\,\overline{x(k+1,\eta)}\right)\sqrt{k+1}\left((k+1)^{2p} - k^{2p}\right) \\ & \le \left|\alpha(t)\right| K \sum_{k\in\mathbb{Z}_+,\eta=\pm} \left|x(k,\eta)\right|^2\left(1 + k^{2p-1/2}\right). \end{aligned} \tag{44}$$

Combining (43) with (44) we get

$$2\Re\left\langle N^{2p}x, G(t)x\right\rangle + \sum_{\ell=1}^{3}\left\|N^p L_\ell x\right\|^2 \le K\left|\alpha(t)\right|\left\|x\right\|_{N^p}^2,$$

and so Condition H4.3 of Hypothesis 4 holds because $\mathfrak{D}$ is a core of N^p. Then, applying Theorem 2.4 of [36] (see also Theorem 4.4) gives Condition H2.3, together with

$$\mathbb{E}\left\|X_t(\xi)\right\|_{N^p}^2 \le K(t)\,\mathbb{E}\left\|\xi\right\|_{N^p}^2; \tag{45}$$

here and subsequently, $X_t(\xi)$ is the strong N^p-solution of (14) with $G(t)$, L_1, L_2, L_3 given by (42) and initial condition $\xi \in L^2_{N^p}(\mathbb{P},\mathfrak{h})$. Thus, $G(t)$, L_1, L_2, L_3, described by (42), satisfy Hypothesis 2 with $C = N^p$.

According to Theorem 3.1 of [35] we have that $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$ for certain $\xi \in L^2_{N^p}(\mathbb{P},\mathfrak{h})$. Using Theorem 4.1 we obtain that $\rho_t := \mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|$ satisfies

$$\begin{cases} \dfrac{d}{dt}\operatorname{Tr}(A\rho_t) = \operatorname{Tr}\left(A\left(G(t)\rho_t + \rho_t G(t)^* + \displaystyle\sum_{\ell=1}^{3} L_\ell\rho_t L_\ell^*\right)\right) & \forall A \in \mathfrak{L}(\mathfrak{h}) \\ \rho_0 = \varrho \end{cases}, \tag{46}$$

as well as the relation (23).

Second, we will prove that (21) has at most one N^p-weak solution provided that the initial condition is N^p-regular. Suppose that (46) holds. Taking $A = |y\rangle\langle x|$ in (46) we get

$$\frac{d}{dt}\langle x, \rho_t y\rangle = \langle G(t)^* x, \rho_t y\rangle + \langle x, \rho_t G(t)^* y\rangle + \sum_{\ell=1}^{3} \langle L_\ell^* x, \rho_t L_\ell^* y\rangle \tag{47}$$

for all $x, y \in \mathcal{D}(N^p)$. Relation (47) coincides with (20) with $\mathcal{A}_t$, $G(t)$, L_1, L_2 and L_3 replaced by ρ_t, $G(t)^*$, L_1^*, L_2^* and L_3^*. This suggests us to apply Theorem 4.3 to (47) in order to prove the uniqueness of the solution of (46). To this end, we next deduce that the linear stochastic Schrödinger equation

$$Y_t(\xi) = \xi + \int_0^t G(s)^* Y_s(\xi)\, ds + \sum_{\ell=1}^{3} \int_0^t L_\ell^* Y_s(\xi)\, dW_s^\ell \tag{48}$$

satisfies Hypothesis 4 with $C = N^p$.

Now, we check Hypothesis 4 with $G(t)$, L_1, L_2 and L_3 replaced by $G(t)^*$, L_1^*, L_2^* and L_3^*. Take $C = N^p$. Since $a^\dagger$ is relatively bounded with respect to N, using analysis similar to that in the second paragraph we can check that $G(t)^* = \mathrm{i}H(t) - \frac{1}{2}\sum_{\ell=1}^{3} L_\ell^* L_\ell$ satisfies Condition H4.1 of Hypothesis 4 with $G(t)$ substituted by $G(t)^*$, as well as Condition H4.2 holds with $L_\ell(t)$ replaced by $L_1^* = \sqrt{2\kappa}a^\dagger$, $L_2^* = \sqrt{\gamma(1-d)}\sigma^+$, $L_3^* = \sqrt{\gamma(1+d)}\sigma^-$. On $\mathfrak{D}$ we have

$$\begin{aligned} G(t)^* + (G(t)^*)^* + \sum_{\ell=1}^{3} (L_\ell^*)^* L_\ell^* &= \sum_{\ell=1}^{3} (L_\ell L_\ell^* - L_\ell^* L_\ell) \\ &= 4\kappa^2 I + 2\gamma^2 \left(1 + d^2\right) \sigma_3, \end{aligned}$$

which gives Condition H4.4. For any $x \in \mathfrak{D}$,

$$\begin{aligned} & 2\Re \left\langle N^{2p} x, \mathrm{i}H(t)x \right\rangle \\ &= \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\Re\left(\alpha(t)\, x(k,\eta)\, \overline{x(k+1,\eta)}\right) \sqrt{k+1}\left((k+1)^{2p} - k^{2p}\right) \end{aligned} \tag{49}$$

and

$$\begin{aligned} & \left\langle x, \left(L_1 N^{2p} L_1^* - \frac{1}{2} L_1^* L_1 N^{2p} - \frac{1}{2} N^{2p} L_1^* L_1\right) x\right\rangle \\ &= \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\kappa\, |x(k,\eta)|^2 \left((k+1)^{2p+1} - k^{2p+1}\right). \end{aligned} \tag{50}$$

Since L_2, L_3 are bounded operators with conmute with N^{2p}, using (49) and (50) yields

$$2\Re\left\langle N^{2p} x, G(t)^* x\right\rangle + \sum_{\ell=1}^{3} \left\| N^p L_\ell^* x\right\|^2 \le K(t)\left\| N^p x\right\|^2$$

and hence Condition H4.3 holds. By Theorem 4.4, (48) has a unique strong N^p-solution whenever $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$. It follows from Theorem 4.3 that (47) has at most one solution

$\varrho_t \in \mathfrak{L}(\mathfrak{h})$ satisfying $\varrho_0 = \varrho$. Thus, (21) has a unique N^p-regular solution, which is equal to $\rho_t := \mathbb{E} \left| X_t(\xi) \right\rangle \left\langle X_t(\xi) \right|$. □

5.2.4. Proof of Theorem 4.6

Proof of Theorem 4.6. In the sequel, $G(t)$, $H(t)$, L_1, L_2, L_3 are described by (42). From the proof of Theorem 4.5 it follows that (14) has a unique strong N^p-solution $X_t(\xi)$ for any initial datum $\xi \in L^2_{N^p}(\mathbb{P}, \mathfrak{h})$. In order to establish (24a) we apply Theorem 4.1 of [36] to obtain

$$\begin{aligned} \operatorname{Tr}(a\rho_t) &= \operatorname{Tr}(a\rho_0) + \sum_{\ell=1}^{3} \int_0^t \mathbb{E} \left\langle L_\ell X_s(\xi), a L_\ell X_s(\xi) \right\rangle ds \\ &\quad + \int_0^t \left(\mathbb{E} \left\langle a^\dagger X_s(\xi), G(s) X_s(\xi) \right\rangle + \mathbb{E} \left\langle G(s) X_s(\xi), a X_s(\xi) \right\rangle \right) ds. \end{aligned} \tag{51}$$

Therefore, $t \mapsto \operatorname{Tr}(a\rho_t)$ is a continuous function. Suppose that $x \in \mathfrak{D}$. Since a conmutes with σ^3 and $\sigma^\pm$, using $\left[a, a^\dagger\right] = I$ we deduce that

$$\begin{aligned} \left\langle a^\dagger x, -\mathrm{i} H(s) x \right\rangle + \left\langle -\mathrm{i} H(s) x, a x \right\rangle &= \left\langle x, \mathrm{i} \left[H(s), a \right] x \right\rangle \\ &= \left\langle x, \left[\mathrm{i}\omega\, a^\dagger a - \alpha(t)\, a^\dagger + \overline{\alpha(t)} a, a \right] x \right\rangle \\ &= \left\langle x, \left(-\mathrm{i}\omega\, a + \alpha(t) \right) x \right\rangle \end{aligned}$$

and

$$\begin{aligned} &\sum_{\ell=1}^{3} \left\langle x, \left(L_\ell^\star a L_\ell - \frac{1}{2} a L_\ell^\star L_\ell - \frac{1}{2} L_\ell^\star L_\ell a \right) x \right\rangle \\ &= \left\langle x, \left(L_1^\star a L_1 - \frac{1}{2} a L_1^\star L_1 - \frac{1}{2} L_1^\star L_1 a \right) x \right\rangle = -\kappa \left\langle x, a x \right\rangle. \end{aligned}$$

Because $\mathfrak{D}$ is a core for N, we obtain that for all $x \in \mathcal{D}(N)$,

$$\left\langle a^\dagger x, G(s) x \right\rangle + \left\langle G(s) x, a x \right\rangle + \sum_{\ell=1}^{3} \left\langle L_\ell x, a L_\ell x \right\rangle = \left\langle x, -(\kappa + \mathrm{i}\omega) a x + \alpha(t) x \right\rangle.$$

Then, from (51) it follows that

$$\operatorname{Tr}(a\rho_t) = \operatorname{Tr}(a\rho_0) + \int_0^t \left(-(\kappa + \mathrm{i}\omega) \operatorname{Tr}(a\rho_s) + \alpha(s) \right) ds,$$

which leads to (24a).

Fix $\eta = -$ or $\eta = 3$. According to (46) we have

$$\frac{d}{dt} \operatorname{Tr}(\rho_t \sigma^\eta) = \operatorname{Tr}\left(\sigma^\eta \left(G(t)\, \rho_t + \rho_t G(t)^* + \sum_{\ell=1}^{3} L_\ell \rho_t L_\ell^* \right) \right),$$

and so applying Theorem 3.2 of [35] we deduce that

$$\begin{aligned}\frac{d}{dt}\operatorname{Tr}\left(\rho_t\sigma^\eta\right) &= \operatorname{Tr}\left(\rho_t\left(\sigma^\eta G\left(t\right)+G\left(t\right)^*\sigma^\eta+\sum_{\ell=1}^{3}L_\ell^*\sigma^\eta L_\ell\right)\right)\\ &= \operatorname{Tr}\left(\rho_t\left(-\mathrm{i}\left[\sigma^\eta, H\left(t\right)\right]+\sum_{\ell=1}^{3}\left(L_\ell^*\sigma^\eta L_\ell-\frac{1}{2}\sigma^\eta L_\ell^*L_\ell-\frac{1}{2}L_\ell^*L_\ell\sigma^\eta\right)\right)\right)\\ &= \operatorname{Tr}\left(-\mathrm{i}\rho_t\left[\sigma^\eta,\frac{\omega}{2}\sigma^3+\mathrm{i}\left(\overline{\beta\left(t\right)}\sigma^- -\beta\left(t\right)\sigma^+\right)\right]\right)\\ &\quad+\sum_{\ell=2}^{3}\operatorname{Tr}\left(\rho_t\left(\left(L_\ell^*\sigma^\eta L_\ell-\frac{1}{2}\sigma^\eta L_\ell^*L_\ell-\frac{1}{2}L_\ell^*L_\ell\sigma^\eta\right)\right)\right).\end{aligned}$$

Now, we use the commutation relations

$$\left[\sigma^+,\sigma^-\right]=\sigma^3,\quad\left[\sigma^3,\sigma^+\right]=2\sigma^+,\quad\left[\sigma^-,\sigma^3\right]=2\sigma^-$$

to derive (24*b*) and (24*c*).

Finally, we deal with (25). Using Theorem 4.1 of [36] yields

$$\begin{aligned}\operatorname{Tr}\left(N\rho_t\right) &= \operatorname{Tr}\left(N\rho_0\right)+\int_0^t\mathbb{E}\left(2\,\Re\left\langle NX_s\left(\xi\right),G\left(s\right)X_s\left(\xi\right)\right\rangle\right)ds\\ &\quad+\sum_{\ell=1}^{3}\int_0^t\mathbb{E}\left\langle N^{1/2}L_\ell X_s\left(\xi\right),N^{1/2}L_\ell X_s\left(\xi\right)\right\rangle ds.\end{aligned}\tag{52}$$

For all $x\in\mathfrak{D}$ we have

$$\begin{aligned}&2\,\Re\left\langle Nx,G\left(t\right)x\right\rangle+\sum_{\ell=1}^{3}\left\langle N^{1/2}L_\ell\,x,N^{1/2}L_\ell\,x\right\rangle\\ &=\left\langle x,\left(\mathrm{i}\left[H\left(t\right),N\right]+\sum_{\ell=1}^{3}\left(\frac{1}{2}\left[L_\ell^*,N\right]L_\ell+\frac{1}{2}L_\ell^*\left[N,L_\ell\right]\right)\right)x\right\rangle\\ &=\left\langle x,\left(-\left[\alpha\left(t\right)a^\dagger-\overline{\alpha\left(t\right)}a,N\right]+\kappa\left[a^\dagger,N\right]a+\kappa\,a^\dagger\left[N,a\right]\right)x\right\rangle,\end{aligned}$$

and so

$$2\,\Re\left\langle Nx,G\left(t\right)x\right\rangle+\sum_{\ell=1}^{3}\left\langle N^{1/2}L_\ell\,x,N^{1/2}L_\ell\,x\right\rangle=\left\langle x,\left(\alpha\left(t\right)a^\dagger+\overline{\alpha\left(t\right)}a-2\kappa N\right)x\right\rangle\tag{53}$$

since $\left[N,a^\dagger\right]=a^\dagger$ and $\left[a,N\right]=a$. By $\mathfrak{D}$ is a core for N, (53) holds for all $x\in\mathcal{D}\left(N\right)$, and hence (52) gives

$$\operatorname{Tr}\left(N\rho_t\right)=\operatorname{Tr}\left(N\rho_0\right)+\int_0^t\left(2\,\Re\left(\overline{\alpha\left(t\right)}\mathbb{E}\left\langle X_s\left(\xi\right),aX_s\left(\xi\right)\right\rangle\right)-2\kappa\,\mathbb{E}\left\langle X_s\left(\xi\right),NX_s\left(\xi\right)\right\rangle\right)ds,$$

which implies

$$\operatorname{Tr}\left(N\rho_t\right)=\operatorname{Tr}\left(N\rho_0\right)+\int_0^t\left(2\,\Re\left(\overline{\alpha\left(s\right)}\,\operatorname{Tr}\left(a\rho_s\right)\right)-2\kappa\,\operatorname{Tr}\left(N\rho_s\right)\right)ds.$$

The continuity of $\alpha\left(t\right)$, $\operatorname{Tr}\left(a\rho_t\right)$ and $\operatorname{Tr}\left(N\rho_t\right)$ yields (25). □

5.2.5. Proof of Theorem 4.7

Lemma 5.3. *For any $\rho \in \mathbb{C}^{2\times 2}$ we define*

$$\begin{aligned}\mathcal{L}_\star^a(\varrho) &= -\frac{\mathrm{i}\omega}{2}\left[\sigma^3, \varrho\right] + \gamma(1-d)\left(\sigma^-\varrho\,\sigma^+ - \frac{1}{2}\sigma^+\sigma^-\varrho - \frac{1}{2}\varrho\,\sigma^+\sigma^-\right)\\ &\quad + \gamma(1+d)\left(\sigma^+\varrho\,\sigma^- - \frac{1}{2}\sigma^-\sigma^+\varrho - \frac{1}{2}\varrho\,\sigma^-\sigma^+\right) + \left[\bar{\beta}\sigma^- - \beta\sigma^+, \rho\right],\end{aligned}$$

with $d, \omega \in \mathbb{R}$, $\gamma > 0$ and $\beta \in \mathbb{C}$. Consider the linear ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}\rho_t^a = \mathcal{L}_\star^a(\rho_t^a) & \forall t \geq 0\\ \rho_0^a = \varrho \end{cases}, \tag{54}$$

where $\rho_t^a \in \mathbb{C}^{2\times 2}$. Then, for all $t \geq 0$ we have

$$\operatorname{Tr}\left(\left|\rho_t^a - \operatorname{Tr}(\varrho)\,\varrho_\infty^a\right|\right) \leq 4\exp(-\gamma\, t)\left(\|\varrho\|_F + |d\ \operatorname{Tr}(\varrho)|\right),$$

where $\|\varrho\|_F$ stands for the Frobenius norm of ϱ, and ϱ_∞^a is as in Theorem 4.7.

Proof. Decomposing ρ_t^a in the canonical basis of $\mathbb{C}^{2\times 2}$ we obtain

$$\rho_t^a = \alpha_{++}(t)\left|e_+\right\rangle\left\langle e_+\right| + \alpha_{+-}(t)\left|e_+\right\rangle\left\langle e_-\right| + \alpha_{-+}(t)\left|e_-\right\rangle\left\langle e_+\right| + \alpha_{--}(t)\left|e_-\right\rangle\left\langle e_-\right|,$$

where $\alpha_{\pm\pm}(t)$ and $\alpha_{\pm\mp}(t)$ belong to $\mathbb{C}$. Then

$$\begin{aligned}\frac{d}{dt}\rho_t^a = \mathcal{L}_\star^a(\rho_t^a) &= \alpha_{++}(t)\,\mathcal{L}_\star^a(\left|e_+\right\rangle\left\langle e_+\right|) + \alpha_{+-}(t)\,\mathcal{L}_\star^a(\left|e_+\right\rangle\left\langle e_-\right|)\\ &\quad + \alpha_{-+}(t)\,\mathcal{L}_\star^a(\left|e_-\right\rangle\left\langle e_+\right|) + \alpha_{--}(t)\,\mathcal{L}_\star^a(\left|e_-\right\rangle\left\langle e_-\right|).\end{aligned}$$

Computing explicitly $\mathcal{L}_\star^a(\left|e_\pm\right\rangle\left\langle e_\pm\right|)$ and $\mathcal{L}_\star^a(\left|e_\pm\right\rangle\left\langle e_\mp\right|)$ yields

$$\begin{cases} \dfrac{d}{dt}\alpha_{++}(t) = -\bar{\beta}\,\alpha_{+-}(t) - \beta\,\alpha_{-+}(t) - \gamma(1-d)\,\alpha_{++}(t) + \gamma(1+d)\,\alpha_{--}(t)\\ \dfrac{d}{dt}\alpha_{--}(t) = \bar{\beta}\,\alpha_{+-}(t) + \beta\,\alpha_{-+}(t) + \gamma(1-d)\,\alpha_{++}(t) - \gamma(1+d)\,\alpha_{--}(t)\\ \dfrac{d}{dt}\alpha_{+-}(t) = -(\gamma + \mathrm{i}\omega)\,\alpha_{+-}(t) + \beta\,\alpha_{++}(t) - \beta\,\alpha_{--}(t)\\ \dfrac{d}{dt}\alpha_{-+}(t) = (-\gamma + \mathrm{i}\omega)\,\alpha_{-+}(t) + \bar{\beta}\,\alpha_{++}(t) - \bar{\beta}\,\alpha_{--}(t) \end{cases}.$$

Adding the first two equations we get

$$\alpha_{++}(t) + \alpha_{--}(t) = \alpha_{++}(0) + \alpha_{--}(0) = \operatorname{Tr}(\varrho), \tag{55}$$

and so subtracting the first two equations we deduce that

$$\begin{aligned}\frac{d}{dt}\left(\alpha_{++}(t) - \alpha_{--}(t)\right) &= -2\bar{\beta}\,\alpha_{+-}(t) - 2\beta\,\alpha_{-+}(t)\\ &\quad - 2\gamma\left(\alpha_{++}(t) - \alpha_{--}(t)\right) + 2\gamma d\ \operatorname{Tr}(\varrho).\end{aligned}$$

Therefore

$$\frac{d}{dt}\begin{pmatrix}\alpha_{++}(t)-\alpha_{--}(t)\\ \alpha_{+-}(t)\\ \alpha_{-+}(t)\end{pmatrix}=A\begin{pmatrix}\alpha_{++}(t)-\alpha_{--}(t)\\ \alpha_{+-}(t)\\ \alpha_{-+}(t)\end{pmatrix}+\begin{pmatrix}2\gamma d\ \mathrm{Tr}(\varrho)\\ 0\\ 0\end{pmatrix}, \tag{56}$$

with $A=\begin{pmatrix}-2\gamma & -2\bar{\beta} & -2\beta\\ \beta & -\gamma-\mathrm{i}\omega & 0\\ \bar{\beta} & 0 & -\gamma+\mathrm{i}\omega\end{pmatrix}$.

Solving explicitly (56) gives

$$\begin{pmatrix}\alpha_{++}(t)-\alpha_{--}(t)\\ \alpha_{+-}(t)\\ \alpha_{-+}(t)\end{pmatrix}=\exp(At)\begin{pmatrix}\alpha_{++}(0)-\alpha_{--}(0)\\ \alpha_{+-}(0)\\ \alpha_{-+}(0)\end{pmatrix}-A^{-1}\begin{pmatrix}2\gamma d\ \mathrm{Tr}(\varrho)\\ 0\\ 0\end{pmatrix}$$
$$+A^{-1}\exp(At)\begin{pmatrix}2\gamma d\ \mathrm{Tr}(\varrho)\\ 0\\ 0\end{pmatrix}.$$

Calculating $A^{-1}\begin{pmatrix}2\gamma d\ \mathrm{Tr}(\varrho)\\ 0\\ 0\end{pmatrix}$ we obtain

$$\begin{aligned}&\begin{pmatrix}\alpha_{++}(t)-\alpha_{--}(t)\\ \alpha_{+-}(t)\\ \alpha_{-+}(t)\end{pmatrix}-\frac{d\ \mathrm{Tr}(\varrho)}{\gamma^2+\omega^2+2|\beta|^2}\begin{pmatrix}\gamma^2+\omega^2\\ \beta(\gamma-\mathrm{i}\omega)\\ \bar{\beta}(\gamma+\mathrm{i}\omega)\end{pmatrix}\\ &=\exp(At)\left(\begin{pmatrix}\alpha_{++}(0)-\alpha_{--}(0)\\ \alpha_{+-}(0)\\ \alpha_{-+}(0)\end{pmatrix}-\frac{d\ \mathrm{Tr}(\varrho)}{\gamma^2+\omega^2+2|\beta|^2}\begin{pmatrix}\gamma^2+\omega^2\\ \beta(\gamma-\mathrm{i}\omega)\\ \bar{\beta}(\gamma+\mathrm{i}\omega)\end{pmatrix}\right).\end{aligned} \tag{57}$$

Consider $v\in\mathbb{C}^{3,3}$ and $M=\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}$. Then

$$\begin{aligned}\frac{d}{dt}\langle\exp(At)v,M\exp(At)v\rangle&=\langle\exp(At)v,(A^{\star}M+MA)\exp(At)v\rangle\\ &=-4\gamma\|\exp(At)v\|^2\leq-2\gamma\langle\exp(At)v,M\exp(At)v\rangle,\end{aligned}$$

and hence for all $t\geq 0$,

$$\|\exp(At)v\|^2\leq\langle\exp(At)v,M\exp(At)v\rangle\leq\exp(-2\gamma t)\langle v,Mv\rangle\leq 2\exp(-2\gamma t)\|v\|^2.$$

From (57) it follows

$$\begin{aligned}&\left\|\begin{pmatrix}\alpha_{++}(t)-\alpha_{--}(t)\\ \alpha_{+-}(t)\\ \alpha_{-+}(t)\end{pmatrix}-\frac{d\ \mathrm{Tr}(\varrho)}{\gamma^2+\omega^2+2|\beta|^2}\begin{pmatrix}\gamma^2+\omega^2\\ \beta(\gamma-\mathrm{i}\omega)\\ \bar{\beta}(\gamma+\mathrm{i}\omega)\end{pmatrix}\right\|\\ &\leq\sqrt{2}\exp(-\gamma t)\left(\left\|\begin{pmatrix}\alpha_{++}(0)-\alpha_{--}(0)\\ \alpha_{+-}(0)\\ \alpha_{-+}(0)\end{pmatrix}\right\|+\frac{|d\ \mathrm{Tr}(\varrho)|\sqrt{\gamma^2+\omega^2}}{\sqrt{\gamma^2+\omega^2+2|\beta|^2}}\right)\end{aligned}$$

$$\leq 2\exp\left(-\gamma\, t\right)\left(\left\|\begin{pmatrix}\alpha_{++}\left(0\right)\\ \alpha_{--}\left(0\right)\\ \alpha_{+-}\left(0\right)\\ \alpha_{-+}\left(0\right)\end{pmatrix}\right\| + \left|d\ \mathrm{Tr}\left(\varrho\right)\right|\right).$$

Using (55) we deduce that

$$\mathrm{Tr}\left(\left|\rho_t^a - \frac{\mathrm{Tr}\left(\varrho\right)}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \frac{d\ \mathrm{Tr}\left(\varrho\right)}{\gamma^2+\omega^2+2\left|\beta\right|^2}\begin{pmatrix}\left(\gamma^2+\omega^2\right)/2 & \beta\left(\gamma-\mathrm{i}\omega\right)\\ \bar{\beta}\left(\gamma+\mathrm{i}\omega\right) & -\left(\gamma^2+\omega^2\right)/2\end{pmatrix}\right|\right)$$

$$\leq \sqrt{2}\left\|\begin{pmatrix}\alpha_{++}\left(t\right)\\ \alpha_{--}\left(t\right)\\ \alpha_{+-}\left(t\right)\\ \alpha_{-+}\left(t\right)\end{pmatrix} - \frac{\mathrm{Tr}\left(\varrho\right)}{2}\begin{pmatrix}1\\ 1\\ 0\\ 0\end{pmatrix} - \frac{d\ \mathrm{Tr}\left(\varrho\right)}{\gamma^2+\omega^2+2\left|\beta\right|^2}\begin{pmatrix}\left(\gamma^2+\omega^2\right)/2\\ -\left(\gamma^2+\omega^2\right)/2\\ \beta\left(\gamma-\mathrm{i}\omega\right)\\ \bar{\beta}\left(\gamma+\mathrm{i}\omega\right)\end{pmatrix}\right\|$$

$$\leq 4\exp\left(-\gamma\, t\right)\left(\left\|\begin{pmatrix}\alpha_{++}\left(0\right)\\ \alpha_{--}\left(0\right)\\ \alpha_{+-}\left(0\right)\\ \alpha_{-+}\left(0\right)\end{pmatrix}\right\| + \left|d\ \mathrm{Tr}\left(\varrho\right)\right|\right). \square$$

Lemma 5.4. *Suppose that* $\left(\rho_t^f\right)_{t\geq 0}$ *is the* N*-weak solution to*

$$\begin{cases}\frac{d}{dt}\rho_t^f = \mathcal{L}_\star^f\left(\rho_t^f\right) & \forall t\geq 0\\ \rho_0^f = \varrho\end{cases}, \tag{58}$$

where $\varrho \in \mathfrak{L}_1^+\left(\ell^2\left(\mathbb{Z}_+\right)\right)$ *is a* N*-regular density operator and*

$$\mathcal{L}_\star^f\left(\rho\right) = \left[-\left(\kappa+\mathrm{i}\omega\right)a^\dagger a + \left(\alpha a^\dagger - \bar{\alpha}a\right), \rho\right] + 2\kappa\, a\,\rho a^\dagger$$

with $\kappa > 0$*,* $\alpha \in \mathbb{C}$ *and* $\omega \in \mathbb{R}$*. Then*

$$\mathrm{Tr}\left(\left|\rho_t^f - \varrho_\infty^f\right|\right) \leq 2\mathrm{e}^{-\kappa t}\left(\sqrt{\mathrm{Tr}\left(\varrho\, N\right)} + \left|\alpha\right|/\sqrt{\kappa^2+\omega^2}\right) \qquad \forall t\geq 0,$$

where $e\left(\zeta\right)$ *is defined by Notation 3.1.*

Proof. Consider the unitary Weyl operator $W\left(u\right)$ defined by

$$W\left(u\right)e\left(z\right) = \exp\left(-\left|u\right|^2/2 - \bar{u}z\right)e\left(z+u\right) \qquad \forall z\in\mathbb{C},$$

where $u\in\mathbb{C}$ and $e\left(\cdot\right)$ is given by Notation 3.1 (see, e.g., [46]). Applying the well-known relations

$$W\left(u\right)W\left(-u\right) = I, \quad W\left(u\right)a\,W\left(-u\right) = a - uI, \quad W\left(u\right)a^\dagger\,W\left(-u\right) = a^\dagger - \bar{u}I$$

we obtain $W\left(u\right)a^\dagger a\,W\left(-u\right) = a^\dagger a - ua^\dagger - \bar{u}a + \left|u\right|^2$. Take $v = \alpha/\left(\kappa+\mathrm{i}\omega\right)$. For any $\xi\in L_N^2\left(\mathbb{P},\mathfrak{h}\right)$, $W\left(-v\right)\mathbb{E}\left|\xi\right\rangle\!\left\langle\xi\right|\ W\left(v\right) = \mathbb{E}\left|W\left(-v\right)\xi\right\rangle\!\left\langle W\left(-v\right)\xi\right|$ and

$$\mathbb{E}\left(\left\|N\,W\left(-v\right)\xi\right\|^2\right) \leq \left\|W\left(-v\right)\right\|^2\mathbb{E}\left\|\left(a^\dagger a - va^\dagger - \bar{v}a + \left|v\right|^2\right)\xi\right\|^2 \leq K\left(\left|v\right|\right)\mathbb{E}\left(\left\|\xi\right\|_N^2\right).$$

Hence, the application $\rho \mapsto W(-v)\,\rho\, W(v)$ preserves the property of being N-regular.

For all x in the domain of $a^\dagger a$ we have

$$\sqrt{2\kappa}\, a\, x = W\left(\frac{\alpha}{\kappa + \mathrm{i}\omega}\right) \widetilde{L}\, W\left(-\frac{\alpha}{\kappa + \mathrm{i}\omega}\right) x$$

and

$$-\left(\kappa + \mathrm{i}\omega\right) a^\dagger a\, x + \left(\alpha a^\dagger - \bar{\alpha} a\right) x = W\left(\frac{\alpha}{\kappa + \mathrm{i}\omega}\right) \widetilde{G}\, W\left(-\frac{\alpha}{\kappa + \mathrm{i}\omega}\right) x,$$

where $\widetilde{L} = \sqrt{2\kappa}\, a + \sqrt{2\kappa}\, v\, I$ and

$$\widetilde{G} = -\left(\kappa + \mathrm{i}\omega\right) a^\dagger a - \frac{2\kappa \bar{\alpha}}{\kappa - \mathrm{i}\omega} a + |\alpha|^2 \left(\frac{1}{\kappa - \mathrm{i}\omega} - \frac{2\kappa}{\kappa^2 + \omega^2}\right) I.$$

This gives

$$\begin{aligned} \mathcal{L}_\star^f(\rho) &= W(v)\, \widetilde{G}\, W(-v)\, \rho + \rho\, W(v)\, \widetilde{G}^*\, W(-v) \\ &\quad + W(v)\, \widetilde{L}\, W(-v)\, \rho\, W(v)\, \widetilde{L}^*\, W(-v), \end{aligned} \tag{59}$$

for any N-regular density operator ρ.

Choose $\widetilde{\rho}_t = W(-v)\, \rho_t^f\, W(v)$. Then, the density operator $\widetilde{\rho}_t$ is N-regular. Using (59) we obtain that $(\widetilde{\rho}_t)_{t \geq 0}$ is the N-weak solution to

$$\begin{cases} \dfrac{d}{dt}\widetilde{\rho}_t = \widetilde{\mathcal{L}_\star}(\widetilde{\rho}_t) & \forall t \geq 0 \\ \widetilde{\rho}_0 = W(-v)\, \varrho W(v) \end{cases}, \tag{60}$$

where $\widetilde{\mathcal{L}_\star}(\rho) = \widetilde{G}\, \rho + \rho\, \widetilde{G}^\star + \widetilde{L}\, \rho\, \widetilde{L}^\star$. A computation yields

$$\left\langle e_j, \widetilde{\mathcal{L}_\star}(\rho)\, e_j \right\rangle = -2\kappa j \left\langle e_j, \rho\, e_j \right\rangle + 2\kappa\,(j+1) \left\langle e_{j+1}, \rho\, e_{j+1} \right\rangle \qquad \forall j \geq 0$$

whenever ϱ is a N-regular density operator. Applying (60) we deduce that the functions $\varphi_j(t) := \langle e_j, \widetilde{\rho}_t\, e_j \rangle$ satisfy

$$\varphi_j'(t) = -2\kappa j \varphi_j(t) + 2\kappa(j+1)\varphi_{j+1}(t), \tag{61}$$

which are the Kolmogorov equations for a pure-death process. In case $\varphi_j(0) = \delta_{jn}$ for all $j \geq 0$, the solution of (61) is

$$\varphi_j(t) = \begin{cases} \dbinom{n}{j} \mathrm{e}^{-2\kappa j t} \left(1 - \mathrm{e}^{-2\kappa t}\right)^{n-j} & \text{if } 0 \leq j \leq n \\ 0 & \text{if } j > n \end{cases}.$$

Therefore, $\langle e_0, \widetilde{\rho}_t\, e_0 \rangle = \varphi_0(t) = \sum_{n \geq 0} \varphi_n(0)\,(1 - \mathrm{e}^{-2\kappa t})^n$.

According to [30] we have $\mathrm{Tr}\left(|\widetilde{\rho}_t - |e_0\rangle \langle e_0||\right) \leq 2\left(1 - \langle e_0, \widetilde{\rho}_t e_0 \rangle\right)^{1/2}$. Using that $\varphi_n(0) \geq 0$ and $\sum_{n \geq 0} \varphi_n(0) = 1$ we obtain

$$0 \leq 1 - \langle e_0, \widetilde{\rho}_t e_0 \rangle = \sum_{n \geq 1} \varphi_n(0) \left(1 - \left(1 - \mathrm{e}^{-2\kappa t}\right)^n\right) \leq \mathrm{e}^{-2\kappa t} \sum_{n \geq 1} n\, \varphi_n(0),$$

because $1-(1-x)^n \leq n\,x$ for any $n \in \mathbb{N}$ and $x \in [0,1]$. Hence

$$\operatorname{Tr}\left(\left|\widetilde{\rho}_t - \left|e_0\right\rangle \left\langle e_0\right|\right|\right) \leq 2\mathrm{e}^{-\kappa t}\left(\sum_{n\geq 0}\left\langle n\, e_n, \widetilde{\rho}_0\, e_n\right\rangle\right)^{1/2} = 2\mathrm{e}^{-\kappa t}\left(\sum_{n\geq 0}\left\langle N\, e_n, \widetilde{\rho}_0\, e_n\right\rangle\right)^{1/2},$$

and so

$$Tr\left(\left|\widetilde{\rho}_t - \left|e_0\right\rangle \left\langle e_0\right|\right|\right) \leq 2\mathrm{e}^{-\kappa t}\left(\operatorname{Tr}\left(\widetilde{\rho}_0 N\right)\right)^{1/2} = 2\mathrm{e}^{-\kappa t}\left(\operatorname{Tr}\left(\varrho W\left(v\right) N\, W\left(-v\right)\right)\right)^{1/2}$$

(see, e.g., Theorem 3.2 of [35]). Then

$$\operatorname{Tr}\left(\left|\widetilde{\rho}_t - \left|e_0\right\rangle \left\langle e_0\right|\right|\right) \leq 2\mathrm{e}^{-\kappa t}\left(\operatorname{Tr}\left(\varrho\left(N - v a^{\dagger} - \bar{v} a + |v|^2\right)\right)\right)^{1/2}. \tag{62}$$

Due to $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$ for certain $\xi \in L^2_N\left(\mathbb{P},\mathfrak{h}\right)$,

$$\left|\operatorname{Tr}\left(\varrho\, a^{\dagger}\right)\right| = \left|\mathbb{E}\langle a\,\xi,\xi\rangle\right| \leq \sqrt{\mathbb{E}\left|a\,\xi\right|^2}\sqrt{\mathbb{E}\left|\xi\right|^2} = \sqrt{\mathbb{E}\left|a\,\xi\right|^2} = \sqrt{\mathbb{E}\langle N\,\xi,\xi\rangle} = \sqrt{\operatorname{Tr}\left(\varrho\, N\right)}$$

and

$$\left|\operatorname{Tr}\left(\varrho\, a\right)\right| = \left|\mathbb{E}\langle a^{\dagger}\,\xi,\xi\rangle\right| \leq \sqrt{\mathbb{E}\left|a\,\xi\right|^2} = \sqrt{\operatorname{Tr}\left(\varrho\, N\right)}$$

(see, e.g., Theorem 3.2 of [35]). From (62) we deduce that

$$\operatorname{Tr}\left(\left|\widetilde{\rho}_t - \left|e_0\right\rangle \left\langle e_0\right|\right|\right) \leq 2\mathrm{e}^{-\kappa t}\left(\sqrt{\operatorname{Tr}\left(\varrho\, N\right)} + |v|\right),$$

and consequently

$$\begin{aligned}\operatorname{Tr}\left(\left|\rho^f_t - W\left(v\right)\left|e_0\right\rangle \left\langle e_0\right| W\left(-v\right)\right|\right) &= \operatorname{Tr}\left(\left|W\left(v\right)\left(\widetilde{\rho}_t - \left|e_0\right\rangle \left\langle e_0\right|\right) W\left(-v\right)\right|\right) \\ &\leq \left\|W\left(v\right)\right\| \left\|W\left(-v\right)\right\| \operatorname{Tr}\left(\left|\widetilde{\rho}_t - \left|e_0\right\rangle \left\langle e_0\right|\right|\right) \\ &= \operatorname{Tr}\left(\left|\widetilde{\rho}_t - \left|e_0\right\rangle \left\langle e_0\right|\right|\right) \leq 2\mathrm{e}^{-\kappa t}\left(\sqrt{\operatorname{Tr}\left(\varrho\, N\right)} + |v|\right).\end{aligned}$$

Since $W\left(v\right) e_0 = W\left(v\right) e\left(0\right) = \exp\left(-\left|v\right|^2/2\right) e\left(v\right)$,

$$W\left(v\right)\left|e_0\right\rangle \left\langle e_0\right| W\left(-v\right) = \exp\left(-\left|\alpha/\left(\kappa + \mathrm{i}\omega\right)\right|^2\right)\left|e\left(\frac{\alpha}{\kappa+\mathrm{i}\omega}\right)\right\rangle\left\langle e\left(\frac{\alpha}{\kappa+\mathrm{i}\omega}\right)\right|. \qquad \square$$

Proof of Theorem 4.7. The solution of (54) is denoted by $\rho^a_t\left(\varrho\right)$, and we write $\left(\rho^f_t\right)_{t\geq 0}$ for the semigroup N-solution of the quantum master equation (58) (see [35] for details). Due to ρ_0 is N-regular,

$$\rho_0 = \mathbb{E}\left|\xi_+ \otimes e_+ + \xi_- \otimes e_-\right\rangle\left\langle \xi_+ \otimes e_+ + \xi_- \otimes e_-\right|$$

with $\xi_{\pm} \in L^2_N\left(\mathbb{P},\mathfrak{h}\right)$, and so

$$\rho_0 = \varrho_{++} \otimes \left|e_+\right\rangle \left\langle e_+\right| + \varrho_{+-} \otimes \left|e_+\right\rangle \left\langle e_-\right| + \varrho_{-+} \otimes \left|e_-\right\rangle \left\langle e_+\right| + \varrho_{--} \otimes \left|e_-\right\rangle \left\langle e_-\right|, \tag{63}$$

where $\rho_{\eta\widetilde{\eta}} = \mathbb{E}\,|\xi_\eta\rangle\langle\xi_{\widetilde{\eta}}|$. Since the right-hand term of (21) is equal to $\mathcal{L}^f_\star \otimes I\left(\rho_t\right) + I \otimes \mathcal{L}^a_\star\left(\rho_t\right)$, where $\mathcal{L}^a_\star$ and $\mathcal{L}^f_\star$ are described by Lemmata 5.3 and 5.4, from (63) we obtain

$$\begin{aligned}\rho_t = \rho^f_t\left(\varrho_{++}\right) \otimes \rho^a_t\left(\left|e_+\right\rangle \left\langle e_+\right|\right) + \rho^f_t\left(\varrho_{+-}\right) \otimes \rho^a_t\left(\left|e_+\right\rangle \left\langle e_-\right|\right) \\ + \rho^f_t\left(\varrho_{-+}\right) \otimes \rho^a_t\left(\left|e_-\right\rangle \left\langle e_+\right|\right) + \rho^f_t\left(\varrho_{--}\right) \otimes \rho^a_t\left(\left|e_-\right\rangle \left\langle e_-\right|\right).\end{aligned} \tag{64}$$

Combining $\mathrm{Tr}\left(|e_\pm\rangle\langle e_\mp|\right)=0$ with Lemma 5.3 we deduce that

$$\begin{aligned}\mathrm{Tr}\left(\left|\rho_t^f\left(\varrho_{\pm\mp}\right)\otimes\rho_t^a\left(|e_\pm\rangle\langle e_\mp|\right)\right|\right)&=\mathrm{Tr}\left(\left|\rho_t^f\left(\varrho_{\pm\mp}\right)\right|\right)\mathrm{Tr}\left(\left|\rho_t^a\left(|e_\pm\rangle\langle e_\mp|\right)\right|\right)\\&\leq\mathrm{Tr}\left(|\varrho_{\pm\mp}|\right)\mathrm{Tr}\left(\left|\rho_t^a\left(|e_\pm\rangle\langle e_\mp|\right)\right|\right)\leq 4\exp\left(-\gamma\, t\right)\mathrm{Tr}\left(|\varrho_{\pm\mp}|\right),\end{aligned}$$

and so

$$\mathrm{Tr}\left(\left|\rho_t^f\left(\varrho_{\pm\mp}\right)\otimes\rho_t^a\left(|e_\pm\rangle\langle e_\mp|\right)\right|\right)\leq 4\exp\left(-\gamma\, t\right)\qquad\forall t\geq 0,\tag{65}$$

because

$$\mathrm{Tr}\left(|\varrho_{\pm\mp}|\right)\leq\mathbb{E}\,\mathrm{Tr}\left(|\xi_\pm\rangle\langle\xi_\mp|\right)=\mathbb{E}\left\|\xi_\pm\right\|\left\|\xi_\mp\right\|\leq\sqrt{\mathbb{E}\left\|\xi_\pm\right\|^2}\sqrt{\mathbb{E}\left\|\xi_\mp\right\|^2}\leq 1.$$

Since

$$\begin{aligned}&\mathrm{Tr}\left(\left|\rho_t^f\left(\varrho_{\pm\pm}\right)\otimes\rho_t^a\left(|e_\pm\rangle\langle e_\pm|\right)-\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\varrho_\infty^f\otimes\varrho_\infty^a\right|\right)\\&\leq\mathrm{Tr}\left(\left|\rho_t^f\left(\varrho_{\pm\pm}\right)\otimes\rho_t^a\left(|e_\pm\rangle\langle e_\pm|\right)-\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\varrho_\infty^f\otimes\rho_t^a\left(|e_\pm\rangle\langle e_\pm|\right)\right|\right)\\&\quad+\mathrm{Tr}\left(\left|\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\varrho_\infty^f\otimes\rho_t^a\left(|e_\pm\rangle\langle e_\pm|\right)-\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\varrho_\infty^f\otimes\varrho_\infty^a\right|\right)\\&=\mathrm{Tr}\left(\left|\rho_t^f\left(\varrho_{\pm\pm}\right)-\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\varrho_\infty^f\right|\right)\mathrm{Tr}\left(\left|\rho_t^a\left(|e_\pm\rangle\langle e_\pm|\right)\right|\right)\\&\quad+\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\mathrm{Tr}\left(\varrho_\infty^f\right)\mathrm{Tr}\left(\left|\rho_t^a\left(|e_\pm\rangle\langle e_\pm|\right)-\varrho_\infty^a\right|\right)\\&=\mathrm{Tr}\left(\left|\rho_t^f\left(\varrho_{\pm\pm}\right)-\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\varrho_\infty^f\right|\right)+\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\mathrm{Tr}\left(\left|\rho_t^a\left(|e_\pm\rangle\langle e_\pm|\right)-\varrho_\infty^a\right|\right),\end{aligned}$$

applying Lemmata 5.3 and 5.4 yields

$$\begin{aligned}&\mathrm{Tr}\left(\left|\rho_t^f\left(\varrho_{\pm\pm}\right)\otimes\rho_t^a\left(|e_\pm\rangle\langle e_\pm|\right)-\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\varrho_\infty^f\otimes\varrho_\infty^a\right|\right)\\&\leq 2\sqrt{\mathrm{Tr}\left(\varrho_{\pm\pm}\right)}\mathrm{e}^{-\kappa t}\sqrt{\mathrm{Tr}\left(\varrho_{\pm\pm}N\right)}+\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\mathrm{e}^{-\kappa t}\frac{2\left|\alpha\right|}{\sqrt{\kappa^2+\omega^2}}+4\mathrm{e}^{-\gamma t}\,\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\left(1+|d|\right)\\&\leq 2\mathrm{e}^{-\kappa t}\sqrt{\mathrm{Tr}\left(\varrho_{\pm\pm}N\right)}+\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\mathrm{e}^{-\kappa t}\frac{2\left|\alpha\right|}{\sqrt{\kappa^2+\omega^2}}+4\mathrm{e}^{-\gamma t}\,\mathrm{Tr}\left(\varrho_{\pm\pm}\right)\left(1+|d|\right).\end{aligned}$$

Now, using (64), (65), $\mathrm{Tr}\left(\varrho_{++}\right)+\mathrm{Tr}\left(\varrho_{--}\right)=1$ and $\mathrm{Tr}\left(\varrho_{\pm\pm}N\right)\leq\mathrm{Tr}\left(\varrho_0 N\right)$ we get (26). □

5.2.6. Proof of Theorem 4.8

Proof of Theorem 4.8. Let ϱ be a non-negative trace-class operator on $\mathfrak{h}$. According to Lemma 7.10 of [35] we have that there exists a sequence of C-regular non-negative operators ϱ_n such as $\lim_{n\to+\infty}\mathrm{Tr}\left(|\varrho-\varrho_n|\right)=0$, where C is a self-adjoint positive operator in $\mathfrak{h}$. Using Theorem 4.3 of [35] yields $\lim_{s\to t}\mathrm{Tr}\left(\left|\rho_t^h\left(\varrho_n\right)-\rho_s^h\left(\varrho_n\right)\right|\right)=0$, and so

$$\begin{aligned}&\mathrm{Tr}\left(\left|\rho_t^h\left(\varrho\right)-\rho_s^h\left(\varrho\right)\right|\right)\\&\leq\mathrm{Tr}\left(\left|\rho_t^h\left(\varrho\right)-\rho_t^h\left(\varrho_n\right)\right|\right)+\mathrm{Tr}\left(\left|\rho_t^h\left(\varrho_n\right)-\rho_s^h\left(\varrho_n\right)\right|\right)+\mathrm{Tr}\left(\left|\rho_s^h\left(\varrho_n\right)-\rho_s^h\left(\varrho\right)\right|\right)\\&\leq 2\ \mathrm{Tr}\left(|\varrho-\varrho_n|\right)+\mathrm{Tr}\left(\left|\rho_t^h\left(\varrho_n\right)-\rho_s^h\left(\varrho_n\right)\right|\right)\end{aligned}$$

leads to

$$\lim_{s\to t} \operatorname{Tr}\left(\left|\rho_t^h(\varrho) - \rho_s^h(\varrho)\right|\right) = 0. \tag{66}$$

Decomposing the real and imaginary parts of an element of $\mathfrak{L}_1(\mathfrak{h})$ into positive and negative parts (see, e.g., proof of Theorem 4.1 of [35]) we find that (66) holds for any $\varrho \in \mathfrak{L}_1(\mathfrak{h})$. Finally, applying Theorem 4.7 gives (28). □

5.2.7. Proof of Theorem 4.9 Throughout the proof, for any $t \geq 0$ we define

$$\mathcal{L}_\star^{full}(t)\ \varrho = \mathcal{L}_\star^h\,\varrho + \left[\alpha(t)\,a^\dagger - \overline{\alpha(t)}a + \overline{\beta(t)}\sigma^- - \beta(t)\,\sigma^+, \varrho\right],$$

where ϱ is a linear combination of a finite number of N-regular density operators in $\mathfrak{L}_1(\mathfrak{h})$.

Lemma 5.5. *Suppose that* $\alpha, \beta : [0, \infty[\to \mathbb{C}$ *are continuous functions, and that* $(\rho_t)_{t\geq 0}$ *is a* N^2*-weak solution of (21). Then, for any* $t \geq 0$ *we have*

$$\mathcal{L}_\star^{full}(s)\ \rho_s \longrightarrow_{s\to t} \mathcal{L}_\star^{full}(t)\ \rho_t \qquad \text{and} \qquad \mathcal{L}_\star^h\,\rho_s \longrightarrow_{s\to t} \mathcal{L}_\star^h\,\rho_t,$$

where both limits are taken in $\mathfrak{L}_1(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$.

Proof. Since ρ_0 is N^2-regular, there exists $\xi \in L^2_{N^2}(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$ such that $\rho_0 = \mathbb{E}\,|\xi\rangle\langle\xi|$ (see, e.g., Theorem 3.1 of [35]). From the proof of Theorem 4.5 we have $\rho_t = \mathbb{E}\,|X_t(\xi)\rangle\,\langle X_t(\xi)|$, where $X_t(\xi)$ is the strong N^2-solution of (14) with $G(t)$, L_1, L_2, L_3 given by (42). Now, applying Theorem 3.2 of [35] we obtain

$$\begin{aligned}\mathcal{L}_\star^{full}(s)\ \rho_s &= \mathbb{E}\,|G(t)\,X_t(\xi)\rangle\langle X_t(\xi)| + \mathbb{E}\,|X_t(\xi)\rangle\langle G(t)\,X_t(\xi)| \\ &\quad + \sum_{\ell=1}^{3} \mathbb{E}\,|L_\ell X_t(\xi)\rangle\langle L_\ell X_t(\xi)|\end{aligned}$$

(see, e.g., proof of Theorem 4.4 of [35]).

Using $X_t(\xi) \in L^2_{N^2}(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$ we deduce that

$$Y_t := N\xi + \int_0^t NG(s)\,X_s(\xi)\,ds + \sum_{\ell=1}^{3} \int_0^t NL_\ell X_s(\xi)\,dW_s^\ell \qquad \forall t \geq 0$$

is a well-defined continuous stochastic process. As N is a closed operator in $\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$ we have $Y_t = NX_t(\xi)$ for all $t \geq 0$ $\mathbb{P}$-a.s. (see, e.g., Proposition 4.15 of [47]). Moreover, $\mathbb{E}\left(\sup_{s\in[0,t+1]} \|Y_s\|^2\right) < \infty$ and $\mathbb{E}\left(\sup_{s\in[0,t+1]} \|X_s(\xi)\|^2\right) < \infty$ (see, e.g., Th. 4.2.5 of [45]). Then, using the dominated convergence theorem gives

$$\lim_{n\to+\infty} \mathbb{E}\,\|NX_{s_n}(\xi) - NX_t(\xi)\|^2 = \lim_{n\to+\infty} \mathbb{E}\,\|Y_{s_n} - Y_t\|^2 = 0$$

and $\lim_{n\to+\infty} \mathbb{E}\,\|X_{s_n}(\xi) - X_t(\xi)\|^2 = 0$, where $s_n \to t$ as $n \to +\infty$. Therefore,

$$\lim_{s\to t} \mathbb{E}\,\|X_s(\xi) - X_t(\xi)\|_N^2 = 0 \qquad \forall t \geq 0. \tag{67}$$

Suppose that A, B are linear operators in $\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2$, which are relatively bounded with respect to N. For any $s, t \geq 0$,

$$
\begin{aligned}
&\operatorname{Tr}\left(\left|\mathbb{E}\,|A\,X_s(\xi)\rangle\langle B\,X_s(\xi)| - \mathbb{E}\,|A\,X_t(\xi)\rangle\langle B\,X_t(\xi)|\right|\right) \\
&\leq \operatorname{Tr}\left(\left|\mathbb{E}\,|A\,X_s(\xi) - A\,X_t(\xi)\rangle\langle B\,X_s(\xi)|\right|\right) + \operatorname{Tr}\left(\left|\mathbb{E}\,|A\,X_t(\xi)\rangle\langle B\,X_s(\xi) - B\,X_t(\xi)|\right|\right) \\
&\leq \mathbb{E}\left(\|A\,X_s(\xi) - A\,X_t(\xi)\|\,\|B\,X_s(\xi)\|\right) + \mathbb{E}\left(\|A\,X_t(\xi)\|\,\|B\,X_s(\xi) - B\,X_t(\xi)\|\right) \\
&\leq \sqrt{\mathbb{E}\left(\|A\,X_s(\xi) - A\,X_t(\xi)\|^2\right)}\sqrt{\mathbb{E}\left(\|B\,X_s(\xi)\|^2\right)} \\
&\quad + \sqrt{\mathbb{E}\left(\|A\,X_t(\xi)\|^2\right)}\sqrt{\mathbb{E}\left(\|B\,X_s(\xi) - B\,X_t(\xi)\|^2\right)}.
\end{aligned}
$$

Combining (67) with $\sup_{s\in[0,t+1]}\mathbb{E}\left(\|X_s(\xi)\|_N^2\right) < \infty$ yields

$$
\operatorname{Tr}\left(\left|\mathbb{E}\,|A\,X_s(\xi)\rangle\langle B\,X_s(\xi)| - \mathbb{E}\,|A\,X_t(\xi)\rangle\langle B\,X_t(\xi)|\right|\right) \longrightarrow_{s\to t} 0. \tag{68}
$$

Since $\alpha, \beta : [0,\infty[\, \to \mathbb{C}$ are continuous functions, it follows from (68) that

$$
t \mapsto \mathbb{E}\,|G(t)\,X_t(\xi)\rangle\langle X_t(\xi)| + \mathbb{E}\,|X_t(\xi)\rangle\langle G(t)\,X_t(\xi)| + \sum_{\ell=1}^{3}\mathbb{E}\,|L_\ell X_t(\xi)\rangle\langle L_\ell X_t(\xi)|
$$

is a continuous function from $[0,\infty[$ to $\mathfrak{L}_1(\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2)$. Hence

$$
\mathcal{L}_\star^{full}(s)\;\rho_s \longrightarrow_{s\to t} \mathcal{L}_\star^{full}(t)\;\rho_t \qquad\qquad \text{in } \mathfrak{L}_1\left(\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2\right).
$$

In the same manner we can see that $t \mapsto \mathcal{L}_\star^h\,\rho_t$ is continuous. □

Lemma 5.6. *Under the assumptions of Lemma 5.5,*

$$
\lim_{s\to t}\frac{\rho_s - \rho_t}{s-t} = \mathcal{L}_\star^{full}(t)\;\rho_t \qquad\qquad \textit{in } \mathfrak{L}_1\left(\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2\right).
$$

Proof. Let $t \geq 0$. From Theorem 4.5 we get

$$
\frac{\rho_t - \rho_s}{t-s} - \mathcal{L}_\star^{full}(t)\;\rho_t = \frac{1}{t-s}\int_s^t\left(\mathcal{L}_\star^{full}(u)\;\rho_u - \mathcal{L}_\star^{full}(t)\;\rho_t\right)du, \tag{69}
$$

where $s \geq 0$, $s \neq t$, and the integral in (69) is understood in the sense of Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$. According to Lemma 5.5 we have that $u \mapsto \mathcal{L}_\star^{full}(u)\;\rho_u - \mathcal{L}_\star^{full}(t)\;\rho_t$ is continuous as a function from $[0,+\infty[$ to $\mathfrak{L}_1(\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2)$, and so

$$
\lim_{s\to t}\frac{1}{t-s}\int_s^t\left(\mathcal{L}_\star^{full}(u)\;\rho_u - \mathcal{L}_\star^{full}(t)\;\rho_t\right)du = 0 \qquad \text{in } \mathfrak{L}_1\left(\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2\right).
$$

Thus, the lemma follows from (69). □

Lemma 5.7. *Assume the hypothesis of Lemma 5.5. Then, for all $t \geq s \geq 0$ we have*

$$
\rho_t = \rho_{t-s}^h(\rho_s) + \int_s^t \rho_{t-u}^h\left(\left[\left(\alpha(u)\,a^\dagger - \overline{\alpha(u)}a\right) + \left(\overline{\beta(u)}\sigma^- - \beta(u)\,\sigma^+\right), \rho_u\right]\right)du.
$$

Proof. Consider $t > s \geq 0$. For any non-zero real number Δ such that $-s \leq \Delta < t - s$ we have

$$
\begin{aligned}
&\frac{1}{\Delta}\left(\rho^h_{t-(s+\Delta)}\left(\rho_{s+\Delta}\right) - \rho^h_{t-s}\left(\rho_s\right)\right) + \mathcal{L}^h_\star\, \rho^h_{t-s}\left(\rho_s\right) - \rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right) \\
&= \rho^h_{t-(s+\Delta)}\left(\frac{1}{\Delta}\left(\rho_{s+\Delta} - \rho_s\right) - \mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right) \\
&\quad + \rho^h_{t-(s+\Delta)}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right) - \rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right) \\
&\quad + \frac{1}{\Delta}\left(\rho^h_{t-(s+\Delta)}\left(\rho_s\right) - \rho^h_{t-s}\left(\rho_s\right)\right) + \mathcal{L}^h_\star\, \rho^h_{t-s}\left(\rho_s\right).
\end{aligned}
$$

Since $\rho^h_{t-(s+\Delta)}$ is a contraction acting on $\mathfrak{L}_1\left(\ell^2\left(\mathbb{Z}_+\right) \otimes \mathbb{C}^2\right)$,

$$
\operatorname{Tr}\left(\left|\rho^h_{t-(s+\Delta)}\left(\frac{1}{\Delta}\left(\rho_{s+\Delta} - \rho_s\right) - \mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right)\right|\right) \leq \operatorname{Tr}\left(\left|\frac{1}{\Delta}\left(\rho_{s+\Delta} - \rho_s\right) - \mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right|\right),
$$

and so applying Lemma 5.6 yields

$$
\operatorname{Tr}\left(\left|\rho^h_{t-(s+\Delta)}\left(\frac{1}{\Delta}\left(\rho_{s+\Delta} - \rho_s\right) - \mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right)\right|\right) \longrightarrow_{\Delta\to 0} 0.
$$

In case $\alpha\left(t\right) = \beta\left(t\right) \equiv 0$, $\mathcal{L}^{full}_\star = \mathcal{L}^h_\star$, and hence using Lemma 5.6 we deduce that $\mathcal{L}^h_\star$ coincides with the infinitesimal generator of the strongly continuous semigroup $\left(\rho^h_u\right)_{u\geq 0}$ on the subset $\mathfrak{L}^+_{1,N^2}\left(\ell^2\left(\mathbb{Z}_+\right) \otimes \mathbb{C}^2\right)$, as well as

$$
\operatorname{Tr}\left(\left|\frac{1}{\Delta}\left(\rho^h_{t-(s+\Delta)}\left(\rho_s\right) - \rho^h_{t-s}\left(\rho_s\right)\right) + \mathcal{L}^h_\star\, \rho^h_{t-s}\left(\rho_s\right)\right|\right) \longrightarrow_{\Delta\to 0} 0.
$$

The strong continuity of the semigroup $\left(\rho^h_u\right)_{u\geq 0}$ implies

$$
\operatorname{Tr}\left(\left|\rho^h_{t-(s+\Delta)}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right) - \rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right)\right|\right) \longrightarrow_{\Delta\to 0} 0.
$$

Therefore, $\frac{1}{\Delta}\left(\rho^h_{t-(s+\Delta)}\left(\rho_{s+\Delta}\right) - \rho^h_{t-s}\left(\rho_s\right)\right) + \mathcal{L}^h_\star\, \rho^h_{t-s}\left(\rho_s\right) - \rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right)$ converges to 0 in $\mathfrak{L}_1\left(\ell^2\left(\mathbb{Z}_+\right) \otimes \mathbb{C}^2\right)$ as $\Delta \to 0$. Thus

$$
\frac{d}{ds}\rho^h_{t-s}\left(\rho_s\right) = \rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right) - \mathcal{L}^h_\star\, \rho^h_{t-s}\left(\rho_s\right) = \rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s\right) - \rho^h_{t-s}\left(\mathcal{L}^h_\star\, \rho_s\right),
$$

and consequently

$$
\frac{d}{ds}\rho^h_{t-s}\left(\rho_s\right) = \rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s - \mathcal{L}^h_\star\, \rho_s\right). \tag{70}
$$

The contraction property of ρ^h_{t-u} leads to

$$
\begin{aligned}
&\operatorname{Tr}\left(\left|\rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s - \mathcal{L}^h_\star\, \rho_s\right) - \rho^h_{t-u}\left(\mathcal{L}^{full}_\star\left(u\right)\, \rho_u - \mathcal{L}^h_\star\, \rho_u\right)\right|\right) \\
&\leq \operatorname{Tr}\left(\left|\rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s - \mathcal{L}^h_\star\, \rho_s\right) - \rho^h_{t-u}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s - \mathcal{L}^h_\star\, \rho_s\right)\right|\right) \\
&\qquad + \operatorname{Tr}\left(\left|\rho^h_{t-u}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s - \mathcal{L}^h_\star\, \rho_s\right) - \rho^h_{t-u}\left(\mathcal{L}^{full}_\star\left(u\right)\, \rho_u - \mathcal{L}^h_\star\, \rho_u\right)\right|\right) \\
&\leq \operatorname{Tr}\left(\left|\rho^h_{t-s}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s - \mathcal{L}^h_\star\, \rho_s\right) - \rho^h_{t-u}\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s - \mathcal{L}^h_\star\, \rho_s\right)\right|\right) \\
&\qquad + \operatorname{Tr}\left(\left|\left(\mathcal{L}^{full}_\star\left(s\right)\, \rho_s - \mathcal{L}^h_\star\, \rho_s\right) - \left(\mathcal{L}^{full}_\star\left(u\right)\, \rho_u - \mathcal{L}^h_\star\, \rho_u\right)\right|\right).
\end{aligned}
$$

According to Lemma 5.5 we have

$$\lim_{u \to s} \operatorname{Tr}\left(\left| \left(\mathcal{L}_{\star}^{full}(s)\ \rho_s - \mathcal{L}_{\star}^{h} \rho_s \right) - \left(\mathcal{L}_{\star}^{full}(u)\ \rho_u - \mathcal{L}_{\star}^{h} \rho_u \right) \right| \right) = 0.$$

The strong continuity of $\left(\rho_r^h\right)_{r \geq 0}$ yields

$$\lim_{u \to s} \operatorname{Tr}\left(\left| \rho_{t-s}^h \left(\mathcal{L}_{\star}^{full}(s)\ \rho_s - \mathcal{L}_{\star}^{h} \rho_s \right) - \rho_{t-u}^h \left(\mathcal{L}_{\star}^{full}(s)\ \rho_s - \mathcal{L}_{\star}^{h} \rho_s \right) \right| \right) = 0.$$

Then, the map $s \mapsto \rho_{t-s}^h \left(\mathcal{L}_{\star}^{full}(s)\ \rho_s - \mathcal{L}_{\star}^{h} \rho_s \right)$ is continuous. By the fundamental theorem of calculus for the Bochner integral, integrating (70) gives

$$\rho_0^h(\rho_t) - \rho_{t-s}^h(\rho_s) = \int_s^t \rho_{t-u}^h \left(\mathcal{L}_{\star}^{full}(u)\ \rho_u - \mathcal{L}_{\star}^{h} \rho_u \right) du,$$

which is the desired conclusion. □

Proof of Theorem 4.9. Since ρ_0 is N-regular, $\rho_0 = \mathbb{E}\,|\nu\rangle\langle\nu|$ for certain ν belonging to $L_N^2\left(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2\right)$ (see, e.g., Theorem 3.1 of [35]). Hence, $\rho_t = \mathbb{E}\,|X_t(\nu)\rangle\,\langle X_t(\nu)|$ (see, e.g., the proof of Theorem 4.5). From now on, $X_t(\xi)$ denotes the strong N-solution of (14) with $\xi \in L_N^2\left(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2\right)$ and $G(t)$, L_1, L_2, L_3 given by (42).

Let pr_n be the orthonormal projection of $\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$ onto the linear span of $e_0 \otimes e_\pm, \ldots, e_n \otimes e_\pm$. As

$$\left\| pr_n(\nu) \right\|^2 + \left\| N^2\, pr_n(\nu) \right\|^2 \leq \|\nu\|^2 + n^2 \sum_{k=0}^{n} \sum_{\eta=\pm} \left| \langle e_k \otimes e_\eta, \xi \rangle \right|^2 \leq 1 + n^2,$$

$pr_n(\nu) \in L_{N^2}^2\left(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2\right)$. There exists $n_0 \in \mathbb{N}$ such that $\mathbb{E}\left\| pr_n(\nu) \right\|^2 > 0$ for all $n \geq n_0$, because the increasing sequence $\mathbb{E}\left\| pr_n(\nu) \right\|^2$ converges to $\mathbb{E}\|\nu\|^2 = 1$ as $n \to +\infty$. For any $n \geq n_0$ we set $\nu_n := pr_n(\nu) / \sqrt{\mathbb{E}\left\| pr_n(\nu) \right\|^2}$. Then, $\rho_t^n = \mathbb{E}\,|X_t(\nu_n)\rangle\,\langle X_t(\nu_n)|$ is a N^2-weak solution of (21) with initial condition $\mathbb{E}\,|\nu_n\rangle\,\langle\nu_n|$ (see, e.g., the proof of Theorem 4.5), and so Lemma 5.7 yields

$$\rho_t^n = \rho_{t-s}^h(\rho_s^n) + \int_s^t \rho_{t-u}^h \left(\left[\left(\alpha(u)\, a^\dagger - \overline{\alpha(u)} a \right) + \left(\overline{\beta(u)} \sigma^- - \beta(u)\, \sigma^+ \right), \rho_u^n \right] \right) du \quad (71)$$

for all $t \geq s \geq 0$ and $n \geq n_0$.

Combining the linearity of (14) with (45) we deduce that

$$\mathbb{E}\left(\left\| X_u(\nu) - X_u(\nu_n) \right\|_N^2 \right) = \mathbb{E}\left(\left\| X_u(\nu - \nu_n) \right\|_N^2 \right) \leq K(u)\, \mathbb{E}\left(\left\| \nu - \nu_n \right\|_N^2 \right)$$

for all $u \geq 0$. Since N commutates with pr_n,

$$\begin{aligned}\left\| \sqrt{\mathbb{E}\left\| pr_n(\nu) \right\|^2} N^p\, \nu - N^p pr_n(\nu) \right\|^2 &= \left(\sqrt{\mathbb{E}\left\| pr_n(\nu) \right\|^2} - 1 \right)^2 \left\| pr_n(N^p\, \nu) \right\|^2 \\ &\quad + \left(\mathbb{E}\left\| pr_n(\nu) \right\|^2 \right) \left\| N^p \nu - pr_n(N^p \nu) \right\|^2\end{aligned}$$

with $p = 0, 1$, and so

$$\mathbb{E}\left(\|\nu - \nu_n\|_N^2\right) \leq \frac{\left(\sqrt{\mathbb{E}\left\|pr_n\left(\nu\right)\right\|^2} - 1\right)^2}{\mathbb{E}\left\|pr_n\left(\nu\right)\right\|^2}\left(\mathbb{E}\left(\|\nu\|^2\right) + \mathbb{E}\left(\|N\nu\|^2\right)\right) + \mathbb{E}\left(\|\nu - pr_n\left(\nu\right)\|^2\right) + \mathbb{E}\left(\|N\nu - pr_n\left(N\nu\right)\|^2\right) \longrightarrow_{n\to+\infty} 0.$$

Therefore,

$$\limsup_{n\to+\infty} \sup_{u\in[0,t]} \mathbb{E}\left(\|X_u\left(\nu\right) - X_u\left(\nu_n\right)\|_N^2\right) \leq \lim_{n\to+\infty} K\left(t\right)\mathbb{E}\left(\|\nu - \nu_n\|_N^2\right) = 0. \qquad (72)$$

Let A, B be linear operators in $\ell^2\left(\mathbb{Z}_+\right) \otimes \mathbb{C}^2$ that are relatively bounded with respect to N. Then

$$\begin{aligned}
&\mathrm{Tr}\left(\left|\mathbb{E}\left|A\,X_u\left(\nu\right)\right\rangle\left\langle B\,X_u\left(\nu\right)\right| - \mathbb{E}\left|A\,X_u\left(\nu_n\right)\right\rangle\left\langle B\,X_u\left(\nu_n\right)\right|\right|\right)\\
&\leq \mathrm{Tr}\left(\left|\mathbb{E}\left|A\,X_u\left(\nu\right) - A\,X_u\left(\nu_n\right)\right\rangle\left\langle B\,X_u\left(\nu\right)\right|\right|\right) + \mathrm{Tr}\left(\left|\mathbb{E}\left|A\,X_u\left(\nu_n\right)\right\rangle\left\langle B\,X_u\left(\nu\right) - B\,X_u\left(\nu_n\right)\right|\right|\right)\\
&\leq \mathbb{E}\left(\|A\,X_u\left(\nu\right) - A\,X_u\left(\nu_n\right)\|\,\|B\,X_u\left(\nu\right)\|\right) + \mathbb{E}\left(\|A\,X_u\left(\nu_n\right)\|\,\|B\,X_u\left(\nu\right) - B\,X_u\left(\nu_n\right)\|\right)\\
&\leq \sqrt{\mathbb{E}\left(\|A\,X_u\left(\nu\right) - A\,X_u\left(\nu_n\right)\|^2\right)}\sqrt{\mathbb{E}\left(\|B\,X_u\left(\nu\right)\|^2\right)}\\
&\quad + \sqrt{\mathbb{E}\left(\|A\,X_u\left(\nu_n\right)\|^2\right)}\sqrt{\mathbb{E}\left(\|B\,X_u\left(\nu\right) - B\,X_u\left(\nu_n\right)\|^2\right)}.
\end{aligned}$$

According to (45) we have

$$\mathbb{E}\left(\|X_u\left(\nu_n\right)\|_N^2\right) \leq K\left(u\right)\mathbb{E}\left(\|\nu_n\|_N^2\right) \leq K\left(u\right)\mathbb{E}\left(\|\nu\|_N^2\right),$$

and so (72) yields

$$\sup_{u\in[0,t]} \mathrm{Tr}\left(\left|\mathbb{E}\left|A\,X_u\left(\nu\right)\right\rangle\left\langle B\,X_u\left(\nu\right)\right| - \mathbb{E}\left|A\,X_u\left(\nu_n\right)\right\rangle\left\langle B\,X_u\left(\nu_n\right)\right|\right|\right) \longrightarrow_{n\to+\infty} 0. \qquad (73)$$

Since ρ^h_{t-s} is a contraction acting on $\mathfrak{L}_1\left(\ell^2\left(\mathbb{Z}_+\right) \otimes \mathbb{C}^2\right)$,

$$\mathrm{Tr}\left(\left|\rho^h_{t-s}\left(\rho_s\right) - \rho^h_{t-s}\left(\rho^n_s\right)\right|\right) \leq \mathrm{Tr}\left(\left|\rho_s - \rho^n_s\right|\right),$$

$$\begin{aligned}
\mathrm{Tr}\left(\left|\rho^h_{t-u}\left(\left[\overline{\beta\left(u\right)}\sigma^- - \beta\left(u\right)\sigma^+, \rho_u - \rho^n_u\right]\right)\right|\right) &\leq \mathrm{Tr}\left(\left|\left[\overline{\beta\left(u\right)}\sigma^- - \beta\left(u\right)\sigma^+, \rho_u - \rho^n_u\right]\right|\right)\\
&\leq 2\left(\left|\beta\left(u\right)\right|\left(\left\|\sigma^-\right\| + \left\|\sigma^+\right\|\right)\right)\mathrm{Tr}\left(\left|\rho_u - \rho^n_u\right|\right),
\end{aligned}$$

and

$$\mathrm{Tr}\left(\left|\rho^h_{t-u}\left(\left[\alpha\left(u\right)a^\dagger - \overline{\alpha\left(u\right)}a, \rho_u - \rho^n_u\right]\right|\right)\right) \leq \mathrm{Tr}\left(\left|\left[\alpha\left(u\right)a^\dagger - \overline{\alpha\left(u\right)}a, \rho_u - \rho^n_u\right]\right|\right). \qquad (74)$$

Applying (73) we obtain

$$\sup_{u\in[0,t]} \mathrm{Tr}\left(\left|\rho_u - \rho^n_u\right|\right) = \mathrm{Tr}\left(\left|\mathbb{E}\left|X_u\left(\nu\right)\right\rangle\left\langle X_u\left(\nu\right)\right| - \mathbb{E}\left|X_u\left(\nu_n\right)\right\rangle\left\langle X_u\left(\nu_n\right)\right|\right|\right) \longrightarrow_{n\to+\infty} 0.$$

This gives $\mathrm{Tr}\left(\left|\rho^h_{t-s}\left(\rho_s\right) - \rho^h_{t-s}\left(\rho^n_s\right)\right|\right) \longrightarrow_{n\to+\infty} 0$, and

$$\sup_{u\in[0,t]} \mathrm{Tr}\left(\left|\rho^h_{t-u}\left(\left[\overline{\beta\left(u\right)}\sigma^- - \beta\left(u\right)\sigma^+, \rho_u - \rho^n_u\right]\right)\right|\right) \longrightarrow_{n\to+\infty} 0.$$

As

$$\left[\alpha\left(u\right)a^{\dagger}-\overline{\alpha\left(u\right)}a,\rho_{u}-\rho_{u}^{n}\right]$$
$$=\mathbb{E}\left|\left(\alpha\left(u\right)a^{\dagger}-\overline{\alpha\left(u\right)}a\right)X_{u}\left(\nu\right)\rangle\langle X_{u}\left(\nu\right)\right|-\mathbb{E}\left|\left(\alpha\left(u\right)a^{\dagger}-\overline{\alpha\left(u\right)}a\right)X_{u}\left(\nu_{n}\right)\rangle\langle X_{u}\left(\nu_{n}\right)\right|$$
$$+\mathbb{E}\left|X_{u}\left(\nu\right)\rangle\langle\left(\alpha\left(u\right)a^{\dagger}-\overline{\alpha\left(u\right)}a\right)X_{u}\left(\nu\right)\right|-\mathbb{E}\left|X_{u}\left(\nu_{n}\right)\rangle\langle\left(\alpha\left(u\right)a^{\dagger}-\overline{\alpha\left(u\right)}a\right)X_{u}\left(\nu_{n}\right)\right|$$

(see, e.g., Theorem 3.2 of [35]), using (73) and (74) we deduce that

$$\sup_{u\in[0,t]}\operatorname{Tr}\left(\left|\rho_{t-u}^{h}\left(\left[\alpha\left(u\right)a^{\dagger}-\overline{\alpha\left(u\right)}a,\rho_{u}-\rho_{u}^{n}\right]\right|\right)\right)\longrightarrow_{n\to+\infty}0.$$

Now, taking the limit as $n\to+\infty$ in (71) we obtain (29). □

5.3. *Proof of Theorem 3.1*

Proof of Theorem 3.1. Let $(A(t),S(t),D(t))$ be the unique global solution of (5) with $A(0)=\operatorname{Tr}(a\varrho)$, $S(0)=\operatorname{Tr}(\sigma^{-}\varrho)$ and $D(0)=\operatorname{Tr}(\sigma^{3}\varrho)$. According to Theorem 4.5 we have that there exists a unique N^{p}-weak solution $(\rho_t)_{t\geq0}$ to (21) with $\alpha(t)=g\,S(t)$, $\beta(t)=g\,A(t)$, and initial datum $\rho_0=\varrho$. Applying Theorem 4.6 we deduce that the evolutions of $\operatorname{Tr}(a\,\rho_t)$, $\operatorname{Tr}(\sigma^{-}\rho_t)$ and $\operatorname{Tr}(\sigma^{3}\rho_t)$ are governed by

$$\begin{cases}\dfrac{d}{dt}\operatorname{Tr}\left(a\,\rho_{t}\right) &=-\left(\kappa+\mathrm{i}\omega\right)\operatorname{Tr}\left(a\,\rho_{t}\right)+g\;S\left(t\right)\\ \dfrac{d}{dt}\operatorname{Tr}\left(\sigma^{-}\rho_{t}\right)&=-\left(\gamma+\mathrm{i}\omega\right)\operatorname{Tr}\left(\sigma^{-}\rho_{t}\right)+g\;A\left(t\right)\operatorname{Tr}\left(\sigma^{3}\rho_{t}\right)\\ \dfrac{d}{dt}\operatorname{Tr}\left(\sigma^{3}\rho_{t}\right)&=-4g\;\Re\left(\overline{A\left(t\right)}\;\operatorname{Tr}\left(\sigma^{-}\rho_{t}\right)\right)-2\gamma\left(\operatorname{Tr}\left(\sigma^{3}\rho_{t}\right)-d\right)\end{cases}. \tag{75}$$

From the uniqueness of solution to (75) we find $\operatorname{Tr}(a\,\rho_t)=A(t)$, $\operatorname{Tr}(\sigma^{-}\rho_t)=S(t)$ and $\operatorname{Tr}(\sigma^{3}\rho_t)=D(t)$, hence

$$\begin{cases}\dfrac{d}{dt}\operatorname{Tr}\left(A\rho_{t}\right)=\operatorname{Tr}\left(A\mathcal{L}_{\star}\left(\rho_{t}\right)\rho_{t}\right) & \forall A\in\mathfrak{L}\left(\mathfrak{h}\right)\\ \rho_{0}=\varrho\end{cases}. \tag{76}$$

On the other hand, suppose that $(\rho_t)_{t\geq0}$ and $(\widetilde{\rho}_t)_{t\geq0}$ are families of N^{p}-regular operators satisfying (76) such that $\rho_0=\widetilde{\rho}_0=\varrho$ and $t\mapsto\operatorname{Tr}(a\,\rho_t)$, $t\mapsto\operatorname{Tr}(a\,\widetilde{\rho}_t)$ are continuous. Then, $(\rho_t)_{t\geq0}$ is a N^{p}-weak solution to (21) with $\alpha(t)=g\operatorname{Tr}(\sigma^{-}\rho_t)$ and $\beta(t)=g\operatorname{Tr}(a\,\rho_t)$, as well as $(\widetilde{\rho}_t)_{t\geq0}$ is a N^{p}-weak solution to (21) with $\alpha(t)=g\operatorname{Tr}(\sigma^{-}\widetilde{\rho}_t)$ and $\beta(t)=g\operatorname{Tr}(a\widetilde{\rho}_t)$. Using Theorem 4.6 we get that $(\operatorname{Tr}(a\,\rho_t),\operatorname{Tr}(\sigma^{-}\rho_t),\operatorname{Tr}(\sigma^{3}\rho_t))$ and $(\operatorname{Tr}(a\,\widetilde{\rho}_t),\operatorname{Tr}(\sigma^{-}\widetilde{\rho}_t),\operatorname{Tr}(\sigma^{3}\widetilde{\rho}_t))$ are solutions of (5) with initial condition $A(0)=\operatorname{Tr}(a\,\varrho)$, $S(0)=\operatorname{Tr}(\sigma^{-}\varrho)$ and $D(0)=\operatorname{Tr}(\sigma^{3}\varrho)$. Applying Theorem 2.1 yields $\operatorname{Tr}(a\,\rho_t)=\operatorname{Tr}(a\,\widetilde{\rho}_t)$, $\operatorname{Tr}(\sigma^{-}\rho_t)=\operatorname{Tr}(\sigma^{-}\widetilde{\rho}_t)$ and $\operatorname{Tr}(\sigma^{3}\rho_t)=\operatorname{Tr}(\sigma^{3}\widetilde{\rho}_t)$. Now, using Theorem 4.5 we can assert that $\rho_t=\widetilde{\rho}_t$ for all $t\geq0$. □

5.4. Proof of Theorem 3.2

Proof of Theorem 3.2. Since $\operatorname{Tr}(\sigma^{-}\varrho_{\infty}) = \frac{d+1}{2}\langle e_{+}, \sigma^{-} e_{+}\rangle + \frac{1-d}{2}\langle e_{-}, \sigma^{-} e_{-}\rangle = 0$ and $\operatorname{Tr}(a\,\varrho_{\infty}) = \langle e_0, a e_0\rangle = 0$,

$$\mathcal{L}_{\star}(\varrho_{\infty})\,\varrho_{\infty} = \mathcal{L}_{\star}^{h}\,\varrho_{\infty},$$

where $\mathcal{L}_{\star}^{h}$ is defined by (22). Using the fact that $A\,|x\rangle\langle y|\,B = |Ax\rangle\langle B^{\star}y|$ for any operators A, B in $\mathfrak{h}$, $x \in \mathcal{D}(A)$ and $y \in \mathcal{D}(B^{\star})$, we obtain $\mathcal{L}_{\star}^{h}\,\varrho_{\infty} = 0$. Hence ϱ_{∞} is a stationary state for (1), which is N^{p}-regular for all $p \in \mathbb{N}$.

Suppose that $\omega \neq 0$. Let $\widetilde{\varrho}$ be a N-regular stationary state for (1). According to Theorem 3.1 we have that the functions $A(t) := \operatorname{Tr}(a\,\widetilde{\varrho})$, $S(t) := \operatorname{Tr}(\sigma^{-}\widetilde{\varrho})$ and $D(t) := \operatorname{Tr}(\sigma^{3}\widetilde{\varrho})$ satisfy (5). From Theorem 2.1 it follows that $\operatorname{Tr}(a\,\widetilde{\varrho}) = \operatorname{Tr}(\sigma^{-}\widetilde{\varrho}) = 0$ and $\operatorname{Tr}(\sigma^{3}\widetilde{\varrho}) = d$. Therefore $\widetilde{\varrho}$ is a stationary state for (27), and so using Theorem 4.8 we obtain that $\widetilde{\varrho}$ coincides with ϱ_{∞}. □

5.5. Proof of Theorem 3.3

Lemma 5.8. *Let $(\rho_t(\varrho))_{t\geq 0}$ be the N-weak solution of (21) with $\alpha(t) \equiv \alpha \in \mathbb{C}$, $\beta(t) \equiv \beta \in \mathbb{C}$ and initial datum $\varrho \in \mathfrak{L}_{1,N}^{+}(\ell^2(\mathbb{Z}_{+}) \otimes \mathbb{C}^2)$. Then $\varrho_{\infty}^{f} \otimes \varrho_{\infty}^{a}$ is the unique operator $\varrho_{\infty} \in \mathfrak{L}_{1,N}^{+}(\ell^2(\mathbb{Z}_{+}) \otimes \mathbb{C}^2)$ for which*

$$\rho_t(\varrho_{\infty}) = \varrho_{\infty} \qquad \forall t \geq 0. \tag{77}$$

Here, ϱ_{∞}^{f} and ϱ_{∞}^{a} are as in Theorem 4.7.

Proof. Using Lemma 5.3 we deduce that $\mathcal{L}_{\star}^{a}(\varrho_{\infty}^{a}) = 0$. Returning to the proof of Lemma 5.4, we verify that $\widetilde{\mathcal{L}_{\star}}(|e_0\rangle\langle e_0|) = 0$, and so (59) leads to

$$\begin{aligned}&\mathcal{L}_{\star}^{f}\left(W\left(\frac{\alpha}{\kappa + \mathrm{i}\omega}\right)|e_0\rangle\langle e_0|\,W\left(-\frac{\alpha}{\kappa + \mathrm{i}\omega}\right)\right)\\ &= W\left(\frac{\alpha}{\kappa + \mathrm{i}\omega}\right)\widetilde{\mathcal{L}_{\star}}(|e_0\rangle\langle e_0|)\,W\left(-\frac{\alpha}{\kappa + \mathrm{i}\omega}\right) = 0.\end{aligned}$$

Since $W\left(\frac{\alpha}{\kappa+\mathrm{i}\omega}\right)e_0 = \exp\left(-\left|\frac{\alpha}{\kappa+\mathrm{i}\omega}\right|^2/2\right)e\left(\frac{\alpha}{\kappa+\mathrm{i}\omega}\right)$, $\mathcal{L}_{\star}^{f}\left(\varrho_{\infty}^{f}\right) = 0$. Therefore,

$$\mathcal{L}_{\star}^{f} \otimes I\left(\varrho_{\infty}^{f} \otimes \varrho_{\infty}^{a}\right) + I \otimes \mathcal{L}_{\star}^{a}\left(\varrho_{\infty}^{f} \otimes \varrho_{\infty}^{a}\right) = 0.$$

This gives $\rho_t\left(\varrho_{\infty}^{f} \otimes \varrho_{\infty}^{a}\right) = \varrho_{\infty}^{f} \otimes \varrho_{\infty}^{a}$ for all $t \geq 0$.

Consider $\varrho_{\infty} \in \mathfrak{L}_{1,N}^{+}(\ell^2(\mathbb{Z}_{+}) \otimes \mathbb{C}^2)$ satisfying (77). Applying Theorem 4.7 yields

$$\varrho_{\infty} = \lim_{t\to+\infty}\rho_t(\varrho_{\infty}) = \varrho_{\infty}^{f} \otimes \varrho_{\infty}^{a} \qquad \text{in } \mathfrak{L}_1\left(\ell^2(\mathbb{Z}_{+}) \otimes \mathbb{C}^2\right).$$ □

Proof of Theorem 3.3. According to the Stone theorem we have that the self-adjoint operator $\frac{\omega}{2}(2N + \sigma^3)$ generates the strongly continuous one-parameter unitary group $\left(\exp\left(\mathrm{i}\frac{\omega}{2}(2N + \sigma^3)\,t\right)\right)_{t\in\mathbb{R}}$. In order to describe the physical system in the interaction picture we set

$$\widetilde{\rho}_t = \exp\left(\mathrm{i}\frac{\omega}{2}\left(2N + \sigma^3\right)t\right)\rho_t\exp\left(-\mathrm{i}\frac{\omega}{2}\left(2N + \sigma^3\right)t\right) \qquad \forall t \geq 0.$$

Since N commutes with σ^3, $\varrho_t \in \mathfrak{L}_{1,N}^{+}\left(\ell^2\left(\mathbb{Z}_+\right)\otimes\mathbb{C}^2\right)$ iff $\widetilde{\rho}_t \in \mathfrak{L}_{1,N}^{+}\left(\ell^2\left(\mathbb{Z}_+\right)\otimes\mathbb{C}^2\right)$. Hence, ϱ_t is a N-regular free interaction solution to (1) iff

$$\widetilde{\rho}_t = \rho_0 \in \mathfrak{L}_{1,N}^{+}\left(\ell^2\left(\mathbb{Z}_+\right)\otimes\mathbb{C}^2\right) \qquad \forall t \geq 0. \tag{78}$$

A careful computation shows that ϱ_t is a N-weak solution to (1) iff $\widetilde{\rho}_t$ is a N-weak solution to

$$\begin{aligned} \frac{d}{dt}\widetilde{\rho}_t =& 2\kappa\left(a\,\widetilde{\rho}_t a^\dagger - \frac{1}{2}a^\dagger a\widetilde{\rho}_t - \frac{1}{2}\widetilde{\rho}_t a^\dagger a\right) \\ &+ \gamma(1-d)\left(\sigma^-\widetilde{\rho}_t\,\sigma^+ - \frac{1}{2}\sigma^+\sigma^-\widetilde{\rho}_t - \frac{1}{2}\widetilde{\rho}_t\,\sigma^+\sigma^-\right) \\ &+ \gamma(1+d)\left(\sigma^+\widetilde{\rho}_t\,\sigma^- - \frac{1}{2}\sigma^-\sigma^+\widetilde{\rho}_t - \frac{1}{2}\,\widetilde{\rho}_t\,\sigma^-\sigma^+\right) \\ &+ g\left[\operatorname{Tr}\left(\sigma^-\widetilde{\rho}_t\right)a^\dagger - \operatorname{Tr}\left(\sigma^+\widetilde{\rho}_t\right)a, \widetilde{\rho}_t\right] + g\left[\operatorname{Tr}\left(a^\dagger\widetilde{\rho}_t\right)\sigma^- - \operatorname{Tr}\left(a\,\widetilde{\rho}_t\right)\sigma^+, \widetilde{\rho}_t\right]. \end{aligned} \tag{79}$$

Therefore, ρ_t is a N-regular free interaction solution to (1) iff ρ_0 is a N-regular stationary state for (79).

We proceed to find all N-regular stationary state for (79). To this end, we suppose that (78) holds. According to Theorem 3.1 we have that the functions $t \mapsto \operatorname{Tr}(a\,\rho_0)$, $t \mapsto \operatorname{Tr}(\sigma^-\rho_0)$ and $t \mapsto \operatorname{Tr}(\sigma^3\rho_0)$ satisfy (2) with $\omega = 0$. It follows that

$$-\kappa\ \operatorname{Tr}(a\,\rho_0) + g\ \operatorname{Tr}\left(\sigma^-\rho_0\right) = 0, \tag{80a}$$

$$-\gamma\ \operatorname{Tr}\left(\sigma^-\rho_0\right) + g\ \operatorname{Tr}(a\,\rho_0)\operatorname{Tr}\left(\sigma^3\rho_0\right) = 0, \tag{80b}$$

$$2g\ \Re\left(\operatorname{Tr}(a\,\rho_0)\ \overline{\operatorname{Tr}(\sigma^-\rho_0)}\right) + \gamma\left(\operatorname{Tr}\left(\sigma^3\rho_0\right) - d\right) = 0. \tag{80c}$$

Combining (80a) with (80b) we obtain

$$\operatorname{Tr}(a\,\rho_0)\left(-\gamma\kappa + g^2\ \operatorname{Tr}\left(\sigma^3\rho_0\right)\right) = 0. \tag{81}$$

Then $\operatorname{Tr}(a\,\rho_0) = 0$ or $g^2\ \operatorname{Tr}(\sigma^3\rho_0) = \gamma\kappa$.

Asume $\operatorname{Tr}(a\,\rho_0) = 0$, together with (78). Then (80a) and (80c) lead to $\operatorname{Tr}(\sigma^-\rho_0) = 0$ and $\operatorname{Tr}(\sigma^3\rho_0) = d$. Therefore

$$\operatorname{Tr}(a\,\widetilde{\rho}_t) = \operatorname{Tr}\left(\sigma^-\widetilde{\rho}_t\right) = 0 \text{ and } \operatorname{Tr}\left(\sigma^3\widetilde{\rho}_t\right) = d, \tag{82}$$

and so ρ_0 is a N-regular stationary state for (27) with $\omega = 0$. Theorem 4.8 gives

$$\rho_0 = \rho_\infty := |e_0\rangle\langle e_0| \otimes \left(\frac{d+1}{2}\,|e_+\rangle\langle e_+| + \frac{1-d}{2}\,|e_-\rangle\langle e_-|\right). \tag{83}$$

On the other hand, $\operatorname{Tr}(a\,\rho_\infty) = \operatorname{Tr}(\sigma^-\rho_\infty) = 0$ and $\operatorname{Tr}(\sigma^3\rho_\infty) = d$, as well as $\mathcal{L}_\star^h\,\varrho_\infty = 0$. Hence, $\widetilde{\rho}_t = \rho_\infty$ solves (79). Summarizing, ρ_∞, given by (83), is the unique N-regular stationary state for (79) satisfying $\operatorname{Tr}(a\,\rho_0) = 0$. We have found the free interaction solution to (1):

$$\begin{aligned} \rho_t &= \exp\left(-\mathrm{i}\frac{\omega}{2}\left(2N+\sigma^3\right)t\right)\widetilde{\rho}_t \exp\left(\mathrm{i}\frac{\omega}{2}\left(2N+\sigma^3\right)t\right) \\ &= \exp\left(-\mathrm{i}\frac{\omega}{2}\left(2N+\sigma^3\right)t\right)\rho_\infty \exp\left(\mathrm{i}\frac{\omega}{2}\left(2N+\sigma^3\right)t\right) \end{aligned}$$

$$= \left| e_0 \right\rangle \left\langle e_0 \right| \otimes \left(\frac{d+1}{2} \left| e_+ \right\rangle \left\langle e_+ \right| + \frac{1-d}{2} \left| e_- \right\rangle \left\langle e_- \right| \right) = \rho_\infty. \tag{84}$$

Let $\mathrm{Tr}\,(a\,\rho_0) \neq 0$, and suppose that (78) holds. Then, (80a) implies that $g \neq 0$, and so (81) gives $\mathrm{Tr}\,(\sigma^3 \rho_0) = \frac{\gamma\kappa}{g^2}$. Using (80$c$) we deduce that

$$\left|\mathrm{Tr}\,(a\,\rho_0)\right|^2 = \frac{\gamma}{2\kappa g^2} \left(dg^2 - \gamma\kappa \right). \tag{85}$$

Therefore $dg^2 > \gamma\kappa$. From the previous paragraph we conclude that the state (83) is the unique N-regular free interaction solution to (1) whenever $dg^2 \leq \gamma\kappa$. In case $dg^2 > \gamma\kappa$, according to (85) we have that there exists $z \in \mathbb{C}$ with $|z| = 1$ such that $\mathrm{Tr}\,(a\,\widetilde{\rho}_t) = z\sqrt{\frac{\gamma}{2\kappa g^2}\,(dg^2 - \gamma\kappa)}$. Using (80$a$) we obtain

$$\mathrm{Tr}\left(\sigma^- \widetilde{\rho}_t\right) = \frac{\kappa\, z}{g} \sqrt{\frac{\gamma}{2\kappa g^2}\,(dg^2 - \gamma\kappa)}.$$

Thus, due to (78) and (79), ρ_0 is a N-regular stationary state for (21) with $\omega = 0$, $\alpha\,(t) \equiv \frac{z\kappa}{g}\sqrt{\frac{\gamma}{2\kappa}\,(dg^2 - \gamma\kappa)}$ and $\beta\,(t) \equiv z\,\sqrt{\frac{\gamma}{2\kappa}\,(dg^2 - \gamma\kappa)}$. Applying Lemma 5.8 gives

$$\begin{aligned} \varrho_0 = \varrho_\infty\,(z) := e^{-\frac{\gamma^2 (C_\mathfrak{b} - 1)}{2g^2}} &\left| e\left(\frac{z\gamma\sqrt{C_\mathfrak{b} - 1}}{\sqrt{2}g} \right) \right\rangle \left\langle e\left(\frac{z\gamma\sqrt{C_\mathfrak{b} - 1}}{\sqrt{2}g} \right) \right| \otimes \\ &\begin{pmatrix} \frac{1}{2}\left(1 + \frac{d}{C_\mathfrak{b}}\right) & \frac{z\kappa\gamma}{\sqrt{2}g^2}\sqrt{C_\mathfrak{b} - 1} \\ \frac{\bar{z}\kappa\gamma}{\sqrt{2}g^2}\sqrt{C_\mathfrak{b} - 1} & \frac{1}{2}\left(1 - \frac{d}{C_\mathfrak{b}}\right) \end{pmatrix}. \end{aligned} \tag{86}$$

From Lemma 5.8 it follows that $\varrho_\infty\,(z)$, given by (86), is the unique N-regular constant solution of (21) with $\omega = 0$, $\alpha\,(t) \equiv \frac{z\kappa}{g}\sqrt{\frac{\gamma}{2\kappa}\,(dg^2 - \gamma\kappa)}$ and

$$\beta\,(t) \equiv z\,\sqrt{\frac{\gamma}{2\kappa}\,(dg^2 - \gamma\kappa)}.$$

Now, Theorem 4.6 implies

$$\begin{cases} -\kappa\ \mathrm{Tr}\,(\rho_0\, a) + \dfrac{z\kappa}{g}\sqrt{\dfrac{\gamma}{2\kappa}\,(dg^2 - \gamma\kappa)} = 0 \\ -\gamma\ \mathrm{Tr}\left(\rho_t\, \sigma^-\right) + z\,\sqrt{\dfrac{\gamma}{2\kappa}\,(dg^2 - \gamma\kappa)}\,\mathrm{Tr}\left(\rho_t\, \sigma^3\right) = 0 \\ 2\,\Re\left(\overline{z}\sqrt{\dfrac{\gamma}{2\kappa}\,(dg^2 - \gamma\kappa)}\,\mathrm{Tr}\left(\rho_t \sigma^-\right) \right) + \gamma\left(\mathrm{Tr}\left(\rho_t \sigma^3\right) - d \right) = 0 \end{cases}.$$

This yields $\mathrm{Tr}\,(a\,\varrho_0) = \frac{z}{g}\sqrt{\frac{\gamma}{2\kappa}\,(dg^2 - \gamma\kappa)}$, $\mathrm{Tr}\,(\sigma^3 \varrho_0) = \frac{\gamma\kappa}{g^2}$, and

$$\mathrm{Tr}\left(\sigma^- \varrho_0\right) = \frac{\kappa\, z}{g^2}\sqrt{\frac{\gamma}{2\kappa}\,(dg^2 - \gamma\kappa)}.$$

Therefore, $\varrho_\infty\,(z)$ is a N-regular stationary state of (79).

In case $dg^2 > \gamma\kappa$, we have proved that in addition to (83) the only N-regular stationary states for (79) are given by (86) for any complex number z with absolute value 1. Since

$$\rho_t = \exp\left(-\mathrm{i}\frac{\omega}{2}\left(2N+\sigma^3\right)t\right)\widetilde{\rho}_t \exp\left(\mathrm{i}\frac{\omega}{2}\left(2N+\sigma^3\right)t\right),$$

all non-constant N-regular free interaction solution to (1) are:

$$e^{-\frac{\gamma^2(C_{\mathfrak{b}}-1)}{2g^2}}\left|e^{-\mathrm{i}\omega Nt}e\left(\frac{z\gamma\sqrt{C_{\mathfrak{b}}-1}}{\sqrt{2}g}\right)\right\rangle\left\langle e^{-\mathrm{i}\omega Nt}e\left(\frac{z\gamma\sqrt{C_{\mathfrak{b}}-1}}{\sqrt{2}g}\right)\right|\otimes$$
$$e^{-\mathrm{i}\frac{\omega}{2}\sigma^3 t}\begin{pmatrix}\frac{1}{2}\left(1+\frac{d}{C_{\mathfrak{b}}}\right) & \frac{z\kappa\gamma}{\sqrt{2}g^2}\sqrt{C_{\mathfrak{b}}-1}\\ \frac{\bar{z}\kappa\gamma}{\sqrt{2}g^2}\sqrt{C_{\mathfrak{b}}-1} & \frac{1}{2}\left(1-\frac{d}{C_{\mathfrak{b}}}\right)\end{pmatrix}e^{\mathrm{i}\frac{\omega}{2}\sigma^3 t},$$

where $|z|=1$, and therefore they are

$$e^{-\frac{\gamma^2(C_{\mathfrak{b}}-1)}{2g^2}}\left|e\left(\frac{z\gamma\sqrt{C_{\mathfrak{b}}-1}}{\sqrt{2}g}e^{-i\omega t}\right)\right\rangle\left\langle e\left(\frac{z\gamma\sqrt{C_{\mathfrak{b}}-1}}{\sqrt{2}g}e^{-i\omega t}\right)\right|\otimes$$
$$\begin{pmatrix}\frac{1}{2}\left(1+\frac{d}{C_{\mathfrak{b}}}\right) & e^{-i\omega t}\frac{z\kappa\gamma}{\sqrt{2}g^2}\sqrt{C_{\mathfrak{b}}-1}\\ e^{i\omega t}\frac{\bar{z}\kappa\gamma}{\sqrt{2}g^2}\sqrt{C_{\mathfrak{b}}-1} & \frac{1}{2}\left(1-\frac{d}{C_{\mathfrak{b}}}\right)\end{pmatrix}$$

for any $|z|=1$. □

5.6. Proof of Theorem 3.4

Proof of Theorem 3.4. According to Theorem 3.1 we have that the evolutions of $\mathrm{Tr}\left(a\,\rho_t\right)$, $\mathrm{Tr}\left(\sigma^-\rho_t\right)$ and $\mathrm{Tr}\left(\sigma^3\rho_t\right)$ are described by the Maxwell-Bloch equations (2). Since $\mathrm{Tr}\left(a\,\varrho\right)=\mathrm{Tr}\left(\sigma^-\varrho\right)=0$, from (2) it follows that $\mathrm{Tr}\left(a\,\rho_t\right)=\mathrm{Tr}\left(\sigma^-\rho_t\right)=0$ for all $t\geq 0$. Therefore, ρ_t solves (27) with initial condition ϱ, and hence $\rho_t=\rho_t^h\left(\varrho\right)$, where $\rho_t^h\left(\varrho\right)$ is the N-weak solution of (27). Applying Theorem 4.8 gives (10). □

5.7. Proof of Theorem 3.5

Lemma 5.9. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Suppose that ϱ is a C-regular density operator in $\mathfrak{h}$. Consider the linear operator $A:\mathcal{D}\left(A\right)\subset\mathfrak{h}\to\mathfrak{h}$. Then:*

- $\mathrm{Tr}\left(|A\varrho|\right)\leq\sqrt{\mathrm{Tr}\left(\varrho A^\star A\right)}$ *whenever* $A, A^\star A\in\mathfrak{L}\left(\left(\mathcal{D}\left(C\right),\|\cdot\|_C\right),\mathfrak{h}\right)$.
- $\mathrm{Tr}\left(|\varrho A|\right)\leq\sqrt{\mathrm{Tr}\left(\varrho AA^\star\right)}$ *provided that* $A^\star, AA^\star\in\mathfrak{L}\left(\left(\mathcal{D}\left(C\right),\|\cdot\|_C\right),\mathfrak{h}\right)$.

Proof. Since $\varrho\in\mathfrak{L}_{1,C}^{+}\left(\mathfrak{h}\right)$, there exists $\xi\in L_C^2\left(\mathbb{P},\mathfrak{h}\right)$ such that $\varrho=\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$ and $\mathbb{E}\left(\|\xi\|^2\right)=1$ (see, e.g., Theorem 3.1 of [35]). If $A, A^\star A\in\mathfrak{L}\left(\left(\mathcal{D}\left(C\right),\|\cdot\|_C\right),\mathfrak{h}\right)$, then using Theorem 3.2 of of [35] we obtain

$$\mathrm{Tr}\left(|A\varrho|\right)=\sup_{\|B\|=1}\left|\mathrm{Tr}\left(BA\varrho\right)\right|=\sup_{\|B\|=1}\left|\mathbb{E}\langle\xi,BA\,\xi\rangle\right|\leq\mathbb{E}\left(\|\xi\|\,\|A\,\xi\|\right),$$

and so

$$\mathbb{E}\left(\|\xi\|\,\|A\,\xi\|\right)\leq\sqrt{\mathbb{E}\left(\|\xi\|^2\right)}\sqrt{\mathbb{E}\left(\|A\,\xi\|^2\right)}=\sqrt{\mathbb{E}\langle A^\star A\,\xi,\xi\rangle}=\sqrt{\mathrm{Tr}\left(\varrho A^\star A\right)},$$

because $\mathbb{E}\left(\|\xi\|^2\right)=1$. Similarly, in case $A^\star, AA^\star \in \mathfrak{L}\left(\left(\mathcal{D}\left(C\right),\|\cdot\|_C\right),\mathfrak{h}\right)$ we have

$$\operatorname{Tr}\left(\left|\varrho A\right|\right)=\sup_{\|B\|=1}\left|\mathbb{E}\langle A^\star\xi,B\xi\rangle\right|=\sqrt{\mathbb{E}\left(\left\|A^\star\xi\right\|^2\right)}=\sqrt{\operatorname{Tr}\left(\varrho AA^\star\right)}. \qquad \square$$

Lemma 5.10. *Suppose that $(\rho_t)_{t\geq 0}$ is a N-weak solution to (1). Consider the solution $(A(t),S(t),D(t))$ to (5) with initial condition $A(0)=\operatorname{Tr}(a\,\rho_0)$, $S(0)=\operatorname{Tr}(\sigma^-\rho_0)$ and $D(0)=\operatorname{Tr}(\sigma^3\rho_0)$. Then, for all $t\geq s\geq 0$ we have*

$$\begin{aligned}\operatorname{Tr}\left(\left|\rho_t-\rho_\infty\right|\right)\leq{}&\operatorname{Tr}\left(\left|\rho^h_{t-s}\left(\rho_s\right)-\rho_\infty\right|\right)+4\left|g\right|\int_s^t\left|S\left(u\right)\right|\sqrt{\operatorname{Tr}\left(\rho_u N\right)+1}\,du\\&+2\left|g\right|\left(\left\|\sigma^-\right\|+\left\|\sigma^+\right\|\right)\int_s^t\left|A\left(u\right)\right|du,\end{aligned}\tag{87}$$

where $\rho_\infty:=|e_0\rangle\langle e_0|\otimes\left(\frac{d+1}{2}\,|e_+\rangle\langle e_+|+\frac{1-d}{2}\,|e_-\rangle\langle e_-|\right)$.

Proof. Since $A(0)=\operatorname{Tr}(a\,\rho_0)$, $S(0)=\operatorname{Tr}(\sigma^-\rho_0)$ and $D(0)=\operatorname{Tr}(\sigma^3\rho_0)$, from Theorems 2.1 and 3.1 we deduce that $A(t)=\operatorname{Tr}(a\,\rho_t)$, $S(t)=\operatorname{Tr}(\sigma^-\rho_t)$ and $D(t)=\operatorname{Tr}(\sigma^3\rho_t)$. Hence, $(\rho_t)_{t\geq 0}$ is the N-weak solution to (21) with $\alpha(t)=g\,S(t)$, $\beta(t)=g\,A(t)$ and initial datum ρ_0. Now, Theorem 4.9 leads to

$$\rho_t=\rho^h_{t-s}\left(\rho_s\right)+g\int_s^t\rho^h_{t-u}\left(\left[\left(S\left(u\right)a^\dagger-\overline{S\left(u\right)}a\right)+\left(\overline{A\left(u\right)}\sigma^- -A\left(u\right)\sigma^+\right),\rho_u\right]\right)du,$$

where $t\geq s\geq 0$. Therefore,

$$\begin{aligned}\operatorname{Tr}\left(\left|\rho_t-\rho_\infty\right|\right)\leq{}&\operatorname{Tr}\left(\left|\rho^h_{t-s}\left(\rho_s\right)-\rho_\infty\right|\right)\\&+\left|g\right|\operatorname{Tr}\left(\left|\int_s^t\rho^h_{t-u}\left(\left[\left(S\left(u\right)a^\dagger-\overline{S\left(u\right)}a\right)+\left(\overline{A\left(u\right)}\sigma^- -A\left(u\right)\sigma^+\right),\rho_u\right]\right)du\right|\right).\end{aligned}$$

Using that ρ^h_{t-u} is a contraction acting on $\mathfrak{L}_1\left(\ell^2\left(\mathbb{Z}_+\right)\otimes\mathbb{C}^2\right)$ we obtain

$$\begin{aligned}\operatorname{Tr}\left(\left|\rho_t-\rho_\infty\right|\right)\leq{}&\operatorname{Tr}\left(\left|\rho^h_{t-s}\left(\rho_s\right)-\rho_\infty\right|\right)\\&+\left|g\right|\int_s^t\operatorname{Tr}\left(\left|\left[\left(S\left(u\right)a^\dagger-\overline{S\left(u\right)}a\right)+\left(\overline{A\left(u\right)}\sigma^- -A\left(u\right)\sigma^+\right),\rho_u\right]\right)\right|du,\end{aligned}$$

and so

$$\begin{aligned}\operatorname{Tr}\left(\left|\rho_t-\rho_\infty\right|\right)\leq{}&\operatorname{Tr}\left(\left|\rho^h_{t-s}\left(\rho_s\right)-\rho_\infty\right|\right)\\&+\left|g\right|\int_s^t\left|S\left(u\right)\right|\left(\operatorname{Tr}\left(\left|a^\dagger\rho_u\right|\right)+\operatorname{Tr}\left(\left|\rho_u\,a^\dagger\right|\right)+\operatorname{Tr}\left(\left|a\,\rho_u\right|\right)+\operatorname{Tr}\left(\left|\rho_u\,a\right|\right)\right)du\\&+\left|g\right|\int_s^t\left|A\left(u\right)\right|\left(\operatorname{Tr}\left(\left|\sigma^-\rho_u\right|\right)+\operatorname{Tr}\left(\left|\rho_u\,\sigma^-\right|\right)+\operatorname{Tr}\left(\left|\sigma^+\rho_u\right|\right)+\operatorname{Tr}\left(\left|\rho_u\sigma^+\right|\right)\right)du.\end{aligned}$$

As $\sigma^\pm$ are bounded operators,

$$\begin{aligned}&\operatorname{Tr}\left(\left|\sigma^-\rho_u\right|\right)+\operatorname{Tr}\left(\left|\rho_u\,\sigma^-\right|\right)+\operatorname{Tr}\left(\left|\sigma^+\rho_u\right|\right)+\operatorname{Tr}\left(\left|\rho_u\,\sigma^+\right|\right)\\&\leq\left(2\left\|\sigma^-\right\|+2\left\|\sigma^+\right\|\right)\operatorname{Tr}\left(\rho_u\right)=2\left\|\sigma^-\right\|+2\left\|\sigma^+\right\|.\end{aligned}$$

By $a^\dagger a = N$, $a\, a^\dagger = N + I$ and $\mathrm{Tr}\,(\rho_u) = 1$, applying Lemma 5.9 yields

$$\begin{aligned}&\mathrm{Tr}\left(\left|a^\dagger \rho_u\right|\right) + \mathrm{Tr}\left(\left|\rho_u\, a^\dagger\right|\right) + \mathrm{Tr}\left(\left|a\, \rho_u\right|\right) + \mathrm{Tr}\left(\left|\rho_u\, a\right|\right)\\ &\leq 2\sqrt{\mathrm{Tr}\left(\rho_u\left(N+I\right)\right)} + 2\sqrt{\mathrm{Tr}\left(\rho_u\, N\right)} \leq 4\sqrt{\mathrm{Tr}\left(\rho_u\, N\right)+1}.\end{aligned}$$

We thus get (87). □

Proof of Theorem 3.5. Combining Lemma 5.10 with Theorem 4.8 we deduce that for all $t \geq s \geq 0$,

$$\begin{aligned}\mathrm{Tr}\left(\left|\rho_t - \rho_\infty\right|\right) &\leq 12\, \mathrm{e}^{-\gamma(t-s)}\left(1+|d|\right) + 4\, \mathrm{e}^{-\kappa(t-s)}\sqrt{\mathrm{Tr}\left(\varrho_s\, N\right)} \\ &+ 4\,|g| \int_s^t \left|S\left(u\right)\right| \sqrt{\mathrm{Tr}\left(\rho_u\, N\right)+1}\, du + 2\,|g| \left(\left\|\sigma^-\right\| + \left\|\sigma^+\right\|\right) \int_s^t \left|A\left(u\right)\right| du,\end{aligned} \tag{88}$$

where $\left(A\left(t\right), S\left(t\right), D\left(t\right)\right)$ be the solution to (5) with $A\left(0\right) = \mathrm{Tr}\left(a\, \rho_0\right)$, $S\left(0\right) = \mathrm{Tr}\left(\sigma^-\, \rho_0\right)$ and $D\left(0\right) = \mathrm{Tr}\left(\sigma^3\, \rho_0\right)$. From Theorems 2.1 and 3.1 we obtain $A\left(t\right) = \mathrm{Tr}\left(a\, \rho_t\right)$, $S\left(t\right) = \mathrm{Tr}\left(\sigma^- \rho_t\right)$ and $D\left(t\right) = \mathrm{Tr}\left(\sigma^3 \rho_t\right)$, and hence $\left(\rho_t\right)_{t\geq 0}$ is the N-weak solution to (21) with $\alpha\left(t\right) = g\, S\left(t\right)$, $\beta\left(t\right) = g\, A\left(t\right)$ and initial datum ρ_0. Applying Theorem 4.6 gives

$$\begin{aligned}\mathrm{Tr}\left(\rho_t\, N\right) &= \mathrm{e}^{-2\kappa\, t}\, \mathrm{Tr}\left(\rho_0\, N\right) + 2g \int_0^t \mathrm{e}^{-2\kappa(t-s)} \Re\left(\overline{S\left(t\right)}\; \mathrm{Tr}\left(\rho_s\, a\right)\right) ds \\ &= \mathrm{e}^{-2\kappa\, t}\, \mathrm{Tr}\left(\rho_0\, N\right) + 2g \int_0^t \mathrm{e}^{-2\kappa(t-s)} \Re\left(\overline{S\left(s\right)} A\left(s\right)\right) ds \\ &\leq \mathrm{e}^{-2\kappa\, t}\, \mathrm{Tr}\left(\rho_0\, N\right) + |g| \int_0^t \mathrm{e}^{-2\kappa(t-s)} \left(\left|S\left(s\right)\right|^2 + \left|A\left(s\right)\right|^2\right) ds.\end{aligned} \tag{89}$$

Since $dg^2/\left(\gamma\kappa\right) < 1$, according to Theorem 2.1 we have that

$$\left|A\left(t\right)\right|^2 \leq K_A \exp\left(-c_s\, t\right) \qquad \text{and} \qquad \left|S\left(t\right)\right|^2 \leq K_S \exp\left(-c_s\, t\right), \tag{90}$$

where $t \geq 0$, $c_s = \begin{cases} 2\, \min\left\{\kappa, \gamma\right\} & \text{if } d < 0 \\ \left(1 - C_{\mathfrak{b}}\right) \min\left\{\kappa, \gamma\right\} & \text{if } d \geq 0 \end{cases}$,

$$K_A = \begin{cases} \left|A\left(0\right)\right|^2 + \left|S\left(0\right)\right|^2 / \left|d\right| + \left(D\left(0\right) - d\right)^2 / \left(4\left|d\right|\right) & \text{if } d < 0 \\ \left|A\left(0\right)\right|^2 + \dfrac{g^2}{\gamma\kappa} \left|S\left(0\right)\right|^2 + \dfrac{g^2}{4\gamma\kappa} \left(D\left(0\right) - d\right)^2 & \text{if } d \geq 0 \end{cases}$$

and $K_S = \begin{cases} \left|d\right| \left|A\left(0\right)\right|^2 + \left|S\left(0\right)\right|^2 + \left(D\left(0\right) - d\right)^2 / 4 & \text{if } d < 0 \\ \dfrac{4\kappa d}{\gamma} \left|A\left(0\right)\right|^2 + \left(\dfrac{4\kappa}{\gamma} + 1\right) \left|S\left(0\right)\right|^2 + \left(\dfrac{\kappa}{\gamma} + \dfrac{1}{4}\right) \left(D\left(0\right) - d\right)^2 & \text{if } d \geq 0 \end{cases}$.

Suppose that either $d \geq 0$ or $d < 0$ with $\kappa > \gamma$. Then $2\kappa > c_s$ and

$$\int_0^t \mathrm{e}^{-2\kappa(t-u)} \left(\left|S\left(u\right)\right|^2 + \left|A\left(u\right)\right|^2\right) du < \frac{K_A + K_S}{2\kappa - c_s} \left(\mathrm{e}^{-c_s\, t} - \mathrm{e}^{-2\kappa\, t}\right) < \frac{K_A + K_S}{2\kappa - c_s} \mathrm{e}^{-c_s\, t}.$$

From (89) it follows that

$$\mathrm{Tr}\left(\rho_t\, N\right) \leq \left(\mathrm{Tr}\left(\rho_0\, N\right) + \frac{|g| \left(K_A + K_S\right)}{2\kappa - c_s}\right) \mathrm{e}^{-c_s\, t}. \tag{91}$$

Consider $t \geq s \geq 0$. Applying (91) we get

$$\operatorname{Tr}(\rho_t N) \leq \operatorname{Tr}(\rho_0 N) + |g| (K_A + K_S) / (2\kappa - c_s),$$

and hence (90) gives

$$\int_s^t |S(u)| \sqrt{\operatorname{Tr}(\rho_u N) + 1}\, du \leq \left(1 + \operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}\right)^{1/2} \int_s^t |S(u)|\, du$$
$$\leq \left(2\sqrt{K_S}\left(1 + \operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}\right)^{1/2}\right) \frac{\mathrm{e}^{-c_s s/2} - \mathrm{e}^{-c_s t/2}}{c_s}.$$

Using (90) we also obtain

$$\left(\left\|\sigma^-\right\| + \left\|\sigma^+\right\|\right) \int_s^t |A(u)|\, du \leq 2 \int_s^t |A(u)|\, du \leq 4\sqrt{K_A} \frac{\mathrm{e}^{-c_s s/2} - \mathrm{e}^{-c_s t/2}}{c_s}. \qquad (92)$$

According to (88) and (91) we have

$$\operatorname{Tr}(|\rho_t - \rho_\infty|) \leq 12\, \mathrm{e}^{-\gamma(t-s)} (1 + |d|) + 4\, \mathrm{e}^{-\kappa(t-s) - \frac{c_s}{2} s} \sqrt{\operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}}$$
$$+ \frac{8|g|}{c_s} \left(\sqrt{K_S}\left(1 + \operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}\right)^{1/2} + \sqrt{K_A}\right) \mathrm{e}^{-\frac{c_s}{2} s}.$$

In case $d \geq 0$, taking $t = 3s/2$ yields

$$\operatorname{Tr}\left(\left|\rho_{3s/2} - \rho_\infty\right|\right) \leq 12\, \mathrm{e}^{-\gamma s/2} (1 + |d|) + 4\, \mathrm{e}^{-\frac{c_s}{2} s} \sqrt{\operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}}$$
$$+ \frac{8|g|}{c_s} \left(\sqrt{K_S}\left(1 + \operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}\right)^{1/2} + \sqrt{K_A}\right) \mathrm{e}^{-\frac{c_s}{2} s},$$

and so for all $t \geq 0$,

$$\operatorname{Tr}(|\rho_t - \rho_\infty|) \leq \mathrm{e}^{-\frac{c_s}{3} t} \left(12(1 + |d|) + 4\sqrt{\operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}} + \frac{8|g|}{c_s}\sqrt{K_A}\right.$$
$$\left. + \frac{8|g|}{c_s}\sqrt{K_S}\left(1 + \operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}\right)^{1/2}\right).$$

On the other hand, in case $d < 0$ with $\kappa > \gamma$, choosing $t = 2s$ we deduce that

$$\operatorname{Tr}(|\rho_{2s} - \rho_\infty|) \leq 12\, \mathrm{e}^{-\gamma s} (1 + |d|) + 4\, \mathrm{e}^{-\kappa s - \frac{c_s}{2} s} \sqrt{\operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}}$$
$$+ \frac{8|g|}{c_s} \left(\sqrt{K_S}\left(1 + \operatorname{Tr}(\rho_0 N) + \frac{|g|(K_A + K_S)}{2\kappa - c_s}\right)^{1/2} + \sqrt{K_A}\right) \mathrm{e}^{-\frac{c_s}{2} s},$$

and consequently

$$\operatorname{Tr}\left(\left|\rho_t-\rho_\infty\right|\right) \leq \mathrm{e}^{-\frac{c_s}{4}t}\left(12\left(1+|d|\right)+4\sqrt{\operatorname{Tr}\left(\rho_0\,N\right)+\frac{|g|\left(K_A+K_S\right)}{2\kappa-c_s}}+\frac{8\,|g|}{c_s}\sqrt{K_A}\right.$$
$$\left.+\frac{8\,|g|}{c_s}\sqrt{K_S}\left(1+\operatorname{Tr}\left(\rho_0\,N\right)+\frac{|g|\left(K_A+K_S\right)}{2\kappa-c_s}\right)^{1/2}\right).$$

for any $t\geq 0$.

Now, we assume that $d<0$ and $\kappa\leq\gamma$. Then

$$\int_0^t \mathrm{e}^{-2\kappa(t-u)}\left(\left|S\left(u\right)\right|^2+\left|A\left(u\right)\right|^2\right)du \leq 2t\,\left(K_A+K_S\right)\exp\left(-2\kappa\,t\right),$$

and so (89) leads to

$$\operatorname{Tr}\left(\rho_t\,N\right)\leq \exp\left(-2\kappa\,t\right)\operatorname{Tr}\left(\rho_0\,N\right)+2\,|g|\left(K_A+K_S\right)t\,\exp\left(-2\kappa\,t\right). \tag{93}$$

Since $t\,\exp\left(-2\kappa\,t\right)\leq 1/\left(2\,\mathrm{e}\,\kappa\right)$, according to (90) we have that for all $t\geq s\geq 0$,

$$\int_s^t \left|S\left(u\right)\right|\sqrt{\operatorname{Tr}\left(\rho_u\,N\right)+1}\,du\leq\sqrt{K_S}\sqrt{\operatorname{Tr}\left(\rho_0\,N\right)+\frac{|g|\left(K_A+K_S\right)}{\kappa\,\mathrm{e}}}\frac{\mathrm{e}^{-\kappa\,s}-\mathrm{e}^{-\kappa\,t}}{\kappa}.$$

Moreover, (92) gives

$$\left(\left\|\sigma^-\right\|+\left\|\sigma^+\right\|\right)\int_s^t\left|A\left(u\right)\right|du\leq 2\sqrt{K_A}\frac{\mathrm{e}^{-\kappa\,s}-\mathrm{e}^{-\kappa\,t}}{\kappa}.$$

Therefore, (88) yields

$$\begin{aligned}\operatorname{Tr}\left(\left|\rho_t-\rho_\infty\right|\right) &\leq 12\,\mathrm{e}^{-\gamma(t-s)}\left(1+|d|\right)+4\,\mathrm{e}^{-\kappa(t-s)}\sqrt{\operatorname{Tr}\left(\rho_0\,N\right)+|g|\left(K_A+K_S\right)/\left(\kappa\mathrm{e}\right)}\\&\quad+4\,|g|\left(\sqrt{K_S}\sqrt{\operatorname{Tr}\left(\rho_0\,N\right)+|g|\left(K_A+K_S\right)/\left(\kappa\mathrm{e}\right)}+\sqrt{K_A}\right)\mathrm{e}^{-\kappa\,s}/\kappa.\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{Tr}\left(\left|\rho_{2s}-\rho_\infty\right|\right) &\leq 12\,\mathrm{e}^{-\gamma s}\left(1+|d|\right)+4\,\mathrm{e}^{-\kappa s}\sqrt{\operatorname{Tr}\left(\rho_0\,N\right)+|g|\left(K_A+K_S\right)/\left(\kappa\mathrm{e}\right)}\\&\quad+4\,|g|\left(\sqrt{K_S}\sqrt{\operatorname{Tr}\left(\rho_0\,N\right)+|g|\left(K_A+K_S\right)/\left(\kappa\mathrm{e}\right)}+\sqrt{K_A}\right)\mathrm{e}^{-\kappa\,s}/\kappa,\end{aligned}$$

which implies

$$\operatorname{Tr}\left(\left|\rho_t-\rho_\infty\right|\right) \leq 4\mathrm{e}^{-\frac{c_s}{4}t}\left(\left(1+\frac{|g|\sqrt{K_S}}{\kappa}\right)\sqrt{\operatorname{Tr}\left(\rho_0\,N\right)+\frac{|g|\left(K_A+K_S\right)}{\kappa\,\mathrm{e}}}\right.$$
$$\left.+3\left(1+|d|\right)+\frac{|g|\sqrt{K_A}}{\kappa}\right).$$

Now, we decompose A as

$$A=A\,P+PA\left(I-P\right)+\left(I-P\right)A\left(I-P\right),$$

where P is the orthogonal projection of $\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2$ onto the linear span of $e_0\otimes e_+$ and $e_0\otimes e_-$, i.e., $P\,x=\langle e_0\otimes e_+,x\rangle\, e_0\otimes e_+ + \langle e_0\otimes e_-,x\rangle\, e_0\otimes e_-$. From (3) it follows

$$\begin{aligned}\operatorname{Tr}(\varrho_\infty\, A\, P) &= \sum_{n=0}^{+\infty}\sum_{\eta=\pm}\langle e_n\otimes e_\eta, \varrho_\infty\, A\, P e_n\otimes e_\eta\rangle = \langle \varrho_\infty e_0\otimes e_\eta, A\, e_0\otimes e_\eta\rangle \\ &= \frac{d+1}{2}\langle e_0\otimes e_+, A\, e_0\otimes e_+\rangle + \frac{1-d}{2}\langle e_0\otimes e_-, A\, e_0\otimes e_-\rangle.\end{aligned}$$

We can extend $PA\,(I-P)$ to the bounded linear operator

$$PA\,(I-P)\,x = \langle A^\star e_0\otimes e_+, (I-P)\,x\rangle\, e_0\otimes e_+ + \langle A^\star e_0\otimes e_-, (I-P)\,x\rangle\, e_0\otimes e_-.$$

Using (3) yields

$$\operatorname{Tr}(\varrho_\infty PA\,(I-P)) = \sum_{n=0}^{+\infty}\sum_{\eta=\pm}\langle \varrho_\infty e_n\otimes e_\eta, PA\,(I-P)\, e_n\otimes e_\eta\rangle = 0.$$

Applying (11) we deduce that for all $t\geq 0$,

$$\begin{aligned}&\left|\operatorname{Tr}(\rho_t\,(A\,P+PA\,(I-P))) - \frac{d+1}{2}\langle e_0\otimes e_+, A\, e_0\otimes e_+\rangle + \frac{1-d}{2}\langle e_0\otimes e_-, A\, e_0\otimes e_-\rangle\right| \\ &\leq (\|A\,P| + \|PA\,(I-P)|)\, K_{sys}(|g|)\exp(-\delta_{sys}\,t)\,. \end{aligned} \tag{94}$$

Since A and $A^\star$ are relatively bounded with respect to N,

$$\begin{aligned}|\langle y, A\,y\rangle| &= \left|\langle y, \frac{1}{2}(A+A^\star)\,y\rangle + i\langle y, \frac{i}{2}(A^\star - A)\,y\rangle\right| \\ &= \sqrt{\langle y, \frac{1}{2}(A+A^\star)\,y\rangle^2 + \langle y, \frac{i}{2}(A^\star - A)\,y\rangle^2} \leq K\left(\|y\|^2 + \langle y, N\,y\rangle\right)\end{aligned}$$

for all $y\in\mathcal{D}(N)$ (see, e.g., Theorem VI.1.38 of [44]). Therefore,

$$\begin{aligned}\langle (I-P)\,x, (I-P)\,A\,(I-P)\,x\rangle &\leq K\left(\|(I-P)\,x\|^2 + \langle (I-P)\,x, N\,(I-P)\,x\rangle\right) \\ &\leq K\,\langle (I-P)\,x, N\,(I-P)\,x\rangle = K\,\langle x, Nx\rangle\end{aligned}$$

for any $x\in\mathcal{D}(N)$, and so (91) gives

$$\operatorname{Tr}(\rho_t\,(I-P)\,A\,(I-P)) \leq K\ \operatorname{Tr}(\rho_t N) \qquad \forall t\geq 0.$$

Then, using (91), (93) and (94) we obtain (12). □

Appendix

A. Proof of Theorem 4.1

Lemma A.1. *Let Hypothesis 2 hold, except Condition H2.4. For any* $\xi\in L^2_C(\mathbb{P},\mathfrak{h})$*, we define*

$$\begin{aligned}\mathcal{L}_*(\xi,t) &= \mathbb{E}\,|G(t)\,X_t(\xi)\rangle\langle X_t(\xi)| + \mathbb{E}\,|X_t(\xi)\rangle\langle G(t)\,X_t(\xi)| \\ &\quad + \sum_{\ell=1}^{\infty}\mathbb{E}\,|L_\ell(t)\,X_t(\xi)\rangle\langle L_\ell(t)\,X_t(\xi)|\,.\end{aligned}$$

Then $\mathcal{L}_*(\xi,t)$ is a trace-class operator on $\mathfrak{h}$ whose trace-norm is uniformly bounded with respect to t on bounded time intervals; the series involved in the definition of $\mathcal{L}_*$ converges in $\mathfrak{L}_1(\mathfrak{h})$.

Proof. By Condition H2.2, using $\left\| |x\rangle\langle y| \right\|_1 = \|x\| \, \|y\|$ and Lemma 7.3 of [35] we get

$$\begin{aligned}
&\left\| \mathbb{E} \left| G(t) X_t(\xi) \right\rangle \left\langle X_t(\xi) \right| \right\|_1 + \left\| \mathbb{E} \left| X_t(\xi) \right\rangle \left\langle G(t) X_t(\xi) \right| \right\|_1 \\
&\qquad\qquad\qquad + \sum_{\ell=1}^{\infty} \left\| \mathbb{E} \left| L_\ell(t) X_t(\xi) \right\rangle \left\langle L_\ell(t) X_t(\xi) \right| \right\|_1 \\
&\leq 4 \mathbb{E} \left(\|X_t(\xi)\| \, \|G(t) X_t(\xi)\| \right) \leq K(t) \sqrt{\mathbb{E} \|\xi\|^2} \sqrt{\mathbb{E} \|X_t(\xi)\|_C^2},
\end{aligned}$$

where the last inequality follows from Condition H2.1. □

Applying Lemma 7.3 of [35] and Lemma 5.2 we easily obtain Lemma A.2.

Lemma A.2. *Suppose that Hypothesis 2 hold, $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$, and $A \in \mathfrak{L}(\mathfrak{h})$. Then, $t \mapsto \operatorname{Tr}(A\mathcal{L}_*(\xi,t))$ is continuous as a function from $[0,\infty[$ to $\mathbb{C}$, and*

$$\begin{aligned}
\operatorname{Tr}(A\mathcal{L}_*(\xi,t)) &= \mathbb{E} \left\langle X_t(\xi), AG(t) X_t(\xi) \right\rangle + \mathbb{E} \left\langle G(t) X_t(\xi), AX_t(\xi) \right\rangle \\
&\quad + \sum_{\ell=1}^{\infty} \mathbb{E} \left\langle L_\ell(t) X_t(\xi), AL_\ell(t) X_t(\xi) \right\rangle .
\end{aligned}$$

Here, $\mathcal{L}_*(\xi,t)$ is as in Lemma A.1.

Lemma A.3. *Adopt Hypothesis 2, together with $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$. Then*

$$\rho_t = \mathbb{E} \left| \xi \right\rangle \left\langle \xi \right| + \int_0^t \mathcal{L}_*(\xi,s) \, ds, \tag{A.1}$$

where $t \geq 0$ and $\mathcal{L}_*(\xi,s)$ is as in Lemma A.1; we understand the above integral in the sense of Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$.

Proof. Fix $x \in \mathfrak{h}$, and choose $\tau_n = \inf\{s \geq 0 : \|X_s(\xi)\| > n\}$, with $n \in \mathbb{N}$. Applying the complex Itô formula we obtain that

$$\left\langle X_{t\wedge\tau_n}(\xi), x \right\rangle X_{t\wedge\tau_n}(\xi) = \left\langle \xi, x \right\rangle \xi + \mathbb{E} \int_0^{t\wedge\tau_n} L_x(X_s(\xi), s) \, ds + M_t, \tag{A.2}$$

where

$$M_t = \sum_{\ell=1}^{\infty} \int_0^{t\wedge\tau_n} \left(\left\langle X_s(\xi), x \right\rangle L_\ell(s) X_s(\xi) + \left\langle L_\ell(s) X_s(\xi), x \right\rangle X_s(\xi) \right) dW_s^\ell$$

and for any $z \in \mathcal{D}(C)$,

$$L_x(z,s) = \left\langle z, x \right\rangle G(s) z + \left\langle G(s) z, x \right\rangle z + \sum_{k=1}^{\infty} \left\langle L_k(s) z, x \right\rangle L_k(s) z.$$

According to Condition H2.2 we have

$$\begin{aligned}&\mathbb{E}\sum_{\ell=1}^{\infty}\int_0^{t\wedge\tau_n}\left\|\langle X_s(\xi),x\rangle L_\ell(s)X_s(\xi)+\langle L_\ell(s)X_s(\xi),x\rangle X_s(\xi)\right\|^2 ds\\ &\leq 4n^3\|x\|^2\,\mathbb{E}\int_0^{t\wedge\tau_n}\|G(s)X_s\|\,ds.\end{aligned}$$

Therefore $\mathbb{E}M_t=0$ by Condition H2.1, and so (A.2) yields

$$\mathbb{E}\langle X_{t\wedge\tau_n}(\xi),x\rangle X_{t\wedge\tau_n}(\xi)=\mathbb{E}\langle\xi,x\rangle\xi+\mathbb{E}\int_0^{t\wedge\tau_n}L_x(X_s(\xi),s)\,ds. \tag{A.3}$$

We will take the limit as $n\to\infty$ in (A.3). Since $X(\xi)$ has continuous sample paths, $\tau_n\nearrow_{n\to\infty}\infty$. By H2.1 and H2.2, applying the dominated convergence yields

$$\lim_{n\to\infty}\mathbb{E}\int_0^{t\wedge\tau_n}L_x(X_s(\xi),s)\,ds=\mathbb{E}\int_0^t L_x(X_s(\xi),s)\,ds.$$

Combining $\mathbb{E}\left(\sup_{s\in[0,t+1]}\|X_s(\xi)\|^2\right)<\infty$ with the dominated convergence theorem gives $\lim_{n\to\infty}\mathbb{E}\langle X_{t\wedge\tau_n}(\xi),x\rangle X_{t\wedge\tau_n}(\xi)=\mathbb{E}\langle X_t(\xi),x\rangle X_t(\xi)$. Then, letting first $n\to\infty$ in (A.3) and then using Fubini's theorem we get

$$\mathbb{E}\langle X_t(\xi),x\rangle X_t(\xi)=\mathbb{E}\langle\xi,x\rangle\xi+\int_0^t\mathbb{E}L_x(X_s(\xi),s). \tag{A.4}$$

By Condition H2.2, the dominated convergence theorem leads to

$$\mathbb{E}\sum_{k=1}^{\infty}\langle L_k(s)X_s(\xi),x\rangle L_k(s)X_s(\xi)=\sum_{k=1}^{\infty}\mathbb{E}\langle L_k(s)X_s(\xi),x\rangle L_k(s)X_s(\xi),$$

and so Lemma 7.3 of [35] yields $\mathbb{E}L_x(X_s(\xi),s)=\mathcal{L}_*(\xi,s)\,x$, hence

$$\int_0^t\mathbb{E}L_x(X_s(\xi),s)=\int_0^t\mathcal{L}_*(\xi,s)\,xds. \tag{A.5}$$

Since the dual of $\mathfrak{L}_1(\mathfrak{h})$ consists in all linear maps $\varrho\mapsto\operatorname{Tr}(A\varrho)$ with $A\in\mathfrak{L}(\mathfrak{h})$, Lemma A.2 implies that $t\mapsto\mathcal{L}_*(\xi,t)$ is measurable as a function from $[0,\infty[$ to $\mathfrak{L}_1(\mathfrak{h})$. Furthermore, using Lemma A.1 we get that $t\mapsto\mathcal{L}_*(\xi,t)$ is a Bochner integrable $\mathfrak{L}_1(\mathfrak{h})$-valued function on bounded intervals. Then (A.4), together with (A.5), gives (A.1). □

Proof of Theorem 4.1. According to Theorem 3.2 of [35] we have

$$AG(t)\,\rho_t=\mathbb{E}\,|AG(t)\,X_t(\xi)\rangle\langle X_t(\xi)|\,.$$

Since $G(t),L_1(t),L_2(t),\ldots$ are closable, $G(t)^*,L_1(t)^*,L_2(t)^*,\ldots$ are densely defined and $G(t)^{**}$, $L_1(t)^{**},\ldots$ coincide with the closures of $G(t),L_1(t),\ldots$ respectively (see, e.g., Theorem III.5.29 of [44]). Now, Theorem 3.2 of [35] yields $A\rho_tG(t)^*=\mathbb{E}\,|AX_t(\xi)\rangle\langle G(t)\,X_t(\xi)|$ and

$$AL_k(t)\,\rho_tL_k(t)^*=\mathbb{E}\,|AL_k(t)\,X_t(\xi)\rangle\langle L_k(t)\,X_t(\xi)|\,.$$

Therefore

$$\mathcal{L}_*(\xi, t) = G(t)\rho_t + \rho_t G(t)^* + \sum_{k=1}^{\infty} L_k(t)\rho_t L_k(t)^*, \tag{A.6}$$

where $\mathcal{L}_*(\xi, t)$ is as in Lemma A.1. Combining (A.6) with Lemma A.3 we get (17), and so $\mathrm{Tr}(A\rho_t) = \mathrm{Tr}(A\varrho) + \int_0^t \mathrm{Tr}(A\mathcal{L}_*(\xi, s))\, ds$ for all $t \geq 0$. Using the continuity of $\mathcal{L}_*(\xi, \cdot)$ we obtain (16). □

B. Proof of Theorem 4.3

Proof. Using Itô's formula we will prove that for all $x, y \in \mathcal{D}(C)$,

$$\mathbb{E}\langle X_t(x), AX_t(y)\rangle = \langle x, \mathcal{A}_t y\rangle. \tag{B.1}$$

This, together with Theorem 4.2, implies $\mathcal{A}_t = \mathcal{T}_t(A)$.

Motivated by $\mathcal{A}_t$ is only a weak solution, we fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{h}$ and consider the function $F_n : [0, t] \times \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ defined by

$$F_n(s, u, v) = \langle R_n \overline{u}, \mathcal{A}_{t-s} R_n v\rangle,$$

where $R_n = n(n + C)^{-1}$ and $\bar{u} = \sum_{n\in\mathbb{N}} \overline{\langle e_n, u\rangle} e_n$. Since the range of R_n is contained in $\mathcal{D}(C)$,

$$\frac{d}{ds} F_n(s, u, v) = -g(s, R_n \overline{u}, R_n v), \tag{B.2}$$

with $g(s, x, y) = \langle x, \mathcal{A}_{t-s} Gy\rangle + \langle Gx, \mathcal{A}_{t-s} y\rangle + \sum_{k=1}^{\infty} \langle L_k x, \mathcal{A}_{t-s} L_k y\rangle$. We have that $t \longmapsto \langle u, \mathcal{A}_t v\rangle$ is continuous for all $u, v \in \mathfrak{h}$, and so combining $CR_n \in \mathfrak{L}(\mathfrak{h})$ with Hypothesis 3 we get the uniformly continuity of $(s, u, v) \longmapsto g(s, R_n \overline{u}, R_n v)$ on bounded subsets of $[0, t] \times \mathfrak{h} \times \mathfrak{h}$. Then, we can apply Itô's formula to $F_n\left(s \wedge \tau_j, \overline{X_s^{\tau_j}(x)}, X_s^{\tau_j}(y)\right)$, with $\tau_j = \inf\{t \geq 0 : \|X_t(x)\| + \|X_t(y)\| > j\}$.

Fix $x, y \in \mathcal{D}(C)$. Combining Itô's formula with (B.2) we deduce that

$$F_n\left(t \wedge \tau_j, \overline{X_t^{\tau_j}(x)}, X_t^{\tau_j}(y)\right) = F_n\left(0, \overline{X_0(x)}, X_0(y)\right) + I^n_{t\wedge\tau_j} + M_t,$$

where for $s \in [0, t]$:

$$M_s = \sum_{k=1}^{\infty} \int_0^{s\wedge\tau_j} \langle R_n X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n L_k X_r^{\tau_j}(y)\rangle\, dW_r^k + \sum_{k=1}^{\infty} \int_0^{s\wedge\tau_j} \langle R_n L_k X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n X_r^{\tau_j}(y)\rangle\, dW_r^k$$

and $I_s^n = \int_0^s (-g(r, R_n X_r(x), R_n X_r(y)) + g_n(r, X_r(x), X_r(y)))\, dr$ with

$$g_n(r, u, v) = \langle R_n u, \mathcal{A}_{t-r} R_n G v\rangle + \langle R_n G u, \mathcal{A}_{t-r} R_n v\rangle + \sum_{k=1}^{\infty} \langle R_n L_k u, \mathcal{A}_{t-r} R_n L_k v\rangle.$$

We next establish the martingale property of M_s. For all $r \in [0,t]$ we have

$$\|R_n X_r^{\tau_j}(x)\|^2 \|\mathcal{A}_{t-r}\|^2 \|R_n L_k X_r^{\tau_j}(y)\|^2 \le j^2 \sup_{s\in[0,t]} \|\mathcal{A}_s\|^2 \|L_k X_r^{\tau_j}(y)\|^2 .$$

By H2.1 and H3.1, $\mathbb{E}\int_0^{t\wedge\tau_j}\sum_{k=1}^{\infty}\left|\left\langle R_n X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n L_k X_r^{\tau_j}(y)\right\rangle\right|^2 ds < \infty$. Thus $\left(\sum_{k=1}^{\infty}\int_0^{s\wedge\tau_j}\left\langle R_n X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n L_k X_r^{\tau_j}(y)\right\rangle dW_r^k\right)_{s\in[0,t]}$ is a martingale. The same conclusion can be draw for

$$\sum_{k=1}^{\infty}\int_0^{s\wedge\tau_j}\left\langle R_n L_k X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n X_r^{\tau_j}(y)\right\rangle dW_r^k,$$

and so $(M_s)_{s\in[0,t]}$ is a martingale. Hence

$$\mathbb{E}\left\langle R_n X_t^{\tau_j}(x), \mathcal{A}_{t-t\wedge\tau_j} R_n X_t^{\tau_j}(y)\right\rangle = \langle R_n x, \mathcal{A}_t R_n y\rangle + \mathbb{E} I^n_{t\wedge\tau_j}. \tag{B.3}$$

We will take the limit as $j \to \infty$ in (B.3). Since $\mathbb{E}\left(\sup_{s\in[0,t]}\|X_s(\xi)\|^2\right) < \infty$ for $\xi = x, y$ (see, e.g., Th. 4.2.5 of [45]), using the dominated convergence theorem, together with the continuity of $t \longmapsto \langle u, \mathcal{A}_t v\rangle$, we get

$$\mathbb{E}\left\langle R_n X_t^{\tau_j}(x), \mathcal{A}_{t-t\wedge\tau_j} R_n X_t^{\tau_j}(y)\right\rangle \to_{j\to\infty} \mathbb{E}\left\langle R_n X_t(x), A R_n X_t(y)\right\rangle .$$

Applying again the dominated convergence theorem yields $\mathbb{E} I^n_{t\wedge\tau_j} \longrightarrow_{j\to\infty} \mathbb{E} I^n_t$, and hence letting $j \to \infty$ in (B.3) we deduce that

$$\begin{aligned} &\mathbb{E}\left\langle R_n X_t(x), A R_n X_t(y)\right\rangle - \langle R_n x, \mathcal{A}_t R_n y\rangle \\ &= \mathbb{E}\int_0^t \left(-g(s, R_n X_s(x), R_n X_s(y)) + g_n(s, X_s(x), X_s(y))\right) ds. \end{aligned} \tag{B.4}$$

Finally, we take the limit as $n \to \infty$ in (B.4). Since $\|R_n\| \le 1$ and R_n tends pointwise to I as $n \to \infty$, the dominated convergence theorem yields

$$\lim_{n\to\infty}\mathbb{E}\int_0^t g_n(s, X_s(x), X_s(y))\, ds = \mathbb{E}\int_0^t g(s, X_s(x), X_s(y))\, ds.$$

For any $x \in \mathcal{D}(C)$, $\lim_{n\to\infty} C R_n x = Cx$. By $\|C R_n x\| \le \|Cx\|$, using the dominated convergence theorem gives

$$\lim_{n\to\infty}\mathbb{E}\int_0^t g(s, R_n X_s(x), R_n X_s(y))\, ds = \mathbb{E}\int_0^t g(s, X_s(x), X_s(y))\, ds.$$

Thus, letting $n \to \infty$ in (B.4) we obtain (B.1). □

References

[1] H.P. Breuer and F. Petruccione. *The theory of open quantum systems.* Oxford University Press, 2002.

[2] H. Spohn. Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Modern Phys.*, 52:569–615, 1980.

[3] K. Hepp and E. H. Lieb. Phase transitions in reservoir-driven open systems with applications to lasers and superconductors. *Helv. Phys. Acta*, 46:573–602, 1974.

[4] H. Haken. *Light Vol. II: Laser light dynamics.* North Holland, 1985.

[5] J. Ohtsubo. *Semiconductor Lasers*, volume 11 of *Springer Series in Optical Sciences.* Springer, Berlin, third edition, 2013.

[6] G. H. M. van Tartwijk and G. P. Agrawal. Laser instabilities: a modern perspective. *Prog. Quant. Electron.*, 22:43–122, 1998.

[7] A. I. Khanin. *Fundamentals of Laser Dynamics*. Cambridge, 2006.

[8] M. Sciamanna and K. A. Shore. Physics and applications of laser diode chaos. *Nat. Photonics*, 9:151–162, 2015.

[9] C. O. Weiss and R. Vilaseca. *Dynamics of Lasers*. Weinheim, 1991.

[10] L. Larger and J. M. Dudley. Optoelectronic chaos. *Nature*, 465:41–42, 2010.

[11] M. Virte, K. Panajotov, H. Thienpont, and M. Sciamanna. Deterministic polarization chaos from a laser diode. *Nat. Photonics*, 7:60–65, 2013.

[12] C. O. Weiss, A. Godone, and A. Olafsson. Routes to chaotic emission in a cw he-ne laser. *Phys. Rev. A*, 28:892–895, 1983.

[13] A. Argyris et al. Chaos-based communications at high bit rates using commercial fibre-optic links. *Nature*, 438:343–346, 2005.

[14] A. Uchida. *Optical Communication with Chaotic Lasers: Applications of Nonlinear Dynamics and Synchronization*. Wiley-VCH, 2012.

[15] I. Kanter, Y. Aviad, I. Reidler, E. Cohen, and M. Rosenbluh. An optical ultrafast random bit generator. *Nat. Photonics*, 4:58–61, 2010.

[16] M. Virte, E. Mercier, H. Thienpont, K. Panajotov, and M. Sciamanna. Physical random bit generation from chaotic solitary laser diode. *Opt. Express*, 22:17271–17280, 2014.

[17] V. V. Korobkin and A. V. Uspensky. On the theory of the pulsations in the output of the ruby laser. *Sov. Phys. JETP*, 18:693–697, 1964. (Russian original - Zh. Eksp. Teor. Fiz. 1963. V. 45. P. 1003).

[18] H. Haken. Analogy between higher instabilities in fluids and lasers. *Phys. Lett. A.*, 53:77–78, 1975.

[19] M. A. Armen and H. Mabuchi. Low-lying bifurcations in cavity quantum electrodynamics. *Phys. Rev. A*, 73:063801, 2006.

[20] A. C. Fowler, J. D. Gibbon, and M. McGuinness. The complex Lorenz equations. *Physica D*, 4:139–163, 1982.

[21] D. Braun. *Dissipative Quantum Chaos and Decoherence*. Springer, 2001.

[22] H. J. Stöckmann. *Quantum Chaos: an Introduction*. Cambridge University Press,, 1999.

[23] S. Wimberger. *Nonlinear Dynamics and Quantum Chaos: An Introduction*. Graduate Texts in Physics. Springer, 1 edition, 2014.

[24] A. Arnold and C. Sparber. Quantum dynamical semigroups for diffusion models with hartree interaction. *Comm. Math. Phys.*, 251:179–207, 2004.

[25] M. V. Ivanchenko, E. A. Kozinov, V. D. Volokitin, A. V. Liniov, I. B. Meyerov, and S. V. Denisov. Classical bifurcation diagrams by quantum means. *Ann. Phys.*, 529:1600402, 2017.

[26] C. P. Meaney, T. Duty, R. H. McKenzie, and G. J. Milburn. The jahn-teller instability in dissipative quantum electromechanical systems. *Phys. Rev. A*, 81:043805, 2010.

[27] F. Fagnola and R. Rebolledo. On the existence of stationary states for quantum dynamical semigroups. *J. Math. Phys.*, 42:11296–1308, 2001.

[28] F. Fagnola and R. Rebolledo. Lectures on the qualitative analysis of quantum markov semigroups. In *Quantum interacting particle systems (Trento, 2000)*, volume 14 of *QP-PQ: Quantum Probab. White Noise Anal.*, pages 197–239. World Sci. Publ., 2002.

[29] F. Fagnola and R. Rebolledo. Subharmonic projections for a quantum markov semigroup. *J. Math. Phys.*, 43:1074–1082, 2002.

[30] L. Accardi, F. Fagnola, and S. Hachicha. Generic q-markov semigroups and speed of convergence of q-algorithms. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 9:567–594, 2006.

[31] R. Carbone and F. Fagnola. Exponential L^2-convergence of quantum markov semigroups on $B(h)$. *Math. Notes*, 68:452–463, 2000.

[32] R. Carbone, F. Fagnola, J. C. Garca, and R. Quezada. Spectral properties of the two-photon absorption and emission process. *J. Math. Phys.*, 49:032106, 2008.

[33] C. Z. Ning and H. Haken. Detuned lasers and the complex Lorenz equations: Subcritical and supercritical Hopf bifurcations. *Phys. Rev. A*, 41:3826–3837, 1990.

[34] A. M. Chebotarev, J. Garcia, and R. Quezada. *A priori estimates and existence theorems for the Lindblad equation with unbounded time-dependent coefficients*, volume 1035, pages 44–65. Publ. Res. Inst. Math. Sci. Kokyuroku, 1998.

[35] C.M. Mora. Regularity of solutions to quantum master equations: a stochastic approach. *Ann. Probab.*, 41:1978–2012, 2013.

[36] F. Fagnola and C. M. Mora. Stochastic Schrödinger equations and applications to Ehrenfest-type theorems. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 10:191–223, 2013.

[37] C.M. Mora. Numerical simulation of stochastic evolution equations associated to quantum Markov semigroups. *Math. Comp.*, 73:1393–1415, 2004.

[38] C.M. Mora and R. Rebolledo. Regularity of solutions to linear stochastic Schrödinger equations. *Infinite Dimens. Anal. Quantum Probab. Rel. Topics*, 10:237–259, 2007.

[39] C.M. Mora. Heisenberg evolution of quantum observables represented by unbounded operators. *J. Funct. Anal.*, 255:3249–3273, 2008.

[40] A.M. Chebotarev and F. Fagnola. Sufficient conditions for conservativity of quantum dynamical semigroups. *J. Funct. Anal.*, 118:131–153, 1993.

[41] A.M. Chebotarev and F. Fagnola. Sufficient conditions for conservativity of minimal quantum dynamical semigroups. *J. Funct. Anal.*, 153:382–404, 1998.

[42] F. Fagnola. Quantum markov semigroups and quantum flows. *Proyecciones*, 18:1–144, 1999.

[43] M.W. Hirsch, S. Smale, and R. L. Devaney. *Differential equations, dynamical systems, and an introduction to chaos. Second edition.* Elsevier, Amsterdam, 2013.

[44] T. Kato. *Perturbation theory for linear operators. Corrected printing of the second edition.* Springer, 1980.

[45] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[46] K. R. Parthasarathy. *An introduction to quantum stochastic calculus.* Birkhäuser, Basel, 1992.

[47] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions.* Cambridge University Press, 1992.

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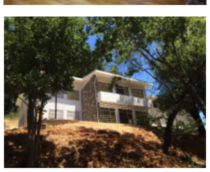

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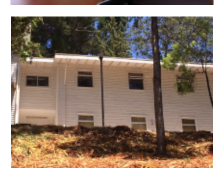

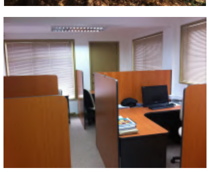

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