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A fully-mixed finite element method for the n-dimensional Boussinesq problem with temperature-dependent parameters*

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Abstract

In this paper, we introduce and analyze a high-order, fully-mixed finite element method for the free convection of n-dimensional fluids, $n \in \{2, 3\}$, with temperature-dependent viscosity and thermal conductivity. The mathematical model is given by the coupling of the equations of continuity, momentum (Navier-Stokes) and energy by means of the Boussinesq approximation, as well as mixed thermal boundary conditions and a Dirichlet condition on the velocity. Because of the dependence on the temperature of the fluid properties, several additional variables are defined, thus resulting in an augmented formulation that seeks the rate of strain, pseudostress and vorticity tensors, velocity, temperature gradient and pseudoheat vectors, and temperature of the fluid. Using a fixed-point approach, smallness-of-data assumptions and a slight higher-regularity assumption for the exact solution provide the necessary well-posedness results at both continuous and discrete levels. In addition, and as a result of the augmentation, no discrete inf-sup conditions are needed for the well-posedness of the Galerkin scheme, which provides freedom of choice with respect to the finite element spaces. In particular, we suggest a combination based on Raviart-Thomas, Lagrange and discontinuous elements for which we derive optimal a priori error estimates. Finally, several numerical examples illustrating the performance of the method and confirming the theoretical rates of convergence are reported.

Key words: Boussinesq equations, augmented fully-mixed formulation, fixed-point theory, finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

Free convective flows can be found in a wide amount of settings throughout nature and industry, for instance, in mantle convection, stratified oceanic flows and the cooling of electronic devices, to name a few. Many of these processes can be modeled by coupling the equations of continuity, momentum (Navier-Stokes) and energy using the Boussinesq approximation, which (in this context) assumes the density of the fluid to be constant in all terms of the equations, except in the buoyancy term of the momentum equation, where a linear dependence is considered. Nevertheless, other properties may also vary with temperature, as is the case of, for example, viscosity and thermal conductivity in oils and nanofluids, which poses a significant effect on the fluid flow. In this regard, several finite

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element methods to approximate the solution to this and related problems, both with constant and temperature-dependent properties have been proposed (see [2, 3, 4, 5, 6, 8, 11, 12, 13, 20, 21, 24, 25, 26] and the references therein).

In particular, the authors in [20, 21] propose finite element methods based on primal formulations of the Boussinesq system. The first one deals with the problem in its primitive variables, while the second one introduces the normal heat flux through the boundary as an additional variable to achieve conformity of the scheme. Nonetheless, both methods are proved to be optimally convergent whenever the exact solution is smooth enough, and the data and the $\mathrm{W}^{1,\infty}$-norm of the velocity and temperature are small enough. More recently in [2], while the authors still consider a primal formulation of the energy equation with a space-dependent thermal conductivity, they also consider a mixed formulation of the momentum and continuity equations but with a temperature-dependent viscosity in a pseudostress-velocity-vorticity formulation. Hence, using fixed-point strategies from [5, 11], and using Raviart-Thomas elements to approximate the pseudostress, Lagrange elements for the velocity and temperature, and discontinuous elements for the vorticity and normal heat flux, they are able to construct an optimally-convergent method whenever the exact solution is smooth enough, and the data is sufficiently small. However, the presence of a variable viscosity leads to restrict the analysis to the two-dimensional case, as it becomes necessary to use Sobolev embeddings into smaller L^{p} spaces. To overcome this drawback, recent work [4] has shown that, by defining the rate of strain tensor as a new variable (in addition to the pseudostress, velocity and vorticity), the analysis is now valid for two and three-dimensional domains.

The purpose of this work is to extend the analysis and results of [4] by deriving now an augmented fully-mixed finite element method for the Boussinesq problem, but considering this time both the viscosity and the thermal conductivity of the fluid as temperature-dependent functions, and mixed thermal boundary conditions. To this end, we consider again the mixed formulation of the Navier-Stokes equations in [4], to then, using this same approach, construct a mixed formulation for the energy equation. More precisely, we consider the temperature gradient and pseudoheat vector as additional variables, which together with the temperature, rate of strain, pseudostress, velocity and vorticity comprise the unknowns of the problem. At this point, we remark that the main difference with respect to [12] is the consideration in this work of temperature-dependent parameters, which also becomes the cause of defining the rate of strain and temperature gradient in addition to the variables defined in the aforementioned work (recall from [2] that the vorticity appears in this formulation because of the consideration of a more physical version of the Cauchy stress tensor). That being said, part of our analysis follows basically the same uncoupling and fixed-point strategies from [2, 4, 5, 11, 12], reason why we do not provide all details but make the proper references when it corresponds. At a continuous level, we prove that the uncoupled problems are well-posed thanks to the Lax-Milgram theorem, and then we prove that the fixed-point operator admits a unique fixed point by means of the Banach fixed-point theorem, whenever the data is sufficiently small and assuming that the exact solution has a slightly higher regularity than the one the well-posedness results provide. Then, following these same steps, we provide a well-posedness result for the Galerkin scheme where one of the key features of this work can be appreciated: there is no need to impose inf-sup conditions on the discrete analysis, which gives us the freedom to choose any combination of finite element subspaces. In particular, we approximate the pseudostress and pseudoheat variables using Raviart-Thomas elements of order k, the velocity and temperature using Lagrange elements of order $k+1$, and the rate of strain, vorticity and temperature gradient using just discontinuous piecewise polynomials of degree $\leq k$. When the data are sufficiently small, optimal a priori error estimates can be derived thanks to the Strang lemma, and these are later verified with numerical examples in two and three-dimensional domains.

Finally, we consider worthwhile to mention other features of this fully-mixed method. First, as Dirichlet conditions appear naturally in mixed formulations, it is not necessary to define boundary unknowns to achieve conformity in the scheme (see the Lagrange multiplier defined in [2, 4]), thus unifying the analysis of the uncoupled problems and simplifying the computational implementation of this method. Also, as not only the velocity and temperature of the fluid are part of the solution but also their gradients, many other physical variables of interest can be computed as a simple post-process without requiring numerical differentiations that could deteriorate the good quality of the results.

1.1 Outline

The rest of this work is organized as follows. First, we end this section by introducing some notation that will be used throughout the paper. Then, in Section 2, we set the Boussinesq problem with temperature-dependent viscosity and thermal conductivity functions, and introduce the new variables that will allow us to construct a fully-mixed formulation. Next, in Section 3, we uncouple the problem using a fixed-point argument. The uncoupled problems are then analyzed by means of the Lax-Milgram theorem, and existence of a unique fixed point is proved by fulfilling the hypotheses of the Banach fixed-point theorem. Later, in Section 4 these techniques are used to prove the well-posedness of the corresponding Galerkin scheme, but this time using the Brouwer fixed-point theorem, and then we make a specific choice of finite element subspaces. Finally, in Section 5 we derive some *a priori* error estimates using Strang's lemmas, to then in Section 6 illustrate the good performance of this augmented fully-mixed finite element method and confirm the theoretical rates of convergence through several numerical examples in two and three dimensions.

1.2 Preliminaries

Let us denote by $\Omega \subset \mathrm{R}^n$, $n \in \{2,3\}$, a given bounded domain with polyhedral boundary Γ, and denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,2}(\Omega) =: \mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions in $\mathrm{H}^1(\Omega)$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,\dots,n}$ and $\mathbf{w} = (w_i)_{i=1,\dots,n}$, we set the gradient, divergence and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,\dots,n}, \quad \mathrm{div}\,\mathbf{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,\dots,n}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,\dots,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,\dots,n}$, we let $\mathbf{div}\,\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,\dots,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\,\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathrm{R}^{n\times n}$. Furthermore, we recall that

$$\mathbb{H}(\mathbf{div};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \ \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) \Big\},$$

equipped with the usual norm

$$\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega},$$

is a standard Hilbert space in the realm of mixed problems. Finally, in what follows, $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathrm{R}^n$. Also, we employ $\mathbf{0}$ to denote a generic null vector and use C, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

2 The Model Problem

In this section, we first introduce the Boussinesq problem with its original unknowns, to then define suitable new variables that will later allow us to construct a fully-mixed formulation.

2.1 The original formulation

Let us consider the flow of a non-isothermal, incompressible, Newtonian fluid with varying viscosity and thermal conductivity within a region Ω. Then, under the Boussinesq approximation, the problem reads: Find a velocity $\mathbf{u}$, a pressure p and a temperature φ such that

$$-\mathbf{div}\,(\mu(\varphi)\mathbf{e}(\mathbf{u})) + (\nabla\mathbf{u})\mathbf{u} + \nabla p - \varphi\mathbf{g} = \mathbf{f}^{\mathtt{m}} \quad \text{in } \Omega, \tag{2.1a}$$

$$\mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.1b}$$

$$-\mathrm{div}\,(k(\varphi)\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi = f^{\mathtt{e}} \quad \text{in } \Omega, \tag{2.1c}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.1d}$$

$$\varphi = \varphi_D \quad \text{on } \Gamma_D, \tag{2.1e}$$

$$k(\varphi)\nabla\varphi\cdot\boldsymbol{\nu} = 0 \quad \text{on } \Gamma_N, \tag{2.1f}$$

where the boundary Γ of Ω is split as $\Gamma = \Gamma_D \cup \Gamma_N$, $-\mathbf{g} \in \mathbf{L}^\infty(\Omega)$ is an external force per unit mass (e.g. gravity force, centrifugal force, Coriolis force), $\mathbf{f}^{\mathtt{m}} \in \mathbf{L}^2(\Omega)$ and $f^{\mathtt{e}} \in \mathrm{L}^2(\Omega)$ are source terms, $\varphi_D \in \mathrm{H}^{1/2}(\Gamma_D)$ is a prescribed temperature and $\mu, k : \mathrm{R} \to \mathrm{R}^+$ are the temperature-dependent viscosity and thermal conductivity functions, respectively, which are assumed to be bounded above and below by positive constants, that is, there exist $\mu_2 \geq \mu_1 > 0$ and $k_2 \geq k_1 > 0$ such that

$$\mu_1 \leq \mu(t) \leq \mu_2 \quad \text{and} \quad k_1 \leq k(t) \leq k_2, \quad \forall\, t \in \mathrm{R}. \tag{2.2}$$

We also assume that μ and k are Lipschitz continuous functions, that is, there exist $L_\mu, L_k > 0$ such that

$$|\mu(s) - \mu(t)| \leq L_\mu |s-t| \quad \text{and} \quad |k(s) - k(t)| \leq L_k |s-t| \quad \forall\, s,t \in \mathrm{R}\,. \tag{2.3}$$

Notice that the difference in (2.1) with respect to the previous work [2] relies on the introduction of a temperature-dependent thermal conductivity, the introduction of mixed boundary conditions for the energy equation, and the presence of source terms, which albeit used in the numerical examples of [2, Section 6], they were not considered in the theoretical results, and therefore, we include them for clarity purposes.

2.2 Introduction of new variables

Let us first consider the spaces

$$\mathbb{L}^2_{\mathtt{tr}}(\Omega) := \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \ \mathbf{s} = \mathbf{s}^{\mathtt{t}} \quad \text{and} \quad \mathrm{tr}(\mathbf{s}) = 0 \Big\}, \tag{2.4}$$

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) := \Big\{ \boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \ \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} = \mathbf{0} \Big\}, \tag{2.5}$$

and, in a similar way to [4], define the following variables, known respectively as the rate of strain, pseudostress and vorticity tensors:

$$\mathbf{t} := \mathbf{e}(\mathbf{u}), \quad \boldsymbol{\sigma} := \mu(\varphi)\mathbf{e}(\mathbf{u}) - \mathbf{u}\otimes\mathbf{u} - p\mathbb{I}, \quad \boldsymbol{\gamma} := \boldsymbol{\omega}(\mathbf{u}), \tag{2.6}$$

where $\mathbf{e}(\mathbf{u})$ and $\boldsymbol{\omega}(\mathbf{u})$ are respectively the symmetric and skew-symmetric parts of the velocity gradient tensor $\nabla\mathbf{u}$. Then, the momentum and continuity equations (2.1a) and (2.1b) can be rewritten as:

$$
\begin{aligned}
-\nabla\mathbf{u} + \mathbf{t} + \boldsymbol{\gamma} &= 0 \quad \text{in } \Omega, && (2.7a)\\
\mu(\varphi)\mathbf{t} - (\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} - \boldsymbol{\sigma}^{\mathtt{d}} &= 0 \quad \text{in } \Omega, && (2.7b)\\
-\mathbf{div}\,\boldsymbol{\sigma} - \varphi\mathbf{g} &= \mathbf{f}^{\mathtt{m}} \quad \text{in } \Omega, && (2.7c)\\
\mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma, && (2.7d)\\
\int_\Omega \operatorname{tr}(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) &= 0. && (2.7e)
\end{aligned}
$$

Notice that the continuity equation is implicitly present in (2.7b), and it suggests us that rate of strain tensor $\mathbf{t}$ must be sought in $\mathbb{L}^2_{\mathtt{tr}}(\Omega)$, whereas (2.7a) suggests the vorticity tensor to be sought in $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$. Consequently, to characterize these tensors in the two-dimensional case, we only need to know two of the four components of the rate of strain tensor, and only one component of the vorticity tensor (similar simplifications hold for the three-dimensional case).

Next, in order to construct a mixed formulation for the energy equation, we follow the approach taken in [9, 12] and define

$$\mathbf{p} := k(\varphi)\nabla\varphi - \varphi\mathbf{u}, \tag{2.8}$$

which from now on will be called "pseudoheat". In addition, analogously to the mixed formulation for the momentum equation, we consider the temperature gradient as another new variable

$$\boldsymbol{\zeta} := \nabla\varphi. \tag{2.9}$$

Therefore, the energy equation (2.1c) can be rewritten in these terms as

$$
\begin{aligned}
-\nabla\varphi + \boldsymbol{\zeta} &= 0 \quad \text{in } \Omega, && (2.10a)\\
k(\varphi)\boldsymbol{\zeta} - \varphi\mathbf{u} - \mathbf{p} &= 0 \quad \text{in } \Omega, && (2.10b)\\
-\operatorname{div}\mathbf{p} &= f^{\mathtt{e}} \quad \text{in } \Omega, && (2.10c)\\
\varphi &= \varphi_D \quad \text{on } \Gamma_D, && (2.10d)\\
\mathbf{p}\cdot\boldsymbol{\nu} &= 0 \quad \text{on } \Gamma_N, && (2.10e)
\end{aligned}
$$

where the Neumann condition (2.1f) has been converted to (2.10e) thanks to the no-slip condition $\mathbf{u} = \mathbf{0}$ on Γ.

In this way, the Boussinesq problem (2.1) can now be seen as the set of equations (2.7) and (2.10). Then, a mixed formulation of each one of them can be constructed upon integration by parts of (2.7a) and (2.10a) when multiplied by proper test functions, which is the purpose of the next section.

3 The continuous problem

We now turn to the construction and analysis of a fully-mixed formulation for the Boussinesq problem introduced in the previous section.

3.1 An augmented fully-mixed formulation

First, we recall from [4] the equations corresponding to the mixed formulation of the momentum equation:

$$\int_\Omega \mathbf{t} : \boldsymbol{\tau}^{\mathtt{d}} + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} + \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} = 0 \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega), \tag{3.1}$$

$$\int_\Omega \mu(\varphi)\mathbf{t} : \mathbf{s} - \int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} : \mathbf{s} - \int_\Omega \boldsymbol{\sigma}^{\mathtt{d}} : \mathbf{s} = 0 \quad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega), \tag{3.2}$$

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} = \int_\Omega \varphi\mathbf{g}\cdot\mathbf{v} + \int_\Omega \mathbf{f}^{\mathtt{m}}\cdot\mathbf{v} \quad \forall\, (\mathbf{v},\boldsymbol{\eta}) \in \mathbf{L}^2(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega), \tag{3.3}$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega) := \left\{ \boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega) : \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \right\}.$$

Then, to obtain a mixed formulation for the energy equation, we multiply (2.10a) by a test function $\mathbf{q}\in\mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega)$, where

$$\mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) := \left\{ \mathbf{q}\in\mathbf{H}(\mathrm{div};\Omega) :\ \mathbf{q}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma_N \right\}, \tag{3.4}$$

and integrate by parts. Using the boundary condition (2.10d), we get

$$\int_\Omega \boldsymbol{\zeta}\cdot\mathbf{q} + \int_\Omega \varphi\ \mathrm{div}\,\mathbf{q} = \langle\, \mathbf{q}\cdot\boldsymbol{\nu}, \varphi_D \,\rangle_{\Gamma_D} \quad \forall\, \mathbf{q}\in\mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega)\,, \tag{3.5}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_D}$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Gamma_D)$ and $\mathrm{H}^{1/2}(\Gamma_D)$. Next, we only multiply (2.10b) and (2.10c) by proper test functions:

$$\int_\Omega k(\varphi)\boldsymbol{\zeta}\cdot\boldsymbol{\chi} - \int_\Omega \varphi\mathbf{u}\cdot\boldsymbol{\chi} - \int_\Omega \mathbf{p}\cdot\boldsymbol{\chi} = 0 \quad \forall\, \boldsymbol{\chi}\in\mathbf{L}^2(\Omega), \tag{3.6}$$

and

$$-\int_\Omega \psi\ \mathrm{div}\,\mathbf{p} = \int_\Omega f^{\mathtt{e}}\psi \quad \forall\, \psi\in\mathrm{L}^2(\Omega). \tag{3.7}$$

Notice that, due to the second term in both (3.2) and (3.6), we require the velocity and temperature to have (weak) bounded derivatives as shown in the following inequalities, which can be obtained by using the Hölder inequality and the continuous injections $i : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$ and $\mathbf{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ (cf. [1, Theorem 4.12], [22, Theorem 1.3.4]):

$$\left| \int_\Omega (\mathbf{u}\otimes\mathbf{w})^{\mathtt{d}} : \mathbf{s} \right| \le c_1(\Omega)\|\,\mathbf{u}\,\|_{1,\Omega}\|\,\mathbf{w}\,\|_{1,\Omega}\|\,\mathbf{s}\,\|_{0,\Omega} \quad \forall\, \mathbf{u},\mathbf{w}\in\mathbf{H}^1(\Omega),\ \mathbf{s}\in\mathbb{L}^2(\Omega); \tag{3.8}$$

$$\left| \int_\Omega \varphi\mathbf{u}\cdot\boldsymbol{\chi} \right| \le c_2(\Omega)\|\,\varphi\,\|_{1,\Omega}\|\,\mathbf{u}\,\|_{1,\Omega}\|\,\boldsymbol{\chi}\,\|_{0,\Omega}, \quad \forall\, \varphi\in\mathrm{H}^1(\Omega),\ \mathbf{u}\in\mathbf{H}^1(\Omega),\ \boldsymbol{\chi}\in\mathbf{L}^2(\Omega), \tag{3.9}$$

where $c_1(\Omega)$ and $c_2(\Omega)$ are positive constants that depend solely on $\|i\|$ and $\|\mathbf{i}\|$. Therefore, at a first glance, the fully-mixed formulation for the Boussinesq problem (2.1) is composed by equations (3.1)-(3.3) and (3.5)-(3.7). Nevertheless, to properly analyze the problem, and to achieve conformity of the scheme, we augment it with redundant Galerkin-type terms that arise from the modelling equations, namely

$$\kappa_1 \int_\Omega \left\{ \boldsymbol{\sigma}^{\mathtt{d}} + (\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} - \mu(\varphi)\mathbf{t} \right\} : \boldsymbol{\tau}^{\mathtt{d}} = 0 \qquad \forall\, \boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega), \tag{3.10}$$

$$\kappa_2 \int_\Omega \left\{ \mathbf{div}\,\boldsymbol{\sigma} + \varphi\mathbf{g} \right\}\cdot\mathbf{div}\,\boldsymbol{\tau} = -\kappa_2 \int_\Omega \mathbf{f}^{\mathtt{m}}\cdot\mathbf{div}\,\boldsymbol{\tau} \quad \forall\, \boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega), \tag{3.11}$$

$$\kappa_3 \int_\Omega \left\{ \mathbf{e}(\mathbf{u}) - \mathbf{t} \right\} : \mathbf{e}(\mathbf{v}) = 0 \qquad \forall\, \mathbf{v}\in\mathbf{H}^1_0(\Omega), \tag{3.12}$$

$$\kappa_4 \int_\Omega \left\{ \boldsymbol{\gamma} - \boldsymbol{\omega}(\mathbf{u}) \right\} : \boldsymbol{\eta} = 0 \qquad \forall\, \boldsymbol{\eta}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega)\,, \tag{3.13}$$

for the momentum and continuity equations, and

$$\kappa_5 \int_\Omega \Big\{ \mathbf{p} + \varphi \mathbf{u} - k(\varphi)\boldsymbol{\zeta} \Big\} \cdot \mathbf{q} = 0 \qquad \forall\, \mathbf{q} \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega), \tag{3.14}$$

$$\kappa_6 \int_\Omega \mathrm{div}\,\mathbf{p}\ \mathrm{div}\,\mathbf{q} = -\kappa_6 \int_\Omega f^{\mathtt{e}}\ \mathrm{div}\,\mathbf{q} \quad \forall\, \mathbf{q} \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega), \tag{3.15}$$

$$\kappa_7 \int_\Omega \Big\{ \nabla\varphi - \boldsymbol{\zeta} \Big\} \cdot \nabla\psi = 0 \qquad \forall\, \psi \in \mathrm{H}^1(\Omega), \tag{3.16}$$

$$\kappa_8 \int_{\Gamma_D} \varphi\ \psi = \kappa_8 \int_{\Gamma_D} \varphi_D\ \psi \qquad \forall\, \psi \in \mathrm{H}^1(\Omega), \tag{3.17}$$

for the energy equations, where κ_j, $j \in \{1,\dots,8\}$, are positive constants to be specified later on. Here, equations (3.10)-(3.13) are extracted directly from [4], whereas (3.14)-(3.17) are constructed using the same principles. In this way, denoting by

$$\mathcal{H} := \mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1_0(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega), \quad \mathcal{Q} := \mathbf{L}^2(\Omega) \times \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) \times \mathrm{H}^1(\Omega),$$

$$\vec{\mathbf{t}} := (\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma}), \quad \vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}), \quad \vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta},\mathbf{p},\varphi), \quad \vec{\boldsymbol{\chi}} := (\boldsymbol{\chi},\mathbf{q},\psi),$$

the augmented fully-mixed formulation for this Boussinesq problem is: Find $(\vec{\mathbf{t}},\vec{\boldsymbol{\zeta}}) \in \mathcal{H} \times \mathcal{Q}$ such that

$$\mathbf{A}_\varphi(\vec{\mathbf{t}},\vec{\mathbf{s}}) + \mathbf{B}_{\mathbf{u}}(\vec{\mathbf{t}},\vec{\mathbf{s}}) = F_\varphi(\vec{\mathbf{s}}) + F_{\mathtt{m}}(\vec{\mathbf{s}}) \qquad \forall\, \vec{\mathbf{s}} \in \mathcal{H}, \tag{3.18a}$$

$$\mathbf{C}_\varphi(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}}) + \mathbf{D}_{\mathbf{u}}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}}) = G_D(\vec{\boldsymbol{\chi}}) + G_{\mathtt{e}}(\vec{\boldsymbol{\chi}}) \quad \forall\, \vec{\boldsymbol{\chi}} \in \mathcal{Q}, \tag{3.18b}$$

where, given $(\mathbf{w},\phi) \in \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$, the forms $\mathbf{A}_\phi$, $\mathbf{B}_{\mathbf{w}}$, $\mathbf{C}_\phi$, $\mathbf{D}_{\mathbf{w}}$ and the functionals F_ϕ, $F_{\mathtt{m}}$, G_D, $G_{\mathtt{e}}$ are defined as:

$$\begin{aligned}\mathbf{A}_\phi(\vec{\mathbf{t}},\vec{\mathbf{s}}) &:= \int_\Omega \mu(\phi)\mathbf{t} : \{\mathbf{s} - \kappa_1\boldsymbol{\tau}^{\mathtt{d}}\} + \int_\Omega \mathbf{t} : \{\boldsymbol{\tau}^{\mathtt{d}} - \kappa_3\mathbf{e}(\mathbf{v})\} - \int_\Omega \boldsymbol{\sigma}^{\mathtt{d}} : \{\mathbf{s} - \kappa_1\boldsymbol{\tau}^{\mathtt{d}}\} \\ &+ \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} - \int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \int_\Omega \boldsymbol{\eta} : \boldsymbol{\sigma} - \kappa_4 \int_\Omega \boldsymbol{\omega}(\mathbf{u}) : \boldsymbol{\eta} \\ &+ \kappa_2 \int_\Omega \mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + \kappa_3 \int_\Omega \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}) + \kappa_4 \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\eta},\end{aligned} \tag{3.19}$$

$$\mathbf{B}_{\mathbf{w}}(\vec{\mathbf{t}},\vec{\mathbf{s}}) := -\int_\Omega (\mathbf{u}\otimes\mathbf{w})^{\mathtt{d}} : \{\mathbf{s} - \kappa_1\boldsymbol{\tau}^{\mathtt{d}}\}, \tag{3.20}$$

for all $\vec{\mathbf{t}},\vec{\mathbf{s}} \in \mathcal{H}$;

$$\begin{aligned}\mathbf{C}_\phi(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}}) &:= \int_\Omega k(\phi)\boldsymbol{\zeta}\cdot\{\boldsymbol{\chi} - \kappa_5\mathbf{q}\} + \int_\Omega \boldsymbol{\zeta}\cdot\{\mathbf{q} - \kappa_7\nabla\psi\} - \int_\Omega \mathbf{p}\cdot\{\boldsymbol{\chi} - \kappa_5\mathbf{q}\} \\ &+ \int_\Omega \varphi\ \mathrm{div}\,\mathbf{q} - \int_\Omega \psi\ \mathrm{div}\,\mathbf{p} + \kappa_6\int_\Omega \mathrm{div}\,\mathbf{p}\ \mathrm{div}\,\mathbf{q} + \kappa_7\int_\Omega \nabla\varphi\cdot\nabla\psi + \kappa_8\int_{\Gamma_D} \varphi\ \psi,\end{aligned} \tag{3.21}$$

$$\mathbf{D}_{\mathbf{w}}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}}) := -\int_\Omega \varphi\mathbf{w}\cdot\{\boldsymbol{\chi} - \kappa_5\mathbf{q}\}, \tag{3.22}$$

for all $\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}} \in \mathcal{Q}$;

$$F_\phi(\vec{\mathbf{s}}) := \int_\Omega \phi\mathbf{g}\cdot\{\mathbf{v} - \kappa_2\mathbf{div}\,\boldsymbol{\tau}\}, \tag{3.23}$$

$$F_{\mathtt{m}}(\vec{\mathbf{s}}) := \int_\Omega \mathbf{f}^{\mathtt{m}}\cdot\{\mathbf{v} - \kappa_2\mathbf{div}\,\boldsymbol{\tau}\}, \tag{3.24}$$

for all $\vec{\mathbf{s}} \in \mathcal{H}$;

$$G_D(\vec{\boldsymbol{\chi}}) := \langle \mathbf{q}\cdot\boldsymbol{\nu}, \varphi_D \rangle_{\Gamma_D} + \kappa_8 \int_{\Gamma_D} \varphi_D\, \psi, \tag{3.25}$$

$$G_{\mathsf{e}}(\vec{\boldsymbol{\chi}}) := \int_\Omega f^{\mathsf{e}} \left\{\psi - \kappa_6 \mathrm{div}\,\mathbf{q}\right\}, \tag{3.26}$$

for all $\vec{\boldsymbol{\chi}} \in \mathcal{Q}$.

Before we continue, let us have a brief look at what the energy equation (2.1c) would have looked like if we had considered instead a heat-temperature mixed formulation. Indeed, if we define $\widetilde{\mathbf{p}} := k(\varphi)\nabla\varphi$ in Ω, (2.1c) can be rewritten as

$$\frac{1}{k(\varphi)}\widetilde{\mathbf{p}} - \nabla\varphi = 0 \quad \text{in } \Omega, \qquad -\mathrm{div}\,\widetilde{\mathbf{p}} + \mathbf{u}\cdot\nabla\varphi = f^{\mathsf{e}} \quad \text{in } \Omega$$
$$\varphi = \varphi_D \quad \text{on } \Gamma_D, \qquad \widetilde{\mathbf{p}}\cdot\boldsymbol{\nu} = 0 \quad \text{on } \Gamma_N.$$

Then, multiplying the first equation by a test function $\mathbf{q} \in \mathbf{Q}_1$ and integrating by parts, and multiplying the second one by a test function $\psi \in \mathrm{Q}_2$ (with $\mathbf{Q}_1$ and Q_2 suitable spaces to be determined), a mixed formulation for this part of the problem reads: Find $(\widetilde{\mathbf{p}}, \varphi) \in \mathbf{Q}_1 \times \mathrm{Q}_2$ such that $\widetilde{\mathbf{p}}\cdot\boldsymbol{\nu} = 0$ on Γ_N and

$$\int_\Omega \frac{1}{k(\varphi)}\widetilde{\mathbf{p}}\cdot\mathbf{q} + \int_\Omega \varphi\,\mathrm{div}\,\mathbf{q} = \langle \mathbf{q}\cdot\boldsymbol{\nu}, \varphi_D \rangle_{\Gamma_D},$$
$$-\int_\Omega \psi\,\mathrm{div}\,\widetilde{\mathbf{p}} = \int_\Omega f^{\mathsf{e}}\,\psi - \int_\Omega \psi\,\mathbf{u}\cdot\nabla\varphi,$$

for all $\mathbf{q} \in \mathbf{Q}_1$ such that $\mathbf{q}\cdot\boldsymbol{\nu} = 0$ on Γ_N, and for all $\psi \in \mathrm{Q}_2$. Thus, if we augment this formulation using the same strategy as in (3.14)-(3.17), in particular (3.15) becomes

$$\widetilde{\kappa}_6 \int_\Omega \mathrm{div}\,\widetilde{\mathbf{p}}\,\mathrm{div}\,\mathbf{q} = -\widetilde{\kappa}_6 \int_\Omega f^{\mathsf{e}}\,\mathrm{div}\,\mathbf{q} + \widetilde{\kappa}_6 \int_\Omega (\mathbf{u}\cdot\nabla\varphi)\,\mathrm{div}\,\mathbf{q} \quad \forall\, \mathbf{q} \in \mathbf{Q}_1,$$

where $\widetilde{\kappa}_6 > 0$. In this way, if we consider $\mathrm{div}\,\mathbf{q} \in \mathrm{L}^2(\Omega)$, the last term in the previous equation is not well-defined, and therefore we need higher regularity in the test functions and more demanding spaces for the unknowns, e.g. $\mathrm{div}\,\mathbf{q} \in \mathrm{L}^4(\Omega)$ or $\varphi \in \mathrm{W}^{1,4}(\Omega)$. Since these changes require a substantial modification of the approach we have taken for this problem, we have left this treatment for a future work.

In the upcoming sections, we analyze the problem (3.18) using fixed-point strategies from [2, 4, 5, 12]. More precisely, in Section 3.2 we rewrite (3.18) as a fixed-point problem, to then in Sections 3.3 and 3.4 establish sufficient conditions for existence and uniqueness of this fixed point.

3.2 The fixed-point approach

First, let us define $\mathbf{H} := \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ and consider the operator $\mathbf{M} : \mathbf{H} \to \mathcal{H}$ defined as

$$\mathbf{M}(\mathbf{w}, \phi) = \Big(\mathbf{M}_1(\mathbf{w}, \phi), \mathbf{M}_2(\mathbf{w}, \phi), \mathbf{M}_3(\mathbf{w}, \phi), \mathbf{M}_4(\mathbf{w}, \phi)\Big) := \vec{\mathbf{t}}, \tag{3.27}$$

where $\vec{\mathbf{t}} \in \mathcal{H}$ is the solution to the mixed formulation of the momentum equation, that is: Find $\vec{\mathbf{t}} \in \mathcal{H}$ such that

$$\mathbf{A}_\phi(\vec{\mathbf{t}}, \vec{\mathbf{s}}) + \mathbf{B}_{\mathbf{w}}(\vec{\mathbf{t}}, \vec{\mathbf{s}}) = F_\phi(\vec{\mathbf{s}}) + F_{\mathtt{m}}(\vec{\mathbf{s}}) \quad \forall\, \vec{\mathbf{s}} \in \mathcal{H}. \tag{3.28}$$

Then, consider the operator $\mathbf{E} : \mathbf{H} \to \mathcal{Q}$ defined as

$$\mathbf{E}(\mathbf{w}, \phi) = \Big(\mathbf{E}_1(\mathbf{w}, \phi), \mathbf{E}_2(\mathbf{w}, \phi), \mathbf{E}_3(\mathbf{w}, \phi)\Big) := \vec{\boldsymbol{\zeta}}, \tag{3.29}$$

where $\vec{\boldsymbol{\zeta}} \in \mathcal{Q}$ is now the solution to the mixed formulation of the energy equation, that is: Find $\vec{\boldsymbol{\zeta}} \in \mathcal{Q}$ such that

$$\mathbf{C}_\phi(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\chi}}) + \mathbf{D}_{\mathbf{w}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\chi}}) = G_D(\vec{\boldsymbol{\chi}}) + G_{\mathbf{e}}(\vec{\boldsymbol{\chi}}) \quad \forall\, \vec{\boldsymbol{\chi}} \in \mathcal{Q}. \tag{3.30}$$

Consequently, we can define the operator $\mathbf{T} : \mathbf{H} \to \mathbf{H}$ as

$$\mathbf{T}(\mathbf{w}, \phi) := \Big(\mathbf{M}_3(\mathbf{w}, \phi), \mathbf{E}_3(\mathbf{M}_3(\mathbf{w}, \phi), \phi)\Big), \tag{3.31}$$

and look at (3.18) as the fixed-point problem: Find $(\mathbf{u}, \varphi) \in \mathbf{H}$ such that

$$\mathbf{T}(\mathbf{u}, \varphi) = (\mathbf{u}, \varphi). \tag{3.32}$$

Therefore, we first focus on proving that $\mathbf{T}$ is well-defined and then we use the Banach fixed-point theorem to show that this operator has a unique fixed point.

3.3 Well-definitiness of the fixed-point operator

As usual, we consider

$$\|\, \vec{\mathbf{t}} \,\| := \Big\{ \|\, \mathbf{t} \,\|^2_{0,\Omega} + \|\, \boldsymbol{\tau} \,\|^2_{\mathbf{div};\Omega} + \|\, \mathbf{v} \,\|^2_{1,\Omega} + \|\, \boldsymbol{\eta} \,\|^2_{0,\Omega} \Big\}^{1/2} \quad \forall\, \vec{\mathbf{t}} \in \mathcal{H},$$

and

$$\|\, \vec{\boldsymbol{\zeta}} \,\| := \Big\{ \|\, \boldsymbol{\zeta} \,\|^2_{0,\Omega} + \|\, \mathbf{q} \,\|^2_{\mathrm{div};\Omega} + \|\, \psi \,\|^2_{1,\Omega} \Big\}^{1/2} \quad \forall\, \vec{\boldsymbol{\zeta}} \in \mathcal{Q}.$$

We begin by stating the well-posedness result corresponding to the mixed formulation of the momentum equation, equivalently, to the well-definitiness of the operator $\mathbf{M}$.

Lemma 3.1. *Assume that for $\delta_1 \in \Big(0, \frac{2}{\mu_2}\Big)$, $\delta_2, \delta_3 \in (0, 2)$ we choose*

$$\kappa_1 \in \left(0, \frac{2\mu_1\delta_1}{\mu_2}\right), \quad \kappa_2 \in (0, \infty),$$
$$\kappa_3 \in \left(0, 2\delta_2\left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right)\right) \quad \text{and} \quad \kappa_4 \in \left(0, 2\delta_3\kappa_0\left(1 - \frac{\delta_2}{2}\right)\kappa_3\right).$$

where κ_0 is a positive constant depending solely on Ω. Then, there exists $r_1 > 0$ such that for each $r \in (0, r_1)$, and for each $(\mathbf{w}, \phi) \in \mathbf{H}$ satisfying $\|\, \mathbf{w} \,\|_{1,\Omega} \le r$, the problem (3.28) has a unique solution $\vec{\mathbf{t}} := \mathbf{M}(\mathbf{w}, \phi) \in \mathcal{H}$. Moreover, there exists a constant $C_{\mathbf{M}} > 0$, independent of $(\mathbf{w}, \phi)$, such that there holds

$$\|\, \mathbf{M}(\mathbf{w}, \phi) \,\| = \|\, \vec{\mathbf{t}} \,\| \le C_{\mathbf{M}} \Big\{ \|\, \mathbf{g} \,\|_{\infty,\Omega} \|\, \phi \,\|_{0,\Omega} + \|\, \mathbf{f}^{\mathtt{m}} \,\|_{0,\Omega} \Big\}. \tag{3.33}$$

Proof. Notice that an analogous result has been proved in [4, Lemma 2.3] for the case where a non-homogeneous velocity boundary condition is imposed, however, since we want to find $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$, augmenting Galerkin-type boundary terms are not needed, thus modifying the forms and the way ellipticity is proved. Nevertheless, the other properties asked by the Lax-Milgram theorem (see, e.g., [15, Theorem 1.1]) can be easily extracted from this result, that is, given $(\mathbf{w}, \phi) \in \mathbf{H}$, $\mathbf{A}_\phi + \mathbf{B}_{\mathbf{w}}$ is a bilinear form and there exists a positive constant denoted by $\|\, \mathbf{A}_\phi + \mathbf{B}_{\mathbf{w}} \,\|$, independent of $(\mathbf{w}, \phi)$, such that

$$|(\mathbf{A}_\phi + \mathbf{B}_{\mathbf{w}})(\vec{\mathbf{t}}, \vec{\mathbf{s}})| \le \|\, \mathbf{A}_\phi + \mathbf{B}_{\mathbf{w}} \,\| \|\, \vec{\mathbf{t}} \,\| \|\, \vec{\mathbf{s}} \,\| \quad \forall\, \vec{\mathbf{t}}, \vec{\mathbf{s}} \in \mathcal{H}\,. \tag{3.34}$$

We begin by analysing the ellipticity of the bilinear form $\mathbf{A}_\phi$. From (3.19), we have for any $\vec{\mathbf{s}} \in \mathcal{H}$ that

$$\begin{aligned}\mathbf{A}_\phi(\vec{\mathbf{s}},\vec{\mathbf{s}}) = & \int_\Omega \mu(\phi)\mathbf{s}:\mathbf{s} - \kappa_1\int_\Omega \mu(\phi)\mathbf{s}:\boldsymbol{\tau}^{\mathtt{d}} - \kappa_3\int_\Omega \mathbf{s}:\mathbf{e}(\mathbf{v}) - \kappa_4\int_\Omega \boldsymbol{\omega}(\mathbf{v}):\boldsymbol{\eta}\\ + & \quad \kappa_1\|\,\boldsymbol{\tau}^{\mathtt{d}}\,\|_{0,\Omega}^2 + \kappa_2\|\,\mathbf{div}\,\boldsymbol{\tau}\,\|_{0,\Omega}^2 + \kappa_3\|\,\mathbf{e}(\mathbf{v})\,\|_{0,\Omega}^2 + \kappa_4\|\,\boldsymbol{\eta}\,\|_{0,\Omega}^2\,.\end{aligned}$$

Then, using the bounds for the viscosity and the Cauchy-Schwarz and Young inequalities, we obtain for any $\delta_1,\delta_2,\delta_3>0$ and any $\vec{\mathbf{s}} \in \mathcal{H}$ that

$$\begin{aligned}\mathbf{A}_\phi(\vec{\mathbf{s}},\vec{\mathbf{s}}) \geq\ & \mu_1\|\,\mathbf{s}\,\|_{0,\Omega}^2 - \frac{\kappa_1\mu_2}{2\delta_1}\|\,\mathbf{s}\,\|_{0,\Omega}^2 - \frac{\kappa_1\mu_2\delta_1}{2}\|\,\boldsymbol{\tau}^{\mathtt{d}}\,\|_{0,\Omega}^2 - \frac{\kappa_3}{2\delta_2}\|\,\mathbf{s}\,\|_{0,\Omega}^2 - \frac{\kappa_3\delta_2}{2}\|\,\mathbf{e}(\mathbf{v})\,\|_{0,\Omega}^2\\ & - \frac{\kappa_4}{2\delta_3}\|\,\boldsymbol{\omega}(\mathbf{v})\,\|_{0,\Omega}^2 - \frac{\kappa_4\delta_3}{2}\|\,\boldsymbol{\eta}\,\|_{0,\Omega}^2 + \kappa_1\|\,\boldsymbol{\tau}^{\mathtt{d}}\,\|_{0,\Omega}^2 + \kappa_2\|\,\mathbf{div}\,\boldsymbol{\tau}\,\|_{0,\Omega}^2 + \kappa_3\|\,\mathbf{e}(\mathbf{v})\,\|_{0,\Omega}^2 + \kappa_4\|\,\boldsymbol{\eta}\,\|_{0,\Omega}^2\\ \geq\ & \left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1} - \frac{\kappa_3}{2\delta_2}\right)\|\,\mathbf{s}\,\|_{0,\Omega}^2 + \kappa_1\left(1-\frac{\mu_2\delta_1}{2}\right)\|\,\boldsymbol{\tau}^{\mathtt{d}}\,\|_{0,\Omega}^2 + \kappa_2\|\,\mathbf{div}\,\boldsymbol{\tau}\,\|_{0,\Omega}^2\\ & + \kappa_3\left(1-\frac{\delta_2}{2}\right)\|\,\mathbf{e}(\mathbf{v})\,\|_{0,\Omega}^2 - \frac{\kappa_4}{2\delta_3}|\mathbf{v}|_{1,\Omega}^2 + \kappa_4\left(1-\frac{\delta_3}{2}\right)\|\,\boldsymbol{\eta}\,\|_{0,\Omega}^2\,.\end{aligned}$$

In this case, the norm of $\mathbf{e}(\mathbf{v})$ can be bounded by using the classical Korn inequality (cf. [19, Theorem 10.1]):

$$\|\,\mathbf{e}(\mathbf{v})\,\|_{0,\Omega}^2 \geq \kappa_0(\Omega)|\,\mathbf{v}\,|_{1,\Omega}^2 \quad \forall\,\mathbf{v}\in\mathbf{H}_0^1(\Omega), \tag{3.35}$$

whereas the norm of $\boldsymbol{\tau}$ can be constructed thanks to the inequality (cf. [7, Proposition 3.1] or [15, Lemma 2.3]):

$$\|\,\boldsymbol{\tau}^{\mathtt{d}}\,\|_{0,\Omega}^2 + \|\,\mathbf{div}\,\boldsymbol{\tau}\,\|_{0,\Omega}^2 \geq c_3(\Omega)\|\,\boldsymbol{\tau}\,\|_{0,\Omega}^2 \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega), \tag{3.36}$$

where κ_0 and c_3 are positive constants depending solely on Ω. Thus, defining the following constants:

$$\alpha_1 := \mu_1 - \frac{\kappa_1\mu_2}{2\delta_1} - \frac{\kappa_3}{2\delta_2}, \quad \alpha_2 := \min\left\{\kappa_1\left(1-\frac{\mu_2\delta_1}{2}\right), \frac{\kappa_2}{2}\right\}, \quad \alpha_3 := \kappa_3\left(1-\frac{\delta_2}{2}\right),$$
$$\alpha_4 := \min\left\{\alpha_2 c_3(\Omega), \frac{\kappa_2}{2}\right\}, \quad \alpha_5 := \alpha_3\kappa_0 - \frac{\kappa_4}{2\delta_3}, \quad \alpha_6 := \kappa_4\left(1-\frac{\delta_3}{2}\right),$$

we deduce the existence of a positive constant $\alpha(\Omega) := \min\{\alpha_1,\alpha_4,\alpha_5,\alpha_6\}$, independent of $(\mathbf{w},\phi)$, such that

$$\mathbf{A}_\phi(\vec{\mathbf{s}},\vec{\mathbf{s}}) \geq \alpha(\Omega)\|\,\vec{\mathbf{s}}\,\|^2 \quad \forall\,\vec{\mathbf{s}}\in\mathcal{H}. \tag{3.37}$$

The rest of the proof is identical to [4, Lemma 2.3], but we recall it for completion purposes. Thus, the foregoing inequality, the definition of $\mathbf{B}_\mathbf{w}$ (cf. (3.20)) and the inequality (3.8) give place to

$$(\mathbf{A}_\phi + \mathbf{B}_\mathbf{w})(\vec{\mathbf{s}},\vec{\mathbf{s}}) \geq \Big(\alpha(\Omega) - c_1(\Omega)(1+\kappa_1^2)^{1/2}\|\,\mathbf{w}\,\|_{1,\Omega}\Big)\|\,\vec{\mathbf{s}}\,\|^2 \quad \forall\,\vec{\mathbf{s}}\in\mathcal{H}\,,$$

and then, we easily see that

$$(\mathbf{A}_\phi + \mathbf{B}_\mathbf{w})(\vec{\mathbf{s}},\vec{\mathbf{s}}) \geq \frac{\alpha(\Omega)}{2}\|\,\vec{\mathbf{s}}\,\|^2 \quad \forall\,\vec{\mathbf{s}}\in\mathcal{H} \tag{3.38}$$

provided that

$$\frac{\alpha(\Omega)}{2} \geq c_1(\Omega)(1+\kappa_1^2)^{1/2}\|\,\mathbf{w}\,\|_{1,\Omega},$$

that is

$$\|\,\mathbf{w}\,\|_{1,\Omega} \leq \frac{\alpha(\Omega)}{2c_1(\Omega)(1+\kappa_1^2)^{1/2}} =: r_1\,, \tag{3.39}$$

thus proving ellipticity for $\mathbf{A}_\phi + \mathbf{B}_{\mathbf{w}}$ under the requirement (3.39). On the other hand, it is not hard to see that $F_\phi + F_{\mathtt{m}}$ is a linear functional, and that there holds

$$|(F_\phi + F_{\mathtt{m}})(\vec{\mathbf{s}})| \leq (1+\kappa_2^2)^{1/2}\Big(\|\,\mathbf{g}\,\|_{\infty,\Omega}\|\,\phi\,\|_{0,\Omega} + \|\,\mathbf{f}^{\mathtt{m}}\,\|_{0,\Omega}\Big)\|\,\vec{\mathbf{s}}\,\| \quad \forall\, \vec{\mathbf{s}} \in \mathcal{H}. \tag{3.40}$$

Therefore, by the Lax-Milgram theorem, there exists a unique $\vec{\mathbf{t}} \in \mathcal{H}$ solution to (3.28), and the continuous dependence result (3.33) is satisfied with $C_{\mathbf{M}} := \frac{2(1+\kappa_2^2)^{1/2}}{\alpha(\Omega)}$. $\square$

Next, and prior to establish an analogous result for the mixed formulation of the energy equation, we recall the following Poincaré-type inequality that will help us to prove the ellipticity of the underlying bilinear form.

Lemma 3.2. *There exists $c_4(\Omega) > 0$ such that*

$$|\,\psi\,|_{1,\Omega}^2 + \|\,\psi\,\|_{0,\Gamma_D}^2 \geq c_4(\Omega)\|\,\psi\,\|_{1,\Omega}^2 \quad \forall\, \psi \in \mathrm{H}^1(\Omega). \tag{3.41}$$

Proof. See [18, Theorem 5.11.2]. $\square$

Lemma 3.3. *Assume that for $\delta_4 \in \left(0, \frac{2}{k_2}\right)$, $\delta_5 \in (0,2)$ we choose*

$$\kappa_5 \in \left(0, \frac{2\delta_4 k_1}{k_2}\right), \quad \kappa_6, \kappa_8 \in (0,\infty) \quad \textit{and} \quad \kappa_7 \in \left(0, 2\delta_5\left(k_1 - \frac{\kappa_5 k_2}{2\delta_4}\right)\right).$$

Then, there exists $r_2 > 0$ such that for each $r \in (0, r_2)$ and for each $(\mathbf{w}, \phi) \in \mathbf{H}$ satisfying $\|\,\mathbf{w}\,\|_{1,\Omega} \leq r$, the problem (3.30) has a unique solution $\vec{\boldsymbol{\zeta}} := \mathbf{E}(\mathbf{w},\phi) \in \mathcal{Q}$. Moreover, there exists a constant $C_{\mathbf{E}} > 0$, independent of $(\mathbf{w},\phi)$, such that there holds

$$\|\,\mathbf{E}(\mathbf{w},\phi)\,\| = \|\,\vec{\boldsymbol{\zeta}}\,\| \leq C_{\mathbf{E}}\Big\{\|\,\varphi_D\,\|_{1/2,\Gamma_D} + \|\,f^{\mathtt{e}}\,\|_{0,\Omega}\Big\}. \tag{3.42}$$

Proof. Let $(\mathbf{w},\phi) \in \mathbf{H}$. It is clear from (3.21) and (3.22) that $\mathbf{C}_\phi$ and $\mathbf{D}_{\mathbf{w}}$ are bilinear forms, and their bounded character can be seen using the upper bound of the thermal conductivity function, the Cauchy-Schwarz inequality and the trace inequality with boundedness constant $c_0(\Omega)$, that is, for the first one

$$\begin{aligned}
|\mathbf{C}_\phi(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}})| \leq\; & k_2(1+\kappa_5^2)^{1/2}\|\,\boldsymbol{\zeta}\,\|_{0,\Omega}\|\,\vec{\boldsymbol{\chi}}\,\| + (1+\kappa_7^2)^{1/2}\|\,\boldsymbol{\zeta}\,\|_{0,\Omega}\|\,\vec{\boldsymbol{\chi}}\,\| \\
&+ (1+\kappa_5^2)^{1/2}\|\,\mathbf{p}\,\|_{0,\Omega}\|\,\vec{\boldsymbol{\chi}}\,\| + \|\,\varphi\,\|_{0,\Omega}\|\,\mathrm{div}\,\mathbf{q}\,\|_{0,\Omega} + \|\,\mathrm{div}\,\mathbf{p}\,\|_{0,\Omega}\|\,\psi\,\|_{0,\Omega} \\
&+ \kappa_6\|\,\mathrm{div}\,\mathbf{p}\,\|_{0,\Omega}\|\,\mathrm{div}\,\mathbf{q}\,\|_{0,\Omega} + \kappa_7|\,\varphi\,|_{1,\Omega}|\,\psi\,|_{1,\Omega} + \kappa_8 c_0(\Omega)^2\|\,\varphi\,\|_{1,\Omega}\|\,\psi\,\|_{1,\Omega},
\end{aligned}$$

thus obtaining the existence of a positive constant $C_{\mathbf{C}} > 0$ such that

$$|\mathbf{C}_\phi(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}})| \leq C_{\mathbf{C}}\|\,\vec{\boldsymbol{\zeta}}\,\|\|\,\vec{\boldsymbol{\chi}}\,\|. \tag{3.43}$$

For $\mathbf{D}_{\mathbf{w}}$, in addition to the above, we use (3.9) to get

$$|\mathbf{D}_{\mathbf{w}}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}})| \leq c_2(\Omega)(1+\kappa_5^2)^{1/2}\|\,\mathbf{w}\,\|_{1,\Omega}\|\,\varphi\,\|_{1,\Omega}\|\,\vec{\boldsymbol{\chi}}\,\|. \tag{3.44}$$

Hence, from the previous two equations, there exists a positive constant denoted by $\|\,\mathbf{C}_\phi + \mathbf{D}_{\mathbf{w}}\,\|$ independent of $(\mathbf{w},\phi)$ such that

$$|(\mathbf{C}_\phi + \mathbf{D}_{\mathbf{w}})(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}})| \leq \|\,\mathbf{C}_\phi + \mathbf{D}_{\mathbf{w}}\,\|\|\,\vec{\boldsymbol{\zeta}}\,\|\|\,\vec{\boldsymbol{\chi}}\,\| \quad \forall\, \vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\chi}} \in \mathcal{Q}. \tag{3.45}$$

Next, to prove that $\mathbf{C}_\phi + \mathbf{D}_\mathbf{w}$ is elliptic, we first prove that $\mathbf{C}_\phi$ has this property. Indeed, for any $\vec{\boldsymbol{\chi}} \in \mathcal{Q}$, we obtain from (3.21) that

$$\begin{aligned}\mathbf{C}_\phi(\vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\chi}}) = &\int_\Omega k(\phi)\boldsymbol{\chi}\cdot\boldsymbol{\chi} - \kappa_5\int_\Omega k(\phi)\boldsymbol{\chi}\cdot\mathbf{q} - \kappa_7\int_\Omega \boldsymbol{\chi}\cdot\nabla\psi \\ &+ \kappa_5\,\|\,\mathbf{q}\,\|_{0,\Omega}^2 + \kappa_6\|\operatorname{div}\mathbf{q}\,\|_{0,\Omega}^2 + \kappa_7|\,\psi\,|_{0,\Omega}^2 + \kappa_8\|\,\psi\,\|_{0,\Gamma_D}^2.\end{aligned}$$

Then, using the bounds for the thermal conductivity function, the Cauchy-Schwarz and Young inequalities, we get for any $\delta_4, \delta_5 > 0$ that

$$\begin{aligned}\mathbf{C}_\phi(\vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\chi}}) \geq\, & k_1\|\,\boldsymbol{\chi}\,\|_{0,\Omega}^2 - \frac{\kappa_5 k_2}{2\delta_4}\|\,\boldsymbol{\chi}\,\|_{0,\Omega}^2 - \frac{\kappa_5 k_2\delta_4}{2}\|\,\mathbf{q}\,\|_{0,\Omega}^2 - \frac{\kappa_7}{2\delta_5}\|\,\boldsymbol{\chi}\,\|_{0,\Omega}^2 - \frac{\kappa_7\delta_5}{2}|\,\psi\,|_{1,\Omega}^2 \\ &+ \kappa_5\|\,\mathbf{q}\,\|_{0,\Omega}^2 + \kappa_6\|\operatorname{div}\mathbf{q}\,\|_{0,\Omega}^2 + \kappa_7|\,\psi\,|_{0,\Omega}^2 + \kappa_8\|\,\psi\,\|_{0,\Gamma}^2 \\ =\, & \left(k_1 - \frac{\kappa_5 k_2}{2\delta_4} - \frac{\kappa_7}{2\delta_5}\right)\|\,\boldsymbol{\chi}\,\|_{0,\Omega}^2 + \kappa_5\left(1 - \frac{k_2\delta_4}{2}\right)\|\,\mathbf{q}\,\|_{0,\Omega}^2 + \kappa_6\|\operatorname{div}\mathbf{q}\,\|_{0,\Omega}^2 \\ &+ \kappa_7\left(1 - \frac{\delta_5}{2}\right)|\,\psi\,|_{1,\Omega}^2 + \kappa_8\|\,\psi\,\|_{0,\Gamma_D}^2.\end{aligned}$$

Hence, applying Lemma 3.2 and defining the constants

$$\beta_1 := k_1 - \frac{\kappa_5 k_2}{2\delta_4} - \frac{\kappa_7}{2\delta_5}, \quad \beta_2 := \min\left\{\kappa_5\left(1 - \frac{k_2\delta_4}{2}\right), \kappa_6\right\} \quad \text{and} \quad \beta_3 := \min\left\{\kappa_7\left(1 - \frac{\delta_5}{2}\right), \kappa_8\right\},$$

there exists $\beta(\Omega) := \min\{\beta_1, \beta_2, c_4(\Omega)\beta_3\}$ such that

$$\mathbf{C}_\phi(\vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\chi}}) \geq \beta(\Omega)\|\,\vec{\boldsymbol{\chi}}\,\|^2, \tag{3.46}$$

which, together with (3.22) and the inequality (3.9), allows us to write

$$(\mathbf{C}_\phi + \mathbf{D}_\mathbf{w})(\vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\chi}}) \geq \Big(\beta(\Omega) - c_2(\Omega)(1+\kappa_5^2)^{1/2}\|\,\mathbf{w}\,\|_{1,\Omega}\Big)\|\,\vec{\boldsymbol{\chi}}\,\|^2.$$

Therefore, we easily see that

$$(\mathbf{C}_\phi + \mathbf{D}_\mathbf{w})(\vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\chi}}) \geq \frac{\beta(\Omega)}{2}\|\,\vec{\boldsymbol{\chi}}\,\|^2 \quad \forall\,\vec{\boldsymbol{\chi}} \in \mathcal{Q}, \tag{3.47}$$

provided that

$$\frac{\beta(\Omega)}{2} \geq c_2(\Omega)(1+\kappa_5^2)^{1/2}\|\,\mathbf{w}\,\|_{1,\Omega},$$

that is

$$\|\,\mathbf{w}\,\|_{1,\Omega} \leq \frac{\beta(\Omega)}{2c_2(\Omega)(1+\kappa_5^2)^{1/2}} =: r_2, \tag{3.48}$$

thus proving ellipticity for $\mathbf{C}_\phi + \mathbf{D}_\mathbf{w}$ under the requirement (3.48). Finally, it is clear from (3.25) and (3.26) that for any $\vec{\boldsymbol{\chi}} \in \mathcal{H}$ there holds

$$|G_D(\vec{\boldsymbol{\chi}})| \leq \|\,\mathbf{q}\,\|_{\operatorname{div},\Omega}\|\,\varphi_D\,\|_{1/2,\Gamma_D} + \kappa_8\|\,\varphi_D\,\|_{0,\Gamma_D}\|\,\psi\,\|_{0,\Gamma_D}$$

and

$$|G_\mathsf{e}(\vec{\boldsymbol{\chi}})| \leq (1+\kappa_6^2)^{1/2}\|\,f^\mathsf{e}\,\|_{0,\Omega}\|\,\vec{\boldsymbol{\chi}}\,\|,$$

and then, using the continuous injection from $\mathrm{H}^{1/2}(\Gamma_D)$ into $\mathrm{L}^2(\Gamma_D)$ with constant $C_{1/2}(\Gamma_D)$, and the trace inequality in $\mathrm{H}^1(\Omega)$ with constant $c_0(\Omega)$, we conclude the existence of a positive constant $C_G := \min\{1 + \kappa_8 c_0(\Omega)C_{1/2}, (1+\kappa_6^2)^{1/2}\}$ such that

$$|(G_D + G_\mathsf{e})(\vec{\boldsymbol{\chi}})| \leq C_G\Big\{\|\,\varphi_D\,\|_{1/2,\Gamma_D} + \|\,f^\mathsf{e}\,\|_{0,\Omega}\Big\}\|\,\vec{\boldsymbol{\chi}}\,\| \quad \forall\,\vec{\boldsymbol{\chi}} \in \mathcal{Q}. \tag{3.49}$$

Since $G_D + G_{\mathtt{e}}$ is also linear, existence and uniqueness of a solution to the problem (3.30) is just a consequence of the Lax-Milgram theorem. Moreover, the continuous dependence result (3.42) holds with $C_{\mathbf{E}} := \frac{2C_G}{\beta(\Omega)}$. □

As a consequence of the previous two lemmas, the operator $\mathbf{T}$ is well-defined in the closed ball of center 0 and radius r, with $r \in (0, \min\{r_1, r_2\})$. More precisely, we have the following result.

Lemma 3.4. *Given $r \in (0, r_0)$ with $r_0 := \min\{r_1, r_2\}$, r_1 as in (3.39) and r_2 as in (3.48), let*

$$\mathbf{W}_r := \Big\{ (\mathbf{w}, \phi) \in \mathbf{H} : \| (\mathbf{w}, \phi) \| \leq r \Big\}. \tag{3.50}$$

Assume that the stabilization parameters κ_j, $j \in \{1, \ldots, 8\}$, are taken as in Lemmas 3.1 and 3.3, and that there holds

$$C_{\mathbf{M}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{0,\Omega} \Big\} < r_2, \tag{3.51}$$

with $C_{\mathbf{M}} > 0$ as in (3.33). Then $\mathbf{T} : \mathbf{W}_r \to \mathbf{H}$ is well-defined and there holds

$$\| \mathbf{T}(\mathbf{w}, \phi) \| \leq C_{\mathbf{M}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{0,\Omega} \Big\} + C_{\mathbf{E}} \Big\{ \| \varphi_D \|_{1/2,\Gamma_D} + \| f^{\mathtt{e}} \|_{0,\Omega} \Big\}. \tag{3.52}$$

Proof. Let $(\mathbf{w}, \phi) \in \mathbf{W}_r$. Since $\mathbf{T}(\mathbf{w}, \phi) := \Big(\mathbf{M}_3(\mathbf{w}, \phi), \mathbf{E}_3(\mathbf{M}_3(\mathbf{w}, \phi), \phi) \Big)$, the norm of the first component $\mathbf{T}$ is bounded above by the right-hand side of (3.33), whereas the norm second one is bounded above by the right-hand side of (3.42), provided (3.51) holds. Therefore, $\mathbf{T}$ is well-defined and (3.52) trivially holds. □

For computational purposes, a particular choice of stabilization parameters has to be made. Hence, we first consider the middle points of the intervals for δ_1, δ_2, δ_3, δ_4, δ_5, κ_1, κ_3, κ_4, κ_5 and κ_7; to then choose κ_2, κ_6 and κ_8 so α_2, β_2 and β_3 can be respectively as large as possible. This results in the following set of values:

$$\kappa_1 = \frac{\mu_1}{\mu_2^2}, \quad \kappa_2 = \frac{\mu_1}{\mu_2^2}, \quad \kappa_3 = \frac{\mu_1}{2}, \quad \kappa_4 = \kappa_0 \frac{\mu_1}{4}, \quad \kappa_5 = \frac{k_1}{k_2^2}, \quad \kappa_6 = \frac{k_1}{2k_2^2}, \quad \kappa_7 = \frac{k_1}{2}, \quad \kappa_8 = \frac{k_1}{4}.$$

Notice that κ_0, the constant that appears in the Korn inequality (3.35), takes the value $\frac{1}{2}$ when $\Omega \subset R^2$.

3.4 Solvability analysis of the fixed-point problem

In this section, we pursue to comply with the hypotheses of the Banach fixed-point theorem to ensure existence and uniqueness of a fixed point. First, we notice from (3.52) that $\mathbf{T}$ maps the ball $\mathbf{W}_r$ into itself whenever the data satisfy

$$C_{\mathbf{M}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{0,\Omega} \Big\} + C_{\mathbf{E}} \Big\{ \| \varphi_D \|_{1/2,\Gamma_D} + \| f^{\mathtt{e}} \|_{0,\Omega} \Big\} \leq r. \tag{3.53}$$

Therefore, it only remains to prove that $\mathbf{T}$ is a Lipschitz-continuous and contracting mapping. As in [2, 4, 5, 9], this analysis will require a further-regularity assumption on the solution of both uncoupled problems. Hence, in what follows we assume that $\mathbf{f}^{\mathtt{m}} \in \mathbf{H}^{\varepsilon}(\Omega)$ and $\varphi_D \in \mathrm{H}^{1/2+\varepsilon}(\Gamma_D)$ for some $\varepsilon \in (0,1)$ when $n = 2$, or $\varepsilon \in \big[\frac{1}{2}, 1\big)$ when $n = 3$, and that for each $(\mathbf{z}, \rho) \in \mathbf{H}$, with $\| \mathbf{z} \|_{1,\Omega} \leq r$, $r > 0$ given, there hold, on the one hand, that $(\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) := \mathbf{M}(\mathbf{z}, \rho) \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) \cap \mathbb{H}^{\varepsilon}(\Omega) \times \mathbb{H}_0(\mathbf{div}; \Omega) \cap \mathbb{H}^{\varepsilon}(\Omega) \times \mathbf{H}^{1+\varepsilon}_0(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{H}^{\varepsilon}(\Omega)$ and (following (3.33))

$$\| \mathbf{s} \|_{\varepsilon,\Omega} + \| \boldsymbol{\tau} \|_{\varepsilon,\Omega} + \| \mathbf{v} \|_{1+\varepsilon,\Omega} + \| \boldsymbol{\eta} \|_{\varepsilon,\Omega} \leq \widetilde{C}_{\mathbf{M}}(r) \Big\{ \| \mathbf{g} \|_{\infty,\Omega} \| \rho \|_{1,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{\varepsilon,\Omega} \Big\}, \tag{3.54}$$

and on the other hand, that $(\boldsymbol{\chi}, \mathbf{q}, \psi) := \mathbf{E}(\mathbf{z}, \rho) \in \mathbf{L}^2(\Omega) \cap \mathbf{H}^\varepsilon(\Omega) \times \mathbf{H}_{\mathrm{N}}(\mathrm{div}; \Omega) \cap \mathbf{H}^\varepsilon(\Omega) \times \mathrm{H}^{1+\varepsilon}(\Omega)$ and (following (3.42))

$$\| \boldsymbol{\chi} \|_{\varepsilon,\Omega} + \| \mathbf{q} \|_{\varepsilon,\Omega} + \| \psi \|_{1+\varepsilon,\Omega} \leq \widetilde{C}_{\mathbf{E}}(r) \Big\{ \| \varphi_D \|_{1/2+\varepsilon,\Gamma_D} + \| f^{\mathtt{e}} \|_{0,\Omega} \Big\}. \tag{3.55}$$

Here, $\widetilde{C}_{\mathbf{M}}(r)$ and $\widetilde{C}_{\mathbf{E}}(r)$ are positive constants independent of $\mathbf{z}$ but depending on the upper bound r of its $\mathbf{H}^1$-norm. A difference with respect to other works, namely [2, 4], is that the presence in this case of a homogeneous boundary condition for the momentum equation leads us to ask for more regularity to the source term, which for this uncoupled problem, becomes $\mathbf{f}^{\mathtt{m}} + \rho\mathbf{g}$, hence the reason why not only we consider the H^ε-norm of $\mathbf{f}^{\mathtt{m}}$ in (3.54) but also the H^ε-norm of ρ, though since we are already assuming that $\rho \in \mathrm{H}^1(\Omega)$, we have used a continuous injection from $\mathrm{H}^1(\Omega)$ into $\mathrm{H}^\varepsilon(\Omega)$ to transform this H^ε-norm into the H^1-norm that appears. That being said, we now proceed to the main results of this section. First, we recall from [4] a preliminary result for $\mathbf{M}$, to then prove the Lipschitz-continuity of this operator.

Lemma 3.5. *Let $r \in (0, r_0)$ with r_0 given as in Lemma 3.4. Then, there exists a positive constant $\widehat{C}_{\mathbf{M}}$, independent of r, such that*

$$\begin{aligned} \| \mathbf{M}(\mathbf{w}_1, \phi_1) - \mathbf{M}(\mathbf{w}_2, \phi_2) \| \leq \widehat{C}_{\mathbf{M}} \Big\{ & \| \mathbf{M}_1(\mathbf{w}_1, \phi_1) \|_{\varepsilon,\Omega} \| \phi_1 - \phi_2 \|_{\mathrm{L}^{n/\varepsilon}(\Omega)} \\ & + \| \mathbf{M}_3(\mathbf{w}_1, \phi_1) \|_{1,\Omega} \| \mathbf{w}_1 - \mathbf{w}_2 \|_{1,\Omega} + \| \mathbf{g} \|_{\infty,\Omega} \| \phi_1 - \phi_2 \|_{0,\Omega} \Big\}, \end{aligned} \tag{3.56}$$

for all $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathbf{H}$ such that $\| \mathbf{w}_1 \|_{1,\Omega}, \| \mathbf{w}_2 \|_{1,\Omega} \leq r$.

Proof. See [4, Lemma 2.6]. $\square$

We stress here that Lemma 3.5 makes use precisely of the regularity assumption (3.54) and the fact, that under the specified range of ε, $H^1(\Omega)$ is continuously embedded in $\mathrm{L}^{n/\varepsilon}(\Omega)$, for $n \in \{2, 3\}$.

Lemma 3.6. *Let $r \in (0, r_0)$, with r_0 given as in Lemma 3.4. Then, there exists $L_{\mathbf{M}} > 0$ such that*

$$\| \mathbf{M}(\mathbf{w}_1, \phi_1) - \mathbf{M}(\mathbf{w}_2, \phi_2) \| \leq L_{\mathbf{M}} \Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{\varepsilon,\Omega} \Big\} \| (\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2) \|, \tag{3.57}$$

for all $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathbf{W}_r$.

Proof. Let $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathbf{W}_r$. Considering that $H^1(\Omega) \hookrightarrow \mathrm{L}^{n/\varepsilon}(\Omega)$ with constant $\widetilde{C}_\varepsilon$, that $\mathrm{H}^{1+\varepsilon}(\Omega) \hookrightarrow \mathrm{H}^1(\Omega)$ with constant $\widehat{C}_\varepsilon$, and that the solution to the problem defining $\mathbf{M}$ verifies the further-regularity assumption (3.54), the following can be inferred from (3.56):

$$\begin{aligned} & \| \mathbf{M}(\mathbf{w}_1, \phi_1) - \mathbf{M}(\mathbf{w}_2, \phi_2) \| \\ & \quad \leq \widehat{C}_{\mathbf{M}} \Big\{ \widetilde{C}_{\mathbf{M}}(r) (\widetilde{C}_\varepsilon^2 + \widehat{C}_\varepsilon^2)^{1/2} \Big(r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{\varepsilon,\Omega} \Big) + \| \mathbf{g} \|_{\infty,\Omega} \Big\} \| (\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2) \|. \end{aligned}$$

Then, defining the constants

$$\widehat{C}_1 = \widetilde{C}_{\mathbf{M}}(r) (\widetilde{C}_\varepsilon^2 + \widehat{C}_\varepsilon^2)^{1/2}, \quad \widehat{C}_2 = 1 + \widehat{C}_1 r,$$

the result (3.57) holds with $L_{\mathbf{M}} := \widehat{C}_{\mathbf{M}} \max \Big\{ \widehat{C}_1, \widehat{C}_2 \Big\}$. $\square$

In a similar way, we obtain the following result for the operator $\mathbf{E}$.

Lemma 3.7. *Let $r \in (0, r_0)$ with r_0 given as in Lemma 3.4. Then, there exists $L_{\mathbf{E}} > 0$, such that*

$$\| \mathbf{E}(\mathbf{w}_1, \phi_1) - \mathbf{E}(\mathbf{w}_2, \phi_2) \| \leq L_{\mathbf{E}} \Big\{ \| \varphi_D \|_{1/2+\varepsilon, \Gamma_D} + \| f^{\mathrm{e}} \|_{0,\Omega} \Big\} \| (\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2) \|, \tag{3.58}$$

for all $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathbf{W}_r$.

Proof. Let $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathbf{H}$ such that $\| \mathbf{w}_1 \|_{1,\Omega}, \| \mathbf{w}_2 \|_{1,\Omega} \leq r$, and let $\vec{\boldsymbol{\zeta}}_j = (\boldsymbol{\zeta}_j, \mathbf{p}_j, \varphi_j) := \mathbf{E}(\mathbf{w}_j, \phi_j)$, $j \in \{1, 2\}$, be the corresponding solutions of (3.30). Then, since for any $\vec{\boldsymbol{\chi}} \in \mathcal{Q}$ there holds

$$\mathbf{C}_{\phi_1}(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\chi}}) + \mathbf{D}_{\mathbf{w}_1}(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\chi}}) = G_D(\vec{\boldsymbol{\chi}}) + G_{\mathsf{e}}(\vec{\boldsymbol{\chi}}),$$

it follows that

$$\begin{aligned} \mathbf{C}_{\phi_2}(\vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2, \vec{\boldsymbol{\chi}}) + \mathbf{D}_{\mathbf{w}_2}(\vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2, \vec{\boldsymbol{\chi}}) \,&=\, \mathbf{C}_{\phi_2}(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\chi}}) + \mathbf{D}_{\mathbf{w}_2}(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\chi}}) - G_D(\vec{\boldsymbol{\chi}}) - G_{\mathsf{e}}(\vec{\boldsymbol{\chi}}) \\ &=\, \mathbf{C}_{\phi_2}(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\chi}}) - \mathbf{C}_{\phi_1}(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\chi}}) + \mathbf{D}_{\mathbf{w}_2}(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\chi}}) - \mathbf{D}_{\mathbf{w}_1}(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\chi}})\,, \end{aligned} \tag{3.59}$$

for any $\vec{\boldsymbol{\chi}} \in \mathcal{Q}$. In this way, using the ellipticity of the bilinear form $\mathbf{C}_{\phi_2} + \mathbf{D}_{\mathbf{w}_2}$ (cf. (3.47)), we see that

$$\begin{aligned} \frac{\beta(\Omega)}{2} \| \vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2 \|^2 &\leq (\mathbf{C}_{\phi_2} + \mathbf{D}_{\mathbf{w}_2})(\vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2, \vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2) \\ &= (\mathbf{C}_{\phi_2} - \mathbf{C}_{\phi_1})(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2) + \mathbf{D}_{\mathbf{w}_2 - \mathbf{w}_1}(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2) \\ &= \int_\Omega \Big\{ k(\phi_2) - k(\phi_1) \Big\} \boldsymbol{\zeta}_1 \cdot \Big\{ (\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) - \kappa_5(\mathbf{p}_1 - \mathbf{p}_2) \Big\} \\ &\quad + \int_\Omega \varphi_1 (\mathbf{w}_1 - \mathbf{w}_2) \cdot \Big\{ (\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) - \kappa_5(\mathbf{p}_1 - \mathbf{p}_2) \Big\}\,. \end{aligned} \tag{3.60}$$

The last term in the previous inequality can be easily split using (3.9), that is

$$\begin{aligned} &\left| \int_\Omega \varphi_1 (\mathbf{w}_1 - \mathbf{w}_2) \cdot \Big\{ (\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) - \kappa_5(\mathbf{p}_1 - \mathbf{p}_2) \Big\} \right| \\ &\qquad \leq\, c_2(\Omega)(1 + \kappa_5^2)^{1/2} \| \varphi_1 \|_{1,\Omega} \| \mathbf{w}_1 - \mathbf{w}_2 \|_{1,\Omega} \| \vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2 \|\,, \end{aligned} \tag{3.61}$$

whereas for the first term we use the Hölder inequality to show that

$$\begin{aligned} &\left| \int_\Omega \Big\{ k(\phi_2) - k(\phi_1) \Big\} \boldsymbol{\zeta}_1 \cdot \Big\{ (\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) - \kappa_5(\mathbf{p}_1 - \mathbf{p}_2) \Big\} \right| \\ &\qquad \leq\, L_k \int_\Omega \left| (\phi_2 - \phi_1) \boldsymbol{\zeta}_1 \cdot \Big\{ (\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) - \kappa_5(\mathbf{p}_1 - \mathbf{p}_2) \Big\} \right| \\ &\qquad \leq\, L_k (1 + \kappa_5^2)^{1/2} \| (\phi_2 - \phi_1) \boldsymbol{\zeta}_1 \|_{0,\Omega} \| \vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2 \| \\ &\qquad \leq\, L_k (1 + \kappa_5^2)^{1/2} \| \phi_2 - \phi_1 \|_{\mathrm{L}^{2q}(\Omega)} \| \boldsymbol{\zeta}_1 \|_{\mathbf{L}^{2p}(\Omega)} \| \vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2 \|\,, \end{aligned} \tag{3.62}$$

with $p, q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Having in mind the further-regularity assumption (3.55), we recall that $\mathrm{H}^\varepsilon(\Omega)$ is continuously embedded into $\mathrm{L}^{2p}(\Omega)$ whenever

$$2p = \begin{cases} \dfrac{2}{1 - \varepsilon} & \text{if } n = 2, \\[2mm] \dfrac{6}{3 - 2\varepsilon} & \text{if } n = 3, \end{cases}$$

and hence, there exists $C_\varepsilon > 0$ such that

$$\| \boldsymbol{\zeta} \|_{\mathbf{L}^{2p}(\Omega)} \leq C_\varepsilon \| \boldsymbol{\zeta} \|_{\varepsilon,\Omega} \quad \forall\, \boldsymbol{\zeta} \in \mathbf{H}^\varepsilon(\Omega). \tag{3.63}$$

In this way,

$$2q = \frac{2p}{p-1} = \begin{cases} \dfrac{2}{\varepsilon} & \text{if } n=2, \\ \dfrac{3}{\varepsilon} & \text{if } n=3 \end{cases} = \frac{n}{\varepsilon},$$

and (3.62) now yields

$$\begin{aligned} &\left| \int_\Omega \Big\{ k(\phi_2) - k(\phi_1) \Big\} \boldsymbol{\zeta}_1 \cdot \Big\{ (\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) - \kappa_5 (\mathbf{p}_1 - \mathbf{p}_2) \Big\} \right| \\ &\qquad \leq C_\varepsilon L_k (1+\kappa_5^2)^{1/2} \| \boldsymbol{\zeta}_1 \|_{\varepsilon,\Omega} \| \phi_1 - \phi_2 \|_{\mathrm{L}^{n/\varepsilon}(\Omega)} \| \vec{\boldsymbol{\zeta}}_1 - \vec{\boldsymbol{\zeta}}_2 \| \,. \end{aligned} \tag{3.64}$$

Since $\boldsymbol{\zeta}_1 = \mathbf{E}_1(\mathbf{w}_1, \phi_1)$ and $\varphi_1 = \mathbf{E}_3(\mathbf{w}_1, \phi_1)$, putting together (3.61) and (3.64) into (3.60), we find that there exists $\widehat{C}_{\mathbf{E}} := \frac{2(1+\kappa_5^2)^{1/2}}{\beta(\Omega)} \max\{C_\varepsilon L_k, c_2(\Omega)\}$ such that

$$\begin{aligned} &\| \mathbf{E}(\mathbf{w}_1, \phi_1) - \mathbf{E}(\mathbf{w}_2, \phi_2) \| \\ &\qquad \leq \widehat{C}_{\mathbf{E}} \Big\{ \| \mathbf{E}_1(\mathbf{w}_1, \phi_1) \|_{\varepsilon,\Omega} \| \phi_1 - \phi_2 \|_{\mathrm{L}^{n/\varepsilon}(\Omega)} + \| \mathbf{E}_3(\mathbf{w}_1, \phi_1) \|_{1,\Omega} \| \mathbf{w}_1 - \mathbf{w}_2 \|_{1,\Omega} \Big\} \,, \end{aligned} \tag{3.65}$$

for any $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathbf{H}$ such that $\| \mathbf{w}_1 \|_{1,\Omega}, \| \mathbf{w}_2 \|_{1,\Omega} \leq r$. Then, considering the same injections as in the proof of Lemma 3.6 and the further-regularity assumption (3.55), we see that for any $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathbf{W}_r$,

$$\begin{aligned} &\| \mathbf{E}(\mathbf{w}_1, \phi_1) - \mathbf{E}(\mathbf{w}_2, \phi_2) \| \\ &\qquad \leq \widehat{C}_{\mathbf{E}} \widetilde{C}_{\mathbf{E}}(r) (\widetilde{C}_\varepsilon^2 + \widehat{C}_\varepsilon^2)^{1/2} \Big(\| \varphi_D \|_{1/2+\varepsilon,\Gamma_D} + \| f^{\mathsf{e}} \|_{0,\Omega} \Big) \| (\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2) \| \,, \end{aligned} \tag{3.66}$$

thus obtaining (3.58) with $L_{\mathbf{E}} := \widehat{C}_{\mathbf{E}} \widetilde{C}_{\mathbf{E}}(r) (\widetilde{C}_\varepsilon^2 + \widehat{C}_\varepsilon^2)^{1/2}$. ◻

Consequently, the proof of Lipschitz-continuity for $\mathbf{T} : \mathbf{W}_r \to \mathbf{H}$ becomes straightforward.

Lemma 3.8. *Let $r \in (0, r_0)$ with r_0 given as in Lemma 3.4. Assume that*

$$C_{\mathbf{M}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathsf{m}} \|_{0,\Omega} \Big\} \leq r, \tag{3.67}$$

with $C_{\mathbf{M}} > 0$ as in (3.42). Then, $\mathbf{T} : \mathbf{W}_r \to \mathbf{H}$ is Lipschitz-continuous and there holds

$$\| \mathbf{T}(\mathbf{w}_1, \phi_1) - \mathbf{T}(\mathbf{w}_2, \phi_2) \| \leq L_{\mathbf{T}} \| (\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2) \| \quad \forall\, (\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathbf{W}_r, \tag{3.68}$$

where

$$L_{\mathbf{T}} := \Big(L_{\mathbf{M}} (\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathsf{m}} \|_{\varepsilon,\Omega}) + 1 \Big) \Big(L_{\mathbf{E}} (\| \varphi_D \|_{1/2+\varepsilon,\Gamma_D} + \| f^{\mathsf{e}} \|_{0,\Omega}) + 1 \Big) - 1 \,. \tag{3.69}$$

Proof. Let $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathbf{W}_r$. Then, according to the definition of $\mathbf{T}$,

$$\begin{aligned} &\| \mathbf{T}(\mathbf{w}_1, \phi_1) - \mathbf{T}(\mathbf{w}_2, \phi_2) \| \\ &\qquad \leq \| \mathbf{M}_3(\mathbf{w}_1, \phi_1) - \mathbf{M}_3(\mathbf{w}_2, \phi_2) \| + \| \mathbf{E}_3(\mathbf{M}_3(\mathbf{w}_1, \phi_1), \phi_1) - \mathbf{E}_3(\mathbf{M}_3(\mathbf{w}_2, \phi_2), \phi_2) \| \,. \end{aligned} \tag{3.70}$$

The bound for the first term comes directly from (3.57), whereas for the second one, we first use the Lipschitz-continuity of $\mathbf{E}$ (notice that the assumption (3.67) is necessary to be able to use this property, i.e., (3.58)) and then the Lipschitz-continuity of $\mathbf{M}$, that is

$$\begin{aligned} &\| \mathbf{E}_3(\mathbf{M}_3(\mathbf{w}_1, \phi_1), \phi_1) - \mathbf{E}_3(\mathbf{M}_3(\mathbf{w}_2, \phi_2), \phi_2) \| \\ &\quad \leq L_{\mathbf{E}} \big(\| \varphi_D \|_{1/2+\varepsilon,\Gamma_D} + \| f^{\mathsf{e}} \|_{0,\Omega} \big) \| \big(\mathbf{M}_3(\mathbf{w}_1, \phi_1), \phi_1 \big) - \big(\mathbf{M}_3(\mathbf{w}_2, \phi_2), \phi_2 \big) \| \\ &\quad \leq L_{\mathbf{E}} \big(\| \varphi_D \|_{1/2+\varepsilon,\Gamma_D} + \| f^{\mathsf{e}} \|_{0,\Omega} \big) \Big(L_{\mathbf{M}} \big(\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathsf{m}} \|_{\varepsilon,\Omega} \big) + 1 \Big) \| (\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2) \| | \,. \end{aligned}$$

◻

Summarizing, Lemma 3.4 ensures us that $\mathbf{T} : \mathbf{H} \to \mathbf{H}$ is well defined in the closed ball $\mathbf{W}_r$, and maps the ball into itself whenever the data satisfy the condition (3.53). Then, this operator is Lipschitz-continuous according to the foregoing Lemma, and it becomes a contracting map whenever $L_{\mathbf{T}} < 1$. Putting together all these results, we get the main result of this section.

Theorem 3.9. *Let $r \in (0, r_0)$ with r_0 as in Lemma 3.4, and let $\mathbf{W}_r$ be the closed ball in $\mathbf{H}$ with center $(\mathbf{0}, 0)$ and radius r (cf. (3.50)). Assume that the stabilization parameters κ_j, $j \in \{1, \ldots, 8\}$, are taken as in Lemmas 3.1 and 3.3, and that the data $\mathbf{g}$, $\mathbf{f}^{\mathtt{m}}$, $f^{\mathtt{e}}$ and φ_D satisfy*

$$C_{\mathbf{M}}\Big\{r\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{0,\Omega}\Big\} + C_{\mathbf{E}}\Big\{\| \varphi_D \|_{1/2,\Gamma_D} + \| f^{\mathtt{e}} \|_{0,\Omega}\Big\} \leq r, \tag{3.71a}$$

$$L_{\mathbf{M}}\Big(\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{\varepsilon,\Omega}\Big) < \sqrt{2} - 1, \tag{3.71b}$$

$$L_{\mathbf{E}}\Big(\| \varphi_D \|_{1/2+\varepsilon,\Gamma_D} + \| f^{\mathtt{e}} \|_{0,\Omega}\Big) < \sqrt{2} - 1, \tag{3.71c}$$

with $C_{\mathbf{M}}$, $C_{\mathbf{E}}$, $L_{\mathbf{M}}$ and $L_{\mathbf{E}}$ respectively as in Lemmas 3.1, 3.3, 3.6 and 3.7. Then, there exists a unique $(\vec{\mathbf{t}}, \vec{\boldsymbol{\zeta}}) \in \mathcal{H} \times \mathcal{Q}$, with $(\mathbf{u}, \varphi) \in \mathbf{W}_r$, solution to the fully-mixed formulation (3.18) of the Boussinesq problem (2.1). Moreover, there holds,

$$\| \vec{\mathbf{t}} \| \leq C_{\mathbf{M}}\Big\{r\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{0,\Omega}\Big\}, \tag{3.72}$$

and

$$\| \vec{\boldsymbol{\zeta}} \| \leq C_{\mathbf{E}}\Big\{\| \varphi_D \|_{1/2,\Gamma_D} + \| f^{\mathtt{e}} \|_{0,\Omega}\Big\}. \tag{3.73}$$

Proof. Notice that (3.71a) ensures that both (3.51) and (3.53) hold, so that $\mathbf{T} : \mathbf{W}_r \to \mathbf{W}_r$ is indeed well-defined. In addition, (3.71b) and (3.71c) imply that $\mathbf{T}$ is a contracting map (that is, $L_{\mathbf{T}} < 1$). Therefore, the result is just a consequence of the Banach fixed-point theorem applied to the operator $\mathbf{T}$ in $\mathbf{W}_r$. $\square$

4 The Galerkin Scheme

We advocate in this section to present the Galerkin scheme for the continuous problem (3.18), whose well-posedness will be proved using the same steps and methods as in the previous section.

4.1 Preliminaries

Let us consider $\mathcal{T}_h$ a regular triangulation of $\bar{\Omega}$ made by triangles T when $n = 2$ (or tetrahedra when $n = 3$) of diameter h_T an define the meshsize $h := \max\{h_T : T \in \mathcal{T}_h\}$. Then, consider arbitrary finite-dimensional subspaces $\mathbb{H}_h^{\mathbf{t}} \subset \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, $\mathbb{H}_h^{\boldsymbol{\sigma}} \subset \mathbb{H}_0(\mathbf{div}; \Omega)$, $\mathbf{H}_h^{\mathbf{u}} \subset \mathbf{H}_0^1(\Omega)$, $\mathbb{H}_h^{\gamma} \subset \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, $\mathbf{H}_h^{\boldsymbol{\zeta}} \subset \mathbf{L}^2(\Omega)$, $\mathbf{H}_h^{\mathbf{p}} \subset \mathbf{H}_{\mathrm{N}}(\mathrm{div}; \Omega)$, $\mathrm{H}_h^{\varphi} \subset \mathrm{H}^1(\Omega)$ and denote by

$$\mathcal{H}_h := \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}, \quad \mathcal{Q}_h := \mathbf{H}_h^{\boldsymbol{\zeta}} \times \mathbf{H}_h^{\mathbf{p}} \times \mathrm{H}_h^{\varphi},$$
$$\vec{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), \quad \vec{\mathbf{s}}_h := (\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h), \quad \vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_h, \mathbf{p}_h, \varphi_h), \quad \vec{\boldsymbol{\chi}}_h := (\boldsymbol{\chi}_h, \mathbf{q}_h, \psi_h).$$

Hence, according to the continuous formulation (3.18), the Galerkin scheme reads: Find $(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\zeta}}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$ such that

$$\mathbf{A}_{\varphi_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) + \mathbf{B}_{\mathbf{u}_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) = F_{\varphi_h}(\vec{\mathbf{s}}_h) + F_{\mathtt{m}}(\vec{\mathbf{s}}_h) \quad \forall\ \vec{\mathbf{s}}_h \in \mathcal{H}_h, \tag{4.1a}$$

$$\mathbf{C}_{\varphi_h}(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\chi}}_h) + \mathbf{D}_{\mathbf{u}_h}(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\chi}}_h) = G_D(\vec{\boldsymbol{\chi}}_h) + G_{\mathtt{e}}(\vec{\boldsymbol{\chi}}_h) \quad \forall\ \vec{\boldsymbol{\chi}}_h \in \mathcal{Q}_h, \tag{4.1b}$$

where the forms $\mathbf{A}_{\varphi_h}$, $\mathbf{B}_{\mathbf{u}_h}$, $\mathbf{C}_{\varphi_h}$ and $\mathbf{D}_{\mathbf{u}_h}$, and the functionals F_{φ_h}, $F_{\mathtt{m}}$, G_D and $G_{\mathtt{e}}$ are defined by (3.19)-(3.26).

We will see that it is possible to establish sufficient conditions for well-posedness of (4.1) in the same form they were established for the continuous problem (3.18). To this end, we now split the discrete formulation into the two corresponding mixed formulations. In fact, we first set $\mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$ and consider the operator $\mathbf{M}_h : \mathbf{H}_h \to \mathcal{H}_h$ defined for any $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h$ by

$$\mathbf{M}_h(\mathbf{w}_h, \phi_h) = \Big(\mathbf{M}_{1,h}(\mathbf{w}_h, \phi_h), \mathbf{M}_{2,h}(\mathbf{w}_h, \phi_h), \mathbf{M}_{3,h}(\mathbf{w}_h, \phi_h), \mathbf{M}_{4,h}(\mathbf{w}_h, \phi_h)\Big) := \vec{\mathbf{t}}_h \,, \tag{4.2}$$

where $\vec{\mathbf{t}}_h \in \mathcal{H}_h$ is the solution to the problem: Find $\vec{\mathbf{t}}_h \in \mathcal{H}_h$ such that

$$\mathbf{A}_{\phi_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) + \mathbf{B}_{\mathbf{w}_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) = F_{\phi_h}(\vec{\mathbf{s}}_h) + F_{\mathtt{m}}(\vec{\mathbf{s}}_h) \quad \forall\, \vec{\mathbf{s}}_h \in \mathcal{H}_h \,. \tag{4.3}$$

In turn, we let $\mathbf{E}_h : \mathbf{H}_h \to \mathcal{Q}_h$ be the operator defined for any $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h$ by

$$\mathbf{E}_h(\mathbf{w}_h, \phi_h) = \Big(\mathbf{E}_{1,h}(\mathbf{w}_h, \phi_h), \mathbf{E}_{2,h}(\mathbf{w}_h, \phi_h), \mathbf{E}_{3,h}(\mathbf{w}_h, \phi_h)\Big) := \vec{\boldsymbol{\zeta}}_h \,, \tag{4.4}$$

where $\vec{\boldsymbol{\zeta}}_h \in \mathcal{Q}_h$ is the solution to the problem: Find $\vec{\boldsymbol{\zeta}}_h \in \mathcal{Q}_h$ such that

$$\mathbf{C}_{\phi_h}(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\chi}}_h) + \mathbf{D}_{\mathbf{w}_h}(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\chi}}_h) = G_D(\vec{\boldsymbol{\chi}}_h) + G_{\mathtt{e}}(\vec{\boldsymbol{\chi}}_h) \quad \forall\, \vec{\boldsymbol{\chi}} \in \mathcal{Q}_h \,. \tag{4.5}$$

Therefore, by introducing the operator $\mathbf{T}_h : \mathbf{H}_h \to \mathbf{H}_h$ as

$$\mathbf{T}_h(\mathbf{w}_h, \phi_h) := \Big(\mathbf{M}_{3,h}(\mathbf{w}_h, \phi_h), \mathbf{E}_{3,h}\big(\mathbf{M}_{3,h}(\mathbf{w}_h, \phi_h), \phi_h\big)\Big), \tag{4.6}$$

we can rewrite (4.1) as the fixed-point problem: Find $(\mathbf{u}_h, \varphi_h) \in \mathbf{H}_h$ such that

$$\mathbf{T}_h(\mathbf{u}_h, \varphi_h) = (\mathbf{u}_h, \varphi_h). \tag{4.7}$$

In this case, existence of a fixed point for this problem will be proved by means of the Brouwer fixed-point theorem, which we recall next.

Theorem 4.1 (Brouwer)**.** *Let W be a compact and convex subset of a finite-dimensional Banach space, and let $T : W \to W$ be a continuous mapping. Then, T has at least one fixed-point.*

Proof. See [10, Theorem 9.9-2]. □

4.2 Solvability analysis

We first study under which conditions $\mathbf{T}_h$ is well-defined by looking at the well-posedness of the uncoupled problems (4.3) and (4.5). It is easy to realize that the analysis comes in a straightforward way from the one realized in Lemmas 3.1, 3.3 and 3.4, thus obtaining the following discrete versions for these results.

Lemma 4.2. *Assume that for $\delta_1 \in \left(0, \frac{2}{\mu_2}\right)$, $\delta_2, \delta_3 \in (0,2)$ we choose*

$$\kappa_1 \in \left(0, \frac{2\mu_1\delta_1}{\mu_2}\right), \quad \kappa_2 \in (0,\infty),$$
$$\kappa_3 \in \left(0, 2\delta_2\left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right)\right) \quad \textit{and} \quad \kappa_4 \in \left(0, 2\delta_3\kappa_0\left(1 - \frac{\delta_2}{2}\right)\kappa_3\right),$$

where κ_0 is the constant in the Korn inequality (3.35). Then, for each $r \in (0, r_1)$, with r_1 as in (3.39), the problem (4.3) has a unique solution $\vec{\mathbf{t}}_h := \mathbf{M}_h(\mathbf{w}_h, \phi_h) \in \mathcal{H}_h$ for each $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h$ such that $\| \mathbf{w}_h \|_{1,\Omega} \leq r$. Moreover, there holds

$$\| \mathbf{M}_h(\mathbf{w}_h, \phi_h) \| = \| \vec{\mathbf{t}}_h \| \leq C_{\mathbf{M}} \Big\{ \| \mathbf{g} \|_{\infty,\Omega} \| \phi_h \|_{0,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{0,\Omega} \Big\}, \tag{4.8}$$

with $C_{\mathbf{M}} > 0$ as in (3.33).

Proof. It follows from a direct application of the Lax-Milgram theorem to (4.3) in the same way it was applied in Lemma 3.1. $\square$

Lemma 4.3. *Assume that for $\delta_4 \in \left(0, \frac{2}{k_2}\right)$, $\delta_5 \in (0, 2)$ we choose*

$$\kappa_5 \in \left(0, \frac{2\delta_4 k_1}{k_2}\right), \quad \kappa_6, \kappa_8 \in (0, \infty) \quad \textit{and} \quad \kappa_7 \in \left(0, 2\delta_5 \left(k_1 - \frac{\kappa_5 k_2}{2\delta_4}\right)\right).$$

Then, for each $r \in (0, r_2)$ with r_2 as in (3.48), the problem (4.5) has a unique solution $\vec{\boldsymbol{\zeta}}_h := \mathbf{E}_h(\mathbf{w}_h, \phi_h) \in \mathcal{Q}_h$ for each $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h$ such that $\| \mathbf{w}_h \|_{1,\Omega} \leq r$. Moreover, there holds

$$\| \mathbf{E}_h(\mathbf{w}_h, \phi_h) \| = \| \vec{\boldsymbol{\zeta}}_h \| \leq C_{\mathbf{E}} \Big\{ \| \varphi_D \|_{1/2,\Gamma_D} + \| f^{\mathtt{e}} \|_{0,\Omega} \Big\}, \tag{4.9}$$

with $C_{\mathbf{E}} > 0$ as in (3.42).

Proof. It also follows from an application of the Lax-Milgram theorem to (4.5) in the same way it was applied in Lemma 3.3. $\square$

Analogously to the continuous case, the previous two lemmas provide the well-definitiness of the operator $\mathbf{T}_h$.

Lemma 4.4. *Given $r \in (0, r_0)$ with $r_0 := \min\{r_1, r_2\}$, r_1 as in (3.39) and r_2 as in (3.48), let*

$$\mathbf{W}_{h,r} := \Big\{ (\mathbf{w}_h, \phi_h) \in \mathbf{H}_h : \| (\mathbf{w}_h, \phi_h) \| \leq r \Big\}. \tag{4.10}$$

Assume that the stabilization parameters κ_j, $j \in \{1, \ldots, 8\}$, are taken as in Lemmas 4.2 and 4.3 and that the data satisfy (3.51). Then $\mathbf{T}_h : \mathbf{W}_{h,r} \to \mathbf{H}_h$ is well-defined and there holds

$$\| \mathbf{T}_h(\mathbf{w}_h, \phi_h) \| \leq C_{\mathbf{M}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{0,\Omega} \Big\} + C_{\mathbf{E}} \Big\{ \| \varphi_D \|_{1/2,\Gamma_D} + \| f^{\mathtt{e}} \|_{0,\Omega} \Big\}. \tag{4.11}$$

Moreover, $\mathbf{T}_h$ maps the ball $\mathbf{W}_{h,r}$ into itself whenever (3.53) holds.

Proof. It comes straightforwardly from Lemmas 4.2 and 4.3. We omit further details. $\square$

We now turn to prove the continuity of $\mathbf{T}_h$. Notice that we are no longer able to use the further-regularity assumptions (3.54) and (3.55), and therefore, we will only obtain results that are analogous to (3.56) and (3.65) in terms of L^4 norms. Indeed, although their bounds are not known, they are finite since the solution of the Galerkin scheme will be formed by piecewise polynomial functions.

Lemma 4.5. *Let $r \in (0, r_0)$ with r_0 as in Lemma 4.4. Then, there exists a positive constant $\overline{C}_{\mathbf{M}}$, independent of r, such that*

$$\begin{aligned} \| \mathbf{M}_h(\mathbf{w}_h^1, \phi_h^1) - \mathbf{M}_h(\mathbf{w}_h^2, \phi_h^2) \| \leq \overline{C}_{\mathbf{M}} \Big\{ & \| \mathbf{M}_{1,h}(\mathbf{w}_h^1, \phi_h^1) \|_{\mathbb{L}^4(\Omega)} \| \phi_h^1 - \phi_h^2 \|_{\mathrm{L}^4(\Omega)} \\ & + \| \mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1) \|_{1,\Omega} \| \mathbf{w}_h^1 - \mathbf{w}_h^2 \|_{1,\Omega} + \| \mathbf{g} \|_{\infty,\Omega} \| \phi_h^1 - \phi_h^2 \|_{0,\Omega} \Big\}, \end{aligned} \tag{4.12}$$

for all $(\mathbf{w}_h^1, \phi_h^1), (\mathbf{w}_h^2, \phi_h^2) \in \mathbf{H}_h$ such that $\| \mathbf{w}_h^1 \|_{1,\Omega}, \| \mathbf{w}_h^2 \|_{1,\Omega} \leq r$.

Proof. It comes from [4, Lemma 2.6], but employing an L^4-L^4-L^2 argument, instead of the L^{2p}-L^{2q}-L^2 argument taken in [4, Eq. (2.52)]. $\square$

Lemma 4.6. *Let $r \in (0, r_0)$ with r_0 as in Lemma 4.4. Then, there exists a positive constant $\overline{C}_{\mathbf{E}}$ such that*

$$\begin{aligned} &\| \mathbf{E}_h(\mathbf{w}_h^1, \phi_h^1) - \mathbf{E}_h(\mathbf{w}_h^2, \phi_h^2) \| \\ &\quad \leq \overline{C}_{\mathbf{E}} \Big\{ \| \mathbf{E}_{1,h}(\mathbf{w}_h^1, \phi_h^1) \|_{\mathbf{L}^4(\Omega)} \| \phi_h^1 - \phi_h^2 \|_{\mathrm{L}^4(\Omega)} + \| \mathbf{E}_{3,h}(\mathbf{w}_h^1, \phi_h^1) \|_{1,\Omega} \| \mathbf{w}_h^1 - \mathbf{w}_h^2 \|_{1,\Omega} \Big\}, \end{aligned} \tag{4.13}$$

for all $(\mathbf{w}_h^1, \phi_h^1), (\mathbf{w}_h^2, \phi_h^2) \in \mathbf{H}_h$ such that $\| \mathbf{w}_h^1 \|_{1,\Omega}, \| \mathbf{w}_h^2 \|_{1,\Omega} \leq r$.

Proof. As in the previous Lemma, the proof is based on the one for its continuous counterpart Lemma 3.7, but just taking $p = q = 2$ in (3.62). $\square$

Consequently, we have the following result for the operator $\mathbf{T}_h$.

Lemma 4.7. *Given $r \in (0, r_0)$ with r_0 as in Lemma 4.4, let $\mathbf{W}_{h,r}$ be the closed ball in $\mathbf{H}_h$ with center $(\mathbf{0}, 0)$ and radius r (cf. (4.10)), and assume that (3.67) holds. Then, $\mathbf{T}_h : \mathbf{W}_{h,r} \to \mathbf{H}_h$ is continuous and there exists a constant $C_{\mathbf{T}_h} > 0$ such that*

$$\begin{aligned} &\| \mathbf{T}_h(\mathbf{w}_h^1, \phi_h^1) - \mathbf{T}_h(\mathbf{w}_h^2, \phi_h^2) \| \\ &\quad \leq C_{\mathbf{T}_h} \Bigg\{ \| \mathbf{E}_{1,h}\big(\mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1), \phi_h^1\big) \|_{\mathbf{L}^4(\Omega)} + \Big(1 + \| \mathbf{E}_{3,h}\big(\mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1), \phi_h^1\big) \|_{1,\Omega} \Big) \\ &\qquad \times \Big(\| \mathbf{M}_{1,h}(\mathbf{w}_h^1, \phi_h^1) \|_{\mathbb{L}^4(\Omega)} + \| \mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1) \|_{1,\Omega} + \| \mathbf{g} \|_{\infty,\Omega} \Big) \Bigg\} \| (\mathbf{w}_h^1, \phi_h^1) - (\mathbf{w}_h^2, \phi_h^2) \|, \end{aligned} \tag{4.14}$$

for all $(\mathbf{w}_h^1, \phi_h^1), (\mathbf{w}_h^2, \phi_h^2) \in \mathbf{W}_{h,r}$.

Proof. Let $(\mathbf{w}_h^1, \phi_h^1), (\mathbf{w}_h^2, \phi_h^2) \in \mathbf{W}_{h,r}$. From the definition of $\mathbf{T}_h$ (cf. (4.6)) we see that

$$\begin{aligned} \| \mathbf{T}_h(\mathbf{w}_h^1, \phi_h^1) - \mathbf{T}_h(\mathbf{w}_h^2, \phi_h^2) \| &\leq \| \mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1) - \mathbf{M}_{3,h}(\mathbf{w}_h^2, \phi_h^2) \|_{1,\Omega} \\ &\quad + \| \mathbf{E}_{3,h}\big(\mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1), \phi_h^1\big) - \mathbf{E}_{3,h}\big(\mathbf{M}_{3,h}(\mathbf{w}_h^2, \phi_h^2), \phi_h^2\big) \|, \end{aligned}$$

and since (3.67) holds, $\| \mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1) \|_{1,\Omega} \leq r$ and we can use (4.13) to bound the second term in the previous inequality. In this way we get

$$\begin{aligned} \| \mathbf{T}_h(\mathbf{w}_h^1, \phi_h^1) - \mathbf{T}_h(\mathbf{w}_h^2, \phi_h^2) \| &\leq \overline{C}_{\mathbf{E}} C_i \| \mathbf{E}_{1,h}\big(\mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1), \phi_h^1\big) \|_{\mathbf{L}^4(\Omega)} \| \phi_h^1 - \phi_h^2 \|_{1,\Omega} \\ &\quad + \Big(1 + \overline{C}_{\mathbf{E}} \| \mathbf{E}_{3,h}\big(\mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1), \phi_h^1\big) \|_{1,\Omega} \Big) \| \mathbf{M}_{3,h}(\mathbf{w}_h^1, \phi_h^1) - \mathbf{M}_{3,h}(\mathbf{w}_h^2, \phi_h^2) \|_{1,\Omega}, \end{aligned} \tag{4.15}$$

where C_i is the boundedness constant of the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^4(\Omega)$. Then, we can use the analogous result for $\mathbf{M}_h$ (cf. (4.12)) and, after some algebraic work, we arrive to (4.14) with

$$C_{\mathbf{T}_h} = \max\{\overline{C}_1, \overline{C}_2\}, \quad \overline{C}_1 = \overline{C}_{\mathbf{M}} \max\{1, \overline{C}_{\mathbf{E}}\} \max\{1, C_i\}, \quad \overline{C}_2 = C_i \overline{C}_{\mathbf{E}}.$$

$\square$

Having proved that $\mathbf{T}_h$ is continuous and that it maps the closed and convex set $\mathbf{W}_{h,r}$ into itself whenever (3.53) holds, existence of at least one fixed point is ensured by the Brouwer fixed-point theorem, equivalently, existence of at least one solution for (4.1); a result that is summarized in the following statement.

Theorem 4.8. *Let $r \in (0, r_0)$ with r_0 as in Lemma 4.4, and let $\mathbf{W}_{h,r}$ be the closed ball of center $(\mathbf{0}, 0)$ and radius r (cf. (4.10)). Assume the stabilization parameters κ_j, $j \in \{1, \ldots, 8\}$, are taken as in Lemmas 4.2 and 4.3 and that the data satisfy (3.53). Then, the Galerkin scheme (4.1) has at least one solution $(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\zeta}}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$ with $(\mathbf{u}_h, \varphi_h) \in \mathbf{W}_{h,r}$. Moreover, there hold*

$$\| \vec{\mathbf{t}}_h \| \le C_{\mathbf{M}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathtt{m}} \|_{0,\Omega} \Big\}, \tag{4.16}$$

and

$$\| \vec{\boldsymbol{\zeta}}_h \| \le C_{\mathbf{E}} \Big\{ \| \varphi_D \|_{1/2,\Gamma_D} + \| f^{\mathtt{e}} \|_{0,\Omega} \Big\}, \tag{4.17}$$

with $C_{\mathbf{M}}$ and $C_{\mathbf{E}}$ as in Lemmas 4.2 and 4.3, respectively.

4.3 Specific finite element subspaces

An interesting point to realize in this fully-mixed approach with respect to the finite-dimensional subspaces is that we have not imposed any kind of inf-sup conditions as in [2, 4], or any other requirement than being finite-dimensional, which give us the chance to freely choose these subspaces. That being said, given an integer $k \ge 0$, for each $T \in \mathcal{T}_h$ we define the local Raviart-Thomas space of order k as

$$\mathbf{RT}_k(T) := \mathbf{P}_k(T) + \mathrm{P}_k(T)\mathbf{x},$$

where according to the terminology described in Section 1, $\mathbf{P}_k(T) := [\mathrm{P}_k(T)]^n$ and $\mathbf{x}$ is a generic vector in $\mathrm{R}^n = \mathbf{R}$. Similarly $\mathbf{C}(\bar{\Omega}) := [\mathrm{C}(\bar{\Omega})]^n$ denotes the space of continuous functions in $\bar{\Omega}$. Thus, we approximate the rate of strain, the vorticity and temperature gradient by discontinuous piecewise polynomial tensors of degree $\le k$, the pseudostress and pseudoheat by Raviart-Thomas elements of order k, and the velocity and temperature by Lagrange elements of order k, that is

$$\mathbb{H}_h^{\mathbf{t}} := \Big\{ \mathbf{s}_h \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) : \mathbf{s}_h|_T \in \mathbb{P}_k(T), \ \forall\, T \in \mathcal{T}_h \Big\}, \tag{4.18}$$

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}; \Omega) : \mathbf{c}^{\mathtt{t}} \boldsymbol{\tau}_h|_T \in \mathbf{RT}_k(T), \ \forall\, \mathbf{c} \in \mathbf{R}, \ T \in \mathcal{T}_h \Big\}, \tag{4.19}$$

$$\mathbf{H}_h^{\mathbf{u}} := \Big\{ \mathbf{v}_h \in \mathbf{C}(\bar{\Omega}) : \mathbf{v}_h|_T \in \mathbf{P}_{k+1}(T), \ \forall\, T \in \mathcal{T}_h \Big\} \cap \mathbf{H}_0^1(\Omega), \tag{4.20}$$

$$\mathbb{H}_h^{\boldsymbol{\gamma}} := \Big\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathtt{skew}}(\Omega) : \boldsymbol{\eta}_h|_T \in \mathbb{P}_k(T), \ \forall\, T \in \mathcal{T}_h \Big\}, \tag{4.21}$$

$$\mathbf{H}_h^{\boldsymbol{\zeta}} := \Big\{ \boldsymbol{\chi}_h \in \mathbf{L}^2(\Omega) : \boldsymbol{\chi}_h|_T \in \mathbf{P}_k(T), \ \forall\, T \in \mathcal{T}_h \Big\}, \tag{4.22}$$

$$\mathbf{H}_h^{\mathbf{p}} := \Big\{ \mathbf{q}_h \in \mathbf{H}_{\mathrm{N}}(\mathrm{div}; \Omega) : \mathbf{q}_h|_T \in \mathbf{RT}_k(T), \ \forall\, T \in \mathcal{T}_h \Big\}, \tag{4.23}$$

$$\mathrm{H}_h^{\varphi} := \Big\{ \psi_h \in C(\bar{\Omega}) : \psi_h|_T \in \mathrm{P}_{k+1}(T), \ \forall\, T \in \mathcal{T}_h \Big\}. \tag{4.24}$$

According to [7, 15], their corresponding approximation properties are:

($\mathbf{AP}_h^{\mathbf{t}}$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\mathbf{t} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, there holds

$$\mathrm{dist}\left(\mathbf{t}, \mathbb{H}_h^{\mathbf{t}}\right) \le C h^s \| \mathbf{t} \|_{s,\Omega}, \tag{4.25}$$

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_0(\mathbf{div}; \Omega)$ with $\mathbf{div}\, \boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, there holds

$$\mathrm{dist}\,(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) \le C h^s \left\{ \| \boldsymbol{\sigma} \|_{s,\Omega} + \| \mathbf{div}\, \boldsymbol{\sigma} \|_{s,\Omega} \right\}, \tag{4.26}$$

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega) \cap \mathbf{H}_0^1(\Omega)$, there holds

$$\mathrm{dist}\,(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) \le C h^s \| \mathbf{u} \|_{s+1,\Omega}, \tag{4.27}$$

$(\mathbf{AP}_h^{\boldsymbol{\gamma}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\gamma} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, there holds

$$\mathrm{dist}\,\big(\boldsymbol{\gamma}, \mathbb{H}_h^{\boldsymbol{\gamma}}\big) \le C h^s \| \boldsymbol{\gamma} \|_{s,\Omega}, \tag{4.28}$$

$(\mathbf{AP}_h^{\boldsymbol{\zeta}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\zeta} \in \mathbb{H}^s(\Omega)$, there holds

$$\mathrm{dist}\left(\boldsymbol{\zeta}, \mathbf{H}_h^{\boldsymbol{\zeta}}\right) \le C h^s \| \boldsymbol{\zeta} \|_{s,\Omega}, \tag{4.29}$$

$(\mathbf{AP}_h^{\mathbf{p}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\mathbf{p} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega)$ with $\mathrm{div}\,\mathbf{p} \in \mathrm{H}^s(\Omega)$, there holds,

$$\mathrm{dist}\,\big(\mathbf{p}, \mathbf{H}_h^{\mathbf{p}}\big) \le C h^s \left\{ \| \mathbf{p} \|_{s,\Omega} + \| \mathrm{div}\,\mathbf{p} \|_{s,\Omega} \right\}, \tag{4.30}$$

$(\mathbf{AP}_h^{\varphi})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\varphi \in \mathrm{H}^{s+1}(\Omega)$, there holds

$$\mathrm{dist}\,\big(\varphi, \mathbf{H}_h^{\mathbf{p}}\big) \le C h^s \| \varphi \|_{s+1,\Omega}. \tag{4.31}$$

5 A priori error analysis

Let $(\vec{\mathbf{t}}, \vec{\boldsymbol{\zeta}}) \in \mathcal{H} \times \mathcal{Q}$ with $(\mathbf{u}, \varphi) \in \mathbf{W}_{r_0}$ (r_0 as in Lemma 3.4) and $(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\zeta}}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$ with $(\mathbf{u}_h, \varphi_h) \in \mathbf{W}_{h,r_0}$ be the solutions to the continuous and discrete problems (3.18) and (4.1), respectively, that is

$$\mathbf{A}_{\varphi}(\vec{\mathbf{t}}, \vec{\mathbf{s}}) + \mathbf{B}_{\mathbf{u}}(\vec{\mathbf{t}}, \vec{\mathbf{s}}) = F_{\varphi}(\vec{\mathbf{s}}) + F_{\mathtt{m}}(\vec{\mathbf{s}}) \quad \forall\, \vec{\mathbf{s}} \in \mathcal{H}, \tag{5.1a}$$

$$\mathbf{A}_{\varphi_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) + \mathbf{B}_{\mathbf{u}_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) = F_{\varphi_h}(\vec{\mathbf{s}}) + F_{\mathtt{m}}(\vec{\mathbf{s}}) \quad \forall\, \vec{\mathbf{s}}_h \in \mathcal{H}_h, \tag{5.1b}$$

and

$$\mathbf{C}_{\varphi}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\chi}}) + \mathbf{D}_{\mathbf{u}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\chi}}) = G_D(\vec{\boldsymbol{\chi}}) + G_{\mathsf{e}}(\vec{\boldsymbol{\chi}}) \quad \forall\, \vec{\boldsymbol{\chi}} \in \mathcal{Q}, \tag{5.2a}$$

$$\mathbf{C}_{\varphi_h}(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\chi}}_h) + \mathbf{D}_{\mathbf{u}_h}(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\chi}}_h) = G_D(\vec{\boldsymbol{\chi}}) + G_{\mathsf{e}}(\vec{\boldsymbol{\chi}}) \quad \forall\, \vec{\boldsymbol{\chi}}_h \in \mathcal{Q}_h. \tag{5.2b}$$

In addition, we denote as usual

$$\mathrm{dist}\,\big(\vec{\mathbf{t}}, \mathcal{H}_h\big) := \inf_{\vec{\mathbf{s}}_h \in \mathcal{H}_h} \| \vec{\mathbf{t}} - \vec{\mathbf{s}}_h \|, \quad \mathrm{dist}\,\big(\vec{\boldsymbol{\zeta}}, \mathcal{Q}_h\big) := \inf_{\vec{\boldsymbol{\chi}}_h \in \mathcal{Q}_h} \| \vec{\boldsymbol{\zeta}} - \vec{\boldsymbol{\chi}}_h \|.$$

Similar to [2, Lemma 5.3] and [4, Lemma 4.2], we will apply the Strang Lemma to the pair of equations (5.1) and (5.2) separately, to then join the resulting estimates to derive a proper Céa estimate. We begin by recalling from [23] and [4] this Lemma and the result regarding the first pair of equations, respectively.

Lemma 5.1 (Strang)**.** *Let V be a Hilbert space, $F \in V'$, and $A : V \times V \to \mathrm{R}$ be a bounded and V-elliptic bilinear form. In addition, let $\{V_h\}_{h>0}$ be a sequence of finite-dimensional subspaces of V, and for each $h > 0$, consider a bounded bilinear form $A_h : V_h \times V_h \to \mathrm{R}$ and a functional $F_h \in V_h'$.*

Assume that the family $\{A_h\}_{h>0}$ is uniformly elliptic in V_h, that is, there exists a constant $\widetilde{\alpha} > 0$, independent of h, such that

$$A_h(v_h, v_h) \geq \widetilde{\alpha} \| v_h \|_V^2 \quad \forall v_h \in V_h, \ \forall h > 0.$$

In turn, let $u \in V$ and $u_h \in V_h$ such that

$$A(u, v) = F(v) \quad \forall\, v \in V \qquad \text{and} \qquad A_h(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h.$$

Then, for each $h > 0$, there holds

$$\begin{aligned} \| u - u_h \|_V \leq C_{ST} \Bigg\{ & \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|F(w_h) - F_h(w_h)|}{\| w_h \|_V} \\ & + \inf_{\substack{v_h \in V_h \\ v_h \neq 0}} \left(\| u - v_h \|_V + \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|A(v_h, w_h) - A_h(v_h, w_h)|}{\| w_h \|_V} \right) \Bigg\}, \end{aligned} \tag{5.3}$$

where $C_{ST} := \widetilde{\alpha}^{-1} \max\{1, \| A \|\}$.

Proof. See [23, Theorem 11.1]. □

Lemma 5.2. *Let $C_{ST}^{\mathbf{M}} := \frac{2}{\alpha(\Omega)} \max\{1, \| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \|\}$, where $\frac{\alpha(\Omega)}{2}$ is the ellipticity constant of $\mathbf{A}_\varphi + \mathbf{B}_\mathbf{u}$ (cf. (3.38)) and $\| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \|$ is its norm, which is independent of $(\mathbf{u}, \varphi)$ (cf. (3.34)). Then, there holds*

$$\begin{aligned} \| \vec{\mathbf{t}} - \vec{\mathbf{t}}_h \| \leq C_{ST}^{\mathbf{M}} \Bigg\{ & \Big(1 + 2 \| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \| \Big) \operatorname{dist}\big(\vec{\mathbf{t}}, \mathcal{H}_h\big) + c_1(\Omega)(1 + \kappa_1^2)^{1/2} \| \mathbf{u} \|_{1,\Omega} \| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} \\ & + \Big(L_\mu C_\varepsilon \widetilde{C}_\varepsilon (1 + \kappa_1^2)^{1/2} \| \mathbf{t} \|_{\varepsilon,\Omega} + (1 + \kappa_2^2)^{1/2} \| \mathbf{g} \|_{\infty,\Omega} \Big) \| \varphi - \varphi_h \|_{1,\Omega} \Bigg\}. \end{aligned} \tag{5.4}$$

Proof. See [4, Lemma 4.2]. □

Lemma 5.3. *Let $C_{ST}^{\mathbf{E}} := \frac{2}{\beta(\Omega)} \max\{1, \| \mathbf{C}_\varphi + \mathbf{D}_\mathbf{u} \|\}$, where $\frac{\beta(\Omega)}{2}$ is the ellipticity constant of $\mathbf{C}_\varphi + \mathbf{D}_\mathbf{u}$ (cf. (3.47)) and $\| \mathbf{C}_\varphi + \mathbf{D}_\mathbf{u} \|$ is its norm, which is independent of $(\mathbf{u}, \varphi)$ (cf. (3.45)). Then, there holds*

$$\begin{aligned} \| \vec{\boldsymbol{\zeta}} - \vec{\boldsymbol{\zeta}}_h \| \leq C_{ST}^{\mathbf{E}} \Bigg\{ & \Big(1 + 2 \| \mathbf{C}_\varphi + \mathbf{D}_\mathbf{u} \| \Big) \operatorname{dist}\big(\vec{\boldsymbol{\zeta}}, \mathcal{Q}_h\big) + c_2(\Omega)(1 + \kappa_5^2)^{1/2} \| \varphi \|_{1,\Omega} \| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} \\ & + L_k C_\varepsilon \widetilde{C}_\varepsilon (1 + \kappa_5^2)^{1/2} \| \boldsymbol{\zeta} \|_{\varepsilon,\Omega} \| \varphi - \varphi_h \|_{1,\Omega} \Bigg\}. \end{aligned} \tag{5.5}$$

Proof. It is clear from (3.42) that $\mathbf{C}_\varphi + \mathbf{D}_\mathbf{u}$ and $\mathbf{C}_{\varphi_h} + \mathbf{D}_{\mathbf{u}_h}$ are bounded bilinear forms (both with constant $\| \mathbf{C}_\varphi + \mathbf{D}_\mathbf{u} \|$, w.l.o.g. since it is independent of $(\mathbf{u}, \varphi)$) and uniformly elliptic with constant $\frac{\beta(\Omega)}{2}$. Also, $G_D + G_\mathbf{e}$ is a bounded linear functional in both $\mathcal{H}$ and $\mathcal{H}_h$. Therefore, a straightforward application of Lemma 5.1 to the pair (5.2) leads us to

$$\| \vec{\boldsymbol{\zeta}} - \vec{\boldsymbol{\zeta}}_h \| \leq C_{ST}^{\mathbf{E}} \inf_{\substack{\vec{\boldsymbol{\xi}}_h \in \mathcal{Q}_h \\ \vec{\boldsymbol{\xi}}_h \neq \vec{\mathbf{0}}}} \left\{ \| \vec{\boldsymbol{\zeta}} - \vec{\boldsymbol{\xi}}_h \| + \sup_{\substack{\vec{\boldsymbol{\chi}}_h \in \mathcal{Q}_h \\ \vec{\boldsymbol{\chi}}_h \neq \vec{\mathbf{0}}}} \frac{|(\mathbf{C}_\varphi + \mathbf{D}_\mathbf{u})(\vec{\boldsymbol{\xi}}_h, \vec{\boldsymbol{\chi}}_h) - (\mathbf{C}_{\varphi_h} + \mathbf{D}_{\mathbf{u}_h})(\vec{\boldsymbol{\xi}}_h, \vec{\boldsymbol{\chi}}_h)|}{\| \vec{\boldsymbol{\chi}}_h \|} \right\}. \tag{5.6}$$

where $C_{ST}^{\mathbf{E}} := \frac{2}{\beta(\Omega)} \max\{1, \| \mathbf{C}_\varphi + \mathbf{D}_{\mathbf{u}} \|\}$. In what follows we make use of the boundedness constants C_ε and $\widetilde{C}_\varepsilon$, which correspond, respectively, to the continuous embedding of $\mathbf{H}^\varepsilon(\Omega)$ in $\mathbf{L}^{2p}(\Omega)$ (cf. (3.63)) and $H^1(\Omega)$ in $\mathrm{L}^{n/\varepsilon}(\Omega)$ (cf. proof of Lemma 3.6). Then, adding and subtracting suitable terms, using the boundedness of the bilinear forms, and the same bounding technique as in (3.62), the numerator of the last term in (5.6) can be treated as follows:

$$
\begin{aligned}
&|(\mathbf{C}_\varphi + \mathbf{D}_{\mathbf{u}})(\vec{\boldsymbol{\xi}}_h, \vec{\boldsymbol{\chi}}_h) - (\mathbf{C}_{\varphi_h} + \mathbf{D}_{\mathbf{u}_h})(\vec{\boldsymbol{\xi}}_h, \vec{\boldsymbol{\chi}}_h)| \\
&\quad \leq \Big|(\mathbf{C}_\varphi + \mathbf{D}_{\mathbf{u}})(\vec{\boldsymbol{\xi}}_h - \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\chi}}_h) + (\mathbf{C}_\varphi - \mathbf{C}_{\varphi_h})(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\chi}}_h) + (\mathbf{D}_{\mathbf{u}} - \mathbf{D}_{\mathbf{u}_h})(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\chi}}_h) \\
&\quad - (\mathbf{C}_{\varphi_h} + \mathbf{D}_{\mathbf{u}_h})(\vec{\boldsymbol{\xi}}_h - \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\chi}}_h)\Big| \\
&\quad \leq 2\| \mathbf{C}_\varphi + \mathbf{D}_{\mathbf{u}} \| \| \vec{\boldsymbol{\zeta}} - \vec{\boldsymbol{\xi}}_h \| \| \vec{\boldsymbol{\chi}}_h \| + \int_\Omega \Big(k(\varphi) - k(\varphi_h)\Big) \boldsymbol{\zeta} \cdot \Big\{ \boldsymbol{\chi}_h - \kappa_5 \mathbf{q}_h \Big\} \\
&\quad - \int_\Omega \varphi(\mathbf{u} - \mathbf{u}_h) \cdot \Big\{ \boldsymbol{\chi}_h - \kappa_5 \mathbf{q}_h \Big\} \\
&\quad \leq 2\| \mathbf{C}_\varphi + \mathbf{D}_{\mathbf{u}} \| \| \vec{\boldsymbol{\zeta}} - \vec{\boldsymbol{\xi}}_h \| \| \vec{\boldsymbol{\chi}}_h \| + L_k (1 + \kappa_5^2)^{1/2} C_\varepsilon \widetilde{C}_\varepsilon \| \boldsymbol{\zeta} \|_{\varepsilon,\Omega} \| \varphi - \varphi_h \|_{1,\Omega} \| \vec{\boldsymbol{\chi}}_h \| \\
&\quad + c_2(\Omega)(1 + \kappa_5^2)^{1/2} \| \varphi \|_{1,\Omega} \| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} \| \vec{\boldsymbol{\chi}}_h \|,
\end{aligned}
\tag{5.7}
$$

which back into (5.6) gives (5.5), concluding this way the proof. $\square$

As a result of the previous two lemmas, we have a preliminary estimate for the error:

$$
\begin{aligned}
\| (\vec{\mathbf{t}}, \vec{\boldsymbol{\zeta}}) - (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\zeta}}_h) \| &\leq C_1 \operatorname{dist}\big(\vec{\mathbf{t}}, \mathcal{H}_h\big) + C_2 \operatorname{dist}\big(\vec{\boldsymbol{\zeta}}, \mathcal{Q}_h\big) + \Big\{ C_3 \widehat{C}_\varepsilon \| \mathbf{u} \|_{1+\varepsilon,\Omega} \\
&+ C_4 \| \mathbf{t} \|_{\varepsilon,\Omega} + C_5 \| \mathbf{g} \|_{\infty,\Omega} + C_6 \widehat{C}_\varepsilon \| \varphi \|_{1+\varepsilon,\Omega} + C_7 \| \boldsymbol{\zeta} \|_{\varepsilon,\Omega} \Big\} \| (\vec{\mathbf{t}}, \vec{\boldsymbol{\zeta}}) - (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\zeta}}_h) \|,
\end{aligned}
\tag{5.8}
$$

where

$$
\begin{gathered}
C_1 := C_{ST}^{\mathbf{M}}\big(1 + 2\| \mathbf{A}_\varphi + \mathbf{B}_{\mathbf{u}} \|\big), \quad C_2 := C_{ST}^{\mathbf{E}}\big(1 + 2\| \mathbf{C}_\varphi + \mathbf{D}_\varphi \|\big), \quad C_3 := C_{ST}^{\mathbf{M}} c_1(\Omega)(1 + \kappa_1^2)^{1/2} \\
C_4 := C_{ST}^{\mathbf{M}} L_\mu C_\varepsilon \widetilde{C}_\varepsilon (1 + \kappa_1^2)^{1/2}, \quad C_5 := C_{ST}^{\mathbf{M}}(1 + \kappa_2^2)^{1/2}, \quad C_6 := C_{ST}^{\mathbf{E}} c_2(\Omega)(1 + \kappa_5^2)^{1/2}, \\
C_7 := C_{ST}^{\mathbf{E}} L_k C_\varepsilon \widetilde{C}_\varepsilon (1 + \kappa_5^2)^{1/2},
\end{gathered}
$$

all being positive constants independent of the discretization parameters. Thus, we first bound the terms $\| \mathbf{u} \|_{1+\varepsilon,\Omega}$ and $\| \mathbf{t} \|_{\varepsilon,\Omega}$ using the further-regularity assumption (3.54), and then we bound $\| \varphi \|_{1+\varepsilon,\Omega}$ and $\| \boldsymbol{\zeta} \|_{\varepsilon,\Omega}$ in a similar way using (3.55), whence denoting by

$$
C_8 := (C_3 \widehat{C}_\varepsilon + C_4) \widetilde{C}_{\mathbf{M}}(r), \quad C_9 := (C_6 \widehat{C}_\varepsilon + C_7) \widetilde{C}_{\mathbf{E}}(r) \quad \text{and} \quad C_0 := \max\{C_8 r + C_5, C_8, C_9\},
$$

inequality (5.8) becomes

$$
\begin{aligned}
\| (\vec{\mathbf{t}}, \vec{\boldsymbol{\zeta}}) - (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\zeta}}_h) \| &\leq C_1 \operatorname{dist}\big(\vec{\mathbf{t}}, \mathcal{H}_h\big) + C_2 \operatorname{dist}\big(\vec{\boldsymbol{\zeta}}, \mathcal{Q}_h\big) \\
&+ C_0 \Big(\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathrm{m}} \|_{\varepsilon,\Omega} + \| \varphi_D \|_{1/2+\varepsilon,\Gamma_D} + \| f^{\mathrm{e}} \|_{0,\Omega} \Big) \| (\vec{\mathbf{t}}, \vec{\boldsymbol{\zeta}}) - (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\zeta}}_h) \|,
\end{aligned}
\tag{5.9}
$$

thus leading us to the main result of this section.

Theorem 5.4. *Assume that the data satisfy*

$$
C_0 \Big(\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{f}^{\mathrm{m}} \|_{\varepsilon,\Omega} + \| \varphi_D \|_{1/2+\varepsilon,\Gamma_D} + \| f^{\mathrm{e}} \|_{0,\Omega} \Big) < \frac{1}{2}.
\tag{5.10}
$$

Then, there exists a constant $C > 0$ depending only on parameters, data, and other constants, all of them independent of h, such that the following Céa estimate holds

$$
\| (\vec{\mathbf{t}}, \vec{\boldsymbol{\zeta}}) - (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\zeta}}_h) \| \leq C \Big\{ \operatorname{dist}\big(\vec{\mathbf{t}}, \mathcal{H}_h\big) + \operatorname{dist}\big(\vec{\boldsymbol{\zeta}}, \mathcal{Q}_h\big) \Big\}.
\tag{5.11}
$$

Proof. Thanks to (5.10), the estimate can be directly obtained from (5.9), thus arriving to $C := 2\max\{C_1, C_2\}$. □

Consequently, when using the finite element subspaces (4.18)-(4.24), the following can be established regarding the rates of convergence of the method.

Lemma 5.5. *In addition to the hypotheses of Theorems 3.9, 4.8 and 5.4, assume that there exists $s > 0$ such that $\mathbf{t} \in \mathbb{H}^s(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$, $\boldsymbol{\gamma} \in \mathbb{H}^s(\Omega)$, $\boldsymbol{\zeta} \in \mathbf{H}^s(\Omega)$, $\mathbf{p} \in \mathbf{H}^s(\Omega)$, $\mathrm{div}\,\mathbf{p} \in H^s(\Omega)$ and $\varphi \in \mathrm{H}^{s+1}(\Omega)$, and that the finite element subspaces are defined by (4.18)-(4.24). Then, there exists a constant $C > 0$, independent of h, such that*

$$\|(\vec{\mathbf{t}}, \vec{\boldsymbol{\zeta}}) - (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\zeta}}_h)\| \le C h^{\min\{s,k+1\}} \Big\{ \|\mathbf{t}\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{u}\|_{s+1,\Omega} \\ + \|\boldsymbol{\gamma}\|_{s,\Omega} + \|\boldsymbol{\zeta}\|_{s,\Omega} + \|\mathbf{p}\|_{s,\Omega} + \|\mathrm{div}\,\mathbf{p}\|_{s,\Omega} + \|\varphi\|_{s+1,\Omega} \Big\}. \tag{5.12}$$

Proof. It follows from the Céa estimate (5.11) and the approximation properties (4.25)-(4.31). □

6 Numerical Results

We now present two examples that will illustrate the performance of the augmented fully-mixed finite element method (4.1) with the subspaces indicated in (4.18)-(4.24) on a set of quasiuniform triangulations. The computational implementation is based on a `FreeFem++` code (cf. [17]) and the iterative method comes straightforward from the uncoupling strategy presented in Section 4.1. Then, as a stopping criteria, we finish the algorithm when the relative error between two consecutive iterations of the complete coefficient vector measured in the discrete ℓ^2 norm is sufficiently small, this is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\ell^2}}{\|\mathbf{coeff}^m\|_{\ell^2}} < \mathtt{tol},$$

where `tol` is a specified tolerance.

Let us first define the error per variable

$$e(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}, \quad e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega}, \quad e(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}, \quad e(p) := \|p - p_h\|_{0,\Omega},$$
$$e(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega}, \quad e(\boldsymbol{\zeta}) := \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{0,\Omega}, \quad e(\mathbf{p}) := \|\mathbf{p} - \mathbf{p}_h\|_{\mathrm{div};\Omega}, \quad e(\varphi) := \|\varphi - \varphi_h\|_{1,\Omega},$$

as well as their corresponding rates of convergence

$$r(\star) := \frac{\log(e(\star)/e'(\star))}{\log(h/h')} \quad \forall\, \star \in \{\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, p, \boldsymbol{\gamma}, \boldsymbol{\zeta}, \mathbf{p}, \varphi\},$$

where h and h' denote two consecutive meshsizes with errors e and e'.

6.1 Example 1: Two-dimensional smooth exact solution

In our first example, we consider $\Omega = (-1,1)^2$ and viscosity, thermal conductivity and body force given by

$$\mu(\varphi) = \exp(-0.25\,\varphi), \quad k(\varphi) = \exp(0.25\,\varphi), \quad \mathbf{g} = (0,1)^{\mathtt{t}}.$$

Dirichlet boundary conditions will be imposed on $\Gamma_D := \Gamma_D^1 \cup \Gamma_D^2$ where $\Gamma_D^1 := [-1,1] \times \{-1\}$ and $\Gamma_D^2 := [-1,1] \times \{1\}$, whereas Neumann boundary conditions will be imposed on the remainder of the border, that is, $\Gamma_N := \Gamma\backslash\Gamma_D$. In this way, we consider boundary data φ_D and source terms $\mathbf{f}^{\mathtt{m}}$ and $f^{\mathtt{e}}$ such that the exact solution to the Boussinesq problem is given by $\mathbf{u}(x,y) = (u_1(x,y), u_2(x,y))^{\mathtt{t}}$ with

$$u_1(x,y) := 2\,y\sin(\pi x)\sin(\pi y)(x^2-1) + \pi\sin(\pi x)\cos(\pi y)(x^2-1)(y^2-1),$$
$$u_2(x,y) := -2\,x\sin(\pi x)\sin(\pi y)(y^2-1) - \pi\cos(\pi x)\sin(\pi y)(x^2-1)(y^2-1),$$

and

$$p(x,y) := y^2 - x^2, \quad \varphi(x,y) := -0.6944\,y^4 + 1.6944\,y^2.$$

Concerning the stabilization parameters, these are taken as pointed out at the end of Section 3.3, with $\kappa_0 = \frac{1}{2}$ and the bounds for the viscosity and thermal conductivity functions are estimated in

$$\mu_1 = 0.5, \quad \mu_2 = 1.25, \quad k_1 = 0.75, \quad k_2 = 1.3.$$

In Figure 6.1 we display part of the solution obtained with fully-mixed finite element method using a first order approximation and 1,409,884 DOF. Notice that not only we are able to recover the original unknowns but also to compute further variables of physical interest. In turn, Tables 6.1 and 6.2 show the convergence history for a sequence of quasi-uniform mesh refinements, thus confirming the rates of convergence predicted by Lemma 5.5, that is, when using first and second order finite elements, and considering that the exact solution is smooth enough, whence the method converges with orders $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$, respectively.

6.2 Example 2: Three-dimensional smooth exact solution

In our second example, we consider $\Omega = (0,1)^3$ and viscosity, thermal conductivity and body force given by

$$\mu(\varphi) = 2.0 - 0.5\,\varphi^2 - 0.5\,\varphi^4, \quad k(\varphi) = -0.5 + 2.0\,\mu(\varphi), \quad \mathbf{g} = (0,0,1)^{\mathtt{t}}.$$

With respect to boundary conditions, we impose Dirichlet conditions on $[0,1]^2 \times \{0\} =: \Gamma_D$ and Neumann conditions on the rest of the boundary, that is, $\Gamma_N := \Gamma\backslash\Gamma_D$. Then, we consider boundary data φ_D and source terms $\mathbf{f}^{\mathtt{m}}$ and $f^{\mathtt{e}}$ such that the exact solution to the Boussinesq problem is given by $\mathbf{u}(x,y,z) = (u_1(x,y,z), u_2(x,y,z), u_3(x,y,z))^{\mathtt{t}}$ with

$$u_1(x,y,z) := 8\,x^2yz(x-1)^2(y-1)(z-1)(y-z)$$
$$u_2(x,y,z) := -8\,xy^2z(x-1)(y-1)^2(z-1)(x-z)$$
$$u_3(x,y,z) := 8\,xyz^2(x-1)(y-1)(z-1)^2(x-y)$$

and

$$\varphi(x,y,z) := \sin(\pi x)^2\sin(\pi y)^2(z-1)^2, \quad p(x,y,z) = (x-0.5)^3\sin(y+z).$$

Concerning the stabilization parameters, we take them again as in Section 3.3, but this time with $\kappa_0 = 1$. In tun, the bounds for the viscosity and thermal conductivity are estimated in

$$\mu_1 = 1.0, \quad \mu_2 = 2.0, \quad k_1 = 1.5, \quad k_2 = 3.5.$$

Part of the solution is shown in Figure 6.2, and a convergence history for a set of quasi-uniform mesh refinements is shown in Table 6.3, thus showing also that, having the problem a smooth exact solution, this fully-mixed finite element method converges optimally with order $\mathcal{O}(h)$ (when using a first order element).

Finite Element: $\mathbb{H}_h^{\mathbf{t}}$ - $\mathbb{H}_h^{\boldsymbol{\sigma}}$ - $\mathbf{H}_h^{\mathbf{u}}$ - $\mathbb{H}_h^{\gamma}$ - $\mathbf{H}_h^{\boldsymbol{\zeta}}$ - $\mathbf{H}_h^{\mathbf{p}}$ - H_h^{φ} with $k=0$								
DOF	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(p)$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\zeta})$	$e(\mathbf{p})$	$e(\varphi)$
1,816	4.5307	15.6976	7.7257	2.3619	14.3622	0.4298	1.5650	0.3698
6,972	2.1933	7.9406	3.3074	1.0874	8.8796	0.2124	0.7864	0.1758
27,052	1.0947	3.9421	1.6088	0.5396	6.0695	0.1022	0.3809	0.0831
107,142	0.5331	2.0064	0.8011	0.2678	2.5649	0.0517	0.1962	0.0426
431,398	0.2581	0.9882	0.3925	0.1216	1.5602	0.0259	0.0970	0.0211
1,707,922	0.1271	0.4937	0.1947	0.0613	0.7027	0.0127	0.0482	0.0105
h	$r(\mathbf{t})$	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(p)$	$r(\boldsymbol{\gamma})$	$r(\boldsymbol{\zeta})$	$r(\mathbf{p})$	$r(\varphi)$
0.4129	-	-	-	-	-	-	-	-
0.1940	0.9605	0.9023	1.1233	1.0270	0.6366	0.9331	0.9111	0.9845
0.0995	1.0402	1.0482	1.0788	1.0490	0.5695	1.0951	1.0851	1.1221
0.0527	1.1333	1.0638	1.0982	1.1032	1.3567	1.0736	1.0453	1.0521
0.0311	1.3744	1.3419	1.3519	1.4961	0.9419	1.3087	1.3350	1.3342
0.0150	0.9729	0.9535	0.9635	0.9411	1.0960	0.9778	0.9600	0.9503

Table 6.1: Convergence history for Example 1, with a uniform mesh refinement and a first order approximation. Here, the simulation with 1,816 DOF took 10 fixed-point iterations, while the rest of them took 9 iterations to achieve a tolerance of $\mathtt{tol} = 10^{-8}$.

Finite Element: $\mathbb{H}_h^{\mathbf{t}}$ - $\mathbb{H}_h^{\boldsymbol{\sigma}}$ - $\mathbf{H}_h^{\mathbf{u}}$ - $\mathbb{H}_h^{\gamma}$ - $\mathbf{H}_h^{\boldsymbol{\zeta}}$ - $\mathbf{H}_h^{\mathbf{p}}$ - H_h^{φ} with $k=1$								
DOF	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(p)$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\zeta})$	$e(\mathbf{p})$	$e(\varphi)$
5,812	0.9568	3.0676	1.3170	0.6622	2.2406	0.0690	0.2052	0.0434
22,564	0.2193	0.7735	0.3002	0.1932	0.6699	0.0154	0.0467	0.0098
88,036	0.0554	0.1939	0.0750	0.0472	0.2187	0.0038	0.0118	0.0023
349,660	0.0140	0.0494	0.0192	0.0119	0.0494	0.0010	0.0031	0.0006
1,409,884	0.0035	0.0121	0.0047	0.0029	0.0152	0.0002	0.0008	0.0001
h	$r(\mathbf{t})$	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(p)$	$r(\boldsymbol{\gamma})$	$r(\boldsymbol{\zeta})$	$r(\mathbf{p})$	$r(\varphi)$
0.4129	-	-	-	-	-	-	-	-
0.1940	1.9507	1.8241	1.9580	1.6310	1.5987	1.9811	1.9595	1.9687
0.0995	2.0591	2.0712	2.0759	2.1084	1.6752	2.1123	2.0631	2.1848
0.0527	2.1678	2.1538	2.1490	2.1666	2.3443	2.0727	2.1265	2.1000
0.0311	2.6452	2.6677	2.6506	2.6598	2.2355	2.6761	2.6587	2.6447

Table 6.2: Convergence history for Example 1, with a uniform mesh refinement and a second order approximation. Here, all the simulations took 9 fixed-point iterations to achieve a tolerance of $\mathtt{tol} = 10^{-8}$.

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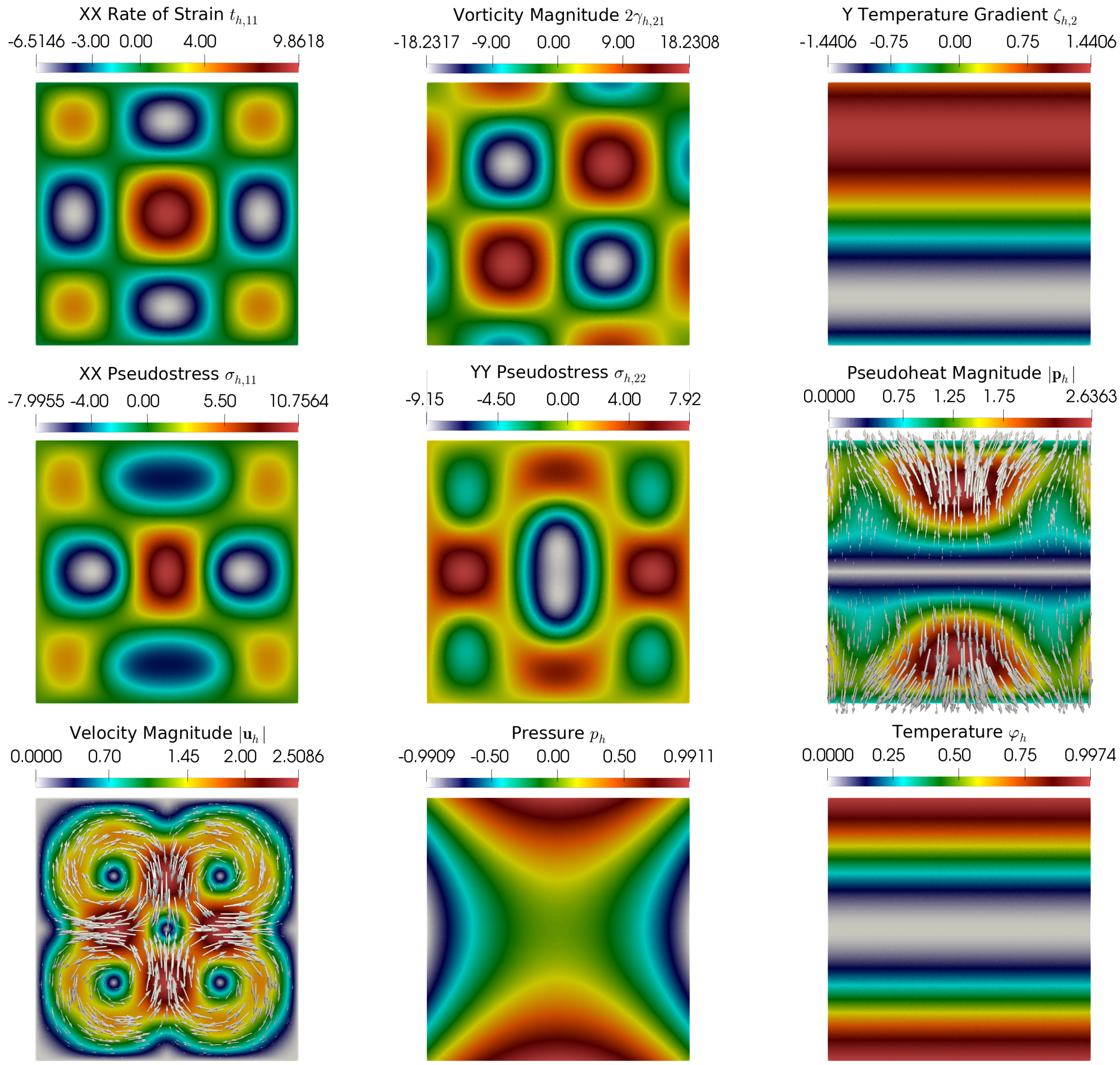

Figure 6.1: Numerical Results for Example 1. From left to right and from up to down: XX-component of the Rate of Strain tensor, true vorticty magnitude (computed as twice the YX-component of the full vorticity tensor), Y-component of the temperature gradient, XX and YY components of the pseudostress tensor, pseudoheat magnitude and its vector field, velocity magnitude and its vector field, pressure and temperature. Snapshots obtained from a simulation with 1,409,884 DOF and a second order approximation.

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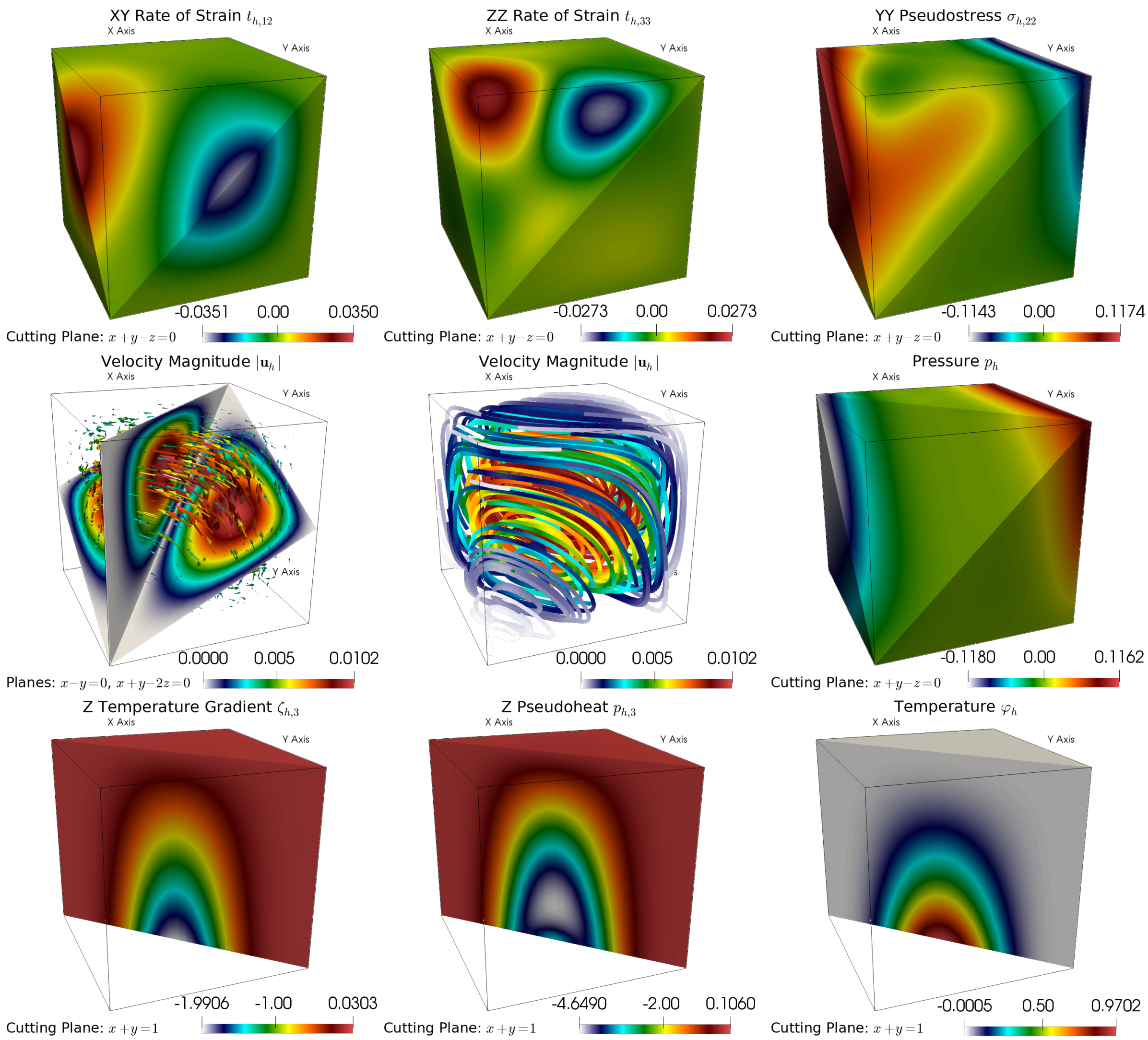

Figure 6.2: Numerical Results for Example 2. From left to right and from up to down: XY and ZZ components of the rate of strain tensor (the last one postprocessed as $-t_{h,11} - t_{h,22}$), YY component of the pseudostress tensor, velocity vector field, flow streamlines, pressure, Z components of the temperature gradient and pseudoheat, and temperature. Snapshots obtained from a simulation with 3,903,877 DOF and a first order approximation.

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Finite Element: $\mathbb{H}_h^{\mathbf{t}}$ - $\mathbb{H}_h^{\boldsymbol{\sigma}}$ - $\mathbf{H}_h^{\mathbf{u}}$ - $\mathbb{H}_h^{\gamma}$ - $\mathbf{H}_h^{\boldsymbol{\zeta}}$ - $\mathbf{H}_h^{\mathbf{p}}$ - H_h^{φ} with $k=0$								
DOF	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(p)$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\zeta})$	$e(\mathbf{p})$	$e(\varphi)$
1,117	0.0219	0.2559	0.0420	0.0249	0.0262	0.7239	17.6230	1.2979
8,181	0.0128	0.1367	0.0265	0.0176	0.0196	0.4249	8.0953	0.6128
62,821	0.0079	0.0700	0.0140	0.0097	0.0132	0.2240	4.3469	0.3044
492,741	0.0042	0.0351	0.0071	0.0047	0.0077	0.1137	2.1971	0.1505
3,903,877	0.0022	0.0175	0.0035	0.0022	0.0040	0.0571	1.1017	0.0750
h	$r(\mathbf{t})$	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(p)$	$r(\boldsymbol{\gamma})$	$r(\boldsymbol{\zeta})$	$r(\mathbf{p})$	$r(\varphi)$
0.7071	-	-	-	-	-	-	-	-
0.3536	0.7778	0.9044	0.6636	0.4972	0.4220	0.7689	1.1223	1.0828
0.1768	0.7024	0.9661	0.9205	0.8660	0.5634	0.9235	0.8971	1.0094
0.0884	0.9094	0.9951	0.9871	1.0518	0.7899	0.9788	0.9844	1.0165
0.0442	0.9513	1.0018	1.0000	1.0742	0.9191	0.9943	0.9958	1.0044

Table 6.3: Convergence history for Example 2, with a uniform mesh refinement and a first order approximation. Here, to achieve a tolerance of $\mathtt{tol} = 10^{-8}$, a number of 6 fixed-point iterations were needed for the first simulation, 7 iterations for the second one, and 8 for the third, fourth and fifth simulations.

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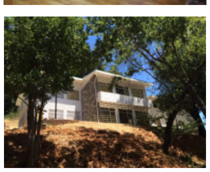

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