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Abstract

This paper deals with the *a posteriori* error analysis for an augmented mixed discontinuous formulation for the stationary Stokes problem. By considering an appropriate auxiliary problem, we derive an *a posteriori* error estimator. We prove that this estimator is reliable and locally efficient, and consists of just five residual terms. Numerical experiments confirming the theoretical properties of the augmented discontinuous scheme as well as of the estimator. They also show the capability of the corresponding adaptive algorithm to localize the singularities and the large stress regions of the solution, are also reported.

Key words: Discontinuous Galerkin, augmented formulation, *a posteriori* residual error estimates.

Mathematics subject classifications (1991): 65N30; 65N12; 65N15

1 Introduction

Recently in [10] we developed an analysis of the DG method for stationary Stokes problem, based on velocity pseudo-stress formulation. The approach there introduce a new unknown called pseudo-stress, which allows us to eliminate the pressure and obtain a nonstandard first order system for the Stokes problem. The well posedness of the resulting DG scheme, in dual-primal mixed formulation, is a straightforward consequence of Babuška-Brezzi theory for high-order approximation spaces. For low-order methods, the use of the Raviart-Thomas projection and the Fredholm's alternative, allows us to conclude the bijectivity of the operator that defines the DG scheme. A priori error estimates are then derived, under the well known assumptions:

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- the piecewise divergence of tensors in the approximation space of stresses, belongs to the discrete space that approximates the velocity,
- the piecewise gradient of elements in the discrete space for velocity, is in the stress approximation space.

Next, in order to relax these two restrictions, we follow the ideas given in previous works (see [3], [4] and [5]) to propose two stabilized schemes: one incorporating a div-div term, and a second one adding a Galerkin least-squares residual. As consequence, for the second stabilized scheme we can establish existence and uniqueness of the discrete scheme thanks to Lax-Milgram's Theorem. This way we circumvent the use of any lifting operators (i.e we do not need the mild condition ([13])), and thus allows us to use any pair of approximation spaces for the unknowns. Furthermore, a cheap and easy element by element post-processing, gives us an approximation of the pressure, which has the optimal expected rate of convergence.

Strongly motivated by the competitive character of our augmented DG formulation, we now believe in the need of deriving the corresponding *a posteriori* error estimator, and thus to give an adaptive refinement strategy. Several kind of *a posteriori* error analyses have been developed for discontinuous Galerkin methods during the last years (see, for e.g. [11], [15], [22], [23], [28]). In the framework of stabilized DG methods, based on an appropiate Helmholtz decomposition, we have developed an *a posteriori* error analysis for linear elliptic problems (see [5]), Darcy flows (see [4]) and Helmholtz equation (see [8]). Therefore, the aim of this paper is to complement the analysis done in [10], by extending our previous works on *a posteriori* error analysis to the Stokes problem.

Now, in order to describe the model of interest, we let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary Γ. Then, given the source terms $\boldsymbol{f} \in [L^2(\Omega)]^2$ and $\boldsymbol{g} \in [H^{1/2}(\Gamma)]^2$, we look for the velocity (vector field) $\boldsymbol{u}$ and the pressure (scalar field) p such that

$$-\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega\,, \qquad \operatorname{div}(\boldsymbol{u}) = 0 \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \Gamma\,, \tag{1.1}$$

where $\nu > 0$ is the viscosity of the flow and the datum $\boldsymbol{g}$ satisfies the compatibility condition $\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{\nu} = 0$, with $\boldsymbol{\nu}$ stands for the unit outward normal at Γ. In addition, for uniqueness purposes, we suppose that $p \in L^2_0(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q = 0\}$.

We remark that problem (1.1) has been already analyzed in [16], applying the local discontinuous Galerkin method. Moreover, in [24] a stabilized DG formulation is developed for Stokes problem. Concerning an *a posteriori* error estimator for Stokes problem, we can refer [22] as one of the first papers in this kind of linear problems. In the current work, we consider the discontinuous approach based on the non-standard velocity-pseudo stress formulation for the Stokes system, previously introduced in [10]. Our purpose is to develop the corresponding *a posteriori* error analysis, which allows us to give an adaptive strategy that automatically generates adapted meshes, localizing inner or boundary layers, or regions where the gradient of the solution is dominant, for example.

The rest of the paper is organized as follows. In Section 2, we review the *a priori* error estimates given in [10] to the augmented discontinuous Galerkin scheme. The main result of the present work is described in Section 3, where we deduce an *a posteriori* error estimator for this scheme. Finally, in Section 4 we give some numerical examples that confirm the theoretical results derived in this work.

We end this section with some notations to be used throughout the paper. Given any Hilbert space H, we denote by H^2 the space of vectors of order 2 with entries in H, and by $H^{2\times 2}$ the space of square tensors of order 2 with entries in H. In particular, given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we write, as usual, $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, $\mathrm{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}$ and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$. For vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^2$, we denote by $\boldsymbol{v}\otimes\boldsymbol{w}$ the matrix whose ij-th entry is $v_i\,w_j$. We also use the standard notations for Sobolev spaces and norms. We denote by $[H^1_{\Gamma_D}(\Omega)]^2 := \{\, \boldsymbol{v} \in [H^1(\Omega)]^2 : \quad \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D \,\}$, by $H = H(\mathrm{div};\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2 \,\}$, and by $H_0 := \{\boldsymbol{\tau} \in H : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\}$. Note that $H = H_0 \oplus \mathbb{R}\,\mathbf{I}$, that is, for any $\boldsymbol{\tau} \in H$ there exist unique $\boldsymbol{\tau}_0 \in H_0$ and $d \in \mathbb{R}$ such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I}$. In addition, we define the deviator of the tensor $\boldsymbol{\tau} \in H$ by $\boldsymbol{\tau}^d := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}$. We remark that $\mathrm{tr}(\boldsymbol{\tau}^d) = 0$ in Ω, then for any $\boldsymbol{\tau} \in H$, $\boldsymbol{\tau}^d \in H_0$. Finally, we use C or c, with or without subscripts, to denote generic constants, independent of the discretization parameters, which may take different values at different occurrences.

2 The augmented DG formulation

In this section, we recall from [10] the augmented discontinuous Galerkin formulations of the corresponding boundary value problem, considering the non-standard velocity-pseudo stress formulation. We review the well posedness of the scheme, and provide the associated *a priori* error estimate.

2.1 Meshes, averages and jumps

We let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape-regular triangulations of $\bar{\Omega}$ (with possible hanging nodes) made up of straight-side triangles T with diameter h_T and unit outward normal to ∂T given by $\boldsymbol{\nu}_T$. As usual, the index h also denotes $h := \max_{T\in\mathcal{T}_h} h_T$. Then, given $\mathcal{T}_h$, its edges are defined as follows. An *interior edge* of $\mathcal{T}_h$ is the (nonempty) interior of $\partial T \cap \partial T'$, where T and T' are two adjacent elements of $\mathcal{T}_h$, not necessarily matching. Similarly, a *boundary edge* of $\mathcal{T}_h$ is the (nonempty) interior of $\partial T \cap \partial\Omega$, where T is a boundary element of $\mathcal{T}_h$. We denote by $\mathcal{E}_I$ the list of all interior edges of $\mathcal{T}_h$ (counted only once) on Ω, and by $\mathcal{E}_\Gamma$ the list of all boundary edges, and set $\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_\Gamma$ the interior grid generated by the triangulation $\mathcal{T}_h$ (also called skeleton). Further, for each $e \in \mathcal{E}$, h_e represents its length. Also, in what follows we assume that $\mathcal{T}_h$ is of *bounded variation*, which means that there exists a constant $l > 1$, independent of the meshsize h, such that $l^{-1} \le \frac{h_T}{h_{T'}} \le l$ for each pair T, $T' \in \mathcal{T}_h$ sharing an interior edge.

Next, to define average and jump operators, we let T and T' be two adjacent elements of $\mathcal{T}_h$ and $\boldsymbol{x}$ be an arbitrary point on the interior edge $e = \partial T \cap \partial T' \in \mathcal{E}_I$. In addition, let q , $\boldsymbol{v}$ and $\boldsymbol{\tau}$ be scalar- vector and matrix-valued functions, respectively, that are smooth inside each element $T \in \mathcal{T}_h$. We denote by $(\boldsymbol{v}_{T,e}, \boldsymbol{\tau}_{T,e})$ the restriction of $(\boldsymbol{v}_T, \boldsymbol{\tau}_T)$ to e. Then, we define the averages at $\boldsymbol{x} \in e$ by:

$$\{\boldsymbol{v}\} := \frac{1}{2}\big(\boldsymbol{v}_{T,e} + \boldsymbol{v}_{T',e}\big)\,, \quad \{\boldsymbol{\tau}\} := \frac{1}{2}\big(\boldsymbol{\tau}_{T,e} + \boldsymbol{\tau}_{T',e}\big)\,.$$

Similarly, the jumps at $\boldsymbol{x} \in e$ are given by

$$[\![\boldsymbol{v}]\!] := \boldsymbol{v}_{T,e} \cdot \boldsymbol{\nu}_T + \boldsymbol{v}_{T',e} \cdot \boldsymbol{\nu}_{T'} \,, \quad \underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v}_{T,e} \otimes \boldsymbol{\nu}_T + \boldsymbol{v}_{T',e} \otimes \boldsymbol{\nu}_{T'} \,, \quad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}_{T,e}\, \boldsymbol{\nu}_T + \boldsymbol{\tau}_{T',e}\, \boldsymbol{\nu}_{T'} \,.$$

On boundary edges e, we set $\{\boldsymbol{v}\} := \boldsymbol{v}$, $\{\boldsymbol{\tau}\} := \boldsymbol{\tau}$, as well as $[\![\boldsymbol{v}]\!] := \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v} \otimes \boldsymbol{\nu}$ and $[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}\, \boldsymbol{\nu}$. Hereafter, as usual $\mathbf{div}_h$ and ∇_h denote the piecewise divergence and gradient operators, respectively.

2.2 The augmented discrete formulation

We begin reformulating the problem (1.1), to this aim we introduce the pseudo stress $\boldsymbol{\sigma} := \nu \nabla \boldsymbol{u} - p\boldsymbol{I}$ in Ω as a new additional unknown. Using the incompressibility condition $\mathrm{div}(\boldsymbol{u}) = 0$ in Ω, it is not difficult to see that $p = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$ in Ω, which implies that $\boldsymbol{\sigma} \in H_0$. This relation allows us rewrite the problem (1.1) as the following linear system of first order in $\bar{\Omega}$: *Find* $(\boldsymbol{\sigma}, \boldsymbol{u}) \in H_0 \times [H^1(\Omega)]^2$

$$\boldsymbol{\sigma}^d = \nu \nabla \boldsymbol{u} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\boldsymbol{\sigma}) = -\boldsymbol{f} \quad \text{in} \quad \Omega, \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \Gamma\,. \tag{2.1}$$

Now, given a mesh $\mathcal{T}_h$, we proceed as in [27] (or [14]) and multiply each one of the equations of (2.1) by suitable test functions. Our purpose is to approximate the exact solution $(\boldsymbol{\sigma}, \boldsymbol{u})$ of (2.1) by discrete functions $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ in appropriate finite element space $\boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ such that for all $T \in \mathcal{T}_h$ we have

$$\begin{aligned} &\frac{1}{\nu}\int_T \boldsymbol{\sigma}_h^{\mathtt{d}} : \boldsymbol{\tau}^d + \int_T \boldsymbol{u}_h \cdot \mathbf{div}\boldsymbol{\tau} - \int_{\partial T} \widehat{\boldsymbol{u}} \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T = 0 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_{h,0}\,, \\ &\int_T \nabla \boldsymbol{v} : \boldsymbol{\sigma}_h - \int_{\partial T} \boldsymbol{v} \cdot \widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T = \int_T \boldsymbol{f} \cdot \boldsymbol{v} \qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h\,, \end{aligned} \tag{2.2}$$

where the *numerical fluxes* $\widehat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{\sigma}}$, which usually depend on $\boldsymbol{u}_h$, $\boldsymbol{\sigma}_h$, and the boundary data, are given by $\widehat{\boldsymbol{u}} := \widehat{\boldsymbol{u}}(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{g})$ and $\widehat{\boldsymbol{\sigma}} := \widehat{\boldsymbol{\sigma}}(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{g})$ for each $T \in \mathcal{T}_h$ such that:

$$\widehat{\boldsymbol{u}}_{T,e} := \begin{cases} \{\boldsymbol{u}_h\} + \underline{[\![\boldsymbol{u}_h]\!]}\boldsymbol{\beta} - \gamma[\![\boldsymbol{\sigma}_h]\!] & \text{if } e \in \mathcal{E}_I, \\ \boldsymbol{g} & \text{if } e \in \mathcal{E}_\Gamma, \end{cases} \tag{2.3}$$

and

$$\widehat{\boldsymbol{\sigma}}_{T,e} := \begin{cases} \{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta} - \alpha\underline{[\![\boldsymbol{u}_h]\!]} & \text{if } e \in \mathcal{E}_I, \\ \boldsymbol{\sigma}_h - \alpha(\boldsymbol{u}_h - \boldsymbol{g}) \otimes \boldsymbol{\nu} & \text{if } e \in \mathcal{E}_\Gamma\,. \end{cases} \tag{2.4}$$

Here, the auxiliary functions α, γ (scalars) and $\boldsymbol{\beta}$ (vector), to be chosen appropriately, are single valued on each edge $e \in \mathcal{E}$ and such that they allow us to prove the optimal rates of convergence of our approximation. The corresponding (and now standard) corresponding analysis let us to set $\alpha := \frac{\widehat{\alpha}}{\mathtt{h}}$, $\gamma := \frac{\widehat{\gamma}}{\mathtt{h}}$, and $\boldsymbol{\beta}$ as an arbitrary vector in $\mathbb{R}^2$. Hereafter, $\widehat{\alpha}$ and $\widehat{\gamma}$ are positive arbitrary constants, while $\mathtt{h}$ is defined by

$$\mathtt{h} := \begin{cases} \max\{h_T, h_{T'}\} & \text{if } e \in \mathcal{E}_I \quad (e = \partial T \cap \partial T')\,, \\ h_T & \text{if } e \in \mathcal{E}_\Gamma \quad (e = \partial T \cap \Gamma)\,. \end{cases}$$

Then, after performing a second integration by parts, summing up over all $T \in \mathcal{T}_h$ and adding appropriate Galerkin-least squares terms, we obtain the following discrete augmented discontinuous Galerkin formulation (see Subsection 3.4 in [10] for more details): *Find* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ *such that*

$$A_{DG}^{stab}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}, \boldsymbol{v})) = F_{DG}^{stab}(\boldsymbol{\tau}, \boldsymbol{v}) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{v}_h\,, \tag{2.5}$$

where the bilinear form $A_{DG}^{stab} : ((H(\text{div}; \mathcal{T}_h) \cap [H^{\varepsilon}(\Omega)]^{2\times 2}) \times H^1(\mathcal{T}_h)) \times ((H(\text{div}; \mathcal{T}_h) \cap [H^{\varepsilon}(\Omega)]^{2\times 2}) \times H^1(\mathcal{T}_h)) \to \mathbb{R}$ and the linear functional $F_{DG}^{stab} : (H(\text{div}; \mathcal{T}_h) \cap [H^{\varepsilon}(\Omega)]^{2\times 2}) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ are defined by

$$\begin{aligned} A_{DG}^{stab}((\boldsymbol{\zeta}, \boldsymbol{w}), (\boldsymbol{\tau}, \boldsymbol{v})) &:= \frac{1}{\nu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \boldsymbol{w} \cdot \mathbf{div}_h(\boldsymbol{\tau}) - \int_{\mathcal{E}_I} \big(\{\boldsymbol{w}\} + \underline{[\![\boldsymbol{w}]\!]}\boldsymbol{\beta}\big) \cdot [\![\boldsymbol{\tau}]\!] \\ &- \int_\Omega \boldsymbol{v} \cdot \mathbf{div}_h(\boldsymbol{\zeta}) + \int_{\mathcal{E}_I} \big(\{\boldsymbol{v}\} + \underline{[\![\boldsymbol{v}]\!]}\boldsymbol{\beta}\big) \cdot [\![\boldsymbol{\zeta}]\!] + \int_{\mathcal{E}_I} \gamma [\![\boldsymbol{\zeta}]\!] \cdot [\![\boldsymbol{\tau}]\!] + \boldsymbol{\alpha}(\boldsymbol{w}, \boldsymbol{v}) \\ &+ \delta_1 \int_\Omega (\nu \nabla_h \boldsymbol{w} - \boldsymbol{\zeta}^d) : (\nu \nabla_h \boldsymbol{v} + \boldsymbol{\tau}^d) + \delta_2 \int_\Omega \mathbf{div}_h(\boldsymbol{\zeta}) \cdot \mathbf{div}_h(\boldsymbol{\tau})\,, \end{aligned} \tag{2.6}$$

with $\boldsymbol{\alpha} : [H^1(\mathcal{T}_h)]^2 \times [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ being the bilinear form defined by:

$$\boldsymbol{\alpha}(\boldsymbol{w}, \boldsymbol{v}) := \int_{\mathcal{E}} \alpha\, \underline{[\![\boldsymbol{v}]\!]} : \underline{[\![\boldsymbol{w}]\!]}\,, \quad \forall\, \boldsymbol{v}, \boldsymbol{w} \in [H^1(\mathcal{T}_h)]^2\,.$$

and

$$F_{DG}^{stab}(\boldsymbol{\tau}, \boldsymbol{v}) := \int_{\mathcal{E}_\Gamma} \boldsymbol{g} \cdot \boldsymbol{\tau}\boldsymbol{\nu} + \int_{\mathcal{E}_\Gamma} \alpha(\boldsymbol{g} \otimes \boldsymbol{\nu}) : (\boldsymbol{v} \otimes \boldsymbol{\nu}) - \delta_2 \int_\Omega \boldsymbol{f} \cdot \mathbf{div}_h(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,,$$

for all $(\boldsymbol{\zeta}, \boldsymbol{w}), (\boldsymbol{\tau}, \boldsymbol{v}) \in (H(\text{div}; \mathcal{T}_h) \cap [H^{\varepsilon}(\Omega)]^{2\times 2}) \times [H^1(\mathcal{T}_h)]^2$, with an appropriate $\varepsilon > 1/2$. The discrete spaces are

$$\boldsymbol{\Sigma}_h := \Big\{ \boldsymbol{\tau}_h \in [L^2(\mathcal{T}_h)]^{2\times 2} : \quad \boldsymbol{\tau}_h\big|_T \in [\mathbf{RT}_r(T)^{\mathrm{t}}]^2 \quad \forall\, T \in \mathcal{T}_h \Big\},$$

$$\boldsymbol{\Sigma}_{h,0} := \Big\{ \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h : \quad \int_\Omega \text{tr}(\boldsymbol{\tau}_h) = 0 \Big\},$$

$$\boldsymbol{V}_h := \Big\{ \boldsymbol{v}_h \in [L^2(\Omega)]^2 : \quad \boldsymbol{v}_h\big|_T \in [\mathbf{P}_k(T)]^2 \quad \forall\, T \in \mathcal{T}_h \Big\},$$

with $k \geq 1$ and $r \geq 0$. Hereafter, given an integer $\kappa \geq 0$ we denote by $\mathbf{P}_\kappa(T)$ the space of polynomials of degree at most κ on T, and for each $T \in \mathcal{T}_h$ we introduce the local Raviart-Thomas space of order κ (cf. [29]), $\mathbf{RT}_\kappa(T) := [\mathbf{P}_\kappa(T)]^2 \oplus \boldsymbol{x}\mathbf{P}_\kappa(T) \subseteq [\mathbf{P}_{\kappa+1}(T)]^2$.

The spaces $\boldsymbol{\Sigma}_h$ and $\boldsymbol{\Sigma}_{h,0}$ are provided with the norm of $\boldsymbol{\Sigma} := H(\text{div}; \mathcal{T}_h) \cap [H^{\epsilon}(\mathcal{T}_h)]^{2\times 2}$, with an appropriate $\epsilon > 1/2$, which is defined by

$$\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}^2 := \|\boldsymbol{\tau}\|_{[L^2(\Omega)]^{2\times 2}}^2 + \|\mathbf{div}_h(\boldsymbol{\tau})\|_{[L^2(\Omega)]^2}^2 + \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|_{[L^2(\mathcal{E}_I)]^2}^2 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}$$

while for $\boldsymbol{V}_h$, we introduce its seminorm $|\cdot|_h : [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ by

$$|\boldsymbol{v}|_h^2 := ||\alpha^{1/2}\underline{[\![\boldsymbol{v}]\!]}||_{[L^2(\mathcal{E}_I)]^{2\times 2}}^2 + ||\alpha^{1/2}\boldsymbol{v} \otimes \boldsymbol{\nu}||_{[L^2(\mathcal{E}_\Gamma)]^{2\times 2}}^2 \quad \forall\, \boldsymbol{v} \in [H^1(\mathcal{T}_h)]^2\,,$$

and the norm $\|\!|\cdot|\!\|_h : [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ as

$$\|\!|\boldsymbol{v}|\!\|_h^2 := ||\nabla_h \boldsymbol{v}||^2_{[L^2(\Omega)]^{2\times 2}} + |\boldsymbol{v}|_h^2 \quad \forall\, \boldsymbol{v} \in [H^1(\mathcal{T}_h)]^2 .$$

In addition, we define the norm $\|(\cdot,\cdot)\|_{DG} : \boldsymbol{\Sigma} \times [H^1(\mathcal{T}_h)]^2 \to \mathbb{R}$ by

$$\|(\boldsymbol{\tau}, \boldsymbol{v})\|^2_{DG} := ||\boldsymbol{\tau}||^2_{\boldsymbol{\Sigma}} + \|\!|\boldsymbol{v}|\!\|_h^2 \quad \forall\,(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma} \times [H^1(\mathcal{T}_h)]^2 .$$

The well posedness as well as the corresponding rate of convergence is summarized in the next theorem.

Theorem 2.1 *Let $(\delta_1, \delta_2) \in \mathbb{R}^2$ such that $0 < \delta_1 < \frac{1}{\nu}$ and $\delta_2 > 0$. Then, problem (2.5) is uniquely solvable, and there exists a positive constant $C_{\mathtt{F}}$, independent of the mesh size, such that there holds*

$$||(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)||_{DG} \leq C_{\mathtt{F}}\, \mathcal{B}(\boldsymbol{f}, \boldsymbol{g}) , \tag{2.7}$$

with $\mathcal{B}(\boldsymbol{f}, \boldsymbol{g}) := \left(\|\boldsymbol{f}\|^2_{[L^2(\Omega)]^2} + \|\alpha^{1/2}\boldsymbol{g}\|^2_{[L^2(\mathcal{E}_\Gamma)]^2}\right)^{1/2}$. In addition, assuming that the exact solution $(\boldsymbol{\sigma} := \nu\nabla\boldsymbol{u} - p\mathbf{I}, \boldsymbol{u})$ of (2.1) is such that $\boldsymbol{\sigma}|_T \in [H^t(T)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}|_T) \in [H^t(T)]^2$ and $\boldsymbol{u}|_T \in [H^{1+t}(T)]^2$ with $t > 1/2$, for all $T \in \mathcal{T}_h$, there exists $C_{\mathtt{err}} > 0$, independent of the mesh size, such that

$$\begin{aligned} &\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h)\|^2_{DG} \\ &\leq C_{\mathtt{err}} \sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k,r+1\}} \Big\{ \|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\sigma})\|^2_{[H^t(T)]^2} + \|\boldsymbol{u}\|^2_{[H^{t+1}(T)]^2} \Big\} . \end{aligned} \tag{2.8}$$

Proof. In [10] we apply the classical Lax–Milgram's Lemma to prove that (2.5) is well posed and to deduce the corresponding rate of convergence. Then, it is enough to show that A^{stab}_{DG} is elliptic on $\boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ (see Lemma 3.13, Theorems 3.5 and 3.6 in [10]). In our case, some slight modifications of the related proofs in [10] are needed for (2.5) since the norm $\|(\cdot,\cdot)\|_{DG}$, now includes the term $\|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|^2_{[L^2(\mathcal{E}_I)]^2}$, which has not been considered in [10]. □

2.3 Oswald interpolation operator

In order to obtain an *a posteriori* error estimate, we will require a suitable continuous function that lives in $\boldsymbol{V}_h$, and approximates the velocity $\boldsymbol{u}_h$. To this end we consider the Oswald interpolation operator, described in [25], among other papers. In our case, we consider its vector-wise version, i.e. the operator $\boldsymbol{I}_{\mathtt{Os}} : \boldsymbol{V}_h \to \boldsymbol{V}_h \cap [H^1(\Omega)]^2$ such that, given a function $\boldsymbol{v} \in \boldsymbol{V}_h$, $\boldsymbol{I}_{\mathtt{Os}}(\boldsymbol{v})$ is the element of $\boldsymbol{V}_h$ that on each Lagrange node $\boldsymbol{x}$ of $\mathcal{T}_h \cap \Omega$, it takes the average of the values of $\boldsymbol{v}$ at this node. This means

$$\boldsymbol{I}_{\mathtt{Os}}(\boldsymbol{v})(\boldsymbol{x}) := \frac{1}{\mathrm{card}(\omega_{\boldsymbol{x}})} \sum_{T\in\omega_{\boldsymbol{x}}} \boldsymbol{v}|_T(\boldsymbol{x}) ,$$

where $\omega_{\boldsymbol{x}} := \{T \in \mathcal{T}_h : \boldsymbol{x} \in T\}$. At boundary (Lagrange) nodes $\boldsymbol{x} \in \mathcal{T}_h \cap \Gamma$, $\boldsymbol{I}_{\mathtt{Os}}(\boldsymbol{v})(\boldsymbol{x}) := \boldsymbol{g}(\boldsymbol{x})$. One important feature of this operator is that the approximation error is bounded by $[\![\boldsymbol{v}]\!]$ on the skeleton of $\mathcal{T}_h$. Then, proceeding component-wise as in Theorem 2.2 in [25], it can be proved that: If $\boldsymbol{g}$ is the trace on Γ of a continuous function $\boldsymbol{v}$ in $\boldsymbol{V}_h$, then

$\boldsymbol{I}_{0s}(\boldsymbol{v}) \in \boldsymbol{V}_h \cap [H^1(\Omega)]^2$, such that $\boldsymbol{I}_{0s}(\boldsymbol{v}) = \boldsymbol{g}$ on Γ. In addition, there exists $C > 0$, independent of the mesh size, such that

$$||\boldsymbol{v} - \boldsymbol{I}_{0s}(\boldsymbol{v})||^2_{0,\mathcal{T}_h} \leq C \left(\sum_{e\in\mathcal{E}_I} h_e \, ||\underline{[\![\boldsymbol{v}]\!]}||^2_{0,e} + \sum_{e\in\mathcal{E}_\Gamma} h_e \, ||\boldsymbol{g} - \boldsymbol{v}||^2_{0,e} \right) , \tag{2.9}$$

$$|\boldsymbol{v} - \boldsymbol{I}_{0s}(\boldsymbol{v})|^2_{1,\mathcal{T}_h} \leq C \left(\sum_{e\in\mathcal{E}_I} h_e^{-1} \, ||\underline{[\![\boldsymbol{v}]\!]}||^2_{0,e} + \sum_{e\in\mathcal{E}_\Gamma} h_e^{-1} \, ||\boldsymbol{g} - \boldsymbol{v}||^2_{0,e} \right) . \tag{2.10}$$

Now, for simplicity we assume that the Dirichlet datum $\boldsymbol{g}$ is the trace of a continuous function in $\boldsymbol{V}_h$. Otherwise, it will appear a high order oscillation term related to $\boldsymbol{g}$. In addition, from here on we set $\widetilde{\boldsymbol{u}}_h := \boldsymbol{I}_{0s}(\boldsymbol{u}_h)$.

3 An a posteriori error analysis

In this section, in order to deduce an *a posteriori* error estimator, we follow the ideas developed in [8], with the help of the Oswald interpolant of $\boldsymbol{u}_h$ in the process. Then, at first, we consider an auxiliar problem to obtain an estimator for the trace of the pseudo stress error, measure in L^2-norm. After that, we apply the well known Helmholtz decomposition, to derive a reliable and efficient *a posteriori* error estimate for (2.5). Hereafter, we denote $\mathbf{curl}(v) := (-\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x})^{\mathtt{t}}$ for any $v \in H^1(\Omega)$, and $\underline{\mathbf{curl}}(\boldsymbol{v}) := \begin{pmatrix} \mathbf{curl}(v_1)^{\mathtt{t}} \\ \mathbf{curl}(v_2)^{\mathtt{t}} \end{pmatrix}$ for all $\boldsymbol{v} := (v_1, v_2)^{\mathtt{t}} \in [H^1(\Omega)]^2$. In what follows, we assume that the solution of the Stokes problem (1.1) has the regularity $(\boldsymbol{u}, p) \in [H^{1+\epsilon}(\Omega)]^2 \times (H^\epsilon(\Omega) \cap L^2_0(\Omega))$, with $\epsilon > 1/2$ (see [18], [19] and [21]). This allows us to ensure that $\boldsymbol{\sigma} \in H_0 \cap [H^\epsilon(\Omega)]^{2\times 2}$, and thus all the integral on the boundary including normal components of $\boldsymbol{\sigma}$ are well defined.

The main result of the present work is summarized in the following theorem.

Theorem 3.1 *Let $(\boldsymbol{\sigma}, \boldsymbol{u})$ be the exact solution of (2.1) and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{V}_h$ the unique solution of* (2.5)*. Then there exists $C_{\mathtt{rel}} > 0$, independent of mesh size, such that*

$$||(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)||_{DG} \leq C_{\mathtt{rel}} \, \eta \, , \tag{3.1}$$

where $\eta^2 := \sum_{T\in\mathcal{T}_h} \eta_T^2$ with η_T^2 given, for each $T \in \mathcal{T}_h$, by

$$\eta_T^2 := \|\boldsymbol{\sigma}_h^{\mathtt{d}} - \nu\nabla\boldsymbol{u}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} + ||\alpha^{1/2}\underline{[\![\boldsymbol{u}_h]\!]}||^2_{[L^2(\partial T\cap\mathcal{E}_I)]^{2\times 2}}$$
$$+ \|\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]||^2_{[L^2(\partial T\cap\mathcal{E}_I)]^2} + ||\alpha^{1/2}(\boldsymbol{g} - \boldsymbol{u}_h) \otimes \boldsymbol{\nu}||^2_{[L^2(\partial T\cap\mathcal{E}_\Gamma)]^{2\times 2}} \, . \tag{3.2}$$

In addition, there exists $C_{\mathtt{eff}} > 0$, independent of mesh size, such that

$$C_{\mathtt{eff}}\eta_T \leq ||(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)||_{DG,T} \, , \tag{3.3}$$

where

$$||(\boldsymbol{\tau}, \boldsymbol{v})||^2_{DG,T} := \|\boldsymbol{\tau}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} + \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]||^2_{[L^2(\partial T\cap\mathcal{E}_I)]^2}$$
$$+ ||\alpha^{1/2}\underline{[\![\boldsymbol{v}]\!]}||^2_{[L^2(\partial T)]^{2\times 2}} + ||\nabla_h \boldsymbol{v}||^2_{[L^2(T)]^{2\times 2}} \, .$$

In what follows, we concentrate in the proof of Theorem 3.1, which is divided in the next subsections. The next subsection will be of great utility in order to prove the reliability of the estimator. In particular, it will be used to deduce an *a posteriori* error estimator for the trace of the pseudo stress error, measured in L^2-norm.

3.1 An auxiliary problem

We consider the following auxiliary problem with null Dirichlet boundary condition: Given the data $\tilde{f}$, we find the vector function $\boldsymbol{v}$ and scalar one q, such that, in the distributional sense holds

$$\begin{aligned} -\nu\Delta\boldsymbol{v} + \nabla q &= \mathbf{0} \quad \text{in } \Omega\,, \\ \operatorname{div}(\boldsymbol{v}) = \tilde{f} \quad \text{in } \Omega, &\quad \boldsymbol{v} = \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

If $\tilde{f} \in L_0^2(\Omega)$ it is very well known that there exists an unique solution $(\boldsymbol{v}, q) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ such that $\|\boldsymbol{v}\|_{1,\Omega} + \|q\|_{0,\Omega} \leq c\|\tilde{f}\|_{0,\Omega}$, with $c > 0$ a constant independent of mesh size (see [19]). In addition, proceeding as in Section 2 in [10], that is, introducing the unknown $\boldsymbol{\tau} = -\nu\nabla\boldsymbol{v} + q\mathbf{I} \in H(\mathbf{div}, \Omega)$ and eliminating the variable $q = \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau}) + \frac{\nu}{2}\tilde{f}$, we obtain that (3.4) can be rewritten as the next first order system

$$\begin{aligned} \frac{1}{\nu}\boldsymbol{\tau}^{\mathrm{d}} + \nabla\boldsymbol{v} = \frac{1}{2}\tilde{f}\mathbf{I} \quad \text{in } \Omega, \\ \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \quad \text{in } \Omega, \quad \boldsymbol{v} = \mathbf{0} \quad \text{on } \partial\Omega\,. \end{aligned} \tag{3.5}$$

On the other hand, setting $\tilde{\boldsymbol{\Sigma}} := H(\mathbf{div}; \mathcal{T}_h)$ and $\tilde{\boldsymbol{\Sigma}}_0 := \{\boldsymbol{\zeta} \in \tilde{\boldsymbol{\Sigma}} : \int_\Omega \operatorname{tr}(\boldsymbol{\zeta}) = 0\}$, the variational formulation associated to (3.5) reads as follow: find $(\boldsymbol{\tau}, \boldsymbol{v}) \in \tilde{\boldsymbol{\Sigma}}_0 \times [L^2(\Omega)]^2$ such that

$$\begin{aligned} \frac{1}{\nu}\int_\Omega \boldsymbol{\tau}^d : \boldsymbol{\zeta} - \int_\Omega \boldsymbol{v}\cdot\mathbf{div}_h(\boldsymbol{\zeta}) = \frac{1}{2}\int_\Omega \tilde{f}\ \operatorname{tr}(\boldsymbol{\zeta}) \quad \forall\,\boldsymbol{\zeta} \in \tilde{\boldsymbol{\Sigma}}_0\,, \\ \int_\Omega \boldsymbol{w}\cdot\mathbf{div}_h(\boldsymbol{\tau}) = 0 \quad \forall\,\boldsymbol{w} \in [L^2(\Omega)]^2\,, \end{aligned} \tag{3.6}$$

Now, in order to circumvent the inf-sup condition we add the stabilization term given by

$$\int_{\mathcal{E}} \alpha\underline{[\![\boldsymbol{v}]\!]} : \underline{[\![\boldsymbol{w}]\!]} = 0\,, \quad \forall\,\boldsymbol{w} \in [H^1(\mathcal{T}_h)]^2\,, \tag{3.7}$$

and the least squares type terms

$$-\tilde{\delta}_1\int_\Omega (\boldsymbol{\tau}^d + \nu\nabla_h\boldsymbol{v}) : (\boldsymbol{\zeta}^d - \nu\nabla_h\boldsymbol{w}) = \frac{\nu\tilde{\delta}_1}{2}\int_\Omega \tilde{f}\ \operatorname{div}_h(\boldsymbol{w}), \quad \forall\,(\boldsymbol{\zeta}, \boldsymbol{w}) \in \boldsymbol{\Sigma}\times[H^1(\mathcal{T}_h)]^2\,, \tag{3.8}$$

and

$$\int_\Omega \mathbf{div}_h(\boldsymbol{\tau})\cdot\mathbf{div}_h(\boldsymbol{\zeta}) = 0 \quad \forall\,\boldsymbol{\zeta} \in H(\mathbf{div}; \mathcal{T}_h)\,. \tag{3.9}$$

where $\tilde{\delta}_1$ is a real parameter at our disposal. Hence, adding (3.6), (3.7), (3.8), (3.9) and penalizing with the term $\int_{\mathcal{E}_I} \gamma[\![\boldsymbol{\tau}]\!]\cdot[\![\boldsymbol{\zeta}]\!]$, we obtain the following stabilized non-comforming dual mixed variational formulation: *Find* $(\boldsymbol{\tau}, \boldsymbol{v}) \in \tilde{\boldsymbol{\Sigma}}_0 \times [H^1(\mathcal{T}_h)]^2$ *such that*

$$B((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\zeta}, \boldsymbol{w})) = G(\boldsymbol{\zeta}, \boldsymbol{w}) \quad \forall\,(\boldsymbol{\zeta}, \boldsymbol{w}) \in \tilde{\boldsymbol{\Sigma}}_0 \times [H^1(\mathcal{T}_h)]^2\,, \tag{3.10}$$

where the bilinear form $B : (\tilde{\boldsymbol{\Sigma}} \times [H^1(\mathcal{T}_h)]^2) \times (\tilde{\boldsymbol{\Sigma}} \times [H^1(\mathcal{T}_h)]^2) \to \mathbb{R}$ and the linear functional $G : (\tilde{\boldsymbol{\Sigma}} \times [H^1(\mathcal{T}_h)]^2) \to \mathbb{R}$ are defined by

$$B((\boldsymbol{\rho}, \boldsymbol{z}), (\boldsymbol{\zeta}, \boldsymbol{w})) := \frac{1}{\nu} \int_\Omega \boldsymbol{\rho}^d : \boldsymbol{\zeta}^d - \int_\Omega \boldsymbol{z} \cdot \mathbf{div}_h(\boldsymbol{\zeta}) + \int_\Omega \boldsymbol{w} \cdot \mathbf{div}_h(\boldsymbol{\rho})$$

$$-\tilde{\delta}_1 \int_\Omega (\boldsymbol{\zeta}^d - \nu \nabla_h \boldsymbol{w}) : (\nu \nabla_h \boldsymbol{z} + \boldsymbol{\rho}^d) + \int_\Omega \mathbf{div}_h(\boldsymbol{\zeta}) \cdot \mathbf{div}_h(\boldsymbol{\rho}) \tag{3.11}$$

$$+ \int_{\mathcal{E}} \alpha \underline{[\![\boldsymbol{z}]\!]} : \underline{[\![\boldsymbol{w}]\!]} + \int_{\mathcal{E}_I} \gamma [\![\boldsymbol{\zeta}]\!] \cdot [\![\boldsymbol{\rho}]\!] ,$$

and

$$G(\boldsymbol{\zeta}, \boldsymbol{w}) := \frac{1}{2} \int_\Omega \tilde{f} \ \mathrm{tr}(\boldsymbol{\zeta}) + \frac{\nu \tilde{\delta}_1}{2} \int_\Omega \tilde{f} \ \mathrm{div}_h(\boldsymbol{w})$$

for all $(\boldsymbol{\rho}, \boldsymbol{z}), (\boldsymbol{\zeta}, \boldsymbol{w}) \in \tilde{\boldsymbol{\Sigma}} \times [H^1(\mathcal{T}_h)]^2$.

Since the norm $\|(\cdot,\cdot)\|_{DG}$ now includes the term $\|\gamma^{1/2} [\![\boldsymbol{\tau}]\!]\|^2_{[L^2(\mathcal{E}_I)]^2}$, a slight modification of the Lemma 3.13 in [10] allows us to establish the coercivity of the bilinear form B, which is included in the next lemma.

Lemma 3.1 *Let $\tilde{\delta}_1 \in \mathbb{R}$ such that $0 < \tilde{\delta}_1 < \frac{1}{\nu}$. Then, there exists a constant $C > 0$, independent of the meshsize, such that*

$$B((\boldsymbol{\zeta}, \boldsymbol{w}), (\boldsymbol{\zeta}, \boldsymbol{w})) \ \geq \ C \, \|(\boldsymbol{\zeta}, \boldsymbol{w})\|^2_{DG} \quad \forall \, (\boldsymbol{\zeta}, \boldsymbol{w}) \in \tilde{\boldsymbol{\Sigma}}_0 \times [H^1(\mathcal{T}_h)]^2 \, . \tag{3.12}$$

Furthermore, the continuity of B relies on a straightforwardly application of Cauchy-Schwarz inequality. Then, existence and uniqueness of solution for the problem (3.10) is guaranteed thanks to Lax-Milgram's Lemma. In addition, there exists $C > 0$, independent of the mesh size, such that

$$\|(\boldsymbol{\tau}, \boldsymbol{v})\|_{DG} \leq C \|\tilde{f}\|_{L^2(\Omega)} \, . \tag{3.13}$$

3.2 Reliability of the estimator

Here we prove the reliability of the estimator η (upper bound in (3.1)). We begin introducing the notation, $e_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ in Ω and $e_{\boldsymbol{u}} := \boldsymbol{u} - \boldsymbol{u}_h$ in Ω. Our first aim is to estimate $\| \mathrm{tr}(e_{\boldsymbol{\sigma}}) \|_{L^2(\Omega)}$.

An *a posteriori* error estimator for $\| \mathrm{tr}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \|_{L^2(\Omega)}$ is given in the following theorem.

Theorem 3.2 *There exists $\hat{C} > 0$, independent of the meshsize, such that*

$$\| \mathrm{tr}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \|^2_{L^2(\Omega)} \leq \hat{C}^2 \, \hat{\eta}^2 := \hat{C}^2 \left\{ \sum_{T \in \mathcal{T}_h} \hat{\eta}_T^2 \right\} ,$$

where, for each $T \in \mathcal{T}_h$, we define

$$\hat{\eta}_T^2 := ||\boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h)||^2_{[L^2(T)]^2} + \left\| \frac{1}{\nu} \boldsymbol{\sigma}_h^{\mathrm{d}} - \nabla \boldsymbol{u}_h \right\|^2_{[L^2(T)]^{2\times 2}}$$

$$||\alpha^{1/2} \underline{[\![\boldsymbol{u}_h]\!]}||^2_{[L^2(\partial T \cap \mathcal{E}_I)]^{2\times 2}} + ||\alpha^{1/2} (\boldsymbol{g} - \boldsymbol{u}_h) \otimes \boldsymbol{\nu}||^2_{[L^2(\partial T \cap \mathcal{E}_\Gamma)]^{2\times 2}} \, . \tag{3.14}$$

Proof. First, since $\mathrm{tr}(e_{\boldsymbol{\sigma}}) \in \boldsymbol{\Sigma}_0$, we set $\tilde{f} := \mathrm{tr}(e_{\boldsymbol{\sigma}})$ in the auxiliary problem (3.10). We note that

$$
\begin{aligned}
\frac{1}{2}\|\,\mathrm{tr}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2_{L^2(\Omega)} = \frac{1}{2}\int_\Omega \mathrm{tr}(e_{\boldsymbol{\sigma}})\,\mathrm{tr}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) &= B((\boldsymbol{\tau},\boldsymbol{v}),(e_{\boldsymbol{\sigma}},e_{\boldsymbol{u}})) \\
&\quad - \frac{\nu\tilde{\delta}_1}{2}\int_\Omega \mathrm{tr}(e_{\boldsymbol{\sigma}})\,\mathrm{div}_h(e_{\boldsymbol{u}}) \\
&= B((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\sigma},\boldsymbol{u})) - B((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)) \\
&\quad - \frac{\nu\tilde{\delta}_1}{2}\int_\Omega \tilde{f}\,\mathrm{div}_h(\boldsymbol{u}_h)\,.
\end{aligned}
\tag{3.15}
$$

Using the definition of bilinear form $B(\cdot,\cdot)$ (cf. (3.11)), and the facts that $\boldsymbol{\sigma}^{\mathtt{d}} = \nu\nabla\boldsymbol{u}$ in Ω, $\mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}$ in Ω, $\mathrm{div}(\boldsymbol{u}) = 0$ in Ω, $[\![\boldsymbol{\sigma}]\!] = 0$ on $\mathcal{E}_I$ and $[\![\boldsymbol{v}]\!] = 0$ on $\mathcal{E}$, we deduce, after integration by parts

$$
\begin{aligned}
B((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\sigma},\boldsymbol{u})) &= \frac{1}{\nu}\int_\Omega \boldsymbol{\tau}^{\mathtt{d}} : \boldsymbol{\sigma}^{\mathtt{d}} - \int_\Omega \boldsymbol{v}\cdot\mathbf{div}_h\boldsymbol{\sigma} = \int_\Omega \boldsymbol{\tau}^{\mathtt{d}} : \nabla\boldsymbol{u} + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \\
&= \int_\Omega \boldsymbol{\tau} : (\nabla\boldsymbol{u})^{\mathtt{d}} + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} = \int_\Omega \boldsymbol{\tau} : \nabla\boldsymbol{u} + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \\
&= \int_{\mathcal{E}_\Gamma} \boldsymbol{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu} + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,.
\end{aligned}
\tag{3.16}
$$

Analogously, using (3.11) and the facts that $\mathbf{div}(\boldsymbol{\tau}) = 0$ in Ω, $[\![\boldsymbol{\tau}]\!] = 0$ on $\mathcal{E}_I$ and $[\![\boldsymbol{v}]\!] = 0$ on $\mathcal{E}$, we obtain

$$
\begin{aligned}
B((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)) &= \frac{1}{\nu}\int_\Omega \boldsymbol{\tau}^{\mathtt{d}} : \boldsymbol{\sigma}_h^{\mathtt{d}} - \int_\Omega \boldsymbol{v}\cdot\mathbf{div}_h(\boldsymbol{\sigma}_h) \\
&\quad - \tilde{\delta}_1\int_\Omega (\boldsymbol{\sigma}_h^{\mathtt{d}} - \nu\nabla_h\boldsymbol{u}_h) : (\nu\nabla_h\boldsymbol{v} + \boldsymbol{\tau}^d) \\
&= \frac{1}{\nu}\int_\Omega \boldsymbol{\tau} : \boldsymbol{\sigma}_h^{\mathtt{d}} - \int_\Omega \boldsymbol{v}\cdot\mathbf{div}_h(\boldsymbol{\sigma}_h) + \int_\Omega \widetilde{\boldsymbol{u}}_h\cdot\mathbf{div}(\boldsymbol{\tau}) \\
&\quad - \tilde{\delta}_1\int_\Omega (\boldsymbol{\sigma}_h^{\mathtt{d}} - \nu\nabla_h\boldsymbol{u}_h) : (\nu\nabla_h\boldsymbol{v} + \boldsymbol{\tau}^d)\,.
\end{aligned}
$$

Here, we recall that $\widetilde{\boldsymbol{u}}_h := \boldsymbol{I}_{0\mathtt{s}}(\boldsymbol{u}_h)$. Now, after integrating by parts the term $\int_\Omega \widetilde{\boldsymbol{u}}_h\cdot\mathbf{div}(\boldsymbol{\tau})$, and recalling that $\widetilde{\boldsymbol{u}}_h = \boldsymbol{g}$ on Γ, results

$$
\begin{aligned}
B((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)) &= \int_\Omega \left(\frac{1}{\nu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla_h\boldsymbol{u}_h\right) : \boldsymbol{\tau} + \int_\Omega (\nabla_h\boldsymbol{u}_h - \nabla\widetilde{\boldsymbol{u}}_h) : \boldsymbol{\tau} \\
&\quad + \int_{\mathcal{E}_\Gamma} \boldsymbol{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu} - \int_\Omega \boldsymbol{v}\cdot\mathbf{div}_h\boldsymbol{\sigma}_h - \tilde{\delta}_1\int_\Omega (\boldsymbol{\sigma}_h^{\mathtt{d}} - \nu\nabla_h\boldsymbol{u}_h) : (\nu\nabla_h\boldsymbol{v} + \boldsymbol{\tau}^d)\,.
\end{aligned}
\tag{3.17}
$$

Then, replacing (3.16) and (3.17) into (3.15) and using the fact that $\frac{1}{\nu}\boldsymbol{\tau}^{\mathtt{d}} = -\nabla\boldsymbol{v} + \frac{1}{2}\tilde{f}\mathbf{I}$ in Ω, we arrive to

$$\begin{aligned}\|\operatorname{tr}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2_{L^2(\Omega)} &= \int_\Omega (\boldsymbol{f} + \mathbf{div}_h\boldsymbol{\sigma}_h)\cdot\boldsymbol{v} \\ &- \int_\Omega \left(\frac{1}{\nu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla_h\boldsymbol{u}_h\right) : \boldsymbol{\tau} - \int_\Omega (\nabla_h\boldsymbol{u}_h - \nabla\widetilde{\boldsymbol{u}}_h) : \boldsymbol{\tau}\,.\end{aligned} \tag{3.18}$$

Next, straightforward applications of the Cauchy-Schwarz inequality imply that

$$\begin{aligned}\left|\int_\Omega (\boldsymbol{f} + \mathbf{div}_h\boldsymbol{\sigma}_h)\cdot\boldsymbol{v}\right| &\le \sum_{T\in\mathcal{T}_h} \|\boldsymbol{f} + \mathbf{div}_h\boldsymbol{\sigma}_h\|_{[L^2(T)]^2}\,\|\boldsymbol{v}\|_{[L^2(T)]^2} \\ &\le \left\{\sum_{T\in\mathcal{T}_h} \|\boldsymbol{f} + \mathbf{div}_h\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2}\right\}^{1/2} \|\boldsymbol{v}\|_{[L^2(\Omega)]^2}\,,\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}&\left|\int_\Omega \left(\frac{1}{\nu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla_h\boldsymbol{u}_h\right) : \boldsymbol{\tau}\right| \\ &\le \sum_{T\in\mathcal{T}_h} \left\|\frac{1}{\nu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla\boldsymbol{u}_h\right\|_{[L^2(T)]^{2\times 2}} \|\boldsymbol{\tau}\|_{[L^2(T)]^{2\times 2}} \\ &\le c\left\{\sum_{T\in\mathcal{T}_h} \left\|\frac{1}{\nu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla\boldsymbol{u}_h\right\|^2_{[L^2(T)]^{2\times 2}}\right\}^{1/2} \|\boldsymbol{\tau}\|_{[L^2(\Omega)]^{2\times 2}}\,.\end{aligned} \tag{3.20}$$

Thanks to the approximation property (2.10), we have

$$\begin{aligned}&\left|\int_\Omega (\nabla\widetilde{\boldsymbol{u}}_h - \nabla_h\boldsymbol{u}_h) : \boldsymbol{\tau}\right| \\ &\le \sum_{T\in\mathcal{T}_h} \|\nabla\widetilde{\boldsymbol{u}}_h - \nabla\boldsymbol{u}_h\|_{[L^2(T)]^{2\times 2}}\|\boldsymbol{\tau}\|_{[L^2(T)]^{2\times 2}} \\ &\le C\left\{\sum_{T\in\mathcal{T}_h} \|\alpha^{1/2}\underline{[\![\widetilde{\boldsymbol{u}}_h - \boldsymbol{u}_h]\!]}\|^2_{[L^2(\partial T)]^{2\times 2}}\right\}^{1/2} \|\boldsymbol{\tau}\|_{[L^2(\Omega)]^{2\times 2}}\,.\end{aligned} \tag{3.21}$$

Then, after replacing the inequalities (3.19)–(3.21) into (3.18), the proof follows from (3.13). □

Now, we are in position to establish an estimator for the velocity. To this aim, we begin with the following Helmholtz decomposition result.

Lemma 3.2 *There exist $\boldsymbol{\psi} \in [H_0^1(\Omega)]^2$ and $\boldsymbol{\chi} \in [H^1(\Omega)]^2$ with $\underline{\mathbf{curl}}(\boldsymbol{\chi})\boldsymbol{\nu} = \mathbf{0}$ on Γ, such that*

$$\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h) = \nabla\boldsymbol{\psi} + \underline{\mathbf{curl}}(\boldsymbol{\chi}) .$$

Furthermore

$$\|\nabla\boldsymbol{\psi}\|_{[L^2(\Omega)]^{2\times 2}} + \|\underline{\mathbf{curl}}(\boldsymbol{\chi})\|_{[L^2(\Omega)]^{2\times 2}} \le \|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{[L^2(\Omega)]^{2\times 2}} .$$

Proof. Let $\boldsymbol{\psi} \in [H^1(\Omega)]^2$ be the unique weak solution of the boundary value problem

$$-\mathbf{div}(\nabla\boldsymbol{\psi}) = -\mathbf{div}(\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)) \quad \text{in} \quad \Omega , \quad \boldsymbol{\psi} = \mathbf{0} \quad \text{on} \quad \Gamma .$$

Since $\mathbf{div}(\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h) - \nabla\boldsymbol{\psi}) = \mathbf{0}$ in Ω in the sense of distributions, and Ω is simply connected, the rest of the proof is consequence of Theorem I.3.1 in [20]. We omit further details. $\square$

Now, for each $\boldsymbol{v} := (v_1, v_2)^{\mathtt{t}} \in [H^1(\Omega)]^2$, let $\mathbf{\Pi}_0\boldsymbol{v} := (\Pi_0 v_1, \Pi_0 v_2)^{\mathtt{t}} \in [L^2(\Omega)]^2$, where Π_0 be a piecewise constant projection from $H^1(\Omega)$ onto $L^2(\Omega)$ such that for all $z \in H^1(\Omega)$, $(\Pi_0 z)|_T := \frac{1}{|T|}\int_T z$ for each $T \in \mathcal{T}_h$ with $\partial T \cap \Gamma = \emptyset$ and $(\Pi_0 z)|_T := 0$ on each $T \in \mathcal{T}_h$ with an edge on Γ.

The following lemmas will be applied in the estimates of $|||\boldsymbol{u} - \boldsymbol{u}_h|||_h$ and $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{\Sigma}}$.

Lemma 3.3 *Let $\boldsymbol{\psi} \in [H_0^1(\Omega)]^2$ be the function from* Lemma 3.2*, and $\mathbf{\Pi}_0$ the piecewise constant projection defined above. Then, there holds*

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\int_T \nu\nabla(\boldsymbol{u} - \boldsymbol{u}_h) : \nabla\boldsymbol{\psi} &= \sum_{T\in\mathcal{T}_h}\int_T \left(\boldsymbol{\sigma}_h^{\mathtt{d}} - \nu\nabla\boldsymbol{u}_h - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\mathbf{I}\right) : \nabla\boldsymbol{\psi} \\ &+ \sum_{T\in\mathcal{T}_h}\int_T (\boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h))\cdot(\boldsymbol{\psi} - \mathbf{\Pi}_0\boldsymbol{\psi}) + \sum_{T\in\mathcal{T}_h}\int_{\partial T\setminus\Gamma}(\boldsymbol{\sigma}_h\boldsymbol{\nu}_T - \widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T)\cdot(\boldsymbol{\psi} - \mathbf{\Pi}_0\boldsymbol{\psi}) .\end{aligned} \tag{3.22}$$

Proof. First, since $\nu\nabla\boldsymbol{u} = \boldsymbol{\sigma}^{\mathtt{d}}$ in Ω, we have on each $T \in \mathcal{T}_h$

$$\int_T \nu\nabla(\boldsymbol{u} - \boldsymbol{u}_h) : \nabla\boldsymbol{\psi} = \int_T \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{d}} - \nu\nabla\boldsymbol{u}_h - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\mathbf{I}\right) : \nabla\boldsymbol{\psi} .$$

We remark that $\boldsymbol{\psi} \in [H_0^1(\Omega)]^2$, and $(\mathbf{\Pi}_0\boldsymbol{\psi})|_T \in [\mathbf{P}_0(T)]^2$. Then, after integration by parts, we obtain that

$$\sum_{T\in\mathcal{T}_h}\int_T (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : \nabla\boldsymbol{\psi} = \sum_{T\in\mathcal{T}_h}\int_T (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : \nabla(\boldsymbol{\psi} - \mathbf{\Pi}_0\boldsymbol{\psi})$$

$$= \sum_{T\in\mathcal{T}_h}\left\{\int_T (\boldsymbol{f} - \mathbf{div}(\boldsymbol{\sigma}_h))\cdot(\boldsymbol{\psi} - \mathbf{\Pi}_0\boldsymbol{\psi}) + \int_{\partial T}(\boldsymbol{\psi} - \mathbf{\Pi}_0\boldsymbol{\psi})\cdot(\boldsymbol{\sigma}\boldsymbol{\nu}_T - \boldsymbol{\sigma}_h\boldsymbol{\nu}_T)\right\} .$$

On the other hand, since $[\![\boldsymbol{\sigma}]\!] = \mathbf{0}$ on $\mathcal{E}_I$, $\boldsymbol{\psi} \in [H_0^1(\Omega)]^2$, and $[\![\widehat{\boldsymbol{\sigma}}]\!] = 0$ on $\mathcal{E}_I$, we find that

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\boldsymbol{\sigma}\boldsymbol{\nu}_T\cdot\boldsymbol{\psi} = \int_{\mathcal{E}_I}\underline{[\![\boldsymbol{\psi}]\!]} : \{\boldsymbol{\sigma}\} + \int_{\mathcal{E}_I}\{\boldsymbol{\psi}\}\cdot[\![\boldsymbol{\sigma}]\!]$$

$$= \int_{\mathcal{E}_I}\underline{[\![\boldsymbol{\psi}]\!]} : \{\widehat{\boldsymbol{\sigma}}\} + \int_{\mathcal{E}_I}\{\boldsymbol{\psi}\}\cdot[\![\widehat{\boldsymbol{\sigma}}]\!] = \sum_{T\in\mathcal{T}_h}\int_{\partial T}\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T\cdot\boldsymbol{\psi} .$$

Next, taking into account that $-\mathbf{div}(\boldsymbol{\sigma}) = \boldsymbol{f}$ in T and the second equation in (2.2), we find that

$$\int_{\partial T} \boldsymbol{\sigma}\boldsymbol{\nu}_T \cdot \boldsymbol{\Pi}_0\boldsymbol{\psi} \;=\; \int_T \mathbf{div}(\boldsymbol{\sigma}) \cdot \boldsymbol{\Pi}_0\boldsymbol{\psi} \;=\; -\int_T \boldsymbol{f} \cdot \boldsymbol{\Pi}_0\boldsymbol{\psi} \;=\; \int_{\partial T} \widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T \cdot \boldsymbol{\Pi}_0\boldsymbol{\psi}\,,$$

which completes the proof. □

Lemma 3.4 *Let* $\boldsymbol{\chi} \in [H_0^1(\Omega)]^2$ *be the function from* Lemma 3.2. *Then, there exists* $c > 0$, *independent of the mesh size, such that*

$$\sum_{T\in\mathcal{T}_h} \int_T \nabla(\boldsymbol{u} - \boldsymbol{u}_h) : \underline{\mathbf{curl}}(\boldsymbol{\chi}) \;\le\; c\,\|\underline{\mathbf{curl}}(\boldsymbol{\chi})\|_{[L^2(\Omega)]^{2\times 2}} ||\alpha^{1/2}\underline{[\![\boldsymbol{u} - \boldsymbol{u}_h]\!]}||_{[L^2(\mathcal{E}_I)]^{2\times 2}}.$$

Proof. First, we define the space $\boldsymbol{\mathcal{W}} := \{\boldsymbol{w} \in [H^1(\Omega)]^2 : \quad \boldsymbol{w} = \boldsymbol{g} \quad \text{on} \quad \Gamma\}$. Then adding and subtracting $\boldsymbol{w} \in \boldsymbol{\mathcal{W}}$, integrating by parts, using that $\underline{\mathbf{curl}}(\boldsymbol{\chi})\boldsymbol{\nu} = \mathbf{0}$ on Γ, we deduce that

$$\sum_{T\in\mathcal{T}_h} \int_T \nabla(\boldsymbol{u}-\boldsymbol{u}_h) : \underline{\mathbf{curl}}(\boldsymbol{\chi}) \;=\; \sum_{T\in\mathcal{T}_h} \left\{ \int_T \nabla(\boldsymbol{u} - \boldsymbol{w}) : \underline{\mathbf{curl}}(\boldsymbol{\chi}) \;+\; \int_T \nabla(\boldsymbol{w} - \boldsymbol{u}_h) : \underline{\mathbf{curl}}(\boldsymbol{\chi}) \right\}$$

$$= \sum_{T\in\mathcal{T}_h} \int_{\partial T\setminus\Gamma} (\boldsymbol{w} - \boldsymbol{u}_h) \cdot \underline{\mathbf{curl}}(\boldsymbol{\chi})\boldsymbol{\nu}_T \;\le\; \hat{c} \sum_{T\in\mathcal{T}_h} \|\boldsymbol{w} - \boldsymbol{u}_h\|_{[H^{1/2}(\partial T\setminus\Gamma)]^2} \|\underline{\mathbf{curl}}(\boldsymbol{\chi})\boldsymbol{\nu}_T\|_{[H^{-1/2}(\partial T)]^2}$$

$$\le \tilde{c} \sum_{T\in\mathcal{T}_h} \|\boldsymbol{w} - \boldsymbol{u}_h\|_{[H^{1/2}(\partial T\setminus\Gamma)]^2} \|\underline{\mathbf{curl}}(\boldsymbol{\chi})\|_{[L^2(T)]^{2\times 2}}$$

$$\le c\|\underline{\mathbf{curl}}(\boldsymbol{\chi})\|_{[L^2(\Omega)]^{2\times 2}} \left(\sum_{T\in\mathcal{T}_h} \|\boldsymbol{w} - \boldsymbol{u}_h\|^2_{[H^{1/2}(\partial T\setminus\Gamma)]^2} \right)^{1/2}.$$

Since $\boldsymbol{w} \in \boldsymbol{\mathcal{W}}$ is arbitrary, we obtain

$$\int_\Omega \nabla_h(\boldsymbol{u} - \boldsymbol{u}_h) : \underline{\mathbf{curl}}(\boldsymbol{\chi}) \;\le\; c\,\|\underline{\mathbf{curl}}(\boldsymbol{\chi})\|_{[L^2(\Omega)]^{2\times 2}} \inf_{\boldsymbol{w}\in\boldsymbol{\mathcal{W}}} \left(\sum_{T\in\mathcal{T}_h} \|\boldsymbol{w} - \boldsymbol{u}_h\|^2_{[H^{1/2}(\partial T\setminus\Gamma)]^2} \right)^{1/2}.$$

Then, the rest follows from the proof of the Lemma 4 in [11]. □

The next theorem establishes an estimate for $||\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)||_{[L^2(\Omega)]^{2\times 2}}$.

Theorem 3.3 *There exists* $C > 0$, *independent of the meshsize, such that*

$$\nu^2\|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_{[L^2(\Omega)]^{2\times 2}} \le C\,\bar{\eta}^2 := C \sum_{T\in\mathcal{T}_h} \bar{\eta}_T^2\,,$$

where, for each $T \in \mathcal{T}_h$, *we define*

$$\bar{\eta}_T^2 := \|\boldsymbol{\sigma}_h^{\mathtt{d}} - \nu\nabla\boldsymbol{u}_h\|^2_{[L^2(T)]^{2\times 2}} \;+\; \|\boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} \;+\; ||\alpha^{1/2}\underline{[\![\boldsymbol{u}_h]\!]}||^2_{[L^2(\partial T\cap\mathcal{E}_I)]^{2\times 2}}$$
$$+\,||\alpha^{1/2}(\boldsymbol{g} - \boldsymbol{u}_h) \otimes \boldsymbol{\nu}||^2_{[L^2(\partial T\cap\mathcal{E}_\Gamma)]^{2\times 2}} \;+\; h_T\,\|\boldsymbol{\sigma}_h\boldsymbol{\nu}_T - \widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T\|^2_{[L^2(\partial T)]^2}\,. \tag{3.23}$$

Proof. Applying Lemmas 3.2 and 3.3, we obtain

$$
\begin{aligned}
\nu\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|^2_{[L^2(\Omega)]^{2\times 2}} &= \int_\Omega \nu\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h):\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\\
&= \sum_{T\in\mathcal{T}_h}\left\{\int_T \nu\nabla(\boldsymbol{u}-\boldsymbol{u}_h):\nabla\boldsymbol{\psi} + \int_T \nu\nabla(\boldsymbol{u}-\boldsymbol{u}_h):\underline{\mathbf{curl}}(\boldsymbol{\chi})\right\}\\
&= \sum_{T\in\mathcal{T}_h}\int_T\left(\boldsymbol{\sigma}_h^{\mathtt{d}}-\nu\nabla\boldsymbol{u}_h-\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\mathbf{I}\right):\nabla\boldsymbol{\psi} + \sum_{T\in\mathcal{T}_h}\int_T(\boldsymbol{f}+\mathbf{div}(\boldsymbol{\sigma}_h))\cdot(\boldsymbol{\psi}-\boldsymbol{\Pi}_0\boldsymbol{\psi})\\
&\quad+\sum_{T\in\mathcal{T}_h}\int_{\partial T\setminus\gamma}(\boldsymbol{\sigma}_h\boldsymbol{\nu}_T-\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T)\cdot(\boldsymbol{\psi}-\boldsymbol{\Pi}_0\boldsymbol{\psi}) + \sum_{T\in\mathcal{T}_h}\int_T\nu\nabla(\boldsymbol{u}-\boldsymbol{u}_h):\underline{\mathbf{curl}}(\boldsymbol{\chi})\,. \qquad (3.24)
\end{aligned}
$$

Next, thanks to Cauchy-Schwarz inequality, we can bound all terms, but the last one, on the right hand side of (3.24).

$$
\begin{aligned}
\left|\sum_{T\in\mathcal{T}_h}\int_T(\boldsymbol{\sigma}_h^{\mathtt{d}}-\nu\nabla\boldsymbol{u}_h):\nabla\boldsymbol{\psi}\right| &\le \sum_{T\in\mathcal{T}_h}\|\boldsymbol{\sigma}_h^{\mathtt{d}}-\nu\nabla\boldsymbol{u}_h\|_{[L^2(T)]^{2\times 2}}\|\nabla\boldsymbol{\psi}\|_{[L^2(T)]^{2\times 2}}\\
&\le\left\{\sum_{T\in\mathcal{T}_h}\|\boldsymbol{\sigma}_h^{\mathtt{d}}-\nu\nabla\boldsymbol{u}_h\|^2_{[L^2(T)]^{2\times 2}}\right\}^{1/2}\|\nabla\boldsymbol{\psi}\|_{[L^2(\Omega)]^{2\times 2}}\,,
\end{aligned}\qquad (3.25)
$$

$$
\begin{aligned}
\left|\sum_{T\in\mathcal{T}_h}\int_T\left(\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\right)\cdot\operatorname{div}(\boldsymbol{\psi})\right| &\le \frac{1}{\sqrt{2}}\sum_{T\in\mathcal{T}_h}\|\operatorname{tr}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L^2(T)}\|\nabla\boldsymbol{\psi}\|_{[L^2(T)]^{2\times 2}}\\
&\le\frac{1}{\sqrt{2}}\left\{\sum_{T\in\mathcal{T}_h}\|\operatorname{tr}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2_{L^2(T)}\right\}^{1/2}\|\nabla\boldsymbol{\psi}\|_{[L^2(\Omega)]^{2\times 2}}\,,
\end{aligned}\qquad (3.26)
$$

$$
\begin{aligned}
\left|\sum_{T\in\mathcal{T}_h}\int_T(\boldsymbol{f}+\mathbf{div}(\boldsymbol{\sigma}_h))\cdot(\boldsymbol{\psi}-\boldsymbol{\Pi}_0\boldsymbol{\psi})\right| &\le \tilde{c}\sum_{T\in\mathcal{T}_h}h_T\|\boldsymbol{f}+\mathbf{div}(\boldsymbol{\sigma}_h)\|_{[L^2(T)]^2}\|\nabla\boldsymbol{\psi}\|_{[L^2(T)]^{2\times 2}}\\
&\le\tilde{c}\left\{\sum_{T\in\mathcal{T}_h}h_T^2\|\boldsymbol{f}+\mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2}\right\}^{1/2}\|\nabla\boldsymbol{\psi}\|_{[L^2(\Omega)]^{2\times 2}}\,,
\end{aligned}\qquad (3.27)
$$

and

$$
\begin{aligned}
\left|\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\boldsymbol{\psi}-\boldsymbol{\Pi}_0\boldsymbol{\psi})(\boldsymbol{\sigma}_h\boldsymbol{\nu}_T-\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T)\right| &\le \hat{c}\sum_{T\in\mathcal{T}_h}h_T^{1/2}\|\boldsymbol{\sigma}_h\boldsymbol{\nu}_T-\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T\|_{[L^2(\partial T)]^2}\|\nabla\boldsymbol{\psi}\|_{[L^2(T)]^{2\times 2}}\\
&\le\hat{c}\left\{\sum_{T\in\mathcal{T}_h}h_T\|\boldsymbol{\sigma}_h\boldsymbol{\nu}_T-\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T\|^2_{[L^2(\partial T)]^2}\right\}^{1/2}\|\nabla\boldsymbol{\psi}\|_{[L^2(\Omega)]^{2\times 2}}\,.
\end{aligned}\qquad (3.28)
$$

Then, using (3.25), (3.26), (3.27), (3.28), Lemmas 3.2, 3.4 and Theorem 3.2 in (3.24) we arrive to

$$\nu \,\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{[L^2(\Omega)]^{2\times 2}} \le \|\boldsymbol{\sigma}_h^{\mathtt{d}} - \nu\nabla_h \boldsymbol{u}_h\|_{[L^2(\Omega)]^{2\times 2}} + ||\alpha^{1/2}[\![\underline{\boldsymbol{u}_h}]\!]||_{[L^2(\mathcal{E}_I)]^{2\times 2}} + \hat{C}\hat{\eta}$$

$$+\,\tilde{c}\left\{\sum_{T\in\mathcal{T}_h} h_T^2\,\|\boldsymbol{f}+\mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2}\right\}^{1/2} + \,\hat{c}\left\{\sum_{T\in\mathcal{T}_h} h_T\,\|\boldsymbol{\sigma}_h\boldsymbol{\nu}_T - \widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T\|^2_{[L^2(\partial T)]^2}\right\}^{1/2}. \tag{3.29}$$

Recalling the definition of the estimator $\hat{\eta}$ (cf. (3.23)), and applying a discrete Cauchy-Schwarz inequality, we obtain

$$\nu\,\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{[L^2(\Omega)]^{2\times 2}} \le c\left\{\sum_{T\in\mathcal{T}_h}\Big(\|\boldsymbol{\sigma}_h^{\mathtt{d}}-\nu\nabla_h\boldsymbol{u}_h\|^2_{[L^2(T)]^{2\times 2}}\right.$$

$$+\,||\alpha^{1/2}(\boldsymbol{g}-\boldsymbol{u}_h)\otimes\boldsymbol{\nu}||^2_{[L^2(\partial T\cap\mathcal{E}_\Gamma)]^{2\times 2}} + ||\alpha^{1/2}[\![\underline{\boldsymbol{u}_h}]\!]||^2_{[L^2(\partial T\cap\mathcal{E}_I)]^{2\times 2}}$$

$$\left.+\|\boldsymbol{f}+\mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} + h_T\,\|\boldsymbol{\sigma}_h\boldsymbol{\nu}_T-\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T\|^2_{[L^2(\partial T)]^2}\Big)\right\}^{1/2},$$

which let us to conclude the proof. □

On the other hand, since $\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\in\boldsymbol{\Sigma}_0$, from Lemma 3.1 in [6] (see also Lemma 3.10 in [10]), using the relations $\boldsymbol{\sigma}^{\mathtt{d}}=\nu\nabla\boldsymbol{u}$ in Ω, $-\mathbf{div}(\boldsymbol{\sigma})=\boldsymbol{f}$ in Ω, and $[\![\boldsymbol{\sigma}]\!]=\mathbf{0}$ on $\mathcal{E}_I$, we deduce that there exists $c_1>0$, independent of the mesh size, such that

$$c_1\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^{2\times 2}} \le \|\boldsymbol{\sigma}^{\mathtt{d}}-\boldsymbol{\sigma}_h^{\mathtt{d}}\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\mathbf{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2_{[L^2(\Omega)]^2} + \|\gamma^{1/2}[\![\boldsymbol{\sigma}-\boldsymbol{\sigma}_h]\!]\|^2_{[L^2(\mathcal{E}_I)]^2}$$

$$\le \sum_{T\in\mathcal{T}_h}\left\{2\|\nu\nabla\boldsymbol{u}_h-\boldsymbol{\sigma}_h^{\mathtt{d}}\|^2_{[L^2(T)]^{2\times 2}} + 2\|\nu\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|^2_{[L^2(T)]^{2\times 2}}\right.$$

$$\left.+\,\|\boldsymbol{f}+\mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} + \|\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]\|^2_{[L^2(\partial T\cap\mathcal{E}_I)]^2}\right\}.$$

Next result helps us to control the last term in (3.23).

Lemma 3.5 *There exists $C_3>0$, independent of the mesh size, such that for any $e\in\mathcal{E}$*

$$h_T\,||\boldsymbol{\sigma}_h\boldsymbol{\nu}_T-\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T||^2_{[L^2(e)]^2} \le C_3\begin{cases} ||\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]||^2_{[L^2(e)]^2} + ||\alpha^{1/2}[\![\underline{\boldsymbol{u}_h}]\!]||^2_{[L^2(e)]^{2\times 2}} & \text{if } e\in\mathcal{E}_I\,,\\[2ex] ||\alpha^{1/2}\underline{[\![\boldsymbol{u}_h-\boldsymbol{g}]\!]}||^2_{[L^2(e)]^{2\times 2}} & \text{if } e\in\mathcal{E}_\Gamma\,.\end{cases} \tag{3.30}$$

Proof. From the definition of $\widehat{\boldsymbol{\sigma}}$ (cf. (2.4)), we deduce

$$\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T-\boldsymbol{\sigma}_h\boldsymbol{\nu}_T = \begin{cases}\left(\dfrac{1}{2}\mathbf{I}+\boldsymbol{\beta}\otimes\boldsymbol{\nu}_T\right)[\![\boldsymbol{\sigma}_h]\!]-\alpha\underline{[\![\boldsymbol{u}_h]\!]}\boldsymbol{\nu}_T & \text{on } \mathcal{E}_I\,,\\[2ex] -\alpha(\boldsymbol{u}_h-\boldsymbol{g}) & \text{on } \mathcal{E}_\Gamma\,.\end{cases}$$

Now, for $e \in \mathcal{E}_I$, we have (after applying Cauchy-Schwarz appropriately)

$$||\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T - \boldsymbol{\sigma}_h\boldsymbol{\nu}_T||^2_{[L^2(e)]^2} \leq C \left(h_e||\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]||^2_{[L^2(e)]^2} + h_e^{-1}||\alpha^{1/2}\underline{[\![\boldsymbol{u}_h]\!]}||^2_{[L^2(e)]^{2\times 2}} \right)$$

and then

$$h_T||\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T - \boldsymbol{\sigma}_h\boldsymbol{\nu}_T||^2_{[L^2(e)]^2} \leq C_1 \left(||\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]||^2_{[L^2(e)]^2} + ||\alpha^{1/2}\underline{[\![\boldsymbol{u}_h]\!]}||^2_{[L^2(e)]^{2\times 2}} \right) .$$

On the other hand, for $e \in \mathcal{E}_\Gamma$ we obtain

$$h_T||\widehat{\boldsymbol{\sigma}}\boldsymbol{\nu}_T - \boldsymbol{\sigma}_h\boldsymbol{\nu}_T||^2_{[L^2(e)]^2} \leq C_2||\alpha^{1/2}\underline{[\![\boldsymbol{u}_h - \boldsymbol{g}]\!]}||^2_{[L^2(e)]^{2\times 2}} ,$$

and we end the proof. □

Finally, we remark that the reliability of (3.1) follows from Theorem 3.3, Lemma 3.5, and the fact that the norms are defined by

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||^2_h = \|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\alpha^{1/2}\underline{[\![\boldsymbol{u} - \boldsymbol{u}_h]\!]}\|^2_{[L^2(\mathcal{E})]^{2\times 2}} .$$

and by

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{\boldsymbol{\Sigma}} = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(\Omega)]^2} + \|\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} .$$

We omit more details.

3.3 Efficiency of the estimator

In this subsection we prove the local efficiency of the estimator η (property (3.3)). We proceed now to estimate the five terms defining the error indicator η_T^2 (cf. (3.2)). Using the fact that $\boldsymbol{\sigma}^{\mathrm{d}} = \nu\nabla\boldsymbol{u}$ in Ω, $\mathbf{div}(\boldsymbol{\sigma}) = -\boldsymbol{f}$ in Ω and $\boldsymbol{u} \in [H^1(\Omega)]^2$, we first observe that

$$\begin{aligned} \|\boldsymbol{\sigma}_h^{\mathrm{d}} - \nu\nabla\boldsymbol{u}_h\|_{[L^2(T)]^{2\times 2}} &\leq \|\boldsymbol{\sigma}^{\mathrm{d}} - \boldsymbol{\sigma}_h^{\mathrm{d}}\|_{[L^2(T)]^{2\times 2}} + \nu\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{[L^2(T)]^{2\times 2}} \\ &\leq C\left\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^{2\times 2}} + \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{[L^2(T)]^{2\times 2}}\right\} , \end{aligned} \tag{3.31}$$

$$\|\boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{[L^2(T)]^2} = \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{[L^2(T)]^2} , \tag{3.32}$$

$$\|\alpha^{1/2}\underline{[\![\boldsymbol{u}_h]\!]}\|_{[L^2(e)]^{2\times 2}} = \|\alpha^{1/2}\underline{[\![\boldsymbol{u} - \boldsymbol{u}_h]\!]}\|_{[L^2(e)]^{2\times 2}} \quad \forall\, e \in \mathcal{E}_I , \tag{3.33}$$

$$\|\alpha^{1/2}(\boldsymbol{g} - \boldsymbol{u}_h) \otimes \boldsymbol{\nu}\|_{[L^2(e)]^{2\times 2}} = \|\alpha^{1/2}(\boldsymbol{u} - \boldsymbol{u}_h) \otimes \boldsymbol{\nu}\|_{[L^2(e)]^{2\times 2}} \quad \forall\, e \in \mathcal{E}_\Gamma , \tag{3.34}$$

and

$$\|\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} = \|\gamma^{1/2}[\![\boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} \quad \forall\, e \in \mathcal{E}_I . \tag{3.35}$$

Finally, we remark that the efficiency of our estimator (cf. (3.3))) follows from (3.31), (3.32), (3.33), (3.34), and (3.35).

Remark 3.1 *The local efficiency of estimator $\bar{\eta}_T$ (cf.* (3.23)*) is proved in analogous way, with the help of Lemma* 3.5.

4 Numerical examples

In this section we include numerical simulations that are in agreement with the theoretical results we have derived in the current work. Since it is not an easy task to find a basis of $\mathbf{\Sigma}_{h,0}$, we incorporate the zero mean value condition of trace of elements in $\mathbf{\Sigma}_h$ using a Lagrange multiplier. This action yields us to prove the following result.

Theorem 4.1 *Consider the problem: Find* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \Lambda) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h \times \mathbb{R}$ *such that*

$$\begin{aligned} A_{DG}^{stab}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}, \boldsymbol{v})) + \Lambda \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) &= F_{DG}((\boldsymbol{\tau}, \boldsymbol{v})) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h \\ \mu \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h) &= 0 \quad \forall\, \mu \in \mathbb{R}\,. \end{aligned} \tag{4.1}$$

Then, we have

1. *If* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \Lambda) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h \times \mathbb{R}$ *is a solution of* (4.1)*, then* $\Lambda = 0$ *and* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{\Sigma}_{h,0} \times \boldsymbol{V}_h$ *is a solution of* (2.5)*.*

2. *If* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{\Sigma}_{h,0} \times \boldsymbol{V}_h$ *is a solution of* (2.5)*, then* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, 0) \in \mathbf{\Sigma}_h \times \boldsymbol{V}_h \times \mathbb{R}$ *is a solution of* (4.1)*.*

Next, we proceed to implement (4.1) in order to obtain numerical approximations of the exact solution. First, we need to introduce some useful notations for the errors and experimental rates of convergence. For simplicity, we consider the approximation spaces of lowest order, i.e. $(\mathbf{\Sigma}_h, \boldsymbol{V}_h) := ([\mathbf{RT}_0(\mathcal{T}_h)]^{2\times 2}, [\mathbf{P}_1(\mathcal{T}_h)]^2)$, in all examples. We remark that numerical examples, with uniform refinement, exhibiting that this scheme does work with a pair of discrete spaces circumventing the mild condition, can be seen in [10].

We let N be the number of degrees of freedom, that in this case corresponds to $N = 12 \times \mathrm{card}(\mathcal{T}_h) + 1$. We also introduce

$$\boldsymbol{e}_{\mathbf{\Sigma}}(\boldsymbol{\sigma}) := \Big(||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{[L^2(\Omega)]^{2\times 2}} + ||\gamma^{1/2} [\![\boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\!]||^2_{[L^2(\mathcal{E}_I)]^2} + ||\mathbf{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||^2_{[L^2(\Omega)]^2} \Big)^{1/2} \,,$$

$\boldsymbol{e}_h(\boldsymbol{u}) := |||\boldsymbol{u} - \boldsymbol{u}_h|||_h$, and $\boldsymbol{e} := ||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)||_{DG}$. Moreover, we set $\boldsymbol{e}_0(\boldsymbol{u}) := ||\boldsymbol{u} - \boldsymbol{u}_h||_{[L^2(\Omega)]^2}$, and $\boldsymbol{e}_0(p) := ||p - p_h||_{L^2(\Omega)}$, where the approximation of pressure, p_h, is obtained by the post-process $p_h := -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)$.

Taking into account that for quasi uniform meshes in 2D, the mesh size h behaves as $\mathcal{O}(N^{-1/2})$, we set the so called experimental rate of convergence of the total error by

$$r := -2 \frac{\log(\boldsymbol{e}/\boldsymbol{e}')}{\log(N/N')}\,,$$

where $\boldsymbol{e}$ and $\boldsymbol{e}'$ denote the corresponding errors at two consecutive triangulations with number of degrees of freedom N and N', respectively. The experimental rates of convergence for the other errors are defined in analogous way.

The parameters that define the numerical fluxes will be defined by $\alpha := \frac{1}{\mathtt{h}}$, $\gamma := \frac{1}{\mathtt{h}}$ and $\boldsymbol{\beta} := (1,1)^{\mathtt{t}}$. Moreover, for all examples we consider, the parameters δ_1 and δ_2 are set as $1/2\nu$ and 1, respectively.

Now, given an *a posteriori* error estimator $\eta := \left(\sum_{T\in\mathcal{T}_h} \eta_T^2\right)^{1/2}$, we use the following adaptive algorithm (see [31]):

1. Start with a coarse mesh $\mathcal{T}_h$.
2. Solve the Galerkin scheme (4.1) for the current mesh $\mathcal{T}_h$.
3. Compute η_T for each triangle $T \in \mathcal{T}_h$.
4. Consider stopping criterion and decide to finish or go to the next step.
5. Use *red-blue-green* procedure to refine each element $T' \in \mathcal{T}_h$ such that
$$\eta_{T'} \geq \frac{1}{2} \max\{\eta_T : T \in \mathcal{T}_h\}.$$
6. Define the resulting mesh as the new $\mathcal{T}_h$ and go to step 2.

We consider two examples. Their domains Ω as well as their corresponding exact solutions $(\boldsymbol{u}, p)$ are given in Table 1. Concerning Example 1, we resume our results in Table 2, where the total error and their components goes to zero as $\mathcal{O}(N^{-1/2})$. This is in agreement with our expectations, thanks to the smoothness of exact solution. In addition, we observe that the L^2 norms of the deviator of stress error $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)$ as well as of the pressure $(p - p_h)$, behaves as $\mathcal{O}(N^{-1/2})$, while the L^2 error norm of the velocity decay to zero as $\mathcal{O}(N^{-1})$.

The analytic solution in Example 2 is taken from [26], where the parameter λ (that acts as Reynold's number) is given by

$$\lambda := -\frac{8\pi^2}{\nu^{-1} + \sqrt{\nu^{-2} + 16\pi^2}}.$$

We remark here that $\bar{p}$ denotes the real value that makes p to have zero mean value in Ω. Our purpose with this example, is to show the robustness of our scheme for moderate values of λ, specially when an adaptive refinement procedure is performed.

In Tables 3, 5 and 7, we display the convergence history and the corresponding rates of convergence of individual errors as well as the total one, on a sequence of uniform triangulation refinements, when the viscosity of the fluid $\nu \in \{1.0, 0.1, 0.059\}$. We notice that for each one of these refinements, the total error behaves as $\mathcal{O}(N^{-1/2})$, which are in agreement with Theorem 2.1, since the exact solution is smooth enough. In addition, as in Example 1, we also observe from these tables, that the L^2-norm of error of the velocity decay to zero as $\mathcal{O}(N^{-1})$. This is not covered by the current theory, but it should be a consequence of duality argument as in [9].

Now, when the estimator η (cf. (3.2)) is taking into account, it helps us to identify the region in $\bar{\Omega}$ where the total error dominates. This procedure generate a sequence of adapted meshes, with a decrease of the total error, faster than when uniform refinement is performed. The history of convergence of the method, for $\nu \in \{1.0, 0.1, 0.059\}$, is summarized in Tables 4, 6 and 8, respectively. From these tables, we observe that the total error behaves as $\mathcal{O}(N^{-1/2})$, and that the index of efficiency in all cases remains bounded. Figures 1, 2 and 3 (in log-log scale) show us that the adaptive refinement improves the quality of approximation, by recognizing the region of $\bar{\Omega}$ where η is dominant. This allows us to confirm that our *a posteriori* error estimator is reliable and (local) efficient, in agreement with Theorem 3.1.

Some of the adapted meshes generated by this procedure, are depicted in Figure 4 ($\nu = 1.0$), Figure 5 ($\nu = 0.1$) and Figure 6 ($\nu = 0.059$). In all cases, we notice that the proposed adaptive refinement algorithm is able to capture a boundary layer close to the line $x_1 = -0.5$, due to the great values of $\nabla \boldsymbol{u}$.

5 Concluding remarks

In this work, we have developed an *a posteriori* error analysis for the Stokes problem, when a DG scheme is applied. It can be seen as the continuation of a previous work, where the same DG approach looks for suitable approximations of the pseudo stress and the velocity.

We have derived an *a posteriori* error estimator is made of only five residual terms. In addition, we have proven that this estimator is reliable and local-efficient. Numerical experiments confirm these properties, and show that the adaptive algorithm based on this estimator, is able to recognize the boundary layer (or region with larger gradients).

Example	Ω	$\boldsymbol{u}(x_1, x_2)$	$p(x_1, x_2)$
1	$]-1,1[^2$	$\begin{pmatrix} -e^{x_1}(x_2\cos(x_2) + \sin(x_2)) \\ e^{x_1} x_2 \sin(x_2) \end{pmatrix}$	$2e^{x_1}\sin(x_2)$
2	$]-1/2, 3/2[\times]0,2[$	$\begin{pmatrix} 1 - e^{\lambda x_1}\cos(2\pi x_2) \\ \frac{\lambda}{2\pi} e^{\lambda x_1}\sin(2\pi x_2) \end{pmatrix}$	$-\frac{1}{2}e^{2\lambda x_1} - \bar{p}$

Table 1: Examples considered for Stokes system

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[3] T.P. Barrios and R. Bustinza: *An augmented discontinuous Galerkin method for elliptic problems*. Comptes Rendus de l'Academie des Sciences, Series I, vol. 344, pp. 53-58, (2007).

dof	$\boldsymbol{e}_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$r_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$\boldsymbol{e}_h(\boldsymbol{u})$	$r_h(\boldsymbol{u})$	$\boldsymbol{e}$	r	$\boldsymbol{e}/\eta$
25	6.0147e+00	——	4.4055e+00	——	7.4555e+00	——	3.820
97	3.4114e+00	0.837	2.6128e+00	0.771	4.2970e+00	0.813	2.552
385	1.9929e+00	0.780	1.4216e+00	0.883	2.4480e+00	0.816	2.155
1537	1.0573e+00	0.916	7.3048e-01	0.962	1.2850e+00	0.931	1.925
6145	5.3305e-01	0.988	3.6743e-01	0.992	6.4741e-01	0.989	1.796
24577	2.6558e-01	1.005	1.8387e-01	0.999	3.2302e-01	1.003	1.736
98305	1.3235e-01	1.005	9.1929e-02	1.000	1.6114e-01	1.003	1.711
dof	$\boldsymbol{e}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$	$\boldsymbol{e}_0(p)$	$r_0(p)$	$\boldsymbol{e}_0(\boldsymbol{\sigma}^{\mathrm{d}})$	$r_0(\boldsymbol{\sigma}^{\mathrm{d}})$	
25	1.4370e+00	——	3.0363e+00	——	4.2083e+00	——	
97	4.4623e-01	1.725	1.6267e+00	0.921	2.5136e+00	0.760	
385	1.4642e-01	1.617	9.4392e-01	0.790	1.4770e+00	0.771	
1537	4.4204e-02	1.730	4.7690e-01	0.986	8.1332e-01	0.862	
6145	1.2174e-02	1.861	2.2616e-01	1.077	4.2627e-01	0.932	
24577	3.1723e-03	1.940	1.0802e-01	1.066	2.1722e-01	0.973	
98305	8.0581e-04	1.977	5.2712e-02	1.035	1.0936e-01	0.990	

Table 2: History of convergence of error terms and total error, as well as corresponding rates of convergence, Example 1, $\nu = 1.0$ (uniform refinement)

dof	$\boldsymbol{e}_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$r_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$\boldsymbol{e}_h(\boldsymbol{u})$	$r_h(\boldsymbol{u})$	$\boldsymbol{e}$	r	$\boldsymbol{e}/\eta$
49	5.2012e+02	——	2.2208e+02	——	5.6555e+02	——	0.956
193	4.0888e+02	0.351	1.0683e+02	1.068	4.2261e+02	0.425	1.005
769	3.0503e+02	0.424	5.4522e+01	0.973	3.0987e+02	0.449	1.027
3073	1.8696e+02	0.707	2.8878e+01	0.918	1.8918e+02	0.712	1.023
12289	1.0024e+02	0.899	1.4691e+01	0.975	1.0131e+02	0.901	1.022
49153	5.1099e+01	0.972	7.3663e+00	0.996	5.1627e+01	0.973	1.021
196609	2.5677e+01	0.993	3.6806e+00	1.001	2.5940e+01	0.993	1.021
dof	$\boldsymbol{e}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$	$\boldsymbol{e}_0(p)$	$r_0(p)$	$\boldsymbol{e}_0(\boldsymbol{\sigma}^{\mathrm{d}})$	$r_0(\boldsymbol{\sigma}^{\mathrm{d}})$	
49	5.1247e+01	——	5.1914e+01	——	1.7751e+02	——	
193	1.4262e+01	1.866	2.5539e+01	1.035	9.3623e+01	0.933	
769	4.0200e+00	1.832	2.6434e+01	—-	5.9649e+01	0.652	
3073	8.9920e-01	2.162	1.5669e+01	0.755	3.4173e+01	0.804	
12289	2.2281e-01	2.013	8.3721e+00	0.904	1.7917e+01	0.932	
49153	5.5005e-02	2.018	4.2736e+00	0.970	9.0961e+00	0.978	
196609	1.3647e-02	2.011	2.1458e+00	0.994	4.5706e+00	0.993	

Table 3: History of convergence of error terms and total error, as well as corresponding rates of convergence, Example 2, $\nu = 1.0$ (uniform refinement)

[4] T.P. Barrios and R. Bustinza: *An augmented DG scheme for porous media equations.* In book: Numerical Mathematics and Advanced Applications, Proceedings ENUMATH-2007, edited by K. Kunisch, G. Of and O. Steinbach, pp. 315-322, Springer Verlag, Heidelberg, 2008.

[5] T.P. Barrios and R. Bustinza: *A priori and a posteriori error analyses of an*

dof	$\boldsymbol{e}_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$r_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$\boldsymbol{e}_h(\boldsymbol{u})$	$r_h(\boldsymbol{u})$	$\boldsymbol{e}$	r	$\boldsymbol{e}/\eta$
49	5.2012e+02	——	2.2208e+02	——	5.6555e+02	——	0.956
133	4.0885e+02	0.482	1.0680e+02	1.466	4.2257e+02	0.584	1.005
229	3.6238e+02	0.444	8.6870e+01	0.760	3.7265e+02	0.463	1.014
301	3.0765e+02	1.198	5.5190e+01	3.319	3.1256e+02	1.286	1.026
637	1.9132e+02	1.267	2.9976e+01	1.628	1.9366e+02	1.277	1.023
1897	1.0860e+02	1.038	1.7076e+01	1.031	1.0993e+02	1.038	1.024
3817	6.9082e+01	1.294	1.1699e+01	1.082	7.0066e+01	1.288	1.029
11593	4.1174e+01	0.932	6.8155e+00	0.973	4.1734e+01	0.933	1.029
27193	2.8156e+01	0.892	5.3354e+00	0.574	2.8657e+01	0.882	1.037
57001	1.9451e+01	1.000	3.3099e+00	1.290	1.9730e+01	1.009	1.030
130561	1.3213e+01	0.933	2.6039e+00	0.579	1.3467e+01	0.922	1.039
235585	9.8470e+00	0.996	1.7536e+00	1.340	1.0002e+01	1.008	1.032
dof	$\boldsymbol{e}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$	$\boldsymbol{e}_0(p)$	$r_0(p)$	$\boldsymbol{e}_0(\boldsymbol{\sigma}^{\mathrm{d}})$	$r_0(\boldsymbol{\sigma}^{\mathrm{d}})$	
49	5.1247e+01	——	5.1914e+01	——	1.7751e+02	——	
133	1.4290e+01	2.558	2.5231e+01	1.445	9.3638e+01	1.281	
229	1.0517e+01	1.128	2.9774e+01	——	7.7711e+01	0.686	
301	4.1536e+00	6.796	2.5820e+01	1.042	5.9057e+01	2.008	
637	1.0054e+00	3.785	1.5534e+01	1.356	3.4881e+01	1.405	
1897	3.8147e-01	1.776	9.2502e+00	0.950	1.9664e+01	1.050	
3817	2.0320e-01	1.802	6.6979e+00	0.923	1.3081e+01	1.166	
11593	1.0821e-01	1.134	3.9887e+00	0.933	7.6651e+00	0.962	
27193	1.0256e-01	0.126	3.0098e+00	0.661	5.8345e+00	0.640	
57001	3.8940e-02	2.617	1.8931e+00	1.253	3.7668e+00	1.182	
130561	3.7621e-02	0.083	1.4313e+00	0.675	2.8693e+00	0.657	
235585	1.3075e-02	3.581	9.8306e-01	1.273	2.0101e+00	1.206	

Table 4: History of convergence of error terms and total error, as well as corresponding rates of convergence, Example 2, $\nu = 1.0$ (adaptive refinement based on η)

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[9] T.P. Barrios, R. Bustinza, G.C. García and E. Hernández: *On stabilized mixed methods for generalized Stokes problem based on the velocity-pseudo stress*

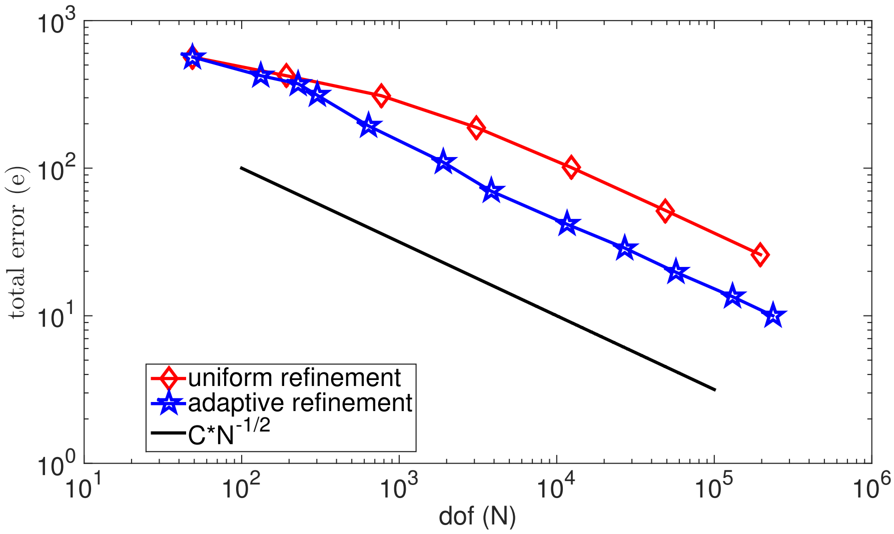

Figure 1: Total error ($\boldsymbol{e}$) vs DOF (N) for uniform and adaptive refinements (Example 2, with $\nu = 1.0$)

dof	$\boldsymbol{e}_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$r_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$\boldsymbol{e}_h(\boldsymbol{u})$	$r_h(\boldsymbol{u})$	$\boldsymbol{e}$	r	$\boldsymbol{e}/\eta$
49	1.7012e+01	—	1.7751e+01	—	2.4586e+01	—	1.342
193	1.3948e+01	0.290	1.7362e+01	0.032	2.2271e+01	0.144	1.379
769	9.8522e+00	0.503	9.9848e+00	0.800	1.4027e+01	0.669	1.358
3073	5.2729e+00	0.902	6.2457e+00	0.677	8.1738e+00	0.780	1.487
12289	2.6930e+00	0.970	3.2281e+00	0.952	4.2039e+00	0.959	1.532
49153	1.3537e+00	0.992	1.6160e+00	0.998	2.1080e+00	0.996	1.539
196609	6.7767e-01	0.998	8.0615e-01	1.003	1.0532e+00	1.001	1.540
dof	$\boldsymbol{e}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$	$\boldsymbol{e}_0(p)$	$r_0(p)$	$\boldsymbol{e}_0(\boldsymbol{\sigma}^{\mathrm{d}})$	$r_0(\boldsymbol{\sigma}^{\mathrm{d}})$	
49	4.0318e+00	—	2.0719e+00	—	3.9510e+00	—	
193	3.5173e+00	0.199	1.8973e+00	0.128	2.8678e+00	0.467	
769	7.6103e-01	2.215	1.1758e+00	0.692	1.8718e+00	0.617	
3073	2.0467e-01	1.896	6.3369e-01	0.892	1.0778e+00	0.797	
12289	5.5038e-02	1.895	3.2486e-01	0.964	5.7140e-01	0.916	
49153	1.4369e-02	1.938	1.6200e-01	1.004	2.9196e-01	0.969	
196609	3.6760e-03	1.967	8.0457e-02	1.010	1.4710e-01	0.989	

Table 5: History of convergence of error terms and total error, as well as corresponding rates of convergence, Example 2, $\nu = 0.1$ (uniform refinement)

formulation: A priori error estimates. Computer Methods in Applied Mechanics and Engineering, vol. 237-240, pp. 78-87, (2012).

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dof	$\boldsymbol{e}_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$r_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$\boldsymbol{e}_h(\boldsymbol{u})$	$r_h(\boldsymbol{u})$	$\boldsymbol{e}$	r	$\boldsymbol{e}/\eta$
49	1.7012e+01	——	1.7751e+01	——	2.4586e+01	——	1.342
133	1.3942e+01	0.399	1.7398e+01	0.040	2.2295e+01	0.196	1.380
301	9.9128e+00	0.835	1.0172e+01	1.314	1.4204e+01	1.104	1.365
889	5.5846e+00	1.060	6.6874e+00	0.775	8.7126e+00	0.903	1.492
2065	3.4250e+00	1.160	4.1875e+00	1.111	5.4097e+00	1.131	1.544
4897	2.2834e+00	0.939	2.8560e+00	0.886	3.6566e+00	0.907	1.567
9769	1.6535e+00	0.935	2.0306e+00	0.988	2.6187e+00	0.967	1.554
20905	1.1657e+00	0.919	1.4314e+00	0.919	1.8460e+00	0.919	1.561
40849	8.4633e-01	0.956	1.0076e+00	1.048	1.3159e+00	1.011	1.535
84529	5.8924e-01	0.996	7.3212e-01	0.879	9.3979e-01	0.926	1.574
164329	4.2818e-01	0.961	5.0490e-01	1.118	6.6201e-01	1.054	1.530
338569	2.9851e-01	0.998	3.6625e-01	0.888	4.7250e-01	0.933	1.565
dof	$\boldsymbol{e}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$	$\boldsymbol{e}_0(p)$	$r_0(p)$	$\boldsymbol{e}_0(\boldsymbol{\sigma}^{\mathrm{d}})$	$r_0(\boldsymbol{\sigma}^{\mathrm{d}})$	
49	4.0318e+00	——	2.0719e+00	——	3.9510e+00	——	
133	3.5091e+00	0.278	1.8670e+00	0.209	2.8636e+00	0.645	
301	8.2330e-01	3.550	1.1690e+00	1.146	1.8771e+00	1.034	
889	3.0236e-01	1.850	6.5085e-01	1.082	1.1329e+00	0.932	
2065	1.6126e-01	1.492	3.9701e-01	1.173	6.9542e-01	1.158	
4897	1.0350e-01	1.027	2.6877e-01	0.904	4.8487e-01	0.835	
9769	5.8741e-02	1.640	1.9328e-01	0.955	3.5715e-01	0.885	
20905	3.1057e-02	1.675	1.3722e-01	0.901	2.5485e-01	0.887	
40849	1.5880e-02	2.003	9.9786e-02	0.951	1.8531e-01	0.951	
84529	1.0563e-02	1.121	6.9398e-02	0.999	1.3095e-01	0.955	
164329	4.0251e-03	2.903	5.0194e-02	0.975	9.4522e-02	0.981	
338569	2.7294e-03	1.075	3.4823e-02	1.012	6.6695e-02	0.965	

Table 6: History of convergence of error terms and total error, as well as corresponding rates of convergence, Example 2, $\nu = 0.1$ (adaptive refinement based on η)

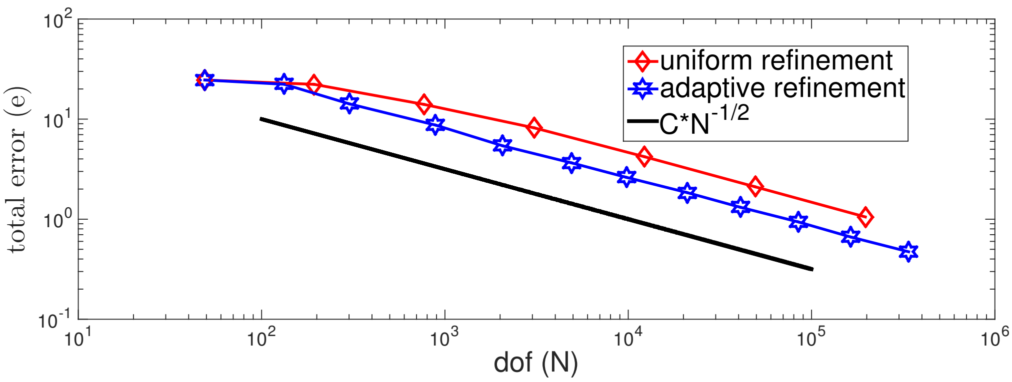

Figure 2: Total error ($\boldsymbol{e}$) vs DOF (N) for uniform and adaptive refinements (Example 2, with $\nu = 0.1$)

dof	$\boldsymbol{e}_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$r_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$\boldsymbol{e}_h(\boldsymbol{u})$	$r_h(\boldsymbol{u})$	$\boldsymbol{e}$	r	$\boldsymbol{e}/\eta$
49	5.5800e+00	——	1.0684e+01	——	1.2054e+01	——	1.832
193	4.0886e+00	0.454	1.0222e+01	0.065	1.1009e+01	0.132	1.875
769	2.9120e+00	0.491	6.5701e+00	0.640	7.1865e+00	0.617	2.150
3073	1.4913e+00	0.966	4.1043e+00	0.679	4.3668e+00	0.719	2.518
12289	7.4960e-01	0.993	2.1141e+00	0.957	2.2431e+00	0.961	2.787
49153	3.7490e-01	1.000	1.0572e+00	1.000	1.1217e+00	1.000	2.866
196609	1.8743e-01	1.000	5.2758e-01	1.003	5.5988e-01	1.003	2.892
dof	$\boldsymbol{e}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$	$\boldsymbol{e}_0(p)$	$r_0(p)$	$\boldsymbol{e}_0(\boldsymbol{\sigma}^{\mathrm{d}})$	$r_0(\boldsymbol{\sigma}^{\mathrm{d}})$	
49	2.4612e+00	——	1.1502e+00	——	1.1670e+00	——	
193	2.2783e+00	0.113	7.8425e-01	0.559	8.8333e-01	0.406	
769	5.6330e-01	2.022	4.1450e-01	0.923	6.0889e-01	0.538	
3073	1.4554e-01	1.954	2.1429e-01	0.952	3.5318e-01	0.786	
12289	3.8589e-02	1.915	1.0824e-01	0.986	1.8799e-01	0.910	
49153	9.9744e-03	1.952	5.3731e-02	1.010	9.6153e-02	0.967	
196609	2.5323e-03	1.978	2.6673e-02	1.010	4.8443e-02	0.989	

Table 7: History of convergence of error terms and total error, as well as corresponding rates of convergence, Example 2, $\nu = 0.059$ (uniform refinement)

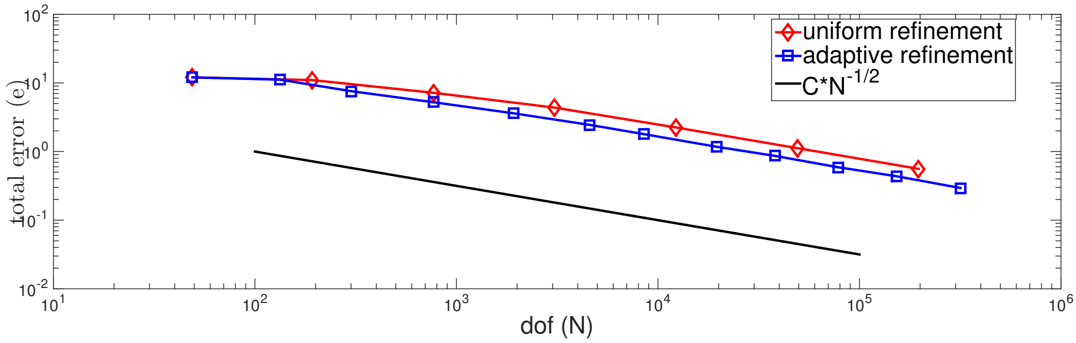

Figure 3: Total error ($\boldsymbol{e}$) vs DOF (N) for uniform and adaptive refinements (Example 2, with $\nu = 0.059$)

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dof	$\boldsymbol{e}_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$r_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$	$\boldsymbol{e}_h(\boldsymbol{u})$	$r_h(\boldsymbol{u})$	$\boldsymbol{e}$	r	$\boldsymbol{e}/\eta$
49	5.5800e+00	——	1.0684e+01	——	1.2054e+01	——	1.832
133	4.0869e+00	0.624	1.0384e+01	0.057	1.1159e+01	0.154	1.894
301	3.0951e+00	0.681	6.9238e+00	0.992	7.5841e+00	0.946	2.151
769	1.9616e+00	0.972	4.8498e+00	0.759	5.2315e+00	0.792	2.337
1921	1.2461e+00	0.991	3.3875e+00	0.784	3.6094e+00	0.811	2.693
4585	8.1739e-01	0.969	2.3075e+00	0.883	2.4480e+00	0.893	2.770
8521	6.0132e-01	0.991	1.6907e+00	1.004	1.7945e+00	1.002	2.832
19585	4.0615e-01	0.943	1.1004e+00	1.032	1.1730e+00	1.022	2.774
38185	2.8755e-01	1.034	8.1422e-01	0.902	8.6350e-01	0.917	2.889
78337	2.0362e-01	0.961	5.5453e-01	1.069	5.9073e-01	1.057	2.808
152257	1.4603e-01	1.001	4.1224e-01	0.892	4.3734e-01	0.905	2.899
317101	1.0217e-01	0.974	2.7604e-01	1.093	2.9434e-01	1.079	2.804
dof	$\boldsymbol{e}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$	$\boldsymbol{e}_0(p)$	$r_0(p)$	$\boldsymbol{e}_0(\boldsymbol{\sigma}^{\mathrm{d}})$	$r_0(\boldsymbol{\sigma}^{\mathrm{d}})$	
49	2.4612e+00	——	1.1502e+00	——	1.1670e+00	——	
133	2.2816e+00	0.152	7.5399e-01	0.846	8.7855e-01	0.569	
301	7.3522e-01	2.773	4.2891e-01	1.381	6.1841e-01	0.860	
769	3.7305e-01	1.447	2.4562e-01	1.189	4.0612e-01	0.897	
1921	1.7552e-01	1.647	1.6208e-01	0.908	2.7489e-01	0.853	
4585	7.3515e-02	2.001	1.0948e-01	0.902	1.9727e-01	0.763	
8521	4.9936e-02	1.248	7.8029e-02	1.093	1.4426e-01	1.010	
19585	1.8493e-02	2.387	5.4182e-02	0.877	1.0155e-01	0.844	
38185	1.2314e-02	1.218	3.7588e-02	1.095	7.1807e-02	1.038	
78337	5.3806e-03	2.304	2.7349e-02	0.885	5.1730e-02	0.913	
152257	3.3131e-03	1.459	1.9111e-02	1.079	3.7122e-02	0.999	
317101	1.2833e-03	2.586	1.3718e-02	0.904	2.6209e-02	0.949	

Table 8: History of convergence of error terms and total error, as well as corresponding rates of convergence, Example 2, $\nu = 0.059$ (adaptive refinement based on η)

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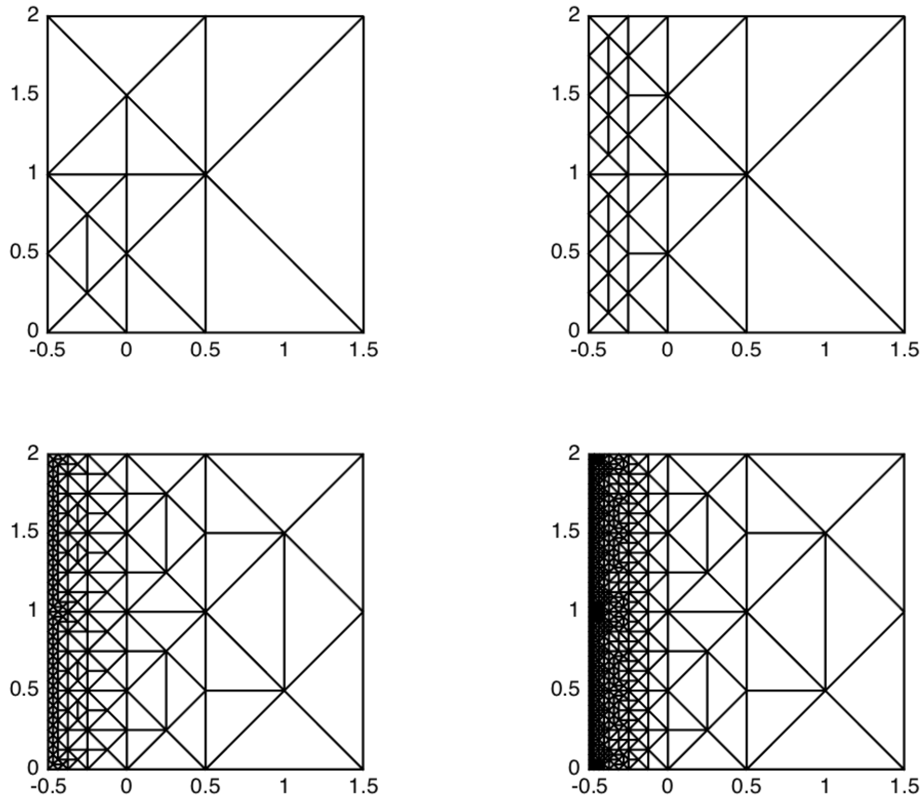

Figure 4: Adaptive refined meshes corresponding to 229, 637, 3817 and 11593 dof (from left to right, top - bottom) (Example 2, with $\nu = 1.0$)

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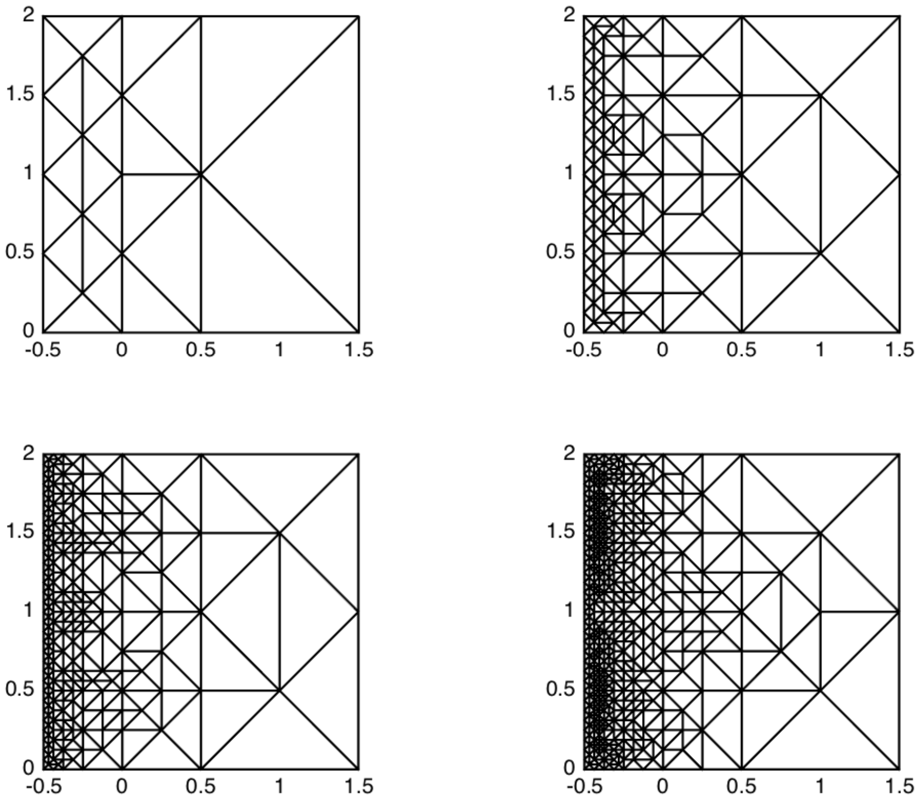

Figure 5: Adaptive refined meshes corresponding to 301, 2065, 4897 and 9769 dof (from left to right, top - bottom) (Example 2, with $\nu = 0.1$)

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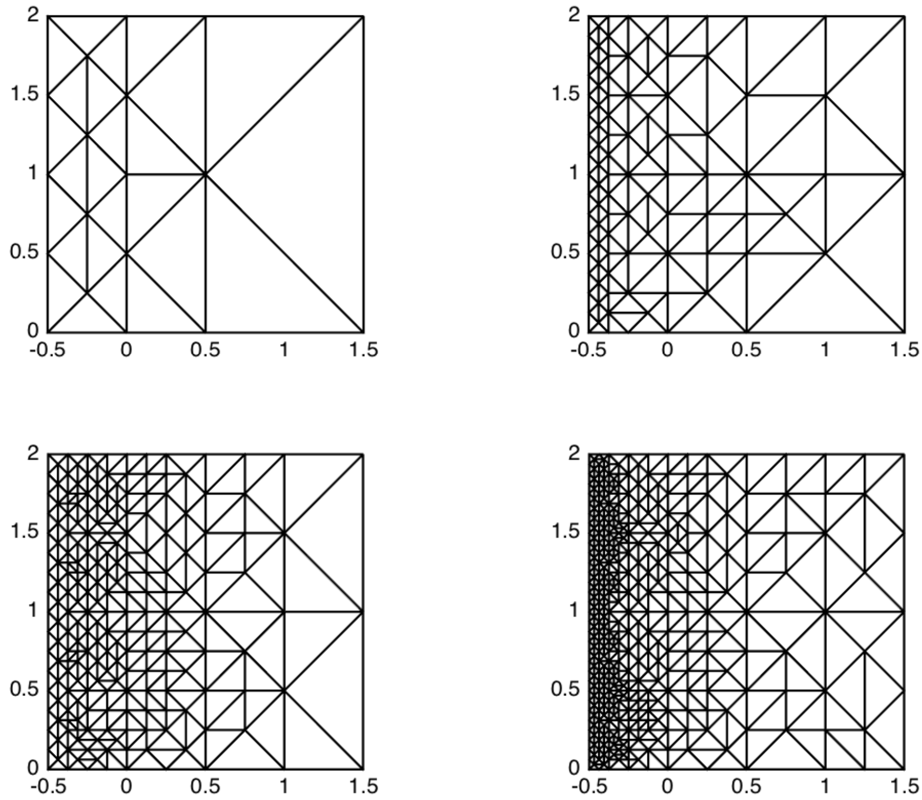

Figure 6: Adaptive refined meshes corresponding to 301, 1921, 4585 and 8521 dof (from left to right, top - bottom) (Example 2, with $\nu = 0.059$)

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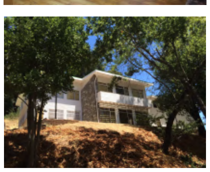

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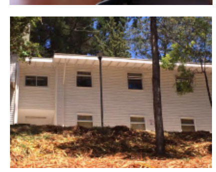

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