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Abstract

We propose and analyze a high order unfitted hybridizable discontinuous Galerkin method to numerically solve Oseen equations in a domain Ω having a curved boundary. The domain is approximated by a polyhedral computational domain not necessarily fitting Ω. The boundary condition is transferred to the computational domain through line integrals over the approximation of the gradient of the velocity and a suitable decomposition of the pressure in the computational domain is employed to obtain an approximation of the pressure having zero-mean in the domain Ω. Under assumptions related to the distance between the computational boundary and the boundary of Ω, we provide stability estimates of the solution that will lead us to the well-posedness of the scheme and also to the error estimates. In particular, we prove that the approximations of the pressure, velocity and its gradient are of order h^{k+1}, where h is the meshsize and k the polynomial degree of the local discrete spaces. We provide numerical experiments validating the theory and also showing the performance of the method when applied to the steady–state incompressible Navier-Stokes equations.

Key words: Oseen equations, curved domains, hybridizable discontinuous Galerkin method, unfitted methods, Navier-Stokes equations

Mathematics subject classifications (2010): 65N30, 65N12, 65N15

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a region not necessarily polygonal ($d = 2$) or polyhedral ($d = 3$) with boundary $\Gamma := \partial\Omega$ compact, Lipschitz and piecewise $\mathcal{C}^2$. Denoting by $\boldsymbol{u}$ the velocity of the fluid, p the pressure, $\nu > 0$ a constant viscosity, $\boldsymbol{\beta}$ the convective velocity, $\boldsymbol{f} \in [L^2(\Omega)]^d$ and $\boldsymbol{g} \in [L^2(\Gamma)]^d$ satisfying $\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} = 0$ ($\boldsymbol{n}$ is the outward unit normal to Ω), the incompressible Oseen equations are:

$$\mathrm{L} - \nabla \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$-\nu \nabla \cdot \mathrm{L} + (\boldsymbol{\beta} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1b}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma, \tag{1.1d}$$

$$\int_\Omega p = 0, \tag{1.1e}$$

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The convective velocity is assumed to satisfy $\boldsymbol{\beta} \in [W^{1,\infty}(\Omega)]^d$ and to be divergence free. The motivation to solve (1.1) arises from the steady–state incompressible Navier–Stokes equations, where the term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ appears instead of $(\boldsymbol{\beta} \cdot \nabla)\boldsymbol{u}$. In fact, the solution of the Navier–Stokes equations can be obtained by using Picard's iteration that consists of solving (1.1) with $\boldsymbol{\beta}$ as the velocity obtained in the previous iteration.

We consider a hybridizable discontinuous Galerkin method (HDG) to approximate the solution of (1.1). Since in this case the domain Ω is not necessarily polyhedral, we approximate Ω by a polyhedral computational domain where the boundary data $\boldsymbol{g}$ is properly transferred to the computational boundary. We follow the approach proposed by [14] and analysed in [13] that consists of integrating the extrapolated discrete gradient along *transferring paths* connecting the computational boundary and Γ. This technique, combined with a high order method in the computational domain, allows us to obtain a high order approximation of the boundary data in the computational domain which leads to a high order accuracy of the discrete solution. In the context of HDG methods, one of the first ideas based on this transferring technique was introduced by [12] for the one-dimensional case and then extended to higher dimensions for pure diffusion ([13, 14]), convection-diffusion [15] equations and interface problems [29]. Recently, [30] analysed this approach applied to Stokes problem. Mixed methods for diffusion problems have also been successfully combined with this transferring technique [25].

Methods for non–polyhedral domains can be classified as *fitted* or *unfitted*. In fitted methods [1, 4, 5, 19, 21], the discretization of the domain resolves the boundary. In general, one of the main advantages of fitted methods is that the prescribed data at the boundary can be easily imposed. However, the construction of the meshes might be difficult, especially in complicated geometries. On the other hand, the attractive feature of unfitted methods [3, 17, 18, 20, 23, 24, 26, 27, 28, 31] is that the mesh is not adjusted to the domain and even Cartesian grids can be considered. However, it is not straightforward to develop a high order unfitted method, mainly because of imposition of the boundary data away from the true boundary. For a more detailed discussion related to fitted and unfitted methods we refer to [14, 30]. It is relevant to mention that the field of developing methods to handle curved boundaries has been quite active recently. In fact, extrapolation and boundary correction techniques can be found in the shifted boundary method [22] and in the cut finite element method [8], among others. In addition, hybrid high-order (HHO) methods have been developed in curved meshes [2] and in polygonal unfitted meshes [7].

Our aim is to develop methods that combine the flexibility of the mesh construction of unfitted methods with the high order accuracy of fitted methods. To that end, we combine the analysis in [9] for HDG methods applied to the Ossen problem in polyhedral domains with the transferring technique analysed in [30].

The rest of the paper is organized as follows. In Section 2, we first recall preliminaries and notation originally presented in [30]. We introduce the method in Section 3 and provide stability estimates leading to the well-posedness of the scheme. In Section 4 the error estimates are stated and their profs are presented in Section 5. Section 6 shows numerical experiments validating the theory and it also includes an application to the steady–state incompressible Navier–Stokes equations through Picard's iteration.

2 Preliminaries

2.1 Computational domain

The computational domain and transferring paths are constructed exactly as in [30]. In order to make this manuscript self–contained, we now present the notation originally introduced there.

Given $h > 0$, we denote by D_h an open polyhedral computational domain, with boundary Γ_h. D_h is partitioned by a triangulation T_h, of meshsize h, with no hanging nodes and made of simplices K uniformly shape-regular, that is,

(D.1) there exists $\gamma > 0$, independent of h, such that $h_K \leq \gamma \rho_K$,

where ρ_K is the radius of the largest ball contained in K and h_K is the diameter of K. We assume

(D.2) $\max_{K \in \mathsf{T}_h} h_K \leq h$ and

(D.3) $\overline{\mathsf{D}_h} \subset \overline{\Omega}$.

This last assumption is for the sake of simplicity of the analysis, however the method can be still considered if $\overline{\mathsf{D}_h} \cup \Omega^c \neq \emptyset$.

Given a simplex K, $\boldsymbol{n}_K$ corresponds to its outward unit normal, writing $\boldsymbol{n}$ instead of $\boldsymbol{n}_K$ when there is no confusion. Similarly, for a face e, we write $\boldsymbol{n}$ instead of $\boldsymbol{n}_e$ to refer to its normal vector. The set of faces and boundary faces of T_h are denoted by $\mathcal{E}_h$ and $\mathcal{E}_h^{\partial}$, respectively, and we define the non-meshed region $\mathsf{D}_h^c := \Omega \setminus \overline{\mathsf{D}_h}$.

2.2 Transferring paths and extrapolation regions

We consider the idea in [14] to transfer the boundary data $\boldsymbol{g}$ from Γ to Γ_h through transferring paths as follows. Let $e \in \mathcal{E}_h^{\partial}$ with normal unit vector $\boldsymbol{n}$. For each $\boldsymbol{x} \in e$, we set $\bar{\boldsymbol{x}} \in \Gamma$ as the closest intersection between Γ and the ray of tangent vector $\boldsymbol{n}$ starting at $\boldsymbol{x}$. We name *transferring path* to the segment $\sigma_{\boldsymbol{n}}(\boldsymbol{x})$ that joins $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$. Its length, $l(\boldsymbol{x}) := |\bar{\boldsymbol{x}} - \boldsymbol{x}|$, is assumed to satisfy

(D.4) $l(\boldsymbol{x}) \lesssim h$ and we suppose that

(D.5) the intersection of the ray $\{\boldsymbol{x} + \eta \boldsymbol{n} : \eta > 0\}$ and Γ is unique.

Now, let $e \in \mathcal{E}_h^{\partial}$ and K^e the elements where it belongs. We define the *extrapolation region* as follows

$$K_{ext}^e := \{\boldsymbol{x} + s\boldsymbol{n} : 0 \leq s \leq l(\boldsymbol{x}),\, \boldsymbol{x} \in e\}.$$

Thus, given a polynomial G defined on the element K^e, we can extrapolate it to K_{ext}^e. We will refer this as *local extrapolation.*

We denote by $H_e^{\perp}$ the largest distance of a point in K_{ext}^e to the plane determined by e and set $h_e^{\perp}$ as the distance between e and the vertex of K^e opposite to e, as Fig. 1 shows. The ratio $r_e := H_e^{\perp}/h_e^{\perp}$ indicates how far is Γ_h from Γ relative to the meshsize. In fact, if the domain is polyhedral and the mesh fits its boundary, this ratio would be zero. We define

$$R = \max_{e \in \mathcal{E}_h^{\partial}} r_e. \tag{2.1}$$

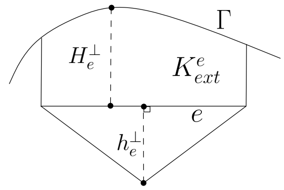

Figure 1: A two-dimensional example, obtained from [30], of K_{ext}^e and the distances $H_e^\perp$ and $h_e^\perp$. Figure obtained from [30].

2.3 Norms and inner products

Given an element K and a non-negative integer r, $\mathcal{P}_r(K)$ denotes the space of polynomials of total degree at most r, and set $\boldsymbol{\mathcal{P}}_r(K) := [\mathcal{P}_r(K)]^d$ and $\mathrm{P}_r(K) := [\mathcal{P}_r(K)]^{d\times d}$. Given a region $D \subset \mathbb{R}^d$, we denote by $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$ the $L^2(D)$ and $L^2(\partial D)$ inner products, respectively. The L^2-norms over D and ∂D will be denoted by $\|\cdot\|_D$ and $\|\cdot\|_{\partial D}$. Vector-valued functions are boldfaced, Tensor-valued functions are written in Roman letters and we define $(\cdot,\cdot)_{\mathsf{T}_h} := \sum_{K\in\mathsf{T}_h}(\cdot,\cdot)_K$ and $\langle\eta,\zeta\rangle_{\partial\mathsf{T}_h} := \sum_{K\in\mathsf{T}_h}\langle\eta,\zeta\rangle_{\partial K}$. We use the standard notation for Sobolev spaces and their associated norms and seminorms. In addition, we consdier the following norms

$$\|\zeta\|_{\mathsf{D}_h^c,h^\perp} := \left\{\sum_{e\in\mathcal{E}_h^\partial} h_e^\perp\|\zeta\|^2_{K_{ext}^e}\right\}^{1/2}, \quad \|\zeta\|_{\partial\mathsf{T}_h,\alpha} := \left\{\sum_{K\in\mathsf{T}_h}\langle\alpha_{\partial K}\zeta,\zeta\rangle_{\partial K}\right\}^{1/2},$$

$$\|\zeta\|_{\Gamma_h,\alpha} := \left\{\sum_{e\in\mathcal{E}_h^\partial}\alpha_e\|\zeta\|_e^2\right\}^{1/2}, \quad \text{and} \quad \|\zeta\|_h := \left\{\sum_{K\in\mathsf{T}_h} h_K\langle\zeta,\zeta\rangle_{\partial K}\right\}^{1/2},$$

where $e\in\mathcal{E}_h$, $\alpha_e := \alpha|_e$ and $\alpha_{\partial K} := \alpha|_{\partial K}$, being α a piecewise positive function on $\mathcal{E}_h$ or $\partial\mathsf{T}_h$, resp.

2.4 Auxiliary estimates

Let $e\in\mathcal{E}_h^\partial$, a point $\boldsymbol{x}$ lying on e and a tensor-valued function G. We define

$$\delta_{\mathrm{G}}(\boldsymbol{x}) := \frac{1}{l(\boldsymbol{x})}\int_0^{l(\boldsymbol{x})}\big(\mathrm{G}(\boldsymbol{x}+s\boldsymbol{n}) - \mathrm{G}(\boldsymbol{x})\big)\boldsymbol{n}\,\mathrm{d}s. \tag{2.2}$$

From Lemma 5.2 of [13], it can be inferred that

$$\|l^{1/2}\delta_{\mathrm{G}}\|_e \le 3^{-1/2} r_e^{3/2} C_{ext}^e C_{inv}^e \|\mathrm{G}\|_{K^e} \quad \forall \mathrm{G}\in\mathrm{P}_r(K), \tag{2.3}$$

where

$$C_{ext}^e := \frac{1}{\sqrt{r_e}}\sup_{\mathrm{G}\in\mathrm{P}_k(K^e)\setminus\{0\}}\frac{\|\mathrm{G}\|_{K_{ext}^e}}{\|\mathrm{G}\|_{K^e}} \tag{2.4}$$

and

$$C_{inv}^e := h_e^\perp \sup_{\mathrm{G}\in\mathrm{P}_k(K^e)\setminus\{0\}}\frac{\|\partial_{\boldsymbol{n}}\mathrm{G}\|_e}{\|\mathrm{G}\|_{K^e}}. \tag{2.5}$$

The constants C^e_{ext} and C^e_{inv} depend on the polynomial degree and shape-regularity constant, but they are independent of h (Lemma A.2 in [13]).

If φ is a scalar, vector or tensor-valued polynomial in K^e, then there exists (Lemma 1.46 in [16]) $C^e_{tr} > 0$, independent of h, such that

$$\|\varphi\|_e \le C^e_{tr} h_e^{-1/2} \|\varphi\|_{K^e}, \tag{2.6}$$

On the other hand, for $K \in \mathsf{T}_h$, let $\boldsymbol{\beta}_0 \in \boldsymbol{\mathcal{P}}_0(K)$ be the function such that

$$\langle (\boldsymbol{\beta} - \boldsymbol{\beta}_0) \cdot \boldsymbol{n}, 1 \rangle_F = 0, \qquad \text{for all faces } F \text{ of } K$$

Observe that $\boldsymbol{\beta}_0$ exists since $\nabla \cdot \boldsymbol{\beta} = 0$. Indeed, it corresponds to the lowest order Raviart–Thomas projection of $\boldsymbol{\beta}$. In addition, $\boldsymbol{\delta\beta} := \boldsymbol{\beta} - \boldsymbol{\beta}_0$ satisfies [6]

$$\|\boldsymbol{\delta\beta}\|_K \le C h_K \|\nabla \boldsymbol{\beta}\|_K, \qquad \forall K \in \mathsf{T}_h. \tag{2.7}$$

We end this section mentioning that, from now on, we suppose Assumptions (D) hold, even if we do not refer to them explicitly. In addition, to avoid proliferation of constants, we will write $a \lesssim b$ instead of $a \le Cb$, where C is a positive constant independent of h.

2.5 Dual problem

In order to estimate the L^2-norm of the error of the velocity, we assume regularity of the solution of the problem: For any given $\boldsymbol{\theta} \in \boldsymbol{L}^2(\Omega)$, let $(\Phi, \boldsymbol{\phi}, \phi)$ be the solution of

$$\Phi - \nabla \boldsymbol{\phi} = 0 \quad \text{in } \Omega, \tag{2.8a}$$
$$-\nu \nabla \cdot \Phi - \nabla \cdot (\boldsymbol{\phi} \otimes \boldsymbol{\beta}) - \nabla \phi = \boldsymbol{\theta} \quad \text{in } \Omega, \tag{2.8b}$$
$$-\nabla \cdot \boldsymbol{\phi} = 0 \quad \text{in } \Omega, \tag{2.8c}$$
$$\boldsymbol{\phi} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{2.8d}$$

Assumption B

$$\nu \|\Phi\|_{\mathrm{H}^1(\Omega)} + \nu \|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)} + \|\phi\|_{H^1(\Omega)} \le C \|\boldsymbol{\theta}\|_\Omega. \tag{2.9}$$

3 The HDG method

First of all, we explain how to construct an approximation of the boundary data at Γ_h, based on integrating the gradient of the velocity. Let $e \in \mathcal{E}_h^\partial$ with outward normal unit vector $\boldsymbol{n}$. Given $\boldsymbol{x} \in e$ and its corresponding transferring segment $\sigma_{\boldsymbol{n}}$, we integrate component-wise $\mathrm{L} = \nabla \boldsymbol{u}$ along $\sigma_{\boldsymbol{n}}(\boldsymbol{x})$ to obtain

$$\widetilde{\boldsymbol{g}}(\boldsymbol{x}) = \boldsymbol{g}(\bar{\boldsymbol{x}}) - \int_0^{l(\boldsymbol{x})} \mathrm{L}(\boldsymbol{x} + \boldsymbol{n}s)\, \boldsymbol{n}\, ds, \tag{3.1}$$

where $\widetilde{\boldsymbol{g}}$ is the trace of $\boldsymbol{u}$ at e and $\boldsymbol{u}(\bar{\boldsymbol{x}}) = \boldsymbol{g}(\bar{\boldsymbol{x}})$. This expression suggests the following approximation $\widetilde{\boldsymbol{g}}_h$ of $\widetilde{\boldsymbol{g}}$:

$$\widetilde{\boldsymbol{g}}_h(\boldsymbol{x}) := \boldsymbol{g}(\bar{\boldsymbol{x}}) - \int_0^{l(\boldsymbol{x})} \mathrm{L}_h(\boldsymbol{x} + \boldsymbol{n}s)\boldsymbol{n}\, ds, \tag{3.2}$$

where L_h is the *local extrapolation* of the HDG approximation of L introduced in Section 2.2.

We follow the same treatment of the pressure introduced in [30], which consists of decomposing $p = \overline{p}^{\mathsf{D}_h} + \widetilde{p}$, where $\overline{p}^{\mathsf{D}_h} := \dfrac{1}{|\mathsf{D}_h|}\displaystyle\int_{\mathsf{D}_h} p$ and $\widetilde{p} \in L^2_0(\mathsf{D}_h) := \left\{ q \in L^2(\mathsf{D}_h) : \displaystyle\int_{\mathsf{D}_h} q = 0 \right\}$. In D_h, we consider the HDG method proposed in [9] but with the approximation of the boundary data (3.2). That is, we seek an approximation $(\mathrm{L}_h, \boldsymbol{u}_h, \widetilde{p}_h, \widehat{\boldsymbol{u}}_h)$ of the solution $(\mathrm{L}, \boldsymbol{u}, \widetilde{p}, \boldsymbol{u}|_{\mathcal{E}_h})$ in the space $\mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$ given by

$$
\begin{aligned}
\mathrm{G}_h &= \{\mathrm{G} \in \mathrm{L}^2(\mathsf{T}_h) : \quad G|_K \in \mathrm{P}_k(K) \quad \forall K \in \mathsf{T}_h\}, && (3.3a)\\
\boldsymbol{V}_h &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathsf{T}_h) : \quad \boldsymbol{v}|_K \in \boldsymbol{\mathcal{P}}_k(K) \quad \forall K \in \mathsf{T}_h\}, && (3.3b)\\
P_h &= \{q \in L^2(\mathsf{T}_h) : \quad q|_K \in \mathcal{P}_k(K) \quad \forall K \in \mathsf{T}_h\}, && (3.3c)\\
\boldsymbol{M}_h &= \{\boldsymbol{\mu} \in \boldsymbol{L}^2(\mathcal{E}_h) : \quad \boldsymbol{\mu}|_e \in \boldsymbol{\mathcal{P}}_k(e) \quad \forall e \in \mathcal{E}_h\}, && (3.3d)
\end{aligned}
$$

such that

$$
\begin{aligned}
(\mathrm{L}_h, \mathrm{G})_{\mathsf{T}_h} + (\boldsymbol{u}_h, \nabla \cdot \mathrm{G})_{\mathsf{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \mathrm{G}\boldsymbol{n}\rangle_{\partial \mathsf{T}_h} &= 0, && (3.4a)\\
(\nu \mathrm{L}_h, \nabla \boldsymbol{v})_{\mathsf{T}_h} - (\boldsymbol{u}_h \otimes \boldsymbol{\beta}, \nabla \boldsymbol{v})_{\mathsf{T}_h} - (\widetilde{p}_h, \nabla \cdot \boldsymbol{v})_{\mathsf{T}_h} & && (3.4b)\\
- \langle \nu \widehat{\mathrm{L}}_h \boldsymbol{n} - \widehat{p}_h \boldsymbol{n} - (\widehat{\boldsymbol{u}_h \otimes \boldsymbol{\beta}})\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial \mathsf{T}_h} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathsf{T}_h}, &&\\
-(\boldsymbol{u}_h, \nabla q)_{\mathsf{T}_h} + \langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, q\rangle_{\partial \mathsf{T}_h} &= 0, && (3.4c)\\
\langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\mu}\rangle_{\Gamma_h} &= \langle \tilde{\boldsymbol{g}}_h, \boldsymbol{\mu}\rangle_{\Gamma_h}, && (3.4d)\\
\langle \nu \widehat{\mathrm{L}}_h \boldsymbol{n} - \widehat{p}_h \boldsymbol{n} - (\widehat{\boldsymbol{u}_h \otimes \boldsymbol{\beta}})\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial \mathsf{T}_h \setminus \Gamma_h} &= 0, && (3.4e)\\
(\widetilde{p}_h, 1)_{\mathsf{D}_h} &= 0, && (3.4f)
\end{aligned}
$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$, where

$$
\nu \widehat{\mathrm{L}}_h \boldsymbol{n} - \widehat{p}_h \boldsymbol{n} - (\widehat{\boldsymbol{u}_h \otimes \boldsymbol{\beta}})\boldsymbol{n} = \nu \mathrm{L}_h \boldsymbol{n} - \widetilde{p}_h \boldsymbol{n} - (\widehat{\boldsymbol{u}}_h \otimes \boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \qquad \text{on } \partial \mathsf{T}_h, \tag{3.4g}
$$

$\mathrm{S} = \nu\tau \mathrm{I}$ and τ is a non-negative piecewise constant stabilization parameter defined on $\partial \mathsf{T}_h$ satisfying

$$
\tau\nu - \frac{1}{2}(\boldsymbol{\beta} \cdot \boldsymbol{n}) > c_\tau \qquad \text{on } \partial K, \tag{3.5}
$$

for all $K \in \mathsf{T}_h$ for some positive constant c_τ. Note that this assumption is equivalent to $\mathrm{S}_{\boldsymbol{\beta}} := \big(\mathrm{S} - \frac{1}{2}(\boldsymbol{\beta} \cdot \boldsymbol{n})\mathrm{I}\big)\big|_{\partial K}$ being positive definite. For the sake of simplicity, we assume τ to be constant on $\partial \mathsf{T}_h$.

3.1 Stability estimate

In this section we establish a stability estimate for the HDG method (3.4) that will lead us to show well-posedness of this scheme and also to the corresponding error estimates.

Given $\mathrm{F}_s \in \mathrm{L}^2(\mathsf{D}_h)$, we consider the following problem: Find $(\mathrm{L}_s, \boldsymbol{u}_s, p_s, \widehat{\boldsymbol{u}}_s)$ in the space $\mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$ such that

$$
\begin{aligned}
(\mathrm{L}_s, \mathrm{G})_{\mathsf{T}_h} + (\boldsymbol{u}_s, \nabla \cdot \mathrm{G})_{\mathsf{T}_h} - \langle \widehat{\boldsymbol{u}}_s, \mathrm{G}\boldsymbol{n}\rangle_{\partial \mathsf{T}_h} &= -(\mathrm{F}_s, \mathrm{G})_{\mathsf{T}_h}, && (3.6a)\\
(\nu \mathrm{L}_s, \nabla \boldsymbol{v})_{\mathsf{T}_h} - (\boldsymbol{u}_s \otimes \boldsymbol{\beta}, \nabla \boldsymbol{v})_{\mathsf{T}_h} - (p_s, \nabla \cdot \boldsymbol{v})_{\mathsf{T}_h} & && (3.6b)\\
- \langle \nu \widehat{\mathrm{L}}_h \boldsymbol{n} - \widehat{p}_h \boldsymbol{n} - (\widehat{\boldsymbol{u}_s \otimes \boldsymbol{\beta}})\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial \mathsf{T}_h} &= 0, &&\\
-(\boldsymbol{u}_s, \nabla q)_{\mathsf{T}_h} + \langle \widehat{\boldsymbol{u}}_s \cdot \boldsymbol{n}, q\rangle_{\partial \mathsf{T}_h} &= 0, && (3.6c)\\
\langle \widehat{\boldsymbol{u}}_s, \boldsymbol{\mu}\rangle_{\Gamma_h} &= \langle \boldsymbol{g}_s, \boldsymbol{\mu}\rangle_{\Gamma_h}, && (3.6d)\\
\langle \nu \widehat{\mathrm{L}}_s \boldsymbol{n} - \widehat{p}_s \boldsymbol{n} - (\widehat{\boldsymbol{u}_s \otimes \boldsymbol{\beta}})\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial \mathsf{T}_h \setminus \Gamma_h} &= 0, && (3.6e)\\
(p_s, 1)_{\mathsf{D}_h} &= 0, && (3.6f)
\end{aligned}
$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$, where

$$\nu\widehat{\mathrm{L}}_s\boldsymbol{n} - \widehat{p}_s\boldsymbol{n} - (\widehat{\boldsymbol{u}_s \otimes \boldsymbol{\beta}})\boldsymbol{n} = \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s) \qquad \text{on } \partial\mathsf{T}_h \tag{3.6g}$$

and

$$\boldsymbol{g}_s(\boldsymbol{x}) := -\int_0^{l(\boldsymbol{x})} (\mathrm{L}_s + \mathrm{F}_s)(\boldsymbol{x} + \boldsymbol{n}t)\boldsymbol{n}\, dt. \tag{3.6h}$$

We now state the main theorem of this section. The proof follows from a lengthy series of estimates and, in order to prioritize clarity of exposition, we postpone it to Section 5. To that end, we introduce further assumptions.

Assumption A For each $e \in \mathcal{E}_h^\partial$,

(A.1) $r_e^3 (C_{ext}^e)^2 (C_{inv}^e)^2 \leq 1/8$,

(A.2) $\nu^{-1} r_e h_e^\perp \left(\nu\tau - \frac{1}{2}(\boldsymbol{\beta}\cdot\boldsymbol{n})\right) \leq 1/8$,

(A.3) $(C_{tr}^e)^2 r_e \gamma \leq 1/6$.

Let $\partial_{\boldsymbol{n}}(\mathrm{F}_s\boldsymbol{n})$ be the directional derivative of each component of $\mathrm{F}_s\boldsymbol{n}$. We define the quantity:

$$\Theta_{\mathrm{F}_s} := \left\{ 8R^2 \|\partial_n(\mathrm{F}_s\boldsymbol{n})\|^2_{\mathsf{D}_h^c,(h^\perp)^2} + 24R\|\mathrm{F}_s\boldsymbol{n}\|^2_{\Gamma_h,h^\perp} + 4\|\mathrm{F}_s\|^2_{\mathsf{D}_h} \right\}^{1/2}, \tag{3.7}$$

where we recall that R has been defined in (2.1).

Theorem 3.1. *Let $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$ an arbitrary function and $H(R,h) := h^{1/2} + R^2h^{3/2} + Rh^{1/2} + R^{1/2} + R^{3/2}h + R$. If Assumptions A hold, then,*

$$\|\mathrm{L}_s\|_{\mathsf{D}_h} + \nu^{-1/2}\langle \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s\rangle^{1/2}_{\partial\mathsf{T}_h} + \|\boldsymbol{g}_s\|_{\Gamma_h,l^{-1}} \lesssim \Theta_{\mathrm{F}_s} + \|\boldsymbol{u}_s\|_{\mathsf{D}_h}, \tag{3.8}$$

$$\|p_s\|_{\mathsf{D}_h} \lesssim \|\boldsymbol{u}_s\|_{\mathsf{D}_h} + \Theta_{\mathrm{F}_s}. \tag{3.9}$$

Moreover, if R holds, then there exists $h_0 > 0$ such that

$$\|\boldsymbol{u}_s\|_{\mathsf{D}_h} \lesssim \left(\|\mathrm{F}_s\|_{\mathsf{D}_h} + \Theta_{\mathrm{F}_s}\right) h^{\min\{k,1\}} + h^{1/2}H(R,h)\Theta_{\mathrm{F}_s} \tag{3.10}$$

$$+ \sup_{\boldsymbol{\theta}\in \boldsymbol{L}^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\nu(\mathrm{F}_s, \nabla\boldsymbol{\phi}_h)_{\mathsf{D}_h}}{\|\boldsymbol{\theta}\|_\Omega} + \sup_{\boldsymbol{\phi}\in \boldsymbol{H}^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{|\langle \mathrm{F}_s\boldsymbol{n}, \mathbf{P}_M(\boldsymbol{\phi})\rangle|}{\|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)}}, \tag{3.11}$$

for all $h < h_0$.

Here,

3.2 Well-posedness

Theorem 3.2. *Let Assumptions B and A hold. The scheme* (3.4) *is well–posed.*

Proof. We observe that $(\mathrm{L}_h, \boldsymbol{u}_h, \widetilde{p}_h, \widehat{\boldsymbol{u}}_h)$ of (3.4) with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{g} = \boldsymbol{0}$ satisfies (3.6) with $\mathrm{F}_s = 0$. Then, by Theorem 3.1 we conclude that $\mathrm{L}_h = 0$, $\boldsymbol{u}_h = \boldsymbol{0}$, $\widehat{\boldsymbol{u}}_h = \boldsymbol{0}$ and $\widetilde{p}_h = 0$. ☐

4 Error estimates

Theorem 4.1. *Suppose that Assumptions A hold, τ is of order one and $k \geq 1$. If $(\mathrm{L}, \boldsymbol{u}, \widetilde{p}) \in H^{k+1}(\Omega) \times \boldsymbol{H}^{k+1}(\Omega) \times H^{k+1}(\Omega)$, then there exists $h_0 > 0$ such that, for all $h < h_0$,*

$$\|\mathrm{L} - \mathrm{L}_h\|_{\mathsf{D}_h} + \|\widetilde{\boldsymbol{g}} - \widetilde{\boldsymbol{g}}_h\|_{\Gamma_h, l^{-1}} + \nu^{-1}\|\widetilde{p} - \widetilde{p}_h\|_{\mathsf{D}_h} \lesssim C_{\mathrm{L},\boldsymbol{g},p} h^{k+1}.$$

Moreover, if Assumption B holds, then $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathsf{D}_h} \lesssim C_{\boldsymbol{u}} h^{k+1}$ and

$$\|\mathbf{P}_M \boldsymbol{u} - \widehat{\boldsymbol{u}}_h\|_{\partial \mathsf{T}_h, h} + \|\boldsymbol{u} - \boldsymbol{u}_h^*\|_{\mathsf{D}_h} \lesssim \begin{cases} C_{\widehat{\mathbf{u}},\boldsymbol{u}^*} h^{k+3/2}, & \text{if } R \text{ is of order } 1, \\ \widetilde{C}_{\widehat{\mathbf{u}},\boldsymbol{u}^*} h^{k+2}, & \text{if } R \text{ is of order } h. \end{cases}$$

Here, P_M is the L^2- projection onto the space of piecewise polynomials of degree k on $\mathcal{E}_h$, denoted by $\boldsymbol{M}_h$. In addition, $C_{\mathrm{L},\boldsymbol{g},p}$, $C_{\boldsymbol{u}}$, $C_{\widehat{\mathbf{u}},\boldsymbol{u}^*}$ and $\widetilde{C}_{\widehat{\mathbf{u}},\boldsymbol{u}^*}$ are positive constants depending on the regularity of the exact solutions L, $\boldsymbol{u}$ and $\widetilde{p}$, the parameters τ and ν, the convective veclocity $\boldsymbol{\beta}$, and on positive powers of h, and $\boldsymbol{u}_h^*$ is an element-by-element postprocessing of $\boldsymbol{u}_h$ computed as follows. For each element $K \in \mathsf{T}_h$, we seek $\boldsymbol{u}_h^* \in \boldsymbol{\mathcal{P}}_{k+1}^0(K) := \{\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_{k+1}(K) : \int_K \boldsymbol{w} = 0\}$ such that

$$(\nabla \boldsymbol{u}_h^*, \nabla \boldsymbol{w}_h)_K = (\mathrm{L}_h, \nabla \boldsymbol{w}_h)_K \quad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{P}}_{k+1}^0(K), \tag{4.1a}$$

$$\int_K \boldsymbol{u}_h^* = \int_K \boldsymbol{u}_h. \tag{4.1b}$$

The first two estimates in Theorem 4.1 imply that the L^2-norm of the errors in L, $\boldsymbol{u}$ and $\widetilde{p}$ are of order h^{k+1} if the solution is smooth enough and if τ is chosen of order one, for instance. The same conclusion was obtained for the case of a polyhedral domain [9]. In addition, the third estimate shows that the error in the numerical trace $\widehat{u}_h$ and the postprocessed velocity $\boldsymbol{u}_h^*$ is of order $h^{k+3/2}$, which is half a power less than the error obtained in the case of a polyhedral domain [9]. However, our numerical experiments show an experimental order of h^{k+2} for these two variables, which suggests that our analysis might not be sharp. The same behavior has been observed for the Poisson's equation [13] and Stokes problem in curved domains [30].

5 Proofs

In this section we prove the error estimates stated in previous sections. We begin by the stability analysis where we employ energy and duality arguments.

5.1 Proof of Theorem 3.1

Step 1: A first energy estimate

Lemma 5.1. *Let* $\mathbb{T}_s := \left\langle \nu \mathrm{L}_s \boldsymbol{n} - p_s \boldsymbol{n} - \frac{1}{2}(\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \widehat{\boldsymbol{u}}_s \right\rangle_{\Gamma_h}$. *It holds,*

$$\nu \|\mathrm{L}_s\|_{\mathsf{D}_h}^2 + \langle \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h} = -(\mathrm{F}_s, \nu \mathrm{L}_s)_{\mathsf{T}_h} + \mathbb{T}_{\mathrm{L},h}.$$

Proof. Taking $\mathrm{G} = \nu \mathrm{L}_s$, $\boldsymbol{v} = \boldsymbol{u}_s$ and $q = p_s$ in (5.15a), (5.15b) and (5.15c), respectively, adding up the resulting equations, canceling and rearranging terms,

$$\begin{aligned} \nu \|\mathrm{L}_s\|_{\mathsf{D}_h}^2 - \langle \widehat{\boldsymbol{u}}_s, \nu \mathrm{L}_s \boldsymbol{n} - p_s \boldsymbol{n} \rangle_{\partial \mathsf{T}_h} + (\nabla \cdot (\boldsymbol{u}_s \otimes \boldsymbol{\beta}), \boldsymbol{u}_s)_{\mathsf{T}_h} & \\ - \left\langle \big((\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s) \otimes \boldsymbol{\beta} \big) \boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{u}_s \right\rangle_{\partial \mathsf{T}_h} &= -(\mathrm{F}_s, \nu \mathrm{L}_s)_{\mathsf{T}_h}. \end{aligned} \tag{5.1}$$

Thanks to (5.15e) with $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_s$, we deduce that

$$\begin{aligned}\langle \nu \mathrm{L}_s \boldsymbol{n} - p_s \boldsymbol{n}, \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h} = {} & \langle \nu \mathrm{L}_s \boldsymbol{n} - p_s \boldsymbol{n} - (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \widehat{\boldsymbol{u}}_s \rangle_{\Gamma_h} \\ & + \langle (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} + \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h}. \end{aligned} \tag{5.2}$$

On the other hand, since $\nabla \cdot \boldsymbol{\beta} = 0$, we have $(\nabla \cdot (\boldsymbol{u}_s \otimes \boldsymbol{\beta}), \boldsymbol{u}_s)_{\mathsf{T}_h} = \big((\nabla \boldsymbol{u}_s)\beta, \boldsymbol{u}_s\big)_{\mathsf{T}_h} = (\nabla \boldsymbol{u}_s, \boldsymbol{u}_s \otimes \boldsymbol{\beta})_{\mathsf{T}_h}$ and integrating by parts the last term we conclude that

$$(\nabla \cdot (\boldsymbol{u}_s \otimes \boldsymbol{\beta}), \boldsymbol{u}_s)_{\mathsf{T}_h} = \frac{1}{2} \langle (\boldsymbol{u}_s \otimes \boldsymbol{\beta})\boldsymbol{n}, \boldsymbol{u}_s \rangle_{\partial \mathsf{T}_h}. \tag{5.3}$$

Using (5.2) and (5.3), we rewrite (5.1) as

$$\begin{aligned} -(\mathrm{F}_s, \nu \mathrm{L}_s)_{\mathsf{T}_h} \;=\; & \nu \|\mathrm{L}_s\|^2_{\mathsf{D}_h} - \langle \nu \mathrm{L}_s \boldsymbol{n} - p_s \boldsymbol{n} - (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \widehat{\boldsymbol{u}}_s \rangle_{\Gamma_h} \\ & - \langle (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} + \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h} + \frac{1}{2} \langle (\boldsymbol{u}_s \otimes \boldsymbol{\beta})\boldsymbol{n}, \boldsymbol{u}_s \rangle_{\partial \mathsf{T}_h} \\ & - \big\langle \big((\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s) \otimes \boldsymbol{\beta}\big)\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{u}_s \big\rangle_{\partial \mathsf{T}_h}, \end{aligned}$$

or equivalently, rearranging terms,

$$\begin{aligned} -(\mathrm{F}_s, \nu \mathrm{L}_s)_{\mathsf{T}_h} = {} & \nu \|\mathrm{L}_s\|^2_{\mathsf{D}_h} - \frac{1}{2} \langle (\boldsymbol{u}_s \otimes \boldsymbol{\beta})\boldsymbol{n}, \boldsymbol{u}_s \rangle_{\partial \mathsf{T}_h} + \langle (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h} \\ & - \langle \nu \mathrm{L}_s \boldsymbol{n} - p_s \boldsymbol{n} - (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \widehat{\boldsymbol{u}}_s \rangle_{\Gamma_h}. \end{aligned}$$

Finally, since $(\boldsymbol{\beta} \cdot \boldsymbol{n})$ is continuous on $\mathcal{E}_h^0$ and $\widehat{\boldsymbol{u}}_s$ is single valued on $\mathcal{E}_h^0$, we have

$$\begin{aligned} -\frac{1}{2} \langle (\boldsymbol{u}_s \otimes \boldsymbol{\beta})\boldsymbol{n}, \boldsymbol{u}_s \rangle_{\partial \mathsf{T}_h} \;+\; & \langle (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h} \\ = {} & -\frac{1}{2} \langle (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n}, \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h} + \langle \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h} \\ = {} & -\frac{1}{2} \langle (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n}, \widehat{\boldsymbol{u}}_s \rangle_{\Gamma_h} + \langle \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h}. \end{aligned}$$

which concludes the proof. □

Lemma 5.2. *Suppose Assumptions A hold. Then,*

$$\begin{aligned} & \|\mathrm{L}_s\|^2_{\mathsf{D}_h} + \nu^{-1} \langle S_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s \rangle_{\partial \mathsf{T}_h} + \|\boldsymbol{g}_s\|^2_{\Gamma_h, l^{-1}} \\ & \qquad \le \Theta^2_{\mathrm{F}_s} + 24 \nu^{-2} \|p_s\|^2_{\Gamma_h, l} + \nu^{-2} \|\boldsymbol{\beta}\|^2_{L^\infty(\Omega)} \|\boldsymbol{u}_s\|^2_{\mathsf{D}_h}. \end{aligned}$$

Proof. Proceeding exactly as in Lemma 5.1 of [13], it is not difficult to see that we can decompose $\boldsymbol{g}_s(\boldsymbol{x}) = -l(\boldsymbol{x}) \big\{ \delta_{\mathrm{F}_s} + \mathrm{F}_s \boldsymbol{n} + \delta_{\mathrm{L}_s} + \mathrm{L}_s \boldsymbol{n} \big\}(\boldsymbol{x})$, which yields $\mathrm{L}_s \boldsymbol{n} = -\boldsymbol{g}_s(\boldsymbol{x})/l - \delta_{\mathrm{F}_s} - \mathrm{F}_s \boldsymbol{n} - \delta_{\mathrm{L}_s}$. Hence, we rewrite $\mathbb{T}_{\mathrm{L},h}$ as

$$\mathbb{T}_s = \left\langle -\nu \left\{ \boldsymbol{g}_s(\boldsymbol{x})/l + \delta_{\mathrm{F}_s} + \mathrm{F}_s \boldsymbol{n} + \delta_{\mathrm{L}_s} \right\} - p_s \boldsymbol{n} - \frac{1}{2} (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \widehat{\boldsymbol{u}}_s \right\rangle_{\Gamma_h}.$$

Observe now that

$$-\left\langle \frac{1}{2} (\widehat{\boldsymbol{u}}_s \otimes \boldsymbol{\beta})\boldsymbol{n} + \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \widehat{\boldsymbol{u}}_s \right\rangle_{\Gamma_h} = -\left\langle \frac{1}{2} (\boldsymbol{u}_s \otimes \boldsymbol{\beta})\boldsymbol{n} + \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \widehat{\boldsymbol{u}}_s \right\rangle_{\Gamma_h}.$$

Then, since $\widehat{\boldsymbol{u}}_s = \mathsf{P}_{\boldsymbol{M}}\boldsymbol{g}_s$ on Γ_h, we can decompose $\mathbb{T}_s = \sum_{i=1}^{7}\mathbb{T}_s^i$, where

$$\begin{array}{rclrcl}
\mathbb{T}_s^1 &=& -\nu\langle \boldsymbol{g}_s/l, \boldsymbol{g}_s\rangle_{\Gamma_h}, & \mathbb{T}_s^2 &=& -\nu\langle \boldsymbol{g}_s, \delta_{\mathrm{F}_s}\rangle_{\Gamma_h},\\
\mathbb{T}_s^3 &=& -\nu\langle \boldsymbol{g}_s, \mathrm{F}_s\boldsymbol{n}\rangle_{\Gamma_h}, & \mathbb{T}_s^4 &=& -\nu\langle \boldsymbol{g}_s, \delta_{\mathrm{L}_s}\rangle_{\Gamma_h},\\
\mathbb{T}_s^5 &=& -\langle \boldsymbol{g}_s, p_s\boldsymbol{n}\rangle_{\Gamma_h}, & \mathbb{T}_s^6 &=& -\langle \boldsymbol{g}_s, \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s)\rangle_{\Gamma_h},\\
\mathbb{T}_s^7 &=& -\dfrac{1}{2}\langle \boldsymbol{g}_s, (\boldsymbol{u}_s\otimes\boldsymbol{\beta})\boldsymbol{n}\rangle_{\Gamma_h}. & & &
\end{array}$$

Applying the Cauchy-Schwarz inequality to each term we obtain

$$\begin{aligned}
\mathbb{T}_s \leq \; & - \; \nu\|\boldsymbol{g}_s\|^2_{\Gamma_h,l^{-1}} + \nu\|\boldsymbol{g}_s\|_{\Gamma_h,l^{-1}}\Big\{\|\delta_{\mathrm{F}_s}\|_{\Gamma_h,l} + \|\mathrm{F}_s\boldsymbol{n}\|_{\Gamma_h,l} + \|\delta_{\mathrm{L}_s}\|_{\Gamma_h,l}\\
& + \; \nu^{-1}\|p_s\|_{\Gamma_h,l} + \nu^{-1}\|\mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s)\|_{\Gamma_h,l} + \frac{1}{2}\nu^{-1}\|(\boldsymbol{u}_s\otimes\boldsymbol{\beta})\boldsymbol{n}\|_{\Gamma_h,l}\Big\}.
\end{aligned}$$

We conclude the proof recalling the estimate in Lemma 5.2 of [13]

$$\|l^{1/2}\delta_{\mathrm{F}_s}\|_e \leq 3^{-\frac{1}{2}}r_e\|h_e^{\perp}\partial_n(\mathrm{F}_s\boldsymbol{n})\|_{K^e_{ext}}, \tag{5.4}$$

gathering the result stated in Lemma 5.1, inequality (5.4), (2.3) with $\mathrm{G} = \mathrm{L}_s$, Young's inequality, definition (3.7), inequality (2.6), Assumptions A and noticing that $\|\mathrm{L}_s\boldsymbol{n}\|^2_{\Gamma_h,l} \leq R\|\mathrm{L}_s\boldsymbol{n}\|^2_{\Gamma_h,h^{\perp}}$. □

Step 2: Estimate for the pressure

At this point, it is convenient to consider auxiliary constants defined in [9]. We denote by $\boldsymbol{P} : \boldsymbol{H}^1(\mathsf{T}_h) \to \boldsymbol{V}_h$ any projection such that

$$(\boldsymbol{P}\boldsymbol{w} - \boldsymbol{w}, \boldsymbol{v})_K = 0, \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K), \tag{5.5}$$

for all $K \in \mathsf{T}_h$. We define

$$\begin{aligned}
H_p^1 &= C\max\Big\{\nu,\; \|\boldsymbol{\beta}_0\|_{L^\infty(\Omega)} \sup_{\boldsymbol{w}\in\boldsymbol{H}_0^1(\mathsf{D}_h)\setminus\{\boldsymbol{0}\}}\frac{\|\boldsymbol{P}\boldsymbol{w}\|_{\mathsf{T}_h}}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathsf{D}_h)}},\\
&\qquad h|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)} \sup_{\boldsymbol{w}\in\boldsymbol{H}_0^1(\mathsf{D}_h)\setminus\{\boldsymbol{0}\}}\frac{\|\boldsymbol{P}\boldsymbol{w}\|_{\mathsf{D}_h}}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathsf{D}_h)}}\Big\},\\
H_p^2 &= C|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)}\max\Big\{h\sup_{\boldsymbol{w}\in\boldsymbol{H}_0^1(\mathsf{D}_h)\setminus\{\boldsymbol{0}\}}\frac{\|\nabla\boldsymbol{P}\boldsymbol{w}\|_{\mathsf{T}_h}}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathsf{D}_h)}},\; \sup_{\boldsymbol{w}\in\boldsymbol{H}_0^1(\mathsf{D}_h)\setminus\{\boldsymbol{0}\}}\frac{\|\boldsymbol{P}\boldsymbol{w}\|_{\mathsf{T}_h}}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathsf{D}_h)}}\Big\},\\
H_p^3 &= \sup_{\boldsymbol{w}\in\boldsymbol{H}_0^1(\mathsf{D}_h)\setminus\{\boldsymbol{0}\}}\frac{\|\boldsymbol{P}\boldsymbol{w} - \mathsf{P}_M\boldsymbol{w}\|_{\partial\mathsf{T}_h,\tau}}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathsf{D}_h)}},
\end{aligned}$$

and $C_{\tau,\nu,\boldsymbol{\beta}} = \nu^{1/2}\tau^{-1/2}\left\{\tau^{1/2}\nu^{1/2} + \|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)}\big(1 + c_\tau^{-1/2}\|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)}\big)\right\}$.

To estimate the term involving the pressure, we assume that, for each $e \in \mathcal{E}_h^{\partial}$,

(A.4) $2\sqrt{6}\kappa\nu^{-1}\big(3H_p^1 + C_{\tau,\nu,\boldsymbol{\beta}}H_p^3\big)C_{tr}^e r_e^{1/2}\gamma^{1/2} < 1/2$, where $\kappa > 0$ is defined below.

Lemma 5.3. *If Assumptions A and (A.4) hold, then*

$$\begin{aligned}
\|p_s\|_{\mathsf{D}_h} \;\leq\; & \kappa\left\{\nu^{-1}\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\big(3H_p^1 + C_{\tau,\nu,\boldsymbol{\beta}}H_p^3\big) + 2H_p^2\right\}\|\boldsymbol{u}_s\|_{\mathsf{D}_h}\\
& + \kappa\big(5H_p^1 + C_{\tau,\nu,\boldsymbol{\beta}}H_p^3\big)\Theta_{\mathrm{F}_s}.
\end{aligned}$$

Proof. We adapt the proof of Proposition 3.4 in [10]to our setting. Since $p_s \in L^2_0(\mathsf{D}_h)$, there exists $\kappa > 0$ such that

$$\|p_s\|_{\mathsf{D}_h} \le \kappa \sup_{\boldsymbol{w}\in \boldsymbol{H}^1_0(\mathsf{D}_h)\setminus\{\boldsymbol{0}\}} \frac{(p_s, \nabla\cdot \boldsymbol{w})_{\mathsf{D}_h}}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathsf{D}_h)}}. \tag{5.6}$$

We can write the numerator as

$$\begin{aligned}(p_s, \nabla\cdot\boldsymbol{w})_{\mathsf{D}_h} \;=\; & (\nu, \mathrm{L}_s), \nabla \boldsymbol{w})_{\mathsf{T}_h} - (\boldsymbol{u}_s \otimes \boldsymbol{\delta\beta}, \nabla \boldsymbol{P}\boldsymbol{w})_{\mathsf{T}_h} \\ & + (\mathrm{F}_s + \mathrm{L}_s, \boldsymbol{P}\boldsymbol{w}\otimes \boldsymbol{\beta}_0)_{\mathsf{T}_h} + \langle \widehat{\boldsymbol{u}}_s, (\boldsymbol{P}\boldsymbol{w}\otimes\boldsymbol{\delta\beta})\boldsymbol{n}\rangle_{\partial\mathsf{T}_h} \\ & + \langle \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{P}\boldsymbol{w} - \mathsf{P}_M \boldsymbol{w}\rangle_{\partial \mathsf{T}_h}.\end{aligned}$$

The result follows by applying the Cauchy-Schwarz inequality, the trace inequality (2.6), Lemma 5.2 and Assumption (A.4). □

Step 3: A duality argument

We proceed to obtain estimates for the velocity by carrying out a duality argument. First we need to define some terms that will appear in the estimates below. For any arbitrary $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$, define

$$H_L := \nu \max\left\{ \sup_{\boldsymbol{\theta}\in L^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\|\Pi^*\Phi - \Phi\|_{\mathsf{T}_h}}{\|\boldsymbol{\theta}\|_\Omega}, \sup_{\boldsymbol{\theta}\in L^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\|\nabla(\boldsymbol{\phi}_h - \boldsymbol{\phi})\|_{\mathsf{T}_h}}{\|\boldsymbol{\theta}\|_\Omega}\right\}, \tag{5.7}$$

$$H_{\boldsymbol{\beta}} := \|\boldsymbol{\delta\beta}\|_{L^\infty(\Omega)} \sup_{\boldsymbol{\theta}\in L^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\|\nabla \boldsymbol{\phi}_h\|_{\mathsf{T}_h}}{\|\boldsymbol{\theta}\|_\Omega}, \tag{5.8}$$

where Π^* is a suitable projection defined in A.

Propositions 3.6 and 3.7 in [9] state that if $\mathrm{tr}(\mathrm{L}) = 0$, then

$$H_L \le C\nu C_{H_L} h^{\min\{k,1\}} \quad \text{and} \quad H_{\boldsymbol{\beta}} \le Ch|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)}, \tag{5.9}$$

where C_{H_L} is of order $\mathcal{O}(1)$ if h is small enough.

Step 1: Estimate of the velocity

Lemma 5.4. *Let $\boldsymbol{\phi}_h$ be an arbitrary element of $\boldsymbol{V}_h$. Then $(\boldsymbol{u}_s, \boldsymbol{\theta})_{\mathsf{T}_h} = T + T_\phi + \mathbb{T}_{\boldsymbol{u}}$, where*

$$\begin{aligned} T &:= \big(\mathrm{F}_s + \mathrm{L}_s, \nu\Pi^*\Phi - \nu\Phi\big)_{\mathsf{T}_h} - \big(\nu \mathrm{F}_s, \nabla(\boldsymbol{\phi}_h - \boldsymbol{\phi})\big)_{\mathsf{T}_h}, \\ T_\phi &:= \nu\big(\mathrm{F}_s, \nabla\boldsymbol{\phi}_h\big)_{\mathsf{T}_h}, \quad \textit{and} \\ \mathbb{T}_{\boldsymbol{u}} &:= \langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - S(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{u}}_s, \nu\Phi\boldsymbol{n} + \phi\boldsymbol{n}\rangle_{\Gamma_h} - \langle (\widehat{\boldsymbol{u}}_s\otimes\boldsymbol{\beta})\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h}. \end{aligned}$$

Proof. Proceeding as in the proof of Lemma 3.4 of [9], we can write $(\boldsymbol{u}_s, \boldsymbol{\theta})_{\mathsf{T}_h} = T + T_\phi + \Lambda$, where

$$\begin{aligned}\Lambda := & -(\nu\nabla\cdot \mathrm{L}_s, \boldsymbol{\phi} - \boldsymbol{\Pi}^*\boldsymbol{\phi})_{\mathsf{T}_h} + \langle \nu \mathrm{L}_s\boldsymbol{n}, \boldsymbol{\phi} - \boldsymbol{\Pi}^*\boldsymbol{\phi}\rangle_{\partial\mathsf{T}_h} - \langle \widehat{\boldsymbol{u}}_s, \nu\Pi^*\Phi\boldsymbol{n}\rangle_{\partial\mathsf{T}_h} \\ & + \langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - (\widehat{\boldsymbol{u}}_s\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{\Pi}^*\boldsymbol{\phi}\rangle_{\partial\mathsf{T}_h} - (\nabla p_s, \boldsymbol{\Pi}^*\boldsymbol{\phi} - \boldsymbol{\phi})_{\mathsf{T}_h} \\ & + \langle p_s\boldsymbol{n}, \boldsymbol{\Pi}^*\boldsymbol{\phi} - \boldsymbol{\phi}\rangle_{\partial\mathsf{T}_h} - (\boldsymbol{u}_s, \nabla \mathit{\Pi}^*\phi)_{\mathsf{T}_h} \\ & - \langle \boldsymbol{u}_s, \nu(\Phi - \Pi^*\Phi)\boldsymbol{n} + (\phi - \mathit{\Pi}^*\phi)\boldsymbol{n} + \big((\boldsymbol{\phi} - \boldsymbol{\Pi}^*\boldsymbol{u})\otimes\boldsymbol{\beta}\big)\boldsymbol{n}\rangle_{\partial\mathsf{T}_h}.\end{aligned}$$

Using (5.15e) with $\boldsymbol{\mu} = \mathsf{P}_M\boldsymbol{\phi}$, we have that

$$\langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - (\widehat{\boldsymbol{u}}_s\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \mathsf{P}_M\boldsymbol{\phi}\rangle_{\partial\mathsf{T}_h} = \langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - (\widehat{\boldsymbol{u}}_s\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \mathsf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h}.$$

Then, adding and subtracting this term, we obtain

$$\begin{aligned}\Lambda = &-(\nu\nabla\cdot \mathrm{L}_s, \boldsymbol{\phi} - \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi})_{\mathsf{T}_h} + \langle \nu \mathrm{L}_s\boldsymbol{n}, \boldsymbol{\phi} - \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi}\rangle_{\partial\mathsf{T}_h} - \langle \widehat{\boldsymbol{u}}_s, \nu\Pi^*\Phi\boldsymbol{n}\rangle_{\partial\mathsf{T}_h}\\ &+ \langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - (\widehat{\boldsymbol{u}}_s\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi} - \mathsf{P}_M\boldsymbol{\phi}\rangle_{\partial\mathsf{T}_h}\\ &- (\nabla p_s, \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi} - \boldsymbol{\phi})_{\mathsf{T}_h} + \langle p_s\boldsymbol{n}, \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi} - \boldsymbol{\phi}\rangle_{\partial\mathsf{T}_h} - (\boldsymbol{u}_s, \nabla \mathit{\Pi}^*\phi)_{\mathsf{T}_h}\\ &- \langle \boldsymbol{u}_s, \nu(\Phi - \Pi^*\Phi)\boldsymbol{n} + (\phi - \mathit{\Pi}^*\phi)\boldsymbol{n} + \big((\boldsymbol{\phi} - \boldsymbol{\mathit{\Pi}}^*\boldsymbol{u})\otimes\boldsymbol{\beta}\big)\boldsymbol{n}\rangle_{\partial\mathsf{T}_h}\\ &+ \langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - (\widehat{\boldsymbol{u}}_s\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \mathsf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h}.\end{aligned}$$

Using the definition of the projections Π_h^* (cf. (A.1)) and P_M, the equations (3.6) and rearranging terms, we rewrite Λ as

$$\begin{aligned}\Lambda = &-\langle \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s, \nu(\Phi - \Pi^*\Phi)\boldsymbol{n} + \big((\boldsymbol{\phi} - \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi})\otimes\boldsymbol{\beta}\big)\boldsymbol{n} - \mathrm{S}(\boldsymbol{\phi} - \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi})\rangle_{\partial\mathsf{T}_h}\\ &- \langle \widehat{\boldsymbol{u}}_s, \mathit{\Pi}^*\phi\boldsymbol{n}\rangle_{\partial\mathsf{T}_h} - \langle \boldsymbol{u}_s, (\phi - \mathit{\Pi}\phi)\boldsymbol{n}\rangle_{\partial\mathsf{T}_h} - \langle \widehat{\boldsymbol{u}}_s, \nu\Phi\boldsymbol{n} + (\boldsymbol{\beta}\cdot\boldsymbol{n})(\boldsymbol{\phi} - \mathsf{P}_M\boldsymbol{\phi})\rangle_{\partial\mathsf{T}_h}\\ &+ \langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - (\widehat{\boldsymbol{u}}_s\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \mathsf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h}.\end{aligned}$$

Since $\widehat{\boldsymbol{u}}_s$ is single valued on $\mathcal{E}_h^0$ and $\nu\Phi\boldsymbol{n} + (\boldsymbol{\beta}\cdot\boldsymbol{n})(\boldsymbol{\phi} - \mathsf{P}_M\boldsymbol{\phi})$ is continuous on $\mathcal{E}_h^0$, then

$$\langle \widehat{\boldsymbol{u}}_s, \nu\Phi\boldsymbol{n} + (\boldsymbol{\beta}\cdot\boldsymbol{n})(\boldsymbol{\phi} - \mathsf{P}_M\boldsymbol{\phi})\rangle_{\partial\mathsf{T}_h} = \langle \widehat{\boldsymbol{u}}_s, \nu\Phi\boldsymbol{n} + (\boldsymbol{\beta}\cdot\boldsymbol{n})(\boldsymbol{\phi} - \mathsf{P}_M\boldsymbol{\phi})\rangle_{\Gamma_h}.$$

Analogously, $\langle \widehat{\boldsymbol{u}}_s, \phi\boldsymbol{n}\rangle_{\partial\mathsf{T}_h} = \langle \widehat{\boldsymbol{u}}_s, \phi\boldsymbol{n}\rangle_{\Gamma_h}$. Adding and subtracting this term, we rewrite Λ as

$$\begin{aligned}\Lambda = &-\langle \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s, \nu(\Phi - \Pi^*\Phi)\boldsymbol{n} + (\phi - \mathit{\Pi}^*\phi)\boldsymbol{n} + \big((\boldsymbol{\phi} - \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi})\otimes\boldsymbol{\beta}\big)\boldsymbol{n} - \mathrm{S}(\boldsymbol{\phi} - \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi})\rangle_{\partial\mathsf{T}_h}\\ &+ \langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - (\widehat{\boldsymbol{u}}_s\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \mathsf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{u}}_s, \nu\Phi\boldsymbol{n}\rangle_{\Gamma_h}\\ &- \langle \widehat{\boldsymbol{u}}_s, (\boldsymbol{\beta}\cdot\boldsymbol{n})(\boldsymbol{\phi} - \mathsf{P}_M\boldsymbol{\phi})\rangle_{\Gamma_h}.\end{aligned}$$

The first term of the right hand side vanishes thanks to (A.1d) with $\boldsymbol{\mu} = \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s$. Then, rearranging terms, we have

$$\Lambda = \langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{u}}_s, \nu\Phi\boldsymbol{n}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{u}}_s, \phi\boldsymbol{n}\rangle_{\Gamma_h} - \langle (\boldsymbol{\beta}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_s, \boldsymbol{\phi}\rangle_{\Gamma_h},$$

which completes the proof. □

Step 2: A new expression for $\mathbb{T}_{\boldsymbol{u}}$

Lemma 5.5. *We have that* $\mathbb{T}_{\boldsymbol{u}} = \sum_{i=1}^{13} \mathbb{T}_{\boldsymbol{u}}^i$, *where*

$$\begin{aligned}\mathbb{T}_{\boldsymbol{u}}^1 &= -\nu\langle \boldsymbol{g}_s/l, \boldsymbol{\phi} + \ l\nabla\boldsymbol{\phi}\boldsymbol{n}\rangle_{\Gamma_h}, \quad \mathbb{T}_{\boldsymbol{u}}^2 = \nu\langle \boldsymbol{g}_s, \nabla\boldsymbol{\phi}\boldsymbol{n} - \mathsf{P}_M(\nabla\boldsymbol{\phi}\boldsymbol{n})\rangle_{\Gamma_h},\\ \mathbb{T}_{\boldsymbol{u}}^3 &= -\nu\langle \delta_{\mathrm{F}_s}, \boldsymbol{\phi}\rangle_{\Gamma_h}, \quad \mathbb{T}_{\boldsymbol{u}}^4 = -\nu\langle \mathrm{F}_s\boldsymbol{n}, \boldsymbol{\phi} - \mathsf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h}, \mathbb{T}_{\boldsymbol{u}}^5 = -\nu\langle \mathrm{F}_s\boldsymbol{n}, \mathsf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h},\\ \mathbb{T}_{\boldsymbol{u}}^8 &= -\nu\langle \delta_{\mathrm{L}_s}, \boldsymbol{\phi}\rangle_{\Gamma_h}, \quad \mathbb{T}_{\boldsymbol{u}}^9 = -\langle p_s\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h},\\ \mathbb{T}_{\boldsymbol{u}}^{10} &= -\langle \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{\phi}\rangle_{\Gamma_h}, \quad \mathbb{T}_{\boldsymbol{u}}^{11} = -\langle \boldsymbol{g}_s, \mathsf{P}_M(\phi\boldsymbol{n})\rangle_{\Gamma_h},\\ \mathbb{T}_{\boldsymbol{u}}^{12} &= -\frac{1}{2}\left\langle \big(\widehat{\boldsymbol{u}}_s\otimes\boldsymbol{\beta}\big)\boldsymbol{n}, \boldsymbol{\phi}\right\rangle_{\Gamma_h}, \quad \mathbb{T}_{\boldsymbol{u}}^{13} = -\frac{1}{2}\left\langle \big(\boldsymbol{u}_s\otimes\boldsymbol{\beta}\big)\boldsymbol{n}, \boldsymbol{\phi}\right\rangle_{\Gamma_h}.\end{aligned}$$

Proof. Let us begin by noting that we can rewrite $\mathbb{T}_{\boldsymbol{u}}$ as follows:

$$\begin{aligned}\mathbb{T}_{\boldsymbol{u}} =& \langle \nu\mathrm{L}_s\boldsymbol{n} - p_s\boldsymbol{n} - \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{u}}_s, \nu\Phi\boldsymbol{n}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{u}}_s, \phi\boldsymbol{n}\rangle_{\Gamma_h}\\ &- \frac{1}{2}\langle (\boldsymbol{\beta}\cdot\boldsymbol{n})\boldsymbol{u}_s, \boldsymbol{\phi}\rangle_{\Gamma_h} - \frac{1}{2}\langle (\boldsymbol{\beta}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_s, \boldsymbol{\phi}\rangle_{\Gamma_h}\end{aligned}$$

We use now that $\widehat{\boldsymbol{u}}_s = \mathbf{P}_M(\boldsymbol{g}_s)$ on $\partial\mathsf{T}_h$ and the fact that $\mathrm{L}_s\boldsymbol{n} = -(\boldsymbol{g}_s)/l - \delta_{\mathrm{F}_s} - \mathrm{F}_s\boldsymbol{n} - \delta_{\mathrm{L}_s}$ (as we already saw in the proof of Lemma 5.2), to obtain

$$\begin{aligned}\mathbb{T}_{\boldsymbol{u}} =& -\nu\langle \boldsymbol{g}_s/l, \boldsymbol{\phi}\rangle_{\Gamma_h} - \nu\langle \delta_{\mathrm{F}_s}, \boldsymbol{\phi}\rangle_{\Gamma_h} - \nu\langle \mathrm{F}_s\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h} - \nu\langle \delta_{\mathrm{L}_s}, \boldsymbol{\phi}\rangle_{\Gamma_h}\\ &- \langle p_s\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \boldsymbol{g}_s, \nu\mathbf{P}_M(\Phi\boldsymbol{n})\rangle_{\Gamma_h}\\ &- \langle \boldsymbol{g}_s, \mathbf{P}_M(\phi\boldsymbol{n})\rangle_{\Gamma_h} - \frac{1}{2}\langle(\boldsymbol{\beta}\cdot\boldsymbol{n})\boldsymbol{u}_s, \boldsymbol{\phi}\rangle_{\Gamma_h} - \frac{1}{2}\langle(\boldsymbol{\beta}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_s, \boldsymbol{\phi}\rangle_{\Gamma_h}.\end{aligned} \tag{5.10}$$

We can group the first and seventh terms. In addition, adding and subtracting $l\nabla\boldsymbol{\phi}\boldsymbol{n}$, and using (2.8a), we have

$$\begin{aligned}-\nu\langle \boldsymbol{g}_s/l, \boldsymbol{\phi} \quad + \quad l\mathbf{P}_M(\Phi\boldsymbol{n})\rangle_{\Gamma_h} &= -\nu\langle \boldsymbol{g}_s/l, \boldsymbol{\phi} + l\nabla\boldsymbol{\phi}\boldsymbol{n} - l\nabla\boldsymbol{\phi}\boldsymbol{n} + l\mathbf{P}_M(\Phi\boldsymbol{n})\rangle_{\Gamma_h}\\ &= -\nu\langle \boldsymbol{g}_s/l, \boldsymbol{\phi} + l\nabla\boldsymbol{\phi}\boldsymbol{n}\rangle_{\Gamma_h} + \nu\left\langle \boldsymbol{g}_s, \nabla\boldsymbol{\phi}\boldsymbol{n} - \mathbf{P}_M\big(\nabla\boldsymbol{\phi}\boldsymbol{n}\big)\right\rangle_{\Gamma_h}.\end{aligned}$$

On the other hand, we can write $\nu\langle \mathrm{F}_s\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h} = \nu\langle \mathrm{F}_s\boldsymbol{n}, \boldsymbol{\phi} - \mathbf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h} + \nu\langle \mathrm{F}_s\boldsymbol{n}, \mathbf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h}$. Finally, replacing these two expressions in (5.10), we obtain

$$\begin{aligned}\mathbb{T}_{\boldsymbol{u}} =& -\nu\langle \boldsymbol{g}_s/l, \boldsymbol{\phi} + l\nabla\boldsymbol{\phi}\boldsymbol{n}\rangle_{\Gamma_h} + \nu\left\langle \boldsymbol{g}_s, \nabla\boldsymbol{\phi}\boldsymbol{n} - \mathbf{P}_M\big(\nabla\boldsymbol{\phi}\boldsymbol{n}\big)\right\rangle_{\Gamma_h} - \nu\langle \delta_{\mathrm{F}_s}, \boldsymbol{\phi}\rangle_{\Gamma_h}\\ &- \nu\langle \mathrm{F}_s\boldsymbol{n}, \boldsymbol{\phi} - \mathbf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h} - \langle Fs\boldsymbol{n}, \mathbf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h}\\ &- \nu\langle \delta_{\mathrm{L}_s}, \boldsymbol{\phi}\rangle_{\Gamma_h} - \langle p_s\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\Gamma_h} - \langle \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s), \boldsymbol{\phi}\rangle_{\Gamma_h}\\ &- \langle \boldsymbol{g}_s, \mathbf{P}_M(\phi\boldsymbol{n})\rangle_{\Gamma_h} - \frac{1}{2}\langle(\boldsymbol{\beta}\cdot\boldsymbol{n})\boldsymbol{u}_s, \boldsymbol{\phi}\rangle_{\Gamma_h} - \frac{1}{2}\langle(\boldsymbol{\beta}\cdot\boldsymbol{n})\widehat{\boldsymbol{u}}_s, \boldsymbol{\phi}\rangle_{\Gamma_h},\end{aligned}$$

which completes the proof. □

Lemma 5.6. *If Assumptions B, A and (A.4) hold, then*

$$\begin{aligned}|T| &\lesssim \Big(\|\mathrm{F}_s\|_{\mathsf{D}_h} + \Theta_{\mathrm{F}_s} + \|\boldsymbol{u}_s\|_{\mathsf{D}_h}\Big)H_L\|\boldsymbol{\theta}\|_\Omega \quad \textit{and}\\ |\mathbb{T}_{\boldsymbol{u}}| &\lesssim h^{1/2}H(R,h)\left(\Theta_{\mathrm{F}_s} + \|\boldsymbol{u}_s\|_{\mathsf{D}_h}\right)\|\boldsymbol{\theta}\|_\Omega + \sup_{\boldsymbol{\phi}\in\boldsymbol{H}^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{|\langle \mathrm{F}_s\boldsymbol{n}, \mathbf{P}_M(\boldsymbol{\phi})\rangle|}{\|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)}}\|\boldsymbol{\theta}\|_\Omega.\end{aligned}$$

Proof. By the Cauchy-Schwarz inequality, the definition of H_L in (5.7) and (3.8), we have

$$|T| \lesssim \Big(\|\mathrm{F}_s\|_{\mathsf{D}_h} + \|\mathrm{L}_s\|_{\mathsf{D}_h}\Big)H_L\|\boldsymbol{\theta}\|_\Omega \lesssim \Big(\|\mathrm{F}_s\|_{\mathsf{D}_h} + \Theta_{\mathrm{F}_s} + \|\boldsymbol{u}_s\|_{\mathsf{D}_h}\Big)H_L\|\boldsymbol{\theta}\|_\Omega.$$

By Lemma 5.5, we know that $\mathbb{T}_{\boldsymbol{u},h} = \sum_{i=1}^{13}\mathbb{T}^i_{\boldsymbol{u},h}$. We first apply the Cauchy-Schwarz inequality to each term $\mathbb{T}^i_{\boldsymbol{u},h}$. The first estimate follows from estimate in (5.4), Lemma A1, Assumption A, estimates (3.9) and (3.8).

□

Step 4: Proof of Theorem 3.1

Proof. Observe that $\|p_s\|_{\Gamma_h,l} \lesssim \|p_s\|_{\mathsf{D}_h}$. Then, from the estimate of Lemma 5.3 we obtain (3.9). Combining these expressions, from Lemma 5.2 we deduce (3.8). In addition, by taking $\boldsymbol{\theta} = \boldsymbol{u}_s$ in D_h

and zero otherwise in Lemmas 5.4 and 5.6; and recalling (5.9), we have

$$\begin{aligned}\|\boldsymbol{u}_s\|_{\mathsf{D}_h} \lesssim& \Big(\|\mathrm{F}_s\|_{\mathsf{D}_h} + \Theta_{\mathrm{F}_s} + \|\boldsymbol{u}_s\|_{\mathsf{D}_h}\Big) H_L + h^{1/2} H(R,h)\,(\Theta_{\mathrm{F}_s} + \|\boldsymbol{u}_s\|_{\mathsf{D}_h})\\
&+ \sup_{\boldsymbol{\theta}\in \boldsymbol{L}^2(\Omega)\setminus\{\mathbf{0}\}} \frac{\nu(\mathrm{F}_s, \nabla\boldsymbol{\phi}_h)_{\mathsf{D}_h}}{\|\boldsymbol{\theta}\|_\Omega} + \sup_{\boldsymbol{\phi}\in \boldsymbol{H}^2(\Omega)\setminus\{\mathbf{0}\}} \frac{|\langle \mathrm{F}_s\boldsymbol{n}, \mathsf{P}_M(\boldsymbol{\phi})\rangle|}{\|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)}}\\
\lesssim& \Big(\|\mathrm{F}_s\|_{\mathsf{D}_h} + \Theta_{\mathrm{F}_s} + \|\boldsymbol{u}_s\|_{\mathsf{D}_h}\Big) h^{\min\{k,1\}} + h^{1/2} H(R,h)\,(\Theta_{\mathrm{F}_s} + \|\boldsymbol{u}_s\|_{\mathsf{D}_h})\\
&+ \sup_{\boldsymbol{\theta}\in \boldsymbol{L}^2(\Omega)\setminus\{\mathbf{0}\}} \frac{\nu(\mathrm{F}_s, \nabla\boldsymbol{\phi}_h)_{\mathsf{D}_h}}{\|\boldsymbol{\theta}\|_\Omega} + \sup_{\boldsymbol{\phi}\in \boldsymbol{H}^2(\Omega)\setminus\{\mathbf{0}\}} \frac{|\langle \mathrm{F}_s\boldsymbol{n}, \mathsf{P}_M(\boldsymbol{\phi})\rangle|}{\|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)}},\end{aligned}$$

which implies the result for h small enough.

□

5.2 Proof of Theorem 4.1

We first introduce the projection defined in [9] which will be used in our analysis. If $(\mathrm{L}, \boldsymbol{u}, \widetilde{p}) \in \mathrm{H}^1(\mathsf{T}_h) \times \boldsymbol{H}^1(\mathsf{T}_h) \times H^1(\mathsf{T}_h)$, we take its projection $\mathit{\Pi}_h(\mathrm{L}, \boldsymbol{u}, \widetilde{p}) := (\Pi\mathrm{L}, \boldsymbol{\mathit{\Pi}} \boldsymbol{u}, \mathit{\Pi}\widetilde{p})$ as the element of $\mathrm{G}_h \times \boldsymbol{V}_h \times P_h$ defined as follows. On an arbitrary element K of the triangulation T_h, the values of the projected function on the simplex K are determined by requiring that

$$(\nu\Pi\mathrm{L}, \mathrm{G})_K - (\boldsymbol{\mathit{\Pi}}\boldsymbol{u}\otimes\boldsymbol{\beta}, \mathrm{G})_K = (\nu\mathrm{L}, \mathrm{G})_K - (\boldsymbol{u}\otimes\boldsymbol{\beta}, \mathrm{G})_K, \tag{5.11a}$$

$$(\boldsymbol{\mathit{\Pi}}\boldsymbol{u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K, \tag{5.11b}$$

$$(\mathit{\Pi}\widetilde{p}, q)_K = (\widetilde{p}, q)_K, \tag{5.11c}$$

$$\langle \nu\Pi\mathrm{L}\boldsymbol{n} - \mathit{\Pi}\widetilde{p}\boldsymbol{n} - (\mathsf{P}_M\boldsymbol{u}\otimes\boldsymbol{\beta})\boldsymbol{n} - \boldsymbol{\mathit{\Pi}}\boldsymbol{u}, \boldsymbol{\mu}\rangle_e = \langle \nu\mathrm{L}\boldsymbol{n} - \widetilde{p}\boldsymbol{n} - (\boldsymbol{u}\otimes\boldsymbol{\beta})\boldsymbol{n} - S\boldsymbol{u}, \boldsymbol{\mu}\rangle_e, \tag{5.11d}$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{P}_{k-1}(K)\times\boldsymbol{\mathcal{P}}_{k-1}(K)\times\mathcal{P}_{k-1}(K)\times\boldsymbol{\mathcal{P}}_k(e)$ and for all faces e of the simplex K. Thus, we define the projection of the errors $\mathrm{E}^{\mathrm{L}} := \Pi\mathrm{L} - \mathrm{L}_h$, $\boldsymbol{\varepsilon}^{\boldsymbol{u}} := \boldsymbol{\mathit{\Pi}}\boldsymbol{u} - \boldsymbol{u}_h$, $\varepsilon^p := \mathit{\Pi}\widetilde{p} - \widetilde{p}_h$, $\boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}} := \mathsf{P}_M\boldsymbol{u} - \widehat{\boldsymbol{u}}_h$; and the interpolation errors $\mathrm{I}^{\mathrm{L}} := \mathrm{L} - \Pi\mathrm{L}$, $\boldsymbol{I}^{\boldsymbol{u}} := \boldsymbol{u} - \boldsymbol{\mathit{\Pi}}\boldsymbol{u}$, $\boldsymbol{I}^{\boldsymbol{u}}_M := \boldsymbol{u} - \mathsf{P}_M\boldsymbol{u}$, $I^p := \widetilde{p} - \mathit{\Pi}\widetilde{p}$. If τ satisfies (3.5), $(\mathrm{L}, \boldsymbol{u}, \widetilde{p})|_K \in \mathrm{H}^{k+1}(K)\times\boldsymbol{H}^{k+1}(K)\times H^{k+1}(K)$ on each element $K \in \mathsf{T}_h$ and $\mathrm{tr}(\mathrm{L}) = 0$, it is known (Theorem 2.3 in [9]) that the above defined projection satisfies the following properties:

$$\|I^p\|_K \lesssim h_K^{k+1}|\widetilde{p}|_{H^{k+1}(K)}, \tag{5.12a}$$

$$\|\boldsymbol{I}^{\boldsymbol{u}}\|_K \lesssim (\tau\nu + h_K) h_K^{k+1}|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(K)} + h_K^{k+1}|\nabla\cdot(\nu\mathrm{L} - \widetilde{p}\mathrm{I})|_{\mathrm{H}^k(K)}, \tag{5.12b}$$

$$\begin{aligned}\nu\|\mathrm{I}^{\mathrm{L}}\|_K \lesssim& \nu h_K^{k+1}|\mathrm{L}|_{\mathrm{H}^{k+1}(K)} + (\tau\nu + h_K) h_K^{k+1}|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(K)} + \|I^p\|_K\\ &+ \big(\tau\nu + (1+\nu)h_K\big)\|\boldsymbol{I}^{\boldsymbol{u}}\|_K.\end{aligned} \tag{5.12c}$$

Moreover, by a standard scaling argument and the fact that $h_e^\perp \le h_K$, we obtain

$$\|\mathrm{I}^{\mathrm{L}}\boldsymbol{n}\|_{e,h_e^\perp} \lesssim \|\mathrm{I}^{\mathrm{L}}\|_K, \quad \|\boldsymbol{I}^{\boldsymbol{u}}\|_{e,h_e^\perp} \lesssim \|\boldsymbol{I}^{\boldsymbol{u}}\|_K, \quad \text{and} \quad \|I^p\|_{e,h_e^\perp} \lesssim \|I^p\|_K. \tag{5.13}$$

If $(\mathrm{L}, \boldsymbol{u}, \widetilde{p}) \in \mathrm{H}^{k+1}(\Omega)\times\boldsymbol{H}^{k+1}(\Omega)\times H^{k+1}(\Omega)$, according to (3.7), Lemma 3.8 in [13] and the approximation properties (5.12) of $\mathit{\Pi}_h$, we have

$$\Theta_{\mathrm{I^L}} \lesssim C_1^{\mathrm{reg}} h^{k+1}, \tag{5.14}$$

where $C_1^{\mathrm{reg}} = |\mathrm{L}|_{\mathrm{H}^{k+1}(\Omega)} + \nu^{-1}|\widetilde{p}|_{H^{k+1}(\Omega)} + \big(\tau + (\nu^{-1}+1)h\big)|\nu\mathrm{L} - \widetilde{p}\mathrm{I}|_{\mathrm{H}^{k+1}(\Omega)}$
$+ \big(\tau\nu + (1+\nu)h + 1\big)(\tau + \nu^{-1}h)|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(\Omega)}$.

Step 1: The projection of the errors

Lemma 5.7. *The projection of the errors satisfy*

$$
\begin{aligned}
(\mathrm{E^L}, \mathrm{G})_{\mathsf{T}_h} + (\boldsymbol{\varepsilon}^{\boldsymbol{u}}, \nabla\cdot \mathrm{G})_{\mathsf{T}_h} - \langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}, \mathrm{G}\boldsymbol{n}\rangle_{\partial \mathsf{T}_h} &= -\,(\mathrm{I^L}, \mathrm{G})_{\mathsf{T}_h}, && (5.15a)\\
-(\nu\nabla\cdot \mathrm{E^L}, \boldsymbol{v})_{\mathsf{T}_h} + (\nabla\cdot(\boldsymbol{\varepsilon}^{\boldsymbol{u}}\otimes\boldsymbol{\beta}), \boldsymbol{v})_{\mathsf{T}_h} + (\nabla\varepsilon^p, \boldsymbol{v})_{\mathsf{T}_h} & && (5.15b)\\
-\Big\langle \big((\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})\otimes\boldsymbol{\beta}\big)\boldsymbol{n} - \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{v}\Big\rangle_{\partial\mathsf{T}_h} &= 0, &&\\
-(\boldsymbol{\varepsilon}^{\boldsymbol{u}}, \nabla q)_{\mathsf{T}_h} + \langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}, q\boldsymbol{n}\rangle_{\partial\mathsf{T}_h} &= 0, && (5.15c)\\
\langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}, \boldsymbol{\mu}\rangle_{\Gamma_h} &= \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \boldsymbol{\mu}\rangle_{\Gamma_h}, && (5.15d)\\
\langle \nu \mathrm{E^L}\boldsymbol{n} - \varepsilon^p\boldsymbol{n} - (\boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{\mu}\rangle_{\partial\mathsf{T}_h\setminus\Gamma_h} &= 0, && (5.15e)\\
(\varepsilon^p, 1)_{\mathsf{D}_h} &= 0, && (5.15f)
\end{aligned}
$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$.

Proof. Observe that if we insert (3.4g) in (3.4b) and (3.4e), we obtain

$$
\begin{aligned}
(\mathrm{L}_h, \mathrm{G})_{\mathsf{T}_h} + (\boldsymbol{u}_h, \nabla\cdot\mathrm{G})_{\mathsf{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathsf{T}_h} &= 0,\\
(\nu\mathrm{L}_h, \nabla\boldsymbol{v})_{\mathsf{T}_h} - (\boldsymbol{u}_h\otimes\boldsymbol{\beta}, \nabla\boldsymbol{v})_{\mathsf{T}_h} - (\widetilde{p}_h, \nabla\cdot\boldsymbol{v})_{\mathsf{T}_h}&\\
-\langle \nu\mathrm{L}_h\boldsymbol{n} - \widetilde{p}_h\boldsymbol{n} - (\widehat{\boldsymbol{u}}_h\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{v}\rangle_{\partial\mathsf{T}_h} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathsf{T}_h},\\
-(\boldsymbol{u}_h, \nabla q)_{\mathsf{T}_h} + \langle \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}, q\rangle_{\partial\mathsf{T}_h} &= 0,\\
\langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\mu}\rangle_{\Gamma_h} &= \langle \tilde{\boldsymbol{g}}_h, \boldsymbol{\mu}\rangle_{\Gamma_h},\\
\langle \nu\mathrm{L}_h\boldsymbol{n} - \widetilde{p}_h\boldsymbol{n} - (\widehat{\boldsymbol{u}}_h\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{\mu}\rangle_{\partial\mathrm{T}_h\setminus\Gamma_h} &= 0,\\
(\widetilde{p}_h, 1)_{\mathsf{D}_h} &= 0,
\end{aligned}
$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$. On the other hand, using the projection (5.11a)-(5.11d), the exact solution satisfies

$$
\begin{aligned}
(\mathrm{L}, \mathrm{G})_{\mathsf{T}_h} + (\boldsymbol{\mathit{\Pi}}\boldsymbol{u}, \nabla\cdot\mathrm{G})_{\mathsf{T}_h} - \langle \mathsf{P}_M\boldsymbol{u}, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathsf{T}_h} &= 0,\\
(\nu\Pi\mathrm{L}, \nabla\boldsymbol{v})_{\mathsf{T}_h} - (\boldsymbol{\mathit{\Pi}}\boldsymbol{u}\otimes\boldsymbol{\beta}, \nabla\boldsymbol{v})_{\mathsf{T}_h} - (\mathit{\Pi}\widetilde{p}, \nabla\cdot\boldsymbol{v})_{\mathsf{T}_h}&\\
-\langle \nu\Pi\mathrm{L}\boldsymbol{n} - \mathit{\Pi}\widetilde{p}\boldsymbol{n} - (\mathsf{P}_M\boldsymbol{u}\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{\mathit{\Pi}}\boldsymbol{u} - \mathsf{P}_M\boldsymbol{u}), \boldsymbol{v}\rangle_{\partial\mathsf{T}_h} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathsf{T}_h},\\
-(\boldsymbol{\mathit{\Pi}}\boldsymbol{u}, \nabla q)_{\mathsf{T}_h} + \langle \mathsf{P}_M\boldsymbol{u}\cdot\boldsymbol{n}, q\rangle_{\partial\mathsf{T}} &= 0,\\
\langle \mathsf{P}_M\boldsymbol{u}, \boldsymbol{\mu}\rangle_{\Gamma_h} &= \langle \tilde{\boldsymbol{g}}, \boldsymbol{\mu}\rangle_{\Gamma_h},\\
\langle \nu\Pi\mathrm{L}\boldsymbol{n} - \mathit{\Pi}\widetilde{p}\boldsymbol{n} - (\mathsf{P}_M\boldsymbol{u}\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{\mathit{\Pi}}\boldsymbol{u} - \mathsf{P}_M\boldsymbol{u}), \boldsymbol{\mu}\rangle_{\partial\mathrm{T}_h\setminus\Gamma_h} &= 0,\\
(\widetilde{p}, 1)_{\mathsf{D}_h} &= 0,
\end{aligned}
$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$. Subtracting both groups of equations we have

$$
\begin{aligned}
(\mathrm{L} - \mathrm{L}_h, \mathrm{G})_{\mathsf{T}_h} + (\boldsymbol{\varepsilon}^{\boldsymbol{u}}, \nabla\cdot\mathrm{G})_{\mathsf{T}_h} - \langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathsf{T}_h} &= 0,\\
(\nu\mathrm{E^L}, \nabla\boldsymbol{v})_{\mathsf{T}_h} - (\boldsymbol{\varepsilon}^{\boldsymbol{u}}\otimes\boldsymbol{\beta}, \nabla\boldsymbol{v})_{\mathsf{T}_h} - (\varepsilon^p, \nabla\cdot\boldsymbol{v})_{\mathsf{T}_h}&\\
-\langle \nu\mathrm{E^L}\boldsymbol{n} - \varepsilon^p\boldsymbol{n} - (\boldsymbol{\varepsilon}^{\hat{\boldsymbol{u}}}\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{v}\rangle_{\partial\mathsf{T}_h} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathsf{T}_h},\\
-(\boldsymbol{\varepsilon}^{\boldsymbol{u}}, \nabla q)_{\mathsf{T}_h} + \langle \boldsymbol{\varepsilon}^{\hat{\boldsymbol{u}}}\cdot\boldsymbol{n}, q\rangle_{\partial\mathsf{T}} &= 0,\\
\langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}, \boldsymbol{\mu}\rangle_{\Gamma_h} &= \langle \tilde{\boldsymbol{g}} - \tilde{\boldsymbol{g}}_h, \boldsymbol{\mu}\rangle_{\Gamma_h},\\
\langle \nu\mathrm{E^L}\boldsymbol{n} - \varepsilon^p\boldsymbol{n} - (\boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}\otimes\boldsymbol{\beta})\boldsymbol{n} - \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{\mu}\rangle_{\partial\mathrm{T}_h\setminus\Gamma_h} &= 0,\\
(\varepsilon^p, 1)_{\mathsf{D}_h} &= 0,
\end{aligned}
$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$. Integrating by parts the first three terms of the second equation, cancelling and rearranging terms, we rewrite that equation as

$$\begin{aligned} -(\nu\nabla\cdot \mathrm{E}^{\mathrm{L}}, \boldsymbol{v})_{\mathsf{T}_h} + (\nabla\cdot(\boldsymbol{\varepsilon}^{\boldsymbol{u}}\otimes\boldsymbol{\beta}), \boldsymbol{v})_{\mathsf{T}_h} + (\nabla\varepsilon^p, \boldsymbol{v})_{\mathsf{T}_h} & \\ -\Big\langle \big((\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})\otimes\boldsymbol{\beta}\big)\boldsymbol{n} - \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{v}\Big\rangle_{\partial\mathsf{T}_h} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathsf{T}_h}, \end{aligned}$$

which ends the proof. □

We observe that E^{L}, $\boldsymbol{\varepsilon}^{\boldsymbol{u}}$, ε^p and $\boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}$ satisfy (3.6) with $\mathrm{F}_s = \mathrm{I}^{\mathrm{L}}$. Then, by Theorem 3.1 we conclude that

$$\|\mathrm{E}^{\mathrm{L}}\|_{\mathsf{D}_h} + \nu^{-1/2}\langle \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}\rangle^{1/2}_{\partial\mathsf{T}_h} + \|\boldsymbol{g}_s\|_{\Gamma_h, l^{-1}} \lesssim \Theta_{\mathrm{I}^{\mathrm{L}}} + \|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h}.$$

Moreover, if the regularity assumption B hold, there exists $h_0 > 0$ such that

$$\begin{aligned} \|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h} \lesssim\; & \Big(\|\mathrm{I}^{\mathrm{L}}\|_{\mathsf{D}_h} + \Theta_{\mathrm{I}^{\mathrm{L}},}\Big) h^{\min\{k,1\}} + h^{1/2} H(R,h)\Theta_{\mathrm{I}^{\mathrm{L}}} \\ & + \sup_{\boldsymbol{\theta}\in \boldsymbol{L}^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\nu(\mathrm{I}^{\mathrm{L}}, \nabla\boldsymbol{\phi}_h)_{\mathsf{D}_h}}{\|\boldsymbol{\theta}\|_{\mathsf{D}_h}} + \sup_{\boldsymbol{\phi}\in \boldsymbol{H}^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{|\langle \mathrm{I}^{\mathrm{L}}\boldsymbol{n}, \mathbf{P}_M(\boldsymbol{\phi})\rangle|}{\|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)}}, \end{aligned}$$

for all $h < h_0$.

By (5.11a), recalling that $\boldsymbol{\delta\beta} := \boldsymbol{\beta} - \boldsymbol{\beta}_0$ with $\boldsymbol{\beta}_0 \in \boldsymbol{\mathcal{P}}_0(K)$, using the properties of the projector (5.11b) and noticing that $\nabla\boldsymbol{\phi}_h \in \boldsymbol{\mathcal{P}}_{k-1}(K)$, on each element K, we obtain

$$\nu(\mathrm{I}^{\mathrm{L}}, \nabla\boldsymbol{\phi}_h)_{\mathsf{D}_h} = (\boldsymbol{I}^{\boldsymbol{u}}\otimes\boldsymbol{\beta}, \nabla\boldsymbol{\phi}_h)_{\mathsf{D}_h} = (\boldsymbol{I}^{\boldsymbol{u}}\otimes\boldsymbol{\delta\beta}, \nabla\boldsymbol{\phi}_h)_{\mathsf{D}_h} \le H_{\boldsymbol{\beta}}\|\boldsymbol{I}^{\boldsymbol{u}}\|_{\mathsf{D}_h},$$

where for the last inequality we used the definition of $H_{\boldsymbol{\beta}}$ given in (5.8). Moreover, by (5.11d), we have

$$|\langle \mathrm{I}^{\mathrm{L}}\boldsymbol{n}, \mathbf{P}_M(\boldsymbol{\phi})\rangle| = \langle I^p\boldsymbol{n}, \mathbf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h} + \nu\langle \tau\boldsymbol{I}^{\boldsymbol{u}}, \mathbf{P}_M\boldsymbol{\phi}\rangle_{\Gamma_h} + \big\langle(\boldsymbol{I}^{\boldsymbol{u}}\otimes\boldsymbol{\beta})\boldsymbol{n}, \mathbf{P}_M\boldsymbol{\phi}\big\rangle_{\Gamma_h}.$$

Then, we by the Cauchy-Schwarz inequality, estimate in (5.4), Lemma A1, Assumption A, the fact that $\|I^p\boldsymbol{n}\|_{\Gamma_h, l^2} \le Rh^{1/2}\|I^p\boldsymbol{n}\|_{\Gamma_h, h^{\perp}}$, the interpolation properties (5.13) and estimates (3.9) and (3.8), we obtain

$$|\langle \mathrm{I}^{\mathrm{L}}\boldsymbol{n}, \mathbf{P}_M(\boldsymbol{\phi})\rangle| \lesssim \Big\{\nu^{-1}Rh^{1/2}\|I^p\|_{\mathsf{D}_h} + \nu Rh\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\|\boldsymbol{I}^{\boldsymbol{u}}\|_{L^2(\Gamma_h)}\Big\}\|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)}$$

Then,

$$\begin{aligned} \|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h} \lesssim & \Big(\|\mathrm{I}^{\mathrm{L}}\|_{\mathsf{D}_h} + \Theta_{\mathrm{I}^{\mathrm{L}}}\Big) h^{\min\{k,1\}} + h^{1/2}H(R,h)\Theta_{\mathrm{I}^{\mathrm{L}}} + H_{\boldsymbol{\beta}}\|\boldsymbol{I}^{\boldsymbol{u}}\|_{\mathsf{D}_h} \\ & + \Big\{\nu^{-1}Rh^{1/2}\|I^p\|_{\mathsf{D}_h} + \nu Rh\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\|\boldsymbol{I}^{\boldsymbol{u}}\|_{L^2(\Gamma_h)}\Big\}. \end{aligned}$$

Thus, we have obtained the following result:

Lemma 5.8. *If Assumptions A hold, then there exists $h_0 > 0$ such that for all $h < h_0$,*

$$\|\mathrm{E}^{\mathrm{L}}\|_{\mathsf{D}_h} + \nu^{-1/2}\langle \mathrm{S}_{\boldsymbol{\beta}}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}\rangle^{1/2}_{\partial\mathsf{T}_h} + \|\boldsymbol{g}_s\|_{\Gamma_h, l^{-1}} \lesssim \Theta_{\mathrm{I}^{\mathrm{L}}} + \|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h},$$

$$\|\varepsilon^p\|_{\mathsf{D}_h} \lesssim \|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h} + \Theta_{\mathrm{I}^{\mathrm{L}}}.$$

Moreover, if the regularity assumption B hold, then there exists $h_0 > 0$ such that

$$\|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h} \lesssim \Big(\|\mathrm{I}^{\mathrm{L}}\|_{\mathsf{D}_h} + \Theta_{\mathrm{I}^{\mathrm{L}}}\Big) h^{\min\{k,1\}} + h^{1/2} H(R,h)\Theta_{\mathrm{I}^{\mathrm{L}}} + H_{\boldsymbol{\beta}}\|\boldsymbol{I}^{\boldsymbol{u}}\|_{\mathsf{D}_h} + \Big\{\nu^{-1} R h^{1/2} \|I^p\|_{\mathsf{D}_h} + \nu R h \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} \|\boldsymbol{I}^{\boldsymbol{u}}\|_{L^2(\Gamma_h)}\Big\},$$

for all $h < h_0$.

This lemma implies that $\|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h} \lesssim h^{k+3/2}$ if the solution is smooth enough and if τ and R are of order one, since the interpolation errors are of order h^{k+1}. In the case of a polyhedral domain where Γ_h fits Γ, we would have $R = 0$, $\widetilde{\boldsymbol{g}} = \widetilde{\boldsymbol{g}}_h$. As a consequence, $H(R,h) = h^{1/2}$ and the estimate in Lemma 5.6 would read $\|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h} \lesssim h\big(\Theta_{\mathrm{I}^{\mathrm{L}}} + \|\boldsymbol{I}^{\boldsymbol{u}}\|_{\mathsf{D}_h}\big)$. Hence, $\|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h}$ would be of order h^{k+2}, which agrees with the estimate stated in Theorem 2.6 in [9] for the polyhedral case.

Step 2: Conclusion of the proof of Theorem 4.1

First of all, adding and subtracting $\Pi \mathrm{L}$ and $\mathit{\Pi}\widetilde{p}$, using the triangle inequality, estimates (3.9) and (3.8), and recalling the definition of $\Theta_{\mathrm{I}^{\mathrm{L}}}$ in (3.7), we get

$$\|\mathrm{L} - \mathrm{L}_h\|_{\mathsf{D}_h} + \|\widetilde{\boldsymbol{g}} - \widetilde{\boldsymbol{g}}_h\|_{\Gamma_h, l^{-1}} + \nu^{-1}\|\widetilde{p} - \widetilde{p}_h\|_{\mathsf{D}_h} \lesssim \Theta_{\mathrm{I}^{\mathrm{L}}} + \|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h} + \nu^{-1}\|I^p\|_{\mathsf{D}_h}.$$

Moreover, if Assumption B holds, by adding and subtracting $\boldsymbol{\Pi u}$, using triangle inequality and Lemma 5.8, we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathsf{D}_h} \lesssim h^{1/2} H_{\boldsymbol{\varepsilon}^{\boldsymbol{u}}}(R,h)\Theta_{\mathrm{I}^{\mathrm{L}}} + \nu^{-1} R h^{1/2}\|I^p\|_{\mathsf{D}_h} + (\tau R h^{1/2} + h + 1)\|\boldsymbol{I}^{\boldsymbol{u}}\|_{\mathsf{D}_h} + \nu R h \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} \|\boldsymbol{I}^{\boldsymbol{u}}_M\|_{L^2(\Gamma_h)}.$$

Lemma 3.7 in [10] states that $\|\boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}\|_h \lesssim h\|\mathrm{I}^{\mathrm{L}}\|_{\mathsf{D}_h} + h\|\mathrm{E}^{\mathrm{L}}\|_{\mathsf{D}_h} + \|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h}$. This, together with the definition of $\Theta_{\mathrm{I}^{\mathrm{L}}}$, estimate (3.8) and Lemma 5.8, implies

$$\|\mathbf{P}_M \boldsymbol{u} - \widehat{\boldsymbol{u}}_h\|_h \lesssim h^{1/2}\Big\{(h^{1/2} + (1+h) H_{\boldsymbol{\varepsilon}^{\boldsymbol{u}}}(R,h)\Big\}\Theta_{\mathrm{I}^{\mathrm{L}}} + h^{1/2}(1+h)\nu^{-1} R \|I^p\|_{\mathsf{D}_h} + h^{1/2}(1+h)\Big\{(\tau R + h^{1/2})\|\boldsymbol{I}^{\boldsymbol{u}}\|_{\mathsf{D}_h} + \nu R h^{1/2}\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\|\boldsymbol{I}^{\boldsymbol{u}}_M\|_{L^2(\Gamma_h)}\Big\}.$$

The error estimate $\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_{\mathsf{D}_h} \leq \|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_{\mathsf{D}_h} + Ch\|\mathrm{L} - \mathrm{L}_h\|_{\mathsf{D}_h} + Ch^{k+2}|\mathrm{L}|_{\mathrm{H}^{k+1}(\Omega)}$ can be found in [11] and, from Lemma 5.8, it follows that

$$\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_{\mathsf{D}_h} \lesssim h^{1/2} H(R,h)\Theta_{\mathrm{I}^{\mathrm{L}}} + h^{1/2}\Big\{\nu^{-1} R\|I^p\|_{\mathsf{D}_h} + (\tau R + h^{1/2})\|\boldsymbol{I}^{\boldsymbol{u}}\|_{\mathsf{D}_h}\Big\} + h^{1/2}\Big\{h^{1/2}\Big(\nu R\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\|\boldsymbol{I}^{\boldsymbol{u}}_M\|_{L^2(\Gamma_h)} + \|\mathrm{L} - \mathrm{L}_h\|_{\mathsf{D}_h} + h|\mathrm{L}|_{\mathrm{H}^{k+1}(\Omega)}\Big)\Big\}.$$

Hence, since τ is of order one, we observe that if R is of order one, then $H_{\boldsymbol{\varepsilon}^{\boldsymbol{u}}}(R,h)$ is also of order one, whereas if R is of order h, $H_{\boldsymbol{\varepsilon}^{\boldsymbol{u}}}(R,h)$ is of order $h^{1/2}$. The estimates of Theorem 4.1 follow from the fact that $\Theta_{\mathrm{I}^{\mathrm{L}}}$ and the interpolation errors are of order h^{k+1}.

6 Numerical results

In this section we present two-dimensional numerical experiments to validate the theoretical orders of convergence of the approximations provided by the HDG method. In order to satisfy (3.5), in all our experiments we choose $\tau = \dfrac{1}{2\nu}\max_{\boldsymbol{x}\in\mathsf{T}_h}\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n} + 1$. We compute the errors $e_p := \|p - p_h\|_\Omega$, $e_{\boldsymbol{u}} := \|\boldsymbol{u} - \boldsymbol{u}_h\|_\Omega$,

$$e_{\mathrm{L}} := \|\mathrm{L} - \mathrm{L}_h\|_\Omega,\ e_{\widehat{\boldsymbol{u}}} := \left\{ \sum_{K\in \mathsf{D}_h} h_K \|\mathbf{P}_M \boldsymbol{u} - \widehat{\boldsymbol{u}}_h\|_{\partial K} \right\}^{1/2} \quad \text{and } e_{\boldsymbol{u}^*} := \left\{ \|\boldsymbol{u} - \boldsymbol{u}_h^*\|^2_{\mathsf{D}_h} + \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{\mathsf{D}_h^c} \right\}^{1/2}$$

and, in addition, for each variable, we calculate the experimental order of convergence e.o.c. $= -2\dfrac{\log\left(e_{\mathcal{T}_1}/e_{\mathcal{T}_2}\right)}{\log(N_{\mathcal{T}_1}/N_{\mathcal{T}_2})}$, where $e_{\mathcal{T}_1}$ and $e_{\mathcal{T}_2}$ are the errors associated to the corresponding variable considering two consecutive meshes with $N_{\mathcal{T}_1}$ and $N_{\mathcal{T}_2}$ elements, respectively.

6.1 Example 1: $\mathrm{dist}(\Gamma_h, \Gamma)$ of order h^2

In this example, we consider the circular domain $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 0.75^2\}$. The computational boundary Γ_h is constructed by interpolating $\partial\Omega$ by a piecewise linear function and D_h is the domain enclosed by Γ_h. In this case Assumptions D and A are satisfied for h small enough since r_e is of order h. The source term $\boldsymbol{f}$ and boundary data $\boldsymbol{g}$ are such that the exact solution is

$$p(x, y) = \sin(x^2 + y^2) + \frac{\cos(0.75^2) - 1}{0.75^2}, \quad \boldsymbol{u}(x, y) = \begin{bmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{bmatrix},$$

the convective velocity is chosen to be $\boldsymbol{\beta}(x, y) = (1, 1)$ and $\nu = 1$. In this case, since we are interpolating the curved boundary with a piecewise linear computational boundary, Theorem 4.1 predicts that the error are of order h^{k+1} for the pressure, the velocity and its gradient, whereas the numerical trace and postprocessed velocity are predicted to converge with order h^{k+2}. In Table 1 we present the experimental rates of convergence, showing an agreement with the theory.

k	N	e_p	order	$e_{\boldsymbol{u}}$	order	e_{L}	order	$e_{\widehat{\boldsymbol{u}}}$	order	$e_{\boldsymbol{u}^*}$	order
1	60	2.94e-02	–	3.66e-03	–	2.97e-02	–	6.95e-03	–	1.95e-03	–
	129	9.48e-03	2.96	1.35e-03	2.60	1.32e-02	2.11	1.78e-03	3.55	4.83e-04	3.65
	234	4.43e-03	2.55	7.09e-04	2.17	7.46e-03	1.92	7.21e-04	3.04	1.91e-04	3.12
	485	2.27e-03	1.83	3.54e-04	1.91	3.85e-03	1.81	2.64e-04	2.75	7.10e-05	2.71
	918	1.15e-03	2.12	1.75e-04	2.21	1.95e-03	2.13	9.52e-05	3.20	2.53e-05	3.24
	1764	5.69e-04	2.17	8.85e-05	2.08	1.01e-03	2.00	3.53e-05	3.04	9.47e-06	3.01
	3546	2.82e-04	2.01	4.46e-05	1.96	5.21e-04	1.91	1.33e-05	2.80	3.60e-06	2.77
	7089	1.44e-04	1.93	2.24e-05	1.99	2.63e-04	1.97	4.79e-06	2.94	1.32e-06	2.90
	14291	7.20e-05	1.98	1.11e-05	1.99	1.33e-04	1.96	1.72e-06	2.93	4.72e-07	2.93
2	60	6.99e-04	–	1.23e-04	–	8.35e-04	–	1.20e-04	–	4.77e-05	–
	129	1.51e-04	4.00	2.61e-05	4.05	1.90e-04	3.87	1.41e-05	5.59	6.62e-06	5.16
	234	4.79e-05	3.87	9.32e-06	3.46	7.05e-05	3.33	3.71e-06	4.49	1.87e-06	4.25
	485	1.47e-05	3.23	2.94e-06	3.16	2.27e-05	3.11	8.56e-07	4.02	4.30e-07	4.03
	918	5.31e-06	3.20	1.09e-06	3.11	8.36e-06	3.12	2.34e-07	4.06	1.20e-07	4.01
	1764	2.00e-06	2.98	4.15e-07	2.96	3.16e-06	2.98	6.46e-08	3.94	3.34e-08	3.91
	3546	7.40e-07	2.85	1.51e-07	2.90	1.17e-06	2.84	1.80e-08	3.66	9.15e-09	3.71
	7089	2.55e-07	3.08	5.25e-08	3.05	4.08e-07	3.05	4.46e-09	4.03	2.25e-09	4.05
	14291	9.16e-08	2.92	1.88e-08	2.93	1.47e-07	2.91	1.19e-09	3.76	5.89e-10	3.82
3	60	6.42e-05	–	6.71e-06	–	5.12e-05	–	5.65e-06	–	2.06e-06	–
	129	8.83e-06	5.18	1.01e-06	4.95	7.58e-06	4.99	5.17e-07	6.25	2.03e-07	6.06
	234	1.87e-06	5.21	2.77e-07	4.34	1.94e-06	4.57	9.09e-08	5.84	3.94e-08	5.50
	485	3.97e-07	4.26	7.19e-08	3.70	4.84e-07	3.81	1.64e-08	4.69	7.74e-09	4.47
	918	1.06e-07	4.15	1.76e-08	4.42	1.20e-07	4.38	2.96e-09	5.38	1.33e-09	5.51
	1764	2.79e-08	4.07	4.37e-09	4.26	3.04e-08	4.20	5.35e-10	5.23	2.35e-10	5.32
	3546	7.65e-09	3.71	1.15e-09	3.83	8.17e-09	3.76	1.10e-10	4.53	4.62e-11	4.66
	7089	1.88e-09	4.05	2.91e-10	3.96	2.07e-09	3.96	1.96e-11	4.98	8.42e-12	4.92
	14291	4.87e-10	3.85	7.29e-11	3.95	5.28e-10	3.90	3.65e-12	4.80	1.53e-12	4.87

Table 1: History of convergence of Example 1.

6.2 Example 2: $\text{dist}(\Gamma_h, \Gamma)$ of order h

In this example we consider $\Omega := \left\{(x,y) \in \mathbb{R}^2_+ : 1.4 < \sqrt{x^2+y^2} < 2\right\}$ and the exact solution

$$p(x,y) = e^{x^2+y^2} - (e^{2^2} - e^{1.4^2})/(2^2 - 1.4^2), \quad \boldsymbol{u}(x,y) = \begin{bmatrix} \sin(3x)e^y \\ -3\cos(3x)e^y \end{bmatrix}.$$

The convective velocity $\boldsymbol{\beta}$ and the viscosity ν are chosen as in Example 1. The computational domain

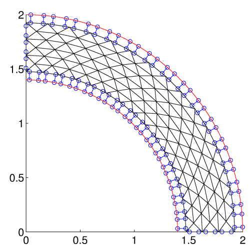

Figure 2: Domain Ω and computational domain of Example 2. Figure obtained from [30].

D_h, as we see in Fig. 2, is constructed in such a way that $r_e = 1$ for all $e \in \Gamma_h \setminus \{(x,y) : x = 0 \vee y = 0\}$. In this case Assumptions D are satisfied and also (A.2) for h small enough. However, it is not possible to ensure that (A.1), (A.3) and (A.4) hold true. If they are, according to Theorem 4.1, the predicted orders of convergence for this example are h^{k+1} for the pressure, the velocity and its gradient, and $h^{k+3/2}$ for the numerical trace and the postprocessed velocity. Actually, in Table 2 we observe this optimal behavior experimentally.

k	N	e_p	order	$e_{\boldsymbol{u}}$	order	e_{L}	order	$e_{\widehat{\boldsymbol{u}}}$	order	$e_{\boldsymbol{u}^*}$	order
1	180	2.30e+00	–	8.38e-02	–	2.09e+00	–	1.71e-01	–	4.73e-02	–
	868	4.32e-01	2.12	1.21e-02	2.46	2.93e-01	2.50	2.44e-02	2.47	7.18e-03	2.40
	3780	9.72e-02	2.03	2.44e-03	2.18	7.27e-02	1.89	3.73e-03	2.56	1.11e-03	2.54
	15748	2.25e-02	2.05	5.46e-04	2.09	1.81e-02	1.95	5.28e-04	2.74	1.58e-04	2.72
	64260	5.28e-03	2.06	1.31e-04	2.03	4.47e-03	1.99	7.16e-05	2.84	2.17e-05	2.83
2	180	2.04e-01	–	5.96e-03	–	1.04e-01	–	1.27e-02	–	3.80e-03	–
	868	2.19e-02	2.84	4.63e-04	3.25	1.30e-02	2.65	1.06e-03	3.16	3.18e-04	3.15
	3780	2.33e-03	3.05	3.90e-05	3.36	1.55e-03	2.89	7.81e-05	3.54	2.35e-05	3.54
	15748	2.54e-04	3.11	3.83e-06	3.25	1.84e-04	2.99	5.40e-06	3.74	1.64e-06	3.73
	64260	2.82e-05	3.13	4.31e-07	3.11	2.18e-05	3.03	3.60e-07	3.85	1.10e-07	3.84
3	180	4.70e-02	–	1.77e-03	–	4.78e-02	–	3.19e-03	–	9.42e-04	–
	868	1.28e-03	4.59	1.97e-05	5.72	6.15e-04	5.53	5.45e-05	5.17	1.63e-05	5.16
	3780	6.86e-05	3.97	7.13e-07	4.51	3.78e-05	3.79	2.02e-06	4.48	6.06e-07	4.48
	15748	3.75e-06	4.08	2.72e-08	4.58	2.33e-06	3.91	7.04e-08	4.70	2.13e-08	4.69

Table 2: History of convergence of Example 2.

6.3 Example 3: Other choice of transferring paths

In this last set of examples, we explore the capabilities of the method in a more general setting where some of the assumption are not necessarily satisfied. We consider a kidney-shaped domain whose boundary satisfies the equation

$$(2[(x+0.5)^2 + y^2] - x - 0.5)^2 - [(x+0.5)^2 + y^2] + 0.1 = 0$$

and a triangulation of a background domain $\mathcal{B}$ such that $\Omega \subset \mathcal{B}$. We set D_h as the union of all the elements inside Ω, as it is shown in Fig. 3 (most-left). In this case, the family of transferring paths is constructed by the procedure in Section 2.4.1 of [14]. We point out that now the tangent vector associated to a transferring path is not, in general, normal to a boundary edge. An example is depicted in Fig. 3. In this case, instead of (3.2), for $\boldsymbol{x}$ in a boundary vertex e, we set

$$\widetilde{\boldsymbol{g}}_h(\boldsymbol{x}) := \boldsymbol{g}(\bar{\boldsymbol{x}}) - \int_0^{l(\boldsymbol{x})} \mathrm{L}_h(\boldsymbol{x} + \boldsymbol{t}(\boldsymbol{x})s\,)\boldsymbol{t}(\boldsymbol{x})\,ds, \tag{6.1}$$

where $\boldsymbol{t}(\boldsymbol{x})$ is the unit vector joining $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$.

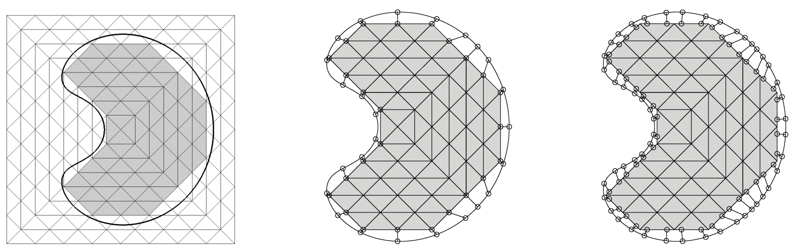

Figure 3: Left: Example of a domain Ω (kidney-shaped), background domain (square) and polygonal subdomain (gray). Middle: *transferring paths* (segments with starting and ending points marked with $\circ$) associated to boundary vertices. Right: *transferring paths* associated to two points on each boundary edge. Figure obtained from [30].

In all the simulations the source term $\boldsymbol{f}$ and boundary data $\boldsymbol{g}$ are such that the exact solution is

$$p(x,y) = \sin(x^2+y^2) - c_\Omega \quad \text{and} \quad \boldsymbol{u}(x,y) = \begin{bmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{bmatrix},$$

where $c_\Omega := \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \sin(x^2+y^2)\,dx\,dy$ was computed numerically considering a extremely fine triangulation that fits the domain.

The results of this experiment, with $\nu = 1$, are displayed in Table 3. We observe that when $k = 2$, the results seem to oscillate. This behavior was also observed in the case of Stokes problem [30]. On the other hand, the orders of convergence of the pressure, velocity and its gradient are $k+1$, and around $k + 3/2$ for the numerical trace and postprocessed velocity.

6.4 Example 4: Application to the the steady–state incompressible Navier–Stokes equations.

In this example, which is not covered by the error estimates of our work, we explore the performance of the method in solving the steady–state incompressible Navier–Stokes equations written as the first order system (1.1), where the second equation is replaced by $-\nu\nabla\cdot\mathrm{L} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \boldsymbol{f}$ in Ω. To that end, we carry out an iterative process. For a fixed mesh, we solve first a Stokes problem and compute the postprocessed velocity, that we denote by $\boldsymbol{u}_h^{*,0}$. Then, for $n = 0, 1, 2, \ldots$, we solve the Oseen equations, using $\boldsymbol{\beta} := \boldsymbol{u}_h^{*,n}$, and compute the postprocessed velocity, denoted by $\boldsymbol{u}_h^{*,n+1}$, until the relative error satisfies $\dfrac{\|\boldsymbol{u}_h^{*,n+1} - \boldsymbol{u}_h^{*,n}\|}{\|\boldsymbol{u}_h^{*,n}\|} < \mathrm{tol}$, where tol is a prescribed value. Once that precision

k	N	e_p	order	$e_{\boldsymbol{u}}$	order	e_{L}	order	$e_{\boldsymbol{u}^*}$	order	$e_{\widehat{\boldsymbol{u}}}$	order
1	28	5.79e-02	–	1.06e-03	–	1.83e-02	–	3.03e-03	–	7.88e-04	–
	154	9.50e-03	2.12	2.00e-04	1.96	4.30e-03	1.70	5.68e-04	1.97	1.52e-04	1.93
	712	1.48e-03	2.43	6.93e-05	1.38	1.22e-03	1.65	6.95e-05	2.74	2.28e-05	2.47
	3054	2.72e-04	2.33	1.53e-05	2.07	2.91e-04	1.97	8.30e-06	2.92	2.79e-06	2.89
	12579	4.49e-05	2.54	3.61e-06	2.04	6.36e-05	2.15	9.19e-07	3.11	3.08e-07	3.11
	50877	1.04e-05	2.09	8.86e-07	2.01	1.54e-05	2.03	1.23e-07	2.88	4.09e-08	2.89
2	28	3.72e-03	–	1.20e-04	–	1.47e-03	–	1.62e-04	–	5.09e-05	–
	154	3.79e-04	2.68	1.22e-05	2.69	1.82e-04	2.45	2.67e-05	2.11	8.59e-06	2.09
	712	3.73e-05	3.03	7.57e-07	3.63	2.28e-05	2.71	1.66e-06	3.63	5.35e-07	3.63
	3054	2.98e-05	0.31	2.72e-07	1.41	1.86e-05	0.28	5.98e-07	1.40	1.91e-07	1.41
	12579	1.19e-07	7.80	2.93e-09	6.40	1.12e-07	7.23	2.54e-09	7.71	8.09e-10	7.72
	50877	1.19e-08	3.30	3.38e-10	3.09	1.13e-08	3.28	1.47e-10	4.08	4.68e-11	4.08
3	28	6.34e-02	–	2.30e-03	–	2.63e-02	–	3.63e-03	–	1.21e-03	–
	154	4.01e-05	8.64	1.04e-06	9.04	2.18e-05	8.32	1.98e-06	8.82	6.46e-07	8.84
	712	1.32e-06	4.46	2.12e-08	5.09	7.91e-07	4.33	4.45e-08	4.96	1.44e-08	4.97
	3054	4.00e-08	4.81	3.72e-10	5.55	2.48e-08	4.75	7.48e-10	5.61	2.39e-10	5.63
	12579	1.93e-09	4.28	1.35e-11	4.69	1.33e-09	4.14	2.85e-11	4.62	8.90e-12	4.65
	50877	1.49e-10	3.67	8.47e-13	3.96	7.67e-11	4.08	1.99e-12	3.81	6.37e-13	3.77

Table 3: History of convergence of Example 3.

is achieved, we move to the next mesh.

We consider the circular domain $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 0.75^2\}$, and the computational domain is constructed by interpolating $\partial\Omega$ by a piecewise linear function, as we did in Example 1. Note that, according to the error estimates stated in Theorem 4.1, the orders of convergence of the velocity and postprocessed velocity when solving Oseen equations are h^{k+1} and h^{k+2}, resp. This implies that $\|\nabla \cdot \boldsymbol{u}_h^*\|_{\mathsf{D}_h}$ converges to zero more quickly than $\|\nabla \cdot \boldsymbol{u}_h\|_{\mathsf{D}_h}$, even though neither $\nabla \cdot \boldsymbol{u}_h^*$ nor $\nabla \cdot \boldsymbol{u}_h^*$ are exactly zero. This is why we use $\boldsymbol{\beta} = \boldsymbol{u}_h^{*,n}$ in the n–th iteration. The orders of convergence are displayed in Table 4. We observe that the results are the optimal for the pressure, velocity and its gradient. Additionally, the expected order h^{k+2} for the numerical trace and postprocessed velocity is also attained.

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A

Lemma A1. *Suppose that the elliptic regularity inequality (2.9) holds. Then*

$$\nu\|\boldsymbol{\phi} - \mathbf{P}_M\boldsymbol{\phi}\|_{\Gamma_h,(h^\perp)^{-1}} \lesssim h\|\boldsymbol{\theta}\|_\Omega, \quad \nu\|\nabla\boldsymbol{\phi}\boldsymbol{n} - \mathbf{P}_M(\nabla\boldsymbol{\phi}\boldsymbol{n})\|_{\Gamma_h,l} \lesssim hR\|\boldsymbol{\theta}\|_\Omega,$$
$$\nu\|\boldsymbol{\phi} + \ l\nabla\boldsymbol{\phi}\boldsymbol{n}\|_{\Gamma_h,l^{-3}} \lesssim \|\boldsymbol{\theta}\|_\Omega, \quad \nu\|\boldsymbol{\phi}\|_{\Gamma_h,l^{-2}} \lesssim \|\boldsymbol{\theta}\|_\Omega, \quad \|\phi\boldsymbol{n}\|_{\Gamma_h,l} \lesssim h^{1/2}R^{1/2}\|\boldsymbol{\theta}\|_\Omega.$$

Proof. Lemma 11 in [30]. □

k	N	it	e_p	order	$e_{\boldsymbol{u}}$	order	e_{L}	order	$e_{\widehat{\boldsymbol{u}}}$	order	$e_{\boldsymbol{u}}^*$	order
1	60	5	9.96e-03	–	3.55e-03	–	1.02e-02	–	2.30e-03	–	6.53e-04	–
	129	4	3.42e-03	2.79	1.46e-03	2.32	4.13e-03	2.37	5.70e-04	3.65	1.49e-04	3.87
	234	4	1.65e-03	2.44	7.99e-04	2.03	2.20e-03	2.11	2.23e-04	3.16	5.61e-05	3.27
	485	4	8.27e-04	1.90	3.98e-04	1.91	1.11e-03	1.88	8.05e-05	2.79	2.05e-05	2.77
	918	3	4.08e-04	2.21	2.02e-04	2.13	5.51e-04	2.20	2.83e-05	3.27	7.19e-06	3.28
	1764	3	2.06e-04	2.09	1.03e-04	2.05	2.83e-04	2.05	1.04e-05	3.08	2.65e-06	3.06
	3546	3	1.02e-04	2.02	5.20e-05	1.96	1.43e-04	1.95	3.80e-06	2.87	9.73e-07	2.86
	7089	2	5.13e-05	1.98	2.61e-05	1.99	7.19e-05	1.98	1.36e-06	2.96	3.54e-07	2.92
2	60	4	4.27e-04	–	1.23e-04	–	4.21e-04	–	5.81e-05	–	1.92e-05	–
	129	3	1.01e-04	3.77	3.13e-05	3.57	9.66e-05	3.85	8.04e-06	5.17	2.71e-06	5.11
	234	3	3.36e-05	3.68	1.13e-05	3.41	3.57e-05	3.34	2.02e-06	4.65	7.36e-07	4.38
	485	3	9.89e-06	3.36	3.73e-06	3.05	1.16e-05	3.09	4.30e-07	4.24	1.69e-07	4.03
	918	2	3.67e-06	3.11	1.40e-06	3.08	4.14e-06	3.22	1.16e-07	4.11	4.50e-08	4.15
	1764	2	1.37e-06	3.03	5.32e-07	2.96	1.50e-06	3.10	3.10e-08	4.04	1.20e-08	4.05
	3546	2	5.22e-07	2.75	1.97e-07	2.85	5.59e-07	2.83	8.85e-09	3.59	3.31e-09	3.68
	7089	2	1.78e-07	3.10	6.88e-08	3.03	1.94e-07	3.05	2.15e-09	4.09	8.11e-10	4.06
3	60	3	5.75e-05	–	9.39e-06	–	4.00e-05	–	5.15e-06	–	1.34e-06	–
	129	2	7.94e-06	5.17	1.52e-06	4.76	5.25e-06	5.31	4.69e-07	6.26	1.19e-07	6.34
	234	2	1.70e-06	5.18	3.98e-07	4.50	1.08e-06	5.30	6.82e-08	6.48	1.65e-08	6.63
	485	2	3.39e-07	4.42	9.40e-08	3.96	2.15e-07	4.44	9.89e-09	5.30	2.49e-09	5.18
	918	1	9.24e-08	4.08	2.47e-08	4.19	5.66e-08	4.19	1.98e-09	5.05	4.69e-10	5.24
	1764	1	2.52e-08	3.98	6.54e-09	4.07	1.50e-08	4.06	3.88e-10	4.99	8.96e-11	5.07
	3546	1	6.96e-09	3.68	1.75e-09	3.78	4.19e-09	3.66	8.12e-11	4.48	1.83e-11	4.55
	7089	1	1.69e-09	4.09	4.33e-10	4.02	1.02e-09	4.08	1.39e-11	5.10	3.20e-12	5.04

Table 4: History of convergence of Example 4. Circular domain.

Additionally, for any element $K \in \mathsf{T}_h$, we introduce the projection $\Pi_h^*(\Phi, \boldsymbol{\phi}, \phi) = (\Pi^*\Phi, \boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi}, \mathit{\Pi}^*\phi) \in \mathrm{P}_k(K) \times \boldsymbol{P}_k(K) \times P_k(K)$ such that

$$\nu(\Pi^*\Phi, \mathrm{G})_K + (\boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi} \otimes \boldsymbol{\beta}, \mathrm{G})_K = \nu(\Phi, \mathrm{G})_K + (\boldsymbol{\phi} \otimes \boldsymbol{\beta}, \mathrm{G})_K, \tag{A.1a}$$

$$(\boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi}, \boldsymbol{v})_K = (\boldsymbol{\phi}, \boldsymbol{v})_K, \tag{A.1b}$$

$$(\mathit{\Pi}^*\phi, q)_K = (\phi, q)_K, \tag{A.1c}$$

$$\langle \nu\Pi^*\Phi\boldsymbol{n} + \mathit{\Pi}^*\phi\boldsymbol{n} + (\boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi} \otimes \boldsymbol{\beta})\boldsymbol{n} - S\boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi}, \boldsymbol{\mu}\rangle_e = \langle \nu\Phi\boldsymbol{n} + \phi\boldsymbol{n} + (\boldsymbol{\phi} \otimes \boldsymbol{\beta})\boldsymbol{n} - S\boldsymbol{\phi}, \boldsymbol{\mu}\rangle_e, \tag{A.1d}$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{P}_{k-1}(K) \times \boldsymbol{\mathcal{P}}_{k-1}(K) \times \mathcal{P}_{k-1}(K) \times \boldsymbol{\mathcal{P}}_k(e)$ and for all the faces e of K. If τ satisfies (3.5), then

$$\|\mathit{\Pi}^*\phi - \phi\|_K \lesssim C h_K^{k+1}|\phi|_{k+1,K}, \tag{A.2a}$$

$$\|\boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi} - \boldsymbol{\phi}\|_K \lesssim (\tau\nu + 1 + h_K)h_K^{k+1}|\boldsymbol{\phi}|_{k+1,K} + h_K^{k+1}|\nabla\cdot(\nu\Phi + \phi\mathrm{I})|_{k,K}, \tag{A.2b}$$

$$\nu\|\Pi^*\Phi - \Phi\|_K \lesssim \nu h_K^{k+1}|\Phi|_{k+1,K} + (\tau\nu + h_K)h_K^{k+1}|\boldsymbol{\phi}|_{k+1,K} + (\tau\nu + (1+\nu) + h_K)\|\boldsymbol{\mathit{\Pi}}^*\boldsymbol{\phi} - \boldsymbol{\phi}\|_K + \|\mathit{\Pi}^*\phi - \phi\|_K. \tag{A.2c}$$

For a proof we refer to Theorem 2.5 in [9].

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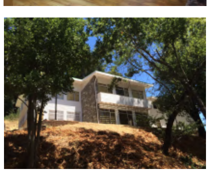

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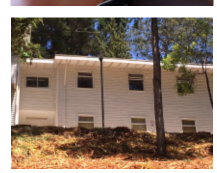
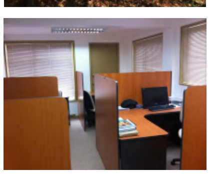

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