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Numerical analysis of a penalty approach for the solution of a transient eddy current problem

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Abstract

The aim of this paper is to propose and analyze a numerical method to solve transient eddy current problems formulated in terms of the magnetic field intensity. Space discretization is based on Nédélec edge elements, while a backward Euler scheme is used for time discretization; the curl-free constraint in the dielectric domain is imposed by means of a penalty strategy. Convergence of the penalized problem as the penalty parameter goes to zero is proved for the continuous and the discrete problems, for the latter uniformly in the discretization parameters. Optimal order error estimates for the convergence of the discrete penalized problem with respect to the penalty and the discretization parameters are also proved. Finally, some numerical tests are reported to assess the performance of this approach.

Keywords: Eddy current problems, transient electromagnetic problems, edge finite elements, penalty formulation.
2010 MSC: 78M10, 65M60

1. Introduction

In this paper, we analyze a finite element method to solve a three-dimensional transient eddy current problem in a bounded domain that contains conductors and dielectrics, when the source current density is known in a part of the domain. This situation arises in many practical applications in which the current density is known for instance in a coil and the main interest is the computation of the induced currents in some neighboring conducting pieces. This problem has been extensively studied in the literature, specially in the harmonic regime; see, for instance, the complete monograph [4], which deals with this situation in quite general topological frameworks and provides the mathematical and numerical analysis for a wide variety of formulations combining vector fields and/or potentials. The present paper is devoted to analyze this problem in the time-dependent case by using a variational formulation based only on the magnetic field $\boldsymbol{H}$ and introducing a penalty approach to deal with the Ampére's law in the non conducting regions.

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Let us remark that the eddy current approximation consists of neglecting the term corresponding to the electric displacement in Ampére's law and, consequently, a distinctive feature of this model is that it is needed to deal with an algebraic constraint on the curl of the magnetic field in the dielectric domain; namely, $\boldsymbol{H}$ must satisfy

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}_{\mathrm{S}}, \tag{1.1}$$

$\boldsymbol{J}_{\mathrm{S}}$ being the known current density whose support is contained in the dielectric domain.

The two most popular options in the literature to deal with this constraint are either to impose it in a weak form by using a Lagrange multiplier [3] (and consequently adding unknowns to the system) or to introduce a scalar potential in the dielectric domain [2, 11]. The last option is clearly less expensive, but it requires two additional steps: first, to determine a particular vector potential $\widehat{\boldsymbol{H}}$ such that $\mathbf{curl}\,\widehat{\boldsymbol{H}} = \boldsymbol{J}_{\mathrm{S}}$ and, secondly, to deal with the constraint $\mathbf{curl}(\boldsymbol{H} - \widehat{\boldsymbol{H}}) = \mathbf{0}$ by introducing a multivalued scalar potential. From the numerical point of view, the first option leads to a discrete mixed problem that can be quite expensive, because it involves several vector fields as unknowns. In its turn, the scalar potential alternative requires an efficient way to deal with the multivalued potential, which is not trivial at all in general topological configurations with non-simply connected dielectric domains [1, 2]. (See also [11] for an alternative approach based on graph techniques.)

To overcome these difficulties, the penalty approach is an alternative that has been also used in fluid dynamics and solid mechanics to deal with incompressibility conditions (see, for instance, [17] and references therein). In the eddy current context, it is based on relaxing the constraint (1.1), which can be interpreted as assuming that the dielectric is not a perfect insulator but a *fake conductor*; namely, a material with a very low conductivity. This approach has been already used in electrical engineering to approximate either magnetostatic [6, 7] or eddy current problems [12, 18]. We also refer the reader to [8], where a similar approach is used in an axisymmetric framework to avoid introducing additional unknowns in the dielectric domain. More recently, it was shown in [10] that this penalty approach is an interesting technique to deal with eddy current problems that involve moving conductors, because, in spite of the fact that the conducting domain changes over the time, it is possible to design an efficient numerical scheme without the need of re-meshing.

The main goal of this paper is the convergence analysis of the proposed penalty scheme in the context of the fully-discrete approximation of transient eddy current problems. Similar analyses have been performed in the case of Navier-Stokes equations in [17] only for a time discretization and in [14] for a fully-discrete scheme. Although our analysis has been inspired by these references, the road taken to perform it is not the same. Our road consists of combining optimal order error estimates of a fully-discrete scheme with respect to the discretization parameters and convergence of the penalty fully-discrete scheme as the penalty parameter goes to zero, the latter uniformly in the discretization parameters. Moreover, in order to obtain an optimal order error estimate in terms of the penalty parameter under appropriate assumptions, we resort again to [17] and adapt to our discrete problem arguments proposed in that reference for the analysis of the penalized continuous Navier-Stokes equations.

The outline of the paper is as follows. In Section 2, we introduce the time-dependent eddy current problem in a bounded domain with essential homogeneous boundary conditions and derive a weak formulation in terms of the magnetic field, which is proved to be well posed. Next, in Section 3, we introduce a mixed formulation to impose the Ampére's law in a weak form. This is the starting point from which we derive the penalty approach that we propose for the continuous model. Proofs of its well-posedness and convergence of the solution of the penalized problem to that of the original one as the penalty parameter ε goes to zero are postponed to an appendix. In Section 4, we introduce full discretizations of the mixed and the penalized problems by means of backward Euler schemes for time discretization, while the magnetic field and the Lagrange multiplier are approximated by Nédélec edge elements and curls of Nédélec elements, respectively. Section 5 is devoted to analyze the convergence of the solution of the discrete mixed problem as the discretization parameters goes to zero. Let us remark that although this is just an intermediate result in our analysis, it is interesting by itself and its convergence analysis had not been previously performed in the time-dependent case. In Section 6, we prove a preliminary result of convergence, which states that the solution of the penalized problem converges to that of the mixed problem with suboptimal rate $\mathcal{O}(\sqrt{\varepsilon})$ as ε tends to zero, uniformly in the discretization parameters. This is improved in Theorem 6 from Section 7 (which is the main result of this paper), where we prove an optimal rate of convergence $\mathcal{O}(\varepsilon)$ in the case of vanishing initial data and a smooth source current. Finally, in Section 8, we report some numerical results obtained with a MATLAB code that implements the penalty technique, which allows us to illustrate the above mentioned convergence results.

Throughout this paper, we use classical Sobolev as well as other well known spaces like, for instance, $\mathrm{H}_0(\mathbf{curl};\omega) := \{\boldsymbol{G} \in \mathrm{H}(\mathbf{curl};\omega) : \ \boldsymbol{G} \times \boldsymbol{n} = \mathbf{0} \text{ on } \partial\omega\}$, $\mathrm{H}_\gamma(\mathrm{div}^0;\omega) := \{\boldsymbol{F} \in \mathrm{H}(\mathrm{div};\omega) : \mathrm{div}\,\boldsymbol{F} = 0 \text{ in } \omega \text{ and } \boldsymbol{F}\cdot\boldsymbol{n} = 0 \text{ on } \gamma\}$, etc., for any domain ω and any connected component γ of $\partial\omega$. Here and thereafter, we use boldface letters to denote vector fields and variables as well as vector-valued operators. Finally, C will denote strictly positive generic constants, not necessarily the same at each occurrence.

2. A magnetic field formulation

Let us consider a coil, which occupies a three-dimensional domain Ω_{S} and carries a given time-dependent current density $\boldsymbol{J}_{\mathrm{S}}$. This current creates a varying electromagnetic field in the whole space, which in turn induces an eddy current in a neighboring conducting piece that occupies another domain Ω_{C} (the *conducting domain*). Let Ω be a simply connected bounded domain with a Lipschitz continuous connected boundary Γ that contains the coil and the conducting piece. We are interested in computing the eddy currents in the conducting piece over time. We assume that $\bar{\Omega}_{\mathrm{S}} \cup \bar{\Omega}_{\mathrm{C}} \subset \Omega$ and $\bar{\Omega}_{\mathrm{S}} \cap \bar{\Omega}_{\mathrm{C}} = \emptyset$. We denote $\Omega_{\mathrm{D}} := \Omega \setminus \bar{\Omega}_{\mathrm{C}}$ (the *dielectric domain*) and notice that $\bar{\Omega}_{\mathrm{S}} \subset \Omega_{\mathrm{D}}$, so that $\boldsymbol{J}_{\mathrm{S}}|_{\Omega_{\mathrm{C}}} = \mathbf{0}$ (see Figure 1).

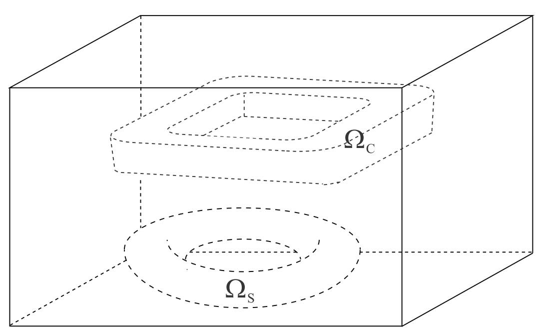

Figure 1: Sketch of the domain.

The time-dependent eddy current equations read as follows:

$$\partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \qquad \text{in } (0,T)\times\Omega, \tag{2.1a}$$

$$\mathbf{curl}\,\boldsymbol{H} = \sigma\boldsymbol{E} + \boldsymbol{J}_{\mathrm{S}} \qquad \text{in } [0,T]\times\Omega, \tag{2.1b}$$

$$\operatorname{div}(\mu\boldsymbol{H}) = 0 \qquad \text{in } [0,T]\times\Omega, \tag{2.1c}$$

where $\boldsymbol{E}(t,\boldsymbol{x})$ is the electric field, $\boldsymbol{H}(t,\boldsymbol{x})$ the magnetic field, $\mu(\boldsymbol{x})$ the magnetic permeability and $\sigma(\boldsymbol{x})$ the electric conductivity. The source current $\boldsymbol{J}_{\mathrm{S}}$ is a given data, that we assume satisfies $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\operatorname{div}^0;\Omega_{\mathrm{S}}))$. This implies that its extension by zero belongs to $\mathrm{H}^1(0,T;\mathrm{H}_0(\operatorname{div}^0;\Omega))$ too. On the other hand, we assume that the coefficients μ and σ are time-independent and that there exist positive constants $\underline{\mu}$, $\overline{\mu}$, $\overline{\sigma}$ and $\underline{\sigma}$ such that

$$0 < \underline{\mu} \leq \mu(\boldsymbol{x}) \leq \overline{\mu}, \qquad \boldsymbol{x}\in\Omega,$$

$$0 < \underline{\sigma} \leq \sigma(\boldsymbol{x}) \leq \overline{\sigma}, \qquad \boldsymbol{x}\in\Omega_{\mathrm{C}} \qquad \text{and} \qquad \sigma = 0 \text{ in } \Omega_{\mathrm{D}}.$$

These equations must be completed with suitable boundary and initial conditions that guarantee the well-posedness of the problem. A similar problem but defined in the whole space $\mathbb{R}^3$ has been dealt with in [15] by using the magnetic field as the main unknown. In our case, we restrict our analysis to a bounded domain Ω and consider the following homogeneous boundary condition:

$$\boldsymbol{H}\times\boldsymbol{n} = \mathbf{0} \qquad \text{in } [0,T]\times\Gamma. \tag{2.2}$$

Let us remark that the modeling error arisen from imposing this approximate boundary condition is negligible, provided the domain Ω is chosen with its boundary sufficiently far from Ω_{S} and Ω_{C}.

Finally, we must add an appropriate initial condition

$$\boldsymbol{H}(0,\boldsymbol{x}) = \boldsymbol{H}_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Omega. \tag{2.3}$$

When the electromagnetic setting is turned on at the initial time $t=0$, the natural initial data is $\boldsymbol{H}_0 = \mathbf{0}$, whereas $\boldsymbol{J}_{\mathrm{S}}(0) = \mathbf{0}$, too. Although this is the most usual case in the applications, almost all the results of this paper remain true for a more general initial data $\boldsymbol{H}_0$ satisfying

$$\boldsymbol{H}_0 \in \mathrm{H}_0(\mathbf{curl};\Omega): \qquad \mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{J}_{\mathrm{S}}(0) \quad \text{in } \Omega_{\mathrm{D}} \qquad \text{and} \qquad \operatorname{div}(\mu\boldsymbol{H}_0) = 0 \quad \text{in } \Omega. \tag{2.4}$$

The first constraint is a compatibility condition among the problem data $\boldsymbol{H}_0$ and $\boldsymbol{J}_{\mathrm{S}}$; it means that Equation (2.1b) holds true in Ω_{D} at the initial time. The second constraint means that Equation (2.1c) holds true at the initial time. Note that by virtue of (2.1a), this constraint is actually equivalent to (2.1c).

For the sake of generality, for most of the paper we will consider as initial data an arbitrary $\boldsymbol{H}_0$ satisfying (2.4). We will assume $\boldsymbol{H}_0 = \mathbf{0}$ only when it will be actually needed. In such a case, we will also report what holds for a more general $\boldsymbol{H}_0$.

To derive a first weak formulation of this problem, we introduce the following space:

$$\boldsymbol{\mathcal{Y}} := \{\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega):\ \mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}}\}.$$

By testing (2.1a) with $\boldsymbol{G}\in\boldsymbol{\mathcal{Y}}$, integrating by parts and using (2.1b) to eliminate $\boldsymbol{E}$ in terms of $\mathbf{curl}\,\boldsymbol{H}$ in Ω_{C}, we arrive at the following.

Problem 1. *Find* $\boldsymbol{H} \in \mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ *such that*

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}_{\mathrm{S}} \qquad \text{in } [0,T]\times\Omega_{\mathrm{D}}, \tag{2.5a}$$

$$\int_\Omega \mu\,\partial_t\boldsymbol{H}\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{G} = 0 \qquad \forall \boldsymbol{G}\in\boldsymbol{\mathcal{Y}},\ \text{a.e. in } [0,T], \tag{2.5b}$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0 \qquad \text{in } \Omega. \tag{2.5c}$$

Theorem 1. *Problem 1 has a unique solution* $\boldsymbol{H}$*. Furthermore,* $\boldsymbol{H} \in \mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ *and there exists* $C>0$ *independent of* $\boldsymbol{H}_0$ *and* $\boldsymbol{J}_{\mathrm{S}}$ *such that*

$$\|\boldsymbol{H}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}(\mathbf{curl};\Omega))} + \|\partial_t\boldsymbol{H}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)} \le C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}\right\}. \tag{2.6}$$

Proof. The proof follows similar lines to those in [9, Section 3.2]. The first step is to build $\widehat{\boldsymbol{H}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ such that $\mathbf{curl}\,\widehat{\boldsymbol{H}} = \boldsymbol{J}_{\mathrm{S}}$. To define it, we use that $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega))$. Then, since Ω is simply connected, for almost all $t\in[0,T]$ there exists a unique divergence-free vector potential $\boldsymbol{Q}(t)\in\mathrm{H}_0(\mathbf{curl};\Omega)$ such that (see [13, Theorem I.3.6])

$$\mathbf{curl}\,\boldsymbol{Q}(t) = \partial_t\boldsymbol{J}_{\mathrm{S}}(t) \qquad \text{in } \Omega \tag{2.7}$$

and there exists $C>0$ such that $\|\boldsymbol{Q}(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le C\,\|\partial_t\boldsymbol{J}_{\mathrm{S}}(t)\|_{\mathrm{L}^2(\Omega_{\mathrm{S}})^3}$ On the other hand, since $\boldsymbol{J}_{\mathrm{S}}(0)\in\mathrm{H}_0(\mathrm{div}^0;\Omega)$ too, there also exists a unique divergence-free vector potential $\boldsymbol{R}\in\mathrm{H}_0(\mathbf{curl};\Omega)$ such that

$$\mathbf{curl}\,\boldsymbol{R} = \boldsymbol{J}_{\mathrm{S}}(0) \qquad \text{in } \Omega \tag{2.8}$$

and $\|\boldsymbol{R}\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le C\,\|\boldsymbol{J}_{\mathrm{S}}(0)\|_{\mathrm{L}^2(\Omega_{\mathrm{S}})^3}$. Then, we define

$$\widehat{\boldsymbol{H}}(t) := \boldsymbol{R} + \int_0^t \boldsymbol{Q}(s)\,ds, \tag{2.9}$$

so that $\partial_t\widehat{\boldsymbol{H}}(t) = \boldsymbol{Q}(t)$ in the sense of distributions in $(0,T)$ (see [19, Remark 131(b)], for instance). Hence, $\int_0^T \|\partial_t\widehat{\boldsymbol{H}}(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,dt = \int_0^T \|\boldsymbol{Q}(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,dt \le C\int_0^T \|\partial_t\boldsymbol{J}_{\mathrm{S}}(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{S}})^3}\,dt$. On the other hand, straightforward computations allow us to bound $\int_0^T \|\widehat{\boldsymbol{H}}(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,dt$ too, so that we conclude that $\widehat{\boldsymbol{H}}\in\mathrm{H}^1(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ and

$$\|\widehat{\boldsymbol{H}}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}(\mathbf{curl};\Omega))} \le C\,\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}. \tag{2.10}$$

Furthermore, from (2.9), (2.8), (2.7) and [19, Theorems 111 & 127] we have that

$$\mathbf{curl}\,\widehat{\boldsymbol{H}}(t) = \mathbf{curl}\,\boldsymbol{R} + \int_0^t \mathbf{curl}\,\boldsymbol{Q}(s)\,ds = \boldsymbol{J}_{\mathrm{S}}(t) \qquad \text{in } [0,T]\times\Omega. \tag{2.11}$$

Next, we define $\mathcal{H}_{\boldsymbol{\mathcal{Y}}}$ as the closure of $\boldsymbol{\mathcal{Y}}$ in $\mathrm{L}^2(\Omega)^3$. It is shown in [9, Lemma 3.2] that $\mathcal{H}_{\boldsymbol{\mathcal{Y}}} = \left\{\boldsymbol{G}\in\mathrm{L}^2(\Omega)^3 : \mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}}\right\}$. Then, by writing $\boldsymbol{H} = \widetilde{\boldsymbol{H}} + \widehat{\boldsymbol{H}}$, it is easy to check that Problem 1 is equivalent to finding $\widetilde{\boldsymbol{H}}\in\mathrm{L}^2(0,T;\boldsymbol{\mathcal{Y}})\cap\mathrm{H}^1(0,T;\mathcal{H}_{\boldsymbol{\mathcal{Y}}})$ such that

$$\int_\Omega \mu\,\partial_t\widetilde{\boldsymbol{H}}\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}\cdot\mathbf{curl}\,\boldsymbol{G} = -\int_\Omega \mu\,\partial_t\widehat{\boldsymbol{H}}\cdot\boldsymbol{G} \qquad \forall\boldsymbol{G}\in\boldsymbol{\mathcal{Y}},$$

$$\widetilde{\boldsymbol{H}}(0) = \boldsymbol{H}_0 - \widehat{\boldsymbol{H}}_0.$$

By applying similar techniques to those used in the proof of [9, Corollary A.2], we are able to prove that this problem has a unique solution $\widetilde{\boldsymbol{H}}$, that this solution belongs to $\mathrm{L}^\infty(0,T;\boldsymbol{\mathcal{Y}})$ and that there exists $C>0$ independent of $\boldsymbol{H}_0$ and $\boldsymbol{J}_{\mathrm{S}}$ such that

$$\|\widetilde{\boldsymbol{H}}\|^2_{\mathrm{L}^\infty(0,T;\boldsymbol{\mathcal{Y}})} + \|\partial_t\widetilde{\boldsymbol{H}}\|^2_{\mathrm{L}^2(0,T;\mathcal{H}_{\boldsymbol{\mathcal{Y}}})} \le C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}\right\}.$$

It is clear that $\boldsymbol{H}=\widetilde{\boldsymbol{H}}+\widehat{\boldsymbol{H}}$ is a solution to Problem 1. Moreover, $\boldsymbol{H}\in\mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ and, as a consequence of the previous inequality and (2.10), we derive (2.6). Since the problem is linear, the uniqueness of the solution is an immediate consequence of this estimate. □

3. Mixed and penalty approaches

To solve Problem 1, it is necessary to impose somehow the constraint (2.5a). A possible way of doing it is by means of a Lagrange multiplier in the dielectric domain, which leads to a mixed problem (see [3] and [15] for further details in the harmonic and time-domain regimes, respectively). To derive this mixed formulation, we test (2.1a) with $\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega)$, integrate by parts and use again (2.1b) to substitute $\boldsymbol{E}$ in terms of $\mathbf{curl}\,\boldsymbol{H}$ in Ω_{C}. The resulting equation combined with a weak form of (2.1b) in Ω_{D} yield the following.

Problem 2. *Find* $\boldsymbol{H}\in\mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))\cap\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ *and* $\boldsymbol{E}\in\mathrm{L}^2(0,T;\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}}))$ *such that*

$$\int_\Omega \mu\,\partial_t\boldsymbol{H}\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{G} + \int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{E} = 0 \qquad \forall\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega),\ \textit{a.e. in } [0,T],$$

$$\int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{F} = \int_{\Omega_{\mathrm{S}}}\boldsymbol{J}_{\mathrm{S}}\cdot\boldsymbol{F} \qquad \forall\boldsymbol{F}\in\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}}),\ \textit{a.e. in } [0,T],$$

$$\boldsymbol{H}(0)=\boldsymbol{H}_0.$$

The choice of the space $\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}})$ for the Lagrange multiplier is justified by the following result, in which we prove that Problem 2 is actually equivalent to Problem 1.

Theorem 2. *Let* $\boldsymbol{H}$ *be the solution to Problem 1. Then, there exists* $\boldsymbol{E}\in\mathrm{L}^2(0,T;\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}}))$ *such that* $(\boldsymbol{H},\boldsymbol{E})$ *is the unique solution of Problem 2. Moreover, there exists* $C>0$ *independent of* $\boldsymbol{H}_0$ *and* $\boldsymbol{J}_{\mathrm{S}}$ *such that*

$$\|\boldsymbol{E}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{D}})^3)} \le C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}\right\}. \tag{3.1}$$

Proof. It is easy to check that the following *inf-sup* condition holds true (see (4.8) in the proof of Theorem 4.3 from [3]):

$$\sup_{\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega)}\frac{\int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{F}}{\|\boldsymbol{G}\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \ge \beta\,\|\boldsymbol{F}\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \qquad \forall\boldsymbol{F}\in\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}}). \tag{3.2}$$

Then, let $\boldsymbol{H}$ be the solution of Problem 1. By virtue of (2.5b) and the above *inf-sup* condition, we are in a position to use [13, Lemma I.4.1(ii)] to derive that for almost all $t\in[0,T]$ there exists a unique $\boldsymbol{E}(t)\in\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}})$ such that

$$\int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{E}(t) = -\int_\Omega\mu\,\partial_t\boldsymbol{H}(t)\cdot\boldsymbol{G} - \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}(t)\cdot\mathbf{curl}\,\boldsymbol{G} \qquad \forall\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega). \tag{3.3}$$

On the other hand, since $\mathbf{curl}\,\boldsymbol{H}|_{\Omega_{\mathrm{D}}}$ and $\boldsymbol{J}_{\mathrm{S}}|_{\Omega_{\mathrm{D}}}$ both belong to $\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}})$, equation (2.5a) is clearly equivalent to the second equation from Problem 2. Consequently, $(\boldsymbol{H},\boldsymbol{E})$ is a solution to Problem 2.

Furthermore, from (3.2) and (3.3), it is immediate to check that there exists $C>0$ such that $\|\boldsymbol{E}(t)\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \le C\left\{\|\partial_t \boldsymbol{H}(t)\|_{\mathrm{L}^2(\Omega)^3} + \|\mathbf{curl}\,\boldsymbol{H}(t)\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\right\}$, which combined with (2.6) yield (3.1).

There only remains to prove that this problem has at most one solution. To do this, let $(\breve{\boldsymbol{H}},\breve{\boldsymbol{E}})$ be a solution of Problem 2 with data $\boldsymbol{J}_{\mathrm{S}}=\mathbf{0}$ and $\boldsymbol{H}_0=\mathbf{0}$. We only have to prove that $(\breve{\boldsymbol{H}},\breve{\boldsymbol{E}})$ vanishes. With this aim, for each $t\in[0,T]$, we take $\boldsymbol{G}=\breve{\boldsymbol{H}}(t)$ and $\boldsymbol{F}=\breve{\boldsymbol{E}}(t)$ in Problem 2. Subtracting the resulting equations we obtain

$$\int_\Omega \mu\,\partial_t \breve{\boldsymbol{H}}(t)\cdot\breve{\boldsymbol{H}}(t) + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\left|\mathbf{curl}\,\breve{\boldsymbol{H}}(t)\right|^2 = 0.$$

Then, using that $\int_\Omega \mu\,\partial_t \breve{\boldsymbol{H}}(t)\cdot\breve{\boldsymbol{H}}(t) = \frac{1}{2}\frac{d}{dt}\left\|\mu^{1/2}\breve{\boldsymbol{H}}(t)\right\|^2_{\mathrm{L}^2(\Omega)^3}$, we have that

$$\frac{1}{2}\left\|\mu^{1/2}\breve{\boldsymbol{H}}(t)\right\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^t \left[\int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\left|\mathbf{curl}\,\breve{\boldsymbol{H}}(s)\right|^2\right] ds = \frac{1}{2}\left\|\mu^{1/2}\breve{\boldsymbol{H}}(0)\right\|^2_{\mathrm{L}^2(\Omega)^3} = 0 \qquad \forall t\in[0,T].$$

Then, it follows that $\breve{\boldsymbol{H}}=\mathbf{0}$. Finally, from (3.2) and (3.3) we derive that $\breve{\boldsymbol{E}}=\mathbf{0}$, too. □

The finite element discretization of Problem 2 looks expensive, since it involves two vector fields: $\boldsymbol{H}$ in the whole domain Ω and $\boldsymbol{E}$ in the dielectric domain. Moreover, it is not simple to find a basis of the finite element subspace used to discretize the space $\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}})$ where the Lagrange multiplier of Problem 2 lies. To avoid these drawbacks, we will introduce an alternative penalty technique to relax the constraint (2.5a). It consists of assuming that the dielectric is not a perfect insulator but a *fake conductor*; namely, a material with a very low conductivity $\varepsilon>0$. In such a case, instead of (2.5a), the solution $(\boldsymbol{H}_\varepsilon,\boldsymbol{E}_\varepsilon)$ of this penalty approach has to satisfy

$$\mathbf{curl}\,\boldsymbol{H}_\varepsilon = \varepsilon\boldsymbol{E}_\varepsilon + \boldsymbol{J}_{\mathrm{S}} \qquad \text{in } \Omega_{\mathrm{D}}.$$

Therefore, for each $\varepsilon>0$, the same steps that lead to Problem 2 but using now the above equation to substitute $\boldsymbol{E}_\varepsilon$ in terms of $\boldsymbol{H}_\varepsilon$ and $\boldsymbol{J}_{\mathrm{S}}$ in Ω_{D}, yield the following penalized form of problem (2.1)–(2.3).

Problem 3. *Find* $\boldsymbol{H}_\varepsilon \in \mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))\cap\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ *such that*

$$\int_\Omega \mu\,\partial_t\boldsymbol{H}_\varepsilon\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}_\varepsilon\cdot\mathbf{curl}\,\boldsymbol{G} + \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{H}_\varepsilon\cdot\mathbf{curl}\,\boldsymbol{G} = \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{S}}}\boldsymbol{J}_{\mathrm{S}}\cdot\mathbf{curl}\,\boldsymbol{G} \qquad \forall\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega),$$

$$\boldsymbol{H}_\varepsilon(0)=\boldsymbol{H}_0.$$

This problem has a unique solution $\boldsymbol{H}_\varepsilon$, which is bounded independently of the penalty parameter ε. We will prove this fact in an appendix, as well as the convergence of the solution of the penalized Problem 3 to that of the mixed Problem 2 with a suboptimal rate $\mathcal{O}(\sqrt{\varepsilon})$. In principle, one could try to adapt the techniques from [17] to our problem in order to improve this and obtain an optimal rate $\mathcal{O}(\varepsilon)$. However, we will not try to do it, since we will not need this optimal rate for the forthcoming analysis.

In fact, in the following sections, we will prove an optimal rate $\mathcal{O}(\varepsilon)$ for the respective discretizations of problems 3 and 2. Therefore, we will only use Problem 3 to derive the discrete penalty scheme. This is the reason why we postpone to an appendix the above mentioned proofs.

4. Discretization

The aim of this section is to introduce a full discretization of Problem 3 in order to numerically solve Problem 1. To prove the convergence of the proposed method, we will use as an intermediate step a discretization of Problem 2, which will be introduced in this section, too.

From now on, we assume that Ω, Ω_{S} and Ω_{C} (and hence Ω_{D}) are polyhedral. We consider a regular family of tetrahedral meshes $\mathcal{T}_h$ of Ω, such that each element $K \in \mathcal{T}_h$ is contained either in $\bar{\Omega}_{\text{S}}$, or in $\bar{\Omega}_{\text{C}}$, or in $\bar{\Omega} \setminus (\Omega_{\text{S}} \cup \Omega_{\text{C}})$ (h stands, as usual, for the corresponding mesh-size). We employ edge finite elements to approximate the magnetic field; more precisely, elements from the lowest-order Nédélec space

$$\boldsymbol{\mathcal{N}}_h(\Omega) := \{\boldsymbol{G}_h \in \mathrm{H}(\mathbf{curl}; \Omega) : \; \boldsymbol{G}_h|_K \in \boldsymbol{\mathcal{N}}(K) \; \forall K \in \mathcal{T}_h\},$$

where

$$\boldsymbol{\mathcal{N}}(K) := \left\{\boldsymbol{G}_h \in \mathbb{P}_1^3 : \; \boldsymbol{G}_h(\boldsymbol{x}) = \mathbf{a} \times \boldsymbol{x} + \mathbf{b}, \; \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \; \boldsymbol{x} \in K\right\}.$$

We also introduce the subspace

$$\boldsymbol{\mathcal{N}}_h^0(\Omega) := \{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega) : \; \boldsymbol{G}_h \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\} \subset \mathrm{H}_0(\mathbf{curl}; \Omega).$$

For any $r > 0$, let $\mathrm{H}^r(\mathbf{curl}; \Omega) := \left\{\boldsymbol{G} \in \mathrm{H}^r(\Omega)^3 : \; \mathbf{curl}\, \boldsymbol{G} \in \mathrm{H}^r(\Omega)^3\right\}$. We recall that the Nédélec interpolant $\mathcal{I}_h^{\mathcal{N}} \boldsymbol{G} \in \boldsymbol{\mathcal{N}}_h(\Omega)$ is well defined for all $\boldsymbol{G} \in \mathrm{H}^r(\mathbf{curl}; \Omega)$ provided $r > \frac{1}{2}$, and there exists a constant $C > 0$ independent of $\boldsymbol{G}$ and h, such that $\left\|\mathcal{I}_h^{\mathcal{N}} \boldsymbol{G}\right\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C \left\|\boldsymbol{G}\right\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}$ (see [5]). Moreover, if $\boldsymbol{G} \in \mathrm{H}_0(\mathbf{curl}; \Omega)$, then $\mathcal{I}_h^{\mathcal{N}} \boldsymbol{G} \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$.

For the subsequent analysis we will need some additional regularity of the solution to Problem 1. In particular, from now on, we assume that the magnetic field satisfies $\boldsymbol{H} \in \mathrm{H}^1(0, T; \mathrm{H}^r(\mathbf{curl}; \Omega))$ for some fixed $r \in \left(\frac{1}{2}, 1\right]$. Notice that this assumption implies that $\boldsymbol{H}(t) \in \mathrm{H}^r(\mathbf{curl}; \Omega)$ for all $t \in [0, T]$, so that its Nédélec interpolant $\mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}(t)$ is well defined. In particular, $\mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0$ is well defined and we use it as the initial data of the discrete problem.

For time discretization, we use a backward Euler scheme on a uniform partition of $[0, T]$: $t_m := m\Delta t$, $m = 0, \ldots, M$, with time-step $\Delta t := \frac{T}{M}$. Thus, a fully-discrete approximation of Problem 3 reads as follows:

Problem 4. *Let $\boldsymbol{H}_{h,\varepsilon}^0 := \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0$. For $m = 1, \ldots, M$, find $\boldsymbol{H}_{h,\varepsilon}^m \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ such that*

$$\begin{aligned}\int_\Omega \mu \frac{\boldsymbol{H}_{h,\varepsilon}^m - \boldsymbol{H}_{h,\varepsilon}^{m-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_{\text{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^m \cdot \mathbf{curl}\, \boldsymbol{G}_h + \frac{1}{\varepsilon} \int_{\Omega_{\text{D}}} \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^m \cdot \mathbf{curl}\, \boldsymbol{G}_h \\ = \frac{1}{\varepsilon} \int_{\Omega_{\text{S}}} \boldsymbol{J}_{\text{S}}(t_m) \cdot \mathbf{curl}\, \boldsymbol{G}_h \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega).\end{aligned}$$

Note that the existence and uniqueness of solution $\boldsymbol{H}_{h,\varepsilon}^m$ at each time step $m = 1, \ldots, M$ follows immediately from the Lax-Milgram Lemma.

The main goal of this paper is to prove that the solution of Problem 4 converges to that of Problem 1 as ε, h and Δt go to zero and to obtain error estimates with respect to these parameters. For this analysis, we will use a full discretization of Problem 2 in which the Lagrange multiplier is sought in the space $\mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\text{D}}))$, where

$$\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\text{D}}) := \left\{\boldsymbol{G}_h|_{\Omega_{\text{D}}} : \; \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega)\right\} = \{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega_{\text{D}}) : \; \boldsymbol{G}_h \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}.$$

A full discretization of Problem 2 based on these finite element spaces and a backward Euler scheme reads as follows:

Problem 5. *Let* $\boldsymbol{H}_h^0 := \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0$. *For* $m = 1, \dots, M$, *find* $\boldsymbol{H}_h^m \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ *and* $\boldsymbol{E}_h^m \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}}))$ *such that*

$$\int_\Omega \mu \frac{\boldsymbol{H}_h^m - \boldsymbol{H}_h^{m-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_h^m \cdot \mathbf{curl}\, \boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{E}_h^m = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega),$$

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{H}_h^m \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \boldsymbol{J}_{\mathrm{S}}(t_m) \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})).$$

To prove that this problem has a unique solution, we resort to the classical theory of mixed problems. In fact, first, by proceeding as in the proof of Theorem 5.2 from [3] (cf. (5.3) in this reference), we derive the following discrete *inf-sup* condition:

$$\sup_{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega)} \frac{\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{F}_h}{\|\boldsymbol{G}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \geq \beta^* \|\boldsymbol{F}_h\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})) \tag{4.1}$$

with a constant $\beta^* > 0$ independent of h. Secondly, it is easy to check that the discrete kernel is in this case

$$\boldsymbol{\mathcal{Y}}_h := \left\{ \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega) : \ \mathbf{curl}\, \boldsymbol{G}_h = \mathbf{0} \text{ in } \Omega_{\mathrm{D}} \right\} \subset \boldsymbol{\mathcal{Y}}.$$

Then, the ellipticity in the discrete kernel follows immediately. Hence, applying [13, Theorem II.1.1], for instance, we derive the existence and uniqueness of solution for each $m = 1, \dots, M$.

Our next goal is to establish appropriate *a priori* estimates for the solutions to Problems 4 and 5. We begin with the latter.

Lemma 1. *Let* $\boldsymbol{H}_h^0$ *and* $(\boldsymbol{H}_h^k, \boldsymbol{E}_h^k)$, $k = 1, \dots, M$, *be the solution to Problem 5. Then, there exists a constant* $C > 0$ *independent of* h, Δt, $\boldsymbol{J}_{\mathrm{S}}$ *and* $\boldsymbol{H}_0$ *such that*

$$\max_{1 \leq k \leq M} \|\boldsymbol{H}_h^k\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^M \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^M \|\boldsymbol{E}_h^k\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2$$
$$\leq C \left\{ \|\boldsymbol{H}_0\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2 \right\}.$$

Proof. The first step is to build $\widehat{\boldsymbol{H}}_h \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{N}}_h^0(\Omega))$ such that

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \boldsymbol{J}_{\mathrm{S}}(t) \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})), \qquad t \in [0,T]. \tag{4.2}$$

To do this, we proceed as in the proof of Theorem 1, but we use now the discrete *inf-sup* condition (4.1) and Lemma I.4.1(iii) from [13] to derive that there exists a unique $\boldsymbol{Q}_h(t) \in \boldsymbol{\mathcal{Y}}_h^{\perp_{\boldsymbol{\mathcal{N}}_h^0(\Omega)}}$ such that

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{Q}_h(t) \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \partial_t \boldsymbol{J}_{\mathrm{S}}(t) \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})), \quad \text{a.e. } t \in [0,T],$$

and $\|\boldsymbol{Q}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le C\,\|\partial_t \boldsymbol{J}_{\mathrm{S}}(t)\|_{\mathrm{L}^2(\Omega_{\mathrm{S}})^3}$. Proceeding analogously, we also derive that there exists a unique $\boldsymbol{R}_h \in \boldsymbol{\mathcal{Y}}_h^{\perp_{\boldsymbol{\mathcal{N}}_h^0(\Omega)}}$ such that

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\,\boldsymbol{R}_h \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \boldsymbol{J}_{\mathrm{S}}(0)\cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^{\Gamma}(\Omega_{\mathrm{D}})) \tag{4.3}$$

and $\|\boldsymbol{R}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le C\,\|\boldsymbol{J}_{\mathrm{S}}(0)\|_{\mathrm{L}^2(\Omega_{\mathrm{S}})^3}$. Then, we define $\widehat{\boldsymbol{H}}_h(t) := \boldsymbol{R}_h + \int_0^t \boldsymbol{Q}_h(s)\,ds$ and, by repeating the same arguments used in the proof of Theorem 1, we are able to show that $\widehat{\boldsymbol{H}}_h \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ satisfies (4.2) and

$$\|\widehat{\boldsymbol{H}}_h\|_{\mathrm{H}^1(0,T;\mathrm{H}(\mathbf{curl};\Omega))} \le C\,\|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}\,. \tag{4.4}$$

Now, if we write $\boldsymbol{H}_h^k = \widetilde{\boldsymbol{H}}_h^k + \widehat{\boldsymbol{H}}_h^k$, where $\widehat{\boldsymbol{H}}_h^k := \widehat{\boldsymbol{H}}_h(t_k)$, Problem 5 is equivalent to find $\widetilde{\boldsymbol{H}}_h^0 := \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}_0 - \widehat{\boldsymbol{H}}_h^0$ and, for $k=1,\dots,M$, $\widetilde{\boldsymbol{H}}_h^k \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and $\boldsymbol{E}_h^k \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^{\Gamma}(\Omega_{\mathrm{D}}))$ such that

$$\begin{aligned}
\int_{\Omega} \mu \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t}\cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^k \cdot \mathbf{curl}\,\boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\,\boldsymbol{G}_h \cdot \boldsymbol{E}_h^k \qquad\qquad & \\
= -\int_{\Omega} \mu \frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t}\cdot \boldsymbol{G}_h - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\widehat{\boldsymbol{H}}_h^k \cdot \mathbf{curl}\,\boldsymbol{G}_h \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega), & \qquad (4.5a)\\
\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^k \cdot \boldsymbol{F}_h = 0 \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^{\Gamma}(\Omega_{\mathrm{D}})), & \qquad (4.5b)
\end{aligned}$$

where we have used (4.2) to derive (4.5b).

By taking $\boldsymbol{G}_h = \widetilde{\boldsymbol{H}}_h^k$ in (4.5a), using (4.5b), the inequality $2\,(p-q)\,p \ge p^2 - q^2$ and Young's inequality, we obtain

$$\begin{aligned}
\int_{\Omega} \mu\,\big|\widetilde{\boldsymbol{H}}_h^k\big|^2 - \int_{\Omega} \mu\,\big|\widetilde{\boldsymbol{H}}_h^{k-1}\big|^2 + \frac{\Delta t}{\overline{\sigma}}\big\|\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^k\big\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} & \\
\le \frac{\Delta t}{2T}\int_{\Omega} \mu\,\big|\widetilde{\boldsymbol{H}}_h^k\big|^2 + C\Delta t \left\{\big\|\,\mathbf{curl}\,\widehat{\boldsymbol{H}}_h^k\big\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \left\|\frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3}\right\}. &
\end{aligned}$$

Summing up from $k=1$ to m ($m \le M$), using

$$\Delta t \sum_{k=1}^{M} \left\|\frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3} \le C\,\big\|\widehat{\boldsymbol{H}}_h\big\|^2_{\mathrm{H}^1(0,T;\mathrm{H}(\mathbf{curl};\Omega))}, \tag{4.6}$$

as well as the estimate (4.4) and the discrete Gronwall's inequality, yield

$$\big\|\widetilde{\boldsymbol{H}}_h^m\big\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{m} \big\|\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^k\big\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \le C\left\{\big\|\mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}_0\big\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}\right\}. \tag{4.7}$$

On the other hand, by taking $\boldsymbol{G}_h = \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t}$ in (4.5a) and using

$$\int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k \cdot \mathbf{curl} \left(\frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right) \geq \frac{1}{2\Delta t} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \left| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k \right|^2 - \frac{1}{2\Delta t} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \left| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} \right|^2$$

and

$$\begin{aligned} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^k \cdot \mathbf{curl} \left(\frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right) &= \frac{1}{\Delta t} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^k \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k \\ - \frac{1}{\Delta t} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^{k-1} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} &- \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl} \left(\frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right) \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1}, \end{aligned}$$

together with (4.5b) and Young's inequality, we obtain

$$\begin{aligned} \Delta t \int_{\Omega} \mu \left| \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right|^2 &+ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \left| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k \right|^2 - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \left| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} \right|^2 \\ &\leq -2 \left\{ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^k \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^{k-1} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} \right\} \\ &\quad + C \Delta t \left\{ \left\| \frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \left\| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \right\}. \end{aligned}$$

Summing up from $k = 1$ to m ($m \leq M$) leads to

$$\begin{aligned} \Delta t \sum_{k=1}^{m} \left\| \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 &+ \left\| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^m \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \\ \leq C \Bigg\{ \left\| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^0 \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 &+ \left\| \mathbf{curl}\, \widehat{\boldsymbol{H}}_h^0 \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 + \left\| \mathbf{curl}\, \widehat{\boldsymbol{H}}_h^m \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \\ &+ \Delta t \sum_{k=1}^{m} \left[\left\| \frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \left\| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \right] \Bigg\}. \end{aligned}$$

Using the estimates (4.6), (4.4) and (4.7), and the fact that $\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^m = \mathbf{0}$ in Ω_{D} (cf. (4.5b), we obtain for all $m = 1, \ldots, M$,

$$\Delta t \sum_{k=1}^{m} \left\| \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \left\| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^m \right\|_{\mathrm{L}^2(\Omega)^3}^2 \leq C \left\{ \left\| \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0 \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \| \boldsymbol{J}_{\mathrm{S}} \|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2 \right\}.$$

Finally, since $\boldsymbol{H}_h^k = \widetilde{\boldsymbol{H}}_h^k + \widehat{\boldsymbol{H}}_h^k$, thanks to the above estimate, (4.7), (4.6) and (4.4), we conclude that

$$\max_{1\leq k\leq M} \left\|\boldsymbol{H}_h^k\right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^M \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 \leq C \left\{ \left\|\mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0\right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2 \right\}.$$

There remains to estimate the terms involving the Lagrange multipliers $\boldsymbol{E}_h^k$. To do this, note that as a consequence of the *inf-sup* condition (4.1) and the first equation of Problem 5 we have

$$\beta^* \left\|\boldsymbol{E}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \leq \sup_{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega)} \frac{\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{E}_h^k}{\|\boldsymbol{G}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \leq C \left\{ \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \left\| \mathbf{curl}\, \boldsymbol{H}_h^k \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \right\}^{1/2}.$$

Then, the last two inequalities yield

$$\Delta t \sum_{k=1}^M \left\|\boldsymbol{E}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \leq C \left\{ \left\|\mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0\right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2 \right\},$$

which allows us to conclude the proof. □

Regarding Problem 4, the following result establishes an *a priori* estimate of its solution.

Lemma 2. *Let $\boldsymbol{H}_{h,\varepsilon}^k$, $k = 0, \dots, M$, be the solution to Problem 4. Then, there exists a constant $C > 0$ independent of ε, h, Δt, $\boldsymbol{J}_{\mathrm{S}}$ and $\boldsymbol{H}_0$ such that*

$$\max_{1\leq k\leq M} \left\|\boldsymbol{H}_{h,\varepsilon}^k\right\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^M \left\| \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^k \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \leq C \left\{ \|\boldsymbol{H}_0\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2 \right\},$$

$$\Delta t \sum_{k=1}^M \left\| \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^k \right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \leq C\varepsilon \left\{ \|\boldsymbol{H}_0\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2 \right\}$$

and

$$\max_{1\leq k\leq M} \left\| \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^M \left\| \frac{\boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_{h,\varepsilon}^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2$$
$$\leq C \left\{ \|\boldsymbol{H}_0\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2 + \frac{1}{\varepsilon} \int_\Omega \left| \mathbf{curl}(\mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0 - \boldsymbol{H}_0) \right|^2 \right\}.$$

Proof. We just give a sketch of the proof since it is very close to that of the previous lemma. Let $\widehat{\boldsymbol{H}}_h \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{N}}_h^0(\Omega))$ be as in that proof, so that (4.2) and (4.4) hold true. We write $\boldsymbol{H}_{h,\varepsilon}^k = \widetilde{\boldsymbol{H}}_{h,\varepsilon}^k + \widehat{\boldsymbol{H}}_h^k$, where $\widehat{\boldsymbol{H}}_h^k := \widehat{\boldsymbol{H}}_h(t_k)$ again. Then, Problem 4 is equivalent to find $\widetilde{\boldsymbol{H}}_{h,\varepsilon}^0 := \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0 - \widehat{\boldsymbol{H}}_h^0$ and, for $k = 1, \dots, M$, $\widetilde{\boldsymbol{H}}_{h,\varepsilon}^k \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ such that

$$\int_\Omega \mu \frac{\widetilde{\boldsymbol{H}}_{h,\varepsilon}^k - \widetilde{\boldsymbol{H}}_{h,\varepsilon}^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \widetilde{\boldsymbol{H}}_{h,\varepsilon}^k \cdot \mathbf{curl}\, \boldsymbol{G}_h + \frac{1}{\varepsilon} \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \widetilde{\boldsymbol{H}}_{h,\varepsilon}^k \cdot \mathbf{curl}\, \boldsymbol{G}_h$$
$$= - \int_\Omega \mu \frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \widehat{\boldsymbol{H}}_h^k \cdot \mathbf{curl}\, \boldsymbol{G}_h \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega).$$

By testing with $\boldsymbol{G}_h = \widetilde{\boldsymbol{H}}^k_{h,\varepsilon}$, using the inequality $2\,(p-q)\,p \geq p^2 - q^2$ and Young's inequality, summing up from $k=1$ to m ($m \leq M$) and using estimates (4.6), (4.4) and the discrete Gronwall's inequality, we obtain

$$\|\widetilde{\boldsymbol{H}}^m_{h,\varepsilon}\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{m} \|\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}^k_{h,\varepsilon}\|^2_{\mathrm{L}^2(\Omega_\mathrm{C})^3} + \frac{\Delta t}{\varepsilon} \sum_{k=1}^{m} \|\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}^k_{h,\varepsilon}\|^2_{\mathrm{L}^2(\Omega_\mathrm{D})^3} \\ \leq C \left\{ \|\mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_\mathrm{S}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^3)} \right\},$$

which together with (4.4) lead to the first two estimates of the lemma.

On the other hand, by taking $\boldsymbol{G}_h = \frac{\widetilde{\boldsymbol{H}}^k_{h,\varepsilon} - \widetilde{\boldsymbol{H}}^{k-1}_{h,\varepsilon}}{\Delta t}$, similar arguments to those used in the proof of Lemma 1 allow us to show that

$$\Delta t \sum_{k=1}^{m} \left\| \frac{\widetilde{\boldsymbol{H}}^k_{h,\varepsilon} - \widetilde{\boldsymbol{H}}^{k-1}_{h,\varepsilon}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} + \|\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}^m_{h,\varepsilon}\|^2_{\mathrm{L}^2(\Omega)^3} \\ \leq C \left\{ \|\mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}_0\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_\mathrm{S}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^3)} + \frac{1}{\varepsilon} \int_\Omega |\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}^0_{h,\varepsilon}|^2 \right\}.$$

Since $\boldsymbol{H}^0_{h,\varepsilon} = \widetilde{\boldsymbol{H}}^0_{h,\varepsilon} + \widehat{\boldsymbol{H}}^0_h$, by using the definition of $\widehat{\boldsymbol{H}}_h$, (4.3) and (2.4), the last term reads

$$\int_\Omega |\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}^0_{h,\varepsilon}|^2 = \int_\Omega \left[\mathbf{curl}(\mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}_0) - \mathbf{curl}\,\widehat{\boldsymbol{H}}_h(0) \right] \cdot \mathbf{curl}\,\widetilde{\boldsymbol{H}}^0_{h,\varepsilon} \\ = \int_\Omega \left[\mathbf{curl}(\mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}_0) - \boldsymbol{J}_\mathrm{S}(0) \right] \cdot \mathbf{curl}\,\widetilde{\boldsymbol{H}}^0_{h,\varepsilon} = \int_\Omega \mathbf{curl}(\mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}_0 - \boldsymbol{H}_0) \cdot \mathbf{curl}\,\widetilde{\boldsymbol{H}}^0_{h,\varepsilon}.$$

Thus, the last estimate of the lemma follows from the above equations, (4.6) and (4.4). □

Let us remark that the last estimate from this lemma is not independent of the penalty parameter ε, because of the term $\frac{1}{\varepsilon} \int_\Omega \left|\mathbf{curl}(\mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}_0 - \boldsymbol{H}_0)\right|^2$. However, this term vanishes for a vanishing initial data $\boldsymbol{H}(0) = \mathbf{0}$.

5. Error estimates for the discretization of the mixed problem

The next step is to obtain error estimates for the solution of Problem 5 as an approximation to that of Problem 2. This result will be used in this paper as an intermediate step of the error analysis for the numerical solution of the penalized problem, although it has an interest by itself.

Let $\boldsymbol{H}$ be the first component of the solution to Problem 2. We proceed as in the proof of Lemma 1, but using $\mathbf{curl}\left(\boldsymbol{H} - \mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}\right)$ instead of $\boldsymbol{J}_\mathrm{S}$. Thus, instead of $\widehat{\boldsymbol{H}}_h$, we have $\boldsymbol{Z}_h \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{N}}^0_h(\Omega))$ which satisfies the analogues to (4.2) and (4.4), namely,

$$\int_{\Omega_\mathrm{D}} \mathbf{curl}\,\boldsymbol{Z}_h(t) \cdot \boldsymbol{F}_h = \int_{\Omega_\mathrm{D}} \mathbf{curl}\left(\boldsymbol{H}(t) - \mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}(t)\right) \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}^\Gamma_h(\Omega_\mathrm{D})), \quad t \in [0,T]$$

and

$$\|\boldsymbol{Z}_h\|_{\mathrm{H}^1(0,T;\mathrm{H}(\mathbf{curl};\Omega))} \leq C \left\|\mathbf{curl}\left(\boldsymbol{H} - \mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}\right)\right\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_\mathrm{D})^3)}.$$

Then, we define $\mathring{\boldsymbol{H}}_h(t) := \boldsymbol{Z}_h(t) + \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}(t)$, so that

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \mathring{\boldsymbol{H}}_h(t) \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{H}(t) \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \boldsymbol{J}_{\mathrm{S}}(t) \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^{\mathrm{T}}(\Omega_{\mathrm{D}})) \tag{5.1}$$

for almost all $t \in [0, T]$, where we have used the second equation from Problem 2 for the last equality. Moreover, by using classical error estimates for the Nédélec interpolant (see, for instance, [16, Theorem 5.41]), we have that

$$\left\| \boldsymbol{H} - \mathring{\boldsymbol{H}}_h \right\|_{\mathrm{H}^1(0,T;\mathrm{H}(\mathbf{curl};\Omega))} \leq C h^r \left\| \boldsymbol{H} \right\|_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}. \tag{5.2}$$

Next, provided $\boldsymbol{H} \in \mathcal{C}^1(0, T; \mathrm{L}^2(\Omega)^3)$, we write

$$\partial_t \boldsymbol{H}(t_k) - \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} = \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} + \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} - \boldsymbol{\tau}^k, \tag{5.3}$$

where

$$\boldsymbol{\rho}_h^k := \boldsymbol{H}(t_k) - \mathring{\boldsymbol{H}}_h(t_k), \qquad \boldsymbol{\delta}_h^k := \mathring{\boldsymbol{H}}_h(t_k) - \boldsymbol{H}_h^k \qquad \text{and} \qquad \boldsymbol{\tau}^k := \frac{\boldsymbol{H}(t_k) - \boldsymbol{H}(t_{k-1})}{\Delta t} - \partial_t \boldsymbol{H}(t_k).$$

Notice that by virtue of (5.1) and the second equation from Problem 5, $\boldsymbol{\delta}_h^k \in \boldsymbol{\mathcal{Y}}_h$, $k = 1, \ldots, M$. Moreover, we have the following auxiliary result.

Lemma 3. *Let $\boldsymbol{H}$ be the first component of the solution to Problem 2 and $\boldsymbol{H}_h^k$, $k = 0, \ldots, M$, that to Problem 5. If $\boldsymbol{H} \in \mathrm{H}^1(0, T; \mathrm{H}^r(\mathbf{curl}; \Omega)) \cap \mathcal{C}^1(0, T; \mathrm{L}^2(\Omega)^3)$ with $r \in \left(\frac{1}{2}, 1\right]$, then there exists a constant $C > 0$ independent of h and Δt such that*

$$\begin{aligned}
\max_{1 \leq k \leq M} \left\| \boldsymbol{\delta}_h^k \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 &+ \Delta t \sum_{k=1}^M \left\| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 \\
&\leq C \left\{ \left\| \boldsymbol{\delta}_h^0 \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \max_{1 \leq k \leq M} \left\| \mathbf{curl}\, \boldsymbol{\rho}_h^k \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \right. \\
&\qquad \left. + \Delta t \sum_{k=1}^M \left\| \boldsymbol{\tau}^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^M \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 \right\}.
\end{aligned}$$

Proof. By testing the first equations of Problem 2 and Problem 5 with $\boldsymbol{G}_h \in \boldsymbol{\mathcal{Y}}_h$, a straightforward computation allows us to show that

$$\begin{aligned}
\int_\Omega \mu \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h &+ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{\delta}_h^k \cdot \mathbf{curl}\, \boldsymbol{G}_h \\
&= \int_\Omega \mu \boldsymbol{\tau}^k \cdot \boldsymbol{G}_h - \int_\Omega \mu \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{\rho}_h^k \cdot \mathbf{curl}\, \boldsymbol{G}_h \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{Y}}_h.
\end{aligned} \tag{5.4}$$

By taking $\boldsymbol{G}_h = \boldsymbol{\delta}_h^k$, using classical inequalities, summing up from $k=1$ to m ($m \le M$) and using the discrete Gronwall's inequality lead to

$$\begin{aligned}
&\|\boldsymbol{\delta}_h^m\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^m \left\| \mathbf{curl}\, \boldsymbol{\delta}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \\
&\quad \le C \left\{ \left\|\boldsymbol{\delta}_h^0\right\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^m \left[\left\|\boldsymbol{\tau}^k\right\|^2_{\mathrm{L}^2(\Omega)^3} + \left\|\mathbf{curl}\, \boldsymbol{\rho}_h^k\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} \right] \right\}. \qquad (5.5)
\end{aligned}$$

On the other hand, by taking $\boldsymbol{G}_h = \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t}$ in (5.4), similar arguments and the fact that

$$\begin{aligned}
\int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{\rho}_h^k \cdot \mathbf{curl}\, \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} &= - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl} \left(\frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right) \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^{k-1} \\
&\quad + \frac{1}{\Delta t} \left\{ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{\rho}_h^k \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^k - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{\rho}_h^{k-1} \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^{k-1} \right\}
\end{aligned}$$

lead to

$$\begin{aligned}
&\Delta t \int_\Omega \mu \left| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right|^2 + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \left| \mathbf{curl}\, \boldsymbol{\delta}_h^k \right|^2 - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \left| \mathbf{curl}\, \boldsymbol{\delta}_h^{k-1} \right|^2 \\
&\qquad \le -2 \left\{ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{\rho}_h^k \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^k - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{\rho}_h^{k-1} \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^{k-1} \right\} \\
&\qquad\quad + C \Delta t \left\{ \left\|\boldsymbol{\tau}^k\right\|^2_{\mathrm{L}^2(\Omega)^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \left\| \mathbf{curl}\, \boldsymbol{\delta}_h^{k-1} \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \right\}.
\end{aligned}$$

Summing up from $k=1$ to m ($m \le M$) and using (5.5) to estimate $\Delta t \sum_{k=2}^m \left\| \mathbf{curl}\, \boldsymbol{\delta}_h^{k-1} \right\|^2_{\mathrm{L}^2(\Omega)^3}$, we obtain

$$\begin{aligned}
&\Delta t \sum_{k=1}^m \left\| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} + \|\mathbf{curl}\, \boldsymbol{\delta}_h^m\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \\
&\qquad \le C \Bigg\{ \Delta t \sum_{k=1}^m \left[\left\|\boldsymbol{\tau}^k\right\|^2_{\mathrm{L}^2(\Omega)^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \left\|\mathbf{curl}\, \boldsymbol{\rho}_h^k\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \right] \\
&\qquad\qquad + \left\|\mathbf{curl}\, \boldsymbol{\delta}_h^0\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \|\mathbf{curl}\, \boldsymbol{\rho}_h^m\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \left\|\mathbf{curl}\, \boldsymbol{\rho}_h^0\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \Bigg\}.
\end{aligned}$$

Thus, the result follows by combining the above inequality with (5.5) and the fact that $\boldsymbol{\delta}_h^k \in \boldsymbol{\mathcal{Y}}_h$. $\square$

Now, we are in a position to prove the following error estimate.

Theorem 3. *Let $\boldsymbol{H}$ be the first component of the solution to Problem 2 and $\boldsymbol{H}_h^k$, $k = 0, \ldots, M$, that to Problem 5. If $\boldsymbol{H} \in \mathrm{H}^1(0, T; \mathrm{H}^r(\mathbf{curl}; \Omega)) \cap \mathrm{H}^2(0, T; \mathrm{L}^2(\Omega)^3)$ with $r \in \left(\frac{1}{2}, 1\right]$, then there exists a constant $C > 0$ independent of h and Δt such that*

$$\max_{1 \leq k \leq M} \left\| \boldsymbol{H}(t_k) - \boldsymbol{H}_h^k \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{M} \left\| \partial_t \boldsymbol{H}(t_k) - \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 \leq C \left\{ (\Delta t)^2 \left\| \boldsymbol{H} \right\|_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)}^2 + h^{2r} \left\| \boldsymbol{H} \right\|_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}^2 \right\}.$$

Proof. A Taylor expansion allows us to show that

$$\sum_{k=1}^{M} \left\| \boldsymbol{\tau}^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 = \sum_{k=1}^{M} \left\| \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} (t_{k-1} - t)\, \partial_{tt} \boldsymbol{H}(t)\, dt \right\|_{\mathrm{L}^2(\Omega)^3}^2 \leq \Delta t \int_0^T \left\| \partial_{tt} \boldsymbol{H}(t) \right\|_{\mathrm{L}^2(\Omega)^3}^2 \, dt.$$

Moreover, from the definition of $\boldsymbol{\rho}_h^k$ and (5.2), we write

$$\Delta t \sum_{k=1}^{M} \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 \leq \int_0^T \left\| \partial_t \big(\boldsymbol{H}(t) - \mathring{\boldsymbol{H}}_h(t) \big) \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 dt \leq C h^{2r} \left\| \boldsymbol{H} \right\|_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}^2$$

and from the definitions of $\boldsymbol{\delta}_h^0$ and $\boldsymbol{\rho}_h^0$, (5.2) and the error estimate for the Nédélec interpolant,

$$\left\| \boldsymbol{\delta}_h^0 \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 \leq \left\{ \left\| \boldsymbol{\rho}_h^0 \right\|_{\mathrm{H}(\mathbf{curl};\Omega)} + \left\| \boldsymbol{H}_0 - \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0 \right\|_{\mathrm{H}(\mathbf{curl};\Omega)} \right\}^2 \leq C h^{2r} \left\| \boldsymbol{H} \right\|_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}^2.$$

Since $\boldsymbol{H}(t_k) - \boldsymbol{H}_h^k = \boldsymbol{\delta}_h^k + \boldsymbol{\rho}_h^k$, the result follows from the previous lemma, (5.2), (5.3) and the above estimates. □

Let us remark that although in our analysis this result is just an intermediate step, as claimed above it is interesting by itself since it yields an optimal order error estimate for the magnetic field computed by solving Problem 5 as an approximation of the solution to Problem 1.

6. Error estimates for the penalized problem

Now, we return to the convergence analysis of the penalty fully-discrete scheme. Taking into account Theorem 3, it is enough to show that the solution of Problem 4 converges to that of Problem 5 as $\varepsilon \to 0$, uniformly in the discretization parameters h and Δt. This is established in the following lemma by means of an error estimate in terms of the penalty parameter ε. This estimate is not of optimal order. In fact, in the next section, we will improve it in the case of vanishing initial data.

Lemma 4. *Let $(\boldsymbol{H}_h^k, \boldsymbol{E}_h^k)$, $k = 1, \ldots, M$, be the solution to Problem 5 and $\boldsymbol{H}_{h,\varepsilon}^k$, $k = 1, \ldots, M$, that to Problem 4. Let $\boldsymbol{E}_{h,\varepsilon}^k := \frac{1}{\varepsilon}\big(\mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^k - \mathbf{curl}\, \boldsymbol{H}_h^k \big)|_{\Omega_{\mathrm{D}}}$. Then, there exists $C > 0$ independent of ε, h, Δt, $\boldsymbol{J}_{\mathrm{S}}$ and $\boldsymbol{H}_0$ such that*

$$\max_{1 \leq k \leq M} \left\| \boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^{M} \left\| \mathbf{curl} \left(\boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k \right) \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \varepsilon \Delta t \sum_{k=1}^{M} \left\| \boldsymbol{E}_{h,\varepsilon}^k - \boldsymbol{E}_h^k \right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \leq C \varepsilon \left\{ \left\| \boldsymbol{H}_0 \right\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}^2 + \left\| \boldsymbol{J}_{\mathrm{S}} \right\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2 \right\}.$$

Proof. First, note that $\boldsymbol{H}_{h,\varepsilon}^k \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and $\boldsymbol{E}_{h,\varepsilon}^k \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}}))$, $k = 1, \dots, M$, satisfy

$$\int_\Omega \mu \frac{\boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_{h,\varepsilon}^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^k \cdot \mathbf{curl}\, \boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{E}_{h,\varepsilon}^k = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega), \tag{6.1a}$$

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^k \cdot \boldsymbol{F}_h - \varepsilon \int_{\Omega_{\mathrm{D}}} \boldsymbol{E}_{h,\varepsilon}^k \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \boldsymbol{J}_{\mathrm{S}}(t_k) \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})), \tag{6.1b}$$

$$\boldsymbol{H}_{h,\varepsilon}^0 = \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0. \tag{6.1c}$$

We denote $\boldsymbol{u}_h^k := \boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k$, $k = 0, \dots, M$, and $\boldsymbol{v}_h^k := \boldsymbol{E}_{h,\varepsilon}^k - \boldsymbol{E}_h^k$, $k = 1, \dots, M$. By subtracting (6.1a)–(6.1c) from the corresponding equations of Problem 5, we obtain

$$\int_\Omega \mu \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{u}_h^k \cdot \mathbf{curl}\, \boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{v}_h^k = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega), \tag{6.2a}$$

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{u}_h^k \cdot \boldsymbol{F}_h - \varepsilon \int_{\Omega_{\mathrm{D}}} \boldsymbol{v}_h^k \cdot \boldsymbol{F}_h = \varepsilon \int_{\Omega_{\mathrm{D}}} \boldsymbol{E}_h^k \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})), \tag{6.2b}$$

$$\boldsymbol{u}_h^0 = \mathbf{0}. \tag{6.2c}$$

By taking $\boldsymbol{G}_h = \boldsymbol{u}_h^k$ and $\boldsymbol{F}_h = \boldsymbol{v}_h^k$ in (6.2a) and (6.2b), respectively, subtracting the resulting expressions, using classical inequalities, summing from $k = 1$ to m ($m \leq M$), using a discrete Gronwall's inequality and the fact that $\boldsymbol{u}_h^0 = \mathbf{0}$, we obtain

$$\|\boldsymbol{u}_h^m\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^m \left\|\mathbf{curl}\, \boldsymbol{u}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 + \varepsilon \Delta t \sum_{k=1}^m \left\|\boldsymbol{v}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \leq C \varepsilon \Delta t \sum_{k=1}^m \left\|\boldsymbol{E}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2.$$

There only remains to estimate $\Delta t \sum_{k=1}^m \left\|\mathbf{curl}\, \boldsymbol{u}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2$. With this end, we note first that it is easy to check that $\mathbf{curl}\, \boldsymbol{u}_h^k = \varepsilon\big(\boldsymbol{v}_h^k + \boldsymbol{E}_h^k\big)$ in Ω_{D}. Then, $\left\|\mathbf{curl}\, \boldsymbol{u}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \leq 2\varepsilon^2 \big\{ \left\|\boldsymbol{v}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 + \left\|\boldsymbol{E}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \big\}$. Hence, from the previous estimate we obtain

$$\Delta t \sum_{k=1}^m \left\|\mathbf{curl}\, \boldsymbol{u}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \leq C \varepsilon^2 \Delta t \sum_{k=1}^m \left\|\boldsymbol{E}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2.$$

Then, from the last two inequalities, we have that

$$\begin{aligned} &\|\boldsymbol{u}_h^m\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^m \left\|\mathbf{curl}\, \boldsymbol{u}_h^k\right\|_{\mathrm{L}^2(\Omega)^3}^2 + \varepsilon \Delta t \sum_{k=1}^m \left\|\boldsymbol{v}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \\ &\qquad \leq C \varepsilon \Delta t \sum_{k=1}^m \left\|\boldsymbol{E}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \leq C \varepsilon \left\{ \|\boldsymbol{H}_0\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2 \right\}, \end{aligned}$$

where we have used Lemma 1 for the last inequality. Thus, the theorem follows from the fact that $\boldsymbol{u}_h^k := \boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k$ and $\boldsymbol{v}_h^k := \boldsymbol{E}_{h,\varepsilon}^k - \boldsymbol{E}_h^k$. $\square$

Now we are in a position to derive the convergence of the solution of the fully-discrete penalty scheme (Problem 4) to that of the continuous problem (Problem 1), as the discretization and penalty parameters h, Δt and ε go to zero.

Theorem 4. *Let $\boldsymbol{H}$ be the solution to Problem 1 and $\boldsymbol{H}^k_{h,\varepsilon}$, $k=1,\ldots,M$, that to Problem 4. If $\boldsymbol{H}\in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))\cap \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)$ with $r\in\left(\frac{1}{2},1\right]$, then there exists a constant $C>0$ independent of h, Δt and ε such that*

$$\max_{1\le k\le M}\left\|\boldsymbol{H}(t_k)-\boldsymbol{H}^k_{h,\varepsilon}\right\|^2_{\mathrm{L}^2(\Omega)^3}+\Delta t\sum_{k=1}^{M}\left\|\mathbf{curl}\left(\boldsymbol{H}(t_k)-\boldsymbol{H}^k_{h,\varepsilon}\right)\right\|^2_{\mathrm{L}^2(\Omega)^3}$$
$$\le C\left\{(\Delta t)^2\left\|\boldsymbol{H}\right\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)}+\left(h^{2r}+\varepsilon\right)\left\|\boldsymbol{H}\right\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}\right\}.$$

Proof. It follows immediately by combining the estimates from the previous lemma with those from Theorem 3 and using Theorem 1 and the fact that $\boldsymbol{J}_{\mathrm{S}}=\mathbf{curl}\,\boldsymbol{H}$ in Ω_{D}. □

This theorem provides an error estimate that is of optimal order in the discretization parameters, $\mathcal{O}(h^r+\Delta t)$, but not in the penalty one, $\mathcal{O}(\sqrt{\varepsilon})$. In the following section, we will derive an improved error estimate $\mathcal{O}(\varepsilon)$ valid in the case of a vanishing initial data and a smoother source term $\boldsymbol{J}_{\mathrm{S}}\in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)$.

7. Improved error estimates

First, we prove an $\mathcal{O}(\varepsilon)$ estimate for the penalty error of the discrete problem in $\ell^2(0,T;\mathrm{L}^2(\Omega)^3)$.

Lemma 5. *Let $(\boldsymbol{H}^k_h,\boldsymbol{E}^k_h)$, $k=1,\ldots,M$, be the solution to Problem 5 and $\boldsymbol{H}^k_{h,\varepsilon}$, $k=1,\ldots,M$, that to Problem 4. If $\boldsymbol{H}_0=\mathbf{0}$, then there exists a positive constant C independent of ε, h, Δt and $\boldsymbol{J}_{\mathrm{S}}$ such that*

$$\Delta t\sum_{k=1}^{M}\left\|\boldsymbol{H}^k_{h,\varepsilon}-\boldsymbol{H}^k_h\right\|^2_{\mathrm{L}^2(\Omega)^3}\le C\varepsilon^2\left\|\boldsymbol{J}_{\mathrm{S}}\right\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.$$

Proof. First, note that as a consequence of the *inf-sup* condition (4.1) and (6.1a) we have

$$\beta^*\left\|\boldsymbol{E}^k_{h,\varepsilon}\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}\le\sup_{\boldsymbol{G}_h\in\boldsymbol{\mathcal{N}}^0_h(\Omega)}\frac{\int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{G}_h\cdot\boldsymbol{E}^k_{h,\varepsilon}}{\left\|\boldsymbol{G}_h\right\|_{\mathrm{H}(\mathbf{curl};\Omega)}}$$
$$\le C\left\{\left\|\frac{\boldsymbol{H}^k_{h,\varepsilon}-\boldsymbol{H}^{k-1}_{h,\varepsilon}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3}+\left\|\mathbf{curl}\,\boldsymbol{H}^k_{h,\varepsilon}\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\right\}^{1/2}.$$

Thus, as a consequence of Lemma 2 and the fact that for $\boldsymbol{H}_0=\mathbf{0}$, $\mathcal{I}^{\mathcal{N}}_h\boldsymbol{H}_0=\mathbf{0}$, too, we have

$$\Delta t\sum_{k=1}^{M}\left\|\boldsymbol{E}^k_{h,\varepsilon}\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}\le C\left\|\boldsymbol{J}_{\mathrm{S}}\right\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.\tag{7.1}$$

Secondly, we use a duality argument. In what follows, we will use the notation introduced in the proof of Lemma 4. In particular, let $\boldsymbol{u}_h^k := \boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k$, $k = 0, \dots, M$, and consider the following problem: Find $\boldsymbol{w}_h^k \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and $\boldsymbol{y}_h^k \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}}))$, $k = M, \dots, 1$, such that

$$\int_\Omega \mu \frac{\boldsymbol{w}_h^k - \boldsymbol{w}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{w}_h^{k-1} \cdot \mathbf{curl}\, \boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{y}_h^{k-1} = \int_\Omega \boldsymbol{u}_h^k \cdot \boldsymbol{G}_h \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega), \tag{7.2a}$$

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{w}_h^{k-1} \cdot \boldsymbol{F}_h = 0 \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})), \tag{7.2b}$$

$$\boldsymbol{w}_h^M = \mathbf{0}. \tag{7.2c}$$

It is easy to prove that there exists a unique solution $(\boldsymbol{w}_h^k, \boldsymbol{z}_h^k)$, $k = M, \dots, 1$, and that there exists a constant $C > 0$, independent of h and Δt, such that

$$\max_{1 \le k \le M} \left\| \boldsymbol{w}_h^{k-1} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^M \left\| \frac{\boldsymbol{w}_h^k - \boldsymbol{w}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^M \left\| \boldsymbol{y}_h^{k-1} \right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \le C \Delta t \sum_{k=1}^M \left\| \boldsymbol{u}_h^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 \tag{7.3}$$

Now, by taking $\boldsymbol{G}_h = \boldsymbol{u}_h^k$ in (7.2a), we obtain

$$\left\| \boldsymbol{u}_h^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 = \int_\Omega \mu \frac{\boldsymbol{w}_h^k - \boldsymbol{w}_h^{k-1}}{\Delta t} \cdot \boldsymbol{u}_h^k - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{w}_h^{k-1} \cdot \mathbf{curl}\, \boldsymbol{u}_h^k + \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{u}_h^k \cdot \boldsymbol{y}_h^{k-1}$$

and summing up from $k = 1$ to M,

$$\sum_{k=1}^M \left\| \boldsymbol{u}_h^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 = \sum_{k=1}^M \left\{ - \int_\Omega \mu \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t} \cdot \boldsymbol{w}_h^{k-1} - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{w}_h^{k-1} \cdot \mathbf{curl}\, \boldsymbol{u}_h^k + \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{u}_h^k \cdot \boldsymbol{y}_h^{k-1} \right\}.$$

On the other hand, by taking $\boldsymbol{G}_h = \boldsymbol{w}_h^{k-1}$ and $\boldsymbol{F}_h = \boldsymbol{y}_h^{k-1}$ in (6.2a) and (6.2b), respectively, subtracting the resulting expressions and replacing in the above equation, we obtain

$$\sum_{k=1}^M \left\| \boldsymbol{u}_h^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 = \sum_{k=1}^M \left\{ \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{w}_h^{k-1} \cdot \boldsymbol{v}_h^k + \varepsilon \int_{\Omega_{\mathrm{D}}} \boldsymbol{v}_h^k \cdot \boldsymbol{y}_h^{k-1} + \varepsilon \int_{\Omega_{\mathrm{D}}} \boldsymbol{E}_h^k \cdot \boldsymbol{y}_h^{k-1} \right\}.$$

Since $\boldsymbol{v}_h^k := \boldsymbol{E}_{h,\varepsilon}^k - \boldsymbol{E}_h^k \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}}))$, from (7.2b) the first term on the right-hand side above vanishes. Moreover, since $\boldsymbol{v}_h^k + \boldsymbol{E}_h^k = \boldsymbol{E}_{h,\varepsilon}^k$, we have that

$$\sum_{k=1}^M \left\| \boldsymbol{u}_h^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 = \varepsilon \sum_{k=1}^M \int_{\Omega_{\mathrm{D}}} \boldsymbol{E}_{h,\varepsilon}^k \cdot \boldsymbol{y}_h^{k-1} \le C \varepsilon \left\{ \sum_{k=1}^M \left\| \boldsymbol{E}_{h,\varepsilon}^k \right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \right\}^{\frac{1}{2}} \left\{ \sum_{k=1}^M \left\| \boldsymbol{y}_h^{k-1} \right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \right\}^{\frac{1}{2}}.$$

Hence, using (7.1) and (7.3),

$$\Delta t \sum_{k=1}^{M} \left\| \boldsymbol{u}_h^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 \leq C\varepsilon \left\{ \Delta t \sum_{k=1}^{M} \left\| \boldsymbol{u}_h^k \right\|_{\mathrm{L}^2(\Omega)^3}^2 \right\}^{\frac{1}{2}} \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.$$

Thus, the lemma follows from the fact that $\boldsymbol{u}_h^k := \boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k$. □

Our next goal is to obtain $\mathcal{O}(\varepsilon)$ estimates in $\ell^\infty(0,T;\mathrm{L}^2(\Omega)^3)$ and $\ell^2(0,T;\mathrm{H}(\mathbf{curl};\Omega))$ for $\sqrt{t_k}(\boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k)$. With this aim, we will use the following auxiliary estimates.

Lemma 6. *Let* $(\boldsymbol{H}_h^k, \boldsymbol{E}_h^k)$*,* $k = 1, \dots, M$*, be the solution to Problem 5 with* $\boldsymbol{H}_0 = \mathbf{0}$ *and* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)$*. Then, there exists a constant* $C > 0$ *independent of* h *and* Δt *such that*

$$\Delta t \sum_{k=1}^{M-1} \left\| \frac{t_{k+1}\boldsymbol{E}_h^{k+1} - t_k \boldsymbol{E}_h^k}{\Delta t} \right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 \leq C \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}^2.$$

Proof. First, we take $m = k+1$ and multiply by t_{k+1} in Problem 5. Then, we take $m = k$ and multiply by t_k in Problem 5. By subtracting the resulting expressions and dividing by Δt, we obtain

$$\int_\Omega \frac{\mu}{\Delta t} \left(t_{k+1} \frac{\boldsymbol{H}_h^{k+1} - \boldsymbol{H}_h^k}{\Delta t} - t_k \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right) \cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl} \left(\frac{t_{k+1}\boldsymbol{H}_h^{k+1} - t_k \boldsymbol{H}_h^k}{\Delta t} \right) \cdot \mathbf{curl}\, \boldsymbol{G}_h$$
$$+ \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \left(\frac{t_{k+1}\boldsymbol{E}_h^{k+1} - t_k \boldsymbol{E}_h^k}{\Delta t} \right) = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega),$$
$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl} \left(\frac{t_{k+1}\boldsymbol{H}_h^{k+1} - t_k \boldsymbol{H}_h^k}{\Delta t} \right) \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \frac{t_{k+1}\boldsymbol{J}_{\mathrm{S}}(t_{k+1}) - t_k \boldsymbol{J}_{\mathrm{S}}(t_k)}{\Delta t} \cdot \boldsymbol{F}_h \quad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})).$$

Consequently, $\boldsymbol{p}_h^k := \dfrac{t_{k+1}\boldsymbol{H}_h^{k+1} - t_k \boldsymbol{H}_h^k}{\Delta t}$, $k = 0, \dots, M-1$, and $\boldsymbol{q}_h^k := \dfrac{t_{k+1}\boldsymbol{E}_h^{k+1} - t_k \boldsymbol{E}_h^k}{\Delta t}$, $k = 1, \dots, M-1$, are the solution of the following problem: Find $\boldsymbol{p}_h^m \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and $\boldsymbol{q}_h^m \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}}))$, $m = 1, \dots, M-1$, such that

$$\int_\Omega \mu \frac{\boldsymbol{p}_h^m - \boldsymbol{p}_h^{m-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{p}_h^m \cdot \mathbf{curl}\, \boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{q}_h^m = \int_\Omega \mu \frac{\boldsymbol{H}_h^m - \boldsymbol{H}_h^{m-1}}{\Delta t} \cdot \boldsymbol{G}_h \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega),$$
$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{p}_h^m \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{D}}} \frac{t_{m+1}\boldsymbol{J}_{\mathrm{S}}(t_{m+1}) - t_m \boldsymbol{J}_{\mathrm{S}}(t_m)}{\Delta t} \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})),$$
$$\boldsymbol{p}_h^0 = \boldsymbol{H}_h^1.$$

Notice that, this problem is similar to Problem 5. Therefore, it has a unique solution $(\boldsymbol{p}_h^k, \boldsymbol{q}_h^k)$, for all $k = 1, \dots, M-1$. Now, to prove the *a priori* estimate, we proceed as was done in the proof of Lemma 1 to derive (4.2)–(4.4) and obtain that there exists a unique $\widehat{\boldsymbol{p}}_h \in \mathrm{H}^1(0,T^*;\boldsymbol{\mathcal{N}}_h^0(\Omega))$ with $T^* := T - \Delta t$ such that

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \widehat{\boldsymbol{p}}_h(t) \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \frac{(t+\Delta t)\boldsymbol{J}_{\mathrm{S}}(t+\Delta t) - t\boldsymbol{J}_{\mathrm{S}}(t)}{\Delta t} \cdot \boldsymbol{F}_h \quad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})), \quad t \in [0,T^*]$$

and there exists a constant $C>0$, independent of h, such that

$$\|\widehat{\boldsymbol{p}}_h\|_{\mathrm{H}^1(0,T^*;\mathrm{H}(\mathbf{curl};\Omega))} \leq C \left\| \frac{(t+\Delta t)\boldsymbol{J}_{\mathrm{S}}(t+\Delta t) - t\boldsymbol{J}_{\mathrm{S}}(t)}{\Delta t} \right\|^2_{\mathrm{H}^1(0,T^*;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.$$

Moreover, by using classical computations we obtain

$$\|\widehat{\boldsymbol{p}}_h\|_{\mathrm{H}^1(0,T^*;\mathrm{H}(\mathbf{curl};\Omega))} \leq C \,\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)},$$

where C is independent of Δt, too. Then, following the steps of the proof of Lemma 1 and using the above estimate, we are able to prove that

$$\max_{1\leq k\leq M-1} \left\|\boldsymbol{p}_h^k\right\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{M-1} \left\| \frac{\boldsymbol{p}_h^k - \boldsymbol{p}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{M-1} \left\|\boldsymbol{q}_h^k\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}$$
$$\leq C \left\{ \left\|\boldsymbol{H}_h^1\right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} + \Delta t \sum_{k=1}^{M-1} \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} \right\}.$$

Thus, using the *a priori* estimates of Lemma 1 and the fact that $\boldsymbol{q}_h^k := \dfrac{t_{k+1}\boldsymbol{E}_h^{k+1} - t_k\boldsymbol{E}_h^k}{\Delta t}$, we conclude the proof. ☐

Lemma 7. *Let* $(\boldsymbol{H}_h^k, \boldsymbol{E}_h^k)$, $k=1,\ldots,M$, *be the solution to Problem 5 with* $\boldsymbol{H}_0 = \mathbf{0}$ *and* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)$ *with* $\boldsymbol{J}_{\mathrm{S}}(0) = \mathbf{0}$, *too. Then, there exists a constant* $C>0$ *independent of* h *and* Δt *such that*

$$\max_{1\leq k\leq M} t_k \left\| \boldsymbol{E}_h^k\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \leq C \,\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.$$

Proof. First, note that by taking $m=k$ and $m=k-1$ in Problem 5, subtracting the resulting expressions and dividing by Δt, we obtain

$$\int_\Omega \frac{\mu}{\Delta t}\left(\frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} - \frac{\boldsymbol{H}_h^{k-1} - \boldsymbol{H}_h^{k-2}}{\Delta t}\right)\cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\left(\frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t}\right)\cdot \mathbf{curl}\,\boldsymbol{G}_h$$
$$+ \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\,\boldsymbol{G}_h \cdot \left(\frac{\boldsymbol{E}_h^k - \boldsymbol{E}_h^{k-1}}{\Delta t}\right) = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega),$$

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\left(\frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t}\right)\cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \frac{\boldsymbol{J}_{\mathrm{S}}(t_k) - \boldsymbol{J}_{\mathrm{S}}(t_{k-1})}{\Delta t}\cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^{\Gamma}(\Omega_{\mathrm{D}})).$$

Consequently, $\boldsymbol{z}_h^k := \dfrac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t}$, $k=1,\ldots,M$, and $\boldsymbol{r}_h^k := \dfrac{\boldsymbol{E}_h^k - \boldsymbol{E}_h^{k-1}}{\Delta t}$, $k=2,\ldots,M$, are the solution of the following problem: Find $\boldsymbol{z}_h^k \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and $\boldsymbol{r}_h^k \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^{\Gamma}(\Omega_{\mathrm{D}}))$, $k=2,\ldots,M$, such that

$$\int_\Omega \mu \frac{\boldsymbol{z}_h^k - \boldsymbol{z}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \boldsymbol{z}_h^k \cdot \operatorname{\mathbf{curl}} \boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \operatorname{\mathbf{curl}} \boldsymbol{G}_h \cdot \boldsymbol{r}_h^k = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega),$$

$$\int_{\Omega_{\mathrm{D}}} \operatorname{\mathbf{curl}} \boldsymbol{z}_h^k \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{D}}} \frac{\boldsymbol{J}_{\mathrm{S}}(t_k) - \boldsymbol{J}_{\mathrm{S}}(t_{k-1})}{\Delta t} \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \operatorname{\mathbf{curl}}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})),$$

$$\boldsymbol{z}_h^1 = \frac{\boldsymbol{H}_h^1}{\Delta t}.$$

Notice that this problem is similar to Problem 5. Therefore, it has a unique solution $(\boldsymbol{z}_h^k, \boldsymbol{r}_h^k)$, $k = 2, \ldots, M$. Then, by proceeding as in the proof of Lemma 1, we derive the analogues to (4.2)–(4.4). In particular, we obtain that there exists a unique $\widehat{\boldsymbol{z}}_h \in \mathrm{H}^1(\Delta t, T; \boldsymbol{\mathcal{N}}_h^0(\Omega))$ such that

$$\int_{\Omega_{\mathrm{D}}} \operatorname{\mathbf{curl}} \widehat{\boldsymbol{z}}_h(t) \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{S}}} \frac{\boldsymbol{J}_{\mathrm{S}}(t) - \boldsymbol{J}_{\mathrm{S}}(t - \Delta t)}{\Delta t} \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \operatorname{\mathbf{curl}}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})), \quad t \in [\Delta t, T]$$

and that there exists a constant $C > 0$, independent of h and Δt, such that

$$\|\widehat{\boldsymbol{z}}_h\|^2_{\mathrm{H}^1(\Delta t, T; \mathrm{H}(\mathbf{curl};\Omega))} \leq C \left\| \frac{\boldsymbol{J}_{\mathrm{S}}(t) - \boldsymbol{J}_{\mathrm{S}}(t - \Delta t)}{\Delta t} \right\|^2_{\mathrm{H}^1(\Delta t, T; \mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \leq C \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}. \tag{7.4}$$

Now, if we write $\boldsymbol{z}_h^k = \widetilde{\boldsymbol{z}}_h^k + \widehat{\boldsymbol{z}}_h^k$ with $\widehat{\boldsymbol{z}}_h^k := \widehat{\boldsymbol{z}}_h(t_k)$, the problem above is equivalent to finding $\widetilde{\boldsymbol{z}}_h^1 := \frac{\boldsymbol{H}_h^1}{\Delta t} - \widehat{\boldsymbol{z}}_h^1$ and, for $k = 2, \ldots, M$, $\widetilde{\boldsymbol{z}}_h^k \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and $\boldsymbol{r}_h^k \in \operatorname{\mathbf{curl}}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}}))$ such that

$$\begin{aligned}\int_\Omega \mu \frac{\widetilde{\boldsymbol{z}}_h^k - \widetilde{\boldsymbol{z}}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h &+ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \widetilde{\boldsymbol{z}}_h^k \cdot \operatorname{\mathbf{curl}} \boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \operatorname{\mathbf{curl}} \boldsymbol{G}_h \cdot \boldsymbol{r}_h^k \\ &= -\int_\Omega \mu \frac{\widehat{\boldsymbol{z}}_h^k - \widehat{\boldsymbol{z}}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \widehat{\boldsymbol{z}}_h^k \cdot \operatorname{\mathbf{curl}} \boldsymbol{G}_h \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega),\end{aligned} \tag{7.5a}$$

$$\int_{\Omega_{\mathrm{D}}} \operatorname{\mathbf{curl}} \widetilde{\boldsymbol{z}}_h^k \cdot \boldsymbol{F}_h = 0 \qquad \forall \boldsymbol{F}_h \in \operatorname{\mathbf{curl}}(\boldsymbol{\mathcal{N}}_h^\Gamma(\Omega_{\mathrm{D}})). \tag{7.5b}$$

By taking $\boldsymbol{G}_h = t_k \widetilde{\boldsymbol{z}}_h^k$ in (7.5a), using (7.5b), the inequality $2\,(p-q)\,p \geq p^2 - q^2$ and Young's inequality, we write

$$\begin{aligned}&\frac{1}{2\Delta t} \int_\Omega \mu\, t_k \left(\left|\widetilde{\boldsymbol{z}}_h^k\right|^2 - \left|\widetilde{\boldsymbol{z}}_h^{k-1}\right|^2 \right) + \frac{t_k}{\overline{\sigma}} \left\| \operatorname{\mathbf{curl}} \widetilde{\boldsymbol{z}}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \\ &\leq \frac{1}{4T} \int_\Omega \mu\, t_k \left|\widetilde{\boldsymbol{z}}_h^k\right|^2 + T \int_\Omega \mu\, t_k \left| \frac{\widehat{\boldsymbol{z}}_h^k - \widehat{\boldsymbol{z}}_h^{k-1}}{\Delta t} \right|^2 + \frac{t_k}{2\overline{\sigma}} \left\| \operatorname{\mathbf{curl}} \widetilde{\boldsymbol{z}}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \frac{t_k\, \overline{\sigma}}{2\underline{\sigma}^2} \left\| \operatorname{\mathbf{curl}} \widehat{\boldsymbol{z}}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}.\end{aligned}$$

Thus, multiplying by $2\Delta t$, summing up from $k = 2$ to m ($m \leq M$) and using the fact that

$$\frac{1}{2\Delta t} \int_\Omega \mu\, t_k \left(\left|\widetilde{\boldsymbol{z}}_h^k\right|^2 - \left|\widetilde{\boldsymbol{z}}_h^{k-1}\right|^2 \right) = \frac{1}{2\Delta t} \left(\int_\Omega \mu\, t_k \left|\widetilde{\boldsymbol{z}}_h^k\right|^2 - \int_\Omega \mu\, t_{k-1} \left|\widetilde{\boldsymbol{z}}_h^{k-1}\right|^2 \right) - \frac{1}{2} \int_\Omega \mu \left|\widetilde{\boldsymbol{z}}_h^k\right|^2,$$

we obtain

$$\int_\Omega \mu\, t_m \left|\widetilde{\boldsymbol{z}}_h^m\right|^2 + \frac{1}{\underline{\sigma}}\Delta t \sum_{k=2}^{m} t_k \left\| \mathbf{curl}\, \widetilde{\boldsymbol{z}}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}$$
$$\leq \frac{1}{2T}\,\Delta t \sum_{k=2}^{m} \int_\Omega \mu\, t_k \left|\widetilde{\boldsymbol{z}}_h^k\right|^2 + 2T^2\,\overline{\mu}\,\Delta t \sum_{k=2}^{m} \left\| \frac{\widehat{\boldsymbol{z}}_h^k - \widehat{\boldsymbol{z}}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3}$$
$$+ \overline{\mu}\,\Delta t \sum_{k=1}^{m} \left\|\widetilde{\boldsymbol{z}}_h^k\right\|^2_{\mathrm{L}^2(\Omega)^3} + \frac{T\,\overline{\sigma}}{\underline{\sigma}^2}\,\Delta t \sum_{k=2}^{m} \left\| \mathbf{curl}\, \widehat{\boldsymbol{z}}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}.$$

Now, taking into account that $\widetilde{\boldsymbol{z}}_h^k = \boldsymbol{z}_h^k - \widehat{\boldsymbol{z}}_h^k = \dfrac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} - \widehat{\boldsymbol{z}}_h(t_k)$ and $\|\widehat{\boldsymbol{z}}_h(t_k)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C\|\widehat{\boldsymbol{z}}_h\|^2_{\mathrm{H}^1(\Delta t,T;\mathrm{H}(\mathbf{curl};\Omega))}$, from Lemma 1 and (7.4), we have

$$\Delta t \sum_{k=1}^{m} \left\|\widetilde{\boldsymbol{z}}_h^k\right\|^2_{\mathrm{L}^2(\Omega)^3} \leq 2\Delta t \sum_{k=1}^{m} \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} + 2\Delta t \sum_{k=1}^{m} \left\|\widehat{\boldsymbol{z}}_h^k\right\|^2_{\mathrm{L}^2(\Omega)^3} \leq C\,\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.$$

Then, replacing this estimate in the above inequality and using the discrete Gronwall's inequality, (7.4) and the fact that

$$\Delta t \sum_{k=2}^{M} \left\| \frac{\widehat{\boldsymbol{z}}_h^k - \widehat{\boldsymbol{z}}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} \leq C\,\|\widehat{\boldsymbol{z}}_h\|^2_{\mathrm{H}^1(\Delta t,T;\mathrm{H}(\mathbf{curl};\Omega))}$$

lead to

$$\max_{2\leq k\leq M} t_k \left\|\widetilde{\boldsymbol{z}}_h^k\right\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=2}^{M} t_k \left\| \mathbf{curl}\, \widetilde{\boldsymbol{z}}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \leq C\,\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.$$

Therefore, using again that $\|\widehat{\boldsymbol{z}}_h(t_k)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C\,\|\widehat{\boldsymbol{z}}_h\|^2_{\mathrm{H}^1(\Delta t,T;\mathrm{H}(\mathbf{curl};\Omega))}$ and (7.4),

$$\max_{2\leq k\leq M} t_k \left\|\boldsymbol{z}_h^k\right\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=2}^{M} t_k \left\| \mathbf{curl}\, \boldsymbol{z}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \leq C\,\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.$$

Hence, since $\boldsymbol{z}_h^k := \dfrac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t}$, we have proved that

$$\max_{2\leq k\leq M} t_k \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} \leq C\,\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.$$

There remains to prove an analogous estimate for $k = 1$. With this end, we proceed as follows. First we use Lemma 1 and the fact that $\boldsymbol{H}_h^0 = \boldsymbol{H}_0 = \mathbf{0}$ to write

$$t_1 \left\| \frac{\boldsymbol{H}_h^1 - \boldsymbol{H}_h^0}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} = \frac{1}{\Delta t} \left\|\boldsymbol{H}_h^1\right\|^2_{\mathrm{L}^2(\Omega)^3} \leq C\frac{1}{\Delta t}\,\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,t_1;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}.$$

Then, standard computations allow us to show that

$$\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{L}^2(0,t_1;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \leq 2\Delta t \left(\|\boldsymbol{J}_{\mathrm{S}}(0)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{S}})^3} + \Delta t \, \|\partial_t \boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{L}^2(0,t_1;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \right) \leq 2\Delta t^2 \, \|\partial_t \boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{L}^2(0,t_1;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \,,$$

where we have used that $\boldsymbol{J}_{\mathrm{S}}(0) = \mathbf{0}$, and, analogously,

$$\|\partial_t \boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{L}^2(0,t_1;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \leq 2\Delta t \left(\|\partial_t \boldsymbol{J}_{\mathrm{S}}(0)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{S}})^3} + \Delta t \, \|\partial_{tt} \boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{L}^2(0,t_1;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \right) \leq C \Delta t \, \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,t_1;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \,.$$

Combining the last four estimates we have that

$$\max_{1 \leq k \leq M} t_k \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} \leq C \, \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \,.$$

Hence, as a consequence of the *inf-sup* condition (4.1) and the first equation of Problem 5 multiplied by t_k, for $k = 1 \ldots, M$ we have that

$$\beta^* \sqrt{t_k} \left\| \boldsymbol{E}_h^k \right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \leq C \left\{ t_k \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} + t_k \left\| \operatorname{\mathbf{curl}} \boldsymbol{H}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \right\}^{1/2} \leq C \, \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \,,$$

where we have also used the *a priori* estimate from Lemma 1. □

Now, we are in a position to obtain $\mathcal{O}(\varepsilon)$ estimates for $\sqrt{t_k}\left(\boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k\right)$.

Theorem 5. *Let* $(\boldsymbol{H}_h^k, \boldsymbol{E}_h^k)$*,* $k = 1, \ldots, M$*, be the solution to Problem 5 and* $\boldsymbol{H}_{h,\varepsilon}^k$*,* $k = 1, \ldots, M$*, that to Problem 4. If* $\boldsymbol{H}_0 = \mathbf{0}$ *and* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)$ *with* $\boldsymbol{J}_{\mathrm{S}}(0) = \mathbf{0}$*, then there exists a constant* $C > 0$*, independent of* ε*,* h *and* Δt*, such that*

$$\max_{1 \leq k \leq M} t_k \left\| \boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k \right\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{M} t_k \left\| \operatorname{\mathbf{curl}} \left(\boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k \right) \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}$$
$$+ \varepsilon \Delta t \sum_{k=1}^{M} t_k \left\| \boldsymbol{E}_{h,\varepsilon}^k - \boldsymbol{E}_h^k \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \leq C \varepsilon^2.$$

Proof. Let $\boldsymbol{u}_h^k$ and $\boldsymbol{v}_h^k$ be defined as in the proof of Lemma 1. Taking $\boldsymbol{G}_h = t_k \boldsymbol{u}_h^k$ and $\boldsymbol{F}_h = t_k \boldsymbol{v}_h^k$ in (6.2a) and (6.2b), respectively, we obtain

$$\int_\Omega \mu t_k \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t} \cdot \boldsymbol{u}_h^k + \int_{\Omega_{\mathrm{C}}} \frac{t_k}{\sigma} \left| \operatorname{\mathbf{curl}} \boldsymbol{u}_h^k \right|^2 + \varepsilon \int_{\Omega_{\mathrm{D}}} t_k \left| \boldsymbol{v}_h^k \right|^2 = -\varepsilon \int_{\Omega_{\mathrm{D}}} t_k \boldsymbol{E}_h^k \cdot \boldsymbol{v}_h^k. \tag{7.6}$$

Note that for all $\boldsymbol{E}_h^k$, $k = 1, \ldots, M$, there exists a unique $\boldsymbol{y}_h^k \in \boldsymbol{\mathcal{Y}}_h^{\perp_{\boldsymbol{\mathcal{N}}_h^0(\Omega)}}$ such that

$$\int_{\Omega_{\mathrm{D}}} \operatorname{\mathbf{curl}} \boldsymbol{y}_h^k \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{D}}} \boldsymbol{E}_h^k \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \operatorname{\mathbf{curl}}(\boldsymbol{\mathcal{N}}_h^{\mathrm{T}}(\Omega_{\mathrm{D}})) \tag{7.7}$$

and there exists a constant $C > 0$, independent of h, such that

$$\|\boldsymbol{y}_h^k\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C \left\|\boldsymbol{E}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \tag{7.8}$$

(see [13, Lemma I.4.1(iii)]). In particular

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\left(\frac{t_{k+1}\boldsymbol{y}_h^{k+1} - t_k\boldsymbol{y}_h^k}{\Delta t}\right)\cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{D}}} \frac{t_{k+1}\boldsymbol{E}_h^{k+1} - t_k\boldsymbol{E}_h^k}{\Delta t}\cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}(\boldsymbol{\mathcal{N}}_h^{\Gamma}(\Omega_{\mathrm{D}}))$$

and

$$\Delta t \sum_{k=1}^{M-1} \left\|\frac{t_{k+1}\boldsymbol{y}_h^{k+1} - t_k\boldsymbol{y}_h^k}{\Delta t}\right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 \leq C\Delta t \sum_{k=1}^{M-1} \left\|\frac{t_{k+1}\boldsymbol{E}_h^{k+1} - t_k\boldsymbol{E}_h^k}{\Delta t}\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2. \tag{7.9}$$

Substituting (7.7) in (7.6) and using (6.2a) we obtain

$$\begin{aligned}
\int_{\Omega} \mu t_k \,\frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t}\cdot \boldsymbol{u}_h^k + \int_{\Omega_{\mathrm{C}}} \frac{t_k}{\sigma}\left|\mathbf{curl}\,\boldsymbol{u}_h^k\right|^2 + \varepsilon \int_{\Omega_{\mathrm{D}}} t_k \left|\boldsymbol{v}_h^k\right|^2 &\\
= \varepsilon \int_{\Omega} \mu t_k \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t}\cdot \boldsymbol{y}_h^k + \varepsilon \int_{\Omega_{\mathrm{C}}} \frac{t_k}{\sigma}\,\mathbf{curl}\,\boldsymbol{u}_h^k \cdot \mathbf{curl}\,\boldsymbol{y}_h^k &\\
\leq \frac{1}{2}\int_{\Omega_{\mathrm{C}}} \frac{t_k}{\sigma}\left|\mathbf{curl}\,\boldsymbol{u}_h^k\right|^2 + \frac{\varepsilon^2}{2}\int_{\Omega_{\mathrm{C}}} \frac{t_k}{\sigma}\left|\mathbf{curl}\,\boldsymbol{y}_h^k\right|^2 + \varepsilon \int_{\Omega} \mu t_k \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t}\cdot \boldsymbol{y}_h^k &
\end{aligned}$$

and, hence,

$$\begin{aligned}
\frac{1}{2\Delta t}\int_{\Omega} \mu t_k \left|\boldsymbol{u}_h^k\right|^2 - \frac{1}{2\Delta t}\int_{\Omega} \mu t_k \left|\boldsymbol{u}_h^{k-1}\right|^2 + \frac{1}{2}\int_{\Omega_{\mathrm{C}}} \frac{t_k}{\sigma}\left|\mathbf{curl}\,\boldsymbol{u}_h^k\right|^2 + \varepsilon t_k \left\|\boldsymbol{v}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 &\\
\leq \frac{\varepsilon^2}{2}\int_{\Omega_{\mathrm{C}}} \frac{t_k}{\sigma}\left|\mathbf{curl}\,\boldsymbol{y}_h^k\right|^2 + \varepsilon \int_{\Omega} \mu\, t_k \,\frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t}\cdot \boldsymbol{y}_h^k. &
\end{aligned}$$

Thus, multiplying by $2\Delta t$ and summing up from $k = 1$ to m ($m \leq M$) and using the fact that

$$\sum_{k=1}^{m} \int_{\Omega} \mu\, t_k \,\frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t}\cdot \boldsymbol{y}_h^k = \frac{1}{\Delta t}\int_{\Omega} \mu t_m \,\boldsymbol{u}_h^m \cdot \boldsymbol{y}_h^m - \sum_{k=1}^{m-1}\int_{\Omega} \mu \,\frac{t_{k+1}\boldsymbol{y}_h^{k+1} - t_k\boldsymbol{y}_h^k}{\Delta t}\cdot \boldsymbol{u}_h^k,$$

we obtain

$$\begin{aligned}
t_m \left\|\boldsymbol{u}_h^m\right\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^{m} t_k \left\|\mathbf{curl}\,\boldsymbol{u}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 + \varepsilon\Delta t \sum_{k=1}^{m} t_k \left\|\boldsymbol{v}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3}^2 &\\
\leq C\Bigg\{ \varepsilon^2 t_m \left\|\boldsymbol{y}_h^m\right\|_{\mathrm{L}^2(\Omega)^3}^2 + \varepsilon^2 \Delta t \sum_{k=1}^{m} \left\|\mathbf{curl}\,\boldsymbol{y}_h^k\right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 + \Delta t \sum_{k=1}^{m-1} \left\|\boldsymbol{u}_h^k\right\|_{\mathrm{L}^2(\Omega)^3}^2 &\\
+ \varepsilon^2 \Delta t \sum_{k=1}^{m-1} \left\|\frac{t_{k+1}\boldsymbol{y}_h^{k+1} - t_k\boldsymbol{y}_h^k}{\Delta t}\right\|_{\mathrm{L}^2(\Omega)^3}^2 \Bigg\}. &
\end{aligned}$$

Using (7.8), Lemma 1, (7.9), Lemma 6, Lemma 7 and Lemma 5, we have

$$t_m \|\boldsymbol{u}_h^m\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{m} t_k \left\|\mathbf{curl}\, \boldsymbol{u}_h^k\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \varepsilon \Delta t \sum_{k=1}^{m} t_k \left\|\boldsymbol{v}_h^k\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} \leq C\varepsilon^2$$

Thus, the estimate of the theorem follows from the fact that $\boldsymbol{u}_h^k := \boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k$ and $\boldsymbol{v}_h^k := \boldsymbol{E}_{h,\varepsilon}^k - \boldsymbol{E}_h^k$. □

Finally, we are in a position to write the main result of this section.

Theorem 6. *Let $\boldsymbol{H}$ and $\boldsymbol{H}_{h,\varepsilon}^k$, $k = 1, \dots, M$, be the solutions to Problem 1 and Problem 4, respectively, with $\boldsymbol{H}_0 = \mathbf{0}$, $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)$ and $\boldsymbol{J}_{\mathrm{S}}(0) = \mathbf{0}$. If $\boldsymbol{H} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega)) \cap \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)$ with $r \in \left(\frac{1}{2}, 1\right]$, then there exists a constant $C > 0$ independent of ε, h and Δt such that*

$$\max_{1\leq k\leq M} t_k \left\|\boldsymbol{H}(t_k) - \boldsymbol{H}_{h,\varepsilon}^k\right\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{M} t_k \left\| \mathbf{curl} \left(\boldsymbol{H}(t_k) - \boldsymbol{H}_{h,\varepsilon}^k\right)\right\|^2_{\mathrm{L}^2(\Omega)^3} \leq C \left\{(\Delta t)^2 + h^{2r} + \varepsilon^2\right\}.$$

Proof. The result follows from the equivalence between primal and mixed formulations of the problem (cf. Theorem 2), Theorem 3 and Theorem 5. □

8. Numerical tests

In this section, we report some numerical results obtained with a MATLAB code that implements the penalty technique described above. First, we have applied it to a realistic test whose geometry fits in the above theoretical framework. The aim of this test is to illustrate the convergence with respect to the penalty parameter ε. Next, in order to illustrate the convergence with respect to the discretization parameters, we have considered an academic example with a known analytical solution. Similar tests but with a moving conducting domain Ω_{C} (which therefore lie beyond the theoretical scope of this paper) can be found in [10].

8.1. Test 1: Convergence with respect to the penalty parameter

Let us consider the geometry sketched in Figure 2, which includes a toroidal coil Ω_{S}, a conducting piece Ω_{C} and the air around.

The source current density, which is supported in Ω_{S}, is given by

$$\boldsymbol{J}_{\mathrm{S}}(t, \boldsymbol{x}) = \frac{I(t)}{\mathrm{meas}(S)} \begin{pmatrix} -\frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ 0 \end{pmatrix} \quad \text{in } \Omega_{\mathrm{S}},$$

where the current intensity $I(t)$ is shown in Figure 3. Concerning the physical parameters, we have taken $\mu = \mu_0 = 4\pi \times 10^{-7}\,\mathrm{Hm}^{-1}$ (the magnetic permeability of vacuum) and $\sigma = 10^6\,(\Omega\mathrm{m})^{-1}$ in the conducting piece.

To solve this problem, we have implemented Problems 4 and 5 and compared the results at each time step for different values of the parameter ε. Notice that the implementation of Problem 5 would in principle require to have a basis of the discrete space $\mathbf{curl}(\boldsymbol{\mathcal{N}}_h^{\Gamma}(\Omega_{\mathrm{D}}))$, which is not easily

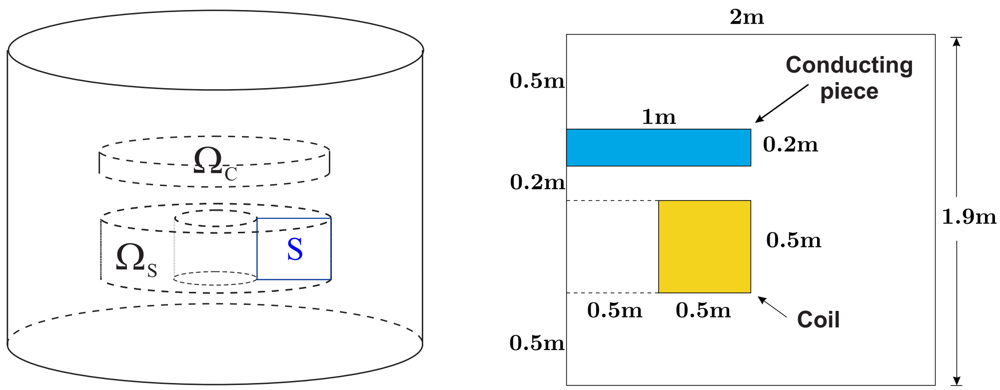

Figure 2: Sketch of the domain for Test 1 (left). Meridian section (right).

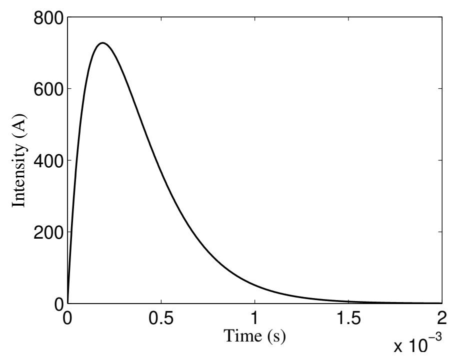

Figure 3: Source current intensity (A) vs. time (s) for Test 1.

available in practice. To circumvent this drawback, we have used the standard basis functions of $\boldsymbol{\mathcal{N}}_h^{\Gamma}(\Omega_{\mathrm{D}})$, to construct with their curls a (not linearly independent) spanning set. By so doing, at each time-step, we have been led to solve a singular system of linear equations, well determined in the sense that it has an (obviously non-unique) solution. The rank-degenerate linear system is solved in the least-square sense, which can be easily done in MATLAB environment.

In this case, we have focused on the penalty error. With this end, we have considered the solutions $\boldsymbol{H}_{h,\varepsilon}^m$ and $\boldsymbol{H}_h^m$ of Problem 4 and 5, respectively, with fixed mesh-size h, fixed time-step Δt and ε varying from 10^0 to 10^{-4}. Notice that the difference between the solutions $\boldsymbol{H}_{h,\varepsilon}^k$ and $\boldsymbol{H}_h^k$ of these two problems is due only to the penalty approach. We have computed the following percentage errors:

$$100\,\frac{\max\limits_{1\le k\le M}\left\|\boldsymbol{H}_{h,\varepsilon}^k-\boldsymbol{H}_h^k\right\|_{\mathrm{L}^2(\Omega)^3}}{\max\limits_{1\le k\le M}\left\|\boldsymbol{H}_h^k\right\|_{\mathrm{L}^2(\Omega)^3}} \qquad \text{and} \qquad 100\,\frac{\left\{\Delta t\sum_{k=1}^M\left\|\boldsymbol{H}_{h,\varepsilon}^k-\boldsymbol{H}_h^k\right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2\right\}^{1/2}}{\left\{\Delta t\sum_{k=1}^M\left\|\boldsymbol{H}_h^k\right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2\right\}^{1/2}},$$

which are time-discrete forms of the errors in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)^3)$ and $\mathrm{L}^2(0,T;\mathrm{H}(\mathbf{curl};\Omega))$ norms, respectively.

Table 1: Test 1. Percentage penalty errors.

ε	ε/σ	$\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)^3)$	$\mathrm{L}^2(0,T;\mathrm{H}(\mathbf{curl};\Omega))$
10^{0}	10^{-6}	0.2690402	0.1184663
10^{-1}	10^{-7}	0.0270514	0.0118915
10^{-2}	10^{-8}	0.0027066	0.0011896
10^{-3}	10^{-9}	0.0002707	0.0001192
10^{-4}	10^{-10}	0.0000272	0.0000166

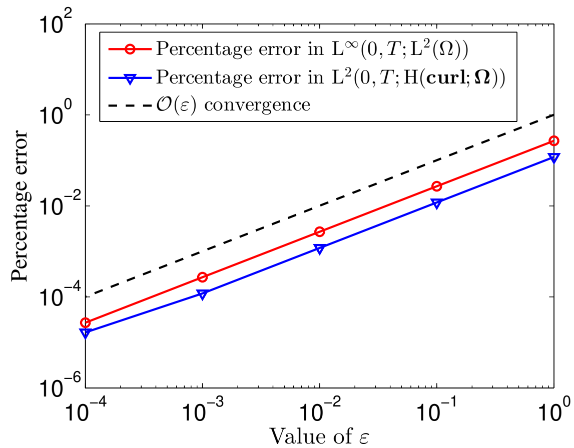

Figure 4: Percentage penalty error curves in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)^3)$ (left) and $\mathrm{L}^2(0,T;\mathrm{H}(\mathbf{curl};\Omega))$ (right) discrete norms, for the numerical solution of Test 1 computed on a mesh with 8448 elements and a time-step $\Delta t = 10^{-4}\,$s.

We report in Table 1 and Figure 4 these errors for a fixed mesh with 8448 elements, a fixed time step $\Delta t = 10^{-4}$ and different values of the penalty parameter $\varepsilon \in (0,1]$. We also include in the table the relative values of ε with respect to the conductivity $\sigma = 10^6$ used in this test. The numerical results show a clear linear convergence with respect to the parameter ε until it becomes too small. Indeed, for values of $\varepsilon/\sigma < 10^{-10}$, the convergence deteriorates due to ill-conditioning of the resulting linear system, but, in such a case, the percentage errors are already extremely small.

Let us remark that we only report results for one fixed mesh and one time-step, because we have checked that the penalty errors do not change significantly when the experiments are repeated with different discretization parameters.

8.2. Test 2: Convergence with respect to the discretization parameters

Let us consider a conducting domain Ω_{C} occupied by the cube $(0,1)\times(0,1)\times(1,2)$, being the whole domain $\Omega := (0,1)\times(0,1)\times(0,3)$ (see Figure 5). We have applied our code to solve the following source problem:

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{H} &= \sigma \boldsymbol{E} \qquad && \text{in } (0,T)\times\Omega, \\ \partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} &= \boldsymbol{f} \qquad && \text{in } (0,T)\times\Omega, \end{aligned}$$

where $\boldsymbol{f}$ is a given data and $T = \frac{1}{2}$.

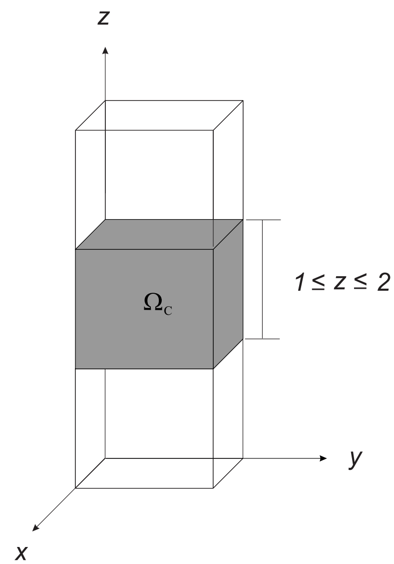

Figure 5: Test 2. Sketch of the domain.

We have used for this test the same physical parameters μ and σ as in the previous one. The data $\boldsymbol{f}$ has been chosen so that the analytical solution be

$$\boldsymbol{H}(t,\boldsymbol{x}) := t^2 \begin{pmatrix} \varphi(z) \\ \varphi(z) \\ z \end{pmatrix} \qquad \text{with} \quad \varphi(z) := \begin{cases} (z-1)^2\,(z-2)^2\,, & z\in[1,2], \\ 0, & z\notin[1,2] \end{cases}$$

and

$$\boldsymbol{E}(t,\boldsymbol{x}) := \begin{cases} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}(t,\boldsymbol{x}) & \text{in } \Omega_{\mathrm{C}}, \\ \mathbf{0} & \text{in } \Omega_{\mathrm{D}}. \end{cases}$$

Notice that $\mathbf{curl}\,\boldsymbol{H}(t) = \mathbf{0}$ in Ω_{D} for all $t\in[0,T]$. This is the constraint that has to be penalized, since there is no source current $\boldsymbol{J}_{\mathrm{S}}$ in this test. Given $\varepsilon > 0$, the corresponding penalized problem reads as follows: find $\boldsymbol{H}_\varepsilon \in \mathrm{L}^2(0,T;\mathrm{H}(\mathbf{curl};\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ such that

$$\frac{d}{dt}\int_\Omega \mu\boldsymbol{H}_\varepsilon\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}_\varepsilon\cdot\mathbf{curl}\,\boldsymbol{G} + \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\,\boldsymbol{H}_\varepsilon\cdot\mathbf{curl}\,\boldsymbol{G} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{G} + \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{G} \qquad \forall \boldsymbol{G}\in\mathrm{H}(\mathbf{curl};\Omega),$$

$$\boldsymbol{H}_\varepsilon(0) = \boldsymbol{H}_0 \qquad \text{in } \Omega,$$

where the exact values of $\boldsymbol{f}$ in Ω and $\boldsymbol{g} := \boldsymbol{E} \times \boldsymbol{n}$ on Γ have been used as problem data.

These equations have been discretized by using Nédélec finite elements in space and the backward Euler method in time, leading to a scheme similar to that in Problem 4. To assess the dependence of the errors on the discretization parameters h and Δt, we have chosen a sufficiently small fixed value of the penalty parameter: $\varepsilon = 10^{-4}$. We have checked that the errors almost do not change when using other values of the penalty parameter in the range between 10^{-3} and 10^{-5}, which is an evidence of the fact that the penalty errors are absolutely negligible for such small values of ε (as it happened in the previous test, too).

For this test, we have computed the actual errors, namely the differences between the obtained numerical solution and the analytical one. To report in the same figure the dependence of the errors with respect to both discretization parameters, we have used different meshes (the coarsest one with 144 elements) and, for each mesh, we have used time steps Δt proportional to the mesh-size h (the coarsest time step being $\Delta t = \frac{1}{20}$). We report in Figure 6 the corresponding error curves, namely log-log plots of the respective errors versus the number of degrees of freedom (d.o.f.) (i.e., the number of unknowns of the linear system to be solved at each time step). A clear linear dependence $\mathcal{O}(h + \Delta t)$ can be easily appreciated from these curves.

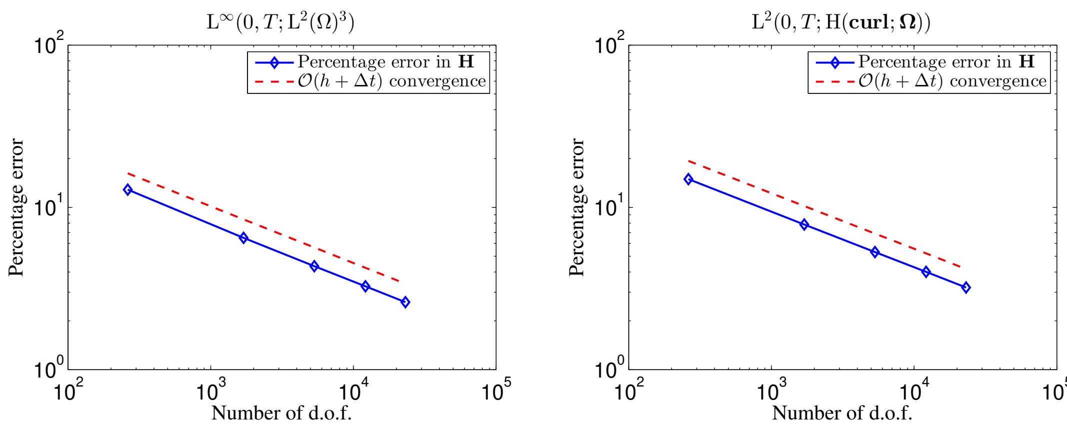

Figure 6: Percentage discretization error curves in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)^3)$ (left) and $\mathrm{L}^2(0,T;\mathrm{H}(\mathbf{curl};\Omega))$ (right) discrete norms for the numerical solution of Test 2 ($\varepsilon = 10^{-4}$).

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Appendix A. Analysis of the penalty technique for the continuous problem

In Section 3, we introduced Problem 3, which is a penalized form of problem (2.1)–(2.3), and we claimed that Problem 3 is well posed and that its solution converges to that of (2.1)–(2.3) as the penalty parameter goes to zero. These two assertions were not proved in that section, because they were not needed for the subsequent analysis. However, they are interesting by themselves. This is the reason why we prove them in this appendix.

First, we show that Problem 3 has a unique solution and that this solution is bounded independently of the penalty parameter ε.

Theorem 7. *Problem 3 has a unique solution that satisfies* $\boldsymbol{H}_\varepsilon \in \mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ *and there exists a constant* $C>0$*, independent of* ε*,* $\boldsymbol{J}_{\mathrm{S}}$ *and* $\boldsymbol{H}_0$*, such that*

$$\|\boldsymbol{H}_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}(\mathbf{curl};\Omega))} + \|\partial_t \boldsymbol{H}_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)} \leq C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}\right\}.$$

Proof. The existence of a unique solution $\boldsymbol{H}_\varepsilon$ follows by applying Corollary A.2 from [9], where, in particular, it is shown that the assumption $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$ yields $\boldsymbol{H}_\varepsilon \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$. The proof of the *a priori* estimate is very similar to that of Theorem 1. Thus, we only give a sketch of this proof. We consider $\widehat{\boldsymbol{H}}$ as in the proof of Theorem 1, so that in particular (2.10) and (2.11) hold true. We define $\widetilde{\boldsymbol{H}}_\varepsilon := \boldsymbol{H}_\varepsilon - \widehat{\boldsymbol{H}}$. Then, the same arguments as in Theorem 1 allow us to show that $\widetilde{\boldsymbol{H}}_\varepsilon \in \mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ and satisfies

$$\int_\Omega \mu\,\partial_t \widetilde{\boldsymbol{H}}_\varepsilon \cdot \boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}_\varepsilon \cdot \mathbf{curl}\,\boldsymbol{G} + \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\,\widetilde{\boldsymbol{H}}_\varepsilon \cdot \mathbf{curl}\,\boldsymbol{G} = -\int_\Omega \mu\,\partial_t \widehat{\boldsymbol{H}} \cdot \boldsymbol{G} \qquad \forall \boldsymbol{G} \in \mathrm{H}_0(\mathbf{curl};\Omega),$$

$$\widetilde{\boldsymbol{H}}_\varepsilon(0) = \boldsymbol{H}_0 - \widehat{\boldsymbol{H}}(0).$$

Therefore, applying similar techniques to those used in the proof of Theorem A.1 and Corollary A.2 from [9], we obtain that $\widetilde{\boldsymbol{H}}_\varepsilon \in \mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ and that there exists a positive constant C independent of ε, $\boldsymbol{J}_{\mathrm{S}}$ and $\boldsymbol{H}_0$ such that

$$\|\widetilde{\boldsymbol{H}}_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}(\mathbf{curl};\Omega))} + \|\partial_t \widetilde{\boldsymbol{H}}_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)} \\ \leq C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} + \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{D}}} \left|\mathbf{curl}\,\widetilde{\boldsymbol{H}}_\varepsilon(0)\right|^2\right\}.$$

Since the last term on the right-hand side above vanishes, because $\mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{J}_{\mathrm{S}}(0) = \mathbf{curl}\,\widehat{\boldsymbol{H}}(0)$ (cf. (2.4) and (2.11)), the result follows from (2.10) and the above estimate. $\square$

Finally, we prove that $\boldsymbol{H}_\varepsilon \to \boldsymbol{H}$ as $\varepsilon \to 0$.

Theorem 8. *Let $(\boldsymbol{H},\boldsymbol{E})$ be the solution to Problem 2. Let $\boldsymbol{H}_\varepsilon$ be the solution to Problem 3 and $\boldsymbol{E}_\varepsilon := \frac{1}{\varepsilon}\left(\mathbf{curl}\,\boldsymbol{H}_\varepsilon - \boldsymbol{J}_{\mathrm{S}}\right)|_{\Omega_{\mathrm{D}}}$. Then, there exists $C>0$ independent of ε, $\boldsymbol{H}_0$ and $\boldsymbol{J}_{\mathrm{S}}$, such that*

$$\|\boldsymbol{H}-\boldsymbol{H}_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)^3)} + \|\mathbf{curl}\,\boldsymbol{H}-\mathbf{curl}\,\boldsymbol{H}_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)} + \sqrt{\varepsilon}\,\|\boldsymbol{E}-\boldsymbol{E}_\varepsilon\|_{L^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{D}})^3)} \\ \leq C\sqrt{\varepsilon}\left\{\|\boldsymbol{H}_0\|_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}\right\}.$$

Proof. According to its definition, $\boldsymbol{E}_\varepsilon \in \mathrm{L}^2(0,T;\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}}))$ and $(\boldsymbol{H}_\varepsilon,\boldsymbol{E}_\varepsilon)$ satisfies

$$\int_\Omega \mu\,\partial_t\boldsymbol{H}_\varepsilon\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}_\varepsilon\cdot\mathbf{curl}\,\boldsymbol{G} + \int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{E}_\varepsilon = 0 \qquad \forall\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega), \tag{A.1a}$$

$$\int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{H}_\varepsilon\cdot\boldsymbol{F} - \varepsilon\int_{\Omega_{\mathrm{D}}}\boldsymbol{E}_\varepsilon\cdot\boldsymbol{F} = \int_{\Omega_{\mathrm{S}}}\boldsymbol{J}_{\mathrm{S}}\cdot\boldsymbol{F} \qquad \forall\boldsymbol{F}\in\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}}), \tag{A.1b}$$

$$\boldsymbol{H}_\varepsilon(0) = \boldsymbol{H}_0. \tag{A.1c}$$

For $\boldsymbol{H}_\varepsilon$ we have already proved an *a priori* estimate in Theorem 7. For $\boldsymbol{E}_\varepsilon$, we use the *inf-sup* condition (3.2), the estimate from Theorem 7 and (A.1a) to write

$$\|\boldsymbol{E}_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{D}})^3)} \leq C\left\{\|\boldsymbol{H}_0\|_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}\right\}. \tag{A.2}$$

Next, we denote $\boldsymbol{u} := \boldsymbol{H}_\varepsilon - \boldsymbol{H}$ and $\boldsymbol{v} := \boldsymbol{E}_\varepsilon - \boldsymbol{E}$. Subtracting (A.1a)–(A.1c) from the corresponding equations of Problem 2 yields

$$\int_\Omega \mu\,\partial_t\boldsymbol{u}\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{G} + \int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{v} = 0 \qquad \forall\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega), \tag{A.3a}$$

$$\int_{\Omega_{\mathrm{D}}}\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{F} - \varepsilon\int_{\Omega_{\mathrm{D}}}\boldsymbol{v}\cdot\boldsymbol{F} = \varepsilon\int_{\Omega_{\mathrm{D}}}\boldsymbol{E}\cdot\boldsymbol{F} \qquad \forall\boldsymbol{F}\in\mathrm{H}_\Gamma(\mathrm{div}^0;\Omega_{\mathrm{D}}), \tag{A.3b}$$

$$\boldsymbol{u}(0) = \boldsymbol{0}. \tag{A.3c}$$

By taking $\boldsymbol{G} = \boldsymbol{u}(t)$ and $\boldsymbol{F} = \boldsymbol{v}(t)$ in (A.3a) and (A.3b), respectively, subtracting the resulting expressions, integrating in time, using (A.3c) and Young's inequality, we obtain

$$\frac{1}{2}\underline{\mu}\,\|\boldsymbol{u}(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \frac{1}{\overline{\sigma}}\int_0^t \|\mathbf{curl}\,\boldsymbol{u}(s)\|^2_{\Omega_{\mathrm{C}}}\,ds + \frac{\varepsilon}{2}\int_0^t \|\boldsymbol{v}(s)\|^2_{\Omega_{\mathrm{D}}}\,ds \leq \frac{\varepsilon}{2}\int_0^t \|\boldsymbol{E}(s)\|^2_{\Omega_{\mathrm{D}}}\,ds \\ \leq C\varepsilon\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)}\right\} \qquad \forall t\in[0,T],$$

where we have used (3.1) for the last inequality.

Thus, there only remains to estimate $\|\operatorname{\mathbf{curl}} \boldsymbol{u}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{D}})^3)}$. With this end, we take $\boldsymbol{F} = \operatorname{\mathbf{curl}} \boldsymbol{u} - \varepsilon \boldsymbol{v} - \varepsilon \boldsymbol{E} = \operatorname{\mathbf{curl}} \boldsymbol{u} - \varepsilon \boldsymbol{E}_\varepsilon$ in (A.3b) to obtain that $\operatorname{\mathbf{curl}} \boldsymbol{u}(t) = \varepsilon \boldsymbol{E}_\varepsilon(t)$ in Ω_{D}. Hence, by using (A.2), we conclude that

$$\|\operatorname{\mathbf{curl}} \boldsymbol{u}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{D}})^3)} = \varepsilon \,\|\boldsymbol{E}_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{D}})^3)} \leq C\varepsilon \left\{ \|\boldsymbol{H}_0\|_{\mathrm{H}(\operatorname{\mathbf{curl}};\Omega)} + \|\boldsymbol{J}_{\mathrm{s}}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^3)} \right\}.$$

Therefore, the theorem follows from the last two inequalities. $\square$

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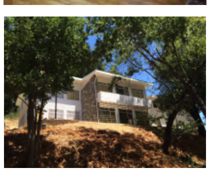

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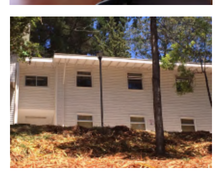

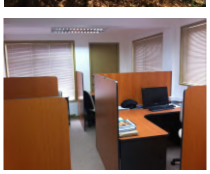

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