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A TREE-COTREE SPLITTING FOR THE CONSTRUCTION OF DIVERGENCE-FREE FINITE ELEMENTS: THE HIGH ORDER CASE

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Abstract. We extend, to Raviart–Thomas finite elements of any degree, two methods for the construction of basis of the space of divergence-free functions that are well established in the case of degree one. The first one computes directly a basis of the kernel of the divergence operator whereas the second one computes a basis of the image of the curl operator that, if the boundary of the domain is not connected, is completed with a basis of the second de Rham cohomology group (namely, the space of divergence-free functions that are not curls). When using the lower order Whitney elements on a tetrahedral mesh, the degrees of freedom are supported on the vertices, edges, faces and tetrahedra of the mesh respectively and, from Stokes theorem, the matrices describing the differential operators gradient, curl and divergence are the transposed of the connectivity matrices of the mesh. This allows the use of a tree-cotree splitting of associated oriented graphs to efficiently construct a basis of either the kernel of the divergence or the image of the curl operator. We prove that these two properties hold true also for $r > 0$ when using as degrees of freedom a particular realization, based on Berstein polynomials, of the moments. In this work we analyze in detail the second method, the one based on the identification of a basis of the space of the curls of Nédélec finite elements. (The first one has been analyzed in [5].)

Key words. High order Raviart–Thomas finite elements, divergence-free finite elements, spanning tree, oriented graph, incidence matrix

AMS subject classifications. 65N30, 05C05

1. Introduction.

Two approaches for the construction of a basis of the divergence-free Raviart–Thomas finite element space, RT^0_{r+1}, in a bounded polyhedral domain, Ω, discretized by a tetrahedral mesh have been presented in the work [6], for the lower order case $r = 0$. There are not restrictions on the topology of Ω. In the first approach, the authors compute directly a basis of the kernel of the divergence operator. In the second one, the construction starts from a basis of the image of the matrix associated with the curl operator. If the boundary of the domain has $p + 1$ connected components with $p > 0$, in this second approach it is necessary to complete the previous set with p discrete representatives of a basis of the second de Rham cohomology group (divergence-free functions that are not curls). These two methods can be extended to the high order case $r > 0$. The extension of the first one has been analyzed in [5]. In this work we analyze in detail the extension of the second approach. However, for the sake of completeness we include in this introduction a brief description of both methods in the high order case.

Let $\mathcal{T}$ be a tetrahedral mesh of a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$. We will denote $\mathcal{P}^-_{r+1}\Lambda^k(\mathcal{T})$ the space of Whitney k-differential forms of degree $r + 1$ (see e.g. [9]). They can be identified with L_{r+1}, the Lagrange finite elements of degree $r+1$, if $k = 0$, with N_{r+1}, the first family of Nédélec finite elements of degree $r + 1$, if $k = 1$,

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with RT_{r+1}, the Raviart–Thomas finite elements of degree $r+1$, if $k=2$, and with P_r, the space of discontinuous piecewise polynomial functions of degree r, if $k=3$. When using the lower order Whitney elements on a simplicial complex, $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$, with $k=0,1,2,3$, the degrees of freedom are supported on the vertices (V), edges (E), faces (F) and tetrahedra (T) of the mesh respectively. It is well known (see e.g. [12]) that given an orientation to edges, faces and tetrahedra of the mesh, the matrices describing the differential operators $\mathrm{d} : \mathcal{P}_1^-\Lambda^k(\mathcal{T}) \to \mathcal{P}_1^-\Lambda^{k+1}(\mathcal{T})$ in terms of the degrees of freedom are the transposed of the matrices of the boundary operators $\partial : \mathcal{C}_{k+1}(\mathcal{T},\mathbb{Z}) \to \mathcal{C}_k(\mathcal{T},\mathbb{Z})$ being $\mathcal{C}_k(\mathcal{T},\mathbb{Z})$ the group of k-chains in $\mathcal{T}$. Figure 1.1 represents de De Rham's complex as in [11]. It summarizes these facts in both the continuous and the discrete case. $\mathcal{H}^k$ denotes the cohomology groups for $k \in \{0,1,2\}$ and M^V denotes the set of degrees of freedom (moments) of the finite elements space V.

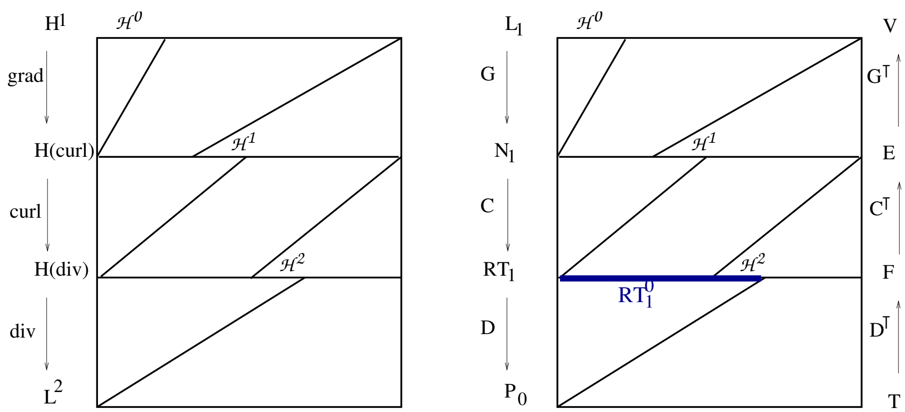

FIG. 1.1. *The De Rham complex for the continuos spaces (left) and for Whitney differential forms (right).*

Since the boundary of an edge consists of two vertices, and any face belongs to the boundary of one or two tetrahedra, from the point of view of graph theory we observe that: i) the transposed of the matrix associated with the gradient, $G^\top$, is the all-nodes incidence matrix of a directed and connected graph having a node for each vertex and an arc for each (oriented) edge of the mesh; ii) the matrix associated with the divergence operator, D, is an incidence matrix of a directed and connected graph having a node for each tetrahedron, plus an additional node associated with the exterior of the domain, and an arc for each face. This fact is used in different contexts as the tree-cotree gauge (see [3], [4], [18], [15]), the construction of bases of the space of divergence-free Raviart–Thomas finite elements (see [8], [20], [6]) or the construction of discrete potentials (see [22], [7]).

One of the goals of this paper is to emphasize that the duality between the differential operators and the boundary operators in the discrete De Rham complex is preserved to some extent when $r>0$ if an appropriate set of degrees of freedom is choosen. For instance the two properties i) and ii) hold true also for $r>0$ when using as degrees of freedom for $u_h \in \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ a particular realizations of the moments,

$$m_S(u_h) = \int_S \mathrm{Tr}_S(u_h) \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-\dim S}\Lambda^{\dim S - k}(S)\,,$$

being S any subsimplex of the mesh with $\dim S \geq k$. (See e.g. [9].) We use Berstein polynomials to identify a basis of $\mathcal{P}_{r+k-\dim S}\Lambda^{\dim S-k}(S)$, following the approach in [1]. (See also [2] where Berstein polynomials are used to express a set of basis of $\mathcal{P}^{-}_{r+1}\Lambda^k(\mathcal{T})$.) In [5] we identify in this way a set of moments in P_r and RT_{r+1} such that the matrix associated with the divergence operator D, when considering cardinal basis, is an incidence matrix of a directed and connected graph having a node for each moment of P_r plus an additional node associated with the exterior of the domain, and an arc for each moment of RT_{r+1}. Proceeding in a similar way, we identify, in Section 3, a set of moments of N_{r+1} and L_{r+1} such that, when considering cardinal basis, the transposed of the matrix associated with the gradient operator, G^T, is the all nodes incidence matrix of a directed and connected graph having a node for each moment of L_{r+1} and an arc for each moment of N_{r+1}.

A way to construct the moments of a basis of RT^0_{r+1}, the divergence-free subspace of RT_{r+1}, is to identify a maximal invertible submatrix of the matrix D associated with the divergence operator. Following the method proposed in [8] for a triangular mesh in $\mathbb{R}^2$, this can be done by choosing the columns corresponding to the arcs in a spanning tree of the associated tetrahedra-faces graph. If we decompose $D = [D_{st}, D_{ct}]$ being D_{st} the columns corresponding to the spanning tree and D_{ct} those of the co-tree, then D_{st} is invertible. Let us set $d_P = \dim P_r$ and $d_{RT} = \dim RT_{r+1}$. For any vector $\mathbf{m}_{ct} \in \mathbb{R}^{d_{RT}-d_P}$, solving $D_{st}\mathbf{m}_{st} = -D_{ct}\mathbf{m}_{ct}$ the entries of the vector $\begin{bmatrix} \mathbf{m}_{st} \\ \mathbf{m}_{ct} \end{bmatrix}$ are the degrees of freedom of a discrete function that is divergence-free. Then it is easy to prove that the columns of $\begin{bmatrix} -D_{st}^{-1}D_{ct} \\ I \end{bmatrix}$, being I the identity matrix in $\mathbb{R}^{(d_{RT}-d_P)\times(d_{RT}-d_P)}$, are the degrees of freedom of the elements of a basis of divergence-free Raviart–Thomas finite elements.

A different approach, that does not require to invert a matrix, is based on the fact that, if the boundary of the domain is connected, the curls of the elements of any basis of N_{r+1} generates the divergence-free subspace of RT_{r+1}. However this set is not linear independent because the kernel of the curl operator is not trivial. To obtain a basis is necessary to disregard some elements of the basis of N_{r+1}, precisely, those elements that generates the kernel of the curl operator. In other words, it is possible to compute the degrees of freedom of the elements of a basis of RT^0_{r+1} by just choosing a maximal linear independent set of the columns of the matrix associated with the curl operator when considering a cardinal basis.

This is done in [20] (see also [19]), for the lower order case $r = 0$ and a simply connected domain, by eliminating the elements of the cardinal basis of N_{r+1} associated to edges in a spanning tree of the graph made up of vertices and edges of the mesh. In this paper we prove that, if the degrees of freedom are the moments, it is possible, to use a procedure analogous also for $r > 0$: the elements to be disregarded are those corresponding to the moments in a *belted* tree (see e.g. [14], [18], [17]) of the graph associated with $G^\top$. If the domain is simply connected then the belted tree is in fact a spanning tree. If the boundary of the domain has $p+1$ connected components with $p > 0$, then it is necessary to complete the previous set with p discrete representatives of a basis of the second de Rham cohomology group (divergence-free functions that are not curls).

This paper is organized as follows. In Section 2 we introduce the notation and recall some elementary results of graph theory. In Section 3 we choose the degrees of freedom that will be used in the sequel. For this choice of degrees of freedom, we compute the matrix associated with the gradient operator when using cardinal basis, and we see that it is the transposed of the all-nodes incidence matrix of a directed graph $\mathcal{M}^G$. The proof of this result can be found in the appendix. We propose also an algorithm to construct a spanning tree of this graph. In Section 4 the spanning tree is used to identify a maximal set of linearly independent curls of Nédélec functions. (If Ω is not simply connected it is used, in fact, a belted tree). Then, if the boundary of Ω is not connected, we complete this set with discrete representatives of a basis of the second de Rham cohomology group (computed again using a tree-cotree decomposition method) to obtain a basis of the subspace of divergence-free elements of RT_{r+1}. Section 5 contains some conclusions and remarks concerning the use of this basis. In particular we notice that the cardinal basis can be constructed from any basis of the polynomial space using a generalized Vandermonde matrix whose entries are the degrees of freedom of the elements of the choosen basis. This is not expensive from the computational point of view because the Vandermonde matrix for the moments is the same for all the elements of the mesh.

2. Notation. Let $\mathcal{T} = (V, E, F, T)$ be a tetrahedral mesh of Ω where V is the set of vertices, E is the set of edges, F is the set of faces, and T is the set of tetrahedra of $\mathcal{T}$. If $\Delta_d(\mathcal{T})$ denotes, for $d = 0, 1, 2, 3$, the set of d-simplex of the mesh then $\Delta_0(\mathcal{T}) = V$, $\Delta_1(\mathcal{T}) = E$, $\Delta_2(\mathcal{T}) = F$, and $\Delta_3(\mathcal{T}) = T$. Let us fix an orientation on each edge, face, and tetrahedron of $\mathcal{T}$. This can be done by choosing a total ordering of the vertices in $V = \{\mathbf{v}_i\}_{i=1}^{n_V}$ and by associating with each d-simplex of the mesh, $S \in \Delta_d(\mathcal{T})$, an increasing function $m_S : \{0, 1, \dots, d\} \to \{1, \dots, n_V\}$: the oriented d-simplex S is hence given by $S = [\mathbf{v}_{m_S(0)}, \dots, \mathbf{v}_{m_{S(d)}}]$.

Analogously, if $S \in \Delta_d(\mathcal{T})$ we denote by $\Delta_\ell(S)$, for $\ell = 0, \dots, d$, the set of ℓ-subsimplices of S. If $\Sigma \in \Delta_\ell(S)$ with $\ell \in \{0, \dots, d-1\}$ then we can write $\Sigma = S - \sigma$ being σ the (oriented) $(d-1-\ell)$-subsimplex of S such that $\Delta_0(\Sigma) \cap \Delta_0(\sigma) = \emptyset$. Moreover there exists a unique increasing map $m_\Sigma^S : \{0, \dots, \ell\} \to \{0, \dots, d\}$ such that, for each $i \in \{0, \dots, \ell\}$, $m_\Sigma(i) = m_S(m_\Sigma^S(i))$.

For any $t \in T$ we denote by $\mathbf{n}_t$ the outward unit vector normal to the boundary of t. For any $f \in F$ we define $\mathbf{n}_f := \frac{(\mathbf{v}_{m_f(1)} - \mathbf{v}_{m_f(0)}) \times (\mathbf{v}_{m_f(2)} - \mathbf{v}_{m_f(0)})}{|(\mathbf{v}_{m_f(1)} - \mathbf{v}_{m_f(0)}) \times (\mathbf{v}_{m_f(2)} - \mathbf{v}_{m_f(0)})|}$ and we denote by $\boldsymbol{\nu}_{\partial f}$ the unit vector normal to the boundary of f in the plane containing f and pointing outward f. We define $\boldsymbol{\tau}_{\partial f} := \mathbf{n}_f \times \boldsymbol{\nu}_{\partial f}$. For any $e \in E$ we define $\mathbf{t}_e = \frac{\mathbf{v}_{m_e(1)} - \mathbf{v}_{m_e(0)}}{|\mathbf{v}_{m_e(1)} - \mathbf{v}_{m_e(0)}|}$.

For ℓ, $d \in \mathbb{N}$ let us set

$$\mathcal{I}(\ell, d+1) = \{\boldsymbol{\eta} = (\eta_0, \dots \eta_d) \in \mathbb{N}^{d+1} \, : \, |\boldsymbol{\eta}| = \ell\} \,,$$

being $|\boldsymbol{\eta}| := \sum_{i=0}^{d} \eta_i$. The cardinality of $\mathcal{I}(\ell, d+1)$ is equal to $\left(\begin{array}{c} \ell + d \\ d \end{array} \right)$.

For $\boldsymbol{\zeta} \in \mathcal{I}(\ell+1, d+1)$ and $j \in \{0, 1, \dots, d\}$ we denote by $\boldsymbol{\zeta} - \mathbf{e}_j$ the vector in $\mathbb{Z}^{d+1}$ with components $(\boldsymbol{\zeta} - \mathbf{e}_j)_i = \zeta_i - \delta_{j,i}$, for $i \in \{0, 1, \dots, d\}$, namely

$$(\boldsymbol{\zeta} - \mathbf{e}_j)_i = \begin{cases} \zeta_i & \text{if } i \neq j \\ \zeta_i - 1 & \text{if } i = j \,. \end{cases} \tag{2.1}$$

Note that $\boldsymbol{\zeta} - \mathbf{e}_j \in \mathcal{I}(\ell, d+1)$ if and only if $\zeta_j > 0$. So we also consider the following set of vectors in $\mathbb{Z}^{d+1}$:

$$\mathcal{J}(\ell, d{+}1) = \{\widetilde{\boldsymbol{\eta}} \in \mathbb{Z}^{d+1} : \widetilde{\boldsymbol{\eta}} = \boldsymbol{\zeta} {-} \mathbf{e}_j \text{ for some } \boldsymbol{\zeta} \in \mathcal{I}(\ell{+}1, d{+}1) \text{ and } j \in \{0, 1, \dots, d\}\} \,.$$

Clearly $\mathcal{I}(\ell, d+1) \subset \mathcal{J}(\ell, d+1)$. For each $\widetilde{\boldsymbol{\eta}} \in \mathcal{J}(\ell, d+1)$ we define

$$a_{\widetilde{\boldsymbol{\eta}}} = \begin{cases} \dfrac{\ell!}{\Pi_{i=0}^{d} \widetilde{\eta}_i!} = \begin{pmatrix} \ell \\ \widetilde{\boldsymbol{\eta}} \end{pmatrix} & \text{if } \widetilde{\boldsymbol{\eta}} \in \mathcal{I}(\ell, d+1) \\ 0 & \text{otherwise.} \end{cases}$$

In this way for each $\boldsymbol{\zeta} \in \mathcal{I}(\ell+1, d+1)$ we have

$$a_{\boldsymbol{\zeta}} \zeta_j = (\ell+1) a_{\boldsymbol{\zeta}-\mathbf{e}_j}, \qquad j \in \{0, 1, \ldots, d\}\,. \tag{2.2}$$

The barycentric coordinates of a point $\mathbf{x} \in \mathbb{R}^3$ with respect to the vertices of $t \in T$ are given by the unique set of scalars $\boldsymbol{\lambda}_t(\mathbf{x}) = (\lambda_{t,0}(\mathbf{x}), \lambda_{t,1}(\mathbf{x}), \lambda_{t,2}(\mathbf{x}), \lambda_{t,3}(\mathbf{x}))$ satisfying

$$\mathbf{x} = \sum_{j=0}^{3} \lambda_{t,j}(\mathbf{x}) \mathbf{v}_{m_t(j)} \ \text{ and } \ \sum_{j=0}^{3} \lambda_{t,j}(\mathbf{x}) = 1\,.$$

For each $i \in \{1, \ldots, n_V\}$, $\lambda_i : \overline{\Omega} \to \mathbb{R}$ denotes the continuous function

$$\lambda_i : \mathbf{x} \mapsto \lambda_i(\mathbf{x}) = \begin{cases} \lambda_{t,j}(\mathbf{x}) & \text{if } \mathbf{x} \in t \text{ and } i = m_t(j) \text{ for some } j \in \{0,1,2,3\} \\ 0 & \text{otherwise}\,; \end{cases}$$

and for $S \in \Delta_d(\mathcal{T})$ and $\widetilde{\boldsymbol{\eta}} \in \mathcal{J}(\ell, d+1)$, $\lambda_S^{\widetilde{\boldsymbol{\eta}}} : \overline{\Omega} \to \mathbb{R}$ denotes the continuous function

$$\lambda_S^{\widetilde{\boldsymbol{\eta}}} : \mathbf{x} \mapsto \lambda_S^{\widetilde{\boldsymbol{\eta}}}(\mathbf{x}) = \begin{cases} \Pi_{j=0}^{d} \left[\lambda_{m_S(j)}(\mathbf{x})\right]^{\widetilde{\eta}_j} & \text{if } \widetilde{\boldsymbol{\eta}} \in \mathcal{I}(\ell, d+1) \\ 0 & \text{otherwise.} \end{cases}$$

For each $\ell \in \mathbb{N}$, $S \in \Delta_d(\mathcal{T})$ and $\mathbf{x} \in S$

$$1 = \left(\sum_{j=0}^{d} \lambda_{m_S(j)}(\mathbf{x}) \right)^{\ell} = \sum_{\boldsymbol{\eta} \in \mathcal{I}(\ell, d+1)} a_{\boldsymbol{\eta}} \lambda_S^{\boldsymbol{\eta}}(\mathbf{x})\,. \tag{2.3}$$

Given $\boldsymbol{\eta} \in \mathcal{I}(\ell, d+1)$, $B_S^{\boldsymbol{\eta}} := a_{\boldsymbol{\eta}} \lambda_S^{\boldsymbol{\eta}}$ is a Berstein polynomial of degree ℓ associated with S, (see, e.g., [1]). Clearly $B_S^{\boldsymbol{\eta}-\mathbf{e}_j} \equiv 0$ if $\eta_j = 0$.

For $t \in T$, $S \in \Delta_d(t)$, let us denote $\nabla_S[\lambda_{m_S(j)|t}] = \mathbf{n}_S \times \nabla[\lambda_{m_S(j)|t}] \times \mathbf{n}_S$ if $d = 2$ and $\nabla_S[\lambda_{m_S(j)|t}] = (\mathbf{t}_S \cdot \nabla[\lambda_{m_S(j)|t}])\, \mathbf{t}_S$ if $d = 1$. Then, for any $\mathbf{x} \in \mathring{t}$

$$\begin{aligned} \nabla_S B_S^{\boldsymbol{\eta}}(\mathbf{x}) = a_{\boldsymbol{\eta}} \nabla_S \lambda_S^{\boldsymbol{\eta}}(\mathbf{x}) &= \textstyle\sum_{j=0}^{d} a_{\boldsymbol{\eta}} \eta_j \lambda_S^{\boldsymbol{\eta}-\mathbf{e}_j}(\mathbf{x}) \nabla_S \lambda_{m_S(j)}(\mathbf{x}) \\ &= |\boldsymbol{\eta}| \textstyle\sum_{j=0}^{d} a_{\boldsymbol{\eta}-\mathbf{e}_j} \lambda_S^{\boldsymbol{\eta}-\mathbf{e}_j}(\mathbf{x}) \nabla_S \lambda_{m_S(j)}(\mathbf{x}) \\ &= |\boldsymbol{\eta}| \textstyle\sum_{j=0}^{d} B_S^{\boldsymbol{\eta}-\mathbf{e}_j}(\mathbf{x}) \nabla_S \lambda_{m_S(j)}(\mathbf{x})\,. \end{aligned}$$

It is worth noting that if $\mathbf{x} \in \mathring{S}$ and $S \in \Delta_d(\mathcal{T})$ with $d = 2$ or $d = 1$ then $\nabla_S \lambda_{m_S(i)}(\mathbf{x})$ is well defined because it is independent of the tetrahedron t used to define it. Moreover, if $\mathbf{x} \in S$ then we have $\sum_{j=0}^{d} \lambda_{m_S(j)}(\mathbf{x}) = 1$ and $\nabla_S \lambda_{m_S(0)} = -\sum_{j=1}^{d} \nabla_S \lambda_{m_S(j)}$. So we can write, for $\mathbf{x} \in \mathring{S}$,

$$\nabla_S B_S^{\boldsymbol{\eta}}(\mathbf{x}) = a_{\boldsymbol{\eta}} \nabla_S \lambda_S^{\boldsymbol{\eta}}(\mathbf{x}) = |\boldsymbol{\eta}| \textstyle\sum_{j=1}^{d} \left[B_S^{\boldsymbol{\eta}-\mathbf{e}_j}(\mathbf{x}) - B_S^{\boldsymbol{\eta}-\mathbf{e}_0}(\mathbf{x}) \right] \nabla_S \lambda_{m_S(j)}\,. \tag{2.4}$$

We denote $R_{d,j} \in \mathbb{Z}^{d \times (d+1)}$ the matrix obtained from the $(d+1) \times (d+1)$ identity matrix omitting the j-th row. Clearly, if $\boldsymbol{\eta} \in \mathcal{I}(\ell, d+1)$ then, for each $j \in \{0, 1, \ldots, d\}$

such that $\eta_j = 0$ we have $R_{d,j}\boldsymbol{\eta} \in \mathcal{I}(\ell, d)$ and $a_{\boldsymbol{\eta}} = a_{R_{d,j}\boldsymbol{\eta}}$. Notice that, if $\Sigma \in \Delta_{d-1}(S)$ then $\Sigma = S - [\mathbf{v}_{m_S(i)}]$ for a certain $i \in \{0, 1, \ldots, d\}$ and $[\lambda_S^{\boldsymbol{\eta}}]_{|\Sigma} = 0$ if $\eta_i \neq 0$, hence

$$[B_S^{\boldsymbol{\eta}}]_{|\Sigma} = \begin{cases} B_\Sigma^{\boldsymbol{\eta}'} \text{ with } \boldsymbol{\eta}' = R_{d,i}\boldsymbol{\eta} \in \mathcal{I}(\ell, d) & \text{if } \eta_i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2.1. A drop of graph theory. We recall some basic definitions and results of graph theory that will be used in the sequel. They can be found, for instance, in [21].

A graph $\mathcal{M} = (\mathcal{N}, \mathcal{A})$ consists of two finite sets: a set $\mathcal{N} = \{\mathfrak{n}_i\}_{i=1}^n$ of nodes and a set $\mathcal{A} = \{\mathfrak{a}_j\}_{j=1}^m$ of arcs. Each arc is identified with a pair of nodes. The two nodes defining an arc need not be distinct. If the arc $\mathfrak{a}_j$ has the two points equal to the node $\mathfrak{n}_i$ then it is called a self-loop at node $\mathfrak{n}_i$. If the arcs of $\mathcal{M}$ are identified with ordered pairs of nodes, then $\mathcal{M}$ is called a directed or an oriented graph. Otherwise $\mathcal{M}$ is called an undirected or a nonoriented graph.

The following definitions concern both directed and undirected graphs.

A walk is a finite alternating sequence of nodes and arcs $\mathfrak{n}_{i_0}$, $\mathfrak{a}_{j_1}$, $\mathfrak{n}_{i_1}$, $\mathfrak{a}_{j_2}$, $\mathfrak{n}_{i_2}, \ldots, \mathfrak{n}_{i_{K-1}}$, $\mathfrak{a}_{j_K}$, $\mathfrak{n}_{i_K}$ such that for $k \in \{1, \ldots, K\}$ the arc $\mathfrak{a}_{j_k}$ is identified with the pair of nodes $\mathfrak{n}_{i_{k-1}}$, $\mathfrak{n}_{i_k}$. This walk is usually called a $\mathfrak{n}_{i_0} - \mathfrak{n}_{i_K}$ walk with $\mathfrak{n}_{i_0}$ and $\mathfrak{n}_{i_K}$ referred to as the end or terminal nodes of this walk. A walk is open if its end nodes, $\mathfrak{n}_{i_0}$, $\mathfrak{n}_{i_K}$ are distinct; otherwise it is closed. A walk is a trail if all its edges are distinct. An open trail is a path if all its vertices are distinct. A closed trail is a circuit if all its vertices except the end vertices are distinct. A graph is said to be acyclic if it has no circuits.

Two nodes $\mathfrak{n}_i$, $\mathfrak{n}_{i'}$ are said to be connected in a graph $\mathcal{M}$ if there exists a $\mathfrak{n}_i - \mathfrak{n}_{i'}$ path in $\mathcal{M}$. A graph $\mathcal{M}$ is connected if there exists a path between every pair of nodes in $\mathcal{M}$.

Finally we recall the definition of spanning tree of a graph $\mathcal{M} = (\mathcal{N}, \mathcal{A})$.

Definition 1. *A tree of a graph $\mathcal{M} = (\mathcal{N}, \mathcal{A})$ is a connected acyclic subgraph of $\mathcal{M}$. A spanning tree $\mathcal{S}$ is a tree of $\mathcal{M}$ containing all its nodes.*

It is worth noting that if $\mathcal{S}$ is a spanning tree of $\mathcal{M} = (\mathcal{N}, \mathcal{A})$, then $\mathcal{S} = (\mathcal{N}, \mathcal{B})$ with $\mathcal{B} \subset \mathcal{A}$. Moreover $\mathcal{B}$ has exactly $n - 1$ arcs. If $\mathcal{M}$ is not connected then it has not spanning trees.

We recall also the definition of the all-nodes incidence matrix of a directed graph.

Definition 2. *The all-nodes incidence matrix $M^e \in \mathbb{Z}^{n \times m}$ of a directed graph $\mathcal{M} = (\mathcal{N}, \mathcal{A})$, with n nodes $\mathcal{N} = \{\mathfrak{n}_i\}_{i=1}^n$, m arcs $\mathcal{A} = \{\mathfrak{a}_j\}_{j=1}^m$ and with no self-loop, is the matrix with entries*

$$[M^e]_{i,j} = \begin{cases} 1 & \text{if } \mathfrak{a}_j \text{ is incident on } \mathfrak{n}_i \text{ and oriented away from it,} \\ -1 & \text{if } \mathfrak{a}_j \text{ is incident on } \mathfrak{n}_i \text{ and oriented toward it,} \\ 0 & \text{if } \mathfrak{a}_j \text{ is not incident on } \mathfrak{n}_i \,. \end{cases}$$

An incidence matrix M of $\mathcal{M}$ is any submatrix of M^e with $n-1$ rows and m columns. The node that corresponds to the row of M^e that is not in M will be called the reference node of M.

The following result, that joins Theorem 6.9 and Theorem 6.12 in [21], is crucial in the tree-cotree decompositions.

Theorem 2.1. *Let $\mathcal{M} = (\mathcal{N}, \mathcal{A})$ be a directed connected graph with no self-loop and $M \in \mathbb{Z}^{(n-1) \times m}$ an incidence matrix of $\mathcal{M}$. Let $\mathcal{S} = (\mathcal{N}, \mathcal{B})$ be a spanning tree of $\mathcal{M}$ and M_{st} the submatrix of order $n - 1$ of M given by the columns of M that* *correspond to the arcs in $\mathcal{B}$. Then M_{st} is invertible and the nonzero elements in each row of M_{st}^{-1} are either all 1 or all -1.*

3. Moments. For a Whitney k-form $u_h \in \mathcal{P}_{r+1}^{-}\Lambda^k(\mathcal{T})$ the moments are defined in the following way:

$$m_S(u_h) = \int_S \mathrm{Tr}_S(u_h) \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-\dim S}\Lambda^{\dim S-k}(S)\,,$$

being S any subsimplex of the mesh with $\dim S \geq k$. (See e.g. [9].)

If $k = 3$ then it must be $\dim S = 3$ and a very natural set of moments in P_r (namely, in $P_{r+1}^{-}\Lambda^3(\mathcal{T})$) is

- $M^P_{t,\boldsymbol{\alpha}}(u_h) = \int_t u_h\, B_t^{\boldsymbol{\alpha}}, \quad t \in T$ and $\alpha \in \mathcal{I}(r,3)$.

If, in particular $u_h = \operatorname{div} \mathbf{z}_h$ with $\mathbf{z}_h \in RT_{r+1}$, then using Stokes theorem we have

$$M^P_{t,\boldsymbol{\alpha}}(\operatorname{div} \mathbf{z}_h) = \int_t \operatorname{div} \mathbf{z}_h\, B_t^{\boldsymbol{\alpha}} = \int_{\partial t} \mathbf{z}_h \cdot \mathbf{n}_t\, B_t^{\boldsymbol{\alpha}} - \int_t \mathbf{z}_h \cdot \nabla B_t^{\boldsymbol{\alpha}}$$

Let us consider the following set of moments in RT_{r+1} (namely, in $P_{r+1}^{-}\Lambda^2(\mathcal{T})$):

- $M^{RT}_{f,\boldsymbol{\alpha}}(\mathbf{z}_h) = \int_f \mathbf{z}_h \cdot \mathbf{n}_f B_f^{\boldsymbol{\alpha}} \quad f \in F$ and $\alpha \in \mathcal{I}(r,2)$;
- $M^{RT}_{t,\boldsymbol{\beta}}(\mathbf{z}_h) = r \int_t \mathbf{z}_h \cdot B_t^{\boldsymbol{\beta}} \nabla\lambda_{m_t(i)} \quad t \in T\,,\ \beta \in \mathcal{I}(r-1,2)$ and $i \in \{1,2,3\}$.

Then it is easy to check that the matrix associated with the divergence operator when using cardinal basis for these moments has all its entries in the set $\{0,1,-1\}$. It can be proved (see [5]) that it is the incidence matrix of an oriented connected graph that has a node for each moment of P_r plus a node corresponding to $\mathbb{R}^3 \setminus \overline{\Omega}$ and an arc for each moment in RT_{r+1}.

Similarly, taking in particular $\mathbf{z}_h = \operatorname{curl} \mathbf{u}_h$ with $\mathbf{u}_h \in N_{r+1}$ and applying Stokes theorem it is easy to check that a set of moments in N_{r+1} (namely, in $P_{r+1}^{-}\Lambda^1(\mathcal{T})$) that leads to a matrix associated with the curl operator with all its entries belonging to $\{0,1,-1\}$ are

- $M^N_{\boldsymbol{\alpha},e}(\mathbf{u}_h) = \int_e \mathbf{u}_h \cdot \mathbf{t}_e\, B_e^{\boldsymbol{\alpha}}$, for each $e \in E$ and $\boldsymbol{\alpha} \in \mathcal{I}(r,2)$.
- $M^N_{\boldsymbol{\beta},f,i}(\mathbf{u}_h) = (-1)^i r \int_f (\mathbf{u}_h \times \mathbf{n}_f) \cdot B_f^{\boldsymbol{\beta}} \nabla_f \lambda_{m_f(j)}$ for each $f \in F$, $\boldsymbol{\beta} \in \mathcal{I}(r-1,3)$ and $i \in \{1,2\}$. Here $j \in \{1,2\}$ and $j \neq i$.
- $M^N_{\boldsymbol{\gamma},t,i}(\mathbf{u}_h) = (-1)^s r(r-1) \int_t \mathbf{u}_h \cdot B_t^{\boldsymbol{\gamma}} \nabla\lambda_{m_t(j)} \times \nabla\lambda_{m_t(k)}$ for each $t \in T$, $\boldsymbol{\gamma} \in \mathcal{I}(r-2,4)$ and $i \in \{1,2,3\}$. Here $j,k \in \{1,2,3\}$, $j<k$, $i \notin \{j,k\}$. Moreover $s = 0$ if the tetrahedra t is positive oriented, namely, if $\mathbf{t}_{e_1} \cdot (\mathbf{t}_{e_2} \times \mathbf{t}_{e_3}) > 0$, being $e_i = [\mathbf{v}_{m_t(0)}, \mathbf{v}_{m_t(i)}]$ for $i \in \{1,2,3\}$, and $s = 1$ otherwise.

Remark 3.1. We recall that

$$\nabla_f \lambda_{m_f(j)} = -\tfrac{|f-[\mathbf{v}_{m_f(j)}]|}{2|f|} \boldsymbol{\nu}_{f|f-[\mathbf{v}_{m_f(j)}]} \text{ and } \nabla\lambda_{m_t(j)} = -\tfrac{|t-[\mathbf{v}_{m_f(j)}]|}{3|t|} \mathbf{n}_{t|t-[\mathbf{v}_{m_t(j)}]}\,.$$

Hence we have for $i,j \in \{1,2\}$ and $i \neq j$

$$\begin{aligned}
(\mathbf{u}_h \times \mathbf{n}_f) \cdot \nabla_f \lambda_{m_f(j)} &= \mathbf{u}_h \cdot \big(\mathbf{n}_f \times \nabla_f \lambda_{m_f(j)}\big) \\
&= -\tfrac{|[\mathbf{v}_{m_f(0)}, \mathbf{v}_{m_f(i)}]|}{2|f|} \mathbf{u}_h \cdot (\mathbf{n}_f \times \boldsymbol{\nu}_f)_{|[\mathbf{v}_{m_f(0)}, \mathbf{v}_{m_f(i)}]} \\
&= (-1)^i \tfrac{|[\mathbf{v}_{m_f(0)}, \mathbf{v}_{m_f(i)}]|}{2|f|} \mathbf{u}_h \cdot \mathbf{t}_{[\mathbf{v}_{m_f(0)}, \mathbf{v}_{m_f(i)}]}\,,
\end{aligned}$$

and for $i,j,k \in \{1,2,3\}$, $j<k$ and $i \notin \{j,k\}$

$$\begin{aligned}
\nabla\lambda_{m_t(j)} \times \nabla\lambda_{m_t(k)} &= \tfrac{|t-[\mathbf{v}_{m_t(j)}]|}{3|t|} \mathbf{n}_{t|t-[\mathbf{v}_{m_t(j)}]} \times \tfrac{|t-[\mathbf{v}_{m_t(k)}]|}{3|t|} \mathbf{n}_{t|t-[\mathbf{v}_{m_t(k)}]} \\
&= (-1)^s \tfrac{|t-[\mathbf{v}_{m_t(j)}]|}{3|t|} \tfrac{|t-[\mathbf{v}_{m_t(k)}]|}{3|t|} \mathbf{t}_{[\mathbf{v}_{m_t(0)}, \mathbf{v}_{m_t(i)}]}\,.
\end{aligned}$$

These two results are mainly Proposition 3 and Proposition 4 in [10].

It is worth noting that $M^N_{\boldsymbol{\beta},f,i}$ reads information in the direction of the edge $[\mathbf{v}_{m_f(0)}, \mathbf{v}_{m_f(i)}]$ and $M^N_{\boldsymbol{\gamma},t,i}$ in the direction of the edge $[\mathbf{v}_{m_t(0)}, \mathbf{v}_{m_t(i)}]$.

If $\mathbf{u}_h = \nabla\varphi_h$ with $\varphi_h \in L_{r+1}$, using Stokes theorem, we can rewrite the moments of $\nabla\varphi_h$ in terms of φ_h and, in this way, to identify appropriate moments in L_{r+1} to obtain a matrix associated with D with all its entries in the set $\{0, 1, -1\}$.

We consider the following moments in L_{r+1} (namely, in $P^-_{r+1}\Lambda^0(\mathcal{T})$):

- $M^L_{\alpha,v}(\varphi_h) = (\varphi_h\, B^{\boldsymbol{\alpha}}_v)(v)$, for each $v \in V$ and $\boldsymbol{\alpha} \in \mathcal{I}(r,1)$ (note that $B^{\boldsymbol{\alpha}}_v(v) = 1$);
- $M^L_{\beta,e}(\varphi_h) = \frac{r}{|e|}\int_e \varphi_h\, B^{\boldsymbol{\beta}}_e$, for each $e \in E$ and $\boldsymbol{\beta} \in \mathcal{I}(r-1, 2)$;
- $M^L_{\gamma,f}(\varphi_h) = \frac{r(r-1)}{2|f|}\int_f \varphi_h\, B^{\boldsymbol{\gamma}}_f$, for each $f \in F$ and $\boldsymbol{\gamma} \in \mathcal{I}(r-2, 3)$;
- $M^L_{\delta,t}(\varphi_h) = \frac{r(r-1)(r-2)}{6|t|}\int_t \varphi_h B^{\boldsymbol{\delta}}_t$, for each $t \in T$ and $\boldsymbol{\delta} \in \mathcal{I}(r-3, 4)$;

or, in a more compact way,

$$M^L_{\boldsymbol{\eta},S}(\varphi_h) = \frac{1}{|S|}\begin{pmatrix} r \\ d \end{pmatrix}\int_S \varphi_h\, B^{\boldsymbol{\eta}}_S\,,\ \text{for each } S \in \Delta_d(\mathcal{T}) \text{ and } \boldsymbol{\eta} \in \mathcal{I}(r-d, d+1)\,. \tag{3.1}$$

Remark 3.2. It is well known (see [16] Proposition 3.5) that for any $S \in \Delta_d(\mathcal{T})$ and $\boldsymbol{\eta} \in \mathcal{I}(l, d+1)$

$$\int_S \lambda^{\boldsymbol{\eta}}_S = |S|\,\frac{d!\prod_{j=0}^d \eta_j!}{(l+d)!}\,, \tag{3.2}$$

hence then Lagrange moments of a constant function $\varphi_h \equiv c$ are equal to that constant. In fact, from (3.1) and recalling that, for any $S \in \Delta_d(\mathcal{T})$ and $\boldsymbol{\eta} \in \mathcal{I}(r-d, d+1)$, $B^{\boldsymbol{\eta}}_S = a_{\boldsymbol{\eta}}\lambda^{\boldsymbol{\eta}}_S = \begin{pmatrix} r-d \\ \boldsymbol{\eta} \end{pmatrix}\lambda^{\boldsymbol{\eta}}_S$, we have

$$M^L_{\boldsymbol{\eta},S}(c) = \frac{1}{|S|}\begin{pmatrix} r \\ d \end{pmatrix} c\int_S a_{\boldsymbol{\eta}}\lambda^{\boldsymbol{\eta}}_S = c\frac{1}{|S|}\begin{pmatrix} r \\ d \end{pmatrix}\begin{pmatrix} r-d \\ \boldsymbol{\eta} \end{pmatrix}|S|\,\frac{d!\prod_{j=0}^d \eta_j!}{r!} = c\,.$$

We extend the definition of $M^L_{\boldsymbol{\eta},S}(\varphi_h)$ to $\mathcal{J}(r-d, d+1)$ in the following way. For any $\widetilde{\boldsymbol{\eta}} \in \mathcal{J}(r-d, d+1) \setminus \mathcal{I}(r-d, d+1)$ there exits a unique $j \in \{0, 1, \dots, d\}$ such that $\widetilde{\eta}_j = -1$ and $\boldsymbol{\zeta} := \widetilde{\boldsymbol{\eta}} + \mathbf{e}_j \in \mathcal{I}(r-d+1, d+1)$. We define

$$M^L_{\widetilde{\boldsymbol{\eta}},S}(\varphi_h) = M^L_{\boldsymbol{\zeta}-\mathbf{e}_j,S}(\varphi_h) = M^L_{R_{d,j}\boldsymbol{\zeta},S-[\mathbf{v}_{m_S(j)}]}(\varphi_h)\,.$$

Then it can be proved (see Appendix) that

$$\begin{aligned} M^N_{\boldsymbol{\alpha},e}(\nabla\varphi_h) &= -(M^L_{\boldsymbol{\alpha}-\mathbf{e}_1,e}(\varphi_h) - M^L_{\boldsymbol{\alpha}-\mathbf{e}_0,e}(\varphi_h)) \\ M^N_{\boldsymbol{\beta},f,i}(\nabla\varphi_h) &= -(M^L_{\boldsymbol{\beta}-\mathbf{e}_i,f}(\varphi_h) - M^L_{\boldsymbol{\beta}-\mathbf{e}_0,f}(\varphi_h)) \\ M^N_{\boldsymbol{\gamma},t,i}(\nabla\varphi_h) &= (-1)^i(M^L_{\boldsymbol{\gamma}-\mathbf{e}_i,t}(\varphi_h) - M^L_{\boldsymbol{\gamma}-\mathbf{e}_0,t}(\varphi_h))\,. \end{aligned} \tag{3.3}$$

Proposition 3.3. *Let us set $d_N = \dim N_{r+1}$ and $d_L = \dim L_{r+1}$. If we denote $G \in \mathbb{R}^{d_N\times d_L}$ the matrix that computes, from the moments of φ_h, the moments of $\nabla\varphi_h$ then $G^\top$ is the all-nodes incidence matrix of a directed graph $\mathcal{M}^G$ with a node for each Lagrange moment and an arc for each Nédélec moment: $\mathcal{M}^G = (M^L, M^N)$. This graph is connected.*

Proof. From (3.3) it follows that the matrix $G \in \mathbb{R}^{d_N \times d_L}$ has, on each row, two elements different from zero, one equal 1 and one equal -1, hence $G^\top$ is the all-nodes incidence matrix of a directed graph $\mathcal{M}^G = (M^L, M^N)$.

The fact that it is connected is quite clear using the visualization of the graph illustrated in Figure 3.1 for a single tetrahedron t in the case $r = 2$. As usually the moments in L_{r+1}, that are the nodes of the graph, are the points of the principal lattice of t of order $r+1$. The moments in N_{r+1}, namely the arcs of the graph, are associated with *some* of the (small) edges connecting the adjacent points in the principal lattice (see [16]). On each face f the graph has only (small) edges parallels to $[\mathbf{v}_{m_f(0)}, \mathbf{v}_{m_f(i)}]$ for $i \in \{1, 2\}$. In the interior of each tetrahedron there are only (small) edges parallels to $[\mathbf{v}_{m_t(0)}, \mathbf{v}_{m_t(i)}]$ for $i \in \{1, 2, 3\}$. For a rigorous proof of the fact that the graph $\mathcal{M}^G = (M^L, M^N)$ is connected (see [13]). ∎

The following algorithm constructs a spanning tree of the graph $\mathcal{M}^G$.

```
1. Initialization:
  Construction of a spanning tree τ of the graph (V, E)
  Loop over the edges e ∈ E
    set macrotree(e) = .false.
    set eVisited(e) = .false.
    if ( e ∈ τ ) then
      set macrotree(e) = .true.
    end if
  end Loop over the edges e ∈ E
  Loop over the faces f ∈ F
    set fVisited(f) = .false.
  end Loop over the faces f ∈ F

2. Construction of a spanning tree τ_s of the fictitious mesh
  Loop over the tetra t ∈ T: let t = [v_1, v_2, v_3, v_4]
    Loop over the faces f ∈ F(t): let f = [v_i, v_j, v_k]
      if ( fVisited(f) = .true. ) cycle Loop over the faces f ∈ F(t)

      Loop over the edges e ∈ E(f): let e = [v_g, v_h]
        if ( eVisited(e) = .true. ) cycle Loop on the edges e ∈ E(f)
        if ( macrotree(e) = .true. ) then
          add to τ_s the edge moments M^N_{α,e} for all α
        else
          add to τ_s the edge moments M^N_{α,e} for the r − 1 first† indices α
        end if
        set eVisited(e) = .true.
      end Loop on the edges e ∈ E(f)

      Loop over the layers ss = r − 1 : −1 : 2 internal to f
        add to τ_s the face moments M^N_{β,f,1} for the ss − 1 first† indices β
      end Loop over the layers ss internal to f
      set fVisited(f) = .true.
    end Loop over the faces f ∈ F(t)

    Loop over the layers ℓ = r − 1 : −1 : 3 internal to t
      Loop over the layers k = ℓ − 1 : −1 : 2 internal to the layer ℓ
        add to τ_s the volume moments M^N_{γ,t,1} for the k − 1 first† indices γ
      end Loop over the layers k internal to ℓ
    end Loop over the layers ℓ internal to t
  end Loop over the tetrahedra t ∈ T


first† is always intended in the reversed lexicographical order
```

It induces a numeration of the moments of N_{r+1} (an analogous numeration can be used for the moments of L_{r+1}) and can be also useful to understand the structure of the graph. The construction is based on a spanning tree of the graph given by vertices and edges of the mesh (the so-called *global* spanning tree). Then a loop by elements *enriches* this initial spanning tree with arcs corresponding to face moments (only those faces of the tetrahedra that have not been visited previously) and with the arcs corresponding to the volume moments (see Figure 3.2). Figure 3.3 shows the spanning tree in two tetrahedra of the mesh.

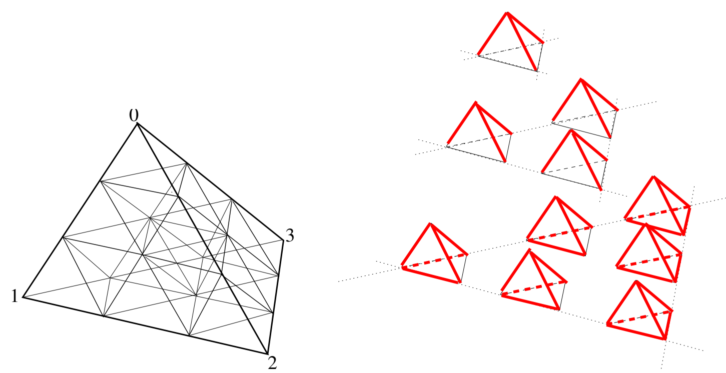

FIG. 3.1. *On the left the decomposition of an element* induced *by the principal lattice. On the right, the arcs of the graph $\mathcal{M}^G$ in red (in thick line), on a fragmented visualization of the decomposition. In this example $r = 2$.*

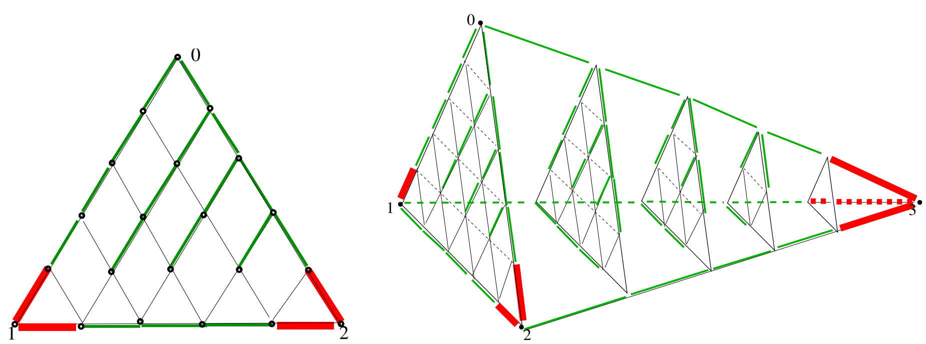

FIG. 3.2. *The spanning tree of the graph $\mathcal{M}^G = (M^L, M^N)$ on a tetrahedron. On the left the subgraph corresponding to edge and face moments of a particular face and the corresponding part of the spanning tree. The red arcs (in very thick line) belong to the spanning tree if and only if its mesh edge belongs to the global spanning tree. The green arcs (less thick but still in thick line) always belong to the spanning tree. On the left the volume moments belonging to the spanning are visualized. In this example $r = 4$.*

4. A basis of the finite element space RT^0_{r+1}. We denote $H^0(\mathrm{div};\Omega) := \{\mathbf{u} \in H(\mathrm{div};\Omega) : \mathrm{div}\,\mathbf{u} = 0\}$ and $RT^0_{r+1} := RT_{r+1} \cap H^0(\mathrm{div};\Omega)$.

In general $\dim RT^0_{r+1} = d_N - (d_L - 1) - g + p$, being $g = \beta_1(\Omega)$ first Betti number of Ω, namely the number of handles of Ω, and $p = \beta_2(\Omega)$ the second Betti number of Ω, namely, the number of connected components of the boundary of Ω minus one.

Let $\{M_i^N\}_{i=1}^{d_N}$ be the set of Nédélec moments and $\{\mathbf{\Phi}_i\}_{i=1}^{d_N}$ be a cardinal basis for those moments. Let $\mathcal{S}^G = (M^L, S^N)$ be a spanning tree of $\mathcal{M}^G$. We consider the set of elements of the cardinal basis corresponding to Nédélec moments that are not in

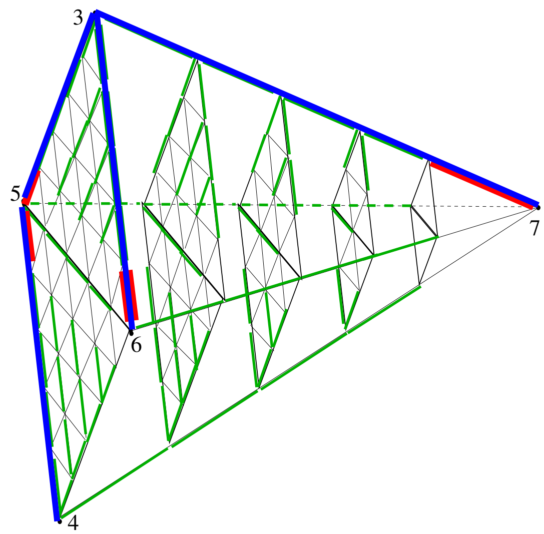

Fig. 3.3. *The spanning tree in two tetrahedra of a mesh ($r = 4$). In blue (long quite thick line) the arcs of the global spanning tree, in green (in less thick line) and red (in thick line, close to the vertices) the arcs of the spanning tree of the graph $\mathcal{M}^G = (M^L, M^N)$. It is worth noting that the red arcs in this figure correspond to the red arcs in Figure 3.2 belonging to an edge of the global spanning tree*

the spannig tree $\mathcal{S}^G$, namely, the set $\{\mathbf{\Phi}_i\}_{i\in K}$ being $K = \{i \in \mathbb{N}, 1 \leq i \leq d_N \,:\, M_i^N \notin S^N\}$. Clearly, denoting d_K the number of elements in K, one has $d_K = d_N - (d_L - 1)$.

Proposition 4.1. *If Ω is simply connected then the set $\{\operatorname{curl}\mathbf{\Phi}_j\}_{j\in K} \subset RT^0_{r+1}$ is linearly independent. Moreover, if we assume also that $\partial\Omega$ is connected then it is a basis of RT^0_{r+1}.*

Proof. Let $A \in \mathbb{R}^{d_N\times d_N}$ be the matrix with entries

$$A_{i,j} = \int_\Omega \operatorname{curl}\mathbf{\Phi}_i \cdot \operatorname{curl}\mathbf{\Phi}_j$$

and $A^0 \in \mathbb{R}^{d_K\times d_K}$ the submatrix of A obtained choosing those rows and columns of A that correspond to indices in K. We will proof that A^0 is non singular. If $K = \{k(1), \ldots, k_(d_K)\} \subset \{1, \ldots, d_N\}$, given a vector $\mathbf{c} \in \mathbb{R}^{d_K}$ with components c_j we denote $\mathbf{c}_h = \sum_{j=1}^{d_K} c_j \mathbf{\Phi}_{k(j)}$. If $A^0\mathbf{c} = 0$ then

$$\mathbf{c}^\top A^0\mathbf{c} = \int_\Omega \operatorname{curl}\mathbf{c}_h \cdot \operatorname{curl}\mathbf{c}_h = 0$$

and, being Ω simply connected, follows that $\mathbf{c}_h = \nabla\phi_h$ for some $\phi_h \in L_{r+1}$. Since the Nédélec moments of $\mathbf{c}_h = \nabla\phi_h$ in a spanning tree of $\mathcal{M}^G$ are equal to zero, hence, from (3.3), all the moments of ϕ_h are equal to the same constant. Due to the unisolvence of the moments, follows that ϕ_h is constant and then $\mathbf{c}_h = \nabla\phi_h = \mathbf{0}$. Since the set of functions $\{\mathbf{\Phi}_j\}_{j=1}^{d_N}$ is linearly independent then $\mathbf{c} = \mathbf{0}$.

If $\mathbf{w}_h = \sum_{j\in K} c_j \operatorname{curl}\mathbf{\Phi}_j = \mathbf{0}$ then, for any $i \in K$ we have

$$0 = \int_\Omega \operatorname{curl}\mathbf{\Phi}_i \cdot \mathbf{w}_h = \sum_{j\in K} c_j \int_\Omega \operatorname{curl}\mathbf{\Phi}_i \cdot \operatorname{curl}\mathbf{\Phi}_j = \sum_{j\in K} c_j A_{i,j} = (A^0\mathbf{c})_i\,.$$

This means that $A^0\mathbf{c} = \mathbf{0}$. Since A^0 is not singular follows that $\mathbf{c} = \mathbf{0}$ hence the set $\{\operatorname{curl}\boldsymbol{\Phi}_j\}_{j\in K} \subset RT^0_{r+1}$ is linearly independent. If Ω is simply connected and $\partial\Omega$ is connected then $\dim RT^0_{r+1} = d_N - (d_L - 1) = d_K$, so $\{\operatorname{curl}\boldsymbol{\Phi}_j\}_{j\in K}$ is a basis. ◻

If Ω is not simply connected then its first Betti number $\beta_1(\Omega) = g$ is not zero. If $\{\sigma_j\}_{j=1}^g$ is a set of 1-cycles representing a basis of $\mathcal{H}_1(\overline{\Omega};\mathbb{Z})$ then any curl-free function with $\oint_{\sigma_j}\mathbf{u}_h = 0$ for $j = 1,\dots,g$ is a gradient. Our aim is to extend to the high order case the notion of belted tree (see e.g. [14], [18], [17]). To this end we assume to know a set of g polygonal loops in $\mathcal{T}$, $\{\sigma_j\}_{j=1}^g$, mutually disjoint and without self-intersection, representing a basis of $\mathcal{H}_1(\overline{\Omega};\mathbb{Z})$. For each $j \in \{1,\dots,g\}$, σ_j is a 1-cycle of the form $\sigma_j = \sum_{e\in E} a_e^j e$ with $a_e^j \in \{-1,0,1\}$. We denote $\operatorname{supp}(\sigma_j) = \{e \in E : a_e^j \neq 0\}$. Then, using (2.3) we have

$$
\begin{aligned}
\oint_{\sigma_j}\mathbf{u}_h &= \textstyle\sum_{e\in\operatorname{supp}(\sigma_j)} a_e^j \int_e \mathbf{u}_h\cdot\boldsymbol{\tau}_e\\
&= \textstyle\sum_{e\in\operatorname{supp}(\sigma_j)} a_e^j \sum_{\boldsymbol{\alpha}\in\mathcal{I}(r,2)} \int_e \mathbf{u}_h\cdot\boldsymbol{\tau}_e B_e^{\boldsymbol{\alpha}}\\
&= \textstyle\sum_{e\in\operatorname{supp}(\sigma_j)} a_e^j \sum_{\boldsymbol{\alpha}\in\mathcal{I}(r,2)} M^N_{\boldsymbol{\alpha},e}\,.
\end{aligned}
\tag{4.1}
$$

We associate with each 1-cycle σ_j in $\mathcal{T}$ a circuit $\mathcal{G}_j$ of the graph $\mathcal{M}^G$ in the following natural way

$$\mathcal{G}_j = \cup_{e\in\operatorname{supp}(\sigma_j)}\mathcal{G}_e^*\,.$$

Each $\mathcal{G}_j$ is a circuit because σ_j has not self-intersections.

For each $j \in \{1,\dots,g\}$, we choose an arc $M^N_{\boldsymbol{\alpha}_j^*,e_j^*}$ of $\mathcal{G}_j$. The new trail $\mathcal{G}_j^-$ obtained from the circuit $\mathcal{G}_j$ removing the arc $M^N_{\boldsymbol{\alpha}_j^*,e_j^*}$ is a path. Since the cycles $\{\sigma_j\}_{j=1}^g$ are mutually disjoint, the subgraph given by the union of the paths $\mathcal{G}_j^-$ has g acyclic connected components, hence it is acyclic. So it is possible to construct a spanning tree $\mathcal{S}^G = (M^L, S^N)$ of $\mathcal{M}^G$ that contains it. We will refer to the subgraph of $\mathcal{M}^G$ given by $\mathcal{B}^G = (M^L, B^N)$ with $B^N = S^N \cup \{M^N_{\boldsymbol{\alpha}_1^*,e_1^*},\dots M^N_{\boldsymbol{\alpha}_g^*,e_g^*}\}$ as a *belted tree.*

PROPOSITION 4.2. *The set* $\{\operatorname{curl}\boldsymbol{\Phi}_j\}_{j\in K^*}$ *being* $K^* = \{j \in \mathbb{N}, 1 \le j \le d_N : M_i^N \notin B^N\}$ *is linearly independent. Moreover if we assume that* $\partial\Omega$ *is connected then it is a basis of* RT^0_{r+1}.

Proof. The proof is similar to the one of Proposition 4.1. Let $A^{0,*} \in \mathbb{R}^{d_{K^*}\times d_{K^*}}$ the submatrix of A obtained by choosing those rows and columns of A that correspond to indices in K^*. (Hence, d_{K^*} denotes the number of elements of the set K^*.) It is easy to verify that $A^{0,*}$ is non singular. In fact, given $\mathbf{c} \in \mathbb{R}^{d_{K^*}}$ and denoting $\mathbf{c}_h = \sum_{j\in K} c_j\boldsymbol{\Phi}_j$, if $A^{0,*}\mathbf{c} = 0$ then

$$\mathbf{c}^\top A^{0,*}\mathbf{c} = \int_\Omega \operatorname{curl}\mathbf{c}_h\cdot\operatorname{curl}\mathbf{c}_h = 0\,,$$

Hence $\operatorname{curl}\mathbf{c}_h = \mathbf{0}$. Moreover since the moments of $\mathbf{c}_h$ in K^* are equal zero, from (4.1) we have $\oint_{\sigma_j}\mathbf{c}_h = 0$ for all $j \in \{1,\dots,g\}$, so $\mathbf{c}_h = \nabla\phi_h$ for some $\phi_h \in L_{r+1}$. Now the result follows as in Proposition 4.1.

If $\partial\Omega$ is connected then $\dim RT^0_{r+1} = d_N - (d_L - 1) - g$. On the other hand the number of elements in a belted tree is equal to the number of elements in a tree plus g hence the number of elements in K^* is equal to $d_N - (d_L - 1 + g) = \dim RT^0_{r+1}$ so $\{\operatorname{curl}\boldsymbol{\Phi}_j\}_{j\in K^*}$ is a basis. ◻

Remark 4.3. Let us denote by $C \in \mathbb{R}^{d_{RT}\times d_N}$ the matrix that from the moments of $\mathbf{u}_h \in N_{r+1}$ computes the moments of $\operatorname{curl}\mathbf{u}_h \in RT_{r+1}$. It is worth noting that if $\{\mathbf{\Phi}_j\}_{j=1}^{d_N}$ is a cardinal basis for the moments in N_{r+1} then for any $j \in \{1,\dots,d_N\}$ the moments of $\operatorname{curl}\mathbf{\Phi}_j$ are the elements of column j of matrix C. Hence the moments of a basis of RT^0_{r+1} are give by the submatrix of C corresponding to the columns of indices $j \in K^*$.

If $\partial\Omega$ has $p+1$ connected components with $p>0$, the space of divergence-free Raviart–Thomas finite elements that are not the curl of Nédélec finite elements is not trivial and has dimension p. To obtain a basis it is necessary to complete the set $\{\operatorname{curl}\mathbf{\Phi}_i\}_{i\in K^*}$ by adding, for each $n=1,\dots,p$, a function $\mathbf{z}_{h,n} \in RT_{r+1}$ such that

$$\begin{cases} \operatorname{div}\mathbf{z}_{h,n} = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_l} \mathbf{z}_{h,n}\cdot\mathbf{n}_\Omega = \delta_{n,l} & l = 1\dots,p\,, \end{cases} \tag{4.2}$$

where $(\partial\Omega)_l$, for $l \in \{0,1,\dots,p\}$ are the connected components of $\partial\Omega$ being $(\partial\Omega)_0$ the external one. Note that, for any choice of $\mathbf{z}_{h,n}$, the set $\{\mathbf{z}_{h,n}\}_{n=1}^p$ is linearly independent. A solution of (4.2) can be computed, for instance, as indicated in [5]. Let us consider the graph $\mathcal{M}^D$ that has $d_P + (p+1)$ nodes, one for each moment of P_r and one for each connected component of $\partial\Omega$, and d_{RT} arcs, one for each moments of RT_{r+1} and for which the divergence matrix is the incidence matrix with reference node the one corresponding to the external connected component of $\partial\Omega$. If we choose a spanning tree of this graph for $n \in \{1,\dots,p\}$ the corresponding column of the matrix $\widetilde{B}_1$ in [5] contains the moments of the unique solution of (4.2) that has all the moments corresponding to the arcs of the cotree equal to zero. This solution can be computed using a very efficient elimination procedure.

Hence, taking into account Proposition 4.2, we obtain the following result that holds for any polyhedral Lipschitz domain:

Theorem 4.4. *The set* $\{\operatorname{curl}\mathbf{\Phi}_j\}_{j\in K^*} \cup \{\mathbf{z}_{h,n}\}_{n=1}^p$ *is a basis of* RT^0_{r+1}.

Proof. We have already proved that both $\{\operatorname{curl}\mathbf{\Phi}_j\}_{j\in K^*}$ and $\{\mathbf{z}_{h,n}\}_{n=1}^p$ are linearly independent. Hence to see that this set is linearly independent it is enough to see that the functions $\mathbf{z}_{h,n}$ are not the curl of any function in $H(\mathrm{curl};\Omega)$. This is clear since $\int_{(\partial\Omega)_l} \mathbf{z}_{h,n}\cdot\mathbf{n}_\Omega \neq 0$ for a connected component $(\partial\Omega)_n$ of $\partial\Omega$ while for any $\mathbf{u} \in H(\mathrm{curl};\Omega)$ and $l \in \{0,1,\dots,p\}$ $\int_{(\partial\Omega)_l} \operatorname{curl}\mathbf{u}\cdot\mathbf{n}_\Omega = 0$. In fact let $\phi_l \in H^1(\Omega)$ be such that $(\phi_l)_{|(\partial\Omega)_m} = \delta_{l,m}$. Then

$$\int_{(\partial\Omega)_l} \operatorname{curl}\mathbf{u}\cdot\mathbf{n}_\Omega = \int_{\partial\Omega} \operatorname{curl}\mathbf{u}\cdot\mathbf{n}_\Omega\,\phi_l = \int_\Omega \operatorname{curl}\mathbf{u}\,\nabla\phi_l = -\int_{\partial\Omega} \mathbf{u}\times\mathbf{n}_\Omega\,\nabla\phi_l = 0$$

because $(\nabla\phi_l)_{|\partial\Omega} = \mathbf{0}$.

The considered set is thus a basis of RT^0_{r+1} because $\dim RT^0_{r+1} = d_N - (d_L-1) - g + p = d_{K^*} + p$. ∎

In Figure 4.1, we summarizes the situation. Due to the property that curl(grad) is zero, we cannot construct a divergence-free basis of RT^0_{r+1} starting from the curl of a basis of N_{r+1} because they are not linear independent. However, the spanning (eventually belted) tree for the gradient of function L_{r+1} allows to identify the set (associated with the corresponding co-tree) K^* of columns in $G^\top$ that will provide a part of this basis, once we apply on them the curl operator (see Proposition 3). If $p>0$, the basis has to be completed *by hands*, by adding the generators of $\mathcal{H}^2$ (see Proposition 7), namely the solution of problem (4.2), for each $n = 1,...,p$.

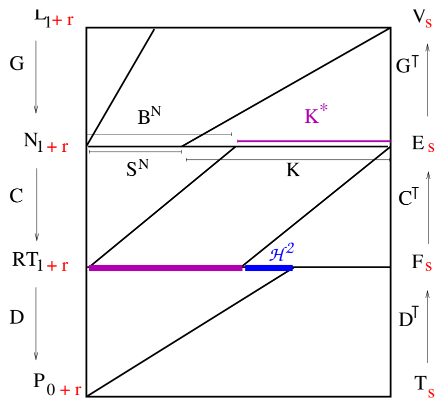

FIG. 4.1. *A graphical summary of the structure of the basis of RT^0_{r+1} and its construction.*

5. Conclusions. We have proved that, for a particular choice of degrees of freedom in the spaces N_{r+1} and L_{r+1} with $r \geq 0$, the matrix associated with the gradient operator is the transposed of the all-nodes incidence matrix of a directed and connected graph. This fact, that was well known when $r = 0$, allows to extend to high order finite elements the construction of a basis of the space of divergence-free finite elements of degree one analyzed in [20] and [6].

This method uses the canonical basis for the chosen degrees of freedom of N_{r+1} that, for $r > 0$, is not explicitly known. However, on each tetrahedron of the mesh, the elements of the canonical basis are related with the elements of a more natural basis by a generalized Vandermonde matrix that is independent of the tetrahedron up to its orientation.

The typical setting in which is convenient to use divergence-free Raviart–Thomas finite elements is the following mixed problem: find $(\mathbf{z}_h, p_h) \in RT_{r+1} \times P_r$ such that

$$\begin{cases} \int_\Omega \mathbf{z}_h \cdot \mathbf{w}_h + \int_\Omega p_h \operatorname{div} \mathbf{w}_h = \int_\Omega \mathbf{f} \cdot \mathbf{w}_h & \forall\, \mathbf{w}_h \in RT_{r+1} \\ \int_\Omega \operatorname{div} \mathbf{z}_h q_h = 0 & \forall\, q_h \in P_r\,. \end{cases}$$

It is clear that $\mathbf{z}_h \in RT^0_{r+1}$ and satisfies

$$\int_\Omega \mathbf{z}_h \cdot \mathbf{w}_h = \int_\Omega \mathbf{f} \cdot \mathbf{w}_h \;\; \forall\, \mathbf{w}_h \in RT^0_{r+1}\,.$$

If $\beta_2(\Omega) = 0$ we have proved that a basis of RT^0_{r+1} is given by $\{\operatorname{curl} \mathbf{\Phi}_i\}_{i \in K^*}$ being $\{\mathbf{\Phi}_i\}_{i=1}^{d_N}$ a cardinal basis of N_{r+1} and K^* the set of indices of the moments associated with arcs not belonging to a given belted tree of the graph of the curl operator. The mass matrix corresponding to this basis of the divergence-free subspace of RT_{r+1} is then the matrix with entries $\int_\Omega \operatorname{curl} \mathbf{\Phi}_j \cdot \operatorname{curl} \mathbf{\Phi}_i$ for i, $j \in K^*$. So it is a submatrix of the stiffness matrix of the cardinal basis of N_{r+1}, the one corresponding to the rows and columns of indices in K^*.

The stiffness matrix for the cardinal basis is computed, as usually, by assembling the elementary matrices $A^t_{card} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with entries $[A^t_{card}]_{k,\ell} = \int_t \operatorname{curl} \mathbf{\Phi}^t_\ell \cdot \operatorname{curl} \mathbf{\Phi}^t_k$, being $\mathtt{n} = (r+1)\begin{pmatrix} r+4 \\ 2 \end{pmatrix}$ the dimension of $\mathcal{P}^-_{r+1}\Lambda^1(t)$. The explicit form of the elements of the local cardinal basis $\{\mathbf{\Phi}^t_k\}_{k=1}^{\mathtt{n}}$ is not known. A basis of $\mathcal{P}^-_{r+1}\Lambda^1(t)$ is given by (see for example [16])

$$\{\lambda_t^{\boldsymbol{\eta}(e')}\boldsymbol{\omega}_{e'} \ : e' \in \Delta_1(t) \text{ and } \boldsymbol{\eta}(e') \in \mathcal{I}^{e'}(r,4)\} \tag{5.1}$$

being $\mathcal{I}^{e'}(r,4) = \{\boldsymbol{\eta} \in \mathcal{I}(r,4) \ : \ \eta_i = 0 \text{ if } i < m^t_{e'}(0)\}$. It is worth noting that the number of elements in $\mathcal{I}^{e'}(r,4)$ is equal to the number of elements in $\mathcal{I}(r,4-m^t_{e'}(0))$. We denote by $\{\boldsymbol{\zeta}_l\}_{l=1}^{\mathtt{n}}$ the elements of this basis and by $\{M_i^{N,t}\}_{i=1}^{\mathtt{n}}$ the set of local Nédélec moments.

Since $\{\boldsymbol{\zeta}_l\}_{l=1}^{\mathtt{n}}$ is a basis, for each $k \in \{1,\dots,\mathtt{n}\}$ there are $\mathtt{n}$ coefficients $c_{j,l}$ (and they are unique) such that $\boldsymbol{\Phi}^t_k = \sum_{l=1}^{\mathtt{n}} c_{k,l}\boldsymbol{\zeta}_l$. Moreover

$$M_i^{N,t}(\boldsymbol{\Phi}^t_k) = \delta_{i,k} = \sum_{l=1}^{\mathtt{n}} c_{k,l} M_i^{N,t}(\boldsymbol{\zeta}_l)\,.$$

Following [10], we introduce the generalized Vandermonde matrix: $V \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with entries $V_{i,l} = M^N_{t,i}(\boldsymbol{\zeta}_l)$. Since $\{\boldsymbol{\Phi}^t_k\}_{k=1}^{\mathtt{n}}$ is the local cardinal basis with respects to the moments introduced in Section 3, we have

$$M_i^{N,t}(\boldsymbol{\Phi}_k) = \delta_{i,k} = \sum_{l=1}^{\mathtt{n}} c_{k,l} M_i^{N,t}(\boldsymbol{\zeta}_l) = \sum_{l=1}^{\mathtt{n}} c_{k,l} V_{i,l} = [VC^\top]_{i,k}\,,$$

being $C \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ the matrix with entries $c_{k,l}$. This means that $VC^\top = I$ and then $C^\top = V^{-1}$. So, it is possible to compute A^t_{card} from the elementary matrices $A^t \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with entries $[A^t]_{i,j} = \int_t \operatorname{curl}\boldsymbol{\zeta}_i \cdot \operatorname{curl}\boldsymbol{\zeta}_j$ because they are linked by V in the following way.

$$A^t_{card} = CA^t\,C^\top = V^{-\top}A^tV^{-1}\,.$$

Moreover it is enough to compute V for the reference element because it is, in fact, independent of the physical element. (See [13]. See also [10] *Property 1.*)

Appendix A. Moments of $\nabla\varphi_h \in N_{r+1}$ being $\varphi_h \in L_{r+1}$.

Let us recall the set of moments that we are considering in the space N_{r+1}:

- $M^N_{\boldsymbol{\alpha},e}(\mathbf{u}_h) = \int_e \mathbf{u}_h \cdot \mathbf{t}_e\, B_e^{\boldsymbol{\alpha}}$, for each $e \in E$ and $\boldsymbol{\alpha} \in \mathcal{I}(r,2)$;
- $M^N_{\boldsymbol{\beta},f,i}(\mathbf{u}_h) = (-1)^i r\int_f (\mathbf{u}_h \times \mathbf{n}_f)\cdot B_f^{\boldsymbol{\beta}}\nabla_f\lambda_{m_f(j)}$ for each $f \in F$, $\boldsymbol{\beta} \in \mathcal{I}(r-1,3)$, $i,j \in \{1,2\}$ and $i \neq j$.
- $M^N_{\boldsymbol{\gamma},t,i}(\mathbf{u}_h) = (-1)^s r(r-1)\int_t \mathbf{u}_h \cdot B_t^{\boldsymbol{\gamma}}\nabla\lambda_{m_t(j)} \times \nabla\lambda_{m_t(k)}$ for each $t \in T$, $\boldsymbol{\gamma} \in \mathcal{I}(r-2,4)$, $i,j,k \in \{1,2,3\}$, $i \notin \{j,k\}$ and $j<k$.

If $\mathbf{u}_h = \nabla\varphi_h$ with $\varphi_h \in L_{r+1}$ we obtain

- $$M^N_{\boldsymbol{\alpha},e}(\nabla\varphi_h) = \int_e \nabla\varphi_h \cdot \mathbf{t}_e\, B_e^{\boldsymbol{\alpha}}$$

$$= \varphi_h(\mathbf{v}_{m_e(1)})B_e^{\boldsymbol{\alpha}}(\mathbf{v}_{m_e(1)}) - \varphi_h(\mathbf{v}_{m_e(0)})B_e^{\boldsymbol{\alpha}}(\mathbf{v}_{m_e(0)}) - \int_e \varphi_h\, \operatorname{grad}_e B_e^{\boldsymbol{\alpha}} \cdot \mathbf{t}_e$$

(since $\operatorname{grad}_e B_e^{\boldsymbol{\alpha}} \cdot \mathbf{t}_e = r\nabla_e\lambda_{m_e(1)}(B_e^{\boldsymbol{\alpha}-\mathbf{e}_1} - B_e^{\boldsymbol{\alpha}-\mathbf{e}_0})$)

$$= \varphi_h(\mathbf{v}_{m_e(1)})B_e^{\boldsymbol{\alpha}}(\mathbf{v}_{m_e(1)}) - \varphi_h(\mathbf{v}_{m_e(0)})B_e^{\boldsymbol{\alpha}}(\mathbf{v}_{m_e(0)}) - r\operatorname{grad}_e\lambda_{m_e(1)} \cdot \mathbf{t}_e \int_e \varphi_h\,(B_e^{\boldsymbol{\alpha}-\mathbf{e}_1} - B_e^{\boldsymbol{\alpha}-\mathbf{e}_0})$$

(using that $\nabla\lambda_{m_e(1)} \cdot \mathbf{t}_e = \frac{1}{|e|}$)

$$= \varphi_h(\mathbf{v}_{m_e(1)})B_e^{\boldsymbol{\alpha}}(\mathbf{v}_{m_e(1)}) - \varphi_h(\mathbf{v}_{m_e(0)})B_e^{\boldsymbol{\alpha}}(\mathbf{v}_{m_e(0)}) - \frac{r}{|e|}\left[\int_e \varphi_h\, B_e^{\boldsymbol{\alpha}-\mathbf{e}_1} - \int_e \varphi_h\, B_e^{\boldsymbol{\alpha}-\mathbf{e}_0}\right]\,.$$

Summing up

$$M^N_{\boldsymbol{\alpha},e}(\nabla\varphi_h) = \begin{cases} -\varphi_h(\mathbf{v}_{m_e(0)}) + \frac{r}{|e|}\int_e \varphi_h\, B_e^{\boldsymbol{\alpha}-\mathbf{e}_0} & \text{if } \boldsymbol{\alpha} = (r,0) \\ \varphi_h(\mathbf{v}_{m_e(1)}) - \frac{r}{|e|}\int_e \varphi_h\, B_e^{\boldsymbol{\alpha}-\mathbf{e}_1} & \text{if } \boldsymbol{\alpha} = (0,r) \\ -\frac{r}{|e|}\left[\int_e \varphi_h\, B_e^{\boldsymbol{\alpha}-\mathbf{e}_1} - \int_e \varphi_h\, B_e^{\boldsymbol{\alpha}-\mathbf{e}_0}\right] & \text{otherwise} \end{cases}$$

where we have used that $\lambda_e^{\boldsymbol{\alpha}}(\mathbf{v}_{m_e(l)}) = \begin{cases} 1 & \text{if } \alpha_i = 0 \\ 0 & \text{otherwise} \end{cases}$ for $i, l \in \{0,1\}$ and $i \neq l$.

- $M^N_{\boldsymbol{\beta},f,i}(\nabla\varphi_h) = (-1)^i r \displaystyle\int_f (\nabla\varphi_h \times \mathbf{n}_f)\cdot B_f^{\boldsymbol{\beta}} \nabla_f \lambda_{m_f(j)}$
(with $i, j \in \{1,2\}$ and $i \neq j$)

$$\begin{aligned} &= (-1)^{i+1} r \int_f (\nabla\varphi_h \times \nabla_f\lambda_{m_f(j)})\cdot \mathbf{n}_f\, B_f^{\boldsymbol{\beta}} \\ &= (-1)^{i+1} r \int_f \operatorname{div}_f(\varphi_h \nabla_f\lambda_{m_f(j)} \times \mathbf{n}_f)\, B_f^{\boldsymbol{\beta}} \\ &= (-1)^{i+1} r \Big[\int_{\partial f} (\varphi_h \nabla_f\lambda_{m_f(j)} \times \mathbf{n}_f)\cdot \boldsymbol{\nu}_f\, B_f^{\boldsymbol{\beta}} \\ &\qquad\qquad - \int_f (\varphi_h \nabla_f\lambda_{m_f(j)} \times \mathbf{n}_f)\cdot \operatorname{grad}_f B_f^{\boldsymbol{\beta}}\Big] \end{aligned}$$

(since $\nabla B_f^{\boldsymbol{\beta}} = (r-1)\sum_{k=1}^2 \nabla_f\lambda_{m_f(k)}\left(B_f^{\boldsymbol{\beta}-\mathbf{e}_k} - B_f^{\boldsymbol{\beta}-\mathbf{e}_0}\right)$)

$$= (-1)^{i+1} r \left[\sum_{l=0}^{2} (\nabla_f\lambda_{m_f(j)} \times \mathbf{n}_f)\cdot \boldsymbol{\nu}_{f|f-\mathbf{v}_l} \int_{f-\mathbf{v}_l} \varphi_h [B_f^{\boldsymbol{\beta}}]_{|f-\mathbf{v}_l}\right.$$

$$\left. -(r-1)\sum_{k=1}^{2} (\nabla_f\lambda_{m_f(j)} \times \mathbf{n}_f)\cdot \nabla_f\lambda_{m_f(k)} \int_f \varphi_h (B_j^{\boldsymbol{\beta}-\mathbf{e}_k} - B_j^{\boldsymbol{\beta}-\mathbf{e}_0})\right]$$

(recalling that $\nabla_f\lambda_{m_f(l)} = -\frac{|f-[\mathbf{v}_l]|}{2|f|}\boldsymbol{\nu}_{f|f-[\mathbf{v}_{m_f(l)}]}$)

$$\begin{aligned} = (-1)^{i+1} r \Big[&-\sum_{l=0}^2 \frac{2|f|}{|f-\mathbf{v}_l|}(\nabla_f\lambda_{m_f(j)} \times \mathbf{n}_f)\cdot \nabla_f\lambda_{m_f(l)} \int_{f-\mathbf{v}_l} \varphi_h [B_f^{\boldsymbol{\beta}}]_{|f-\mathbf{v}_l} \\ &-(r-1)\sum_{k=1}^2 (\nabla_f\lambda_{m_f(j)} \times \mathbf{n}_f)\cdot \nabla_f\lambda_{m_f(k)} \int_f \varphi_h \left(B_f^{\boldsymbol{\beta}-\mathbf{e}_k} - B_f^{\boldsymbol{\beta}-\mathbf{e}_0}\right)\Big], \end{aligned}$$

$$\begin{aligned} &= (-1)^{i+1} r\, (\nabla_f\lambda_{m_f(j)} \times \mathbf{n}_f)\cdot \nabla_f\lambda_{m_f(i)} \left[\frac{2|f|}{|f-\mathbf{v}_0|}\int_{f-\mathbf{v}_0} \varphi_h [B_f^{\boldsymbol{\beta}}]_{|f-\mathbf{v}_0}\right. \\ &\left. -\frac{2|f|}{|f-\mathbf{v}_i|}\int_{f-\mathbf{v}_i} \varphi_h [B_f^{\boldsymbol{\beta}}]_{|f-\mathbf{v}_i} - (r-1)\int_f \varphi_h \left(B_f^{\boldsymbol{\beta}-\mathbf{e}_i} - B_f^{\boldsymbol{\beta}-\mathbf{e}_0}\right)\right] \end{aligned}$$

(using that $(\nabla_f\lambda_{m_f(i)} \times \nabla_f\lambda_{m_f(j)})\cdot \mathbf{n}_f = (-1)^{i+1}\frac{1}{2|f|}$)

$$\begin{aligned} = r \Big[&\frac{1}{|e_{12}|}\int_{e_{12}} \varphi_h [B_f^{\boldsymbol{\beta}}]_{|e_{12}} - \frac{1}{|e_{0j}|}\int_{e_{0j}} \varphi_h [B_f^{\boldsymbol{\beta}}]_{|e_{0j}} \\ &\qquad -\frac{r-1}{2|f|}\int_f \varphi_h \left(B_f^{\boldsymbol{\beta}-\mathbf{e}_i} - B_f^{\boldsymbol{\beta}-\mathbf{e}_0}\right)\Big]\,. \end{aligned}$$

Summing up

$$M^N_{\boldsymbol{\beta},f,i}(\nabla\varphi_h) = \begin{cases} \frac{r}{|e_{12}|}\int_{e_{12}}\varphi_h[B_f^{\boldsymbol{\beta}}]_{|e_{12}} - \frac{r}{|e_{0j}|}\int_{e_{0j}}\varphi_h[B_f^{\boldsymbol{\beta}}]_{|e_{0j}} \\ \qquad\qquad\qquad \text{if } \beta_0 = 0 \text{ and } \beta_i = 0 \\ -\frac{r}{|e_{0j}|}\int_{e_{0j}}\varphi_h[B_f^{\boldsymbol{\beta}}]_{|e_{0j}} + \frac{r(r-1)}{2|f|}\int_f \varphi_h B_f^{\boldsymbol{\beta}-\mathbf{e}_0} \\ \qquad\qquad\qquad \text{if } \beta_0 \neq 0 \text{ and } \beta_i = 0 \\ \frac{r}{|e_{12}|}\int_{e_{12}}\varphi_h[B_f^{\boldsymbol{\beta}}]_{|e_{12}} - \frac{r(r-1)}{2|f|}\int_f \varphi_h B_f^{\boldsymbol{\beta}-\mathbf{e}_i} \\ \qquad\qquad\qquad \text{if } \beta_0 = 0 \text{ and } \beta_i \neq 0 \\ -\frac{r(r-1)}{2|f|}\int_f \varphi_h \left(B_f^{\boldsymbol{\beta}-\mathbf{e}_i} - B_f^{\boldsymbol{\beta}-\mathbf{e}_0}\right) \\ \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Finally we recall that if $e = f - [\mathbf{v}_{m_f(l)}]$ then

$$[B_f^{\boldsymbol{\beta}}]_{|e} = \begin{cases} B_e^{R_{2,l}\boldsymbol{\beta}} & \text{if } \beta_l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

- $M^N_{\boldsymbol{\gamma},t,i}(\nabla\varphi_h) = (-1)^s r(r-1)\int_t \nabla\varphi_h \cdot B_t^{\boldsymbol{\gamma}}\nabla\lambda_{m_t(j)} \times \nabla\lambda_{m_t(k)}$ (with $i,j,k \in \{1,2,3\}$, $j<k$, and $i \notin \{j,k\}$)

$$= (-1)^s r(r-1)\left[\int_{\partial t}\varphi_h\, B_t^{\boldsymbol{\gamma}}\nabla\lambda_{m_t(j)} \times \nabla\lambda_{m_t(k)} \cdot \mathbf{n}_t - \int_t \varphi_h \operatorname{div}\left(B_t^{\boldsymbol{\gamma}}\nabla\lambda_{m_t(j)} \times \nabla\lambda_{m_t(k)}\right)\right]$$

$$= (-1)^s r(r-1)\left[\int_{\partial t}\varphi_h\, B_t^{\boldsymbol{\gamma}}\nabla\lambda_{m_t(j)} \times \nabla\lambda_{m_t(k)} \cdot \mathbf{n}_t - \int_t \varphi_h \nabla B_t^{\boldsymbol{\gamma}} \cdot \left(\nabla\lambda_{m_t(j)} \times \nabla\lambda_{m_t(k)}\right)\right]$$

(since $\nabla B_t^{\boldsymbol{\gamma}} = (r-2)\sum_{l=1}^3 \nabla\lambda_{m_t(l)}\left(B_t^{\boldsymbol{\gamma}-\mathbf{e}_l} - B_t^{\boldsymbol{\gamma}-\mathbf{e}_0}\right)$)

$$= (-1)^s r(r-1)\Big[\sum_{f\in\Delta_2(t)} \nabla\lambda_{m_t(j)} \times \nabla\lambda_{m_t(k)} \cdot \mathbf{n}_{t|f}\int_f \varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f} - (r-2)\sum_{l=1}^3 \nabla\lambda_{m_t(l)}\cdot\left(\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\right)\int_t\varphi_h\left(B_t^{\boldsymbol{\gamma}-\mathbf{e}_l} - B_t^{\boldsymbol{\gamma}-\mathbf{e}_0}\right)\Big]$$

(denoting $f_l = t - \mathbf{v}_l$)

$$= (-1)^s r(r-1)\Big[\sum_{l=0}^3 \nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\cdot\mathbf{n}_{t|f_l}\int_{f_l}\varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f_l} - (r-2)\nabla\lambda_{m_t(i)}\cdot\left(\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\right)\int_t\varphi_h\left(B_t^{\boldsymbol{\gamma}-\mathbf{e}_i} - B_t^{\boldsymbol{\gamma}-\mathbf{e}_0}\right)\Big]$$

(recalling that $\mathbf{n}_{t|f_l} = -\frac{3|t|}{|f_l|}\nabla\lambda_{m_t(l)}$)

$$= (-1)^s r(r-1)\left[\sum_{l=0}^3 -\frac{3|t|}{|f_l|}\nabla\lambda_{m_t(l)}\cdot\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\int_{f_l}\varphi_h[B_t^{\boldsymbol{\gamma}}]_{|f_l} - (r-2)\nabla\lambda_{m_t(i)}\cdot\left(\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\right)\int_t\varphi_h\left(B_t^{\boldsymbol{\gamma}-\mathbf{e}_i} - B_t^{\boldsymbol{\gamma}-\mathbf{e}_0}\right)\right]$$

$$= (-1)^s r(r-1)\left[-\frac{3|t|}{|f_0|}\nabla\lambda_{m_t(0)}\cdot\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\int_{f_0}\varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f_0} - \frac{3|t|}{|f_i|}\nabla\lambda_{m_t(i)}\cdot\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\int_{f_i}\varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f_l} - (r-2)\nabla\lambda_{m_t(i)}\cdot\left(\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\right)\int_t\varphi_h\left(B_t^{\boldsymbol{\gamma}-\mathbf{e}_i} - B_t^{\boldsymbol{\gamma}-\mathbf{e}_0}\right)\right]$$

(since $\nabla\lambda_{m_t(0)} = -\sum_{n=1}^{3}\nabla\lambda_{m_t(n)}$)

$$= (-1)^s r(r-1)\left[\frac{3|t|}{|f_0|}\nabla\lambda_{m_t(i)}\cdot\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\int_{f_0}\varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f_0}\right.$$
$$-\frac{3|t|}{|f_i|}\nabla\lambda_{m_t(i)}\cdot\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\int_{f_i}\varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f_i}$$
$$\left.-(r-2)\nabla\lambda_{m_t(i)}\cdot\left(\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\right)\int_t\varphi_h\left(B_t^{\boldsymbol{\gamma}-\mathbf{e}_i}-B_t^{\boldsymbol{\gamma}-\mathbf{e}_0}\right)\right]$$

$$= (-1)^s r(r-1)\nabla\lambda_{m_t(i)}\cdot\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\left[\frac{3|t|}{|f_0|}\int_{f_0}\varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f_0}\right.$$
$$\left.-\frac{3|t|}{|f_i|}\int_{f_i}\varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f_i}-(r-2)\int_t\varphi_h\left(B_t^{\boldsymbol{\gamma}-\mathbf{e}_i}-B_t^{\boldsymbol{\gamma}-\mathbf{e}_0}\right)\right.$$

(using that $(-1)^s\nabla\lambda_{m_t(i)}\cdot\left(\nabla\lambda_{m_t(j)}\times\nabla\lambda_{m_t(k)}\right) = (-1)^{i+1}\frac{1}{6|t|}$)

$$= (-1)^{i+1}\frac{r(r-1)}{6|t|}\left[\frac{3|t|}{|f_0|}\int_{f_0}\varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f_0}-\frac{3|t|}{|f_i|}\int_{f_i}\varphi_h\,[B_t^{\boldsymbol{\gamma}}]_{|f_i}\right.$$
$$\left.-(r-2)\int_t\varphi_h\left(B_t^{\boldsymbol{\gamma}-\mathbf{e}_i}-B_t^{\boldsymbol{\gamma}-\mathbf{e}_0}\right)\right].$$

Summing up $M^N_{\boldsymbol{\gamma},t,i}(\nabla\varphi_h)$

$$= \begin{cases} (-1)^i r(r-1)\left(-\frac{1}{2|f_{123}|}\int_{f_{123}}\varphi_h[B_t^{\boldsymbol{\gamma}}]_{|f_{123}}+\frac{1)}{2|f_{0jk}|}\int_{f_{0jk}}\varphi_h[B_t^{\boldsymbol{\gamma}}]_{|f_{0jk}}\right) \\ \qquad\qquad \text{if } \gamma_0=0 \text{ and } \gamma_i=0 \\ (-1)^i r(r-1)\left(\frac{1}{2|f_{0jk}|}\int_{f_{0jk}}\varphi_h[B_t^{\boldsymbol{\gamma}}]_{|f_{0jk}}-\frac{r-2}{6|t|}\int_t\varphi_h B_f^{\boldsymbol{\gamma}-\mathbf{e}_0}\right) \\ \qquad\qquad \text{if } \gamma_0\neq 0 \text{ and } \gamma_i=0 \\ (-1)^i r(r-1)\left(-\frac{1}{2|f_{123}|}\int_{f_{123}}\varphi_h[B_t^{\boldsymbol{\gamma}}]_{|f_{123}}+\frac{r-2}{6|t|}\int_t\varphi_h B_t^{\boldsymbol{\gamma}-\mathbf{e}_i}\right) \\ \qquad\qquad \text{if } \gamma_0=0 \text{ and } \gamma_i\neq 0 \\ (-1)^i\,\frac{r(r-1)(r-2)}{6|t|}\int_t\varphi_h\left(B_t^{\boldsymbol{\gamma}-\mathbf{e}_i}-B_f^{\boldsymbol{\gamma}-\mathbf{e}_0}\right) \text{ otherwise.} \end{cases}$$

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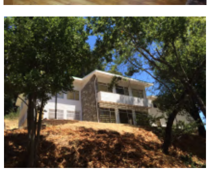

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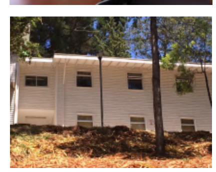
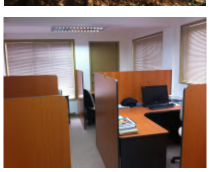

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