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Sergio Caucao, Gabriel N. Gatica,
Ricardo Oyarzúa, Felipe Sandoval

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Residual-based a posteriori error analysis for the coupling of the Navier–Stokes and Darcy–Forchheimer equations*

SERGIO CAUCAO[†] GABRIEL N. GATICA[‡] RICARDO OYARZÚA[§] FELIPE SANDOVAL[¶]

Abstract

In this paper we consider two mixed variational formulations that have been recently proposed for the coupling of the Navier–Stokes and Darcy–Forchheimer equations, and derive reliable and efficient residual-based a posteriori error estimators suitable for adaptive mesh-refinement methods. For the reliability analysis of both schemes we make use of the inf-sup condition and the strict monotonicity of the operators involved, suitable Helmholtz decomposition in nonstandard Banach space in the porous medium, local approximation properties of the Clément interpolant and Raviart–Thomas operator, and a smallness assumption on the data. In turn, inverse inequalities, the localization technique based on triangle-buble and edge-buble functions in local L^p spaces, are the main tools for study the efficiency estimate. In addition, for one of the schemes, we derive two estimators, one obtained as a direct consequence of the Cauchy–Schwarz inequality and the other one employing a Helmholtz decomposition. Finally, several numerical results confirming the properties of the estimators and illustrating the performance of the associated adaptive algorithm are reported.

Key words: Navier–Stokes problem, Darcy–Forchheimer problem, primal-mixed finite element methods, fully-mixed finite element methods, a posteriori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

We have recently introduced two mixed finite element methods to numerically approximate the fluid flow between porous media and free-flow zones described by the coupling of the Navier–Stokes and Darcy–Forchheimer equations with the mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman condition on the interface [7, 10]. In particular, a primal-mixed formulation was derived and analyzed in [7], that is, the standard velocity-pressure mixed formulation in the Navier–Stokes domain and the dual-mixed one in the Darcy–Forchheimer region, which yields the introduction

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†Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: scaucao@ci2ma.udec.cl. Present address: Department of Mathematics, University of Pittsburgh, PA 15260, USA, email: sac304@pitt.edu

‡CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: ggatica@ci2ma.udec.cl.

§GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: royarzua@ubiobio.cl.

¶CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: fsandovals@udec.cl.

of the trace of the porous medium presure as a suitable Lagrange multiplier. The well-posedness of the problem is achieved by combining a fixed-point strategy, classical results on nonlinear monotone operators and the well-known Schauder and Banach fixed-point theorems. A feasible choice of finite element subspaces for the formulation introduced in [7] is given by Bernardi–Raugel and Raviart–Thomas elements for the velocities, and piecewise constant elements for the pressures and the Lagrange multiplier. On the other hand, a fully-mixed formulation, that is, dual-mixed formulations in both domains, which yields the introduction of two additional Lagrange mulipliers: the trace of the porous media pressure and the fluid velocity on the interface, was derived in [10]. The resulting augmented variational system of equations is then ordered so that it shows a twofold saddle point structure. The well-posedness of the problem is obtained by combining a fixed-point argument, an abstract theory for twofold saddle point problems, classical results on nonlinear monotone operators and the well-known Schauder and Banach fixed-point theorems. The corresponding mixed finite element scheme employs Raviart–Thomas element, continuous piecewise polynomials and piecewise polynomials for the pseudostress tensor, velocity and vorticity in the free fluid, whereas Raviart–Thomas and piecewise polynomials for the velocity and pressure in the porous medium. For both formulations, sub-optimal a priori error estimates were also derived.

Now, it is well known that under the eventual presence of singularities, as well as when dealing with nonlinear problems, as in the present case, most of the standard Galerkin procedures such as finite element and mixed finite element methods inevitably lose accuracy, and hence one usually tries to recover it by applying an adaptive algorithm based on a posteriori error estimates. For example, residual-based a posteriori error analyses for the Stokes–Darcy and Navier–Stokes/Darcy coupled problems have been developed in [3] and [8] for the associated primal-mixed and fully-mixed formulations, respectively. In fact, standard arguments relying on duality techniques, suitable decompositions and classical approximation properties, are combined there with corresponding small data assumptions to derive the reliability of the estimators. In turn, inverse inequalities and the usual localization technique based on bubble functions are employed in both works to prove the corresponding efficiency estimates. On the other hand, and concerning quasi-Newtonian fluid flows obeying to the power law, as in the case of the Darcy–Forchheimer model, not much has been done and we just refer to [16, 13, 17], where different contributions addressing this interesting issue can be found. In particular, an a posteriori error estimator defined via a non-linear projection of the residues of the variational equations for a three-field model of a generalized Stokes problem was proposed and analyzed in [16]. We remark that the non-linear projections do not need to be explicitly computed to construct the a posteriori error estimates. In turn, a fully local residual-based a posteriori error estimator for the mixed formulation of the p-Laplacian problem in a polygonal domain, was derived in [13]. In this case, the authors study the reliability of the estimator defining two residues and then bounding the norm of the errors in terms of the norms of these residues. Moreover, the discretized dual-mixed formulation is hybridized and provide several tests for p = 1.8 and p = 3 to experimentally verify the reliability of the estimator. We remark that up to the authors' knowledge, there is not works dealing with the a posteriori error analysis for the coupling of the Navier–Stokes (or Stokes) and the Darcy–Forchheimer models.

According to the above discussion and aiming to complement the results on the numerical analysis of the coupled Navier–Stokes and Darcy–Forchheimer equations, in this paper we proceed similarly to [16, 13, 17, 19, 3, 20, 21, 8] and [9], and develop reliable and efficient residual-based a posteriori error estimators for the mixed finite element methods introduced and analyzed in [7] and [10]. More precisely, starting from the inf-sup condition and the strict monotonicity of the operators involved, and employing suitable Helmholtz decompositions in nonstandard Banach spaces, we prove the reliability of residual-based estimators under a smallness condition on the data, . In turn, the efficiency estimate is consequence of standard arguments such as inverse inequalities and the localization technique based on triangle-buble and edge-buble functions in local L^{p} spaces. Alternatively, and similarly to [9], a second reliable and efficient residual-based a posteriori error estimator not making use of any Helmholtz decomposition in the Navier–Stokes region is also proposed for the fully-mixed formulation.

The rest of this work is organized as follows. The remainder of this section introduces some standard notations and definition of functional spaces. In Section 2 we recall from [7] and [10] the model problem. Next, in Section 3 we describe the continuous formulations and the corresponding primal-mixed and fully-mixed finite element methods, whereas some preliminary results necessary to the a posteriori error analysis are established in Section 4. Then, in Sections 5 and 6 we introduce a posteriori error indicators and, assuming small data, we derive the corresponding theoretical bounds yielding reliability and efficiency of each estimator. Finally, some numerical results confirming the theoretical sub-optimal order of convergence and at the same time suggesting an optimal rate of convergence as in [7, 10] are presented in Section 7. Additionally, these numerical essays illustrate the efficiency and reliability of the a posteriori error estimators, and show the good performance of the associated adaptive algorithm for the finite element methods.

We end this section by introducing some definitions and fixing some notations. Let $\mathcal{O} \subseteq \mathbb{R}^n$, $n \in \{2,3\}$, denote a domain with Lipschitz boundary Γ. For $\mathrm{s} \geq 0$ and $\mathrm{p} \in [1,+\infty]$ we denote by $\mathrm{L}^{\mathrm{p}}(\mathcal{O})$ and $\mathrm{W}^{\mathrm{s},\mathrm{p}}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^{\mathrm{p}}(\mathcal{O})}$ and $\|\cdot\|_{\mathrm{s},\mathrm{p};\mathcal{O}}$, respectively. Note that $\mathrm{W}^{0,\mathrm{p}}(\mathcal{O}) = \mathrm{L}^{\mathrm{p}}(\mathcal{O})$. If $\mathrm{p}=2$, we write $\mathrm{H}^{\mathrm{s}}(\mathcal{O})$ in place of $\mathrm{W}^{\mathrm{s},2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{\mathrm{s},\mathcal{O}}$, respectively, and the seminorm by $|\cdot|_{\mathrm{s},\mathcal{O}}$. In addition, we denote by $\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Gamma)$ the trace space of $\mathrm{W}^{1,\mathrm{p}}(\mathcal{O})$, and let $\mathrm{W}^{-1/\mathrm{q},\mathrm{q}}(\Gamma)$ be the dual space of $\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Gamma)$ endowed with the norms $\|\cdot\|_{1/\mathrm{q},\mathrm{p};\Gamma}$ and $\|\cdot\|_{-1/\mathrm{q},\mathrm{q};\Gamma}$, respectively, with $\mathrm{p},\mathrm{q} \in (1,+\infty)$ satisfying $1/\mathrm{p} + 1/\mathrm{q} = 1$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Furthermore, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij} \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{n\times n}$. In what follows, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$. Additionally, we recall that

$$\mathbb{H}(\mathbf{div};\mathcal{O}) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\mathcal{O}) : \quad \mathbf{div}\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O}) \Big\},$$

equipped with the usual norm $\|\boldsymbol{\tau}\|^2_{\mathbf{div};\mathcal{O}} := \|\boldsymbol{\tau}\|^2_{0,\mathcal{O}} + \|\mathbf{div}\boldsymbol{\tau}\|^2_{0,\mathcal{O}}$, is a standard Hilbert space in the realm of mixed problems. On the other hand, the following symbol for the $\mathrm{L}^2(\Gamma)$ inner product

$$\langle \xi, \lambda \rangle_\Gamma := \int_\Gamma \xi\, \lambda \quad \forall\, \xi, \lambda \in \mathrm{L}^2(\Gamma),$$

will also be employed for their respective extension as the duality parity between $\mathrm{W}^{-1/\mathrm{q},\mathrm{q}}(\Gamma)$ and $\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Gamma)$. Furthermore, given an integer $k \geq 0$ and a set $S \subseteq \mathbb{R}^n$, $\mathrm{P}_k(S)$ denotes the space of polynomial functions on S of degree $\leq k$. In addition, and coherently with previous notations, we set $\mathbf{P}_k(S) := [\mathrm{P}_k(S)]^n$ and $\mathbb{P}_k(S) := [\mathrm{P}_k(S)]^{n\times n}$. Finally, we end this section by mentioning that, throughout the rest of the paper, we employ $\mathbf{0}$ to denote a generic null vector (or tensor), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The model problem

For simplicity of exposition we set the problem in $\mathbb{R}^2$. In order to describe the geometry under consideration we let Ω_S and Ω_D be two bounded and simply connected polygonal domains in $\mathbb{R}^2$ such that $\partial\Omega_\mathrm{S} \cap \partial\Omega_\mathrm{D} = \Sigma \neq \emptyset$ and $\Omega_\mathrm{S} \cap \Omega_\mathrm{D} = \emptyset$. Then, let $\Gamma_\mathrm{S} := \partial\Omega_\mathrm{S} \setminus \overline{\Sigma}$, $\Gamma_\mathrm{D} := \partial\Omega_\mathrm{D} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_\mathrm{S} \cup \Sigma \cup \Omega_\mathrm{D}$ and Ω_S (and hence inward to Ω_D when seen on Σ). On Σ we also consider a unit tangent vector $\mathbf{t}$ (see Figure 2.1 below).

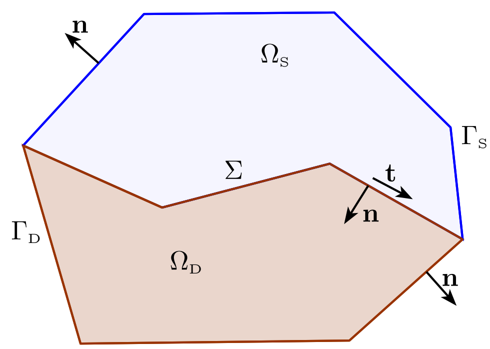

Figure 2.1: Sketch of a 2D geometry of our Navier–Stokes/Darcy–Forchheimer models.

The problem we are interested in consists of the movement of an incompressible viscous fluid occupying Ω_S which flows towards and from a porous medium Ω_D through Σ, where Ω_D is saturated with the same fluid. The mathematical model is defined by two separate groups of equations and by a set of coupling terms. In the free fluid domain Ω_S, the motion of the fluid can be described by the incompressible Navier–Stokes equations:

$$-2\mu\,\mathbf{div}\,\mathbf{e}(\mathbf{u}_\mathrm{S}) + \rho(\nabla\mathbf{u}_\mathrm{S})\mathbf{u}_\mathrm{S} + \nabla p_\mathrm{S} = \mathbf{f}_\mathrm{S} \quad \text{in} \quad \Omega_\mathrm{S}, \quad \operatorname{div}\mathbf{u}_\mathrm{S} = 0 \quad \text{in} \quad \Omega_\mathrm{S}, \quad \mathbf{u}_\mathrm{S} = \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{S}, \tag{2.1}$$

where $\mathbf{e}(\mathbf{u}_\mathrm{S}) := \dfrac{1}{2}\Big\{\nabla\mathbf{u}_\mathrm{S} + (\nabla\mathbf{u}_\mathrm{S})^\mathrm{t}\Big\}$ stands for the strain tensor of small deformations, μ is the viscosity of the fluid, ρ is the density, and $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S})$ is a given external force.

On the other hand, as was explained in [7, 10], given functions $\mathbf{f}_\mathrm{D} \in \mathbf{L}^{3/2}(\Omega_\mathrm{D})$ and $g_\mathrm{D} \in \mathrm{L}^2(\Omega_\mathrm{D})$, in the porous medium Ω_D we consider the Darcy–Forchheimer equations:

$$\frac{\mu}{\rho}\mathbf{K}^{-1}\mathbf{u}_\mathrm{D} + \frac{\mathtt{F}}{\rho}|\mathbf{u}_\mathrm{D}|\mathbf{u}_\mathrm{D} + \nabla p_\mathrm{D} = \mathbf{f}_\mathrm{D} \quad \text{in} \quad \Omega_\mathrm{D}, \quad \operatorname{div}\mathbf{u}_\mathrm{D} = g_\mathrm{D} \quad \text{in} \quad \Omega_\mathrm{D}, \quad \mathbf{u}_\mathrm{D}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_\mathrm{D}, \tag{2.2}$$

where $\mathtt{F}$ represents the Forchheimer number of the porous medium, and $\mathbf{K} \in \mathbb{L}^\infty(\Omega_\mathrm{D})$ is a symmetric tensor in Ω_D representing the intrinsic permeability $\boldsymbol{\kappa}$ of the porous medium divided by the viscosity

μ of the fluid. We assume that there exists $C_{\mathbf{K}} > 0$ such that $\mathbf{w} \cdot \mathbf{K}^{-1}(\mathbf{x})\mathbf{w} \geq C_{\mathbf{K}}|\mathbf{w}|^2$, for almost all $\mathbf{x} \in \Omega_{\mathrm{D}}$, and for all $\mathbf{w} \in \mathbb{R}^2$. Finally, the transmission conditions coupling (2.1) and (2.2) will be described below.

3 The variational formulations

In this section we introduce two variational formulations for the coupling of the Navier–Stokes and Darcy–Forchheimer equations proposed in [7, Section 2.2] and [10, Section 3.1], and recall the respective solvability results.

3.1 Preliminaries

We first introduce further notations and definitions. In what follows, given $\star \in \{\mathrm{S}, \mathrm{D}\}$, we set

$$(u, v)_\star := \int_{\Omega_\star} u\, v, \quad (\mathbf{u}, \mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u} \cdot \mathbf{v}, \quad \text{and} \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}.$$

Furthermore, given $\mathrm{p} \in [2, +\infty)$, in the sequel we will employ the following Banach space,

$$\mathbf{H}^{\mathrm{p}}(\mathrm{div}\,; \Omega_{\mathrm{D}}) := \Big\{ \mathbf{v}_{\mathrm{D}} \in \mathbf{L}^{\mathrm{p}}(\Omega_{\mathrm{D}}) : \quad \mathrm{div}\, \mathbf{v}_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}}) \Big\},$$

endowed with the norm

$$\|\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^{\mathrm{p}}(\mathrm{div}\,;\Omega_{\mathrm{D}})} := \Big(\|\mathbf{v}_{\mathrm{D}}\|^{\mathrm{p}}_{\mathbf{L}^{\mathrm{p}}(\Omega_{\mathrm{D}})} + \|\mathrm{div}\, \mathbf{v}_{\mathrm{D}}\|^{\mathrm{p}}_{0,\Omega_{\mathrm{D}}} \Big)^{1/\mathrm{p}},$$

and the following subspaces of $\mathbb{L}^2(\Omega_{\mathrm{S}})$, $\mathbf{H}^{\mathrm{p}}(\mathrm{div}\,; \Omega_{\mathrm{D}})$ and $\mathbf{H}^1(\Omega_{\mathrm{S}})$, respectively

$$\begin{aligned}
\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) &:= \Big\{ \boldsymbol{\eta}_{\mathrm{S}} \in \mathbb{L}^2(\Omega_{\mathrm{S}}) : \quad \boldsymbol{\eta}_{\mathrm{S}}^{\mathrm{t}} = -\boldsymbol{\eta}_{\mathrm{S}} \Big\}, \\
\mathbf{H}^{\mathrm{p}}_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,; \Omega_{\mathrm{D}}) &:= \Big\{ \mathbf{v}_{\mathrm{D}} \in \mathbf{H}^{\mathrm{p}}(\mathrm{div}\,; \Omega_{\mathrm{D}}) : \quad \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} \Big\}, \\
\mathrm{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) &:= \Big\{ v_{\mathrm{S}} \in \mathrm{H}^1(\Omega_{\mathrm{S}}) : \quad v_{\mathrm{S}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{S}} \Big\}, \quad \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) := [\mathrm{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})]^2.
\end{aligned}$$

In addition, we need to introduce the space of traces $\mathbf{H}^{1/2}_{00}(\Sigma) := \Big[\mathrm{H}^{1/2}_{00}(\Sigma) \Big]^2$, where

$$\mathrm{H}^{1/2}_{00}(\Sigma) := \Big\{ v|_\Sigma : \quad v \in \mathrm{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \Big\}.$$

Observe that, if $\mathrm{E}_{0,\mathrm{S}} : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{L}^2(\partial\Omega_{\mathrm{S}})$ is the extension operator defined by

$$\mathrm{E}_{0,\mathrm{S}}(\psi) := \begin{cases} \psi & \text{on} \quad \Sigma \\ 0 & \text{on} \quad \Gamma_{\mathrm{S}} \end{cases} \quad \forall\, \psi \in \mathrm{H}^{1/2}(\Sigma),$$

we have that

$$\mathrm{H}^{1/2}_{00}(\Sigma) = \Big\{ \psi \in \mathrm{H}^{1/2}(\Sigma) : \quad \mathrm{E}_{0,\mathrm{S}}(\psi) \in \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{S}}) \Big\},$$

which is endowed with the norm $\|\psi\|_{1/2,00;\Sigma} := \|\mathrm{E}_{0,\mathrm{S}}(\psi)\|_{1/2,\partial\Omega_{\mathrm{S}}}$. The dual space of $\mathbf{H}^{1/2}_{00}(\Sigma)$ is denoted by $\mathbf{H}^{-1/2}_{00}(\Sigma)$.

3.2 The primal-mixed approach

Let us consider the Cauchy stress tensor $\widetilde{\boldsymbol{\sigma}}_{\mathrm{S}} := -p_{\mathrm{S}}\mathbb{I} + 2\mu\,\mathbf{e}(\mathbf{u}_{\mathrm{S}})$ in Ω_{S}. In this way, the Navier–Stokes problem 2.1 can be rewritten as

$$-\,\mathbf{div}\widetilde{\boldsymbol{\sigma}}_{\mathrm{S}} + \rho(\nabla\mathbf{u}_{\mathrm{S}})\mathbf{u}_{\mathrm{S}} \,=\, \mathbf{f}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad \operatorname{div}\mathbf{u}_{\mathrm{S}} \,=\, 0 \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad \mathbf{u}_{\mathrm{S}} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}. \tag{3.1}$$

In turn, in the porous media we consider the equations (2.2), whereas the transmission conditions are given by

$$\mathbf{u}_{\mathrm{S}}\cdot\mathbf{n} \,=\, \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \quad \text{on} \quad \Sigma \quad \text{and} \quad \widetilde{\boldsymbol{\sigma}}_{\mathrm{S}}\mathbf{n} + \frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}\,(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t})\,\mathbf{t} \,=\, -p_{\mathrm{D}}\mathbf{n} \quad \text{on} \quad \Sigma, \tag{3.2}$$

where α_d is a dimensionless positive constant which depends only on the geometrical characteristics of the porous medium and usually assumes values between 0.8 and 1.2.

3.2.1 The continuous formulation

In this section we introduce the weak formulation derived for the coupled problem given by (3.1), (2.2), and (3.2) (see [7, Section 2.2] for details). In fact, we first group the spaces and unknowns as follows:

$$\mathbf{H} := \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\operatorname{div};\Omega_{\mathrm{D}}), \quad \mathbf{Q} := \mathrm{L}^2_0(\Omega) \times \mathrm{W}^{1/3,3/2}(\Sigma),$$
$$\mathbf{u} := (\mathbf{u}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}}) \in \mathbf{H}, \quad (p,\lambda) \in \mathbf{Q},$$

where $p := p_{\mathrm{S}}\chi_{\mathrm{S}} + p_{\mathrm{D}}\chi_{\mathrm{D}}$, with $\chi_\star$ being the characteristic function for $\star \in \{\mathrm{S},\mathrm{D}\}$, and $\lambda := p_{\mathrm{D}}|_\Sigma \in \mathrm{W}^{1/3,3/2}(\Sigma)$ is an additional unknown. Thus, we arrive at the mixed variational formulation: Find $(\mathbf{u},(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$, such that

$$\begin{aligned}
[\mathbf{a}(\mathbf{u}_{\mathrm{S}})(\mathbf{u}),\mathbf{v}] + [\mathbf{b}(\mathbf{v}),(p,\lambda)] &= [\mathbf{f},\mathbf{v}] \qquad \forall\,\mathbf{v} := (\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{H},\\
[\mathbf{b}(\mathbf{u}),(q,\xi)] &= [\mathbf{g},(q,\xi)] \quad \forall\,(q,\xi) \in \mathbf{Q},
\end{aligned} \tag{3.3}$$

where, given $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, the operator $\mathbf{a}(\mathbf{w}_{\mathrm{S}}) : \mathbf{H} \to \mathbf{H}'$ is defined by

$$[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}),\mathbf{v}] \,:=\, [\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] + [\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] + [\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}),\mathbf{v}_{\mathrm{D}}],$$

with

$$\begin{aligned}
[\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] &:= 2\mu(\mathbf{e}(\mathbf{u}_{\mathrm{S}}),\mathbf{e}(\mathbf{v}_{\mathrm{S}}))_{\mathrm{S}} + \left\langle \frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}\,\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t}, \mathbf{v}_{\mathrm{S}}\cdot\mathbf{t} \right\rangle_\Sigma,\\
[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] &:= \rho((\nabla\mathbf{u}_{\mathrm{S}})\mathbf{w}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}},\\
[\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}),\mathbf{v}_{\mathrm{D}}] &:= \frac{\mu}{\rho}\left(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}\right)_{\mathrm{D}} + \frac{\mathtt{F}}{\rho}\left(|\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}\right)_{\mathrm{D}},
\end{aligned}$$

whereas the operator $\mathbf{b} : \mathbf{H} \to \mathbf{Q}'$ is given by

$$[\mathbf{b}(\mathbf{v}),(q,\xi)] := -(\operatorname{div}\mathbf{v}_{\mathrm{S}},q)_{\mathrm{S}} - (\operatorname{div}\mathbf{v}_{\mathrm{D}},q)_{\mathrm{D}} + \langle \mathbf{v}_{\mathrm{S}}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_\Sigma\,.$$

In turn, the functionals $\mathbf{f}$ and $\mathbf{g}$ are defined by

$$[\mathbf{f},\mathbf{v}] := (\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \quad \text{and} \quad [\mathbf{g},(q,\xi)] := -(g_{\mathrm{D}},q)_{\mathrm{D}}.$$

In all the terms above, $[\,\cdot,\cdot\,]$ denotes the duality pairing induced by the corresponding operators. Further details for the solvability of (3.3) follows from the fixed-point strategy developed in [7, Theorem 3.12].

3.2.2 The finite element method

Let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D} formed by shape-regular triangles of diameter h_T and denote by h_{S} and h_{D} their corresponding mesh sizes. Assume that they match on Σ so that $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. Hereafter $h := \max\big\{h_{\mathrm{S}}, h_{\mathrm{D}}\big\}$. For each $T \in \mathcal{T}_h^{\mathrm{D}}$ we consider the local Raviart–Thomas space of the lowest order:

$$\mathbf{RT}_0(T) := \operatorname{span}\Big\{(1,0),(0,1),(x_1,x_2)\Big\}.$$

In addition, for each $T \in \mathcal{T}_h^{\mathrm{S}}$ we denote by $\mathbf{BR}(T)$ the local Bernardi–Raugel space:

$$\mathbf{BR}(T) := \mathbf{P}_1(T) \oplus \operatorname{span}\Big\{\eta_2\eta_3\mathbf{n}_1, \eta_1\eta_3\mathbf{n}_2, \eta_1\eta_2\mathbf{n}_3\Big\},$$

where $\big\{\eta_1,\eta_2,\eta_3\big\}$ are the baricentric coordinates of T, and $\big\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\big\}$ are the unit outward normals to the opposite sides of the corresponding vertices of T. Hence, the finite element subspaces for the velocities and pressure are, respectively,

$$\begin{aligned}
\mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) &:= \Big\{\mathbf{v} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) : \quad \mathbf{v}|_T \in \mathbf{BR}(T), \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\Big\},\\
\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) &:= \Big\{\mathbf{v} \in \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\operatorname{div};\Omega_{\mathrm{D}}) : \quad \mathbf{v}|_T \in \mathbf{RT}_0(T), \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}\Big\},\\
\mathrm{L}_{h,0}(\Omega) &:= \Big\{q \in \mathrm{L}^2_0(\Omega) : \quad q|_T \in \mathrm{P}_0(T), \quad \forall\, T \in \mathcal{T}_h\Big\}.
\end{aligned}$$

Next, for introducing the finite element subspace of $\mathrm{W}^{1/3,3/2}(\Sigma)$, we denote by Σ_h the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{D}}$ (or $\mathcal{T}_h^{\mathrm{S}}$), which is formed by edges e of length h_e, and set $h_\Sigma := \max\big\{h_e : \ e \in \Sigma_h\big\}$. Therefore, we can define (see [7, Section 4] for details):

$$\Lambda_h(\Sigma) := \Big\{\xi_h : \Sigma \to \mathbb{R} : \quad \xi_h|_e \in \mathrm{P}_0(e) \quad \forall \text{ edge } e \in \Sigma_h\Big\}. \tag{3.4}$$

In this way, grouping the discrete spaces and unknowns as follows:

$$\mathbf{H}_h := \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}), \quad \mathbf{Q}_h := \mathrm{L}_{h,0}(\Omega) \times \Lambda_h(\Sigma),$$

$$\mathbf{u}_h := (\mathbf{u}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}) \in \mathbf{H}_h, \quad (p_h, \lambda_h) \in \mathbf{Q}_h,$$

where $p_h := p_{\mathrm{S},h}\chi_{\mathrm{S}} + p_{\mathrm{D},h}\chi_{\mathrm{D}}$, the Galerkin approximation of (3.3) reads: Find $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$, such that

$$\begin{aligned}
[\mathbf{a}_h(\mathbf{u}_{\mathrm{S},h})(\mathbf{u}_h), \mathbf{v}_h] + [\mathbf{b}(\mathbf{v}_h), (p_h, \lambda_h)] &= [\mathbf{f}, \mathbf{v}_h] && \forall\, \mathbf{v}_h := (\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{H}_h,\\
[\mathbf{b}(\mathbf{u}_h), (q_h, \xi_h)] &= [\mathbf{g}, (q_h, \xi_h)] && \forall\, (q_h, \xi_h) \in \mathbf{Q}_h.
\end{aligned} \tag{3.5}$$

Here, $\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h}) : \mathbf{H}_h \to \mathbf{H}_h'$ is the discrete version of $\mathbf{a}(\mathbf{w}_{\mathrm{S}})$ (with $\mathbf{w}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ in place of $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$) defined by

$$[\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_h), \mathbf{v}_h] := [\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] + [\mathcal{B}^h_{\mathrm{S}}(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] + [\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D},h}), \mathbf{v}_{\mathrm{D},h}],$$

where $\mathcal{B}^h_{\mathrm{S}}(\mathbf{w}_{\mathrm{S},h})$ is the well-known skew-symmetric convective form:

$$[\mathcal{B}^h_{\mathrm{S}}(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] := \rho\,((\nabla \mathbf{u}_{\mathrm{S},h})\mathbf{w}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h})_{\mathrm{S}} + \frac{\rho}{2}\,(\operatorname{div} \mathbf{w}_{\mathrm{S},h} \mathbf{u}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h})_{\mathrm{S}},$$

for all $\mathbf{u}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h}, \mathbf{w}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$. The solvability analysis and *a priori* error bounds for (3.5) are established in [7, Theorems 4.9 and 5.2], respectively.

3.3 The fully-mixed approach

Now, we consider the nonstandard pseudostress and vorticity tensors, respectively,

$$\boldsymbol{\sigma}_{\mathrm{S}} \,=\, -p_{\mathrm{S}}\mathbb{I} + 2\mu\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - \rho(\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}}) \quad\text{and}\quad \boldsymbol{\gamma}_{\mathrm{S}} \,=\, \frac{1}{2}\Big\{\nabla\mathbf{u}_{\mathrm{S}} - (\nabla\mathbf{u}_{\mathrm{S}})^{\mathrm{t}}\Big\} \quad\text{in}\quad \Omega_{\mathrm{S}}.$$

Thus, the Navier–Stokes problem 2.1 can be rewritten as

$$\begin{gathered}\frac{1}{2\mu}\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}} \,=\, \nabla\mathbf{u}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S}} - \frac{\rho}{2\mu}(\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}} \quad\text{in}\quad \Omega_{\mathrm{S}}, \quad -\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}} \,=\, \mathbf{f}_{\mathrm{S}} \quad\text{in}\quad \Omega_{\mathrm{S}},\\ p_{\mathrm{S}} \,=\, -\frac{1}{n}\operatorname{tr}\left(\boldsymbol{\sigma}_{\mathrm{S}} + \rho(\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})\right) \quad\text{in}\quad \Omega_{\mathrm{S}}, \quad \mathbf{u}_{\mathrm{S}} \,=\, \mathbf{0} \quad\text{on}\quad \Gamma_{\mathrm{S}}.\end{gathered} \tag{3.6}$$

Note that the third equation in (3.6) allows us to eliminate the pressure p_{S} from the system and compute it as a simple post-process of the solution. In turn, in the porous media we consider the equations 2.2, whereas the transmission conditions are given by

$$\mathbf{u}_{\mathrm{S}}\cdot\mathbf{n} \,=\, \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \quad\text{on}\quad \Sigma \quad\text{and}\quad \boldsymbol{\sigma}_{\mathrm{S}}\mathbf{n} + \omega_1^{-1}(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t})\mathbf{t} \,=\, -p_{\mathrm{D}}\mathbf{n} \quad\text{on}\quad \Sigma, \tag{3.7}$$

where ω_1 is a positive frictional constant that can be determined experimentally.

3.3.1 The continuous formulation

In this section we introduce the weak formulation for the coupled problem given by (3.6), (2.2), and (3.7). For this case, we need to add two auxiliary unknowns: $\boldsymbol{\varphi} := -\mathbf{u}_{\mathrm{S}}|_{\Sigma} \in \mathbf{H}_{00}^{1/2}(\Sigma)$ and $\lambda := p_{\mathrm{D}}|_{\Sigma} \in \mathrm{W}^{1/3,3/2}(\Sigma)$. In turn, for uniqueness of solution we will require $p_{\mathrm{D}} \in \mathrm{L}_0^2(\Omega_{\mathrm{D}})$ (see [10, Section 3.2] for details). Then, we group the spaces, unknowns, and test functions as follows:

$$\begin{gathered}\mathbf{X}_1 := \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{S}}}^1(\Omega_{\mathrm{S}}) \times \mathbb{L}_{\mathrm{skew}}^2(\Omega_{\mathrm{S}}), \quad \mathbf{X}_2 := \mathbf{H}_{\Gamma_{\mathrm{D}}}^3(\mathrm{div}\,;\Omega_{\mathrm{D}}),\\ \mathbf{X} := \mathbf{X}_1\times\mathbf{X}_2, \quad \mathbf{Y} := \mathbf{H}_{00}^{1/2}(\Sigma)\times \mathrm{W}^{1/3,3/2}(\Sigma),\\ \mathbb{H} := \mathbf{X}\times\mathbf{Y} \quad\text{and}\quad \mathbb{Q} := \mathrm{L}_0^2(\Omega_{\mathrm{D}}),\end{gathered}$$

$$\begin{gathered}\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\gamma}_{\mathrm{S}}) \in \mathbf{X}_1, \quad \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}) \in \mathbf{X}_1\,,\\ \underline{\mathbf{t}} := (\underline{\boldsymbol{\sigma}}, \mathbf{u}_{\mathrm{D}}) \in \mathbf{X}, \quad \underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi}, \lambda) \in \mathbf{Y}, \quad p_{\mathrm{D}} \in \mathbb{Q},\\ \underline{\mathbf{r}} := (\underline{\boldsymbol{\tau}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{X}, \quad \underline{\boldsymbol{\psi}} := (\boldsymbol{\psi}, \xi) \in \mathbf{Y}, \quad q_{\mathrm{D}} \in \mathbb{Q},\end{gathered}$$

where $\mathbf{X}$, $\mathbf{Y}$, $\mathbb{H}$ and $\mathbb{H}\times\mathbb{Q}$ are respectively endowed with the norms

$$\begin{aligned}&\|\underline{\mathbf{r}}\|_{\mathbf{X}} \,:=\, \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} + \|\boldsymbol{\eta}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})},\\ &\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} \,:=\, \|\boldsymbol{\psi}\|_{1/2,00;\Sigma} + \|\xi\|_{1/3,3/2;\Sigma}, \quad \|(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}} \,:=\, \|\underline{\mathbf{r}}\|_{\mathbf{X}} + \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}},\\ &\|((\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),q_{\mathrm{D}})\|_{\mathbb{H}\times\mathbb{Q}} \,:=\, \|(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}} + \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}.\end{aligned}$$

Hence, the augmented fully-mixed variational formulation for the system (3.6), (2.2) and (3.7) reads: Find $((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),p_{\mathrm{D}}) \in \mathbb{H}\times\mathbb{Q}$ such that

$$\begin{aligned}{}[\mathbf{A}(\mathbf{u}_{\mathrm{S}})(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),p_{\mathrm{D}}] \quad &= \quad [\mathbf{F},(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] \quad \forall\,(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}) \in \mathbb{H},\\ [\mathbf{B}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),q_{\mathrm{D}}] \quad &= \quad [\mathbf{G},q_{\mathrm{D}}] \qquad \forall\,q_{\mathrm{D}} \in \mathbb{Q},\end{aligned} \tag{3.8}$$

where, given $\mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, the operator $\mathbf{A}(\mathbf{w}_S) : \mathbb{H} \to \mathbb{H}'$ is defined by

$$[\mathbf{A}(\mathbf{w}_S)(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] := [\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}}), \underline{\mathbf{r}}] + [\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\varphi}}] + [\mathbf{b}(\underline{\mathbf{t}}), \underline{\boldsymbol{\psi}}] - [\mathbf{c}(\underline{\boldsymbol{\varphi}}), \underline{\boldsymbol{\psi}}],$$

and $\mathbf{a}(\mathbf{w}_S) : \mathbf{X} \to \mathbf{X}'$ is given by

$$[\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}}), \underline{\mathbf{r}}] := [\mathcal{A}_S(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] + [\mathcal{B}_S(\mathbf{w}_S)(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] + [\mathcal{A}_D(\mathbf{u}_D), \mathbf{v}_D],$$

with

$$\begin{aligned}
[\mathcal{A}_S(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] &:= \frac{1}{2\mu}(\boldsymbol{\sigma}_S^d, \boldsymbol{\tau}_S^d)_S + \kappa_1(\mathbf{div}\boldsymbol{\sigma}_S, \mathbf{div}\boldsymbol{\tau}_S)_S + (\mathbf{u}_S, \mathbf{div}\boldsymbol{\tau}_S)_S - (\mathbf{div}\boldsymbol{\sigma}_S, \mathbf{v}_S)_S \\
&\quad + (\boldsymbol{\gamma}_S, \boldsymbol{\tau}_S)_S - (\boldsymbol{\sigma}_S, \boldsymbol{\eta}_S)_S + \kappa_2\left(\mathbf{e}(\mathbf{u}_S) - \frac{1}{2\mu}\boldsymbol{\sigma}_S^d, \mathbf{e}(\mathbf{v}_S)\right)_S \\
&\quad + \kappa_3\left(\boldsymbol{\gamma}_S - \frac{1}{2}(\nabla\mathbf{u}_S - (\nabla\mathbf{u}_S)^t), \boldsymbol{\eta}_S\right)_S, \\
[\mathcal{B}_S(\mathbf{w}_S)(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] &:= \frac{\rho}{2\mu}\left((\mathbf{w}_S \otimes \mathbf{u}_S)^d, \boldsymbol{\tau}_S - \kappa_2\mathbf{e}(\mathbf{v}_S)\right)_S, \\
[\mathcal{A}_D(\mathbf{u}_D), \mathbf{v}_D] &:= \frac{\mu}{\rho}(\mathbf{K}^{-1}\mathbf{u}_D, \mathbf{v}_D)_D + \frac{\mathtt{F}}{\rho}(|\mathbf{u}_D|\mathbf{u}_D, \mathbf{v}_D)_D,
\end{aligned}$$

whereas the operators $\mathbf{b} : \mathbf{X} \to \mathbf{Y}'$, $\mathbf{c} : \mathbf{Y} \to \mathbf{Y}'$ and $\mathbf{B} : \mathbb{H} \to \mathbb{Q}'$ are defined, respectively, by

$$\begin{aligned}
[\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}] &:= \langle \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma - \langle \mathbf{v}_D \cdot \mathbf{n}, \xi\rangle_\Sigma, \\
[\mathbf{c}(\underline{\boldsymbol{\varphi}}), \underline{\boldsymbol{\psi}}] &:= \omega_1^{-1}\langle \boldsymbol{\varphi}\cdot\mathbf{t}, \boldsymbol{\psi}\cdot\mathbf{t}\rangle_\Sigma + \langle \boldsymbol{\varphi}\cdot\mathbf{n}, \xi\rangle_\Sigma - \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda\rangle_\Sigma, \\
[\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), q_D] &:= -(q_D, \operatorname{div}\mathbf{v}_D)_D.
\end{aligned}$$

In turn, the functionals $\mathbf{F}$ and $\mathbf{G}$ are set as

$$[\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] := -\kappa_1(\mathbf{f}_S, \mathbf{div}\boldsymbol{\tau}_S)_S + (\mathbf{f}_S, \mathbf{v}_S)_S + (\mathbf{f}_D, \mathbf{v}_D)_D \quad \text{and} \quad [\mathbf{G}, q_D] := -(g_D, q_D)_D,$$

where κ_1, κ_2, and κ_3 are suitable positive parameters described in [10, Section 3.2], which will be chosen explicitly for the numerical experiments in Section 7.

We remark here that (3.8) is equivalent to the variational formulation defined in [10, Section 3.2], in which $\boldsymbol{\sigma}_S$ is decomposed as $\boldsymbol{\sigma}_S = \boldsymbol{\sigma}_S + \ell\mathbb{I}$, with $\boldsymbol{\sigma}_S \in \mathbb{H}_0(\mathbf{div}; \Omega_S)$ and $\ell \in \mathbb{R}$, where

$$\mathbb{H}_0(\mathbf{div}; \Omega_S) := \Big\{\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}; \Omega_S) : \quad (\operatorname{tr}\boldsymbol{\tau}_S, 1)_S = 0\Big\}.$$

For details of the well-posedness of (3.8) we refer the reader to [10, Section 4].

3.3.2 The finite element method

Now, considering the same notations stated in Section 3.2.2, we recall from [10, Section 5] the finite element subspaces:

$$\begin{aligned}
\mathbb{H}_h(\Omega_S) &:= \Big\{\boldsymbol{\tau}_{S,h} \in \mathbb{H}(\mathbf{div}; \Omega_S) : \quad \mathbf{c}^t\boldsymbol{\tau}_{S,h} \in \mathbf{RT}_0(T) \quad \forall\, \mathbf{c} \in \mathbb{R}^n\Big\}, \\
\mathbf{H}^1_{h,\Gamma_S}(\Omega_S) &:= \Big\{\mathbf{v}_{S,h} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S) : \quad \mathbf{v}_{S,h}|_T \in \mathbf{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^S\Big\}, \\
\mathbb{L}_h(\Omega_S) &:= \Big\{\boldsymbol{\eta}_{S,h} \in \mathbb{L}^2_{\text{skew}}(\Omega_S) : \quad \boldsymbol{\eta}_{S,h}|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^S\Big\}, \\
\mathbf{H}_{h,\Gamma_D}(\Omega_D) &:= \Big\{\mathbf{v}_{D,h} \in \mathbf{H}^3_{\Gamma_D}(\operatorname{div}; \Omega_D) : \quad \mathbf{v}_{D,h}|_T \in \mathbf{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^D\Big\}, \\
\mathrm{L}_{h,0}(\Omega_D) &:= \Big\{q_{D,h} \in \mathrm{L}^2_0(\Omega_D) : \quad q_{D,h}|_T \in \mathrm{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^D\Big\}.
\end{aligned}$$

In turn, we also consider the subspaces $\Lambda_h^{\mathrm{D}}(\Sigma) := \Lambda_h(\Sigma)$ (cf. (3.4)) and $\mathbf{\Lambda}_h^{\mathrm{S}}(\Sigma) := [\Lambda_h^{\mathrm{S}}(\Sigma)]^2$, given by

$$\Lambda_h^{\mathrm{S}}(\Sigma) := \Big\{\psi_h \in \mathcal{C}(\Sigma) : \quad \psi_h|_e \in \mathrm{P}_1(e) \quad \forall \text{ edge } e \in \Sigma_{2h}, \quad \psi_h(x_0) = \psi_h(x_N) = 0\Big\},$$

where Σ_{2h} is the partition of Σ arising by joining pairs of adjacent edges of Σ_h (if the number of edges of Σ_h is odd, we simply reduce to the even case by joining any pair of two adjacent elements and then construct Σ_{2h} from this reduced partition) and x_0 and x_N are the extreme points of Σ.

Then, defining the global subspaces, unknowns, and test functions as follows

$$\mathbf{X}_{h,1} := \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbb{L}_h(\Omega_{\mathrm{S}}), \quad \mathbf{X}_{h,2} := \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}),$$
$$\mathbf{X}_h := \mathbf{X}_{h,1} \times \mathbf{X}_{h,2}, \quad \mathbf{Y}_h := \mathbf{\Lambda}_h^{\mathrm{S}}(\Sigma) \times \Lambda_h^{\mathrm{D}}(\Sigma),$$
$$\mathbb{H}_h := \mathbf{X}_h \times \mathbf{Y}_h \quad \text{and} \quad \mathbb{Q}_h := \mathrm{L}_{h,0}(\Omega_{\mathrm{D}}),$$

$$\underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{S},h}, \boldsymbol{\gamma}_{\mathrm{S},h}) \in \mathbf{X}_{h,1}, \quad \underline{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h}) \in \mathbf{X}_{h,1}\,,$$
$$\underline{\mathbf{t}}_h := (\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\varphi}}_h := (\boldsymbol{\varphi}_h, \lambda_h) \in \mathbf{Y}_h, \quad p_{\mathrm{D},h} \in \mathbb{Q}_h,$$
$$\underline{\mathbf{r}}_h := (\underline{\boldsymbol{\tau}}_h, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h, \xi_h) \in \mathbf{Y}_h, \quad q_{\mathrm{D},h} \in \mathbb{Q}_h,$$

the Galerkin scheme associated with problem (3.8) reads: Find $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), p_{\mathrm{D},h}) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\begin{aligned}
[\mathbf{A}(\mathbf{u}_{\mathrm{S},h})(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), p_{\mathrm{D},h}] &= [\mathbf{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \quad \forall\, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{H}_h, \\
[\mathbf{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), q_{\mathrm{D},h}] &= [\mathbf{G}, q_{\mathrm{D},h}] \qquad \forall\, q_{\mathrm{D},h} \in \mathbb{Q}_h.
\end{aligned} \tag{3.9}$$

The solvability analysis and *a priori* error bounds for (3.9) are established in [10, Theorems 4.10 and 6.1], respectively.

4 Preliminaries for the a posteriori error analysis

Now we introduce a few useful notations for describing local information on elements and edges. First, given $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, we let $\mathcal{E}(T)$ be the set of edges of T, and denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, subdivided as follows:

$$\mathcal{E}_h = \mathcal{E}_h(\Gamma_{\mathrm{S}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{D}}) \cup \mathcal{E}_h(\Omega_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{D}}) \cup \mathcal{E}_h(\Sigma),$$

where $\mathcal{E}_h(\Gamma_\star) := \big\{e \in \mathcal{E}_h : e \subseteq \Gamma_\star\big\}$, $\mathcal{E}_h(\Omega_\star) := \big\{e \in \mathcal{E}_h : e \subseteq \Omega_\star\big\}$, for $\star \in \{\mathrm{S}, \mathrm{D}\}$, and the edges of $\mathcal{E}_h(\Sigma)$ are exactly those forming the previously defined partition Σ_h, that is $\mathcal{E}_h(\Sigma) := \big\{e \in \mathcal{E}_h : \ e \subseteq \Sigma\big\}$. Moreover, h_e stands for the length of a given edge e. Also for each edge $e \in \mathcal{E}_h$ we fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\mathrm{t}}$, and let $\mathbf{t}_e := (-n_2, n_1)^{\mathrm{t}}$ be the corresponding fixed unit tangential vector along e. Now, let $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$ such that $\mathbf{v}|_T \in \mathbf{C}(T)$ on each $T \in \mathcal{T}_h^\star$. Then, given $e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_\star)$, we denote by $[\![\mathbf{v} \cdot \mathbf{t}_e]\!]$ the tangential jump of $\mathbf{v}$ across e, that is, $[\![\mathbf{v} \cdot \mathbf{t}_e]\!] := \big\{(\mathbf{v}|_T)|_e - (\mathbf{v}|_{T'})|_e\big\} \cdot \mathbf{t}_e$, where T and T' are the triangles of $\mathcal{T}_h^\star$ having e as a common edge. In addition, for $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$ such that $\boldsymbol{\tau}|_T \in \mathbb{C}(T)$, we let $[\![\boldsymbol{\tau}\, \mathbf{n}_e]\!]$ be the normal jump of $\boldsymbol{\tau}$ across e, that is, $[\![\boldsymbol{\tau}\, \mathbf{n}_e]\!] := \big\{(\boldsymbol{\tau}|_T)|_e - (\boldsymbol{\tau}|_{T'})|_e\big\}\mathbf{n}_e$ and we let $[\![\boldsymbol{\tau}\, \mathbf{t}_e]\!]$ be the tangential jump of $\boldsymbol{\tau}$ across e, that is, $[\![\boldsymbol{\tau}\, \mathbf{t}_e]\!] := \big\{(\boldsymbol{\tau}|_T)|_e - (\boldsymbol{\tau}|_{T'})|_e\big\}\mathbf{t}_e$. From now on, when no confusion arises, we simply write $\mathbf{n}$ and $\mathbf{t}$ instead of $\mathbf{n}_e$ and $\mathbf{t}_e$, respectively. Finally, given scalar and vector valued fields ϕ, $\mathbf{v} = (v_1, v_2)^{\mathrm{t}}$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we set

$$\mathbf{curl}\,(\phi) := \left(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}\right)^{\mathrm{t}}, \quad \underline{\mathbf{curl}}(\mathbf{v}) := \begin{pmatrix} \mathbf{curl}\,(v_1)^{\mathrm{t}} \\ \mathbf{curl}\,(v_2)^{\mathrm{t}} \end{pmatrix},$$

$$\mathrm{rot}\,(\mathbf{v}) := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \text{and} \quad \mathbf{rot}\,(\boldsymbol{\tau}) := \begin{pmatrix} \mathrm{rot}\,(\tau_{11}, \tau_{12}) \\ \mathrm{rot}\,(\tau_{21}, \tau_{22}) \end{pmatrix},$$

where the derivatives involved are taken in the distributional sense.

Let us now recall the main properties of the Raviart–Thomas interpolator (see [18, 22]) and the Clément operator onto the space of continuous piecewise linear functions [12, 27]. We begin with the former, denoted $\Pi_h^\star : \mathbf{H}^1(\Omega_\star) \to \mathbf{H}_h(\Omega_\star)$, $\star \in \{\mathrm{S}, \mathrm{D}\}$, which is characterized by the identity

$$\int_e \Pi_h^\star(\mathbf{v}) \cdot \mathbf{n} = \int_e \mathbf{v} \cdot \mathbf{n} \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h^\star. \tag{4.1}$$

As consequence of (4.1), there holds: $\operatorname{div}(\Pi_h^\star(\mathbf{v})) = \mathcal{P}_h^\star(\operatorname{div}\mathbf{v})$, where $\mathcal{P}_h^\star$ is the $\mathrm{L}^2(\Omega_\star)$-orthogonal projector onto the piecewise constant functions on $\Omega_\star$. A tensor version of $\Pi_h^\star$, say $\mathbf{\Pi}_h^\star : \mathbb{H}^1(\Omega_\star) \to \mathbb{H}_h(\Omega_\star)$, which is defined row-wise by $\Pi_h^\star$, and a vector version of $\mathcal{P}_h^\star$, say $\mathbf{P}_h^\star$, which is the $\mathbf{L}^2(\Omega_\star)$-orthogonal projector onto piecewise constant vectors on $\Omega_\star$, might also be required. The local approximation properties of $\Pi_h^\star$ (and hence of $\mathbf{\Pi}_h^\star$) are established in the following lemma. For the corresponding proof we refer the reader to [18] (see also [4]).

Lemma 4.1 *For each $\star \in \{\mathrm{S}, \mathrm{D}\}$ there exist constants $c_1, c_2 > 0$, independent of h, such that for all $\mathbf{v} \in \mathbf{H}^1(\Omega_\star)$ there holds*

$$\|\mathbf{v} - \Pi_h^\star \mathbf{v}\|_{0,T} \,\le\, c_1\, h_T\, \|\mathbf{v}\|_{1,T} \quad \forall\, T \in \mathcal{T}_h^\star$$

and

$$\|\mathbf{v}\cdot\mathbf{n} - \Pi_h^\star \mathbf{v}\cdot\mathbf{n}\|_{0,e} \,\le\, c_2\, h_e^{1/2}\, \|\mathbf{v}\|_{1,T_e} \quad \forall\, e \in \mathcal{E}_h,$$

where T_e is a triangle of $\mathcal{T}_h^\star$ containing the edge e on its boundary.

In turn, given $\mathrm{p} \in (1, +\infty)$, we make use of the Clément interpolation operator $I_h^\star : \mathrm{W}^{1,\mathrm{p}}(\Omega_\star) \to \mathrm{X}_h(\Omega_\star)$, with $\star \in \{\mathrm{S}, \mathrm{D}\}$, where

$$\mathrm{X}_h(\Omega_\star) \,:=\, \Big\{ v \in \mathcal{C}(\overline{\Omega}_\star) : \quad v|_T \in \mathrm{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^\star \Big\}.$$

The local approximation properties of this operator are established in the following lemma (see [27, Lemma 3.1] for details):

Lemma 4.2 *For each $\star \in \{\mathrm{S}, \mathrm{D}\}$ there exists constants $c_3, c_4 > 0$, independent of $h_\star$, such that for all $v \in \mathrm{W}^{1,p}(\Omega_\star)$ there hold*

$$\|v - I_h^\star v\|_{\mathrm{L}^\mathrm{p}(T)} \,\le\, c_3\, h_T\, \|v\|_{1,\mathrm{p};\Delta_\star(T)} \quad \forall\, T \in \mathcal{T}_h^\star,$$

and

$$\|v - I_h^\star v\|_{\mathrm{L}^\mathrm{p}(e)} \,\le\, c_4\, h_e^{1-1/\mathrm{p}}\, \|v\|_{1,\mathrm{p};\Delta_\star(e)} \quad \forall\, e \in \mathcal{E}_h,$$

where

$$\Delta_\star(T) \,:=\, \cup\Big\{ T' \in \mathcal{T}_h^\star :\; T' \cap T \neq \emptyset \Big\} \quad \textit{and} \quad \Delta_\star(e) \,:=\, \cup\Big\{ T' \in \mathcal{T}_h^\star :\; T' \cap e \neq \emptyset \Big\}.$$

In particular, for $\mathrm{p} = 2$ a vector version of I_h^{S}, say $\mathbf{I}_h^{\mathrm{S}} : \mathbf{H}^1(\Omega_\mathrm{S}) \to \mathbf{X}_h(\Omega_\mathrm{S})$, which is defined componentwise by I_h^{S}, will be needed as well.

For the forthcoming analysis we will also utilize a stable Helmholtz decomposition for $\mathbf{H}^3_{\Gamma_\mathrm{D}}(\operatorname{div}; \Omega_\mathrm{D})$. In this regard, and in order to analyze a more general result, given $\mathrm{p} \in [2, +\infty)$ we will consider the Banach space $\mathbf{H}^\mathrm{p}_{\Gamma_\mathrm{D}}(\operatorname{div}; \Omega_\mathrm{D})$ introduced in Section 3.1, and analogously to [2] we remark in advance that the decomposition for $\mathbf{H}^\mathrm{p}_{\Gamma_\mathrm{D}}(\operatorname{div}; \Omega_\mathrm{D})$ will require the boundary Γ_D to lie in a "*convex part*" of

Ω_D, which means that there exists a convex domain containing Ω_D, and whose boundary contains Γ_D. We begin by introducing the following subspaces of $W^{1,p}(\Omega_D)$,

$$W^{1,p}_{\Gamma_D}(\Omega_D) := \Big\{ \eta_D \in W^{1,p}(\Omega_D) : \quad \eta_D = 0 \quad \text{on} \quad \Gamma_D \Big\},$$

and establishing a suitable Helmholtz decomposition of our space $\mathbf{H}^p_{\Gamma_D}(\mathrm{div}\,;\Omega_D)$.

Lemma 4.3 *Assume that Ω_D is a connected domain and that Γ_D is contained in the boundary of a convex part of Ω_D, that is there exists a convex domain Ξ such that $\overline{\Omega}_D \subseteq \Xi$ and $\Gamma_D \subseteq \partial\Xi$. Then, for each $\mathbf{v}_D \in \mathbf{H}^p_{\Gamma_D}(\mathrm{div}\,;\Omega_D)$ with $p \in [2,+\infty)$, there exist $\mathbf{w}_D \in \mathbf{H}^1(\Omega_D)$ and $\beta_D \in W^{1,p}_{\Gamma_D}(\Omega_D)$ such that*

$$\mathbf{v}_D \;=\; \mathbf{w}_D + \mathbf{curl}\,\beta_D \quad \textit{in} \quad \Omega_D, \tag{4.2}$$

and

$$\|\mathbf{w}_D\|_{1,\Omega_D} + \|\beta_D\|_{1,p;\Omega_D} \;\leq\; C_{\mathrm{hel}}\,\|\mathbf{v}_D\|_{\mathbf{H}^p(\mathrm{div}\,;\Omega_D)}, \tag{4.3}$$

where C_{hel} is a positive constant independent of all the foregoing variables.

Proof. Since $\mathrm{div}\,\mathbf{v}_D \in L^2(\Omega_D)$ for each $\mathbf{v}_D \in \mathbf{H}^p_{\Gamma_D}(\mathrm{div}\,;\Omega_D)$, the first part of the proof proceeds similarly as the proof of [2, Lemma 3.9]. In fact, given $\mathbf{v}_D \in \mathbf{H}^p_{\Gamma_D}(\mathrm{div}\,;\Omega_D)$, we let $z \in H^2(\Xi)$ be the unique weak solution of the boundary value problem:

$$\Delta z = \left\{ \begin{array}{cl} \mathrm{div}\,\mathbf{v}_D & \text{in} \quad \Omega_D \\ -\dfrac{1}{|\Xi \setminus \overline{\Omega}_D|}\displaystyle\int_{\Omega_D} \mathrm{div}\,\mathbf{v}_D & \text{in} \quad \Xi \setminus \overline{\Omega}_D \end{array} \right. , \quad \nabla z \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Xi, \quad \int_{\Xi} z = 0. \tag{4.4}$$

Thanks to the elliptic regularity result of (4.4) we have that $z \in H^2(\Xi)$ and

$$\|z\|_{2,\Xi} \;\leq\; c\,\|\mathrm{div}\,\mathbf{v}_D\|_{0,\Omega_D},$$

where $c > 0$ is independent of z. In addition, it is clear that $\mathbf{w}_D := (\nabla z)|_{\Omega_D} \in \mathbf{H}^1(\Omega_D)$, $\mathrm{div}\,\mathbf{w}_D = \Delta z = \mathrm{div}\,\mathbf{v}_D$ in Ω_D, $\mathbf{w}_D \cdot \mathbf{n} = 0$ on $\partial\Xi$ (which implies $\mathbf{w}_D \cdot \mathbf{n} = 0$ on Γ_D), and

$$\|\mathbf{w}_D\|_{1,\Omega_D} \;\leq\; \|z\|_{2,\Omega_D} \leq \|z\|_{2,\Xi} \;\leq\; c\,\|\mathrm{div}\,\mathbf{v}_D\|_{0,\Omega_D}. \tag{4.5}$$

On the other hand, let us set $\boldsymbol{\phi}_D := \mathbf{v}_D - \mathbf{w}_D$ and notice that $\boldsymbol{\phi}_D$ is a divergence-free vector field in Ω_D. Then, using the continuous injection from $H^1(\Omega_D)$ into $L^p(\Omega_D)$ with $p \in [2,+\infty)$, and the estimate (4.5), we deduce that $\boldsymbol{\phi}_D \in \mathbf{L}^p(\Omega_D)$ and

$$\|\boldsymbol{\phi}_D\|_{\mathbf{L}^p(\Omega_D)} \;\leq \widehat{c}\,\Big\{ \|\mathbf{v}_D\|_{\mathbf{L}^p(\Omega_D)} + \|\mathbf{w}_D\|_{1,\Omega_D} \Big\} \leq \widetilde{c}\,\|\mathbf{v}_D\|_{\mathbf{H}^p(\mathrm{div}\,;\Omega_D)}. \tag{4.6}$$

In this way, as a consequence of [22, Theorem I.3.1], given $\boldsymbol{\phi}_D \in \mathbf{L}^p(\Omega_D)$ with $p \in [2,+\infty)$ satisfing $\mathrm{div}\,\boldsymbol{\phi}_D = 0$ in Ω_D, and Ω_D connected, there exists $\beta_D \in W^{1,p}(\Omega_D)$ such that $\boldsymbol{\phi}_D = \mathbf{curl}\,\beta_D$ in Ω_D, or equivalently

$$\mathbf{v}_D - \mathbf{w}_D = \mathbf{curl}\,\beta_D \quad \text{in} \quad \Omega_D. \tag{4.7}$$

In turn, noting that $0 = (\mathbf{v}_D - \mathbf{w}_D)\cdot\mathbf{n} = \mathbf{curl}\,\beta_D \cdot \mathbf{n} = \frac{d\beta_D}{d\mathbf{t}}$ on Γ_D, we deduce that β_D is constant on Γ_D, and therefore β_D can be chosen so that $\beta_D \in W^{1,p}_{\Gamma_D}(\Omega_D)$, which, together with (4.7), complete the Helmholtz decomposition (4.2). Finally, as a consequence of the generalized Poincaré inequality, it is easy to see that the norms $\|\beta_D\|_{1,p;\Omega_D}$ and $|\beta_D|_{1,p;\Omega_D} = \|\mathbf{curl}\,\beta_D\|_{\mathbf{L}^p(\Omega_D)}$ are equivalent (see [22, Lemma I.3.1] for details), and employing (4.6), we obtain

$$\|\beta_D\|_{1,p;\Omega_D} \leq c\,\|\mathbf{curl}\,\beta_D\|_{\mathbf{L}^p(\Omega_D)} = c\,\|\boldsymbol{\phi}_D\|_{\mathbf{L}^p(\Omega_D)} \leq C\,\|\mathbf{v}_D\|_{\mathbf{H}^p(\mathrm{div}\,;\Omega_D)}. \tag{4.8}$$

Then, it is clear that (4.5) and (4.8) imply (4.3) and conclude the proof. □

Now, we establish an integration by parts formula which is an extension of [14, Lemma 3.5].

Lemma 4.4 *Let* p *and* q *two fixed real numbers such that* $1/\mathrm{p}+1/\mathrm{q}=1$*. Let* Ω *be a bounded domain with Lipschitz-continuous boundary* $\partial\Omega$*. Then there holds*

$$\langle \mathbf{curl}\,\chi\cdot\mathbf{n},\varphi\rangle_{\partial\Omega} = -\left\langle \frac{d\varphi}{d\mathbf{t}},\chi\right\rangle_{\partial\Omega} \qquad \forall\,\chi\in \mathrm{W}^{1,\mathrm{p}}(\Omega),\ \forall\,\varphi\in \mathrm{W}^{1,\mathrm{q}}(\Omega). \tag{4.9}$$

Proof. We first recall from [15, Corollaries B.57 and B.58] (see also [22, eq. (2.17) and Theorem 2.11]) that the Green formulae in $\mathbf{H}^{\mathrm{p}}(\mathrm{div}_{\,\mathrm{p}};\Omega)$ and $\mathbf{H}^{\mathrm{q}}(\mathrm{rot}_{\,\mathrm{q}};\Omega)$ establish, respectively, that

$$\int_\Omega \phi\,\mathrm{div}\,\mathbf{v} + \int_\Omega \mathbf{v}\cdot\nabla\phi = \langle\mathbf{v}\cdot\mathbf{n},\phi\rangle_{\partial\Omega} \quad \forall\,\mathbf{v}\in\mathbf{H}^{\mathrm{p}}(\mathrm{div}_{\,\mathrm{p}};\Omega),\ \forall\,\phi\in\mathrm{W}^{1,\mathrm{q}}(\Omega), \tag{4.10}$$

$$\int_\Omega \phi\,\mathrm{rot}\,\mathbf{v} - \int_\Omega \mathbf{v}\cdot\mathbf{curl}\,\phi = \langle\mathbf{v}\cdot\mathbf{t},\phi\rangle_{\partial\Omega} \quad \forall\,\mathbf{v}\in\mathbf{H}^{\mathrm{q}}(\mathrm{rot}_{\,\mathrm{q}};\Omega),\ \forall\,\phi\in\mathrm{W}^{1,\mathrm{p}}(\Omega), \tag{4.11}$$

where

$$\mathbf{H}^{\mathrm{p}}(\mathrm{div}_{\,\mathrm{p}};\Omega) := \Big\{\mathbf{v}\in\mathbf{L}^{\mathrm{p}}:\quad \mathrm{div}\,\mathbf{v}\in\mathrm{L}^{\mathrm{p}}(\Omega)\Big\},\quad \mathbf{H}^{\mathrm{q}}(\mathrm{rot}_{\,\mathrm{q}};\Omega) := \Big\{\mathbf{v}\in\mathbf{L}^{\mathrm{q}}:\quad \mathrm{rot}\,\mathbf{v}\in\mathrm{L}^{\mathrm{q}}(\Omega)\Big\}.$$

Then, given now $\chi\in\mathrm{W}^{1,\mathrm{p}}(\Omega)$ and $\varphi\in\mathrm{W}^{1,\mathrm{q}}(\Omega)$, we first apply (4.10) with $\mathbf{v}:=\mathbf{curl}\,\chi\in\mathbf{H}^{\mathrm{p}}(\mathrm{div}_{\,\mathrm{p}};\Omega)$ and $\phi:=\varphi\in\mathrm{W}^{1,\mathrm{q}}(\Omega)$, and then employ (4.11) with $\mathbf{v}:=\nabla\varphi\in\mathbf{H}^{\mathrm{q}}(\mathrm{rot}_{\,\mathrm{q}};\Omega)$ and $\phi:=\chi\in\mathrm{W}^{1,\mathrm{p}}(\Omega)$, to obtain

$$\langle\mathbf{curl}\,\chi\cdot\mathbf{n},\varphi\rangle_{\partial\Omega} = \int_\Omega \mathbf{curl}\,\chi\cdot\nabla\varphi = \int_\Omega \chi\,\mathrm{rot}\,(\nabla\varphi) - \langle\nabla\varphi\cdot\mathbf{t},\chi\rangle_{\partial\Omega} = -\left\langle\frac{d\varphi}{d\mathbf{t}},\chi\right\rangle_{\partial\Omega},$$

which shows (4.9) and completes the proof. □

Finally, we end this section with a lemma providing estimates in terms of local quantities for the $\mathrm{W}^{-1/\mathrm{q},\mathrm{q}}(\Sigma)$ norms of functions in particular subspaces of $\mathrm{L}^{\mathrm{q}}(\Sigma)$, with $1<\mathrm{p}<2$ and $1/\mathrm{p}+1/\mathrm{q}=1$. More precisely, having in mind the definition of $\Lambda_h(\Sigma)$ (cf. (3.4)), which is subspace of $\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Sigma)$, we introduce the orthogonal-type space

$$\Lambda_h^\perp(\Sigma) := \Big\{\lambda\in\mathrm{W}^{-1/\mathrm{q},\mathrm{q}}(\Sigma)\cap\mathrm{L}^{\mathrm{q}}(\Sigma):\quad \langle\lambda,\xi_h\rangle_\Sigma = 0\quad\forall\,\xi_h\in\Lambda_h(\Sigma)\Big\}. \tag{4.12}$$

Then, the announced lemma is stated as follows.

Lemma 4.5 *Let* p *and* q *two fixed real numbers with* $1<\mathrm{p}<2$ *and* $1/\mathrm{p}+1/\mathrm{q}=1$*. Then, there exists* $C>0$*, independent of the meshsizes, such that*

$$\|\lambda\|_{-1/\mathrm{q},\mathrm{q};\Sigma} \le C\left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\lambda\|^{\mathrm{q}}_{\mathrm{L}^{\mathrm{q}}(e)}\right\}^{1/\mathrm{q}} \qquad \forall\,\lambda\in\Lambda_h^\perp(\Sigma). \tag{4.13}$$

Proof. Given $\lambda\in\Lambda_h^\perp(\Sigma)$, we first observe that $\lambda\in\mathrm{W}^{-1/\mathrm{q},\mathrm{q}}(\Sigma)$ and that

$$\|\lambda\|_{-1/\mathrm{q},\mathrm{q};\Sigma} = \sup_{\substack{\xi\in\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Sigma)\\ \xi\neq 0}} \frac{\langle\lambda,\xi\rangle_\Sigma}{\|\xi\|_{1/\mathrm{q},\mathrm{p};\Sigma}}. \tag{4.14}$$

Then, since $\mathcal{P}_\Sigma(\xi)\in\Lambda_h(\Sigma)$ $\forall\,\xi\in\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Sigma)$, with $\mathcal{P}_\Sigma$ being the $\mathrm{L}^2(\Sigma)$-orthogonal projection onto $\Lambda_h(\Sigma)$, it follows from (4.12), (4.14), and Hölder's inequality, that

$$\|\lambda\|_{-1/\mathrm{q},\mathrm{q};\Sigma} = \sup_{\substack{\xi\in\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Sigma)\\ \xi\neq 0}} \frac{\langle\lambda,\xi-\mathcal{P}_\Sigma(\xi)\rangle_\Sigma}{\|\xi\|_{1/\mathrm{q},\mathrm{p};\Sigma}} \le \sup_{\substack{\xi\in\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Sigma)\\ \xi\neq 0}} \frac{\displaystyle\sum_{e\in\mathcal{E}_h(\Sigma)}\|\lambda\|_{\mathrm{L}^{\mathrm{q}}(e)}\|\xi-\mathcal{P}_e(\xi)\|_{\mathrm{L}^{\mathrm{p}}(e)}}{\|\xi\|_{1/\mathrm{q},\mathrm{p};\Sigma}}, \tag{4.15}$$

where $\mathcal{P}_e(\xi) := \mathcal{P}_\Sigma(\xi)|_e$ on each $e \in \mathcal{E}_h(\Sigma)$. In turn, from the local approximation estimates of $\mathcal{P}_e$, we have

$$\|\xi - \mathcal{P}_e(\xi)\|_{\mathrm{L}^{\mathrm{p}}(e)} \leq c h_e^0 \|\xi\|_{\mathrm{L}^{\mathrm{p}}(e)} \quad \forall\, \xi \in \mathrm{L}^{\mathrm{p}}(e) \quad \text{and} \quad \|\xi - \mathcal{P}_e(\xi)\|_{\mathrm{L}^{\mathrm{p}}(e)} \leq c h_e \|\xi\|_{1,\mathrm{p};e} \quad \forall\, \xi \in \mathrm{W}^{1,\mathrm{p}}(e),$$

and then, by interpolation arguments, we find that

$$\|\xi - \mathcal{P}_e(\xi)\|_{\mathrm{L}^{\mathrm{p}}(e)} \leq c h_e^{1/\mathrm{q}} \|\xi\|_{1/\mathrm{q},\mathrm{p};e} \quad \forall\, \xi \in \mathrm{W}^{1/\mathrm{q},\mathrm{p}}(e), \tag{4.16}$$

with $1/\mathrm{p} + 1/\mathrm{q} = 1$. Thus, the estimate (4.16) combined with (4.15), yields

$$\begin{aligned} \sum_{e\in\mathcal{E}_h(\Sigma)} \|\lambda\|_{\mathrm{L}^{\mathrm{q}}(e)} \|\xi - \mathcal{P}_e(\xi)\|_{\mathrm{L}^{\mathrm{p}}(e)} \,&\leq\, c \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/\mathrm{q}} \|\lambda\|_{\mathrm{L}^{\mathrm{q}}(e)} \|\xi\|_{1/\mathrm{q},\mathrm{p};e} \\ \leq\, C \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\lambda\|^{\mathrm{q}}_{\mathrm{L}^{\mathrm{q}}(e)} \right\}^{1/\mathrm{q}} \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \|\xi\|^{\mathrm{p}}_{1/\mathrm{q},\mathrm{p};e} \right\}^{1/\mathrm{p}} &\leq\, C \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\lambda\|^{\mathrm{q}}_{\mathrm{L}^{\mathrm{q}}(e)} \right\}^{1/\mathrm{q}} \|\xi\|_{1/\mathrm{q},\mathrm{p};\Sigma}. \end{aligned}$$

Notice that in the last inequality we have used the fact that the space $\prod_{e\in\mathcal{E}_h(\Sigma)} \mathrm{W}^{1/\mathrm{q},\mathrm{p}}(e)$ coincides with $\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Sigma)$, without extra conditions when $1 < \mathrm{p} < 2$ [23, Theorem 1.5.2.3-(a)], to obtain the norm $\|\xi\|_{1/\mathrm{q},\mathrm{p};\Sigma}$, which combined with (4.15) imply (4.13) and conclude the proof. □

5 A posteriori error analysis: The primal-mixed approach

In what follows we assume that the hypotheses of Theorems 3.12 and 4.9 in [7] hold. Let $\vec{\mathbf{u}} := (\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ and $\vec{\mathbf{u}}_h := (\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ be the unique solutions of problems (3.3) and (3.5), respectively. In addition, let us denote $p_{\mathrm{S},h} := p_h|_{\Omega_\mathrm{S}}$ and $p_{\mathrm{D},h} := p_h|_{\Omega_\mathrm{D}}$. Then, we define for each $T \in \mathcal{T}_h^\mathrm{S}$ the local error indicator

$$\begin{aligned} \Theta^2_{\mathrm{S},T} \,:=\, & \|\mathrm{div}\,\mathbf{u}_{\mathrm{S},h}\|^2_{0,T} + h_T^2 \left\| \mathbf{f}_\mathrm{S} + \mathbf{div}\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h} - \rho(\nabla \mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h} - \frac{\rho}{2}\mathrm{div}\,\mathbf{u}_{\mathrm{S},h}\mathbf{u}_{\mathrm{S},h} \right\|^2_{0,T} \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_\mathrm{S})} h_e \big\| \,[\![\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}\mathbf{n}]\!]\, \big\|^2_{0,e} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \left\| \widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}\mathbf{n} + \lambda_h \mathbf{n} + \frac{\alpha_d \mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}(\mathbf{u}_{\mathrm{S},h}\cdot\mathbf{t})\mathbf{t} \right\|^2_{0,e}, \end{aligned} \tag{5.1}$$

where

$$\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h} \,:=\, -p_{\mathrm{S},h}\mathbb{I} + 2\mu\mathbf{e}(\mathbf{u}_{\mathrm{S},h}) \quad \text{on each} \quad T \in \mathcal{T}_h^\mathrm{S}. \tag{5.2}$$

Similarly, for each $T \in \mathcal{T}_h^\mathrm{D}$ we set

$$\widehat{\Theta}^2_{\mathrm{D},T} \,:=\, \|g_\mathrm{D} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|^2_{0,T} + h_T^2\, \|\mathbf{f}_\mathrm{D} - \mathbf{U}_{\mathrm{D},h}\|^2_{0,T} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \|\lambda_h - p_{\mathrm{D},h}\|^2_{0,e} \tag{5.3}$$

and

$$\begin{aligned} \widetilde{\Theta}^{3/2}_{\mathrm{D},T} \,:=\, & h_T^{3/2} \big\|\mathrm{rot}\,\left(\mathbf{f}_\mathrm{D} - \mathbf{U}_{\mathrm{D},h}\right)\big\|^{3/2}_{\mathrm{L}^{3/2}(T)} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_\mathrm{D})} h_e \big\| \,[\![(\mathbf{f}_\mathrm{D} - \mathbf{U}_{\mathrm{D},h})\cdot\mathbf{t}]\!]\, \big\|^{3/2}_{\mathrm{L}^{3/2}(e)} \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \big\| \left(\mathbf{f}_\mathrm{D} - \mathbf{U}_{\mathrm{D},h}\right)\cdot\mathbf{t} \big\|^{3/2}_{\mathrm{L}^{3/2}(e)}, \end{aligned} \tag{5.4}$$

where

$$\mathbf{U}_{\mathrm{D},h} \,:=\, \frac{\mu}{\rho}\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h} + \frac{\mathtt{F}}{\rho}|\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \quad \text{on each} \quad T \in \mathcal{T}_h^\mathrm{D}. \tag{5.5}$$

Finally, for each $e \in \mathcal{E}_h(\Sigma)$ we define

$$\Theta_{\Sigma,e}^3 := h_e \left\| \mathbf{u}_{\mathrm{S},h} \cdot \mathbf{n} - \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} \right\|_{\mathrm{L}^3(e)}^3, \tag{5.6}$$

so that the global a posteriori error estimator is given by:

$$\Theta := \left\{ \sum_{T \in \mathcal{T}_h^{\mathrm{S}}} \Theta_{\mathrm{S},T}^2 + \sum_{T \in \mathcal{T}_h^{\mathrm{D}}} \widehat{\Theta}_{\mathrm{D},T}^2 \right\}^{1/2} + \left\{ \sum_{T \in \mathcal{T}_h^{\mathrm{D}}} \widetilde{\Theta}_{\mathrm{D},T}^{3/2} \right\}^{2/3} + \left\{ \sum_{e \in \mathcal{E}_h(\Sigma)} \Theta_{\Sigma,e}^3 \right\}^{1/3}. \tag{5.7}$$

Notice that the second term defining $\widehat{\Theta}_{\mathrm{D},T}^2$ requires that $\mathbf{f}_\mathrm{D} \in \mathbf{L}^2(T)$ for each $T \in \mathcal{T}_h^{\mathrm{D}}$. This is ensured below by assuming that $\mathbf{f}_\mathrm{D}$ lives now in $\mathbf{L}^2(\Omega_\mathrm{D})$ in place of $\mathbf{L}^{3/2}(\Omega_\mathrm{D})$.

The main goal of the present section is to establish, under suitable assumptions, the existence of positive constants C_{rel} and C_{eff}, independent of the meshsizes and the continuous and discrete solutions, such that

$$C_{\mathrm{eff}}\, \Theta + \text{h.o.t.} \le \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{H} \times \mathbf{Q}} \le C_{\mathrm{rel}}\, \Theta^{1/2}, \tag{5.8}$$

where h.o.t. stands, eventually, for one or several terms of higher order. The upper and lower bounds in (5.8), which are known as the reliability of $\Theta^{1/2}$ and efficiency of Θ, are derived below in Sections 5.1 and 5.2, respectively.

5.1 Reliability

First, we recall from [7] the following notation

$$\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) := \max \left\{ \mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})^{1/2}, \mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}), \mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})^2 \right\},$$

where $\mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) := \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{\mathbf{L}^{3/2}(\Omega_\mathrm{D})} + \|g_\mathrm{D}\|_{0,\Omega_\mathrm{D}} + \|g_\mathrm{D}\|_{0,\Omega_\mathrm{D}}^2$. Then, we establish the main result of this section.

Theorem 5.1 *Assume that Ω_D is a connected domain and that Γ_D is contained in the boundary of a convex part of Ω_D, that is there exists a convex domain Ξ such that $\overline{\Omega}_\mathrm{D} \subseteq \Xi$ and $\Gamma_\mathrm{D} \subseteq \partial\Xi$. In addition, assume that the data $\mathbf{f}_\mathrm{S}$, $\mathbf{f}_\mathrm{D}$ and g_D, satisfy:*

$$\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) \le \frac{1}{2} \min \left\{ r, \widetilde{r} \right\}, \tag{5.9}$$

where r and $\widetilde{r}$ are the positive constants, independent of the data, provided by [7, Lemma 3.11 and Theorem 4.9], respectively. Assume further that $\mathbf{f}_\mathrm{D} \in \mathbf{L}^2(\Omega_\mathrm{D})$. Then, there exists a constant $C_{\mathrm{rel}} > 0$, independent of h, such that

$$\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{H} \times \mathbf{Q}} \le C_{\mathrm{rel}}\, \Theta^{1/2}. \tag{5.10}$$

We begin the proof of (5.10) with a preliminary estimate for the total error $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{H} \times \mathbf{Q}}$. In fact, proceeding analogously to [13, Section 1] (see also [16, 17]), we first define two residues $\mathcal{R}_\mathbf{f}$ and $\mathcal{R}_\mathbf{g}$ on $\mathbf{H}$ and $\mathbf{Q}$, respectively, by

$$\mathcal{R}_\mathbf{f}(\mathbf{v}) := [\mathbf{f}, \mathbf{v}] - \left\{ [\mathbf{a}_h(\mathbf{u}_{\mathrm{S},h})(\mathbf{u}_h), \mathbf{v}] + [\mathbf{b}(\mathbf{v}), (p_h, \lambda_h)] \right\} \quad \forall\, \mathbf{v} := (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{H}, \tag{5.11}$$

and

$$\mathcal{R}_\mathbf{g}(q, \xi) := [\mathbf{g}, (q, \xi)] - [\mathbf{b}(\mathbf{u}_h), (q, \xi)] \quad \forall\, (q, \xi) \in \mathbf{Q}. \tag{5.12}$$

Then we are able to establish the following preliminary a posteriori error estimate.

Lemma 5.2 *Assume that the data* $\mathbf{f}_{\mathrm{S}}$, $\mathbf{f}_{\mathrm{D}}$ *and* g_{D}, *satisfy* (5.9). *Then, there exists a constant* $C > 0$, *depending only on parameters and other constants, all them independent of* h, *such that*

$$\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{H}\times\mathbf{Q}} \,\leq\, C\,\max\Big\{\|\mathcal{R}\|^{1/2}_{(\mathbf{H}\times\mathbf{Q})'},\, \|\mathcal{R}\|^{2/3}_{(\mathbf{H}\times\mathbf{Q})'},\, \|\mathcal{R}\|^{3/4}_{(\mathbf{H}\times\mathbf{Q})'},\, \|\mathcal{R}\|_{(\mathbf{H}\times\mathbf{Q})'},\, \|\mathcal{R}\|^{3/2}_{(\mathbf{H}\times\mathbf{Q})'}\Big\}, \tag{5.13}$$

where $\mathcal{R} : \mathbf{H}\times\mathbf{Q} \to \mathbb{R}$ *is the residual functional given by* $\mathcal{R}(\vec{\mathbf{v}}) := \mathcal{R}_{\mathbf{f}}(\mathbf{v}) + \mathcal{R}_{\mathbf{g}}(q,\xi) \quad \forall\,\vec{\mathbf{v}} := (\mathbf{v},(q,\xi)) \in \mathbf{H}\times\mathbf{Q}$ *(cf.* (5.11) *and* (5.12)*), which satisfies*

$$\mathcal{R}(\vec{\mathbf{v}}_h) \,=\, 0 \quad \forall\,\vec{\mathbf{v}}_h := (\mathbf{v}_h,(q_h,\xi_h)) \in \mathbf{H}_h\times\mathbf{Q}_h. \tag{5.14}$$

Proof. First, from the assumption (5.9) and the a priori estimates [7, Theorems 3.12 and 4.9], we obtain

$$\begin{aligned} \|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})},\ \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \,&\leq\, c_{\mathbf{T}}\,\mathcal{M}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}}),\\ \|\mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})},\ \|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \,&\leq\, \widetilde{c}_{\mathbf{T}}\,\mathcal{M}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}}). \end{aligned} \tag{5.15}$$

In addition, since the exact solution $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ satisfies $\mathrm{div}\,\mathbf{u}_{\mathrm{S}} = 0$ in Ω_{S}, we have

$$[\mathcal{B}^h_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S},h}] = [\mathcal{B}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S},h}] \quad \forall\,\mathbf{v}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}).$$

Consequently, from the continuous problem (3.3), and the definition of the residual functionals $\mathcal{R}_{\mathbf{f}}$ and $\mathcal{R}_{\mathbf{g}}$ (cf. (5.11) and (5.12)), it is clear that

$$[\mathbf{a}_h(\mathbf{u}_{\mathrm{S},h})(\mathbf{u}) - \mathbf{a}_h(\mathbf{u}_{\mathrm{S},h})(\mathbf{u}_h),\mathbf{v}] + [\mathbf{b}(\mathbf{v}),(p-p_h,\lambda-\lambda_h)] = \mathcal{R}_{\mathbf{f}}(\mathbf{v}) - [\mathcal{B}^h_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}], \tag{5.16}$$

and

$$[\mathbf{b}(\mathbf{u}-\mathbf{u}_h),(q,\xi)] = \mathcal{R}_{\mathbf{g}}(q,\xi), \tag{5.17}$$

for all $\mathbf{v} \in \mathbf{H}$ and $(q,\xi) \in \mathbf{Q}$. Thus, from the inf-sup condition of $\mathbf{b}$ (cf. [7, Lemma 3.5]), the identity (5.16), and the continuity of $\mathbf{a}_h$ and $\mathcal{B}^h_{\mathrm{S}}$ (cf. [7, Lemma 4.3 and eq. (4.4)]), we deduce that

$$\begin{aligned} \beta\,\|(p-p_h,\lambda-\lambda_h)\|_{\mathbf{Q}} \,&\leq\, \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{H}} \frac{[\mathbf{b}(\mathbf{v}),(p-p_h,\lambda-\lambda_h)]}{\|\mathbf{v}\|_{\mathbf{H}}}\\ &\leq\, \|\mathcal{R}_{\mathbf{f}}\|_{\mathbf{H}'} + C_1\Big(1 + \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}\Big)\|\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}\\ &\quad + C_2\Big(1 + \|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} + \|\mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}\Big)\|\mathbf{u}_{\mathrm{D}}-\mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}, \end{aligned}$$

which together with (5.15) and assumption (5.9), implies that there exists $C > 0$, depending only on parameters and other constants, all of them independent of h, such that

$$\|(p-p_h,\lambda-\lambda_h)\|_{\mathbf{Q}} \,\leq\, C\,\Big\{\|\mathcal{R}_{\mathbf{f}}\|_{\mathbf{H}'} + \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}}\Big\}. \tag{5.18}$$

In turn, taking $\mathbf{v} = \mathbf{u}-\mathbf{u}_h$ and $(q,\xi) = (p-p_h,\lambda-\lambda_h)$ in (5.16) and (5.17), respectively, gives

$$\begin{aligned} &[\mathbf{a}_h(\mathbf{u}_{\mathrm{S},h})(\mathbf{u}) - \mathbf{a}_h(\mathbf{u}_{\mathrm{S},h})(\mathbf{u}_h),\mathbf{u}-\mathbf{u}_h]\\ &\quad = \mathcal{R}_{\mathbf{f}}(\mathbf{u}-\mathbf{u}_h) - \mathcal{R}_{\mathbf{g}}(p-p_h,\lambda-\lambda_h) - [\mathcal{B}^h_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S}}),\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}]. \end{aligned}$$

Hence, employing the strict monotonicity of $\mathbf{a}_h$ (cf. [7, Lemma 4.4]), the continuity of $\mathcal{B}^h_{\mathrm{S}}$ (cf. [7, eq. (4.4)]), and the estimates (5.9), (5.15) and (5.18), we deduce the existence of a constant $C > 0$, independent of meshsizes, such that

$$\|\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\|^2_{1,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}-\mathbf{u}_{\mathrm{D},h}\|^3_{\mathbf{L}^3(\Omega_{\mathrm{D}})} \,\leq\, C\,\Big\{\|\mathcal{R}\|_{(\mathbf{H}\times\mathbf{Q})'}\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}} + \|\mathcal{R}_{\mathbf{f}}\|_{\mathbf{H}'}\|\mathcal{R}_{\mathbf{g}}\|_{\mathbf{Q}'}\Big\}. \tag{5.19}$$

Moreover, from the identity (5.17) and the definition of $\mathbf{b}$ we find that the term $\|\mathrm{div}\,(\mathbf{u}_\mathrm{D}-\mathbf{u}_{\mathrm{D},h})\|_{0,\Omega_\mathrm{D}}$ can be bounded by $\|\mathcal{R}_\mathbf{g}\|_{\mathbf{Q}'}$, which, combined with (5.19) and some algebraic manipulations, implies

$$\|\mathbf{u}-\mathbf{u}_h\|_\mathbf{H} \,\leq\, C\,\max\Big\{\|\mathcal{R}\|^{1/2}_{(\mathbf{H}\times\mathbf{Q})'},\,\|\mathcal{R}\|^{2/3}_{(\mathbf{H}\times\mathbf{Q})'},\,\|\mathcal{R}\|^{3/4}_{(\mathbf{H}\times\mathbf{Q})'},\,\|\mathcal{R}\|_{(\mathbf{H}\times\mathbf{Q})'},\,\|\mathcal{R}\|^{3/2}_{(\mathbf{H}\times\mathbf{Q})'}\Big\}. \tag{5.20}$$

Therefore, the estimate (5.13) follows from (5.18) and (5.20). Finally, from the discrete problem (3.5) we deduce that $\mathcal{R}_\mathbf{f}$ and $\mathcal{R}_\mathbf{g}$ vanish on $\mathbf{H}_h$ and $\mathbf{Q}_h$, respectively, which clearly implies (5.14) and conclude the proof. □

We remark here that when $\|\mathcal{R}\|_{(\mathbf{H}\times\mathbf{Q})'}\to 0$, the dominant term in (5.13) is $\|\mathcal{R}\|^{1/2}_{(\mathbf{H}\times\mathbf{Q})'}$. In this way, it only remains now to estimate $\|\mathcal{R}\|_{(\mathbf{H}\times\mathbf{Q})'}$. To this end, we first observe that the functional $\mathcal{R}$ can be decomposed as:

$$\mathcal{R}(\vec{\mathbf{v}}) = \mathcal{R}_1(\mathbf{v}_\mathrm{S}) + \mathcal{R}_2(\mathbf{v}_\mathrm{D}) + \mathcal{R}_3(q) + \mathcal{R}_4(\xi), \tag{5.21}$$

for all $\vec{\mathbf{v}} := ((\mathbf{v}_\mathrm{S},\mathbf{v}_\mathrm{D}),(q,\xi))\in\mathbf{H}\times\mathbf{Q}$, where

$$\begin{aligned}
\mathcal{R}_1(\mathbf{v}_\mathrm{S}) \;:=\;& (\mathbf{f}_\mathrm{S},\mathbf{v}_\mathrm{S})_\mathrm{S} - 2\mu(\mathbf{e}(\mathbf{u}_{\mathrm{S},h}),\mathbf{e}(\mathbf{v}_\mathrm{S}))_\mathrm{S} - \rho((\nabla\mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h},\mathbf{v}_\mathrm{S})_\mathrm{S} - \frac{\rho}{2}(\mathrm{div}\,\mathbf{u}_{\mathrm{S},h}\mathbf{u}_{\mathrm{S},h},\mathbf{v}_\mathrm{S})_\mathrm{S}\\
&+ (\mathrm{div}\,\mathbf{v}_\mathrm{S},p_h)_\mathrm{S} - \left\langle\frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}\mathbf{u}_{\mathrm{S},h}\cdot\mathbf{t},\mathbf{v}_\mathrm{S}\cdot\mathbf{t}\right\rangle_\Sigma - \langle\mathbf{v}_\mathrm{S}\cdot\mathbf{n},\lambda_h\rangle_\Sigma\,,\\
\mathcal{R}_2(\mathbf{v}_\mathrm{D}) \;:=\;& (\mathbf{f}_\mathrm{D},\mathbf{v}_\mathrm{D})_\mathrm{D} - \frac{\mu}{\rho}(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h},\mathbf{v}_\mathrm{D})_\mathrm{D} - \frac{\mathtt{F}}{\rho}(|\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h},\mathbf{v}_\mathrm{D})_\mathrm{D}\\
&+ (\mathrm{div}\,\mathbf{v}_\mathrm{D},p_h)_\mathrm{D} + \langle\mathbf{v}_\mathrm{D}\cdot\mathbf{n},\lambda_h\rangle_\Sigma\,,\\
\mathcal{R}_3(q) \;:=\;& (\mathrm{div}\,\mathbf{u}_{\mathrm{S},h},q)_\mathrm{S} - (g_\mathrm{D}-\mathrm{div}\,\mathbf{u}_{\mathrm{D},h},q)_\mathrm{D},\\
\mathcal{R}_4(\xi) \;:=\;& -\langle\mathbf{u}_{\mathrm{S},h}\cdot\mathbf{n}-\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n},\xi\rangle_\Sigma\,.
\end{aligned}$$

In this way, it follows that

$$\|\mathcal{R}\|_{(\mathbf{H}\times\mathbf{Q})'} \,\leq\, \Big\{\|\mathcal{R}_1\|_{\mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})'} + \|\mathcal{R}_2\|_{\mathbf{H}^3_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D})'} + \|\mathcal{R}_3\|_{\mathrm{L}^2_0(\Omega)'} + \|\mathcal{R}_4\|_{\mathrm{W}^{-1/3,3}(\Sigma)}\Big\}, \tag{5.22}$$

and hence our next purpose is to derive suitable upper bounds for each one of the terms on the right-hand side of (5.22). We start with the following lemma, which is a direct consequence of the Cauchy–Schwarz inequality.

Lemma 5.3 *There holds*

$$\|\mathcal{R}_3\|_{\mathrm{L}^2_0(\Omega)'} \,\leq\, \left\{\sum_{T\in\mathcal{T}_h^\mathrm{S}}\|\mathrm{div}\,\mathbf{u}_{\mathrm{S},h}\|^2_{0,T} + \sum_{T\in\mathcal{T}_h^\mathrm{D}}\|g_\mathrm{D}-\mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|^2_{0,T}\right\}^{1/2}.$$

We now adapt a result taken from [3] in order to obtain an upper bound for $\mathcal{R}_1$.

Lemma 5.4 *There exists $C>0$, independent of the meshsizes, such that*

$$\|\mathcal{R}_1\|_{\mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})'} \,\leq\, C\left\{\sum_{T\in\mathcal{T}_h^\mathrm{S}}\widehat{\Theta}^2_{\mathrm{S},T}\right\}^{1/2},$$

where

$$\widehat{\Theta}^2_{S,T} := h_T^2 \left\| \mathbf{f}_S + \mathbf{div}\widetilde{\boldsymbol{\sigma}}_{S,h} - \rho(\nabla \mathbf{u}_{S,h})\mathbf{u}_{S,h} - \frac{\rho}{2}\mathrm{div}\,\mathbf{u}_{S,h}\mathbf{u}_{S,h} \right\|^2_{0,T}$$

$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_S)} h_e \| [\![\widetilde{\boldsymbol{\sigma}}_{S,h}\mathbf{n}]\!] \|^2_{0,e} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \left\| \widetilde{\boldsymbol{\sigma}}_{S,h}\mathbf{n} + \lambda_h \mathbf{n} + \frac{\alpha_d \mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}(\mathbf{u}_{S,h}\cdot\mathbf{t})\mathbf{t} \right\|^2_{0,e},$$

and $\widetilde{\boldsymbol{\sigma}}_{S,h}$ *is given by* (5.2).

Proof. We proceed similarly as in the proof of [3, Lemma 3.4], by replacing $\mathbf{f}_1, \boldsymbol{\sigma}_{1,h}, \mathbf{u}_{1,h}, \Omega_1$, and Γ_2 by $\mathbf{f}_S - \rho(\nabla\mathbf{u}_{S,h})\mathbf{u}_{S,h} - \frac{\rho}{2}\mathrm{div}\,\mathbf{u}_{S,h}\mathbf{u}_{S,h}, \widetilde{\boldsymbol{\sigma}}_{S,h}, \mathbf{u}_{S,h}, \Omega_S$, and Σ, respectively, and employing the local approximation properties of the Clément interpolation operator $\mathbf{I}_h^S : \mathbf{H}^1(\Omega_S) \to \mathbf{X}_h(\Omega_S)$ provided by Lemma 4.2 with p = 2. We omit further details. □

Next, we derive the upper bound for $\mathcal{R}_4$, the functional acting on the interface Σ.

Lemma 5.5 *There exists* $C > 0$*, independent of the meshsizes, such that*

$$\|\mathcal{R}_4\|_{W^{-1/3,3}(\Sigma)} \le C \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\mathbf{u}_{S,h}\cdot\mathbf{n} - \mathbf{u}_{D,h}\cdot\mathbf{n}\|^3_{L^3(e)} \right\}^{1/3}.$$

Proof. We recall from the definition of $\mathcal{R}_4$ (cf. (5.21)) that

$$\mathcal{R}_4(\xi) = -\langle \mathbf{u}_{S,h}\cdot\mathbf{n} - \mathbf{u}_{D,h}\cdot\mathbf{n}, \xi\rangle_\Sigma \quad \forall\, \xi \in W^{1/3,3/2}(\Sigma),$$

which certainly yields

$$\|\mathcal{R}_4\|_{W^{-1/3,3}(\Sigma)} = \|\mathbf{u}_{S,h}\cdot\mathbf{n} - \mathbf{u}_{D,h}\cdot\mathbf{n}\|_{-1/3,3;\Sigma}\,. \tag{5.23}$$

In turn, taking $\xi_h \in \Lambda_h(\Sigma)$ and then $(\mathbf{0},(0,\xi_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ in (5.14), we deduce that

$$\langle \mathbf{u}_{S,h}\cdot\mathbf{n} - \mathbf{u}_{D,h}\cdot\mathbf{n}, \xi_h\rangle_\Sigma = 0 \quad \forall\, \xi_h \in \Lambda_h(\Sigma),$$

which says that $\mathbf{u}_{S,h}\cdot\mathbf{n} - \mathbf{u}_{D,h}\cdot\mathbf{n}$ belongs to $\Lambda_h^\perp(\Sigma)$ (cf. (4.12)). In this way, the proof follows from (5.23) and a direct application of (4.13) with p = 3/2 and q = 3 (cf. Lemma 4.5). □

Finally, we focus on deriving the upper bound for $\mathcal{R}_2$, for which, given $\mathbf{v}_D \in \mathbf{H}^3_{\Gamma_D}(\mathrm{div}\,;\Omega_D)$, we consider its Helmholtz decomposition provided by Lemma 4.3 with p = 3. More precisely, we let $\mathbf{w}_D \in \mathbf{H}^1(\Omega_D)$ and $\beta_D \in W^{1,3}_{\Gamma_D}(\Omega_D)$ be such that $\mathbf{v}_D = \mathbf{w}_D + \mathbf{curl}\,\beta_D$ in Ω_D, and

$$\|\mathbf{w}_D\|_{1,\Omega_D} + \|\beta_D\|_{1,3;\Omega_D} \le C_{\mathrm{hel}}\,\|\mathbf{v}_D\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)}. \tag{5.24}$$

In turn, similarly to [2], we consider the finite element subspace of $W^{1,3}_{\Gamma_D}(\Omega_D)$ given by

$$X_{h,\Gamma_D} := \Big\{ v \in \mathcal{C}(\overline{\Omega}_D) : \quad v|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h^D, \quad v = 0 \quad \text{on} \quad \Gamma_D \Big\},$$

and introduce the Clément interpolator $I_h^D : W^{1,3}_{\Gamma_D}(\Omega_D) \to X_{h,\Gamma_D}$. In addition, recalling the Raviart–Thomas interpolator $\Pi_h : \mathbf{H}^1(\Omega_D) \to \mathbf{H}_h(\Omega_D)$ introduced in Section 4, we are able to define

$$\mathbf{v}_{D,h} := \Pi_h(\mathbf{w}_D) + \mathbf{curl}\,(I_h^D\beta_D) \in \mathbf{H}_{h,\Gamma_D}(\Omega_D),$$

which can be seen as a discrete Helmholtz decomposition of $\mathbf{v}_{\mathrm{D},h}$. Then, noting from (5.14) that $\mathcal{R}_2(\mathbf{v}_{\mathrm{D},h}) = 0$, we can write

$$\mathcal{R}_2(\mathbf{v}_{\mathrm{D}}) = \mathcal{R}_2(\mathbf{v}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D},h}) = \mathcal{R}_2(\mathbf{w}_{\mathrm{D}} - \Pi_h(\mathbf{w}_{\mathrm{D}})) + \mathcal{R}_2(\mathbf{curl}\,(\beta_{\mathrm{D}} - I_h^{\mathrm{D}}\beta_{\mathrm{D}})).$$

Next, in order to simplify the subsequent writing, we define $\widehat{\mathbf{w}}_{\mathrm{D}} := \mathbf{w}_{\mathrm{D}} - \Pi_h(\mathbf{w}_{\mathrm{D}})$ and $\widehat{\beta}_{\mathrm{D}} := \beta_{\mathrm{D}} - I_h^{\mathrm{D}}\beta_{\mathrm{D}}$. In this way, according to the definition of $\mathcal{R}_2$ (cf. (5.21)), we find that

$$\mathcal{R}_2(\widehat{\mathbf{w}}_{\mathrm{D}}) = (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}, \widehat{\mathbf{w}}_{\mathrm{D}})_{\mathrm{D}} + (\mathrm{div}\,\widehat{\mathbf{w}}_{\mathrm{D}}, p_h)_{\mathrm{D}} + \langle \widehat{\mathbf{w}}_{\mathrm{D}} \cdot \mathbf{n}, \lambda_h \rangle_{\Sigma}\,, \tag{5.25}$$

and

$$\mathcal{R}_2(\mathbf{curl}\,\widehat{\beta}_{\mathrm{D}}) = \Big(\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}, \mathbf{curl}\,\widehat{\beta}_{\mathrm{D}}\Big)_{\mathrm{D}} + \Big\langle \mathbf{curl}\,\widehat{\beta}_{\mathrm{D}} \cdot \mathbf{n}, \lambda_h \Big\rangle_{\Sigma}\,, \tag{5.26}$$

with $\mathbf{U}_{\mathrm{D},h}$ given by (5.5). The following lemma establishes the estimate for $\mathcal{R}_2$.

Lemma 5.6 *Assume that there exists a convex domain Ξ such that $\Omega_{\mathrm{D}} \subseteq \Xi$ and $\Gamma_{\mathrm{D}} \subseteq \partial\Xi$. Assume further that $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$. Then there exist $C_1, C_2 > 0$, independent of the meshsizes, such that*

$$\|\mathcal{R}_2\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})'} \,\le\, C_1 \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \overline{\Theta}_{\mathrm{D},T}^2 \right\}^{1/2} + \,C_2 \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \widetilde{\Theta}_{\mathrm{D},T}^{3/2} \right\}^{2/3}, \tag{5.27}$$

where $\widetilde{\Theta}_{\mathrm{D},T}$ is defined in (5.4)*, and*

$$\overline{\Theta}_{\mathrm{D},T}^2 \,:=\, h_T^2\,\|\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \|\lambda_h - p_{\mathrm{D},h}\|_{0,e}^2$$

with $\mathbf{U}_{\mathrm{D},h}$ given by (5.5)*.*

Proof. We begin by estimating $\mathcal{R}_2(\widehat{\mathbf{w}}_{\mathrm{D}})$ (cf. (5.25)). To that end, and proceeding as in [3, Lemma 3.12], we note first that $\widehat{\mathbf{w}}_{\mathrm{D}} \cdot \mathbf{n} \in \mathrm{L}^2(\Sigma)$, which follows from the fact that $\mathbf{w}_{\mathrm{D}} \in \mathbf{H}^1(\Omega_{\mathrm{D}})$ and $\Pi_h(\mathbf{w}_{\mathrm{D}}) \cdot \mathbf{n}$ is piecewise constant on Σ. In addition, noting that $p_{\mathrm{D},h} := p_h|_{\Omega_{\mathrm{D}}}$ is also piecewise constant on Σ, integrating by parts the second term in (5.25), recalling that $\mathbf{w}_{\mathrm{D}} \cdot \mathbf{n} = 0$ on Γ_{D} (cf. Lemma 4.3), and using (4.1), we find that

$$\mathcal{R}_2(\widehat{\mathbf{w}}_{\mathrm{D}}) \,=\, (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}, \widehat{\mathbf{w}}_{\mathrm{D}})_{\mathrm{D}} + \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e (\lambda_h - p_{\mathrm{D},h})\widehat{\mathbf{w}}_{\mathrm{D}} \cdot \mathbf{n}.$$

In this way, assuming $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$, and applying the Cauchy–Schwarz inequality and the approximation properties of the Raviart–Thomas interpolation operator Π_h (cf. Lemma 4.1), we deduce that

$$\big|\mathcal{R}_2(\widehat{\mathbf{w}}_{\mathrm{D}})\big| \,\le\, \widehat{C}_1 \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T^2\,\|\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\lambda_h - p_{\mathrm{D},h}\|_{0,e}^2 \right\}^{1/2} \|\mathbf{w}_{\mathrm{D}}\|_{1,\Omega_{\mathrm{D}}}. \tag{5.28}$$

On the other hand, in order to bound $\mathcal{R}_2(\mathbf{curl}\,\widehat{\beta}_{\mathrm{D}})$ (cf. (5.26)), we now notice from the integration by parts formula (4.9) with $\mathrm{p} = 3$ and $\mathrm{q} = 3/2$ (cf. Lemma 4.4), and the fact that λ_h is piecewise constant on Σ, that the second term in (5.26) vanishes. Then, integrating by parts the first term in (5.26) on each $T \in \mathcal{T}_h^{\mathrm{D}}$ and using the fact that $\beta_{\mathrm{D}}|_{\Gamma_{\mathrm{D}}} = I_h^{\mathrm{D}} \beta_{\mathrm{D}}|_{\Gamma_{\mathrm{D}}} = 0$, we obtain

$$\begin{aligned} \mathcal{R}_2(\mathbf{curl}\,\widehat{\beta}_{\mathrm{D}}) \,=\, & \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \int_T \mathrm{rot}\,(\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h})\,\widehat{\beta}_{\mathrm{D}} \\ & + \sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{D}})} \int_e [\![(\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h})\cdot\mathbf{t}]\!]\,\widehat{\beta}_{\mathrm{D}} + \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \big\{ (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h})\cdot\mathbf{t} \big\}\,\widehat{\beta}_{\mathrm{D}}, \end{aligned}$$

which, together with the Clément interpolator estimates (cf. Lemma 4.2 with p = 3), implies

$$\begin{aligned} \big|\mathcal{R}_2(\mathbf{curl}\,\widehat{\beta}_{\mathrm{D}})\big| \,\leq\, \widehat{C}_2 \Bigg\{ & \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T^{3/2}\,\big\|\mathrm{rot}\,(\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h})\big\|_{\mathrm{L}^{3/2}(T)}^{3/2} \\ + \sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{D}})} h_e \big\| [\![(\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h})\cdot\mathbf{t}]\!] \big\|_{\mathrm{L}^{3/2}(e)}^{3/2} & + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \big\| (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h})\cdot\mathbf{t} \big\|_{\mathrm{L}^{3/2}(e)}^{3/2} \Bigg\}^{2/3} \|\beta_{\mathrm{D}}\|_{1,3;\Omega_{\mathrm{D}}}. \end{aligned} \tag{5.29}$$

Therefore, as a direct consequence of estimates (5.28) and (5.29), and the stability estimate (5.24) for the Helmholtz decomposition, we get (5.27) and conclude the proof. □

We end this section by stressing that the estimate (5.10) is a straightforward consequence of Lemmas 5.2 and 5.3–5.6, and the definition of the global estimator Θ (cf. (5.7)), when $h \to 0$.

5.2 Efficiency

The following theorem is the main result of this section.

Theorem 5.7 *Suppose that the data* $\mathbf{f}_{\mathrm{S}}$, $\mathbf{f}_{\mathrm{D}}$ *and* g_{D} *satisfy* (5.9), *and that* $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$. *Then, there exists a constant* $\widehat{C}_{\mathrm{eff}} > 0$, *independent of* h, *such that*

$$\widehat{C}_{\mathrm{eff}}\,\Theta \,\leq\, \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{H}\times\mathbf{Q}} + \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T^2\, \big\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \big\|_{0,T}^2 \right\}^{1/2} + \mathrm{h.o.t.}, \tag{5.30}$$

where h.o.t. *stands for one or several terms of higher order. Moreover, assuming that* $\mathbf{u}_{\mathrm{D}} \in \mathbf{L}^6(\Omega_{\mathrm{D}})$, *there exists a constant* $C_{\mathrm{eff}} > 0$, *depending only on parameters,* $\|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{L}^6(\Omega_{\mathrm{D}})}$, *and other constants, all them independent of* h, *such that*

$$C_{\mathrm{eff}}\,\Theta \,\leq\, \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{H}\times\mathbf{Q}} + \mathrm{h.o.t.} \tag{5.31}$$

Throughout this section we assume, without loss of generality, that $\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}$, $\mathbf{f}_{\mathrm{S}}$, and $\mathbf{f}_{\mathrm{D}}$, are all piecewise polynomials. Otherwise, if $\mathbf{K}$, $\mathbf{f}_{\mathrm{S}}$, and $\mathbf{f}_{\mathrm{D}}$ are sufficiently smooth, and proceeding similarly as in [11, Section 6.2], higher order terms given by the errors arising from suitable polynomial approximation of these expressions and functions would appear in (5.30) and (5.31), which explains the eventual h.o.t. in these inequalities. In this regard, analogously to [2, Section 3.3], we remark that (5.30) constitutes what we call a *quasi-efficiency* estimate for the global residual error estimator Θ (cf. (5.7)), in the sense that the expression appearing on the right-hand side of (5.30) is the error plus the nonlinear term given by $\left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T^2\, \big\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \big\|_{0,T}^2 \right\}^{1/2}$. However, assuming additionally that $\mathbf{u}_{\mathrm{D}} \in \mathbf{L}^6(\Omega_{\mathrm{D}})$, we show at the end of this section that the latter can be bounded by $\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}$, thus yielding the efficiency estimate given by (5.31).

In order to prove (5.30) and (5.31), we need first to introduce the Banach space

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega_\mathrm{S}) := \Big\{ \boldsymbol{\tau}_\mathrm{S} \in \mathbb{L}^2(\Omega_\mathrm{S}) : \quad \mathbf{div}\boldsymbol{\tau}_\mathrm{S} \in \mathbf{L}^{4/3}(\Omega_\mathrm{S}) \Big\}.$$

Then, we state the following result, which basically follows by applying integration by parts backwardly in the formulation (3.3), and proceeding as in [6, Remark 2.1] for the Navier–Stokes terms.

Theorem 5.8 *Let* $(\mathbf{u},(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$ *be the unique solution of* (3.3)*. Then* $\mathrm{div}\,\mathbf{u}_\mathrm{S} = 0$ *in* Ω_S*,* $\mathrm{div}\,\mathbf{u}_\mathrm{D} = g_\mathrm{D}$ *in* Ω_D*, and* $\mathbf{u}_\mathrm{D}\cdot\mathbf{n} = \mathbf{u}_\mathrm{S}\cdot\mathbf{n}$ *on* Σ*. In addition, defining* $p_\mathrm{S} := p|_{\Omega_\mathrm{S}}$*,* $p_\mathrm{D} := p|_{\Omega_\mathrm{D}}$*,* $\widetilde{\boldsymbol{\sigma}}_\mathrm{S} := -p_\mathrm{S}\mathbb{I} + 2\mu\mathbf{e}(\mathbf{u}_\mathrm{S})$*, and* $\mathbf{U}_\mathrm{D} := \frac{\mu}{\rho}\mathbf{K}^{-1}\mathbf{u}_\mathrm{D} + \frac{\mathtt{F}}{\rho}|\mathbf{u}_\mathrm{D}|\mathbf{u}_\mathrm{D}$*, there hold* $p_\mathrm{D} \in \mathrm{W}^{1,3/2}(\Omega_\mathrm{D})\cap\mathrm{L}^2(\Omega_\mathrm{D})$*,* $\lambda = p_\mathrm{D}$ *on* Σ*,* $\mathbf{div}\widetilde{\boldsymbol{\sigma}}_\mathrm{S} = \rho(\nabla\mathbf{u}_\mathrm{S})\mathbf{u}_\mathrm{S} - \mathbf{f}_\mathrm{S}$ *in* Ω_S *(which yields* $\widetilde{\boldsymbol{\sigma}}_\mathrm{S} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_\mathrm{S})$*),* $\mathbf{U}_\mathrm{D} + \nabla p_\mathrm{D} = \mathbf{f}_\mathrm{D}$ *in* Ω_D*, and* $\widetilde{\boldsymbol{\sigma}}_\mathrm{S}\mathbf{n} + \lambda\mathbf{n} + \frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}(\mathbf{u}_\mathrm{S}\cdot\mathbf{t})\mathbf{t} = \mathbf{0}$ *on* Σ*.*

We begin the derivation of the efficiency estimates with the following result.

Lemma 5.9 *There hold*

$$\|\mathrm{div}\,\mathbf{u}_{\mathrm{S},h}\|_{0,T} \le |\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}|_{1,T} \quad \forall\, T \in \mathcal{T}_h^\mathrm{S}$$

and

$$\|g_\mathrm{D} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T} \le \|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;T)} \quad \forall\, T \in \mathcal{T}_h^\mathrm{D}.$$

Proof. It suffices to use from Theorem 5.8 that $\mathrm{div}\,\mathbf{u}_\mathrm{S} = 0$ in Ω_S and $\mathrm{div}\,\mathbf{u}_\mathrm{D} = g_\mathrm{D}$ in Ω_D. Further details are omitted. □

In order to derive the upper bounds for the remaining terms defining the global a posteriori error estimator Θ (cf. (5.7)), we proceed similarly as in [19, 3, 11, 2, 8], and apply results ultimately based on inverse inequalities and the localization technique based on triangle-bubble and edge-buble functions. To this end, we now recall some notation and introduce further preliminary results. Given $T \in \mathcal{T}_h := \mathcal{T}_h^\mathrm{S} \cup \mathcal{T}_h^\mathrm{D}$, and $e \in \mathcal{E}(T)$, we let ψ_T and ψ_e be the usual triangle-bubble and edge-bubble functions, respectively (see [26, eqs. (1.5) and (1.6)]), which satisfy:

(i) $\psi_T \in \mathrm{P}_3(T)$, $\mathrm{supp}(\psi_T) \subseteq T$, $\psi_T = 0$ on ∂T, and $0 \le \psi_T \le 1$ in T,

(ii) $\psi_e|_T \in \mathrm{P}_2(T)$, $\mathrm{supp}(\psi_e) \subseteq \omega_e$, $\psi_e = 0$ on $\partial T\setminus e$, and $0 \le \psi_e \le 1$ in $\omega_e := \cup\big\{T' \in \mathcal{T}_h : e \in \mathcal{E}(T')\big\}$.

In addition, we also recall from [26] that, given $k \in \mathbb{N}\cup\{0\}$, there exists an extension operator $L : \mathcal{C}(e) \to \mathcal{C}(\omega_e)$ that satisfies $L(\sigma) \in \mathrm{P}_k(T)$ and $L(\sigma)|_e = \sigma \;\forall\, \sigma \in \mathrm{P}_k(e)$. A corresponding vectorial version of L, that is, the componentwise application of L, is denoted by $\mathbf{L}$. Additional properties of ψ_T, ψ_e, and L are collected in the following lemma. Regarding the corresponding proof we refer to [26, Lemma 3.3] for details.

Lemma 5.10 *Let* p *and* q *two fixed real numbers with* $\mathrm{p} \in [1,+\infty]$ *and* $1/\mathrm{p} + 1/\mathrm{q} = 1$*. Given* $T \in \mathcal{T}_h$ *and* $e \in \mathcal{E}(T)$*, let* $V_T \subset \mathrm{L}^\infty(T)$ *and* $V_e \subset \mathrm{L}^\infty(e)$ *two arbitrary finite dimensional spaces. Then, there exist positive constants* c_i *with* $i \in \{1,\dots,7\}$*, depending only on* p*,* q*, the spaces* V_T *and* V_e*, and the shape-regularity of the triangulations (minimum angle condition), such that for each* $u \in V_T$ *and* $\sigma \in V_e$*, there hold*

$$c_1\,\|u\|_{\mathrm{L}^\mathrm{p}(T)} \le \sup_{v\in V_T} \frac{\displaystyle\int_T u\,\psi_T v}{\|v\|_{\mathrm{L}^\mathrm{q}(T)}} \le \|u\|_{\mathrm{L}^\mathrm{p}(T)}, \tag{5.32}$$

$$c_2\,\|\sigma\|_{\mathrm{L}^{\mathrm{p}}(e)} \,\le\, \sup_{\tau\in V_e} \frac{\displaystyle\int_e \sigma\,\psi_e\tau}{\|\tau\|_{\mathrm{L}^{\mathrm{q}}(e)}} \,\le\, \|\sigma\|_{\mathrm{L}^{\mathrm{p}}(e)}, \tag{5.33}$$

$$c_3\,h_T^{-1}\|\psi_T u\|_{\mathrm{L}^{\mathrm{q}}(T)} \,\le\, \|\nabla(\psi_T u)\|_{\mathrm{L}^{\mathrm{q}}(T)} \,\le\, c_4\,h_T^{-1}\|\psi_T u\|_{\mathrm{L}^{\mathrm{q}}(T)}, \tag{5.34}$$

$$c_5\,h_T^{-1}\|\psi_e L(\sigma)\|_{\mathrm{L}^{\mathrm{q}}(T)} \,\le\, \|\nabla(\psi_e L(\sigma))\|_{\mathrm{L}^{\mathrm{q}}(T)} \,\le\, c_6\,h_T^{-1}\|\psi_e L(\sigma)\|_{\mathrm{L}^{\mathrm{q}}(T)}, \tag{5.35}$$

and

$$\|\psi_e L(\sigma)\|_{\mathrm{L}^{\mathrm{q}}(T)} \,\le\, c_7\,h_e^{1/\mathrm{q}}\|\sigma\|_{\mathrm{L}^{\mathrm{q}}(e)}. \tag{5.36}$$

As stated in [26, Remark 3.2], V_T and V_e can be chosen as suitable spaces of polynomials. Thus, in what follows we will choose V_T as $\mathrm{P}_k(T)$ and V_e as $\mathrm{P}_k(e)$ for a given $k\in\mathbb{N}\cup\{0\}$. In addition, and coherently with previous notations, we set $\mathbf{V}_T$ and $\mathbf{V}_e$, respectively, as the corresponding vectorial counterpart. The following inverse estimate will be also used. We refer the reader to [15, Lemma 1.138] for its proof.

Lemma 5.11 *Let $k\in\mathbb{N}\cup\{0\}$, $n\in\{2,3\}$, $l,m\ge 0$ such that $m\le l$, and $\mathrm{p},\mathrm{q}\in[1,+\infty]$. Then, there exists $c>0$, depending only on k,l,m and the shape regularity of the triangulations, such that, for each triangle (tetrahedron) $T\in\mathcal{T}_h$, there holds*

$$\|v\|_{l,\mathrm{p},T} \,\le\, c\,h_T^{m-l+n(1/\mathrm{p}-1/\mathrm{q})}\,\|v\|_{m,\mathrm{q},T} \quad \forall\,v\in\mathrm{P}_k(T). \tag{5.37}$$

We point out that through this section each proof done in 2D can be easily extended to its three-dimensional counterpart considering $n=3$ when we apply (5.37). In that case, other positive power of the meshsizes $h_{T_\star}$, with $\star\in\{\mathrm{S},\mathrm{D}\}$, will be appear on the right-hand side of the efficiency estimates which anyway are bounded. Next, we continue providing the corresponding upper bounds for the remaining three terms defining $\Theta^2_{\mathrm{S},T}$ (cf. (5.1)), which are adaptations of the proof of [3, Lemmas 4.4, 4.5, and 4.6], respectively, to our configuration.

Lemma 5.12 *There exists $c>0$, independent of h, such that for each $T\in\mathcal{T}_h^{\mathrm{S}}$ there holds*

$$\begin{aligned} h_T^2\left\|\mathbf{f}_\mathrm{S}+\mathbf{div}\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}-\rho(\nabla\mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}-\frac{\rho}{2}\mathrm{div}\,\mathbf{u}_{\mathrm{S},h}\mathbf{u}_{\mathrm{S},h}\right\|_{0,T}^2 \,\le\, c\Big\{\|p_\mathrm{S}-p_{\mathrm{S},h}\|_{0,T}^2+|\mathbf{u}_\mathrm{S}-\mathbf{u}_{\mathrm{S},h}|_{1,T}^2 \\ +\,h_T\big\|(\nabla\mathbf{u}_\mathrm{S})\mathbf{u}_\mathrm{S}-(\nabla\mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\big\|_{\mathbf{L}^{4/3}(T)}^2+h_T\|\mathrm{div}\,(\mathbf{u}_\mathrm{S}-\mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\|_{\mathbf{L}^{4/3}(T)}^2\Big\}. \end{aligned} \tag{5.38}$$

Proof. Given $T\in\mathcal{T}_h^{\mathrm{S}}$ we define $\boldsymbol{\chi}_T:=\mathbf{f}_\mathrm{S}+\mathbf{div}\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}-\rho(\nabla\mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}-\frac{\rho}{2}\mathrm{div}\,\mathbf{u}_{\mathrm{S},h}\mathbf{u}_{\mathrm{S},h}$ in T. Then, applying (5.32) to $\|\boldsymbol{\chi}_T\|_{0,T}$, we obtain

$$c_1\,\|\boldsymbol{\chi}_T\|_{0,T} \,\le\, \sup_{\mathbf{v}\in\mathbf{V}_T}\frac{\displaystyle\int_T \boldsymbol{\chi}_T\cdot\psi_T\mathbf{v}}{\|\mathbf{v}\|_{0,T}}\,. \tag{5.39}$$

Then, thanks to the identities $\mathbf{f}_\mathrm{S}=-\mathbf{div}\widetilde{\boldsymbol{\sigma}}_\mathrm{S}+\rho(\nabla\mathbf{u}_\mathrm{S})\mathbf{u}_\mathrm{S}$ and $\mathrm{div}\,\mathbf{u}_\mathrm{S}=0$ in Ω_S (cf. Theorem 5.8), and integrating by parts, we deduce that

$$\begin{aligned} \int_T \boldsymbol{\chi}_T\cdot\psi_T\mathbf{v} &= \int_T(\widetilde{\boldsymbol{\sigma}}_\mathrm{S}-\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}):\nabla(\psi_T\mathbf{v}) \\ &\quad+\,\rho\int_T\Big\{(\nabla\mathbf{u}_\mathrm{S})\mathbf{u}_\mathrm{S}-(\nabla\mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}+\frac{1}{2}\mathrm{div}\,(\mathbf{u}_\mathrm{S}-\mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\Big\}\cdot\psi_T\mathbf{v}\,, \end{aligned}$$

from which, using Cauchy–Schwarz and Hölder's inequalities, applying (5.34) to $\|\nabla(\psi_T \mathbf{v})\|_{0,T}$, employing the fact that $0 \le \psi_T \le 1$, and bearing in mind the definitions of $\widetilde{\boldsymbol{\sigma}}_{\mathrm{S}}$ and $\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}$ (cf. Theorem 5.8 and (5.2)), we arrive at

$$\begin{aligned}\int_T \boldsymbol{\chi}_T \cdot \psi_T \mathbf{v} \le C_1 \Big\{ h_T^{-1} \|p_{\mathrm{S}} - p_{\mathrm{S},h}\|_{0,T} + h_T^{-1} |\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T} \Big\} \|\mathbf{v}\|_{0,T} \\ + C_2 \Big\{ \big\|(\nabla \mathbf{u}_{\mathrm{S}})\mathbf{u}_{\mathrm{S}} - (\nabla \mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\big\|_{\mathbf{L}^{4/3}(T)} + \|\mathrm{div}\,(\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\|_{\mathbf{L}^{4/3}(T)} \Big\} \|\mathbf{v}\|_{\mathbf{L}^4(T)}.\end{aligned} \tag{5.40}$$

Thus, the local inverse estimate $\|\mathbf{v}\|_{\mathbf{L}^4(T)} \le c\, h_T^{-1/2} \|\mathbf{v}\|_{0,T}$ (cf. (5.37)) in combination with (5.40) and (5.39), imply (5.38) and complete the proof. $\square$

Lemma 5.13 *There exists $c > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Omega_{\mathrm{S}})$ there holds*

$$\begin{aligned} h_e \big\| [\![\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}\mathbf{n}]\!] \big\|_{0,e}^2 \le c \sum_{T \subseteq \omega_e} \Big\{ \|p_{\mathrm{S}} - p_{\mathrm{S},h}\|_{0,T}^2 + |\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T}^2 \\ + h_T \big\|(\nabla \mathbf{u}_{\mathrm{S}})\mathbf{u}_{\mathrm{S}} - (\nabla \mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\big\|_{\mathbf{L}^{4/3}(T)}^2 + h_T \|\mathrm{div}\,(\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\|_{\mathbf{L}^{4/3}(T)}^2 \Big\},\end{aligned} \tag{5.41}$$

where ω_e is the union of the two triangles in $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge.

Proof. Since $\widetilde{\boldsymbol{\sigma}}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_{\mathrm{S}})$ (cf. Theorem 5.8), it follows that $[\![\widetilde{\boldsymbol{\sigma}}_{\mathrm{S}}\mathbf{n}]\!] = \mathbf{0}$ on each $e \in \mathcal{E}_h(\Omega_{\mathrm{S}})$. In this way, applying (5.33) to $\big\| [\![\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}\mathbf{n}]\!] \big\|_{0,e}$, we get

$$c_2 \big\| [\![\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}\mathbf{n}]\!] \big\|_{0,e} \le \sup_{\boldsymbol{\tau} \in \mathbf{V}_e} \frac{\displaystyle\int_e [\![(\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{S}})\mathbf{n}]\!] \cdot \psi_e \mathbf{L}(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{0,e}}, \tag{5.42}$$

from which, integrating by parts on each $T \subseteq \omega_e$, we deduce that

$$\int_e [\![(\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{S}})\mathbf{n}]\!] \cdot \psi_e \mathbf{L}(\boldsymbol{\tau}) = \sum_{T \subseteq \omega_e} \Big\{ \int_T \mathbf{div}(\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{S}}) \cdot \psi_e \mathbf{L}(\boldsymbol{\tau}) + \int_T (\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{S}}) : \nabla\left(\psi_e \mathbf{L}(\boldsymbol{\tau})\right) \Big\}.$$

Next, employing the identities $\mathbf{div}\widetilde{\boldsymbol{\sigma}}_{\mathrm{S}} = \rho(\nabla \mathbf{u}_{\mathrm{S}})\mathbf{u}_{\mathrm{S}} - \mathbf{f}_{\mathrm{S}}$ and $\mathrm{div}\,\mathbf{u}_{\mathrm{S}} = 0$ in Ω_{S} (cf. Theorem 5.8), and the Cauchy–Schwarz and Hölder inequalities, and applying (5.37) and (5.35) to $\psi_e \mathbf{L}(\boldsymbol{\tau})$ and $\|\nabla\left(\psi_e \mathbf{L}(\boldsymbol{\tau})\right)\|_{0,T}$, respectively, we obtain

$$\begin{aligned}\int_e [\![(\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h} - \widetilde{\boldsymbol{\sigma}}_{\mathrm{S}})\mathbf{n}]\!] \cdot \psi_e \mathbf{L}(\boldsymbol{\tau}) \;\le\; C \sum_{T \subseteq \omega_e} \Big\{ \Big\| \mathbf{f}_{\mathrm{S}} + \mathbf{div}\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h} - \rho(\nabla \mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h} - \frac{\rho}{2}\mathrm{div}\,\mathbf{u}_{\mathrm{S},h}\mathbf{u}_{\mathrm{S},h} \Big\|_{0,T} \\ + h_T^{-1/2} \big\|(\nabla \mathbf{u}_{\mathrm{S}})\mathbf{u}_{\mathrm{S}} - (\nabla \mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\big\|_{\mathbf{L}^{4/3}(T)} + h_T^{-1/2} \|\mathrm{div}\,(\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\|_{\mathbf{L}^{4/3}(T)} \\ + h_T^{-1} \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T} \Big\} \|\psi_e \mathbf{L}(\boldsymbol{\tau})\|_{0,T}.\end{aligned} \tag{5.43}$$

Then, applying (5.36) to $\|\psi_e \mathbf{L}(\boldsymbol{\tau})\|_{0,T}$ in combination with (5.43) and (5.42), using the fact that $h_e \le h_T$, the estimate (5.38) and the definitions of $\widetilde{\boldsymbol{\sigma}}_{\mathrm{S}}$ and $\widetilde{\boldsymbol{\sigma}}_{\mathrm{S},h}$ (cf. Theorem 5.8 and (5.2)), we derive (5.41) and conclude the proof. $\square$

Before establishing the following lemma, we need to recall a local trace inequality [1, Theorem 3.10]. Indeed, there exists $c > 0$, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ and $e \in \mathcal{E}(T)$, there holds

$$h_e \|v\|_{0,e}^2 \le c \Big\{ \|v\|_{0,T}^2 + h_T^2 |v|_{1,T}^2 \Big\} \quad \forall\, v \in \mathrm{H}^1(T). \tag{5.44}$$

Lemma 5.14 *There exists $c > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e \left\| \widetilde{\boldsymbol{\sigma}}_{S,h}\mathbf{n} + \lambda_h \mathbf{n} + \frac{\alpha_d \mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}(\mathbf{u}_{S,h}\cdot\mathbf{t})\mathbf{t} \right\|_{0,e}^2 \le c \Big\{ \|p_S - p_{S,h}\|_{0,T}^2 + \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{1,T}^2 \\ + h_e\|\lambda - \lambda_h\|_{0,e}^2 + h_T \big\|(\nabla\mathbf{u}_S)\mathbf{u}_S - (\nabla\mathbf{u}_{S,h})\mathbf{u}_{S,h}\big\|_{\mathbf{L}^{4/3}(T)}^2 + h_T\|\mathrm{div}\,(\mathbf{u}_S - \mathbf{u}_{S,h})\mathbf{u}_{S,h}\|_{\mathbf{L}^{4/3}(T)}^2 \Big\}, \tag{5.45}$$

where T is the triangle of $\mathcal{T}_h^S$ having e as an edge.

Proof. Given $e \in \mathcal{E}_h(\Sigma)$ we let $\boldsymbol{\chi}_e := \widetilde{\boldsymbol{\sigma}}_{S,h}\mathbf{n} + \lambda_h\mathbf{n} + \frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}(\mathbf{u}_{S,h}\cdot\mathbf{t})\mathbf{t}$ on e. Then, applying (5.33) to $\|\boldsymbol{\chi}_e\|_{0,e}$, yields

$$c_2\,\|\boldsymbol{\chi}_e\|_{0,e} \le \sup_{\boldsymbol{\tau}\in\mathbf{V}_e} \frac{\displaystyle\int_e \boldsymbol{\chi}_e\cdot\psi_e\boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{0,e}}, \tag{5.46}$$

where, using the fact that $\widetilde{\boldsymbol{\sigma}}_S\mathbf{n} + \lambda\mathbf{n} + \frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}(\mathbf{u}_S\cdot\mathbf{t})\mathbf{t} = \mathbf{0}$ on Σ (cf. Theorem 5.8), recalling that $\psi_e = 0$ on $\partial T\setminus e$ (T being the triangle of $\mathcal{T}_h^S$ having e as an edge), and integrating by parts on T, we obtain that

$$\int_e \boldsymbol{\chi}_e\cdot\psi_e\boldsymbol{\tau} = \int_T (\widetilde{\boldsymbol{\sigma}}_{S,h} - \widetilde{\boldsymbol{\sigma}}_S) : \nabla(\psi_e\mathbf{L}(\boldsymbol{\tau})) + \int_T \mathbf{div}(\widetilde{\boldsymbol{\sigma}}_{S,h} - \widetilde{\boldsymbol{\sigma}}_S)\cdot\psi_e\mathbf{L}(\boldsymbol{\tau}) \\ + \int_e \Big\{(\lambda_h - \lambda)\mathbf{n} + \frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}((\mathbf{u}_{S,h} - \mathbf{u}_S)\cdot\mathbf{t})\mathbf{t}\Big\}\cdot\psi_e\boldsymbol{\tau}.$$

Hence, using again the identities $\mathbf{div}\widetilde{\boldsymbol{\sigma}}_S = \rho(\nabla\mathbf{u}_S)\mathbf{u}_S - \mathbf{f}_S$ and $\mathrm{div}\,\mathbf{u}_S = 0$ in Ω_S (cf. Theorem 5.8), and the Cauchy–Schwarz and Hölder inequalities, applying (5.35) to $\|\nabla(\psi_e\mathbf{L}(\boldsymbol{\tau}))\|_{0,T}$, noticing that $\|\psi_e\mathbf{L}(\boldsymbol{\tau})\|_{\mathbf{L}^4(T)} \le c h_T^{-1/2}\|\psi_e\mathbf{L}(\boldsymbol{\tau})\|_{0,T}$ (cf. (5.37)), and recalling that $0 \le \psi_e \le 1$, we deduce

$$\int_e \boldsymbol{\chi}_e\cdot\psi_e\boldsymbol{\tau} \le C\Big\{h_T^{-1}\|\widetilde{\boldsymbol{\sigma}}_S - \widetilde{\boldsymbol{\sigma}}_{S,h}\|_{0,T} + \Big\|\mathbf{f}_S + \mathrm{div}\,\widetilde{\boldsymbol{\sigma}}_{S,h} - \rho(\nabla\mathbf{u}_{S,h})\mathbf{u}_{S,h} - \frac{\rho}{2}\mathrm{div}\,\mathbf{u}_{S,h}\mathbf{u}_{S,h}\Big\|_{0,T} \\ + h_T^{-1/2}\big\|(\nabla\mathbf{u}_S)\mathbf{u}_S - (\nabla\mathbf{u}_{S,h})\mathbf{u}_{S,h}\big\|_{\mathbf{L}^{4/3}(T)} + h_T^{-1/2}\|\mathrm{div}\,(\mathbf{u}_S - \mathbf{u}_{S,h})\mathbf{u}_{S,h}\|_{\mathbf{L}^{4/3}(T)}\Big\}\|\psi_e\mathbf{L}(\boldsymbol{\tau})\|_{0,T} \\ + C\Big\{\|\lambda - \lambda_h\|_{0,e} + \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,e}\Big\}\|\boldsymbol{\tau}\|_{0,e}.$$

Now, from the estimate (5.36) we see that $\|\psi_e\mathbf{L}(\boldsymbol{\tau})\|_{0,T} \le c_7\,h_e^{1/2}\,\|\boldsymbol{\tau}\|_{0,e}$, which combined with the above inequality and (5.46), yields

$$\|\boldsymbol{\chi}_e\|_{0,e} \le C\,h_e^{1/2}\Big\{h_T^{-1}\|\widetilde{\boldsymbol{\sigma}}_S - \widetilde{\boldsymbol{\sigma}}_{S,h}\|_{0,T} + \Big\|\mathbf{f}_S + \mathrm{div}\,\widetilde{\boldsymbol{\sigma}}_{S,h} - \rho(\nabla\mathbf{u}_{S,h})\mathbf{u}_{S,h} - \frac{\rho}{2}\mathrm{div}\,\mathbf{u}_{S,h}\mathbf{u}_{S,h}\Big\|_{0,T} \\ + h_T^{-1/2}\big\|(\nabla\mathbf{u}_S)\mathbf{u}_S - (\nabla\mathbf{u}_{S,h})\mathbf{u}_{S,h}\big\|_{\mathbf{L}^{4/3}(T)} + h_T^{-1/2}\|\mathrm{div}\,(\mathbf{u}_S - \mathbf{u}_{S,h})\mathbf{u}_{S,h}\|_{\mathbf{L}^{4/3}(T)}\Big\} \\ + C\Big\{\|\lambda - \lambda_h\|_{0,e} + \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,e}\Big\}. \tag{5.47}$$

Thus, using that $h_e \le h_T$, applying the local trace inequality (5.44) to $\|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,e}^2$, and employing (5.38), and the definitions of $\widetilde{\boldsymbol{\sigma}}_S$ and $\widetilde{\boldsymbol{\sigma}}_{S,h}$ (cf. Theorem 5.8 and (5.2)), we conclude (5.45). □

The second and third residuals expression defining $\widehat{\Theta}_{D,T}^2$ (cf. (5.3)), that is, the one containing the nonlinear Darcy–Forchheimer term, as well as one term acting on Σ, are estimated now. To that end, we adapt the proofs of [5, Lemma 6.3] and [3, Lemma 4.12], respectively, to our context.

Lemma 5.15 *There exists $c > 0$, independent of h, such that for each $T \in \mathcal{T}_h^{\mathrm{D}}$ there holds*

$$\begin{aligned} & h_T^2 \, \|\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}\|_{0,T}^2 \\ & \quad \leq c \Big\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2 \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathbf{L}^3(T)}^2 + h_T^{4/3} \big\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \big\|_{\mathbf{L}^{3/2}(T)}^2 \Big\}. \end{aligned} \tag{5.48}$$

Proof. Given $T \in \mathcal{T}_h^{\mathrm{D}}$, we apply (5.32) to $\|\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}\|_{0,T}$, that is

$$c_1 \, \|\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}\|_{0,T} \; \leq \; \sup_{\mathbf{v} \in \mathbf{V}_T} \frac{\displaystyle\int_T (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \psi_T \mathbf{v}}{\|\mathbf{v}\|_{0,T}}, \tag{5.49}$$

from which, using the identity $\mathbf{f}_{\mathrm{D}} = \mathbf{U}_{\mathrm{D}} + \nabla p_{\mathrm{D}}$ in Ω_{D} (cf. Theorem 5.8), noting that $\nabla p_{\mathrm{D},h} = 0$ on T, and integrating by parts, we find that

$$\begin{aligned} \int_T (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \psi_T \mathbf{v} &= \int_T \Big\{ \nabla(p_{\mathrm{D}} - p_{\mathrm{D},h}) + (\mathbf{U}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \Big\} \cdot \psi_T \mathbf{v} \\ &= - \int_T (p_{\mathrm{D}} - p_{\mathrm{D},h}) \mathrm{div}\,(\psi_T \mathbf{v}) + \int_T (\mathbf{U}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \psi_T \mathbf{v}. \end{aligned}$$

In this way, from the definitions of $\mathbf{U}_{\mathrm{D}}$ and $\mathbf{U}_{\mathrm{D},h}$ (cf. Theorem 5.8 and (5.5)), using the Cauchy–Schwarz and Hölder inequalities, applying (5.34) to $\|\nabla(\psi_T \mathbf{v})\|_{0,T}$, and recalling that $0 \leq \psi_T \leq 1$, we get

$$\begin{aligned} \int_T (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \psi_T \mathbf{v} \; &\leq \; C \Big\{ h_T^{-1} \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T} + \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T} \Big\} \|\mathbf{v}\|_{0,T} \\ & \quad + \big\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \big\|_{\mathbf{L}^{3/2}(T)} \|\mathbf{v}\|_{\mathbf{L}^3(T)}. \end{aligned} \tag{5.50}$$

Then, replacing (5.50) back into (5.49), and then applying Hölder's inequality and the local inverse estimate $\|\mathbf{v}\|_{\mathbf{L}^3(T)} \leq c \, h_T^{-1/3} \, \|\mathbf{v}\|_{0,T}$ (cf. (5.37)), we arrive at (5.48) and complete the proof. □

Lemma 5.16 *Assume that $p_{\mathrm{D}}|_T \in \mathrm{H}^1(T)$ for each $T \in \mathcal{T}_h^{\mathrm{D}}$. Then there exists $c > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$\begin{aligned} h_e \, \|\lambda_h - p_{\mathrm{D},h}\|_{0,e}^2 &\leq c \Big\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2 \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathbf{L}^3(T)}^2 + h_e \|\lambda - \lambda_h\|_{0,e}^2 \\ & \quad + h_T^{4/3} \big\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \big\|_{\mathbf{L}^{3/2}(T)}^2 + h_T^2 \big\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \big\|_{0,T}^2 \Big\}, \end{aligned} \tag{5.51}$$

where T is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge.

Proof. Since $p_{\mathrm{D},h} := p_h|_{\Omega_{\mathrm{D}}}$ is piecewise constant and $\nabla p_{\mathrm{D}} = \mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D}}$ in Ω_{D} (cf. Theorem 5.8), we deduce that for each $T \subseteq \omega_e$ there holds

$$\begin{aligned} h_T^2 |p_{\mathrm{D}} - p_{\mathrm{D},h}|_{1,T}^2 &= h_T^2 \|\nabla p_{\mathrm{D}}\|_{0,T}^2 = h_T^2 \|\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D}}\|_{0,T}^2 \\ &\leq 2 \Big\{ h_T^2 \|\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2 \|\mathbf{U}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}\|_{0,T}^2 \Big\}. \end{aligned} \tag{5.52}$$

Next, using that $\lambda = p_{\mathrm{D}}$ on Σ for each $e \in \mathcal{E}_h(\Sigma)$ (cf. Theorem 5.8), and employing the local trace inequality (5.44), we obtain

$$\begin{aligned} h_e \|\lambda_h - p_{\mathrm{D},h}\|_{0,e}^2 &\leq 2 \, h_e \Big\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,e}^2 + \|\lambda - \lambda_h\|_{0,e}^2 \Big\} \\ &\leq C \Big\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2 |p_{\mathrm{D}} - p_{\mathrm{D},h}|_{1,T}^2 + h_e \|\lambda - \lambda_h\|_{0,e}^2 \Big\}. \end{aligned}$$

In this way, combining the foregoing inequality with (5.52), recalling the definitions of $\mathbf{U}_{\rm D}$ and $\mathbf{U}_{{\rm D},h}$ (cf. Theorem 5.8 and (5.5)), and using (5.48), we are lead to (5.51), thus concluding the proof. □

Now we turn to provide the corresponding estimates for the three terms defining $\widetilde{\Theta}_{{\rm D},T}^{3/2}$ (cf. (5.4)). To that end, we adapt the proofs of [5, Lemmas 6.1, 6.2] and [19, Lemma 20], respectively, to our present spaces configuration.

Lemma 5.17 *There exists $c > 0$, independent of h, such that for each $T \in \mathcal{T}_h^{\rm D}$ there holds*

$$h_T^{3/2} \big\| \text{rot } (\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \big\|_{{\rm L}^{3/2}(T)}^{3/2} \,\leq\, c \Big\{ \|\mathbf{u}_{\rm D} - \mathbf{u}_{{\rm D},h}\|_{\mathbf{L}^3(T)}^{3/2} + \big\| |\mathbf{u}_{\rm D}|\mathbf{u}_{\rm D} - |\mathbf{u}_{{\rm D},h}|\mathbf{u}_{{\rm D},h} \big\|_{\mathbf{L}^{3/2}(T)}^{3/2} \Big\}. \tag{5.53}$$

Proof. Given $T \in \mathcal{T}_h^{\rm D}$, we begin by applying (5.32) to $\big\| \text{rot } (\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \big\|_{{\rm L}^{3/2}(T)}$, which gives

$$c_1 \big\| \text{rot } (\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \big\|_{{\rm L}^{3/2}(T)} \,\leq\, \sup_{v \in V_T} \frac{\displaystyle\int_T \text{rot } (\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h})\, \psi_T v}{\|v\|_{{\rm L}^3(T)}} \,. \tag{5.54}$$

Then, using that $\mathbf{f}_{\rm D} = \nabla p_{\rm D} + \mathbf{U}_{\rm D}$ in $\Omega_{\rm D}$ (cf. Theorem 5.8), and integrating by parts, we readily find that

$$\int_T \text{rot } (\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h})\, \psi_T v = \int_T \text{rot } (\mathbf{U}_{\rm D} - \mathbf{U}_{{\rm D},h})\, \psi_T v = \int_T (\mathbf{U}_{\rm D} - \mathbf{U}_{{\rm D},h}) \cdot \mathbf{curl}\, (\psi_T v) \,,$$

which, together with (5.34) applied to $\|\nabla(\psi_T v)\|_{\mathbf{L}^3(T)}$ and the fact that $0 \leq \psi_T \leq 1$, implies from (5.54) that

$$\|\text{rot } (\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h})\|_{{\rm L}^{3/2}(T)} \leq C\, h_T^{-1}\, \|\mathbf{U}_{\rm D} - \mathbf{U}_{{\rm D},h}\|_{\mathbf{L}^{3/2}(T)}.$$

Then, according to the definitions of $\mathbf{U}_{\rm D}$ and $\mathbf{U}_{{\rm D},h}$ (cf. Theorem 5.8 and (5.5)), and using the triangle and Hölder inequalities, we deduce (5.53) and conclude the proof. □

Lemma 5.18 *There exists $c > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Omega_{\rm D})$ there holds*

$$h_e \big\| [\![(\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \cdot \mathbf{t}]\!] \big\|_{{\rm L}^{3/2}(e)}^{3/2} \leq C \sum_{T \subseteq \omega_e} \Big\{ \|\mathbf{u}_{\rm D} - \mathbf{u}_{{\rm D},h}\|_{\mathbf{L}^3(T)}^{3/2} + \big\| |\mathbf{u}_{\rm D}|\mathbf{u}_{\rm D} - |\mathbf{u}_{{\rm D},h}|\mathbf{u}_{{\rm D},h} \big\|_{\mathbf{L}^{3/2}(T)}^{3/2} \Big\}, \tag{5.55}$$

where ω_e is the union of the two triangles in $\mathcal{T}_h^{\rm D}$ having e as an edge.

Proof. Given $e \in \mathcal{E}_h(\Omega_{\rm D})$, we first apply (5.33) to $\big\| [\![(\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \cdot \mathbf{t}]\!] \big\|_{{\rm L}^{3/2}(e)}$, which gives

$$c_2 \big\| [\![(\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \cdot \mathbf{t}]\!] \big\|_{{\rm L}^{3/2}(e)} \,\leq\, \sup_{\tau \in V_e} \frac{\displaystyle\int_e [\![(\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \cdot \mathbf{t}]\!]\, \psi_e L(\tau)}{\|\tau\|_{{\rm L}^3(e)}} \,. \tag{5.56}$$

Then, integrating by parts on each $T \subseteq \omega_e$, using the fact that $\mathbf{f}_{\rm D} = \nabla p_{\rm D} + \mathbf{U}_{\rm D}$ in $\Omega_{\rm D}$ (cf. Theorem 5.8), similarly to Lemma 5.13, and applying the estimates (5.35) and (5.36) to $\|\nabla(\psi_e L(\tau))\|_{\mathbf{L}^3(T)}$, we find that

$$\begin{aligned}
\int_e [\![(\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \cdot \mathbf{t}]\!]\, \psi_e L(\tau) &= \sum_{T \subseteq \omega_e} \Big\{ \int_T (\mathbf{U}_{\rm D} - \mathbf{U}_{{\rm D},h}) \cdot \mathbf{curl}\, (\psi_e L(\tau)) + \int_T \psi_e L(\tau)\, \text{rot } (\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \Big\} \\
&\leq C\, h_e^{1/3} \sum_{T \subseteq \omega_e} \Big\{ h_T^{-1}\, \|\mathbf{U}_{\rm D} - \mathbf{U}_{{\rm D},h}\|_{\mathbf{L}^{3/2}(T)} + \big\| \text{rot } (\mathbf{f}_{\rm D} - \mathbf{U}_{{\rm D},h}) \big\|_{{\rm L}^{3/2}(T)} \Big\} \|\tau\|_{{\rm L}^3(e)} \,,
\end{aligned}$$

which, replaced back into (5.56), and after using (5.53), Hölder's inequality and the fact that $h_e \leq h_T$ for each $T \subseteq \omega_e$, yields (5.55) and concludes the proof. □

Lemma 5.19 *There exists $c > 0$, independent of h, such that*

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\| (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \mathbf{t} \right\|^{3/2}_{\mathrm{L}^{3/2}(e)} \\ \leq C \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \left(\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|^{3/2}_{\mathbf{L}^3(T_e)} + \left\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \right\|^{3/2}_{\mathbf{L}^{3/2}(T_e)} \right) + \|\lambda - \lambda_h\|^{3/2}_{1/3,3/2;\Sigma} \right\}, \tag{5.57}$$

where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge.

Proof. Given $e \in \mathcal{E}_h(\Sigma)$, the application of (5.33) to $\left\| (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \mathbf{t} \right\|_{\mathrm{L}^{3/2}(e)}$ gives

$$c_2 \left\| (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \mathbf{t} \right\|_{\mathrm{L}^{3/2}(e)} \leq \sup_{\tau\in V_e} \frac{\displaystyle\int_e \left\{ (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \mathbf{t} \right\} \psi_e \tau}{\|\tau\|_{\mathrm{L}^3(e)}}. \tag{5.58}$$

Next, proceeding similarly to Lemma 5.18, using the extension operator L, integrating by parts on the right-hand side of (5.58), employing the identities $\mathbf{f}_{\mathrm{D}} = \nabla p_{\mathrm{D}} + \mathbf{U}_{\mathrm{D}}$ in Ω_{D} and $\lambda = p_{\mathrm{D}}$ on Σ (cf. Theorem 5.8), noting that λ_h is piecewise constant on Σ, and then summing up over all $e \in \mathcal{E}_h(\Sigma)$, we deduce from (5.58) that

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\| (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \mathbf{t} \right\|^{3/2}_{\mathrm{L}^{3/2}(e)} \leq C \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \left(\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|^{3/2}_{\mathbf{L}^3(T_e)} \right.\right. \\ \left.\left. + \left\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \right\|^{3/2}_{\mathbf{L}^{3/2}(T_e)} \right) + \sum_{e\in\mathcal{E}_h(\Sigma)} \left(\sup_{\tau\in V_e} \frac{h_e^{2/3}}{\|\tau\|_{\mathrm{L}^3(e)}} \left\langle \frac{d}{d\mathbf{t}}(\lambda - \lambda_h), \psi_e \tau \right\rangle_e \right)^{3/2} \right\}, \tag{5.59}$$

where $\langle \cdot, \cdot \rangle_e$ stands for the duality pairing between $\left(\mathrm{W}^{2/3,3}_{00}(e)\right)'$ and $\mathrm{W}^{2/3,3}_{00}(e)$. Here, $\mathrm{W}^{2/3,3}_{00}(e)$ denotes the space of traces on e of those elements in $\mathrm{W}^{1,3}(T_e)$ whose traces vanish on $\partial T_e \setminus e$. Now, analogously to [19, Lemma 20], and since $\psi_e \tau \in \mathrm{W}^{2/3,3}_{00}(e)$ for each $e \in \mathcal{E}_h(\Sigma)$, we see that the third term on the right-hand side of (5.59) can be bounded by

$$\left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \sup_{\tau\in V_e} \frac{h_e^{2/3}}{\|\tau\|_{\mathrm{L}^3(e)}} \left\langle \frac{d}{d\mathbf{t}}(\lambda - \lambda_h), \psi_e \tau \right\rangle_e \right\}^{3/2} \leq \left\{ \left\langle \frac{d}{d\mathbf{t}}(\lambda - \lambda_h), \widehat{\tau} \right\rangle_\Sigma \right\}^{3/2},$$

where $\widehat{\tau}|_e := h_e^{2/3} \psi_e \widetilde{\tau}_e$ on each $e \in \mathcal{E}_h(\Sigma)$, with $\|\widetilde{\tau}_e\|_{\mathrm{L}^3(e)} = 1$. Then, applying the boundedness of the tangential derivative $\frac{d}{d\mathbf{t}} : \mathrm{W}^{1/3,3/2}(\Sigma) \to \mathrm{W}^{-2/3,3/2}(\Sigma)$ (see [23, Section I.1.5]), the inverse estimate $\|\widehat{\tau}\|_{2/3,3;\Sigma} \leq c h^{-2/3} \|\widehat{\tau}\|_{\mathrm{L}^3(\Sigma)}$ [15, Corollary 1.141], and the fact that $h_e \leq h$ and $0 \leq \psi_e \leq 1$, we find that

$$\sum_{e\in\mathcal{E}_h(\Sigma)} \left(\sup_{\tau\in V_e} \frac{h_e^{2/3}}{\|\tau\|_{\mathrm{L}^3(e)}} \left\langle \frac{d}{d\mathbf{t}}(\lambda - \lambda_h), \psi_e \tau \right\rangle_e \right)^{3/2} \leq C \|\lambda - \lambda_h\|^{3/2}_{1/3,3/2;\Sigma} \left(h^{-2/3} \|\widehat{\tau}\|_{\mathrm{L}^3(\Sigma)} \right)^{3/2} \\ \leq C \|\lambda - \lambda_h\|^{3/2}_{1/3,3/2;\Sigma} \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \|\widetilde{\tau}_e\|^3_{\mathrm{L}^3(e)} \right\}^{1/2} \leq C \|\lambda - \lambda_h\|^{3/2}_{1/3,3/2;\Sigma},$$

which, combined with (5.59), leads to (5.57) and completes the proof. □

Finally, we provide the upper bound for the term defining $\Theta^3_{\Sigma,e}$ (cf. (5.6)).

Lemma 5.20 *There exists $c > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$\begin{aligned} h_e \, \|\mathbf{u}_{\mathrm{S},h} \cdot \mathbf{n} - \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n}\|^3_{\mathrm{L}^3(e)} \,\leq\, c \Big\{ \|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|^3_{\mathbf{L}^3(T_\mathrm{S})} + h^2_{T_\mathrm{S}} |\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}|^3_{1,T_\mathrm{S}} \\ + \|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|^3_{\mathbf{L}^3(T_\mathrm{D})} + h^2_{T_\mathrm{D}} \|\mathrm{div}\,(\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h})\|^3_{0,T_\mathrm{D}} \Big\}, \end{aligned} \tag{5.60}$$

where T_S and T_D are the triangles of $\mathcal{T}_h^\mathrm{S}$ and $\mathcal{T}_h^\mathrm{D}$, respectively, having e as an edge.

Proof. Given $e \in \mathcal{E}_h(\Sigma)$, we let T_S and T_D be the triangles of $\mathcal{T}_h^\mathrm{S}$ and $\mathcal{T}_h^\mathrm{D}$, respectively, having e as an edge, which means that $\omega_e := T_\mathrm{S} \cup T_\mathrm{D}$, and define $\chi_e := \mathbf{u}_{\mathrm{S},h} \cdot \mathbf{n} - \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n}$ on e. Then, applying (5.33) to $\|\chi_e\|_{\mathrm{L}^3(e)}$, we have

$$c_2 \, \|\chi_e\|_{\mathrm{L}^3(e)} \,\leq\, \sup_{\tau \in V_e} \frac{\displaystyle\int_e \chi_e \, \psi_e \tau}{\|\tau\|_{\mathrm{L}^{3/2}(e)}}, \tag{5.61}$$

Next, setting $\psi_{e,\star} := \psi_e|_{T_\star}$, with $\star \in \{\mathrm{S}, \mathrm{D}\}$, using the identity $\mathbf{u}_\mathrm{D} \cdot \mathbf{n} = \mathbf{u}_\mathrm{S} \cdot \mathbf{n}$ on Σ (cf. Theorem 5.8), recalling that $\psi_{e,\star} = 0$ on $\partial T_\star \setminus e$, and integrating by parts on $T_\star$, we obtain

$$\begin{aligned} \int_e \chi_e \, \psi_e \tau = \int_{T_\mathrm{S}} (\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}) \cdot \nabla(\psi_{e,\mathrm{S}} L(\tau)) + \int_{T_\mathrm{S}} \psi_{e,\mathrm{S}} L(\tau) \,\mathrm{div}\,(\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}) \\ + \int_{T_\mathrm{D}} (\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}) \cdot \nabla(\psi_{e,\mathrm{D}} L(\tau)) + \int_{T_\mathrm{D}} \psi_{e,\mathrm{D}} L(\tau) \,\mathrm{div}\,(\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}). \end{aligned}$$

Thus, using the Cauchy–Schwarz and Hölder inequalities, applying (5.35) to $\|\nabla(\psi_{e,\star} L(\tau))\|_{\mathbf{L}^{3/2}(T_\star)}$, and utilizing the local inverse estimate $\|\psi_{e,\star} L(\tau)\|_{0,T_\star} \leq c\, h_{T_\star}^{-1/3} \|\psi_{e,\star} L(\tau)\|_{\mathrm{L}^{3/2}(T_\star)}$ (cf. (5.37)), and the fact that $0 \leq \psi_e \leq 1$ in ω_e, we find that

$$\begin{aligned} \int_e \chi_e \, \psi_e \tau \leq C \Big\{ h_{T_\mathrm{S}}^{-1} \|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|_{\mathbf{L}^3(T_\mathrm{S})} + h_{T_\mathrm{S}}^{-1/3} \|\mathrm{div}\,(\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h})\|_{0,T_\mathrm{S}} \Big\} \|\psi_{e,\mathrm{S}} L(\tau)\|_{\mathrm{L}^{3/2}(T_\mathrm{S})} \\ + C \Big\{ h_{T_\mathrm{D}}^{-1} \|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|_{\mathbf{L}^3(T_\mathrm{D})} + h_{T_\mathrm{D}}^{-1/3} \|\mathrm{div}\,(\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h})\|_{0,T_\mathrm{D}} \Big\} \|\psi_{e,\mathrm{D}} L(\tau)\|_{\mathrm{L}^{3/2}(T_\mathrm{D})} \,. \end{aligned} \tag{5.62}$$

Finally, applying now (5.36) to $\|\psi_{e,\star} L(\tau)\|_{\mathrm{L}^{3/2}(T_\star)}$, combining the resulting estimate with (5.62) and (5.61), and using that $h_e \leq h_{T_\star}$, we arrive at (5.60) and conclude the proof. □

In order to complete the global efficiency given by (5.30) (cf. Theorem 5.7), we now need to estimate the terms $\|\lambda - \lambda_h\|^2_{0,e}$, $\|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|^3_{\mathbf{L}^3(T)}$, $\|\mathrm{div}\,(\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}) \mathbf{u}_{\mathrm{S},h}\|^2_{\mathbf{L}^{4/3}(T)}$, $\big\|(\nabla \mathbf{u}_\mathrm{S})\mathbf{u}_\mathrm{S} - (\nabla \mathbf{u}_{\mathrm{S},h})\mathbf{u}_{\mathrm{S},h}\big\|^2_{\mathbf{L}^{4/3}(T)}$, and $\big\||\mathbf{u}_\mathrm{D}|\mathbf{u}_\mathrm{D} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h}\big\|^{3/2}_{\mathbf{L}^{3/2}(T)}$ appearing in the upper bounds provided by Lemmas 5.12–5.20. To this end, we first recall that $\mathrm{W}^{1/3,3/2}(\Sigma)$ is continuously embedded into $\mathrm{L}^2(\Sigma)$, whence

$$\sum_{e \in \mathcal{E}_h(\Sigma)} \|\lambda - \lambda_h\|^2_{0,e} \,\leq\, \|\lambda - \lambda_h\|^2_{0,\Sigma} \,\leq\, C \, \|\lambda - \lambda_h\|^2_{1/3,3/2;\Sigma} \,. \tag{5.63}$$

In turn, we make use of the continuity of the injection $\mathbf{i} : \mathbf{H}^1(\Omega_\mathrm{S}) \to \mathbf{L}^3(\Omega_\mathrm{S})$ to obtain

$$\sum_{T \in \mathcal{T}_h^\mathrm{S}} \|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|^3_{\mathbf{L}^3(T)} \,=\, \|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|^3_{\mathbf{L}^3(\Omega_\mathrm{S})} \,\leq\, C \, \|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|^3_{1,\Omega_\mathrm{S}}. \tag{5.64}$$

In addition, applying Hölder's inequality, the continuous injection $\mathbf{i} : \mathbf{H}^1(\Omega_S) \to \mathbf{L}^4(\Omega_S)$, and the a priori bounds of $\|\mathbf{u}_{S,h}\|_{1,\Omega_S}$ (cf. (5.15)) combined with (5.9), we deduce that

$$\begin{aligned} \sum_{T\in\mathcal{T}_h^S} \|\mathrm{div}\,(\mathbf{u}_S - \mathbf{u}_{S,h})\mathbf{u}_{S,h}\|^2_{\mathbf{L}^{4/3}(T)} &\le \sum_{T\in\mathcal{T}_h^S} \|\mathbf{u}_{S,h}\|^2_{\mathbf{L}^4(T)}\|\mathrm{div}\,(\mathbf{u}_S - \mathbf{u}_{S,h})\|^2_{0,T} \\ &\le C\,\|\mathbf{u}_{S,h}\|^2_{1,\Omega_S}\|\mathbf{u}_S - \mathbf{u}_{S,h}\|^2_{1,\Omega_S} \le C\,\|\mathbf{u}_S - \mathbf{u}_{S,h}\|^2_{1,\Omega_S}\,. \end{aligned} \tag{5.65}$$

Similarly, adding and subtracting suitable terms, applying Hölder's inequality, and using the continuous injection $\mathbf{i} : \mathbf{H}^1(\Omega_S) \to \mathbf{L}^4(\Omega_S)$, and the a priori bounds of $\|\mathbf{u}_S\|_{1,\Omega_S}$ and $\|\mathbf{u}_{S,h}\|_{1,\Omega_S}$ (cf. (5.15)) combined with (5.9), we are able to show that

$$\sum_{T\in\mathcal{T}_h^S} \big\|(\nabla\mathbf{u}_S)\mathbf{u}_S - (\nabla\mathbf{u}_{S,h})\mathbf{u}_{S,h}\big\|^2_{\mathbf{L}^{4/3}(T)} \le C\,\|\mathbf{u}_S - \mathbf{u}_{S,h}\|^2_{1,\Omega_S}. \tag{5.66}$$

Finally, applying again the Cauchy–Schwarz inequality, and the a priori bounds of $\|\mathbf{u}_D\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)}$ and $\|\mathbf{u}_{D,h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)}$ (cf. (5.15)) combined with (5.9), we find that

$$\begin{aligned} \sum_{T\in\mathcal{T}_h^D} \big\||\mathbf{u}_D|\mathbf{u}_D - |\mathbf{u}_{D,h}|\mathbf{u}_{D,h}\big\|^{3/2}_{\mathbf{L}^{3/2}(T)} &\le C \sum_{T\in\mathcal{T}_h^D} \Big\{\|\mathbf{u}_D\|^{3/2}_{\mathbf{L}^3(T)} + \|\mathbf{u}_{D,h}\|^{3/2}_{\mathbf{L}^3(T)}\Big\}\|\mathbf{u}_D - \mathbf{u}_{D,h}\|^{3/2}_{\mathbf{L}^3(T)} \\ \le C\Big\{\|\mathbf{u}_D\|^{3/2}_{\mathbf{L}^3(\Omega_D)} + \|\mathbf{u}_{D,h}\|^{3/2}_{\mathbf{L}^3(\Omega_D)}\Big\}&\|\mathbf{u}_D - \mathbf{u}_{D,h}\|^{3/2}_{\mathbf{L}^3(\Omega_D)} \le C\,\|\mathbf{u}_D - \mathbf{u}_{D,h}\|^{3/2}_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)}. \end{aligned} \tag{5.67}$$

Consequently, it is not difficult to see that (5.30) follows from the definition of Θ (cf. (5.7)), Lemmas 5.9, 5.12–5.20, and the estimates (5.63)–(5.67). Furthermore, proceeding similarly to (5.67) and employing that $\|\mathbf{u}_{D,h}\|_{\mathbf{L}^6(T)} \le c\,h_T^{-1/3}\|\mathbf{u}_{D,h}\|_{\mathbf{L}^3(T)}$, which follows from the inverse inequality (5.37), we obtain

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h^D} h_T^2\,\big\||\mathbf{u}_D|\mathbf{u}_D - |\mathbf{u}_{D,h}|\mathbf{u}_{D,h}\big\|^2_{0,T} \\ &\quad\le C \sum_{T\in\mathcal{T}_h^D} h_T^2\Big\{\|\mathbf{u}_D\|^2_{\mathbf{L}^6(T)} + h_T^{-2/3}\,\|\mathbf{u}_{D,h}\|^2_{\mathbf{L}^3(T)}\Big\}\|\mathbf{u}_D - \mathbf{u}_{D,h}\|^2_{\mathbf{L}^3(T)} \\ &\quad\le C \sum_{T\in\mathcal{T}_h^D} h_T^{4/3}\Big\{\|\mathbf{u}_D\|^2_{\mathbf{L}^6(T)} + \|\mathbf{u}_{D,h}\|^2_{\mathbf{L}^3(T)}\Big\}\|\mathbf{u}_D - \mathbf{u}_{D,h}\|^2_{\mathbf{L}^3(T)} \\ &\quad\le C\Big\{\|\mathbf{u}_D\|^2_{\mathbf{L}^6(\Omega_D)} + \|\mathbf{u}_{D,h}\|^2_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)}\Big\}\,\|\mathbf{u}_D - \mathbf{u}_{D,h}\|^2_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)}\,, \end{aligned} \tag{5.68}$$

from which, using the a priori bound of $\|\mathbf{u}_{D,h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)}$ (cf. (5.15)) and the hypothesis on $\mathbf{u}_D$ stated in Theorem 5.7, it easily follows (5.31), thus concluding the proof of this theorem. We stress here that requiring $\mathbf{u}_D \in \mathbf{L}^6(\Omega_D)$ is coherent with Lemma 5.16 in the sense that, under the assumption that $\mathbf{f}_D \in \mathbf{L}^2(\Omega_D)$, and due to the identity $\nabla p_D = \mathbf{f}_D - \mathbf{U}_D$ in Ω_D (cf. Theorem 5.8), there holds $p_D|_T \in \mathrm{H}^1(T)$ for each $T \in \mathcal{T}_h^D$.

6 A posteriori error analysis: The fully-mixed approach

In what follows we assume that the hypotheses from [10, Theorems 4.10 and 5.3] hold. Let $\vec{\mathbf{t}} := ((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), p_D) \in \mathbb{H}\times\mathbb{Q}$ and $\vec{\mathbf{t}}_h := ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), p_{D,h}) \in \mathbb{H}_h\times\mathbb{Q}_h$ be the unique solutions of problems (3.8) and (3.9), respectively. Then, we define for each $T \in \mathcal{T}_h^S$ the local error indicators

$$\begin{aligned} \Psi^2_{S,T} := &\ \|\mathbf{f}_S + \mathbf{div}\boldsymbol{\sigma}_{S,h}\|^2_{0,T} + \|\boldsymbol{\sigma}^{\mathrm{d}}_{S,h} - 2\mu\mathbf{e}(\mathbf{u}_{S,h}) + \rho(\mathbf{u}_{S,h}\otimes\mathbf{u}_{S,h})^{\mathrm{d}}\|^2_{0,T} + \|\boldsymbol{\sigma}_{S,h} - \boldsymbol{\sigma}^{\mathrm{t}}_{S,h}\|^2_{0,T} \\ &+ \Big\|\gamma_{S,h} - \frac{1}{2}\Big(\nabla\mathbf{u}_{S,h} - (\nabla\mathbf{u}_{S,h})^{\mathrm{t}}\Big)\Big\|^2_{0,T} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\|\boldsymbol{\sigma}_{S,h}\mathbf{n} + \omega_1^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t} + \lambda_h\mathbf{n}\|^2_{0,e} \end{aligned} \tag{6.1}$$

and

$$\begin{aligned}\widehat{\Psi}_{\mathrm{S},T}^2 \;:=\; & \Psi_{\mathrm{S},T}^2 \,+\, \|\mathbf{f}_\mathrm{S} - \mathbf{P}_h^\mathrm{S}(\mathbf{f}_\mathrm{S})\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_\mathrm{S})} h_e \left\| [\![(\mathbf{e}(\mathbf{u}_{\mathrm{S},h}) + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t}]\!] \right\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_\mathrm{S})} h_e \|(\mathbf{e}(\mathbf{u}_{\mathrm{S},h}) + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t}\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\|\boldsymbol{\varphi}_h + \mathbf{u}_{\mathrm{S},h}\|_{0,e}^2 + h_e\left\|(\mathbf{e}(\mathbf{u}_{\mathrm{S},h}) + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t} + \frac{d\boldsymbol{\varphi}_h}{d\mathbf{t}}\right\|_{0,e}^2 \right\}.\end{aligned} \tag{6.2}$$

In addition, for each $e \in \mathcal{E}_h(\Sigma)$ we define

$$\Psi_{\Sigma,e}^3 := h_e\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{\mathrm{L}^3(e)}^3, \tag{6.3}$$

and for each $T \in \mathcal{T}_h^\mathrm{D}$ we consider $\widehat{\Theta}_{\mathrm{D},T}^2$ and $\widetilde{\Theta}_{\mathrm{D},T}^{3/2}$ as in (5.3) and (5.4), respectively, so that the global a posteriori error estimates are given, respectively, by:

$$\Psi_1 \;:=\; \left\{ \sum_{T\in\mathcal{T}_h^\mathrm{S}} \Psi_{\mathrm{S},T}^2 + \sum_{T\in\mathcal{T}_h^\mathrm{D}} \widehat{\Theta}_{\mathrm{D},T}^2 + \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}^2 \right\}^{1/2} + \left\{ \sum_{T\in\mathcal{T}_h^\mathrm{D}} \widetilde{\Theta}_{\mathrm{D},T}^{3/2} \right\}^{2/3} + \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \Psi_{\Sigma,e}^3 \right\}^{1/3} \tag{6.4}$$

and

$$\Psi_2 \;:=\; \left\{ \sum_{T\in\mathcal{T}_h^\mathrm{S}} \widehat{\Psi}_{\mathrm{S},T}^2 + \sum_{T\in\mathcal{T}_h^\mathrm{D}} \widehat{\Theta}_{\mathrm{D},T}^2 \right\}^{1/2} + \left\{ \sum_{T\in\mathcal{T}_h^\mathrm{D}} \widetilde{\Theta}_{\mathrm{D},T}^{3/2} \right\}^{2/3} + \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \Psi_{\Sigma,e}^3 \right\}^{1/3}. \tag{6.5}$$

Now, similarly to the Section 5, and under suitable assumptions, we will focus on establishing the existence of positive constants $C_\mathrm{rel}, c_\mathrm{rel}, C_\mathrm{eff}$ and c_eff, independent of the meshsizes and the continuous and discrete solutions, such that

$$C_\mathrm{eff}\,\Psi_1 + \text{h.o.t.} \;\le\; \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbb{Q}} \;\le\; C_\mathrm{rel}\,\Psi_1^{1/2} \tag{6.6}$$

and

$$c_\mathrm{eff}\,\Psi_2 + \text{h.o.t.} \;\le\; \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbb{Q}} \;\le\; c_\mathrm{rel}\,\Psi_2^{1/2}, \tag{6.7}$$

where h.o.t. stands, eventually, for one or several terms of higher order. The upper and lower bounds in (6.6) and (6.7), which are known as the reliability of $\Psi_1^{1/2}$ and $\Psi_2^{1/2}$, and efficiency of Ψ_1 and Ψ_2, are derived below in Sections 6.1 and 6.2, respectively.

6.1 Reliability

Similarly to 5.1, we recall from [10] the following notation

$$\begin{aligned}\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) := \max\Big\{&\mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})^{1/8}, \mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})^{1/4}, \mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})^{1/2}, \\ &\mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}), \mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})^2, \mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})^4\Big\},\end{aligned}$$

where $\mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) := \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{\mathbf{L}^{3/2}(\Omega_\mathrm{D})} + \|g_\mathrm{D}\|_{0,\Omega_\mathrm{D}} + \|g_\mathrm{D}\|_{0,\Omega_\mathrm{D}}^2$. Next, we provide the main result of this section, whose proof follows analogously to Theorem 5.1.

Theorem 6.1 *Assume that Ω_D is a connected domain and that Γ_D is contained in the boundary of a convex part of Ω_D, that is there exists a convex domain Ξ such that $\overline{\Omega}_\mathrm{D} \subseteq \Xi$ and $\Gamma_\mathrm{D} \subseteq \partial\Xi$. In addition, assume that the data $\mathbf{f}_\mathrm{S}$, $\mathbf{f}_\mathrm{D}$ and g_D, satisfy:*

$$\widetilde{c}_\mathbf{T}\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) \;\le\; r, \quad \text{and} \quad c_\mathbf{T}\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) \;\le\; \frac{r}{2} \tag{6.8}$$

with $r \in (0, r_0)$*, where* r_0*,* $c_{\mathbf{T}}$*, and* $\widetilde{c}_{\mathbf{T}}$ *are the positive constants, independent of the data, provided by [10, Lemmas 4.3, 4.5 and 5.2], respectively. Assume further that* $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$*. Then, there exist constants* $C_{\mathrm{rel}}, c_{\mathrm{rel}} > 0$*, independent of* h*, such that*

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbb{Q}} \,\le\, C_{\mathrm{rel}}\,\Psi_1^{1/2} \tag{6.9}$$

and

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbb{Q}} \,\le\, c_{\mathrm{rel}}\,\Psi_2^{1/2}. \tag{6.10}$$

We begin the proof of (6.9)–(6.10) by noticing first that $[\mathbf{B}(\underline{\mathbf{r}}), p_{\mathrm{D},h}] = [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), p_{\mathrm{D},h}]$ and that $\mathbf{F}$ can be decomposed as $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, with $[\mathbf{F}_1, \underline{\mathbf{r}}] = [\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})]$ and $[\mathbf{F}_2, \underline{\boldsymbol{\psi}}] = \mathbf{0}$. Thus, we define the residues $\mathcal{R}_{\mathbf{F}_1}$, $\mathcal{R}_{\mathbf{F}_2}$ and $\mathcal{R}_{\mathbf{G}}$ on $\mathbf{X}$, $\mathbf{Y}$ and $\mathbb{Q}$, respectively, by

$$\mathcal{R}_{\mathbf{F}_1}(\underline{\mathbf{r}}) \,:=\, [\mathbf{F}_1, \underline{\mathbf{r}}] - \Big\{[\mathbf{a}(\mathbf{u}_{\mathrm{S},h})(\underline{\mathbf{t}}_h), \underline{\mathbf{r}}] + [\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\varphi}}_h] + [\mathbf{B}(\underline{\mathbf{r}}), p_{\mathrm{D},h}]\Big\} \quad \forall\, \underline{\mathbf{r}} \in \mathbf{X}, \tag{6.11}$$

$$\mathcal{R}_{\mathbf{F}_2}(\underline{\boldsymbol{\psi}}) \,:=\, -\Big\{[\mathbf{b}(\underline{\mathbf{t}}_h), \underline{\boldsymbol{\psi}}] - [\mathbf{c}(\underline{\boldsymbol{\varphi}}_h), \underline{\boldsymbol{\psi}}]\Big\} \quad \forall\, \underline{\boldsymbol{\psi}} \in \mathbf{Y}, \tag{6.12}$$

and

$$\mathcal{R}_{\mathbf{G}}(q_{\mathrm{D}}) \,:=\, [\mathbf{G}, q_{\mathrm{D}}] - [\mathbf{B}(\underline{\mathbf{t}}_h), q_{\mathrm{D}}] \quad \forall\, q_{\mathrm{D}} \in \mathbb{Q}. \tag{6.13}$$

Then, proceeding as in Lemma 5.2 and [10, Theorem 5.3], and employing the strict monotonicity and continuity of the operator $\mathbf{A}$, the positive semidefinite of $\mathbf{c}$, and the inf-sup conditions of the operators $\mathbf{b}$ and $\mathbf{B}$, we are able to establish the following preliminary a posteriori error estimate.

Lemma 6.2 *Assume that the data* $\mathbf{f}_{\mathrm{S}}$*,* $\mathbf{f}_{\mathrm{D}}$ *and* g_{D} *satisfy* (6.8)*. Then, there exists a constant* $C > 0$*, depending only on parameters and other constants, all them independent of* h*, such that*

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbb{Q}} \,\le\, C\,\max\Big\{\|\mathcal{R}\|_{(\mathbb{H}\times\mathbb{Q})'}^{1/2}, \|\mathcal{R}\|_{(\mathbb{H}\times\mathbb{Q})'}^{2/3}, \|\mathcal{R}\|_{(\mathbb{H}\times\mathbb{Q})'}^{3/4}, \|\mathcal{R}\|_{(\mathbb{H}\times\mathbb{Q})'}, \|\mathcal{R}\|_{(\mathbb{H}\times\mathbb{Q})'}^{3/2}\Big\}, \tag{6.14}$$

where, $\mathcal{R} : \mathbb{H}\times\mathbb{Q} \to \mathbb{R}$ *is the functional given by* $\mathcal{R}(\vec{\mathbf{r}}) := \mathcal{R}_{\mathbf{F}_1}(\underline{\mathbf{r}}) + \mathcal{R}_{\mathbf{F}_2}(\underline{\boldsymbol{\psi}}) + \mathcal{R}_{\mathbf{G}}(q_{\mathrm{D}}) \quad \forall\, \vec{\mathbf{r}} := ((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), q_{\mathrm{D}}) \in \mathbb{H}\times\mathbb{Q}$*, which satisfies*

$$\mathcal{R}(\vec{\mathbf{r}}_h) = 0 \quad \forall\, \vec{\mathbf{r}}_h \in \mathbb{H}_h \times \mathbb{Q}_h.$$

According to the upper bound (6.14) provided in Lemma 6.2, it only remains to estimate $\|\mathcal{R}\|_{\mathbb{H}'\times\mathbb{Q}'}$. To this end, we now notice that the functional $\mathcal{R}$ can be decomposed as:

$$\mathcal{R}(\underline{\vec{\mathbf{r}}}) \,=\, \mathcal{R}_1(\boldsymbol{\tau}_{\mathrm{S}}) + \mathcal{R}_2(\mathbf{v}_{\mathrm{S}}) + \mathcal{R}_3(\boldsymbol{\eta}_{\mathrm{S}}) + \mathcal{R}_4(\mathbf{v}_{\mathrm{D}}) + \mathcal{R}_5(\boldsymbol{\psi}) + \mathcal{R}_6(\xi) + \mathcal{R}_7(q_{\mathrm{D}}),$$

for all $\underline{\vec{\mathbf{r}}} := ((\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi), q_{\mathrm{D}}) \in \mathbb{H}\times\mathbb{Q}$, where

$$\begin{aligned}
\mathcal{R}_1(\boldsymbol{\tau}_{\mathrm{S}}) \,:=\, & -\kappa_1(\mathbf{f}_{\mathrm{S}} + \mathbf{div}\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} - \frac{1}{2\mu}(\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{d}} + \rho(\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}, \boldsymbol{\tau}_{\mathrm{S}}^{\mathrm{d}})_{\mathrm{S}} \\
& -(\boldsymbol{\gamma}_{\mathrm{S},h}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} - (\mathbf{u}_{\mathrm{S},h}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} - \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma\,, \\
\mathcal{R}_2(\mathbf{v}_{\mathrm{S}}) \,:=\, & (\mathbf{f}_{\mathrm{S}} + \mathbf{div}\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + \frac{\kappa_2}{2\mu}(\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{d}} - 2\mu\mathbf{e}(\mathbf{u}_{\mathrm{S},h}) + \rho(\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}, \mathbf{e}(\mathbf{v}_{\mathrm{S}}))_{\mathrm{S}}, \\
\mathcal{R}_3(\boldsymbol{\eta}_{\mathrm{S}}) \,:=\, & (\boldsymbol{\sigma}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} - \kappa_3\Big(\boldsymbol{\gamma}_{\mathrm{S},h} - \frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{S},h} - (\nabla\mathbf{u}_{\mathrm{S},h})^{\mathrm{t}}\right), \boldsymbol{\eta}_{\mathrm{S}}\Big)_{\mathrm{S}}, \\
\mathcal{R}_4(\mathbf{v}_{\mathrm{D}}) \,:=\, & \Big(\mathbf{f}_{\mathrm{D}} - \frac{\mu}{\rho}\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h} - \frac{\mathsf{F}}{\rho}|\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h}, \mathbf{v}_{\mathrm{D}}\Big)_{\mathrm{D}} + (p_{\mathrm{D},h}, \mathrm{div}\,\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma\,, \\
\mathcal{R}_5(\boldsymbol{\psi}) \,:=\, & -\langle \boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \omega_1^{-1}\langle \boldsymbol{\varphi}_h\cdot\mathbf{t}, \boldsymbol{\psi}\cdot\mathbf{t}\rangle_\Sigma - \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma\,, \\
\mathcal{R}_6(\xi) \,:=\, & \langle \mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}, \xi\rangle_\Sigma\,, \\
\mathcal{R}_7(q_{\mathrm{D}}) \,:=\, & -(g_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}, q_{\mathrm{D}})_{\mathrm{D}}.
\end{aligned}$$

In this way, it readily follows that

$$\|\mathcal{R}\|_{\mathbb{H}'\times\mathbb{Q}'} \,\le\, \Big\{\|\mathcal{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'} \,+\, \|\mathcal{R}_2\|_{\mathbf{H}^1_{\Gamma_S}(\Omega_S)'} \,+\, \|\mathcal{R}_3\|_{\mathbb{L}^2_{\rm skew}(\Omega_S)'} \\ +\, \|\mathcal{R}_4\|_{\mathbf{H}^3_{\Gamma_D}(\mathrm{div};\Omega_D)'} \,+\, \|\mathcal{R}_5\|_{\mathbf{H}^{-1/2}_{00}(\Sigma)} \,+\, \|\mathcal{R}_6\|_{\mathbf{W}^{-1/3,3}(\Sigma)} \,+\, \|\mathcal{R}_7\|_{\mathrm{L}^2_0(\Omega_D)'}\Big\}, \tag{6.15}$$

where the upper bounds of $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_7$, obtained as a direct application of the Cauchy–Schwarz inequality, provide the first four terms of $\Psi_{S,T}$ (cf. (6.1)) and first term of $\widehat{\Theta}_{D,T}$ (cf. (5.3)). In addition, $\mathcal{R}_4$ follows from Lemma 5.6 and implies the remaining terms of $\widehat{\Theta}_{D,T}$ and $\widetilde{\Theta}_{D,T}$ (cf. (5.3), (5.4)). In turn, for $\mathcal{R}_5$ we refer the reader to [20, Lemma 3.2], which generates the last term of $\Psi_{S,T}$, whereas $\mathcal{R}_6$ can be bounded from a slight adaptation of Lemma 5.5, thus obtaining $\Psi_{\Sigma,e}$ (cf. (6.3)).

Now, we aim to bound the norm of the functional $\mathcal{R}_1$. Analogously to [9], this task is actually performed in two different ways, which leads to the reliability of Ψ_1 (cf. (6.9)) and Ψ_2 (cf. (6.10)). We begin with the bound for $\mathcal{R}_1$, which yields the remaining terms of Ψ_1 (cf. (6.4)).

Lemma 6.3 *There exists $C > 0$, independent of meshsizes, such that*

$$\|\mathcal{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'} \,\le\, C\Big\{\|\mathbf{f}_S + \mathbf{div}\boldsymbol{\sigma}_{S,h}\|^2_{0,\Omega_S} + \big\|\boldsymbol{\sigma}^{\rm d}_{S,h} - 2\mu\mathbf{e}(\mathbf{u}_{S,h}) + \rho(\mathbf{u}_{S,h}\otimes\mathbf{u}_{S,h})^{\rm d}\big\|^2_{0,\Omega_S} \\ +\, \Big\|\boldsymbol{\gamma}_{S,h} - \frac{1}{2}\left(\nabla\mathbf{u}_{S,h} - (\nabla\mathbf{u}_{S,h})^{\rm t}\right)\Big\|^2_{0,\Omega_S} + \|\mathbf{u}_{S,h} + \boldsymbol{\varphi}_h\|^2_{1/2,00;\Sigma}\Big\}^{1/2}. \tag{6.16}$$

Proof. Similarly to [9, Theorem 3.7], it suffices to integrate by parts the fourth term of $\mathcal{R}_1$, add and subtract $(\mathbf{e}(\mathbf{u}_{S,h}), \boldsymbol{\tau}^{\rm d}_S)_S$, and then employ the Cauchy–Schwarz and trace inequalities. Further details are omitted. □

We now establish the reliability of the remaining terms of Ψ_2 (cf. (6.5)), which is accomplished by applying the Helmholtz decompositions provided by [20, Lemma 3.3] to $\mathcal{R}_1$. In this way, we proceed as in [21, Lemma 3.10] and [8, Lemma 3.9] to bound $\|\mathcal{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'}$.

Lemma 6.4 *There exists $C > 0$, independent of meshsizes, such that*

$$\|\mathcal{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'} \le C\left\{\sum_{T\in\mathcal{T}^S_h}\widetilde{\Psi}^2_{S,T}\right\}^{1/2}, \tag{6.17}$$

where

$$\widetilde{\Psi}^2_{S,T} \,:=\, \|\mathbf{f}_S - \mathbf{P}^S_h(\mathbf{f}_S)\|^2_{0,T} + \big\|\boldsymbol{\sigma}^{\rm d}_{S,h} - 2\mu\mathbf{e}(\mathbf{u}_{S,h}) + \rho(\mathbf{u}_{S,h}\otimes\mathbf{u}_{S,h})^{\rm d}\big\|^2_{0,T} \\ +\, h_T^2\Big\|\boldsymbol{\gamma}_{S,h} - \frac{1}{2}\left(\nabla\mathbf{u}_{S,h} - (\nabla\mathbf{u}_{S,h})^{\rm t}\right)\Big\|^2_{0,T} \\ +\, \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_S)} h_e\big\|[\![(\mathbf{e}(\mathbf{u}_{S,h}) + \boldsymbol{\gamma}_{S,h})\mathbf{t}]\!]\big\|^2_{0,e} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_S)} h_e\|(\mathbf{e}(\mathbf{u}_{S,h}) + \boldsymbol{\gamma}_{S,h})\mathbf{t}\|^2_{0,e} \\ +\, \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)}\left\{h_e\|\boldsymbol{\varphi}_h + \mathbf{u}_{S,h}\|^2_{0,e} + h_e\Big\|(\mathbf{e}(\mathbf{u}_{S,h}) + \boldsymbol{\gamma}_{S,h})\mathbf{t} + \frac{d\boldsymbol{\varphi}_h}{d\mathbf{t}}\Big\|^2_{0,e}\right\}.$$

We end this section by concluding that the estimates (6.9) and (6.10) in Theorem 6.1 follow straightforwardly from Lemma 6.2, the definition of the global estimators Ψ_1 and Ψ_2 (cf. (6.4), (6.5)), and Lemmas 6.3 and 6.4, respectively.

6.2 Efficiency

In this section we provide the efficiency estimate associated to our estimators Ψ_1 and Ψ_2 (cf. (6.4), (6.5)). We begin with the main result of this section, which follows from a slight adaptation of Theorem 5.7.

Theorem 6.5 *Suppose that the data* $\mathbf{f}_{\mathrm{S}}$, $\mathbf{f}_{\mathrm{D}}$ *and* g_{D} *satisfy* (6.8) *and that* $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$. *Then, there exist constants* $\widehat{C}_{\mathrm{eff}}, \widehat{c}_{\mathrm{eff}} > 0$, *independent of* h, *such that*

$$\widehat{C}_{\mathrm{eff}}\, \Psi_1 \leq \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbb{Q}} + \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T^2\, \big\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \big\|_{0,T}^2 \right\}^{1/2} + \text{h.o.t.}, \tag{6.18}$$

and

$$\widehat{c}_{\mathrm{eff}}\, \Psi_2 \leq \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbb{Q}} + \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T^2\, \big\| |\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} - |\mathbf{u}_{\mathrm{D},h}|\mathbf{u}_{\mathrm{D},h} \big\|_{0,T}^2 \right\}^{1/2} + \text{h.o.t.}, \tag{6.19}$$

where h.o.t. *stands for one or several terms of higher order. Moreover, assuming that* $\mathbf{u}_{\mathrm{D}} \in \mathbf{L}^6(\Omega_{\mathrm{D}})$, *there exists constants* $C_{\mathrm{eff}}, c_{\mathrm{eff}} > 0$, *depending only on parameters*, $\|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{L}^6(\Omega_{\mathrm{D}})}$, *and other constants, all them independent of* h, *such that*

$$C_{\mathrm{eff}}\, \Psi_1 \leq \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbb{Q}} + \text{h.o.t.} \tag{6.20}$$

and

$$c_{\mathrm{eff}}\, \Psi_2 \leq \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbb{Q}} + \text{h.o.t.} \tag{6.21}$$

We remark here that the estimates (6.18) and (6.19) follows straightforwardly from the definition of Ψ_1 and Ψ_2 (cf. (6.4), (6.5)), using similar arguments to those from Section 5.2 to bound the Darcy–Forchheimer and interface terms, and appealing to results from previous works [21, 19, 20] to bound the Navier–Stokes terms. In addition, similarly to Theorem 5.7, the extra assumption on $\mathbf{u}_{\mathrm{D}}$, and the estimate (5.68) imply (6.20) and (6.21).

7 Numerical results

This section serves to illustrate the performance and accuracy of our mixed finite element schemes (3.5) and (3.9) along with the reliability and efficiency properties of the a posteriori error estimators Θ, Ψ_1 and Ψ_2 (cf. (5.7), (6.4), (6.5)) derived in Sections 5 and 6. In this regard, we remark that for purposes of adaptivity, which requires to have locally computable indicators, we use that

$$\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}^2 \,\leq\, C\, \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{1,\Sigma}^2 \,=\, C \sum_{e\in\mathcal{E}_h(\Sigma)} \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{1,e}^2$$

and redefine Ψ_1 as

$$\Psi_1 \,:=\, \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \Psi_{\mathrm{S},T}^2 + \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \widehat{\Theta}_{\mathrm{D},T}^2 + \sum_{e\in\mathcal{E}_h(\Sigma)} \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{1,e}^2 \right\}^{1/2} + \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \widetilde{\Theta}_{\mathrm{D},T}^{3/2} \right\}^{2/3} + \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \Psi_{\Sigma,e}^3 \right\}^{1/3}.$$

Under this redefinition Ψ_1 is certainly still reliable, but efficient only up to all terms, except the new term associated to the interface Σ. Nevertheless, the numerical results to be displayed below allow us to conjecture that this modified Ψ_1 actually verifies both properties.

Our implementation is based on a FreeFem++ code [24]. Regarding the implementation of the Newton iterative method associated to (3.5) and (3.9) (see [7, Section 6] and [10, Section 7], respectively, for details), the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, i.e.,

$$\frac{\|\mathsf{coeff}^{m+1} - \mathsf{coeff}^{m}\|_{\ell^2}}{\|\mathsf{coeff}^{m+1}\|_{\ell^2}} \leq \mathsf{tol},$$

where $\|\cdot\|_{\ell^2}$ is the standard ℓ^2-norm in $\mathbb{R}^{\mathrm{N}}$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$ in the primal-mixed scheme (respectively $\mathbb{H}_h$ and $\mathbb{Q}_h$ for the fully-mixed scheme), and tol is a fixed tolerance chosen as $\mathsf{tol} = 1E-06$. As usual, the individual errors are denoted by:

$$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}) := \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}, \quad \mathsf{e}(\mathbf{u}_{\mathrm{S}}) := \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}, \quad \mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}}) := \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}},$$

$$\mathsf{e}(p_{\mathrm{S}}) := \|p_{\mathrm{S}} - p_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}, \quad \mathsf{e}(\mathbf{u}_{\mathrm{D}}) := \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}, \quad \mathsf{e}(p_{\mathrm{D}}) := \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}},$$

$$\mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{(0,1),\Sigma}, \quad \mathsf{e}(\lambda) := \|\lambda - \lambda_h\|_{\mathrm{L}^{3/2}(\Sigma)}.$$

Notice that for the fully-mixed formulation, the Navier–Stokes pressure is calculated through the post-process formula

$$p_{\mathrm{S},h} := -\frac{1}{n}\,\mathrm{tr}\left(\boldsymbol{\sigma}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})\right) \quad \text{in} \quad \Omega_{\mathrm{S}}.$$

Also, since the natural norms to measure the error of the interface unknowns $\|\lambda - \lambda_h\|_{1/3,3/2;\Sigma}$ and $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}$ are not computable, we have decided to replace them respectively by $\|\cdot\|_{\mathrm{L}^{3/2}(\Sigma)}$ and $\|\cdot\|_{(0,1),\Sigma}$, where the last one is defined based on the fact that $\mathbf{H}^{1/2}(\Sigma)$ is the interpolation space with index 1/2 between $\mathbf{H}^1(\Sigma)$ and $\mathbf{L}^2(\Sigma)$:

$$\|\boldsymbol{\psi}\|_{(0,1),\Sigma} := \|\boldsymbol{\psi}\|_{0,\Sigma}^{1/2}\,\|\boldsymbol{\psi}\|_{1,\Sigma}^{1/2} \quad \forall\,\boldsymbol{\psi}\in\mathbf{H}^1(\Sigma).$$

Then, the global errors are computed, respectively, as

$$\mathsf{e}(\vec{\mathbf{u}}) := \mathsf{e}(\mathbf{u}_{\mathrm{S}}) + \mathsf{e}(\mathbf{u}_{\mathrm{D}}) + \mathsf{e}(p_{\mathrm{S}}) + \mathsf{e}(p_{\mathrm{D}}) + \mathsf{e}(\lambda)$$

and

$$\mathsf{e}(\vec{\mathbf{t}}) := \mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}) + \mathsf{e}(\mathbf{u}_{\mathrm{S}}) + \mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}}) + \mathsf{e}(\mathbf{u}_{\mathrm{D}}) + \mathsf{e}(p_{\mathrm{D}}) + \mathsf{e}(\boldsymbol{\varphi}) + \mathsf{e}(\lambda).$$

In turn, the efficiency and reliability indexes with respect to Θ are given by

$$\mathsf{eff}(\Theta) := \frac{\mathsf{e}(\vec{\mathbf{u}})}{\Theta} \quad \text{and} \quad \mathsf{rel}(\Theta^{1/2}) := \frac{\mathsf{e}(\vec{\mathbf{u}})}{\Theta^{1/2}}.$$

Analogue definitions hold for Ψ_1 and Ψ_2 with $\vec{\mathbf{t}}$ instead of $\vec{\mathbf{u}}$. Regarding these indexes, we observe from (5.8) (after discarding the higher order terms there) that

$$C_{\mathtt{eff}} \leq \mathsf{eff}(\Theta) \leq C_{\mathtt{rel}}\,\Theta^{-1/2}, \qquad \text{and} \qquad C_{\mathtt{eff}}\,\Theta^{1/2} \leq \mathsf{rel}(\Theta^{1/2}) \leq C_{\mathtt{rel}}, \tag{7.1}$$

which says that, while $\mathsf{eff}(\Theta)$ and $\mathsf{rel}(\Theta^{1/2})$ are below and above bounded, respectively, $\mathsf{eff}(\Theta)$ could become above unbounded whereas $\mathsf{rel}(\Theta^{1/2})$ could very well approaches 0 as Θ goes to 0. Nevertheless, the numerical results to be displayed below show that $\mathsf{eff}(\Theta)$ remains always above bounded as well, whereas $\mathsf{rel}(\Theta^{1/2})$ does in fact decreases as Θ goes to 0. The same remarks hold for the efficiency and reliability indexes with respect to Ψ_1 and Ψ_2 (cf. (6.6) and (6.7)).

In addition, we define the experimental rates of convergence

$$\mathsf{r}(\diamond) := \frac{\log(\mathsf{e}(\diamond)/\mathsf{e}'(\diamond))}{\log(h/h')} \quad \text{for each } \diamond \in \Big\{\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\gamma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, p_{\mathrm{S}}, p_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda, \vec{\mathbf{u}}, \vec{\mathbf{t}}\Big\},$$

where h and h' denote two consecutive mesh sizes, taken accordingly from $\{h_{\mathrm{S}}, h_{\mathrm{D}}, \widehat{h}_\Sigma, h_\Sigma\}$, with their respective errors e and e'. However, when the adaptive algorithm is applied, the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(\mathrm{N}/\mathrm{N}')$, where N and N′ denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. In all of them, for the sake of simplicity, we choose the parameters $\mu = 1$, $\rho = 1$, $\mathtt{F} = 1$, $\alpha_d = 1$, $\omega_1 = 1$, $\boldsymbol{\kappa} = \mathbb{I}$, and $\mathbf{K} = \mathbb{I}$, and the stabilization parameters are taken as $\kappa_1 = 1/(2\mu)$, $\kappa_2 = 2\mu$ and $\kappa_3 = C_{\mathrm{Ko}}\,\mu$, where we choose heuristically $C_{\mathrm{Ko}} = 1/2$. Furthermore, the conditions $(p_h, 1)_\Omega = 0$, $(\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{S},h}, 1)_{\mathrm{S}} = 0$ and $(p_{\mathrm{D},h}, 1)_{\mathrm{D}} = 0$ are imposed via a penalization strategy.

Example 1 is used to corroborate the reliability and efficiency of the a posteriori error estimators Θ, Ψ_1 and Ψ_2, whereas Example 2 is utilized to illustrate the behavior of the associated adaptive algorithm, which applies the following procedure from [26]:

(1) Start with a coarse mesh $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$.

(2) Solve the Newton iterative method associated to (3.5) (respectively (3.9)) for the current mesh $\mathcal{T}_h$.

(3) Compute the local indicator Θ_T (respectively $\Psi_{1,T}$ and $\Psi_{2,T}$) for each $T \in \mathcal{T}_h := \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ and $e \in \mathcal{E}_h(\Sigma)$, where

$$\Theta_T := \Big\{\Theta_{\mathrm{S},T}^2 + \widehat{\Theta}_{\mathrm{D},T}^2\Big\}^{1/2} + \widetilde{\Theta}_{\mathrm{D},T} + \Theta_{\Sigma,e}, \qquad \text{(cf. (5.1), (5.3), (5.4), (5.6))}$$

$$\Psi_{1,T} := \Big\{\Psi_{\mathrm{S},T}^2 + \widehat{\Theta}_{\mathrm{D},T}^2 + \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{1,e}^2\Big\}^{1/2} + \widetilde{\Theta}_{\mathrm{D},T} + \Psi_{\Sigma,e}, \qquad \text{(cf. (6.1), (5.3), (5.4), (6.3))}$$

$$\Psi_{2,T} := \Big\{\widehat{\Psi}_{\mathrm{S},T}^2 + \widehat{\Theta}_{\mathrm{D},T}^2\Big\}^{1/2} + \widetilde{\Theta}_{\mathrm{D},T} + \Psi_{\Sigma,e}. \qquad \text{(cf. (6.2), (5.3), (5.4), (6.3))}$$

(4) Check the stopping criterion and decide whether to finish or go to next step.

(5) Generate an adapted mesh through a variable metric/Delaunay automatic meshing algorithm (see [25, Section 9.1.9]).

(6) Define resulting meshes as current meshes $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$, and go to step (2).

Example 1: Accuracy assessment with a smooth solution in a rectangular domain.

In our first example we consider a rectangle domain divided in two coupled squares, i.e., $\Omega_{\mathrm{S}} := (0,1)\times(1,2)$, $\Omega_{\mathrm{D}} := (0,1)^2$ and $\Sigma := (0,1)\times\{1\}$. The data $\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}$, and g_{D} are chosen so that the exact solution in the rectangle domain $\Omega = \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ is given by the smooth functions

$$\mathbf{u}_{\mathrm{S}} := \frac{1}{2}\begin{pmatrix} -\sin(\pi x_1)\cos(\pi x_2) \\ \cos(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \quad \mathbf{u}_{\mathrm{D}} := \frac{\pi}{2}\begin{pmatrix} \sin(\pi x_1)\exp(x_2) \\ \sin(\pi x_2)\exp(x_1) \end{pmatrix},$$

$$p_\star := x_1\cos(\pi x_2) \quad \text{in} \quad \Omega_\star, \qquad \text{with } \star \in \{\mathrm{S}, \mathrm{D}\}.$$

Notice that this solution satisfies $\mathbf{u}_{\mathrm{S}} \cdot \mathbf{n} = \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n}$ on Σ. However, the Beavers–Joseph–Saffman conditions (3.2) and (3.7) are not satisfied, whereas the Dirichlet boundary condition for the Navier–Stokes velocity on Γ_{S} and the Neumann boundary condition for the Darcy–Forchheimer velocity on Γ_{D} are both non-homogeneous. In this way, the right-hand side of the resulting system must be modified accordingly as well as the global estimators Θ, Ψ_1 and Ψ_2 (cf. (5.7), (6.4), (6.5)). The results reported in Tables 7.1 and 7.2 are in accordance with the theoretical sub-optimal rate of convergence $O(h^{1/3})$ provided by [7, Theorem 5.2] and [10, Theorem 6.1]. Actually, they are better than expected since they suggest that only technical difficulties stop us of proving an optimal rate of convergence $O(h)$, which is in fact observed there. In addition, we notice that the behaviors predicted by (7.1) and the remarks right after it, are also observed in the tables, in the sense that the efficiency indexes remain above and below bounded and the reliability indexes, while bounded as well, decrease as the estimators approach 0.

Example 2: Adaptivity in a 2D helmet-shaped domain.

In our second example, we consider a 2D helmet-shaped domain. More precisely, we consider the domain $\Omega = \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$, where $\Omega_{\mathrm{S}} := (-1, -0.75) \times (0, 1.25) \cup (-0.75, 0.75) \times (0, 0.25) \cup (0.75, 1) \times (0, 1.25)$, $\Omega_{\mathrm{D}} := (-1, 1) \times (-0.5, 0)$ and $\Sigma := (-1, 1) \times \{0\}$. The data $\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}$, and g_{D} are chosen so that the exact solution in the 2D helmet-shaped domain Ω is given by the smooth functions

$$\mathbf{u}_{\mathrm{S}} := \begin{pmatrix} \dfrac{(x_2 - 0.27)}{r_1(x_1, x_2)} + \dfrac{(x_2 - 0.27)}{r_2(x_1, x_2)} \\ -\dfrac{(x_1 + 0.73)}{r_1(x_1, x_2)} - \dfrac{(x_1 - 0.73)}{r_2(x_1, x_2)} \end{pmatrix}, \quad \mathbf{u}_{\mathrm{D}} := \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_1)\exp(x_2) \\ \sin(\pi x_2)\cos(\pi x_2)\exp(x_1) \end{pmatrix},$$

$$p_\star := x_2 \sin(\pi x_1) \quad \text{in} \quad \Omega_\star, \qquad \text{with } \star \in \{\mathrm{S}, \mathrm{D}\},$$

where

$$r_1(x_1, x_2) := 4\sqrt{(x_1 + 0.73)^2 + (x_2 - 0.27)^2} \quad \text{and} \quad r_2(x_1, x_2) := 4\sqrt{(x_1 - 0.73)^2 + (x_2 - 0.27)^2}.$$

Figure 7.1, summarizes the convergence history of the methods applied to a sequence of quasi-uniformly and adaptively refined triangulation of the domain. Sub-optimal rates are observed in the first case, whereas adaptive refinements according to any of the a posteriori error indicators: $\Theta, \Theta^{1/2}, \Psi_1$, $\Psi_1^{1/2}$, Ψ_2, and $\Psi_2^{1/2}$, yield optimal convergence. In particular, Tables 7.3 and 7.5 summarizes the errors, rates of convergence, efficiency and reliability indexes, and Newton iterations of the methods applied to a sequence of quasi-uniform refinement triangulation of the domain. In turn, and for the sake of simplicity, we only show Tables 7.4 and 7.6, which summarizes the convergence history of the primal-mixed and fully-mixed schemes after Θ and $\Psi_1^{1/2}$, respectively. Notice that in all the examples, when $\Theta < 1$ (respectively Ψ_1 and Ψ_2) and $h \to 0$, the rate of convergence of the total error and the efficiency and reliability indexes have the behavior that we expected. Notice also how the adaptive algorithms improves the efficiency of the method by delivering quality solutions at a lower computational cost, to the point that it is possible to get a better one (in terms of $\mathsf{e}(\vec{\mathbf{u}})$, respectively $\mathsf{e}(\vec{\mathbf{t}})$) with approximately only the 25% of the degrees of freedom of the last quasi-uniform mesh for the primal-mixed scheme (respectively fully-mixed scheme).

On the other hand, in Figure 7.2 we show the domain configuration in the initial mesh, the second component of velocity in the whole domain obtained through the primal-mixed scheme (via the indicator Θ), and the first row of the pseudostress tensor streamlines obtained through the fully-mixed scheme (via the indicator $\Psi_1^{1/2}$). In particular, we notice that the Navier–Stokes velocity exhibit high gradients near the vertices $(-0.75, 0.25)$ and $(0.75, 0.25)$, and the first row of the pseudostress tensor streamlines show vortices in the same region. In turn, examples of some adapted meshes generated using Θ and $\Psi_1^{1/2}$ are collected in Figure 7.3. We can observe a clear clustering of elements near the vertices in Ω_{S} of the 2D helmed-shaped domain as we expected. Notice also a clustering of elements on the interface Σ. This is justified by the fact that $\frac{d\lambda_h}{d\mathbf{t}} = 0$ since λ_h is piecewise constant on Σ and then the local estimator $\widetilde{\Theta}_{\mathrm{D},T}$ (cf. (5.4)) computes $h_e \| (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \mathbf{t} \|^{3/2}_{\mathrm{L}^{3/2}(e)}$ instead of $h_e \| (\mathbf{f}_{\mathrm{D}} - \mathbf{U}_{\mathrm{D},h}) \cdot \mathbf{t} - \frac{d\lambda_h}{d\mathbf{t}} \|^{3/2}_{\mathrm{L}^{3/2}(e)}$ for each $e \in \mathcal{E}_h(\Sigma)$ (cf. Lemma 5.6).

N	h_{S}	h_{D}	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$\mathsf{r}(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(p_{\mathrm{S}})$	$\mathsf{r}(p_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$\mathsf{r}(\mathbf{u}_{\mathrm{D}})$	$\mathsf{e}(p_{\mathrm{D}})$	$\mathsf{r}(p_{\mathrm{D}})$
279	0.373	0.373	0.2853	–	0.1628	–	1.7893	–	0.1226	–
1061	0.196	0.190	0.1320	1.204	0.0601	1.556	0.8346	1.133	0.0503	1.324
3877	0.103	0.097	0.0653	1.081	0.0274	1.208	0.4119	1.043	0.0206	1.320
15057	0.051	0.057	0.0336	0.950	0.0134	1.024	0.2126	1.268	0.0101	1.363
59203	0.027	0.026	0.0164	1.143	0.0069	1.060	0.1057	0.877	0.0049	0.898
236687	0.014	0.013	0.0082	1.000	0.0034	1.039	0.0527	1.068	0.0025	1.072

h_Σ	$\mathsf{e}(\lambda)$	$\mathsf{r}(\lambda)$	$\mathsf{e}(\vec{\mathbf{u}})$	$\mathsf{r}(\vec{\mathbf{u}})$	Θ	$\mathsf{eff}(\Theta)$	$\mathsf{rel}(\Theta^{1/2})$	iter
1/4	0.0866	–	2.4466	–	20.2522	0.1208	0.5437	5
1/8	0.0358	1.274	1.1128	1.230	10.0338	0.1109	0.3513	6
1/16	0.0177	1.020	0.5429	1.104	5.1863	0.1047	0.2384	6
1/32	0.0083	1.083	0.2780	1.151	2.6229	0.1060	0.1716	6
1/64	0.0041	1.019	0.1380	0.936	1.3279	0.1040	0.1198	6
1/128	0.0020	1.004	0.0687	1.005	0.6683	0.1028	0.0840	6

Table 7.1: EXAMPLE 1, $\mathbf{BR} - \mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_0$ primal-mixed scheme with quasi-uniform refinement.

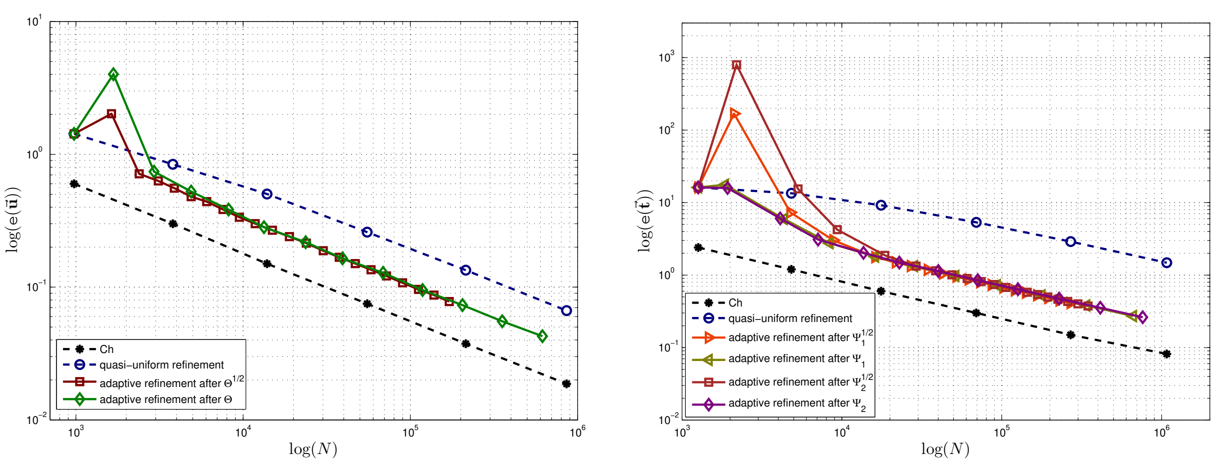

Figure 7.1: Example 2, Log-log plot of $\mathsf{e}(\vec{\mathbf{u}})$ (respectively $\mathsf{e}(\vec{\mathbf{t}})$) vs. N for quasi-uniform/adaptive mixed schemes.

N	h_S	$\mathsf{e}(\boldsymbol{\sigma}_S)$	$\mathsf{r}(\boldsymbol{\sigma}_S)$	$\mathsf{e}(\mathbf{u}_S)$	$\mathsf{r}(\mathbf{u}_S)$	$\mathsf{e}(\boldsymbol{\gamma}_S)$	$\mathsf{r}(\boldsymbol{\gamma}_S)$	$\mathsf{e}(p_S)$	$\mathsf{r}(p_S)$
358	0.373	2.2325	–	0.7314	–	0.4206	–	0.2897	–
1341	0.190	0.9447	1.277	0.2481	1.606	0.2282	0.908	0.1346	1.139
4877	0.102	0.4756	1.103	0.1194	1.176	0.1200	1.033	0.0636	1.204
18874	0.051	0.2335	1.016	0.0589	1.008	0.0638	0.901	0.0296	1.095
74123	0.027	0.1176	1.099	0.0296	1.103	0.0306	1.176	0.0155	1.032
295926	0.014	0.0586	1.001	0.0146	1.016	0.0155	0.982	0.0075	1.038

h_D	$\widehat{h}_\Sigma$	h_Σ	$\mathsf{e}(\mathbf{u}_D)$	$\mathsf{r}(\mathbf{u}_D)$	$\mathsf{e}(p_D)$	$\mathsf{r}(p_D)$	$\mathsf{e}(\boldsymbol{\varphi})$	$\mathsf{r}(\boldsymbol{\varphi})$	$\mathsf{e}(\lambda)$	$\mathsf{r}(\lambda)$
0.373	1/2	1/4	1.7888	–	0.1394	–	0.2324	–	0.0688	–
0.190	1/4	1/8	0.8346	1.132	0.0525	1.450	0.1221	0.929	0.0346	0.992
0.097	1/8	1/16	0.4119	1.043	0.0208	1.366	0.0557	1.132	0.0174	0.993
0.057	1/16	1/32	0.2126	1.237	0.0102	1.350	0.0261	1.092	0.0083	1.064
0.026	1/32	1/64	0.1057	0.882	0.0049	0.907	0.0123	1.082	0.0041	1.017
0.013	1/64	1/128	0.0527	1.082	0.0024	1.087	0.0062	0.984	0.0020	1.003

$\mathsf{e}(\vec{\mathbf{t}})$	$\mathsf{r}(\vec{\mathbf{t}})$	Ψ_1	$\mathsf{eff}(\Psi_1)$	$\mathsf{rel}(\Psi_1^{1/2})$	Ψ_2	$\mathsf{eff}(\Psi_2)$	$\mathsf{rel}(\Psi_2^{1/2})$	iter
5.6139	–	20.3330	0.2761	1.2450	21.3233	0.2633	1.2157	5
2.4348	1.223	10.0583	0.2450	0.7772	10.6751	0.2309	0.7544	6
1.2208	1.130	5.1317	0.2379	0.5389	5.5172	0.2213	0.5197	6
0.6134	1.167	2.5801	0.2378	0.3819	2.8311	0.2167	0.3646	6
0.3050	0.951	1.3070	0.2333	0.2668	1.4719	0.2072	0.2514	6
0.1521	1.001	0.6605	0.2303	0.1872	0.7780	0.1955	0.1725	6

Table 7.2: EXAMPLE 1, $\mathbb{RT}_0 - \mathbf{P}_1 - \mathbb{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{P}_1 - \mathrm{P}_0$ fully-mixed scheme with quasi-uniform refinement.

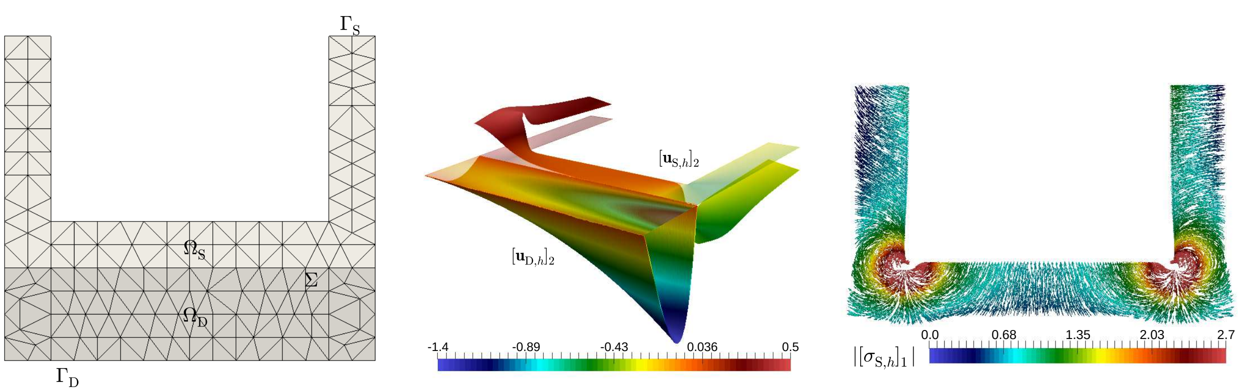

Figure 7.2: Example 2, domain configuration in the initial mesh, second velocity component on the whole domain and first row of the Navier–Stokes pseudostress tensor streamlines.

References

[1] S. AGMON, *Lectures on Elliptic Boundary Value Problems.* Van Nostrand, Princeton, NJ, 1965.

[2] M. ALVAREZ, G.N. GATICA, AND R. RUIZ-BAIER, *A posteriori error analysis for a viscous*

N	h_{S}	h_{D}	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$\mathsf{r}(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(p_{\mathrm{S}})$	$\mathsf{r}(p_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$\mathsf{r}(\mathbf{u}_{\mathrm{D}})$	$\mathsf{e}(p_{\mathrm{D}})$	$\mathsf{r}(p_{\mathrm{D}})$
975	0.188	0.200	0.4994	–	0.2520	–	0.6380	–	0.0279	–
3803	0.100	0.095	0.3710	0.473	0.1476	0.852	0.3067	0.984	0.0135	0.971
13907	0.050	0.049	0.2486	0.578	0.0908	0.700	0.1558	1.037	0.0067	1.064
55232	0.026	0.026	0.1278	1.005	0.0500	0.903	0.0783	1.081	0.0034	1.089
214793	0.014	0.013	0.0668	1.111	0.0265	1.087	0.0392	0.968	0.0017	0.963
859813	0.007	0.007	0.0033	0.962	0.0134	0.917	0.0196	1.204	0.0009	1.202

h_Σ	$\mathsf{e}(\lambda)$	$\mathsf{r}(\lambda)$	$\mathsf{e}(\vec{\mathbf{u}})$	$\mathsf{r}(\vec{\mathbf{u}})$	Θ	$\mathsf{eff}(\Theta)$	$\mathsf{rel}(\Theta^{1/2})$	iter
1/8	0.0088	–	1.4261	–	5.9111	0.2413	0.5866	5
1/16	0.0021	2.088	0.8409	0.776	3.9451	0.2131	0.4234	5
1/32	0.0005	2.032	0.5025	0.794	2.2787	0.2205	0.3329	5
1/64	0.0001	1.974	0.2595	0.958	1.3192	0.1967	0.2260	5
1/128	3.9 e-5	2.052	0.1342	0.971	0.6619	0.2027	0.1649	5
1/256	8.1 e-6	1.934	0.0665	1.013	0.3315	0.2004	0.1154	5

Table 7.3: EXAMPLE 2, $\mathbf{BR} - \mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_0$ primal-mixed scheme with quasi-uniform refinement.

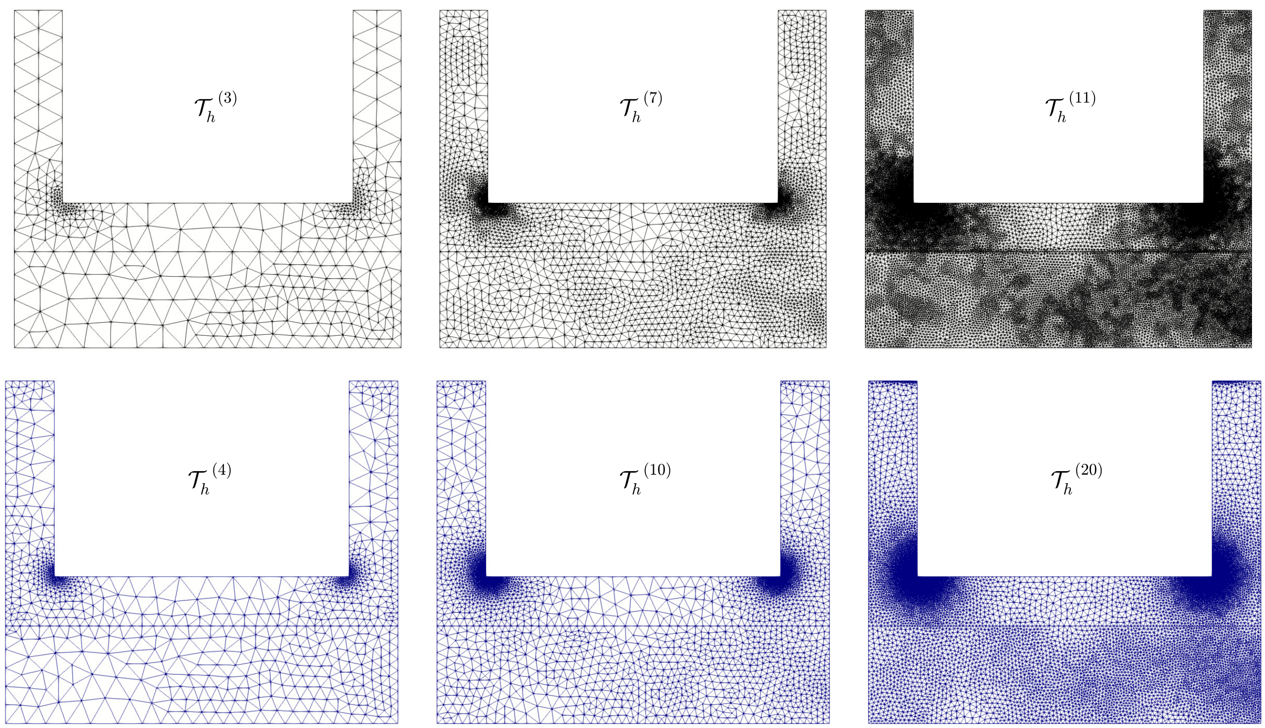

Figure 7.3: Example 2, three snapshots of adapted meshes according to the indicators Θ and $\Psi_1^{1/2}$ (top and bottom plots, respectively).

flow-transport problem. ESAIM Math. Model. Numer. Anal. 50 (2016), no. 6, 1789–1816.

[3] I. Babuška and G.N. Gatica, *A residual-based a posteriori error estimator for the Stokes–Darcy coupled problem.* SIAM J. Numer. Anal. 48 (2010), no. 2, 498–523.

N	$\mathsf{e}(\mathbf{u}_S)$	$\mathsf{r}(\mathbf{u}_S)$	$\mathsf{e}(p_S)$	$\mathsf{r}(p_S)$	$\mathsf{e}(\mathbf{u}_D)$	$\mathsf{r}(\mathbf{u}_D)$	$\mathsf{e}(p_D)$	$\mathsf{r}(p_D)$	$\mathsf{e}(\lambda)$	$\mathsf{r}(\lambda)$	$\mathsf{e}(\vec{\mathbf{u}})$	$\mathsf{r}(\vec{\mathbf{u}})$	Θ	$\mathsf{eff}(\Theta)$	iter
975	0.4994	–	0.2520	–	0.6380	–	0.0279	–	0.0088	–	1.4261	–	5.9111	0.2413	5
1676	0.3082	1.782	0.8858	–	0.5468	0.570	0.8742	–	1.3937	–	4.0087	–	3.0611	1.3096	5
2934	0.1448	2.699	0.0953	7.963	0.4158	0.978	0.0356	11.434	0.0496	11.911	0.7411	6.029	1.9837	0.3736	5
4890	0.0981	1.522	0.0596	1.839	0.3107	1.141	0.0242	1.507	0.0320	1.718	0.5247	1.352	1.4902	0.3521	5
8193	0.0714	1.230	0.0356	1.997	0.2533	0.791	0.0146	1.961	0.0090	4.916	0.3840	1.210	1.1424	0.3361	5
13330	0.0579	0.861	0.0286	0.901	0.1831	1.333	0.0091	1.943	0.0039	3.440	0.2826	1.259	0.8612	0.3282	5
23684	0.0427	1.066	0.0191	1.402	0.1474	0.755	0.0078	0.544	0.0009	5.133	0.2178	0.906	0.6691	0.3256	5
39355	0.0330	1.015	0.0150	0.958	0.1104	1.139	0.0058	1.147	0.0004	3.028	0.1646	1.105	0.5023	0.3276	5
68710	0.0248	1.015	0.0111	1.086	0.0872	0.848	0.0045	0.902	0.0004	0.195	0.1280	0.902	0.3902	0.3280	5
118428	0.0187	1.038	0.0082	1.081	0.0646	1.101	0.0034	1.003	0.0001	5.688	0.0951	1.091	0.2944	0.3230	5
205068	0.0142	1.007	0.0062	1.058	0.0502	0.917	0.0026	1.022	4.7 e-5	2.032	0.0732	0.951	0.2266	0.3233	5
617001	0.0082	1.015	0.0035	1.092	0.0294	0.895	0.0016	0.846	1.5 e-5	2.078	0.0427	0.934	0.1323	0.3228	5

Table 7.4: EXAMPLE 2, $\mathbf{BR} - \mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_0$ primal-mixed scheme with adaptive refinement via Θ.

[4] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*. Springer Series in Computational Mathematics, 15. Springer-Verlag, New York, 1991.

[5] C. CARSTENSEN, *A posteriori error estimate for the mixed finite element method*. Math. Comp. 66 (1997), no. 218, 465–476.

[6] J. CAMAÑO, C. GARCÍA, AND R. OYARZÚA, *Analysis of a conservative mixed-FEM for the stationary Navier–Stokes problem*. Preprint 2018-25, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, Chile, (2018).

[7] S. CAUCAO, M. DISCACCIATI, G.N. GATICA, AND R. OYARZÚA, *A conforming mixed finite element method for the Navier–Stokes/Darcy–Forchheimer coupled problem*. Preprint 2017-29, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, Chile, (2017).

[8] S. CAUCAO, G.N. GATICA, AND R. OYARZÚA, *A posteriori error analysis of a fully-mixed formulation for the Navier–Stokes/Darcy coupled problem with nonlinear viscosity*. Comput. Methods Appl. Mech. Engrg. 315 (2017), 943–971.

N	h_S	e($\boldsymbol{\sigma}_S$)	r($\boldsymbol{\sigma}_S$)	e($\mathbf{u}_S$)	r($\mathbf{u}_S$)	e($\boldsymbol{\gamma}_S$)	r($\boldsymbol{\gamma}_S$)	e(p_S)	r(p_S)
1264	0.188	14.1621	–	0.8298	–	0.3896	–	0.4866	–
4833	0.100	12.2457	0.231	0.4982	0.812	0.3119	0.354	0.2872	0.840
17586	0.150	8.4284	0.539	0.3172	0.651	0.2442	0.353	0.2267	0.341
69327	0.026	4.9025	0.818	0.1817	0.842	0.1486	0.750	0.1423	0.704
269604	0.014	2.6577	1.049	0.0959	1.095	0.0912	0.836	0.0785	1.019
1076768	0.007	1.3576	0.901	0.0487	0.908	0.0475	0.875	0.0406	0.885

h_D	$\widehat{h}_\Sigma$	h_Σ	e($\mathbf{u}_D$)	r($\mathbf{u}_D$)	e(p_D)	r(p_D)	e($\boldsymbol{\varphi}$)	r($\boldsymbol{\varphi}$)	e(λ)	r(λ)
0.200	1/4	1/8	0.6380	–	0.0279	–	0.1328	–	0.0092	–
0.095	1/8	1/16	0.3067	0.984	0.0135	0.970	0.0719	0.885	0.0026	1.834
0.049	1/16	1/32	0.1558	1.037	0.0067	1.064	0.0372	0.951	0.0007	1.800
0.026	1/32	1/64	0.0783	1.081	0.0034	1.089	0.0176	1.078	0.0002	1.754
0.013	1/64	1/128	0.0392	0.968	0.0017	0.963	0.0088	1.005	5.2 e-5	2.075
0.007	1/128	1/256	0.0196	1.204	0.0009	1.202	0.0043	1.046	1.2 e-5	2.104

e($\vec{\mathbf{t}}$)	r($\vec{\mathbf{t}}$)	Ψ_1	eff(Ψ_1)	rel($\Psi_1^{1/2}$)	Ψ_2	eff(Ψ_2)	rel($\Psi_2^{1/2}$)	iter
16.1894	–	15.4476	1.0480	4.1191	21.3174	0.7594	3.5064	5
13.4505	0.267	12.8583	1.0461	3.7510	17.9569	0.7490	3.1741	5
9.1902	0.549	8.7631	1.0487	3.1045	12.3001	0.7472	2.6204	5
5.3323	0.840	5.0686	1.0520	2.3685	7.1428	0.7465	1.9952	5
2.8945	1.022	2.7435	1.0550	1.7475	3.8832	0.7454	1.4688	5
1.4786	0.972	1.4006	1.0557	1.2494	1.9925	0.7421	1.0475	5

Table 7.5: EXAMPLE 2, $\mathbb{RT}_0 - \mathbf{P}_1 - \mathbb{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{P}_1 - \mathrm{P}_0$ fully-mixed scheme with quasi-uniform refinement.

[9] S. CAUCAO, G.N. GATICA, AND R. OYARZÚA, *A posteriori analysis of an augmented fully mixed formulation for the nonisothermal Oldroyd-Stokes problem.* Numer. Methods Partial Differential Equations 35 (2019), no. 1, 295–324.

[10] S. CAUCAO, G.N. GATICA, AND F. SANDOVAL, *A fully-mixed finite element method for the coupling of the Navier-Stokes and Darcy-Forchheimer equations.* Preprint 2018-49, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, Chile, (2018).

[11] S. CAUCAO, D. MORA, AND R. OYARZÚA, *A priori and a posteriori error analysis of a pseudostress-based mixed formulation of the Stokes problem with varying density.* IMA J. Numer. Anal. 36 (2016), no. 2, 947–983.

[12] P. CLÉMENT, *Approximation by finite element functions using local regularisation.* RAIRO Modélisation Mathématique et Analyse Numérique 9 (1975), 77–84.

[13] E. CREUSE, M. FARHLOUL, AND L. PAQUET, *A posteriori error estimation for the dual mixed finite element method for the p-Laplacian in a polygonal domain.* Comput. Methods Appl. Mech. Engrg. 196 (2007), no. 25-28, 2570–2582.

[14] C. DOMÍNGUEZ, G.N. GATICA, AND S. MEDDAHI, *A posteriori error analysis of a fully-mixed finite element method for a two-dimensional fluid-solid interaction problem.* J. Comput. Math. 33 (2015), no. 6, 606–641.

N	e($\boldsymbol{\sigma}_S$)	r($\boldsymbol{\sigma}_S$)	e($\mathbf{u}_S$)	r($\mathbf{u}_S$)	e($\boldsymbol{\gamma}_S$)	r($\boldsymbol{\gamma}_S$)	e(p_S)	r(p_S)	e($\mathbf{u}_D$)	r($\mathbf{u}_D$)
1264	14.1621	–	0.8298	–	0.3896	–	0.4866	–	0.6380	–
2114	167.0755	–	0.4080	2.761	0.3219	0.743	117.9178	–	0.5499	0.578
4774	6.1524	8.127	0.2920	0.823	0.2888	0.267	0.7017	12.614	0.5163	0.155
8836	2.3575	3.106	0.1456	2.253	0.1498	2.125	0.1379	5.267	0.3181	1.568
16162	1.1808	2.290	0.1179	0.700	0.1177	0.798	0.1211	0.432	0.2991	0.204
34393	0.7740	1.082	0.0820	0.671	0.0864	0.650	0.0803	0.484	0.2020	0.891
61968	0.5668	1.055	0.0569	1.095	0.0623	0.854	0.0510	1.272	0.1535	0.952
102107	0.4396	1.022	0.0436	1.203	0.0472	1.106	0.0394	1.199	0.1185	0.993
269005	0.2684	1.014	0.0263	0.972	0.0285	1.020	0.0234	0.901	0.0742	1.011

e(p_D)	r(p_D)	e($\boldsymbol{\varphi}$)	r($\boldsymbol{\varphi}$)	e(λ)	r(λ)	e($\vec{\mathbf{t}}$)	r($\vec{\mathbf{t}}$)	$\Psi_1^{1/2}$	rel($\Psi_1^{1/2}$)	iter
0.0279	–	0.1328	–	0.0092	–	16.1894	–	3.9303	4.1191	5
0.0264	0.214	0.0828	1.834	0.0057	1.905	168.4701	–	11.4076	14.7683	4
0.0249	0.146	0.0427	1.633	0.0040	0.835	7.3211	7.719	2.6533	2.7593	5
0.0171	1.209	0.0347	0.669	0.0019	2.494	3.0247	2.862	1.7252	1.7532	5
0.0152	0.396	0.0342	0.048	0.0016	0.493	1.7665	1.781	1.3396	1.3187	4
0.0106	0.734	0.0291	0.692	0.0009	2.700	1.1850	0.979	1.0934	1.0837	4
0.0077	1.276	0.0214	1.250	0.0005	2.104	0.8691	1.033	0.9408	0.9238	4
0.0062	0.836	0.0167	0.582	0.0003	1.648	0.6721	1.023	0.8310	0.8088	4
0.0039	0.926	0.0102	1.157	0.0001	2.002	0.4116	1.014	0.6535	0.6299	4

Table 7.6: Example 2, $\mathbb{RT}_0 - \mathbf{P}_1 - \mathbb{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{P}_1 - \mathrm{P}_0$ fully-mixed scheme with adaptive refinement via $\Psi_1^{1/2}$, we show the results for the meshes 1-2-3-4-5-8-11-14-17-20.

[15] A. Ern and J.-L. Guermond, *Theory and Practice of Finite Elements.* Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.

[16] V.J. Ervin and T.N. Phillips, *Residual a posteriori error estimator for a three-field model of a non-linear generalized Stokes problem.* Comput. Methods Appl. Mech. Engrg. 195 (2006), no. 19-22, 2599–2610.

[17] M. Farhloul and A.M. Zine, *A posteriori error estimation for a dual mixed finite element approximation of non-Newtonian fluid flow problems.* Int. J. Numer. Anal. Model. 5 (2008), no. 2, 320–330.

[18] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications.* Springer Briefs in Mathematics. Springer, Cham, 2014.

[19] G.N. Gatica, G.C. Hsiao and S. Meddahi, *A residual-based a posteriori error estimator for a two-dimensional fluid-solid interaction problem.* Numer. Math. 114 (2009), no. 1, 63–106.

[20] G.N. Gatica, R. Oyarzúa, and F.-J. Sayas, *A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes–Darcy coupled problem.* Comput. Methods in Appl. Mech. and Engrg, 200 (2011), 1877–1891.

[21] G.N. Gatica, R. Ruiz-Baier, and G. Tierra, *A posteriori error analysis of an augmented mixed method for the Navier-Stokes equations with nonlinear viscosity.* Comput. Mathematics with Appl., vol. 72 (2016), 2289-2310.

[22] V. Girault and P.A. Raviart, *Finite Element Methods for Navier–Stokes Equations. Theory and Algorithms.* Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, 1986.

[23] P. Grisvard, *Elliptic Problems in Nonsmooth Domains.* Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[24] F. Hecht, *New development in freefem++.* J. Numer. Math. 20 (2012), no. 3-4, 251–265.

[25] F. Hecht, *Freefem++.* Third Edition, Version 3.58-1. Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris, 2018. [available in `http://www.freefem.org/ff++`].

[26] R. Verfürth, *A Review of A-Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques.* Wiley Teubner, Chichester, 1996.

[27] R. Verfürth, *A Posteriori Error Estimation Techniques for Finite Element Methods.* Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, 2013.

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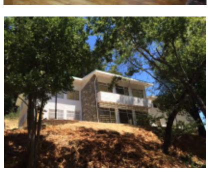

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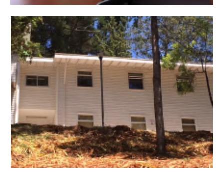
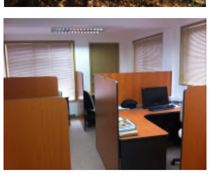

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