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A fully-mixed finite element method for the steady state Oberbeck-Boussinesq system*

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Abstract

We propose a new fully-mixed formulation for the stationary Oberbeck-Boussinesq problem when viscosity depends on both temperature and concentration. Following similar ideas applied previously to the Boussinesq and Navier-Stokes equations, we incorporate the velocity gradient and the Bernoulli stress tensor as auxiliary unknowns of the fluid equations. In turn, the gradients of temperature and of concentration, in addition to a Bernoulli vector, are introduced as further variables of the heat and mass transfer equations. Consequently, a dual-mixed approach with Dirichlet data is defined in each sub-system, and the well-known Banach and Brouwer theorems are combined with Babuška-Brezzi's theory in each independent set of equations, yielding the solvability of the continuous and discrete schemes. Next, we describe specific finite element subspaces satisfying appropriate stability requirements, and derive optimal a priori error estimates. Finally, several numerical examples illustrating the performance of the fully-mixed scheme and confirming the theoretical rates of convergence are presented.

Key words: Oberbeck-Boussinesq equations, fully–mixed formulation, fixed-point theory, finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76D05, 76R10

1 Introduction

Natural convection in porous media is of paramount interest due to its applicability in many environmental and technological processes. Typical examples include seawater flow, mantle flow in the earths crust, water movement in geothermal reservoirs, underground spreading of chemical wastes and other pollutants, grain storage, thermal insulation, evaporative cooling and solidification [19, 40, 41], among others. When both the temperature and concentration differences occur simultaneously, the flow can

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become quite complex. A mathematical description of this kind of flows, in the so-called Oberbeck-Boussinesq approximation framework, is given by the incompressible Navier-Stokes/Brinkman equations for describing the underlying hydrodynamic in the porous media, and the advection-diffusion equations for both the substance concentration and the temperature, nonlinearly coupled via convective mass and heat transfer.

Motivated by the vast possible applications and the challenging mathematical structure of the nonlinearly coupled system, the interest in analyzing and developing efficient numerical techniques to simulate this and related phenomena has significantly increased (see [1, 2, 3, 5, 6, 8, 10, 13, 15, 21, 23, 22, 26, 28, 29, 31, 32, 34, 36, 39, 42, 44, 46, 47, 48], and the references therein). Those include numerical algorithms based on finite volume approaches, standard finite element techniques, parallel and projection-based stabilization methods, spectral collocation methods, control theory, and mixed finite element methods; and concentrate on heat-driven flows and double-diffusion convection, including cases in which the phenomena occurs in porous enclosures, with either constant or variable physical parameters and even time-dependent models.

More closely related contributions dealing with the phenomenon we address in the present work are [15, 21]. In [21], the authors propose a projection-based stabilization method for the Darcy-Brinkman system in double-diffusive convection in unsteady state. They focus on the convergence of the velocity, temperature and concentration in the semi-discrete case and present some numerical experiments to compare with previous studies. There, it is also confirmed that the proposed method provide optimal order of errors and that the results are in agreement with benchmark data. On the other hand, in [15] a divergence-conforming primal formulation for double-diffusive viscous flow in porous media is constructed and analyzed. The well-posedness of the respective discrete scheme and convergence properties are derived rigorously. In particular, the authors present numerical examples confirming the predicted rates of error and state that their scheme produces exactly divergence-free velocity approximations.

In certain applications some additional physically relevant variables, such as the gradient of the fluid velocity or the concentration and the temperature variations, might reveal specific mechanisms of the phenomena, and hence become of primary interest. Whilst these variables could be obtained via numerical integration of the discrete solutions provided by standard methods, this certainly would lead to a loss of accuracy or deteriorate the expected convergence order. In light of this, the purpose of this work is precisely to construct, analyze and implement a new high-order optimally convergent technique based on mixed finite elements for simulating double-diffusive convection in porous media, in which the velocity gradient, the temperature gradient and the concentration gradient are primary unknowns of interest.

To that end, we extend the theory developed in [22], for heat driven flows, and introduce the stress and the velocity gradient as auxiliary variables in the fluid equations, whereas in the temperature and the concentration equations are introduced the respective gradients and a vector version of the Bernoulli tensor that combines advective and diffusive heat and concentration fluxes, respectively. As a consequence, the resulting formulation retains the same saddle-point structure on reflexive Banach spaces for both the Navier-Stokes/Darcy and the thermal energy conservation equations. The latter feature constitutes a clear advantage (essentially from the theoretical point of view) since the continuous and discrete analyses for the two sub-models can be carried out separately and very much in the same way. Indeed, the well-known Banach and Brouwer theorems, combined with the application of the Babuška-Brezzi theory to each independent equation, lead to the solvability of the continuous and discrete schemes. It is further shown that Raviart-Thomas spaces of order $k \geq n-1$ for the Bernoulli tensor and its vector version, and piecewise polynomials of degree $\leq k$ for the velocity, the temperature, the concentration, and all gradients, are an adequate choice. This implies in particular, that the primal unknowns of velocity, pressure, temperature, and concentration end up being approximated by discontinuous functions, which can be appealing when rough solutions are expected.

Some advantages that this new method provides include:

(a) the pressure is eliminated by its own definition of associated function spaces and can be recovered by a simple postprocessing calculation,

(b) a reduced regularity requirement on the temperature and the concentration fields, allowing for more flexibility when choosing particular finite element spaces,

(c) the trace-free velocity gradient and the temperature and the concentration gradients become primary unknowns,

(d) differently from the methods constructed in [10, 15, 23, 25, 42], the Dirichlet boundary conditions for the temperature and concentration are naturally introduced into the formulation, avoiding the use of either an extension or a Lagrange multiplier on the boundary via a weak imposition,

(e) this scheme does not involve any augmentation term (as done, e.g. in [4, 5, 6, 8, 9]), avoiding restrictions regarding stabilization parameters for the well-posedness of the continuous and discrete problem as well as the convergence of the method.

(f) the analysis developed in this work can be adapted to a more general model in which cross-diffusion effects take place.

The rest of this work is organized as follows. At the end of the present section we describe some standard notations and functional spaces. In Section 2 we state the governing equations in strong primal form and in strong mixed form. Next, the continuous variational formulation is derived in Section 3, which, after decoupling the fluid equation from the heat and mass transfer equations, is rewritten as a fixed-point operator equation. The solvability analysis is performed by means of the Banach version of the classical Babuška-Brezzi theory, and the Banach fixed-point theorem. In Section 4 we define the Galerkin scheme with arbitrary finite element subspaces satisfying suitable assumptions, and follow basically the same techniques employed in Section 3 to analyze its solvability. We then specify finite element subspaces satisfying the assumptions stipulated in Section 4. Our analysis makes use of a sufficiency result developed in [22] (see, also [35]) for the occurrence of inf-sup conditions on products of reflexive Banach spaces. Furthermore, in Section 5 we assume sufficiently small data to derive an a priori error estimate for the Galerkin scheme with arbitrary finite element subspaces verifying the hypotheses from Section 4. Finally, several numerical examples illustrating the performance of our fully-mixed formulation with the particular subspaces proposed in Section 4, are reported in Section 6.

Recurrent notation and preliminaries

Let us denote by $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$ a given bounded domain with polyhedral boundary Γ, and denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. Given a generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Furthermore, as usual $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in R^n. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$ we set the tensor product operator as $\mathbf{v}\otimes\mathbf{w} := (v_i\, w_j)_{i,j=1,n}$. In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and denote by $\boldsymbol{\tau}^{\mathsf{t}}$, $\mathrm{tr}(\boldsymbol{\tau})$, and $\boldsymbol{\tau}^{\mathsf{d}}$, the transpose, the trace, and the deviatoric tensor of $\boldsymbol{\tau}$, respectively, and define the tensor inner product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$ as $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij}\zeta_{ij}$. Next, we introduce the Banach spaces $\mathbf{H}(\mathrm{div}_{4/3};\Omega) := \Big\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^{4/3}(\Omega)\Big\}$ and $\mathbb{H}(\mathbf{div}_{4/3};\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^{4/3}(\Omega)\Big\}$, equipped with the natural norms

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_{4/3};\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega} \quad \text{and} \quad \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}\,.$$

In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ is its dual. Also, by $\langle\cdot,\cdot\rangle_\Gamma$ we will denote the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$ (and also between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$).

2 Governing equations

The stationary Oberbeck-Boussinesq problem is constituted by the incompressible Navier-Stokes-Brinkman equation coupled with the heat and mass transfer equations through a convective term and a buoyancy term acting in opposite direction to gravity. The problem of interest (without dimensionless numbers for readability purposes) reduces to: Find a velocity field $\mathbf{u}$, a pressure field p, a temperature field φ_1 and a concentration field φ_2, both defining a vector $\boldsymbol{\varphi} := \big(\varphi_1,\varphi_2\big)$, such that

$$\begin{aligned}
\gamma\,\mathbf{u} - 2\mathbf{div}(\mu(\boldsymbol{\varphi})\,\boldsymbol{e}(\mathbf{u})) + (\nabla\mathbf{u})\mathbf{u} + \nabla p - (\boldsymbol{\vartheta}\cdot\boldsymbol{\varphi})g &= \mathbf{0} \quad \text{in} \quad \Omega\,,\\
\mathrm{div}\,\mathbf{u} &= 0 \quad \text{in} \quad \Omega\,,\\
-\mathrm{div}\big(\mathbb{K}_1\nabla\varphi_1\big) + \mathbf{u}\cdot\nabla\varphi_1 &= 0 \quad \text{in} \quad \Omega\,,\\
-\mathrm{div}\big(\mathbb{K}_2\nabla\varphi_2\big) + \mathbf{u}\cdot\nabla\varphi_2 &= 0 \quad \text{in} \quad \Omega\,,
\end{aligned} \tag{2.1}$$

where γ is a positive constant given by the reciprocal of the Darcy number Da, $\mu : \mathrm{R}\times\mathrm{R}^+ \longrightarrow \mathrm{R}^+$ is the viscosity of the fluid, which is assumed to depend on both the temperature and the concentration of mass, $\boldsymbol{e}(\mathbf{u}) := \dfrac{1}{2}\Big\{\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{t}}\Big\}$ is the rate of strain tensor, $\boldsymbol{\vartheta} := \big(\vartheta_1,\vartheta_2\big)$ is a vector containing expansion coefficients, $\boldsymbol{g} \in \mathbf{L}^\infty(\Omega)$ is an external force per unit mass, and $\mathbb{K}_j \in \mathbb{L}^\infty(\Omega)$, $j \in \big\{1,2\big\}$, are uniformly positive definite tensors allowing the possibility of anisotropy (cf. [37]). In addition, μ is assumed bounded and Lipschitz continuous, i.e., there exist μ_1, μ_2, $L_\mu > 0$, such that

$$\mu_1 \le \mu(\boldsymbol{\phi}) \le \mu_2 \quad \text{and} \quad |\mu(\boldsymbol{\phi}) - \mu(\boldsymbol{\psi})| \le L_\mu\,|\boldsymbol{\phi}-\boldsymbol{\psi}| \qquad \forall\,\boldsymbol{\phi},\,\boldsymbol{\psi} \in \mathrm{R}\times\mathrm{R}^+\,, \tag{2.2}$$

where $|\cdot|$ denotes from on the euclidean norm of R^n, $n \in \big\{1,2,3\big\}$. Equations (2.1) are complemented with Dirichlet boundary conditions for the velocity, the temperature, and the concentration, that is

$$\mathbf{u} = \mathbf{u}_D\,, \qquad \varphi_1 = \varphi_{1,D}\,, \qquad \text{and} \quad \varphi_2 = \varphi_{2,D} \quad \text{on} \quad \Gamma\,, \tag{2.3}$$

with given data $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, $\varphi_{1,D} \in \mathrm{H}^{1/2}(\Gamma)$ and $\varphi_{2,D} \in \mathrm{H}^{1/2}(\Gamma)$. Owing to the incompressibility of the fluid and the Dirichlet boundary condition for $\mathbf{u}$, the datum $\mathbf{u}_D$ must satisfy the compatibility condition $\int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\nu} = 0$. In addition, due to the first equation of (2.1), and in order to guarantee uniqueness of the pressure, this unknown will be sought in the space

$$\mathrm{L}^2_0(\Omega) := \Big\{q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0\Big\}\,.$$

On the other hand, in order to derive a fully-mixed formulation for (2.1) - (2.3), in which the Dirichlet boundary conditions become natural ones, we now proceed as in [22, Section 2] (see similar approaches in [4], [6], [24], and [25]), and introduce the velocity gradient and the Bernoulli stress tensor as further unknowns, that is

$$\mathbf{t} := \nabla \mathbf{u} \quad \text{and} \quad \boldsymbol{\sigma} := 2\mu(\boldsymbol{\varphi})\mathbf{t}_{sym} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u}) - p\mathbf{I}\,, \tag{2.4}$$

where $\mathbf{t}_{sym} := \dfrac{1}{2}\{\mathbf{t}+\mathbf{t}^{\mathtt{t}}\}$ is the symmetric part of $\mathbf{t}$. In this way, and noting thanks to the incompressibility condition that $\mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = (\nabla\mathbf{u})\mathbf{u}$, we find that the first equation of (2.1) becomes

$$\gamma\mathbf{u} - \mathbf{div}\,\boldsymbol{\sigma} + \frac{1}{2}\,\mathbf{t}\mathbf{u} - (\boldsymbol{\vartheta}\cdot\boldsymbol{\varphi})\boldsymbol{g} = \mathbf{0}\,.$$

In turn, applying the matrix trace to the expression defining $\boldsymbol{\sigma}$ and using that $\operatorname{tr}(\mathbf{t}_{sym}) = \operatorname{div}\mathbf{u} = 0$, one arrives at

$$p = -\frac{1}{2n}\operatorname{tr}\big(2\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}\big)\,, \tag{2.5}$$

which, replaced back into the second equation of (2.4), yields what we call from now on the new constitutive law of the fluid, namely

$$\boldsymbol{\sigma}^{\mathtt{d}} = 2\,\mu(\boldsymbol{\varphi})\mathbf{t}_{sym} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}}\,. \tag{2.6}$$

Conversely, starting from (2.5) and (2.6) we readily recover the incompressibility condition and the original definition of $\boldsymbol{\sigma}$, whence these pair of equations are actually equivalent. Furthermore, for the heat and mass transfer equations we proceed similarly as for the fluid, so that following now [22, eq. (2.7)], we introduce for each $j \in \{1,2\}$ the auxiliary unknowns

$$\widetilde{\mathbf{t}}_j := \nabla\varphi_j \quad \text{and} \quad \widetilde{\boldsymbol{\sigma}}_j := \mathbb{K}_j\widetilde{\mathbf{t}}_j - \frac{1}{2}\varphi_j\,\mathbf{u}\,. \tag{2.7}$$

They represent respectively the gradients and the total (diffusive plus advective) fluxes for temperature and concentration of solutes. Observing again from the incompressibility condition that in this case there holds $\operatorname{div}(\varphi_j\,\mathbf{u}) = \nabla\varphi_j\cdot\mathbf{u} = \widetilde{\mathbf{t}}_j\cdot\mathbf{u}$, our model problem (2.1) is re-stated as follows: Find $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma})$ and $(\varphi_j,\widetilde{\mathbf{t}}_j,\widetilde{\boldsymbol{\sigma}}_j)$, $j \in \{1,2\}$, in suitable spaces to be indicated below such that

$$\begin{aligned}
\nabla\mathbf{u} &= \mathbf{t} && \text{in } \Omega\,,\\
\gamma\mathbf{u} - \mathbf{div}\,\boldsymbol{\sigma} + \frac{1}{2}\,\mathbf{t}\mathbf{u} - (\boldsymbol{\vartheta}\cdot\boldsymbol{\varphi})\boldsymbol{g} &= \mathbf{0} && \text{in } \Omega\,,\\
2\mu(\boldsymbol{\varphi})\mathbf{t}_{sym} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} &= \boldsymbol{\sigma}^{\mathtt{d}} && \text{in } \Omega\,,\\
\nabla\varphi_j &= \widetilde{\mathbf{t}}_j && \text{in } \Omega\,,\\
\mathbb{K}_j\widetilde{\mathbf{t}}_j - \frac{1}{2}\varphi_j\,\mathbf{u} &= \widetilde{\boldsymbol{\sigma}}_j && \text{in } \Omega\,,\\
-\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}_j + \frac{1}{2}\widetilde{\mathbf{t}}_j\cdot\mathbf{u} &= 0 && \text{in } \Omega\,,\\
\mathbf{u} = \mathbf{u}_D \quad \text{and} \quad \boldsymbol{\varphi} &= \boldsymbol{\varphi}_D && \text{on } \Gamma\,,\\
\int_\Omega \operatorname{tr}(2\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) &= 0\,,
\end{aligned} \tag{2.8}$$

where the Dirichlet datum for $\boldsymbol{\varphi}$ is certainly given by $\boldsymbol{\varphi}_D := \big(\varphi_{1,D}, \varphi_{2,D}\big)$. At this point we stress that, as suggested by (2.5), p is eliminated from the present formulation and computed afterwards in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$ by using that identity. This fact justifies the last equation in (2.8), which aims to ensure that the resulting p does belong to $\mathrm{L}^2_0(\Omega)$.

3 Well-posedness of the continuous problem

In this section we derive a weak formulation for (2.8) and analyze its properties reusing most of the arguments that hold for the Boussinesq model in [22, eq. (2.8)], that is, we decouple the advection-diffusion equations from the fluid equations using a fixed-point argument. The main differences reside in the presence of an extra first order term $\gamma\mathbf{u}$ and the mass transfer equation, but the overall structure of the problem remains unchanged.

3.1 The fully-mixed formulation

Proceeding in a standard manner, we arrive at the following weak form of (2.8): Find $(\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}) \in \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and $(\varphi_j, \widetilde{\mathbf{t}}_j, \widetilde{\boldsymbol{\sigma}}_j) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, $j \in \big\{1,2\big\}$, such that $\displaystyle\int_\Omega \mathrm{tr}(2\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) = 0$, and

$$
\begin{aligned}
\int_\Omega \gamma\mathbf{u}\cdot\mathbf{v} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) + \frac{1}{2}\int_\Omega \mathbf{t}\mathbf{u}\cdot\mathbf{v} &= \int_\Omega (\boldsymbol{\vartheta}\cdot\boldsymbol{\varphi})\boldsymbol{g}\cdot\mathbf{v} && \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,,\\
\int_\Omega 2\mu(\boldsymbol{\varphi})\mathbf{t}_{sym} : \mathbf{s} - \frac{1}{2}\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \mathbf{s}^{\mathrm{d}} &= \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s}^{\mathrm{d}} && \forall\,\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,,\\
\int_\Omega \boldsymbol{\tau} : \mathbf{t} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) &= \langle\boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D\rangle_\Gamma && \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)\,,\\
-\int_\Omega \psi_j\,\mathrm{div}(\widetilde{\boldsymbol{\sigma}}_j) + \frac{1}{2}\int_\Omega \psi_j\widetilde{\mathbf{t}}_j\cdot\mathbf{u} &= 0 && \forall\,\psi_j\in\mathrm{L}^4(\Omega)\,,\\
\int_\Omega \mathbb{K}_j\widetilde{\mathbf{t}}_j\cdot\widetilde{\mathbf{s}}_j - \frac{1}{2}\int_\Omega \varphi_j\mathbf{u}\cdot\widetilde{\mathbf{s}}_j &= \int_\Omega \widetilde{\boldsymbol{\sigma}}_j\cdot\widetilde{\mathbf{s}}_j && \forall\,\widetilde{\mathbf{s}}_j\in\mathbf{L}^2(\Omega)\,,\\
\int_\Omega \widetilde{\boldsymbol{\tau}}_j\cdot\widetilde{\mathbf{t}}_j + \int_\Omega \varphi_j\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}_j) &= \langle\widetilde{\boldsymbol{\tau}}_j\cdot\boldsymbol{\nu}, \varphi_{j,D}\rangle_\Gamma && \forall\,\widetilde{\boldsymbol{\tau}}_j\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)\,,
\end{aligned}
\tag{3.1}
$$

where the Dirichlet boundary conditions for $\mathbf{u}$ and $\boldsymbol{\varphi}$ has been employed in the derivation of the foregoing weak formulation. Note here that the continuous injection of $\mathbf{H}^1(\Omega)$ in $\mathbf{L}^4(\Omega)$ (resp. the continuous injection of $\mathrm{H}^1(\Omega)$ in $\mathrm{L}^4(\Omega)$) guarantees that $\boldsymbol{\tau}\boldsymbol{\nu}$ (resp. $\widetilde{\boldsymbol{\tau}}_j\cdot\boldsymbol{\nu}$) is well defined and belongs to $\mathbf{H}^{-1/2}(\Gamma)$ (resp. $\mathrm{H}^{-1/2}(\Gamma)$) when $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ (resp. $\widetilde{\boldsymbol{\tau}}_j\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)$). On the other hand, notice that we look for $\mathbf{t}$ in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$ due to the incompressibility condition, where

$$
\mathbb{L}^2_{\mathrm{tr}}(\Omega) := \Big\{\mathbf{s}\in\mathbb{L}^2 : \quad \mathrm{tr}(\mathbf{s}) = 0\Big\}.
$$

We now consider, as in [22, eqs. (3.8) and (3.9)], the orthogonal decomposition (cf., e.g. [30, 43])

$$
\mathbb{H}(\mathbf{div}_{4/3};\Omega) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\mathbb{I}\,,\ \text{with}\ \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) := \Big\{\boldsymbol{\zeta}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega) : \int_\Omega \mathrm{tr}(\boldsymbol{\zeta}) = 0\Big\}, \tag{3.2}
$$

and (3.2) together with $\displaystyle\int_\Omega \mathrm{tr}(2\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) = 0$, imply that $\boldsymbol{\sigma}$ can be uniquely decomposed as

$$
\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0\,\mathbb{I}\,, \qquad \text{with} \quad \boldsymbol{\sigma}_0\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad c_0 := -\frac{1}{2n|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\,. \tag{3.3}
$$

Making abuse of notation, we will continue to denote $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and instead of (3.1) consider the equivalent formulation: Find $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and $(\varphi_j,\widetilde{\mathbf{t}}_j,\widetilde{\boldsymbol{\sigma}}_j) \in \mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, $j\in\{1,2\}$, such that (3.1) holds for all $(\mathbf{v},\mathbf{s},\boldsymbol{\tau}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and $(\boldsymbol{\psi}_j,\widetilde{\mathbf{s}}_j,\widetilde{\boldsymbol{\tau}}_j)\in \mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, $j\in\{1,2\}$. For sake of clarity in the presentation we introduce the following vector quantities

$$\vec{\mathbf{u}} := (\mathbf{u},\mathbf{t})\,,\quad \vec{\mathbf{v}} := (\mathbf{v},\mathbf{s})\,,\quad \vec{\mathbf{u}}_0 := (\mathbf{u}_0,\mathbf{t}_0) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,,$$

and

$$\vec{\boldsymbol{\varphi}}_j := (\varphi_j,\widetilde{\mathbf{t}}_j)\,,\quad \vec{\boldsymbol{\psi}}_j := (\psi_j,\widetilde{\mathbf{s}}_j) \in \mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\,,$$

with their corresponding norms given by

$$\|\vec{\mathbf{u}}\| := \|\mathbf{u}\|_{0,4;\Omega} + ||\mathbf{t}||_{0,\Omega} \qquad \forall\,\vec{\mathbf{u}}\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.4}$$

$$\|\vec{\boldsymbol{\varphi}}_j\| := \|\varphi_j\|_{0,4;\Omega} + ||\widetilde{\mathbf{t}}_j||_{0,\Omega} \qquad \forall\,\vec{\boldsymbol{\varphi}}_j\in\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\,. \tag{3.5}$$

Then, the fully-mixed formulation for the coupled problem reads: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\big)\times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $(\vec{\boldsymbol{\varphi}}_j,\widetilde{\boldsymbol{\sigma}}_j) \in \big(\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\big)\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, $j\in\{1,2\}$, such that

$$\begin{aligned}
a_{\boldsymbol{\varphi}}(\vec{\mathbf{u}},\vec{\mathbf{v}}) + c(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) + b(\vec{\mathbf{v}},\boldsymbol{\sigma}) &= F_{\boldsymbol{\varphi}}(\vec{\mathbf{v}}) && \forall\,\vec{\mathbf{v}}\in\big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\big)\,,\\
b(\vec{\mathbf{u}},\boldsymbol{\tau}) &= G(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\
\widetilde{a}_j(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j) + \widetilde{c}_{\mathbf{u}}(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j) + \widetilde{b}(\vec{\boldsymbol{\psi}}_j,\widetilde{\boldsymbol{\sigma}}_j) &= 0 && \forall\,\vec{\boldsymbol{\psi}}_j\in\big(\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\big)\,,\\
\widetilde{b}(\vec{\boldsymbol{\varphi}}_j,\widetilde{\boldsymbol{\tau}}_j) &= \widetilde{G}_j(\widetilde{\boldsymbol{\tau}}_j) && \forall\,\widetilde{\boldsymbol{\tau}}_j\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)\,,
\end{aligned} \tag{3.6}$$

where, given arbitrary $(\mathbf{w},\boldsymbol{\phi}) \in \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$, the forms $a_{\boldsymbol{\phi}}$, b, $c(\mathbf{w};\cdot,\cdot)$, $\widetilde{a}_j$, $\widetilde{b}$, and $\widetilde{c}_{\mathbf{w}}$, and the functionals $F_{\boldsymbol{\phi}}$, G, and $\widetilde{G}_j$, are defined by

$$a_{\boldsymbol{\phi}}(\vec{\mathbf{u}},\vec{\mathbf{v}}) := \int_\Omega \gamma\mathbf{u}\cdot\mathbf{v} + \int_\Omega 2\mu(\boldsymbol{\phi})\mathbf{t}_{sym}:\mathbf{s}\,,\quad b(\vec{\mathbf{v}},\boldsymbol{\tau}) := -\int_\Omega \boldsymbol{\tau}:\mathbf{s} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \tag{3.7}$$

$$c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) := \frac{1}{2}\left\{\int_\Omega \mathbf{t}\mathbf{w}\cdot\mathbf{v} - \int_\Omega (\mathbf{u}\otimes\mathbf{w})^{\mathrm{d}}:\mathbf{s}^{\mathrm{d}}\right\}\,, \tag{3.8}$$

for all $\vec{\mathbf{u}} := (\mathbf{u},\mathbf{t})$, $\vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, for all $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$,

$$\begin{gathered}
\widetilde{a}_j(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j) := \int_\Omega \mathbb{K}_j\widetilde{\mathbf{t}}_j\cdot\widetilde{\mathbf{s}}_j\,,\quad \widetilde{b}(\vec{\boldsymbol{\psi}}_j,\widetilde{\boldsymbol{\tau}}_j) := -\int_\Omega \widetilde{\boldsymbol{\tau}}_j\cdot\widetilde{\mathbf{s}}_j - \int_\Omega \psi_j\mathrm{div}(\widetilde{\boldsymbol{\tau}}_j)\,,\\
\widetilde{c}_{\mathbf{w}}(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j) := \frac{1}{2}\left\{\int_\Omega \boldsymbol{\psi}_j\widetilde{\mathbf{t}}_j\cdot\mathbf{w} - \int_\Omega \varphi_j\mathbf{w}\cdot\widetilde{\mathbf{s}}_j\right\}\,,
\end{gathered} \tag{3.9}$$

for all $\vec{\boldsymbol{\varphi}}_j := (\varphi_j,\widetilde{\mathbf{t}}_j)$, $\vec{\boldsymbol{\psi}}_j := (\psi_j,\widetilde{\mathbf{s}}_j)\in\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)$, for all $\widetilde{\boldsymbol{\tau}}_j\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, and

$$F_{\boldsymbol{\phi}}(\vec{\mathbf{v}}) := \int_\Omega (\boldsymbol{\vartheta}\cdot\boldsymbol{\phi})\boldsymbol{g}\cdot\mathbf{v}\,,\quad G(\boldsymbol{\tau}) := -\langle\boldsymbol{\tau}\,\nu,\mathbf{u}_D\rangle_\Gamma\,,\quad \widetilde{G}_j(\widetilde{\boldsymbol{\tau}}_j) := -\langle\widetilde{\boldsymbol{\tau}}_j\cdot\nu,\varphi_{j,D}\rangle_\Gamma\,, \tag{3.10}$$

for all $\vec{\mathbf{v}} := (\mathbf{v},\mathbf{s})\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, for all $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, for all $\widetilde{\boldsymbol{\tau}}_j\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)$.

In what follows we proceed similarly as in [9, 23] to prove that problem (3.6) is well-posed. More precisely, in Section 3.2 we will reformulate (3.6) as an equivalent fixed-point equation in terms of a suitable operator T. Then, in Section 3.3 we show that T is well-defined, and finally in Section 3.4 we apply the classical Banach theorem to conclude that T has a unique fixed point.

3.2 The fixed-point approach

We first let $S : \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega) \longrightarrow \mathbf{L}^4(\Omega)$ be the operator defined by

$$S(\mathbf{w}, \boldsymbol{\phi}) \,:=\, \mathbf{u} \qquad \forall\,(\mathbf{w}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)\,,$$

where $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) := \big((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}\big) \in \big(\mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)\big) \times \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ is the unique solution (to be confirmed below) of the problem:

$$\begin{aligned} a_{\boldsymbol{\phi}}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) \,+\, c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) \,+\, b(\vec{\mathbf{v}}, \boldsymbol{\sigma}) \;&=\; F_{\boldsymbol{\phi}}(\vec{\mathbf{v}}) \qquad \forall\,\vec{\mathbf{v}} \in \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \\ b(\vec{\mathbf{u}}, \boldsymbol{\tau}) \;&=\; G(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)\,. \end{aligned} \tag{3.11}$$

In turn, for each $j \in \big\{1, 2\big\}$ we let $\widetilde{S}_j : \mathbf{L}^4(\Omega) \longrightarrow \mathrm{L}^4(\Omega)$ be the operator given by

$$\widetilde{S}_j(\mathbf{w}) \,:=\, \varphi_j \qquad \forall\,\mathbf{w} \in \mathbf{L}^4(\Omega)\,,$$

where $(\vec{\boldsymbol{\varphi}}_j, \widetilde{\boldsymbol{\sigma}}_j) \,:=\, \big((\varphi_j, \widetilde{\mathbf{t}}_j), \widetilde{\boldsymbol{\sigma}}_j\big) \;\in\; \big(\mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega)\big) \times \mathbf{H}(\mathrm{div}_{4/3}; \Omega)$ is the unique solution (to be confirmed below) of the problem:

$$\begin{aligned} \widetilde{a}_j(\vec{\boldsymbol{\varphi}}_j, \vec{\boldsymbol{\psi}}_j) \,+\, \widetilde{c}_{\mathbf{w}}(\vec{\boldsymbol{\varphi}}_j, \vec{\boldsymbol{\psi}}_j) \,+\, \widetilde{b}(\vec{\boldsymbol{\psi}}_j, \widetilde{\boldsymbol{\sigma}}_j) \;&=\; 0 \qquad \forall\,\vec{\boldsymbol{\psi}}_j \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega)\,, \\ \widetilde{b}(\vec{\boldsymbol{\varphi}}_j, \widetilde{\boldsymbol{\tau}}_j) \;&=\; \widetilde{G}_j(\widetilde{\boldsymbol{\tau}}_j) \qquad \forall\,\widetilde{\boldsymbol{\tau}}_j \in \mathbf{H}(\mathrm{div}_{4/3}; \Omega)\,, \end{aligned} \tag{3.12}$$

so that we can introduce $\widetilde{S}(\mathbf{w}) \,:=\, \big(\widetilde{S}_1(\mathbf{w}), \widetilde{S}_2(\mathbf{w})\big) \in \mathbf{L}^4(\Omega)$ for all $\mathbf{w} \in \mathbf{L}^4(\Omega)$. Having defined the mappings S and $\widetilde{S}$, we now set $T : \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega) \longrightarrow \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$ as

$$T(\mathbf{w}, \boldsymbol{\phi}) \,:=\, \big(S(\mathbf{w}, \boldsymbol{\phi}), \widetilde{S}\big(S(\mathbf{w}, \boldsymbol{\phi})\big)\big) \qquad \forall\,(\mathbf{w}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)\,, \tag{3.13}$$

and realize that solving (3.6) is equivalent to finding $(\mathbf{u}, \boldsymbol{\varphi}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$ such that

$$T(\mathbf{u}, \boldsymbol{\varphi}) \,=\, (\mathbf{u}, \boldsymbol{\varphi})\,.$$

3.3 Well-definedness of the fixed-point operator

In what follows we show that T is well-defined, reducing to prove that the uncoupled problems (3.11) and (3.12) are well-posed. These results will be straightforward consequences of the Banach version of the Babuška-Brezzi theory (cf. [27, Theorem 2.34]). Note that the problems in (3.12) only differ in the bilinear forms $\widetilde{a}_j$ and the functionals $\widetilde{G}_j$ on the right-hand side of the second equation. However, since the tensors $\mathbb{K}_j$ defining the forms $\widetilde{a}_j$ satisfy exactly the same properties, the required hypotheses need to be checked only for a generic $\widetilde{a}_j$ and for $\widetilde{b}$.

We begin our analysis by observing, as in [22, eqs. (3.30), (3.31)], that the kernels of the operators induced by the bilinear forms b and $\widetilde{b}$, are given by $\mathbf{V}$ and $\widetilde{\mathbf{V}}$, respectively, where

$$\mathbf{V} := \Big\{\vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \nabla\mathbf{v} = \mathbf{s} \quad \text{and} \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega)\Big\}\,, \tag{3.14}$$

and

$$\widetilde{\mathbf{V}} := \Big\{\vec{\boldsymbol{\psi}}_j = (\psi_j, \widetilde{\mathbf{s}}_j) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) : \quad \nabla\psi_j = \widetilde{\mathbf{s}}_j \quad \text{and} \quad \psi_j \in \mathrm{H}^1_0(\Omega)\Big\}\,. \tag{3.15}$$

Next, we introduce the spaces $\mathbf{H} := \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)$ and $\widetilde{\mathbf{H}} := \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega)$, with norms given by (3.4) and (3.5), and readily establish the boundedness of $a_{\boldsymbol{\phi}}$, b, $\widetilde{a}_j$, and $\widetilde{b}$, by using the Cauchy-Schwarz inequality, the bound for μ (cf. (2.2)), and the fact that $\mathbb{K}_j \in \mathbb{L}^\infty(\Omega)$. More precisely, there hold

$$
\begin{aligned}
a_{\boldsymbol{\phi}}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) &\le (|\Omega|^{1/2}\gamma + 2\mu_2)\,\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \qquad \forall\, \boldsymbol{\phi} \in \mathbf{L}^4(\Omega)\,, \quad \forall\, \vec{\mathbf{u}},\, \vec{\mathbf{v}} \in \mathbf{H}\,, &(3.16)\\
b(\vec{\mathbf{v}}, \boldsymbol{\tau}) &\le \|\vec{\mathbf{v}}\|\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, &(3.17)\\
\widetilde{a}_j(\vec{\boldsymbol{\varphi}}_j, \vec{\boldsymbol{\psi}}_j) &\le \|\mathbb{K}_j\|_{0,\infty;\Omega}\,\|\vec{\boldsymbol{\varphi}}_j\|\,\|\vec{\boldsymbol{\psi}}_j\| \qquad \forall\, \vec{\boldsymbol{\varphi}}_j,\, \vec{\boldsymbol{\psi}}_j \in \widetilde{\mathbf{H}}\,, &(3.18)\\
\widetilde{b}(\vec{\boldsymbol{\psi}}_j, \widetilde{\boldsymbol{\tau}}_j) &\le \|\vec{\boldsymbol{\psi}}_j\|\,\|\widetilde{\boldsymbol{\tau}}_j\|_{\mathrm{div}_{4/3};\Omega} \qquad \forall\, \vec{\boldsymbol{\psi}}_j \in \widetilde{\mathbf{H}}\,, \quad \forall\, \widetilde{\boldsymbol{\tau}}_j \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,. &(3.19)
\end{aligned}
$$

In turn, the following lemma establishes the ellipticity of the bilinear forms a_ϕ and $\widetilde{a}_j$.

Lemma 3.1 *There exist positive constants α and $\widetilde{\alpha}_j$ such that*

$$a_{\boldsymbol{\phi}}(\vec{\mathbf{v}}, \vec{\mathbf{v}}) \ge \alpha\,\|\vec{\mathbf{v}}\|^2 \qquad \forall\, \boldsymbol{\phi} \in \mathbf{L}^4(\Omega)\,, \quad \forall \vec{\mathbf{v}} \in \mathbf{V}\,, \tag{3.20}$$

and

$$\widetilde{a}_j(\vec{\boldsymbol{\psi}}_j, \vec{\boldsymbol{\psi}}_j) \ge \widetilde{\alpha}_j\,\|\vec{\boldsymbol{\psi}}_j\|^2 \qquad \forall \vec{\boldsymbol{\psi}}_j \in \widetilde{\mathbf{V}}\,. \tag{3.21}$$

Proof. Given $\vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{V}$ and $\boldsymbol{\phi} \in \mathbf{L}^4(\Omega)$, we know from (3.14) that $\nabla \mathbf{v} = \mathbf{s}$ and $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$, which yields $\mathbf{e}(\mathbf{v}) = \mathbf{s}_{sym}$. Hence, applying the lower bound of μ (cf. (2.2)), the Korn inequality in $\mathbf{H}^1_0(\Omega)$, the continuous injection $\mathbf{i} : \mathbf{H}^1(\Omega) \longrightarrow \mathbf{L}^4(\Omega)$, and the Friedrichs-Poincaré inequality with constant c_p, we obtain

$$
\begin{aligned}
a_{\boldsymbol{\phi}}(\vec{\mathbf{v}}, \vec{\mathbf{v}}) &= \int_\Omega \gamma \mathbf{v}\cdot\mathbf{v} + \int_\Omega 2\mu(\boldsymbol{\phi})\mathbf{s}_{sym} : \mathbf{s}_{sym} \ge 2\mu_1\,\|\mathbf{s}_{sym}\|^2_{0,\Omega} = 2\mu_1\,\|\mathbf{e}(\mathbf{v})\|^2_{0,\Omega}\\
&\ge \mu_1\,|\mathbf{v}|^2_{1,\Omega} = \frac{\mu_1}{2}\,|\mathbf{v}|^2_{1,\Omega} + \frac{\mu_1}{2}\,\|\mathbf{s}\|^2_{0,\Omega} \ge \frac{\mu_1 c_p}{2\|\mathbf{i}\|^2}\,\|\mathbf{v}\|^2_{0,4;\Omega} + \frac{\mu_1}{2}\,\|\mathbf{s}\|^2_{0,\Omega}\,,
\end{aligned}
$$

which implies (3.20) with α depending on μ_1, c_p, and $\|\mathbf{i}\|$. The proof of (3.21), using that $\mathbb{K}_j$ is a uniformly positive definite tensor, and proceeding analogously to the one of (3.20), is omitted. $\square$

We find it important to remark that the $\mathbf{V}$-ellipticity of $a_{\boldsymbol{\phi}}$ does not depend on γ. This property will remain valid for the discrete case, and therefore this constant could be chosen arbitrarily small. In particular, while γ is related to Darcy's number, it could also arise from time discretization of the evolutionary problem. Next, we recall from [22] that b and $\widetilde{b}$ (cf. (3.7) and (3.9)) verify the inf-sup condition corresponding to the Banach version of the Babuška-Brezzi theory.

Lemma 3.2 *There exist positive constants β and $\widetilde{\beta}$ such that*

$$\sup_{\substack{\vec{\mathbf{v}} \in \mathbf{H}\\ \vec{\mathbf{v}} \neq 0}} \frac{b(\vec{\mathbf{v}}, \boldsymbol{\tau})}{\|\vec{\mathbf{v}}\|} \ge \beta\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,$$

and

$$\sup_{\substack{\vec{\boldsymbol{\psi}} \in \widetilde{\mathbf{H}}\\ \vec{\boldsymbol{\psi}} \neq 0}} \frac{\widetilde{b}(\vec{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\tau}})}{\|\vec{\boldsymbol{\psi}}\|} \ge \widetilde{\beta}\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_{4/3};\Omega} \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,.$$

Proof. See [22, Lemma 3.3]. $\square$

Furthermore, in what follows we collect also from [22] various fundamental properties of the forms $c(\mathbf{w};\cdot,\cdot)$ and $\widetilde{c}_{\mathbf{w}}$ that are instrumental for the forthcoming analysis.

Lemma 3.3 *The bilinear forms* $c(\mathbf{w};\cdot,\cdot):\mathbf{H}\times\mathbf{H}\to\mathrm{R}$ *and* $\widetilde{c}_{\mathbf{w}}:\widetilde{\mathbf{H}}\times\widetilde{\mathbf{H}}\to\mathrm{R}$ *are bounded for each* $\mathbf{w}\in\mathbf{L}^4(\Omega)$ *with boundedness constants given in both cases by* $\|\mathbf{w}\|_{0,4;\Omega}$*. Moreover:*

$$c(\mathbf{w};\vec{\mathbf{v}},\vec{\mathbf{v}})=0 \quad \textit{and} \quad \widetilde{c}_{\mathbf{w}}(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\varphi}}_j)=0 \qquad \forall\,\mathbf{w}\in\mathbf{L}^4(\Omega)\,,\quad \forall\,\vec{\mathbf{v}}\in\mathbf{H}\,,\quad \forall\,\vec{\boldsymbol{\varphi}}_j\in\widetilde{\mathbf{H}}, \tag{3.22}$$

$$\big|c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}})\,-\,c(\mathbf{z};\vec{\mathbf{u}},\vec{\mathbf{v}})\big|\,\le\,\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \qquad \forall\,\mathbf{w},\,\mathbf{z}\in\mathbf{L}^4(\Omega)\,,\quad \forall\,\vec{\mathbf{u}},\,\vec{\mathbf{v}}\in\mathbf{H}\,, \tag{3.23}$$

$$\big|\widetilde{c}_{\mathbf{w}}(\vec{\boldsymbol{\phi}}_j,\vec{\boldsymbol{\psi}}_j)\,-\,\widetilde{c}_{\mathbf{w}}(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j)\big|\,\le\,\|\mathbf{w}\|_{0,4;\Omega}\,\|\vec{\boldsymbol{\phi}}_j-\vec{\boldsymbol{\varphi}}_j\|\,\|\vec{\boldsymbol{\psi}}_j\| \quad \forall\,\mathbf{w}\in\mathbf{L}^4(\Omega)\,,\quad \forall\,\vec{\boldsymbol{\phi}}_j,\,\vec{\boldsymbol{\varphi}}_j,\,\vec{\boldsymbol{\psi}}_j\in\widetilde{\mathbf{H}}\,, \tag{3.24}$$

$$\big|\widetilde{c}_{\mathbf{w}}(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j)\,-\,\widetilde{c}_{\mathbf{z}}(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j)\big|\,\le\,\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\boldsymbol{\varphi}}_j\|\,\|\vec{\boldsymbol{\psi}}_j\| \quad \forall\,\mathbf{w},\,\mathbf{z}\in\mathbf{L}^4(\Omega)\,,\quad \forall\,\vec{\boldsymbol{\varphi}}_j,\,\vec{\boldsymbol{\psi}}_j\in\widetilde{\mathbf{H}}. \tag{3.25}$$

Proof. See [22, Lemma 3.4]. $\square$

Given $(\mathbf{w},\boldsymbol{\phi})\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$, we adopt a similar notation as in [22, Lemma 3.5, 3.6] and introduce the bilinear forms $\mathcal{A}_{\mathbf{w},\boldsymbol{\phi}}:\mathbf{H}\times\mathbf{H}\longrightarrow\mathrm{R}$ and $\widetilde{\mathcal{A}}_{\mathbf{w},j}:\widetilde{\mathbf{H}}\times\widetilde{\mathbf{H}}\longrightarrow\mathrm{R}$ defined by

$$\mathcal{A}_{\mathbf{w},\boldsymbol{\phi}}(\vec{\mathbf{u}},\vec{\mathbf{v}}):=a_{\boldsymbol{\phi}}(\vec{\mathbf{u}},\vec{\mathbf{v}})+c(\mathbf{w},\vec{\mathbf{u}},\vec{\mathbf{v}}) \quad \forall\,\vec{\mathbf{u}},\vec{\mathbf{v}}\in\mathbf{H} \tag{3.26}$$

$$\widetilde{\mathcal{A}}_{\mathbf{w},j}(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j):=\widetilde{a}_j(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j)+c_{\mathbf{w}}(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j) \qquad \forall\,\vec{\boldsymbol{\varphi}}_j,\,\vec{\boldsymbol{\psi}}_j\in\widetilde{\mathbf{H}}\,, \tag{3.27}$$

which, thanks to (3.16), (3.18) and Lemma 3.3, satisfy

$$\big|\mathcal{A}_{\mathbf{w},\boldsymbol{\phi}}(\vec{\mathbf{u}},\vec{\mathbf{v}})\big|\,\le\,\big(|\Omega|^{1/2}\gamma+2\mu_2+\|\mathbf{w}\|_{0,4;\Omega}\big)\,\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \qquad \forall\,\vec{\mathbf{u}},\,\vec{\mathbf{v}}\in\mathbf{H}\,, \tag{3.28}$$

$$|\widetilde{\mathcal{A}}_{\mathbf{w},j}(\vec{\boldsymbol{\varphi}}_j,\vec{\boldsymbol{\psi}}_j)|\,\le\,\big(\|\mathbb{K}_j\|_{0,\infty;\Omega}+\|\mathbf{w}\|_{0,4;\Omega}\big)\,\|\vec{\boldsymbol{\varphi}}_j\|\,\|\vec{\boldsymbol{\psi}}_j\| \qquad \forall\,\vec{\boldsymbol{\varphi}}_j,\,\vec{\boldsymbol{\psi}}_j\in\widetilde{\mathbf{H}}\,. \tag{3.29}$$

In addition, in virtue of Lemma 3.1 and (3.22), we readily see that $\mathcal{A}_{\mathbf{w},\boldsymbol{\phi}}$ and $\widetilde{\mathcal{A}}_{\mathbf{w},j}$ are $\mathbf{V}$-elliptic and $\widetilde{\mathbf{V}}$-elliptic, respectively, with the same constants α and $\widetilde{\alpha}_j$ from Lemma 3.1. According to these results and the inf-sup conditions satisfied by b and $\widetilde{b}$ (cf. Lemma 3.2), straightforward applications of the Babuška-Brezzi theory in Banach spaces imply that (3.11) and (3.12) are well-posed, equivalently that the operators S and $\widetilde{S}_j$, $j\in\{1,2\}$ (and hence $\widetilde{S}$), are all well-defined. More precisely, denoting $\|\mathbb{K}\|_{0,\infty;\Omega}:=\|\mathbb{K}_1\|_{0,\infty;\Omega}+\|\mathbb{K}_2\|_{0,\infty;\Omega}$, we are now in position to state the following lemmas.

Lemma 3.4 *For each* $(\mathbf{w},\boldsymbol{\phi})\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$*, problem* (3.11) *has a unique solution* $(\vec{\mathbf{u}},\boldsymbol{\sigma}):=\big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big)\in\mathbf{H}\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$*. Moreover, there exists* $C_S>0$*, independent of* $(\mathbf{w},\boldsymbol{\phi})$*, such that*

$$\|S(\mathbf{w},\boldsymbol{\phi})\|\,:=\,\|\mathbf{u}\|_{0,4;\Omega}\,\le\,C_S\left\{\|\boldsymbol{\phi}\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{0,\infty;\Omega}\,+\,\big(1\,+\,\|\mathbf{w}\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\right\}. \tag{3.30}$$

Lemma 3.5 *For each* $\mathbf{w}\in\mathbf{L}^4(\Omega)$*, and* $j\in\{1,2\}$*, problem* (3.12) *has a unique solution* $(\vec{\boldsymbol{\varphi}}_j,\widetilde{\boldsymbol{\sigma}}_j):=\big((\varphi_j,\widetilde{\mathbf{t}}_j),\widetilde{\boldsymbol{\sigma}}_j\big)\in\widetilde{\mathbf{H}}\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$*. Moreover, there exists* $C_{\widetilde{S}}>0$*, independent of* $\mathbf{w}$*, such that*

$$||\widetilde{S}(\mathbf{w})||:=||\big(\widetilde{S}_1(\mathbf{w}),\widetilde{S}_2(\mathbf{w})\big)||\,=\,\|(\varphi_1,\varphi_2)\|\le C_{\widetilde{S}}\left\{1\,+\,\|\mathbb{K}\|_{0,\infty;\Omega}\,+\,\|\mathbf{w}\|_{0,4;\Omega}\right\}\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}. \tag{3.31}$$

We refer to [22, Lemmas 3.5 and 3.6] for similar algebraic details on the a priori estimates (3.30) and (3.31), as well as for the explicit expressions for the constants C_S and $C_{\widetilde{S}}$.

3.4 Solvability of the fixed-point equation

Having proved that the operators S, $\widetilde{S}$, and hence T, are well-defined, we now follow [22] to establish the existence of a unique fixed point for T. For sake of simplicity of the remaining analysis, we consider a constant viscosity, but should μ depend on $\boldsymbol{\varphi}$, we would only need to assume further regularity on the solution of the problem defining S, exactly as we did in [22, Section 3.4]. In any case, the most distinctive aspects of our subsequent mathematical discussion will remain unchanged.

We begin by observing from (3.13), the a priori bounds for $\widetilde{S}$ (cf. Lemma 3.5) and S (cf. Lemma 3.4), and some algebraic manipulations, that for all $(\mathbf{w},\boldsymbol{\phi}) \in \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ there holds

$$\begin{aligned} \|T(\mathbf{w},\boldsymbol{\phi})\| \,&:=\, \|\big(S(\mathbf{w},\boldsymbol{\phi}),\widetilde{S}\big(S(\mathbf{w},\boldsymbol{\phi})\big)\big)\| \,=\, \|S(\mathbf{w},\boldsymbol{\phi})\| \,+\, \|\widetilde{S}\big(S(\mathbf{w},\boldsymbol{\phi})\big)\| \\ &\le\, \big(1 + C_{\widetilde{S}}\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\big)\,\|S(\mathbf{w},\boldsymbol{\phi})\| + C_{\widetilde{S}}\,\big(1 \,+\, \|\mathbb{K}\|_{0,\infty;\Omega}\big)\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma} \\ &\le\, C_S\,\max\big\{1,C_{\widetilde{S}}\big\}\,\big(1+\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\big)\,\big(\|\boldsymbol{g}\|_{0,\infty;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big)\,\big(1+\|(\mathbf{w},\boldsymbol{\phi})\|\big) \\ &\quad + C_{\widetilde{S}}\,\big(1 \,+\, \|\mathbb{K}\|_{0,\infty;\Omega}\big)\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\,, \end{aligned} \tag{3.32}$$

from which, assuming that $\|(\mathbf{w},\boldsymbol{\phi})\| \le r$, with $r>0$ given, we get

$$\|T(\mathbf{w},\boldsymbol{\phi})\| \le\, C(r)\,\Big\{\big(1+\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\big)\,\big(\|\boldsymbol{g}\|_{0,\infty;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big) \,+\, \big(1 \,+\, \|\mathbb{K}\|_{0,\infty;\Omega}\big)\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\Big\}, \tag{3.33}$$

with $C(r) := C_S\,\max\big\{1,C_{\widetilde{S}}\big\}\,(r+1)+C_{\widetilde{S}}$. In this way, denoting by W the closed ball of $\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ with radius r, we conclude from the foregoing estimate that if the data satisfy the assumption

$$\Big\{\big(1+\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\big)\,\big(\|\boldsymbol{g}\|_{0,\infty;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big) \,+\, \big(1 \,+\, \|\mathbb{K}\|_{0,\infty;\Omega}\big)\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\Big\} \,\le\, \frac{r}{C(r)}\,, \tag{3.34}$$

then the operator T maps W into itself.

In the following lemmas we establish the continuity of the operators S and $\widetilde{S}$.

Lemma 3.6 *Let α be the $\mathbf{V}$-ellipticity constant provided by Lemma 3.1 and let $L_S := \alpha^{-1}$. Then*

$$\|S(\mathbf{w},\boldsymbol{\phi}) - S(\mathbf{z},\boldsymbol{\psi})\| \,\le\, L_S\,\Big\{\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|S(\mathbf{z},\boldsymbol{\psi})\| \,+\, \|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{0,\infty;\Omega}\Big\}\,, \tag{3.35}$$

for all $(\mathbf{w},\boldsymbol{\phi})$, $(\mathbf{z},\boldsymbol{\psi}) \in \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$.

Proof. It proceeds similarly as in [22, Lemma 3.8 and eq. (3.64)]. We omit further details. □

Lemma 3.7 *There exists a positive constant $L_{\widetilde{S}}$, depending on $\widetilde{\alpha}$ and $C_{\widetilde{S}}$ (cf. Lemma 3.5), such that*

$$\|\widetilde{S}(\mathbf{w}) - \widetilde{S}(\mathbf{z})\| \le\, L_{\widetilde{S}}\,\|\mathbf{z}-\mathbf{w}\|_{0,4;\Omega}\,\Big\{\big(1+\|\mathbb{K}\|_{0,\infty;\Omega}\big)\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma} \,+\, \|\mathbf{z}\|_{0,4;\Omega}\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\Big\}, \tag{3.36}$$

for all $\mathbf{w},\ \mathbf{z} \in \mathbf{L}^4(\Omega)$.

Proof. Given $\mathbf{w},\ \mathbf{z} \in \mathbf{L}^4(\Omega)$, it suffices to recall that $\widetilde{S}(\mathbf{w}) - \widetilde{S}(\mathbf{z}) \,=\, \big(\widetilde{S}_1(\mathbf{w}) - \widetilde{S}_1(\mathbf{z}), \widetilde{S}_2(\mathbf{w}) - \widetilde{S}_2(\mathbf{z})\big)$, and then apply the continuity for each $\widetilde{S}_j$, $j \in \big\{1,2\big\}$, provided by [22, Lemma 3.9]. □

We are now in a position to establish the continuity of T as a consequence of Lemmas 3.6 and 3.7.

Lemma 3.8 *There holds*

$$\begin{aligned} \|T(\mathbf{w},\boldsymbol{\phi}) - T(\mathbf{z},\boldsymbol{\psi})\| \,&\le\, L_S\,\Big\{1 + L_{\widetilde{S}}\,\Big(1+\|\mathbb{K}\|_{0,\infty;\Omega} + \|S(\mathbf{z},\boldsymbol{\psi})\|\Big)\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\Big\} \\ &\qquad\times\Big\{\|(S(\mathbf{z},\boldsymbol{\psi})\| \,+\, \|\boldsymbol{g}\|_{0,\infty;\Omega}\Big\}\,\|(\mathbf{w},\boldsymbol{\phi}) - (\mathbf{z},\boldsymbol{\psi})\| \end{aligned} \tag{3.37}$$

for all $(\mathbf{w},\boldsymbol{\phi}),(\mathbf{z},\boldsymbol{\psi}) \in \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$.

Proof. According to the definition of T (cf. (3.13)), and employing the continuity estimate (3.36) for $\widetilde{S}$ (cf. Lemma 3.7), we readily find first that

$$\begin{aligned}\|T(\mathbf{w},\boldsymbol{\phi}) - T(\mathbf{z},\boldsymbol{\psi})\| \,&=\, \|S(\mathbf{w},\boldsymbol{\phi}) - S(\mathbf{z},\boldsymbol{\psi})\| \,+\, \|\widetilde{S}\big(S(\mathbf{w},\boldsymbol{\phi})\big) - \widetilde{S}\big(S(\mathbf{z},\boldsymbol{\psi})\big)\| \\ &\le\, \Big\{1 + L_{\widetilde{S}}\Big(1 + \|\mathbb{K}\|_{0,\infty;\Omega} + \|S(\mathbf{z},\boldsymbol{\psi})\|\Big)\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\Big\}\|S(\mathbf{w},\boldsymbol{\phi}) - S(\mathbf{z},\boldsymbol{\psi})\|\,,\end{aligned}$$

from which, appealing to the continuity estimate (3.35) for S (Lemma 3.6), we conclude the proof. □

Next, given $(\mathbf{z},\boldsymbol{\psi}) \in \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ such that $\|(\mathbf{z},\boldsymbol{\psi})\| \le r$, with $r > 0$ given, we deduce from the a priori estimate (3.30) for S (cf. 3.4) that

$$\|S(\mathbf{z},\boldsymbol{\psi})\| \,\le\, C_S\,(1+r)\Big\{\|\boldsymbol{g}\|_{0,\infty;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,.$$

In this way, inserting the foregoing estimate back into (3.37), and performing several suitable inequalities, we are able to show the Lipschitz-continuity of T, that is

$$\|T(\mathbf{w},\boldsymbol{\phi}) - T(\mathbf{z},\boldsymbol{\psi})\| \le\, L_T\,(1+r)^2\,C(\mathbb{K},\boldsymbol{g},\mathbf{u}_D,\boldsymbol{\varphi}_D)\,\Big(\|\boldsymbol{g}\|_{0,\infty;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big)\,\|(\mathbf{w},\boldsymbol{\phi}) - (\mathbf{z},\boldsymbol{\psi})\|, \tag{3.38}$$

for all $(\mathbf{w},\boldsymbol{\phi}) \in \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$, where $L_T \,:=\, L_S\,\max\big\{1, L_{\widetilde{S}}\big\}\,\big(\max\big\{1, C_S\big\}\big)^2$, and

$$C(\mathbb{K},\boldsymbol{g},\mathbf{u}_D,\boldsymbol{\varphi}_D) \,:=\, \Big\{1 + \Big(1 + \|\mathbb{K}\|_{0,\infty;\Omega} + \|\boldsymbol{g}\|_{0,\infty;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big)\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\Big\}\,. \tag{3.39}$$

We now establish sufficient conditions for the existence of a unique fixed point of T (equivalently, for the well-posedness of the coupled problem (3.6)). More precisely, we have the following result.

Theorem 3.9 *Given $r > 0$, let W be the closed ball in $\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ with center at the origin and radius r, and assume that the data satisfy* (3.34) *and*

$$L_T\,(1+r)^2\,C(\mathbb{K},\boldsymbol{g},\mathbf{u}_D,\boldsymbol{\varphi}_D)\,\Big(\|\boldsymbol{g}\|_{0,\infty;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big) \,<\, 1\,. \tag{3.40}$$

Then, the operator T has a unique fixed point $(\mathbf{u},\boldsymbol{\varphi}) \in W$. Equivalently, the coupled problem (3.6) *has a unique solution $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{H}\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $(\vec{\boldsymbol{\varphi}}_j,\widetilde{\boldsymbol{\sigma}}_j) := \big((\varphi_j,\widetilde{\mathbf{t}}_j),\widetilde{\boldsymbol{\sigma}}_j\big) \in \widetilde{\mathbf{H}}\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, $j \in \big\{1,2\big\}$, with $(\mathbf{u},\boldsymbol{\varphi}) := \big(\mathbf{u},(\varphi_1,\varphi_2)\big) \in W$. Moreover, there exist positive constants C_i, $i \in \big\{1,2,\dots,6\big\}$, depending on C_S, r, $C_{\widetilde{S}}$, $\|\mathbb{K}\|_{0,\infty;\Omega}$, $|\Omega|$, γ, μ_2, α, ϑ_1, ϑ_2, β, $\widetilde{\beta}$, and $\widetilde{\alpha}_j$, $j \in \big\{1,2\big\}$, such that the following a priori estimates hold*

$$\begin{aligned}\|\vec{\mathbf{u}}\| \,&\le\, C_1\,\|\boldsymbol{g}\|_{0,\infty;\Omega} \,+\, C_2\,\|\mathbf{u}_D\|_{1/2,\Gamma}\,, &(3.41)\\ \|\boldsymbol{\sigma}\| \,&\le\, C_3\,\|\boldsymbol{g}\|_{0,\infty;\Omega} \,+\, C_4\,\|\mathbf{u}_D\|_{1/2,\Gamma}\,, &(3.42)\\ \|\vec{\boldsymbol{\varphi}}_j\| \,&\le\, C_5\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\,, &(3.43)\\ \|\widetilde{\boldsymbol{\sigma}}_j\| \,&\le\, C_6\,\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\,. &(3.44)\end{aligned}$$

Proof. Let us recall from the first part of the present Subsection 3.4 that, under the assumption (3.34), T maps the ball W into itself. Then, thanks to (3.38) and (3.40), a straightforward application of Banach fixed-point theorem implies the existence of a unique fixed point $(\mathbf{u},\boldsymbol{\varphi}) \in W$ of T. In turn, the estimates (3.41), (3.43), (3.42), and (3.44) follow similarly to the derivation of the a priori estimates [22, eqs. (3.74), (3.75), (3.76) and (3.77), Theorem 3.11]. □

3.5 A case of cross-diffusion

In this section we briefly describe a related model to (2.1), which, on one hand is a particular case of that problem, and on the other hand constitutes a slight modification of it. More precisely, the temperature and concentration equations can accommodate cross-diffusion (see, for instance [15])

$$
\begin{aligned}
-\mathrm{div}\big(\mathrm{K}_{11}\nabla\varphi_1 + \mathrm{K}_{12}\nabla\varphi_2\big) + \mathbf{u}\cdot\nabla\varphi_1 &= 0 \quad \text{in} \quad \Omega\,,\\
-\mathrm{div}\big(\mathrm{K}_{21}\nabla\varphi_1 + \mathrm{K}_{22}\nabla\varphi_2\big) + \mathbf{u}\cdot\nabla\varphi_2 &= 0 \quad \text{in} \quad \Omega\,,
\end{aligned}
\tag{3.45}
$$

Here, the coefficients $\mathrm{K}_{ij}\in \mathrm{L}^\infty(\Omega),\ i,\,j\in\{1,2\}$, are appropriate scalar functions that need to satisfy adequate properties so that the equations remain well-defined. Introducing the tensor

$$
\mathbb{K} := \begin{pmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & \mathrm{K}_{22} \end{pmatrix} \in \mathbb{L}^\infty(\Omega)\,,
\tag{3.46}
$$

we realize that (3.45) can be rewritten as the system

$$
\begin{aligned}
-\mathbf{div}\big(\mathbb{K}\nabla\boldsymbol{\varphi}\big) + \big(\nabla\boldsymbol{\varphi}\big)\mathbf{u} &= 0 && \text{in} \quad \Omega\,,\\
\boldsymbol{\varphi} &= (\varphi_{1,D},\varphi_{2,D}) && \text{on} \quad \Gamma\,,
\end{aligned}
\tag{3.47}
$$

including the Dirichlet boundary conditions for the vector temperature-concentration. In this way, proceeding as in Sections 2 and 3.1, but instead of (2.7), setting

$$
\widetilde{\mathbf{t}} := \nabla\boldsymbol{\varphi} \quad \text{and} \quad \widetilde{\boldsymbol{\sigma}} := \mathbb{K}\widetilde{\mathbf{t}} - \frac{1}{2}\boldsymbol{\varphi}\otimes\mathbf{u}\,,
$$

we arrive at the following variational formulation for the coupling of (3.47) with the momentum and mass balance equations: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma}) := \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big) \in \big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\big)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $(\vec{\boldsymbol{\varphi}},\widetilde{\boldsymbol{\sigma}}) := \big((\boldsymbol{\varphi},\widetilde{\mathbf{t}}),\widetilde{\boldsymbol{\sigma}}\big) \in \big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2(\Omega)\big)\times\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ such that

$$
\begin{aligned}
a_{\boldsymbol{\varphi}}(\vec{\mathbf{u}},\vec{\mathbf{v}}) + c(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) + b(\vec{\mathbf{v}},\boldsymbol{\sigma}) &= F_{\boldsymbol{\varphi}}(\vec{\mathbf{v}}) && \forall\,\vec{\mathbf{v}}\in\big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\big)\,,\\
b(\vec{\mathbf{u}},\boldsymbol{\tau}) &= G(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\
\widetilde{a}(\vec{\boldsymbol{\varphi}},\vec{\boldsymbol{\psi}}) + \widetilde{c}_{\mathbf{u}}(\vec{\boldsymbol{\varphi}},\vec{\boldsymbol{\psi}}) + \widetilde{b}(\vec{\boldsymbol{\psi}},\widetilde{\boldsymbol{\sigma}}) &= 0 && \forall\,\vec{\boldsymbol{\psi}}\in\big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2(\Omega)\big)\,,\\
\widetilde{b}(\vec{\boldsymbol{\varphi}},\widetilde{\boldsymbol{\tau}}) &= \widetilde{G}(\widetilde{\boldsymbol{\tau}}) && \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)\,,
\end{aligned}
\tag{3.48}
$$

where, for a given $(\mathbf{w},\boldsymbol{\phi})\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$, the forms $a_{\boldsymbol{\phi}}$, b, and $c(\mathbf{w};\cdot,\cdot)$, and the functionals $F_{\boldsymbol{\phi}}$ and G, are defined as in (3.7),(3.8),(3.10), whereas $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}_{\mathbf{w}}$, and $\widetilde{G}$, are specified as

$$
\begin{aligned}
\widetilde{a}(\vec{\boldsymbol{\varphi}},\vec{\boldsymbol{\psi}}) &:= \int_\Omega \mathbb{K}\widetilde{\mathbf{t}}:\widetilde{\mathbf{s}}\,, \qquad \widetilde{b}(\vec{\boldsymbol{\psi}},\widetilde{\boldsymbol{\tau}}) := -\int_\Omega \widetilde{\boldsymbol{\tau}}:\widetilde{\mathbf{s}} - \int_\Omega \boldsymbol{\psi}\cdot\mathbf{div}(\widetilde{\boldsymbol{\tau}})\,,\\
\widetilde{c}_{\mathbf{w}}(\vec{\boldsymbol{\varphi}},\vec{\boldsymbol{\psi}}) &:= \frac{1}{2}\left\{\int_\Omega \widetilde{\mathbf{t}}\mathbf{w}\cdot\boldsymbol{\psi} - \int_\Omega (\boldsymbol{\varphi}\otimes\mathbf{w}):\widetilde{\mathbf{s}}\right\}\,, \qquad \widetilde{G}(\widetilde{\boldsymbol{\tau}}) := -\langle\widetilde{\boldsymbol{\tau}}\nu,\boldsymbol{\varphi}_D\rangle_\Gamma\,,
\end{aligned}
$$

for all $\vec{\boldsymbol{\varphi}} := (\boldsymbol{\varphi},\widetilde{\mathbf{t}})$, $\vec{\boldsymbol{\psi}} := (\boldsymbol{\psi},\widetilde{\mathbf{s}})\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2(\Omega)$, for all $\widetilde{\boldsymbol{\tau}}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$. Note that the well-posedness analysis for (3.48) follows almost verbatim to that in Sections 3.2-3.5 with a single $j\in\{1,2\}$ in (3.6), upon the assumption that $\mathbb{K}$ (cf. (3.46)) is uniformly positive definite. We omit further details.

4 The Galerkin scheme

We now devote ourselves to constructing a Galerkin method for (3.6). The solvability of this scheme is addressed following similar techniques as those employed throughout Section 3.

4.1 Preliminaries

Let us consider arbitrary finite dimensional subspaces $\mathbf{H}_h^{\mathbf{u}} \subseteq \mathbf{L}^4(\Omega)$, $\mathbb{H}_h^{\mathbf{t}} \subseteq \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}_h^{\boldsymbol{\sigma}} \subseteq \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathrm{H}_h^{\varphi} \subseteq \mathrm{L}^4(\Omega)$, $\mathbf{H}_h^{\widetilde{\mathbf{t}}} \subseteq \mathbf{L}^2(\Omega)$, and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \subseteq \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, whose specific choices are postponed to Section 4.3, below. Hereafter, $h := \max\big\{h_K : K \in \mathcal{T}_h\big\}$ stands for the size of a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ formed by triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K. Next, we denote

$$
\begin{gathered}
\vec{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h)\,, \quad \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h)\,, \quad \vec{\mathbf{u}}_{0,h} := (\mathbf{u}_{0,h}, \mathbf{t}_{0,h}) \,\in\, \mathbf{H}_h \,:=\, \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}},\\
\vec{\boldsymbol{\varphi}}_{j,h} := (\varphi_{j,h}, \widetilde{\mathbf{t}}_{j,h})\,, \quad \vec{\boldsymbol{\psi}}_{j,h} := (\psi_{j,h}, \widetilde{\mathbf{s}}_{j,h}) \,\in\, \widetilde{\mathbf{H}}_h := \mathrm{H}_h^{\varphi} \times \mathbf{H}_h^{\widetilde{\mathbf{t}}}\,.
\end{gathered}
$$

The Galerkin scheme associated with (3.6) reads: Find $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h}) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$, $j \in \big\{1,2\big\}$, such that

$$
\begin{aligned}
a_{\boldsymbol{\varphi}_h}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, c(\mathbf{u}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, b(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) \;&=\; F_{\boldsymbol{\varphi}_h}(\vec{\mathbf{v}}_h) && \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h\,,\\
b(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) \;&=\; G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,,\\
\widetilde{a}_j(\vec{\boldsymbol{\varphi}}_{j,h}, \vec{\boldsymbol{\psi}}_{j,h}) \,+\, \widetilde{c}_{\mathbf{u}_h}(\vec{\boldsymbol{\varphi}}_{j,h}, \vec{\boldsymbol{\psi}}_{j,h}) \,+\, \widetilde{b}(\vec{\boldsymbol{\psi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h}) \;&=\; 0 && \forall\, \vec{\boldsymbol{\psi}}_{j,h} \in \widetilde{\mathbf{H}}_h\,,\\
\widetilde{b}(\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\tau}}_{j,h}) \;&=\; \widetilde{G}_j(\widetilde{\boldsymbol{\tau}}_{j,h}) && \forall\, \widetilde{\boldsymbol{\tau}}_{j,h} \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\,.
\end{aligned}
\tag{4.1}
$$

We now follow a discrete analogue of the fixed-point approach developed in Section 3.2. To this end, we first let $\mathbf{H}_h^{\boldsymbol{\varphi}} := \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\varphi}$ and introduce the operator $S_h : \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}} \to \mathbf{H}_h^{\mathbf{u}}$ defined by

$$
S_h(\mathbf{w}_h, \boldsymbol{\phi}_h) \,:=\, \mathbf{u}_h \qquad \forall\, (\mathbf{w}_h, \boldsymbol{\phi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}}\,,
$$

where $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) = \big((\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\sigma}_h\big) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ is the unique solution (to be confirmed below) of the problem

$$
\begin{aligned}
a_{\boldsymbol{\phi}_h}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, c(\mathbf{w}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, b(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) \;&=\; F_{\boldsymbol{\phi}_h}(\vec{\mathbf{v}}_h) && \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h\,,\\
b(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) \;&=\; G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,.
\end{aligned}
\tag{4.2}
$$

In turn, for each $j \in \big\{1,2\big\}$ we let $\widetilde{S}_{j,h} : \mathbf{H}_h^{\mathbf{u}} \to \mathrm{H}_h^{\varphi}$ be the operator given by

$$
\widetilde{S}_{j,h}(\mathbf{w}_h) \,:=\, \varphi_{j,h} \qquad \forall\, \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}\,,
$$

where $(\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h}) = \big((\varphi_{j,h}, \widetilde{\mathbf{t}}_{j,h}), \widetilde{\boldsymbol{\sigma}}_{j,h}\big) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$ is the unique solution (to be confirmed below) of the problem

$$
\begin{aligned}
\widetilde{a}_j(\vec{\boldsymbol{\varphi}}_{j,h}, \vec{\boldsymbol{\psi}}_{j,h}) \,+\, \widetilde{c}_{\mathbf{w}_h}(\vec{\boldsymbol{\varphi}}_{j,h}, \vec{\boldsymbol{\psi}}_{j,h}) \,+\, \widetilde{b}(\vec{\boldsymbol{\psi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h}) \;&=\; 0 && \forall\, \vec{\boldsymbol{\psi}}_{j,h} \in \widetilde{\mathbf{H}}_h\,,\\
\widetilde{b}(\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\tau}}_{j,h}) \;&=\; \widetilde{G}_j(\widetilde{\boldsymbol{\tau}}_{j,h}) && \forall\, \widetilde{\boldsymbol{\tau}}_{j,h} \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\,,
\end{aligned}
\tag{4.3}
$$

and then we introduce $\widetilde{S}_h(\mathbf{w}_h) := \big(\widetilde{S}_{1,h}(\mathbf{w}_h), \widetilde{S}_{2,h}(\mathbf{w}_h)\big) \in \mathbf{H}_h^{\boldsymbol{\varphi}}$ for all $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$. Hence, defining $T_h : \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}} \to \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}}$ as

$$T_h(\mathbf{w}_h, \boldsymbol{\phi}_h) := \big(S_h(\mathbf{w}_h, \boldsymbol{\phi}_h), \widetilde{S}_h\big(S_h(\mathbf{w}_h, \boldsymbol{\phi}_h)\big)\big) \qquad \forall\, (\mathbf{w}_h, \boldsymbol{\phi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}}\,, \tag{4.4}$$

we realize that solving (4.1) is equivalent to seeking a fixed point of T_h, that is: Find $(\mathbf{u}_h, \boldsymbol{\varphi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}}$ such that

$$T_h(\mathbf{u}_h, \boldsymbol{\varphi}_h) = (\mathbf{u}_h, \boldsymbol{\varphi}_h)\,. \tag{4.5}$$

4.2 Solvability of the discrete problem

We now aim to establish the well-posedness of (4.1) by studying the solvability of the equivalent equation (4.5) using Brouwer's fixed-point theorem (cf. [20, Theorem 9.9-2]). Exactly as for the continuous case, we begin by showing that S_h and $\widetilde{S}_{j,h}$, $j \in \{1,2\}$, and hence $\widetilde{S}_h$ and T_h, are well-defined. For this purpose, we need to establish hypotheses on the (so far, arbitrary) discrete spaces. Subsequently we will specify suitable finite element spaces satisfying these conditions.

In what follows, we let $\mathbf{V}_h$ and $\widetilde{\mathbf{V}}_h$ be the discrete kernels of b and $\widetilde{b}$, respectively, that is

$$\mathbf{V}_h := \Big\{ \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{H}_h : \quad \int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_h + \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) = 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \Big\}\,,$$

$$\widetilde{\mathbf{V}}_h := \Big\{ \vec{\boldsymbol{\psi}}_{j,h} := (\boldsymbol{\psi}_{j,h}, \widetilde{\mathbf{s}}_{j,h}) \in \widetilde{\mathbf{H}}_h : \quad \int_\Omega \widetilde{\boldsymbol{\tau}}_{j,h} \cdot \widetilde{\mathbf{s}}_{j,h} + \int_\Omega \psi_{j,h} \operatorname{div}(\widetilde{\boldsymbol{\tau}}_{j,h}) = 0 \qquad \forall\, \widetilde{\boldsymbol{\tau}}_{j,h} \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \Big\}\,.$$

In addition, for each $\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}}$ we denote by $\mathbf{s}_{h,sym}$ and $\mathbf{s}_{h,skw}$ its symmetric and skew-symmetric parts, respectively.

Then, we consider the following hypotheses on the discrete subspaces employed:

Assumption **4.1** *There exists a positive constant $\beta_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\vec{\mathbf{v}}_h \in \mathbf{H}_h \\ \vec{\mathbf{v}}_h \neq \mathbf{0}}} \frac{b(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h)}{\|\vec{\mathbf{v}}_h\|} \geq \beta_{\mathtt{d}}\, \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,. \tag{4.6}$$

Assumption **4.2** *There exists a positive constant $C_{\mathtt{d}}$, independent of h, such that*

$$\|\mathbf{s}_{h,sym}\|_{0,\Omega} \geq C_{\mathtt{d}}\, \|(\mathbf{v}_h, \mathbf{s}_{h,skw})\| \qquad \forall\, \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h\,. \tag{4.7}$$

Assumption **4.3** *There exists a positive constant $\widetilde{\beta}_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\vec{\boldsymbol{\psi}}_{j,h} \in \widetilde{\mathbf{H}}_h \\ \vec{\boldsymbol{\psi}}_{j,h} \neq \mathbf{0}}} \frac{\widetilde{b}(\vec{\boldsymbol{\psi}}_{j,h}, \widetilde{\boldsymbol{\tau}}_{j,h})}{\|\vec{\boldsymbol{\psi}}_{j,h}\|} \geq \widetilde{\beta}_{\mathtt{d}}\, \|\widetilde{\boldsymbol{\tau}}_{j,h}\|_{\operatorname{div}_{4/3};\Omega} \qquad \forall\, \widetilde{\boldsymbol{\tau}}_{j,h} \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\,. \tag{4.8}$$

Assumption **4.4** *There exists a positive constant $\widetilde{C}_{\mathtt{d}}$, independent of h, such that*

$$\|\widetilde{\mathbf{s}}_{j,h}\|_{0,\Omega} \geq \widetilde{C}_{\mathtt{d}}\, \|\psi_{j,h}\|_{0,4;\Omega} \qquad \forall\, \vec{\boldsymbol{\psi}}_{j,h} := (\psi_{j,h}, \widetilde{\mathbf{s}}_{j,h}) \in \widetilde{\mathbf{V}}_h\,. \tag{4.9}$$

As consequence of ASSUMPTIONS 4.1 and 4.2, and following basically the same procedure and notations from [22, Lemma 4.2], we are able to establish next the well-definedness of S_h, which constitutes the discrete analogue of Lemma 3.4.

Lemma 4.1 *For each $(\mathbf{w}_h, \boldsymbol{\phi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}}$, (4.2) has a unique solution $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) := \big((\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\sigma}_h\big) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$. Moreover there exists a positive constant $C_{S,\mathtt{d}}$, independent of h and $(\mathbf{w}_h, \boldsymbol{\phi}_h)$, such that*

$$\|S_h(\mathbf{w}_h, \boldsymbol{\phi}_h)\| \,:=\, \|\mathbf{u}_h\| \,\le\, C_{S,\mathtt{d}} \Big\{ \|\boldsymbol{\phi}_h\|_{0,4;\Omega} \, \|\boldsymbol{g}\|_{0,\infty;\Omega} \,+\, \big(1 + \|\mathbf{w}_h\|_{0,4;\Omega}\big) \, \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\}\,. \tag{4.10}$$

Proof. Given $(\mathbf{w}_h, \boldsymbol{\phi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}}$, we first recall from (3.26) and (3.28) that $\mathcal{A}_{\mathbf{w}_h,\boldsymbol{\phi}_h}$ is bounded. Then, for each $\vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h$ we easily deduce from (2.2), (4.7) (cf. ASSUMPTION 4.2), and a simple algebraic manipulation, that

$$a_{\boldsymbol{\phi}_h}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \,=\, \int_\Omega \gamma \mathbf{v}_h \cdot \mathbf{v}_h + \int_\Omega 2\mu(\boldsymbol{\phi}_h)\, \mathbf{s}_{h,sym} : \mathbf{s}_{h,sym} \,\ge\, \mu_1 \, \min\big\{1, C_{\mathtt{d}}^2\big\} \, \|\vec{\mathbf{v}}_h\|^2\,,$$

which, together with the fact that $c(\mathbf{w}_h; \vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) = 0 \;\forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h$ (cf. (3.22)), yields the $\mathbf{V}_h$-ellipticity of both $a_{\boldsymbol{\phi}_h}$ and $\mathcal{A}_{\mathbf{w}_h,\boldsymbol{\phi}_h}$ with constant $\alpha_{\mathtt{d}} \,:=\, \mu_1 \, \min\big\{1, C_{\mathtt{d}}^2\big\}$. In turn, it is clear from ASSUMPTION 4.1 that b satisfies the discrete inf-sup condition required by the Babuška-Brezzi theorem in Banach spaces. Invoking then that theorem we readily obtain both the unique solvability of (4.2) and the a priori estimate (4.10), with a positive constant $C_{S,\mathtt{d}}$ depending on Ω, μ_2, $\boldsymbol{\vartheta}$, γ, $\alpha_{\mathtt{d}}$ and $\beta_{\mathtt{d}}$. $\square$

Similarly as we did in the continuous case, we remark that the $\mathbf{V}_h$-ellipticity of $a_{\boldsymbol{\phi}_h}$, and hence of $\mathcal{A}_{\mathbf{w}_h,\boldsymbol{\phi}_h}$, does not depend on γ, certainly yielding the same appealing features mentioned in Section 3.3.

Next, as consequence of ASSUMPTIONS 4.3 and 4.4, we provide the well-definedness of $\widetilde{S}_{j,h}$, $j \in \big\{1, 2\big\}$, and hence of $\widetilde{S}_h$, thus establishing the discrete analogue of Lemma 3.5.

Lemma 4.2 *For each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, and for each $j \in \big\{1, 2\big\}$, (4.3) has a unique solution $(\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h}) := \big((\varphi_{j,h}, \widetilde{\mathbf{t}}_{j,h}), \widetilde{\boldsymbol{\sigma}}_{j,h}\big) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$. Moreover, there exists a positive constant $C_{\widetilde{S},\mathtt{d}}$, independent of h and $\mathbf{w}_h$, such that*

$$\begin{aligned} \|\widetilde{S}_h(\mathbf{w}_h)\| &:= \|\big(\widetilde{S}_{1,h}(\mathbf{w}_h), \widetilde{S}_{2,h}(\mathbf{w}_h)\big)\| \,=\, \|(\varphi_{1,h}, \varphi_{2,h})\| \\ &\le\, C_{\widetilde{S},\mathtt{d}} \Big\{ 1 \,+\, \|\mathbb{K}\|_{0,\infty;\Omega} \,+\, \|\mathbf{w}_h\|_{0,4;\Omega} \Big\} \, \|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\,. \end{aligned} \tag{4.11}$$

Proof. Given $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, we know from (3.27) and (3.29) that each $\widetilde{\mathcal{A}}_{\mathbf{w}_h,j}$ is bounded. In addition, it is easy to see, thanks to the uniform positive definiteness of $\mathbb{K}_j$, the ASSUMPTION 4.4, and the fact that $\widetilde{c}_{\mathbf{w}_h}(\vec{\boldsymbol{\varphi}}_{j,h}, \vec{\boldsymbol{\varphi}}_{j,h}) = 0 \;\forall\, \vec{\boldsymbol{\varphi}}_{j,h} \in \widetilde{\mathbf{H}}_h$ (cf. (3.22)), that $\widetilde{a}_j$ and $\widetilde{\mathcal{A}}_{\mathbf{w}_h,j}$ are $\widetilde{\mathbf{V}}_h$-elliptic with a positive constant $\widetilde{\alpha}_{j,\mathtt{d}}$. In turn, it is clear from ASSUMPTION 4.3 that $\widetilde{b}$ satisfies an adequate discrete inf-sup condition and then the Banach version of the Babuška-Brezzi theory implies unique solvability of (4.3) for each $j \in \big\{1, 2\big\}$. Moreover, a priori estimates for each $\widetilde{S}_{j,h}(\mathbf{w}_h)$ imply (4.11) with a positive constant $C_{\widetilde{S},\mathtt{d}}$ depending on $\widetilde{\alpha}_{\mathtt{d}}$ and $\widetilde{\beta}_{\mathtt{d}}$. $\square$

Proceeding as in the beginning of Section 3.4 but now for T_h (cf. (4.4)), we employ the a priori bounds (4.10) and (4.11), and denote by W_h the closed ball of $\mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}}$ with center at the origin and radius r. We find that for each $(\mathbf{w}_h, \boldsymbol{\phi}_h) \in W_h$ there holds

$$\|T_h(\mathbf{w}_h, \boldsymbol{\phi}_h)\| \le\, C_{\mathtt{d}}(r) \Big\{ \big(1 + \|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\big) \big(\|\boldsymbol{g}\|_{0,\infty;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big) \,+\, \big(1 \,+\, \|\mathbb{K}\|_{0,\infty;\Omega}\big) \, \|\boldsymbol{\varphi}_D\|_{1/2,\Gamma} \Big\}\,,$$

with $C_{\mathtt{d}}(r) := C_{S,\mathtt{d}} \max\big\{1, C_{\widetilde{S},\mathtt{d}}\big\}(r+1) + C_{\widetilde{S},\mathtt{d}}$. It readily follows that, under the assumption

$$\Big\{\big(1+\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\big)\big(\|\boldsymbol{g}\|_{0,\infty;\Omega}+\|\mathbf{u}_D\|_{1/2,\Gamma}\big) + \big(1+\|\mathbb{K}\|_{0,\infty;\Omega}\big)\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\Big\} \le \frac{r}{C_{\mathtt{d}}(r)}, \tag{4.12}$$

the operator T_h maps W_h into itself.

In analogy with the continuous case, the continuity of T_h follows from that of S_h, $\widetilde{S}_{j,h}$, $j \in \big\{1,2\big\}$, and hence $\widetilde{S}_h$. Proceeding as in [22, Lemmas 4.5 and 4.6], we prove the discrete analogues of Lemmas 3.6 and 3.7. More precisely, there exist positive constants $L_{S,\mathtt{d}}$ and $L_{\widetilde{S},\mathtt{d}}$, both independent of h, the first one given by $\alpha_{\mathtt{d}}^{-1}$ (cf. proof of Lemma 4.1), and the second one depending on $\widetilde{\alpha}_{\mathtt{d}}$ and $C_{\widetilde{S},\mathtt{d}}$ (cf. proof of Lemma 4.2), such that

$$\|S_h(\mathbf{w}_h,\boldsymbol{\phi}_h) - S_h(\mathbf{z}_h,\boldsymbol{\psi}_h)\| \le L_{S,\mathtt{d}}\Big\{\|\mathbf{w}_h-\mathbf{z}_h\|_{0,4;\Omega}\|S_h(\mathbf{z}_h,\boldsymbol{\psi}_h)\| + \|\boldsymbol{\phi}_h-\boldsymbol{\psi}_h\|_{0,4;\Omega}\|\boldsymbol{g}\|_{0,\infty;\Omega}\Big\}, \tag{4.13}$$

for all $(\mathbf{w}_h,\boldsymbol{\phi}_h)$, $(\mathbf{z}_h,\boldsymbol{\psi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}}$, and

$$\|\widetilde{S}_h(\mathbf{w}_h) - \widetilde{S}_h(\mathbf{z}_h)\| \le L_{\widetilde{S},\mathtt{d}}\|\mathbf{z}_h-\mathbf{w}_h\|_{0,4;\Omega}\Big\{\big(1+\|\mathbb{K}\|_{0,\infty;\Omega}\big)\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma} + \|\mathbf{z}_h\|_{0,4;\Omega}\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}\Big\}, \tag{4.14}$$

for all $\mathbf{w}_h,\ \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$. Then, as a straightforward consequence of (4.13) and (4.14), and following the same steps from the second half of Section 3.4, we arrive at the discrete analogue of (3.38), that is

$$\begin{aligned}&\|T_h(\mathbf{w}_h,\boldsymbol{\phi}_h) - T_h(\mathbf{z}_h,\boldsymbol{\psi}_h)\| \\ &\quad \le L_{T,\mathtt{d}}(1+r)^2 C(\mathbb{K},\boldsymbol{g},\mathbf{u}_D,\boldsymbol{\varphi}_D)\Big(\|\boldsymbol{g}\|_{0,\infty;\Omega}+\|\mathbf{u}_D\|_{1/2,\Gamma}\Big)\|(\mathbf{w}_h,\boldsymbol{\phi}_h)-(\mathbf{z}_h,\boldsymbol{\psi}_h)\|\end{aligned} \tag{4.15}$$

for all $(\mathbf{w}_h,\boldsymbol{\phi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\varphi}}$, where

$$L_{T,\mathtt{d}} := L_{S,\mathtt{d}} \max\big\{1, L_{\widetilde{S},\mathtt{d}}\big\}\big(\max\big\{1, C_{S,\mathtt{d}}\big\}\big)^2,$$

and $C(\mathbb{K},\boldsymbol{g},\mathbf{u}_D,\boldsymbol{\varphi}_D)$ is given by (3.39).

Then, we are in position to establish our main result.

Theorem 4.3 *Assume that the data satisfy* (4.12) *and*

$$L_{T,\mathtt{d}}(1+r)^2 C(\mathbb{K},\boldsymbol{g},\mathbf{u}_D,\boldsymbol{\varphi}_D)\Big(\|\boldsymbol{g}\|_{0,\infty;\Omega}+\|\mathbf{u}_D\|_{1/2,\Gamma}\Big) < 1. \tag{4.16}$$

Then, the operator T_h has a unique fixed point $(\mathbf{u}_h,\boldsymbol{\varphi}_h) \in W_h$. Equivalently, the coupled problem (4.1) *has a unique solution $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\vec{\boldsymbol{\varphi}}_{j,h},\widetilde{\boldsymbol{\sigma}}_{j,h}) := \big((\varphi_{j,h},\widetilde{\mathbf{t}}_j),\widetilde{\boldsymbol{\sigma}}_{j,h}\big) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$, $j \in \big\{1,2\big\}$, with $(\mathbf{u}_h,\boldsymbol{\varphi}_h) := \big(\mathbf{u}_h,(\varphi_{1,h},\varphi_{2,h})\big) \in W_h$. Moreover, there exist positive constants $C_{i,\mathtt{d}}$, $i \in \big\{1,2,\dots,6\big\}$, depending on $C_{S,\mathtt{d}}$, r, $C_{\widetilde{S},\mathtt{d}}$, $\|\mathbb{K}\|_{0,\infty;\Omega}$, $|\Omega|$, γ, μ_2, $\alpha_{\mathtt{d}}$, ϑ_1, ϑ_2, $\beta_{\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, and $\widetilde{\alpha}_{j,\mathtt{d}}$, $j \in \big\{1,2\big\}$, such that the following a priori estimates hold*

$$\|\vec{\mathbf{u}}_h\| \le C_{1,\mathtt{d}}\|\boldsymbol{g}\|_{0,\infty;\Omega} + C_{2,\mathtt{d}}\|\mathbf{u}_D\|_{1/2,\Gamma}, \tag{4.17}$$

$$\|\boldsymbol{\sigma}_h\| \le C_{3,\mathtt{d}}\|\boldsymbol{g}\|_{0,\infty;\Omega} + C_{4,\mathtt{d}}\|\mathbf{u}_D\|_{1/2,\Gamma}, \tag{4.18}$$

$$\|\vec{\boldsymbol{\varphi}}_{j,h}\| \le C_{5,\mathtt{d}}\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}, \tag{4.19}$$

$$\|\widetilde{\boldsymbol{\sigma}}_{j,h}\| \le C_{6,\mathtt{d}}\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}. \tag{4.20}$$

Proof. We first recall that, under the assumption (4.12), T_h maps W_h into itself. Then, (4.15), (4.16), and the Banach fixed-point theorem conclude the proof. The a priori estimates (4.17), (4.19), (4.18), and (4.20) are derived similarly as for [22, eqs. (4.26)-(4.29), Theorem 4.8]. $\square$

We end this section by remarking that if the viscosity depends on temperature and concentration, it is not possible to establish the Lipschitz-continuity of T_h (cf. (4.15)), but just continuity. Consequently, instead of the Banach theorem, the Brouwer fixed-point theorem is applied, thus yielding only existence of the discrete solution. For related details, we refer to [22, Section 4.2].

4.3 Specific finite element subspaces

In this section we specify finite element subspaces $\mathbf{H}_h^{\mathbf{u}} \subseteq \mathbf{L}^4(\Omega)$, $\mathbb{H}_h^{\mathbf{t}} \subseteq \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}_h^{\boldsymbol{\sigma}} \subseteq \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathrm{H}_h^{\varphi} \subseteq \mathrm{L}^4(\Omega)$, $\mathbf{H}_h^{\widetilde{\mathbf{t}}} \subseteq \mathbf{L}^2(\Omega)$, and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \subseteq \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, satisfying the crucial discrete inf-sup conditions given by Assumptions 4.1, 4.2, 4.3, and 4.4. These discrete spaces arise naturally as consequence of the same analysis developed in [22, Section 5], which is based on stable finite element subspaces for the primal formulation of the Stokes problem (see also [11] for the case of linear elasticity). In particular, here we propose those obtained by considering the Scott-Vogelius pair (cf. [45]). Given a positive integer ℓ and a set $\mathcal{O} \subseteq \mathrm{R}^n$, $\mathrm{P}_\ell(\mathcal{O})$ stands for the space of polynomials of degree $\leq \ell$ defined on $\mathcal{O}$, with vector and tensorial versions denoted by $\mathbf{P}_\ell(\mathcal{O}) := [\mathrm{P}_\ell(\mathcal{O})]^n$ and $\mathbb{P}_\ell(\mathcal{O}) := [\mathrm{P}_\ell(\mathcal{O})]^{n\times n}$, respectively. In addition, given a regular partition $\mathcal{T}_h$ of $\overline{\Omega}$ into triangles (in R^2) or tetrahedra (in R^3), we denote by $\mathcal{T}_h^{\mathrm{b}}$ its barycentric refinement, and let $\mathbf{RT}_\ell(K) := \mathbf{P}_\ell(K) \oplus \mathrm{P}_\ell(K)\,\mathbf{x}$ be the local Raviart-Thomas space of order ℓ for each $K \in \mathcal{T}_h^{\mathrm{b}}$, where $\mathbf{x}$ denotes a generic vector in Ω.

We deduce that, in order to guarantee the well-posedness of our Galerkin scheme (4.1), it suffices to define for each integer k such that $k+1 \geq n$, the finite element subspaces

$$\mathbf{H}_h^{\mathbf{u}} := \Big\{ \mathbf{v}_h \in \mathbf{L}^4(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathrm{b}} \Big\}, \tag{4.21}$$

$$\mathbb{H}_h^{\mathbf{t}} := \Big\{ \mathbf{s}_h \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \mathbf{s}_h|_K \in \mathbb{P}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathrm{b}} \Big\}, \tag{4.22}$$

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) : \quad \mathbf{c}^{\mathtt{t}}\, \boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, \mathbf{c} \in \mathrm{R}^n, \quad \forall\, K \in \mathcal{T}_h^{\mathrm{b}} \Big\}, \tag{4.23}$$

$$\mathrm{H}_h^{\varphi} := \Big\{ \psi_h \in \mathrm{L}^4(\Omega) : \quad \psi_h|_K \in \mathrm{P}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathrm{b}} \Big\}, \tag{4.24}$$

$$\mathbf{H}_h^{\widetilde{\mathbf{t}}} := \Big\{ \widetilde{\mathbf{s}}_h \in \mathbf{L}^2(\Omega) : \quad \widetilde{\mathbf{s}}_h|_K \in \mathbf{P}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathrm{b}} \Big\}, \tag{4.25}$$

$$\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} := \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}(\mathrm{div}_{4/3};\Omega) : \quad \widetilde{\boldsymbol{\tau}}_h|_K \in \mathbf{RT}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathrm{b}} \Big\}. \tag{4.26}$$

We end this section by collecting next the approximation properties of the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, H_h^{φ}, $\mathbf{H}_h^{\widetilde{\mathbf{t}}}$, and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$, which basically follow from interpolation estimates of Sobolev spaces and the approximation properties provided by the projector $\boldsymbol{\mathcal{P}}_h^k$ (see [22, eq. (5.37)]), and the Raviart-Thomas interpolation operator (see [22, eq. (5.41)] and also [12, 14, 17, 30]).

($\mathbf{AP}_h^{\mathbf{u}}$) there exists $C>0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$ there holds

$$\mathrm{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) := \inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \,\leq\, C\, h^l\, \|\mathbf{v}\|_{l,4;\Omega}\,. \tag{4.27}$$

($\mathbf{AP}_h^{\mathbf{t}}$) there exists $C>0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{s} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$ there holds

$$\mathrm{dist}(\mathbf{s}, \mathbb{H}_h^{\mathbf{t}}) := \inf_{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \,\leq\, C\, h^l\, \|\mathbf{s}\|_{l,\Omega}\,. \tag{4.28}$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,4/3}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}) := \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \le C\, h^l \left\{ \|\boldsymbol{\tau}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,4/3;\Omega} \right\}. \tag{4.29}$$

$(\mathbf{AP}_h^{\varphi})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\psi \in \mathrm{W}^{l,4}(\Omega)$ there holds

$$\operatorname{dist}(\psi, \mathrm{H}_h^{\varphi}) := \inf_{\psi_h \in \mathrm{H}_h^{\varphi}} \|\psi - \psi_h\|_{0,4;\Omega} \le C\, h^l\, \|\psi\|_{l,4;\Omega}. \tag{4.30}$$

$(\mathbf{AP}_h^{\widetilde{\mathbf{t}}})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\widetilde{\mathbf{s}} \in \mathbf{H}^l(\Omega)$ there holds

$$\operatorname{dist}(\widetilde{\mathbf{s}}, \mathbf{H}_h^{\widetilde{\mathbf{t}}}) := \inf_{\widetilde{\mathbf{s}}_h \in \mathbf{H}_h^{\widetilde{\mathbf{t}}}} \|\widetilde{\mathbf{s}} - \widetilde{\mathbf{s}}_h\|_{0,\Omega} \le C\, h^l\, \|\widetilde{\mathbf{s}}\|_{l,\Omega}. \tag{4.31}$$

$(\mathbf{AP}_h^{\widetilde{\boldsymbol{\sigma}}})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}^l(\Omega) \cap \mathbf{H}(\operatorname{div}_{4/3};\Omega)$ with $\operatorname{div}(\widetilde{\boldsymbol{\tau}}) \in \mathrm{W}^{l,4/3}(\Omega)$, there holds

$$\operatorname{dist}(\widetilde{\boldsymbol{\tau}}, \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}) := \inf_{\widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}} \|\widetilde{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\tau}}_h\|_{\operatorname{div}_{4/3};\Omega} \le C\, h^l \left\{ \|\widetilde{\boldsymbol{\tau}}\|_{l,\Omega} + \|\operatorname{div}(\widetilde{\boldsymbol{\tau}})\|_{l,4/3;\Omega} \right\}. \tag{4.32}$$

5 A priori error analysis

The first objective here is to derive a Céa estimate. Let $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{H} \times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $(\vec{\boldsymbol{\varphi}}_j, \widetilde{\boldsymbol{\sigma}}_j) := \big((\varphi_j, \widetilde{\mathbf{t}}_j), \widetilde{\boldsymbol{\sigma}}_j\big) \in \widetilde{\mathbf{H}} \times \mathbf{H}(\operatorname{div}_{4/3};\Omega)$, $j \in \{1,2\}$, with $(\mathbf{u}, \boldsymbol{\varphi}) := \big(\mathbf{u}, (\varphi_1, \varphi_2)\big) \in W$, be the unique solution of the coupled problem (3.6), and let $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h}) := \big((\varphi_{j,h}, \widetilde{\mathbf{t}}_{j,h}), \widetilde{\boldsymbol{\sigma}}_{j,h}\big) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$, with $(\mathbf{u}_h, \boldsymbol{\varphi}_h) := \big(\mathbf{u}_h, (\varphi_{1,h}, \varphi_{2,h})\big) \in W_h$, be a solution of the discrete coupled problem (4.1). Then, we first rewrite (3.6) and (4.1) in terms of the forms (3.26) and (3.27), that is

$$\begin{aligned} \mathcal{A}_{\mathbf{u},\boldsymbol{\varphi}}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) + b(\vec{\mathbf{v}}, \boldsymbol{\sigma}) &= F_{\boldsymbol{\varphi}}(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}, \\ b(\vec{\mathbf{u}}, \boldsymbol{\tau}) &= G(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \end{aligned} \tag{5.1}$$

$$\begin{aligned} \widetilde{\mathcal{A}}_{\mathbf{u},j}(\vec{\boldsymbol{\varphi}}_j, \vec{\boldsymbol{\psi}}_j) + \widetilde{b}(\vec{\boldsymbol{\psi}}_j, \widetilde{\boldsymbol{\sigma}}_j) &= 0 \qquad \forall\, \vec{\boldsymbol{\psi}}_j \in \widetilde{\mathbf{H}}, \\ \widetilde{b}(\vec{\boldsymbol{\varphi}}_j, \widetilde{\boldsymbol{\tau}}_j) &= \widetilde{G}_j(\widetilde{\boldsymbol{\tau}}_j) \qquad \forall\, \widetilde{\boldsymbol{\tau}}_j \in \mathbf{H}(\operatorname{div}_{4/3};\Omega), \end{aligned} \tag{5.2}$$

$$\begin{aligned} \mathcal{A}_{\mathbf{u}_h,\boldsymbol{\varphi}_h}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + b(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= F_{\varphi_h}(\vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h, \\ b(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= G(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \widetilde{\mathcal{A}}_{\mathbf{u}_h,j}(\vec{\boldsymbol{\varphi}}_{j,h}, \vec{\boldsymbol{\psi}}_{j,h}) + \widetilde{b}(\vec{\boldsymbol{\psi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h}) &= 0 \qquad \forall\, \vec{\boldsymbol{\psi}}_{j,h} \in \widetilde{\mathbf{H}}_h, \\ \widetilde{b}(\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\tau}}_{j,h}) &= \widetilde{G}_j(\widetilde{\boldsymbol{\tau}}_{j,h}) \qquad \forall\, \widetilde{\boldsymbol{\tau}}_{j,h} \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}. \end{aligned} \tag{5.4}$$

Applying the Strang lemma stated in [22, Lemma 6.1] to the context given by problems (5.1) and (5.3) (resp. problems (5.2) and (5.4)), and bearing in mind similar consistency estimates to those provided in [22, eqs. (6.16) and (6.18)] (resp. [22, eq. (6.17)]), we find, respectively, that

$$\begin{aligned} \|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h)\| &\le \bar{C}_{S,1} \operatorname{dist}\big(\vec{\mathbf{u}}, \mathbf{H}_h\big) + \bar{C}_{S,2} \operatorname{dist}\big(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}\big) \\ &+ \bar{C}_{S,3}\, c(\boldsymbol{g}, \mathbf{u}_D) \left\{ \|\varphi_1 - \varphi_{1,h}\|_{0,4;\Omega} + \|\varphi_2 - \varphi_{2,h}\|_{0,4;\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \right\}, \end{aligned} \tag{5.5}$$

and for each $j \in \{1,2\}$, the bound

$$\|(\vec{\boldsymbol{\varphi}}_j, \widetilde{\boldsymbol{\sigma}}_j) - (\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h})\| \,\le\, \widehat{C}_{S,1}\,\mathrm{dist}\big(\vec{\boldsymbol{\varphi}}_j, \widetilde{\mathbf{H}}_h\big) \,+\, \widehat{C}_{S,2}\,\mathrm{dist}\big(\widetilde{\boldsymbol{\sigma}}_j, \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\big) \,+\, \widehat{C}_{S,3}\,c(\boldsymbol{\varphi}_D)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,. \quad (5.6)$$

For the remaining expressions in (5.5)-(5.6), note that $c(\boldsymbol{g},\mathbf{u}_D)$ depends linearly on $\|\boldsymbol{g}\|_{0,\infty;\Omega}$ and $\|\mathbf{u}_D\|_{1/2,\Gamma}$, whereas $\bar{C}_{S,1}$, $\bar{C}_{S,2}$, and $\bar{C}_{S,3}$ are positive constants computed using [22, eq. (6.4)] and depending on $\mu_2, \boldsymbol{\vartheta}, r, \alpha_{\mathtt{d}}, \beta_{\mathtt{d}}$. After using (3.28), these constants are used to bound both $\|\mathcal{A}_{\mathbf{u},\varphi}\|$ and $\|\mathcal{A}_{\mathbf{u}_h,\varphi_h}\|$ by $\big(|\Omega|^{1/2}\gamma + 2\mu_2 + r\big)$. In turn, $c(\boldsymbol{\varphi}_D)$ is a constant multiple of $\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}$, and $\widehat{C}_{S,1}$, $\widehat{C}_{S,2}$. Also, $\widehat{C}_{S,3}$ are positive constants defined in terms of $\|\mathbb{K}\|_{0,\infty;\Omega}$, r, $\widetilde{\alpha}_{\mathtt{d}}$, and $\widetilde{\beta}_{\mathtt{d}}$, which are computed according to [22, eq. (6.4)], after using (3.29) to bound both $\|\widetilde{\mathcal{A}}_{\mathbf{u},j}\|$ and $\|\widetilde{\mathcal{A}}_{\mathbf{u}_h,j}\|$ by $\big(\|\mathbb{K}\|_{0,\infty;\Omega} + r\big)$.

Next we can insert (5.6) into (5.5), which leads to

$$\begin{aligned}
\|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h)\| \,&\le\, \bar{C}_{S,1}\,\mathrm{dist}\big(\vec{\mathbf{u}}, \mathbf{H}_h\big) \,+\, \bar{C}_{S,2}\,\mathrm{dist}\big(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}\big) \\
&+\, \bar{C}_{S,3}\,c(\boldsymbol{g},\mathbf{u}_D)\,\widehat{C}_{S,1}\,\sum_{j=1}^{2}\mathrm{dist}\big(\vec{\boldsymbol{\varphi}}_j, \widetilde{\mathbf{H}}_h\big) \\
&+\, \bar{C}_{S,3}\,c(\boldsymbol{g},\mathbf{u}_D)\,\widehat{C}_{S,2}\,\sum_{j=1}^{2}\mathrm{dist}\big(\widetilde{\boldsymbol{\sigma}}_j, \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\big) \\
&+\, \bar{C}_{S,3}\,c(\boldsymbol{g},\mathbf{u}_D)\,\big\{1 + 2\,\widehat{C}_{S,3}\,c(\boldsymbol{\varphi}_D)\big\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,.
\end{aligned} \qquad (5.7)$$

Imposing the constant multiplying $\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}$ in (5.7) to be sufficiently small, say $\le 1/2$, we derive the a priori upper bound for $\|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h)\|$. Hence, employing this latter estimate to bound the third term on the right-hand side of (5.6), we deduce the corresponding upper bound for $\|(\vec{\boldsymbol{\varphi}}_j, \widetilde{\boldsymbol{\sigma}}_j) - (\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h})\|$, $j \in \{1,2\}$. We have thus demonstrated the following result.

Theorem 5.1 *Assume that the data $\boldsymbol{g}$, $\mathbf{u}_D$, and $\boldsymbol{\varphi}_D$ satisfy*

$$\bar{C}_{S,3}\,c(\boldsymbol{g},\mathbf{u}_D)\,\big\{1 + 2\,\widehat{C}_{S,3}\,c(\boldsymbol{\varphi}_D)\big\} \,\le\, \frac{1}{2}\,.$$

Then, there exists a positive constant C, independent of h, but depending on μ_2, $\boldsymbol{\vartheta}$, r, $\alpha_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, $\|\mathbb{K}\|_{0,\infty;\Omega}$, $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $\|\boldsymbol{g}\|_{0,\infty;\Omega}$, $\|\mathbf{u}_D\|_{1/2,\Gamma}$, and $\|\boldsymbol{\varphi}_D\|_{1/2,\Gamma}$, such that

$$\begin{aligned}
&\|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h)\| \,+\, \sum_{j=1}^{2}\|(\vec{\boldsymbol{\varphi}}_j, \widetilde{\boldsymbol{\sigma}}_j) - (\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h})\| \\
&\quad\le\, C\,\Bigg\{\mathrm{dist}\big(\vec{\mathbf{u}}, \mathbf{H}_h\big) \,+\, \mathrm{dist}\big(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}\big) \,+\, \sum_{j=1}^{2}\Big(\mathrm{dist}\big(\vec{\boldsymbol{\varphi}}_j, \widetilde{\mathbf{H}}_h\big) \,+\, \mathrm{dist}\big(\widetilde{\boldsymbol{\sigma}}_j, \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\big)\Big)\Bigg\}.
\end{aligned} \qquad (5.8)$$

We are now able to provide the rates of convergence of the Galerkin Scheme (4.1) when the finite element subspaces specified in Section 4.3 are employed.

Theorem 5.2 *Assume that there exists $l \in [0, k+1]$ such that $\mathbf{u} \in \mathbf{W}^{l,4}(\Omega)$, $\mathbf{t} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$, $\varphi_j \in \mathrm{W}^{l,4}(\Omega)$, $\widetilde{\mathbf{t}}_j \in \mathbf{H}^l(\Omega)$, $\widetilde{\boldsymbol{\sigma}}_j \in \mathbf{H}^l(\Omega) \cap \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, and $\mathrm{div}(\widetilde{\boldsymbol{\sigma}}_j) \in \mathrm{W}^{l,4/3}(\Omega)$, for $j \in \{1,2\}$. Then, there exists $C > 0$, independent of h, such that*

$$\begin{aligned}
&\|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h)\| \,+\, \sum_{j=1}^{2}\|(\vec{\boldsymbol{\varphi}}_j, \widetilde{\boldsymbol{\sigma}}_j) - (\vec{\boldsymbol{\varphi}}_{j,h}, \widetilde{\boldsymbol{\sigma}}_{j,h})\| \,\le\, C\,h^l\,\Bigg\{\|\mathbf{u}\|_{l,4;\Omega} + \|\mathbf{t}\|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega} \\
&\quad+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega} + \sum_{j=1}^{2}\Big\{\|\varphi_j\|_{l,4;\Omega} + \|\widetilde{\mathbf{t}}_j\|_{l,\Omega} + \|\widetilde{\boldsymbol{\sigma}}_j\|_{l,\Omega} + \|\mathrm{div}(\widetilde{\boldsymbol{\sigma}}_j)\|_{l,4/3;\Omega}\Big\}\Bigg\}.
\end{aligned} \qquad (5.9)$$

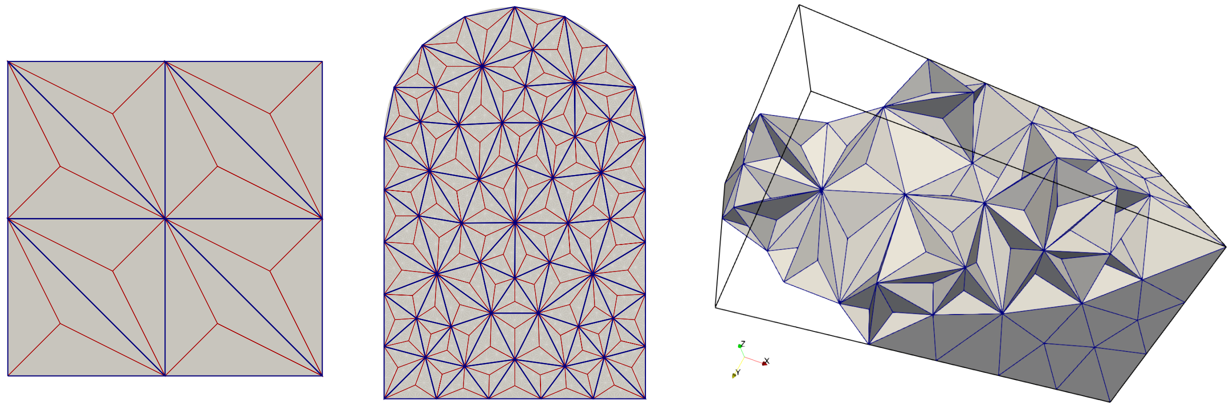

Figure 6.1: Example of Alfeld splits for a coarse 2D uniform mesh used in Example 1 (left), a coarse unstructured grid for Example 2 (center), and for a 3D non-uniform mesh used in Example 3 (crinkle clip on the right panel).

Proof. It follows straightforwardly from (5.8) and the approximation properties from Section 4.3. $\square$

We end this section with the derivative-free postprocessing of the pressure. From the orthogonal decomposition for the pseudostress tensor (3.3) (which yielded the new tensor unknown $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$), we deduce that (2.5) becomes

$$p \,=\, -\frac{1}{2n}\mathrm{tr}\big(2\boldsymbol{\sigma} + 2c\mathbb{I} + \mathbf{u}\otimes\mathbf{u}\big)\,, \quad \text{with} \quad c \,:=\, -\frac{1}{2n|\Omega|}\int_\Omega \mathrm{tr}\big(\mathbf{u}\otimes\mathbf{u}\big)\,.$$

And therefore the discrete pressure will be defined as

$$p_h \,:=\, -\frac{1}{2n}\mathrm{tr}\big(2\boldsymbol{\sigma}_h + 2c_h\mathbb{I} + \mathbf{u}_h\otimes\mathbf{u}_h\big)\,, \quad \text{with} \quad c_h \,:=\, -\frac{1}{2n|\Omega|}\int_\Omega \mathrm{tr}\big(\mathbf{u}_h\otimes\mathbf{u}_h\big)\,.$$

Moreover, it is easy to prove that there exists a positive constant C, independent of h, such that

$$\|p - p_h\|_{0,\Omega} \,\leq\, C\Big\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\Big\}\,,$$

whence the rate of convergence of p_h coincides with the one established by (5.9).

6 Numerical results

In this section we present several numerical examples confirming the good performance of the fully-mixed finite element method (4.1) with the subspaces indicated in Section 4.3. As required for the stability of the Scott-Vogelius pair, the computations are performed on barycentric refined meshes $\mathcal{T}_h^{\mathrm{b}}$ created from regular partitions $\mathcal{T}_h$ of Ω, illustrated for 2D and 3D in Figure 6.1. All initial grids and Alfeld splits (barycentric refinements) are generated with the open-source mesh manipulator GMSH [33] and the computational implementation has been carried out using the open-source finite element library FEniCS [7]. A Newton-Raphson algorithm with null initial guesses is used for the resolution of the nonlinear problem (4.1). As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^{m+1}$ and $\mathbf{coeff}^{m}$, is sufficiently small, that is,

$$\frac{||\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}||_{\ell^2}}{||\mathbf{coeff}^{m+1}||_{\ell^2}} < \texttt{tol}\,,$$

where `tol` is a specified tolerance and $\|\cdot\|_{\ell^2}$ is the standard ℓ^2-norm in $\mathrm{R}^{\mathrm{DoF}}$ with DoF denoting the total number of degrees of freedom generated by the finite element subspaces. The condition of zero-average pressure (translated in terms of the trace of $2\boldsymbol{\sigma}+\mathbf{u}\otimes\mathbf{u}$) is imposed through a real Lagrange multiplier. The solution of all linear systems is carried out with the multifrontal massively parallel sparse direct solver MUMPS.

Errors between exact and approximate solutions are denoted as

$$e(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,, \quad e(\mathbf{t}) := \|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega} \quad e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}\,, \quad e(p) := \|p-p_h\|_{0,\Omega}\,,$$

$$e(\boldsymbol{\varphi}) := \sum_{j=1}^{2}\|\varphi_j-\varphi_{j,h}\|_{0,4;\Omega}\,, \quad e(\widetilde{\mathbf{t}}) := \sum_{j=1}^{2}\|\widetilde{\mathbf{t}}_j-\widetilde{\mathbf{t}}_{j,h}\|_{0,4;\Omega}\,, \quad e(\widetilde{\boldsymbol{\sigma}}) := \sum_{j=1}^{2}\|\widetilde{\boldsymbol{\sigma}}_j-\widetilde{\boldsymbol{\sigma}}_{j,h}\|_{\mathrm{div}_{4/3};\Omega}\,.$$

In turn, we let $r(\star)$ be their corresponding rates of convergence, that is

$$r(\star) := \frac{\log(\mathsf{e}(\star)/\mathsf{e}'(\star))}{\log(h/h')} \qquad \forall\star\in\left\{\mathbf{u},\mathbf{t},\boldsymbol{\sigma},p,\boldsymbol{\varphi},\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\sigma}}\right\},$$

where h and h' denote two consecutive mesh sizes with errors $\mathsf{e}(\star)$ and $\mathsf{e}'(\star)$, respectively.

Example 1: Convergence against smooth exact solutions. In our first example we study the accuracy of the approximations by manufacturing an exact solution of (3.6) in the domain $\Omega := (-1,1)^2$ with the constant and variable coefficients

$$\mu(\boldsymbol{\varphi}) = e^{-\varphi_1}\,, \quad \boldsymbol{\vartheta} = (1,0.5)^{\mathtt{t}}\,, \quad \gamma = 10^{-3}\,, \quad \mathbb{K}_1(\mathbf{x}) = \begin{pmatrix}\exp(-x_1) & x_1/10\\ x_2/10 & \exp(-x_2)\end{pmatrix}\,,$$
$$\mathbb{K}_2(\mathbf{x}) = \begin{pmatrix}\exp(-x_1) & 0\\ 0 & \exp(-x_2)\end{pmatrix}\,, \quad \text{and} \quad \boldsymbol{g}(\mathbf{x}) = (0,-1)^{\mathtt{t}} \quad \forall\,\mathbf{x} := (x_1,x_2)^{\mathtt{t}}\in\Omega\,.$$

Then, the Dirichlet data $\mathbf{u}_D$ and $\boldsymbol{\varphi}_D$, and the terms on the right-hand sides, are imposed according to the exact solutions given by the smooth functions

$$\mathbf{u}(\mathbf{x}) \;=\; \begin{pmatrix}\cos(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2)\\ -\sin(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)\end{pmatrix}, \quad p(\mathbf{x}) = (x_1-0.5)(x_2-0.5)-0.25\,,$$

$$\varphi_1(\mathbf{x}) = \exp(-x_1^2-x_2^2)-\frac{1}{2}, \quad \text{and} \quad \varphi_2(\mathbf{x}) = \exp(-x_1x_2[x_1-1][x_2-1]) \quad \forall\,\mathbf{x} := (x_1,x_2)^{\mathtt{t}}\in\Omega\,.$$

Values of errors and corresponding convergence rates associated with the approximations with the finite element family $\mathbf{P}_1-\mathbb{P}_1-\mathbb{RT}_1-\mathbf{P}_1-\mathbb{P}_1-\mathbb{RT}_1$ are summarized in Table 6.1. As expected, we observe there that the convergence rates are quadratic with respect to h for all the unknowns in their respective norms. Sample solutions of approximate velocity magnitude, temperature, concentration, and postprocessed pressure computed with our fully-mixed method are depicted in Figure 6.2.

Example 2: Manufactured solutions on a different domain. We now perform an accuracy test for (3.6) on the tombstone-shaped domain (see [16])

$$\Omega := \{\mathbf{x} : -0.5 < x_1 < 0.5, -0.5 < x_2 < 0.5\} \cup \{\mathbf{x} : -0.5 < x_1 < 0.5, 0.5 < x_2 < \sin(\pi x_2)\},$$

and consider the same specification as in Example 1 for viscosity, gravity, exact temperature, and diffusivity of the concentration. The modified parameters are the thermal and mass expansions, the

Finite Element Family: $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$								
DoF	h	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	
1305	1.4140	0.09025	–	0.5561	–	0.93790	–	
5153	0.7071	0.01935	2.222	0.1996	1.682	0.27852	1.752	
20481	0.3536	0.00420	2.203	0.0622	1.778	0.07634	1.867	
81665	0.1768	0.00097	2.114	0.0184	1.859	0.02052	1.895	
326145	0.0884	0.00023	2.054	0.0051	1.949	0.00536	1.936	
1303553	0.0442	0.00014	2.003	0.0014	1.984	0.00164	1.979	

$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(\widetilde{\mathbf{t}})$	$r(\widetilde{\mathbf{t}})$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	$e(p)$	$r(p)$	It.
0.0769	–	0.5343	–	0.74246	–	0.17954	–	4
0.01586	2.278	0.1681	1.668	0.20943	1.826	0.04699	1.934	4
0.00334	2.247	0.0474	1.827	0.05768	1.868	0.00976	2.267	4
0.00076	2.126	0.0127	1.898	0.01519	1.925	0.00214	2.185	4
0.00018	2.048	0.0033	1.938	0.00381	1.965	0.00050	2.081	4
0.00011	2.002	0.0009	1.966	0.00143	1.993	0.00013	2.031	4

Table 6.1: Example 1: Convergence history and Newton iteration count for the fully-mixed $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$ approximation. DoF stands for the number of degrees of freedom associated with each barycentric refined mesh $\mathcal{T}_h^{\mathtt{b}}$.

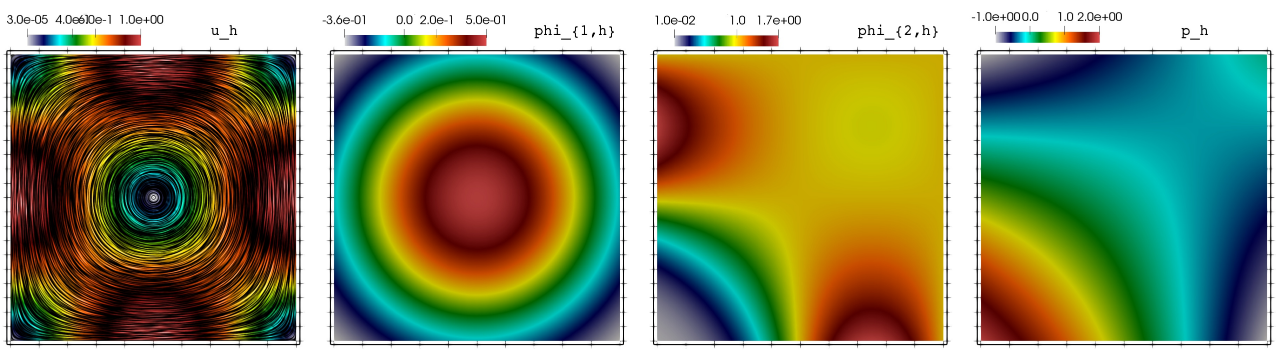

Figure 6.2: Example 1: Approximate velocity magnitude, temperature, concentration, and postprocessed pressure, obtained using $k = 1$ and a barycentrically refined mesh with 19110 elements.

inverse permeability (which is now seven orders of magnitude higher), and the thermal conductivity (which is now isotropic)

$$\boldsymbol{\vartheta} = (0.75, 0.25)^{\mathtt{t}}, \quad \gamma = 1.0678 \cdot 10^4, \quad \mathbb{K}_1(\mathbf{x}) = e^{x+y}\,\mathbb{I}.$$

Again, the right-hand sides and the boundary Dirichlet data are adjusted in terms of the manufactured exact solutions, which are in this case

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} 2\pi\cos(\pi x_2)\sin(\pi x_1)\sin(\pi x_1)\sin(\pi x_2) \\ -2\pi\cos(\pi x_1)\sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_2) \end{pmatrix}, \quad p(\mathbf{x}) = 5x_1\sin(x_2),$$

$$\varphi_1(\mathbf{x}) = \exp(-x^2 - y^2) - \frac{1}{2}, \quad \text{and} \quad \varphi_2(\mathbf{x}) = 15 - 15\exp(-x_1 x_2[x_1 - 1][x_2 - 1]) \quad \forall\,\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega.$$

In Table 6.2 we present errors for each variable with respect to DoF, the experimental convergence rates, and the number of Newton iterations per mesh refinement. This time the computations were

Finite Element Family: $\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathrm{P}_2 - \mathbf{P}_2 - \mathbf{RT}_2$								
DoF	h	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	
2421	0.7224	1.3763e-01	–	1.8751e00	–	4.0911e00	–	
10089	0.4234	2.6073e-02	2.3312	4.5061e-01	1.9979	1.0707e00	1.8784	
39258	0.2460	2.9780e-03	3.1936	7.1098e-02	2.7181	1.6243e-01	2.7759	
158652	0.1397	3.7560e-04	2.9651	1.2822e-02	2.4530	2.5362e-02	2.6594	
633555	0.0763	5.0175e-05	2.9076	1.6559e-03	2.9565	3.4396e-03	2.8857	
$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(\widetilde{\mathbf{t}})$	$r(\widetilde{\mathbf{t}})$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	$e(p)$	$r(p)$	It.
1.8294e-02	–	3.0672e-01	–	6.2336e-01	–	6.4242e00	–	5
2.6228e-03	2.7217	6.8840e-02	2.0937	1.2471e-01	2.2548	5.0722e-01	3.5576	5
2.1645e-04	3.6620	8.7779e-03	3.0316	1.4310e-02	3.1868	4.6338e-02	3.5224	5
2.8554e-05	2.9007	1.4012e-03	2.6277	2.1211e-03	2.7339	7.0811e-03	2.6902	5
3.7241e-06	2.9422	1.8828e-04	2.8991	2.8610e-04	2.8937	1.0393e-03	2.7716	5

Table 6.2: Example 2: Convergence history and Newton iteration count for the fully-mixed $\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathrm{P}_2 - \mathbb{P}_2 - \mathbf{RT}_2$ approximation of the Oberbeck-Boussinesq equations on a tombstone-shaped domain with different model parameters.

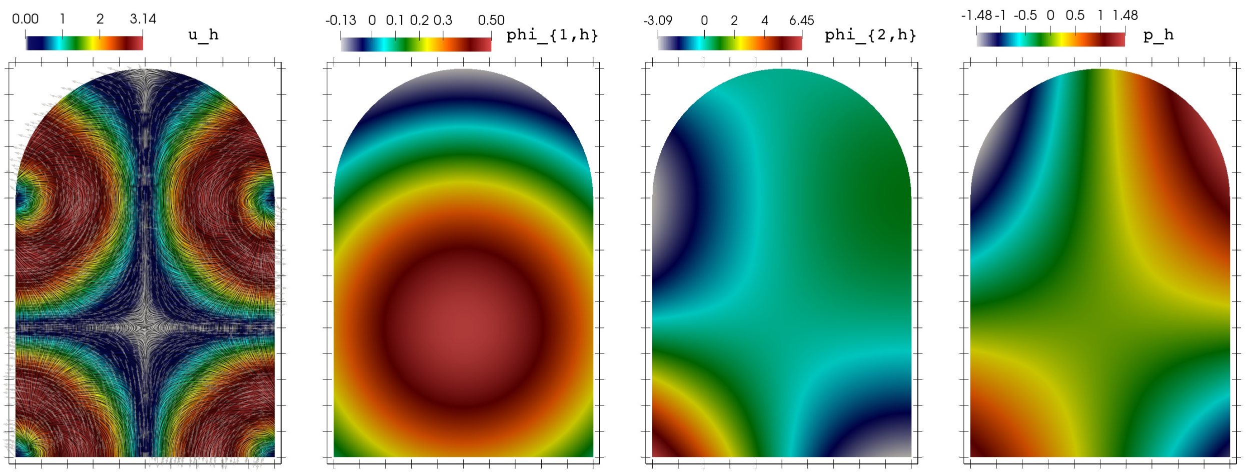

Figure 6.3: Example 2: Approximate velocity magnitude, temperature, concentration, and postprocessed pressure, using $k = 2$ and a barycentric refinement with 11534 triangular elements.

done with the finite element family $\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathrm{P}_2 - \mathbf{P}_2 - \mathbf{RT}_2$ ($k = 2$). In concordance with the theoretical estimates from Section 5, the computational results confirm an error decay with rate $O(h^3)$. A total of 5 Newton iterations were required to reach a tolerance `tol` = 1E-08. In Figure 6.3 we display the velocity magnitude, the temperature, and the concentration produced with our fully-mixed scheme on a barycentric refined mesh that, for $k = 2$, generates 633555 DoFs.

Example 3: Error decay in the 3D case. Verification of the convergence of the method in 3D is provided with a simple test employing the following closed-form solutions

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix}, \qquad p(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3),$$

$$\varphi_1(\mathbf{x}) = 1 - \sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3), \quad \varphi_2(\mathbf{x}) = \exp(-(x_1-0.5)^2 - (x_2-0.25)^2 - (x_3-0.25)^2),$$

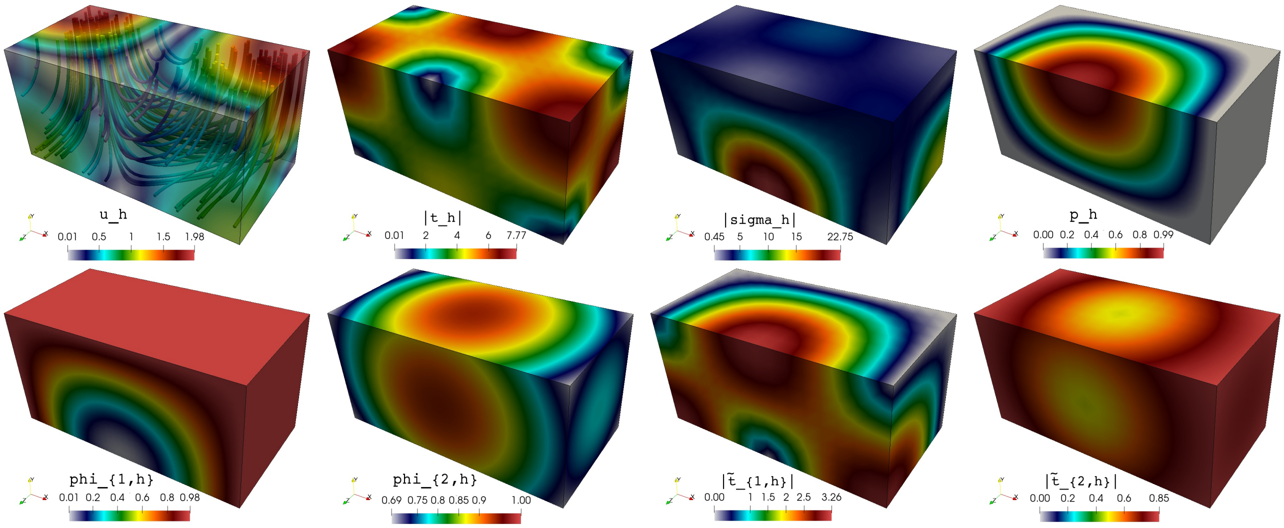

Figure 6.4: Example 3: Approximate velocity magnitude and streamlines, velocity gradient, Bernoulli tensor, postprocessed pressure, temperature, concentration, temperature gradient, and concentration gradient, obtained using $k = 2$ and a barycentrically refined tetrahedral mesh with 24576 elements.

for $\mathbf{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \Omega$. The manufactured velocity is divergence free and we use it to impose the Dirichlet condition on Γ. The exact concentration and temperature are uniformly bounded in Ω and we consider the following constant and variable coefficients

$$\gamma = 1, \quad \boldsymbol{\vartheta} = (1, 0.5)^{\mathtt{t}}, \quad \mathbb{K}_1(\mathbf{x}) = \begin{pmatrix} \exp(-x_1) & 0 & 0 \\ 0 & \exp(-x_2) & 0 \\ 0 & 0 & \exp(-x_3) \end{pmatrix}, \quad \mathbb{K}_2 = \mathbb{I},$$

We recall that the solvability of the discrete problem requires that, for dimension $n = 3$, the finite element spaces made precise in (4.21)-(4.26) should use a polynomial degree $k \geq 2$. The error history is shown in Table 6.3, where the tabulated convergence rates with respect to DoF indicate that all individual fields have optimal error decay as predicted by (5.9). In all cases the number of Newton iterations needed to reach convergence was 4. The solutions on a coarse mesh with 7521 vertices and 24576 tetrahedral elements (actually representing 957121 DoFs for $k = 2$), are displayed in Figure 6.4.

Example 4: Simulating exothermic flows. We finalize with a time-dependent problem that has relevance in the modeling of exothermic reaction-diffusion fronts in porous media. The problem configuration is adapted from that in [39], where apart from advection and diffusion, a reaction term is present in the right-hand sides of the temperature and concentration equation. More precisely, they are $\mathrm{Da} f(\varphi_2)$ in the equation for φ_1 and $-\mathrm{Da}\, f(\varphi_2)$ in the equation for φ_2, where Da$= 0.001$ is the dimensionless Darcy number and the concentration-dependent nonlinear reaction is $f(\varphi_2) := \varphi_2(1 + 7\varphi_2)(1 - \varphi_2)^2$. The buoyancy term is characterized by $\boldsymbol{\vartheta} = (5, -1)^{\mathtt{t}}$, and we simply consider a constant viscosity $\mu = 1$ and a constant permeability $\gamma = 1$. The diffusivities are isotropic and constant $\mathbb{K}_1 = 8\mathbb{I}, \mathbb{K}_2 = 2.5\mathbb{I}$, and the domain is the rectangle $\Omega = (0, 2000) \times (-1000, 0)$. Further differences with respect to the original system (2.1) include boundary conditions: we now set $\mathbf{u}_D = \mathbf{0}$ on the whole boundary whereas we put $\varphi_j = 1$ on the top edge of the domain, $\varphi_j = 0$ on the bottom surface, and on the vertical walls we impose zero flux conditions, which in the context of our mixed formulation are implemented as essential conditions for each $\widetilde{\mathbf{t}}_j$. A barycentric refinement is applied

Finite Element Family: $\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathrm{P}_2 - \mathbf{P}_2 - \mathbf{RT}_2$								
DoF	h	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	
7621	1.225	0.03165	–	0.55572	–	2.4535	–	
15181	0.866	0.01095	3.063	0.18479	3.178	0.8475	3.066	
120241	0.433	0.00143	2.935	0.02785	2.829	0.1554	2.773	
957121	0.2165	0.00026	2.893	0.00484	2.881	0.0293	2.835	
7965323	0.1083	0.00007	2.970	0.00063	2.992	0.0046	2.916	
$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(\widetilde{\mathbf{t}})$	$r(\widetilde{\mathbf{t}})$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	$e(p)$	$r(p)$	It.
0.01043	–	0.10795	–	0.22103	–	0.23510	–	4
0.00397	2.766	0.03389	3.305	0.06753	3.421	0.03295	2.895	4
0.00049	3.029	0.00518	2.705	0.00932	2.860	0.00464	2.974	4
5.99e-05	3.032	0.00069	2.814	0.00120	2.897	0.00087	3.012	4
7.45e-06	2.989	8.17e-05	2.909	1.62e-04	2.936	1.13e-04	3.003	4

Table 6.3: Example 3: Convergence history and Newton iteration count for the fully-mixed $\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathrm{P}_2 - \mathbb{P}_2 - \mathbf{RT}_2$ approximation of the Oberbeck-Boussinesq equations on a 3D box.

on an unstructured triangulation of the domain and the resulting grid has 32491 elements. In the bilinear forms $\widetilde{\mathcal{A}}_{\mathbf{u},j}$ we add the term

$$\int_\Omega \frac{1}{\Delta t}(\varphi_j^{\ell+1} - \varphi_\ell)\psi_j,$$

accounting for the backward Euler time discretization of $\partial_t \varphi_j$, $j \in \{1,2\}$. The same is done to add an acceleration term to the momentum equation. We use a uniform partition of the time domain (from 0 to 2000) and use a constant stepsize of $\Delta t = 20$. The fully mixed scheme is defined by (4.21)-(4.26) with $k = 1$, and the initial conditions for the solutal concentration and high temperature near the domain top surface are uniformly distributed random perturbations, whereas the initial velocity is the zero vector.

We run the system until 2000 time units and show in Figure 6.5 snapshots of concentration of the solute at three different times, together with the postprocessed pressure. As a result of the nonlinear interaction between the change of temperature and the high solute concentration, density-driven instabilities start to form and the solute fingers commence to move downwards also due to gravitational effects. Throughout the computation the Newton-Raphson method took at most five iterations to reach the desired tolerance.

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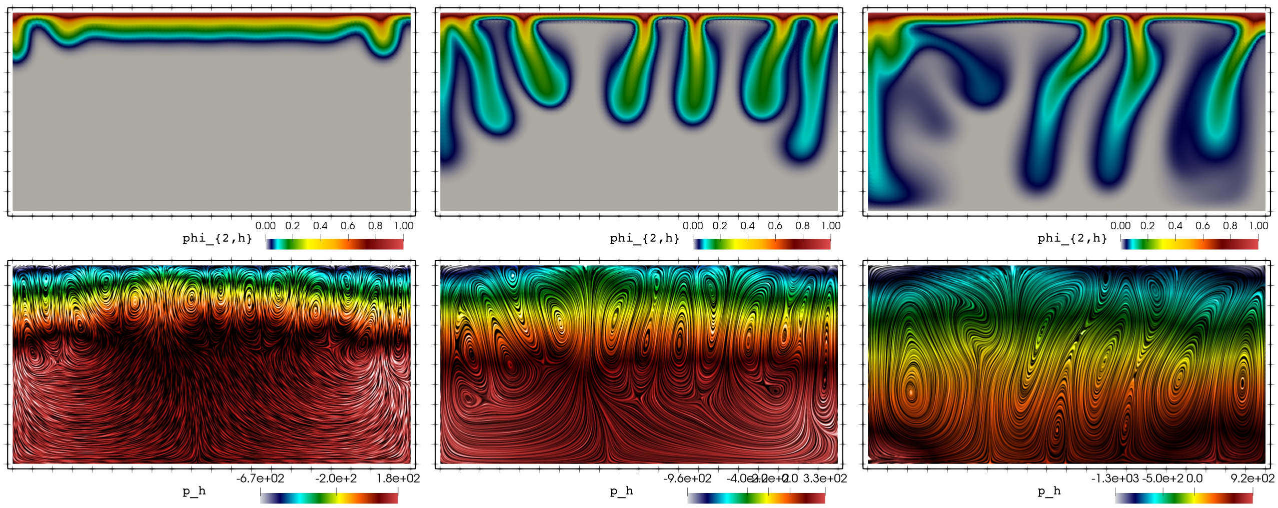

Figure 6.5: Example 4: Evolution of the solute concentration (top) and the postprocessed pressure (bottom, showing also line integral contours of velocity) computed with a method using $k = 1$, and recorded at times $800, 1200, 2000$.

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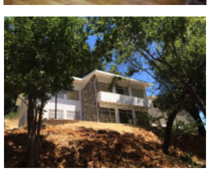

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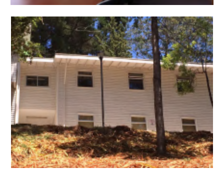
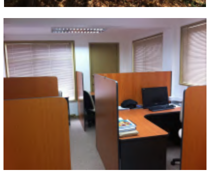

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