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Analysis of an augmented fully-mixed finite element method for a bioconvective flows model*

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Abstract

In this paper we study a stationary generalized bioconvection problem given by the Navier-Stokes equations coupled to a cell conservation equation for describing the hydrodynamic and micro-organisms concentration of a culture fluid, assumed to be viscous and incompressible, and in which the viscosity might depend on the concentration. The model is rewritten in terms of a first-order system based on the introduction of the shear-stress, the vorticity, and the pseudo-stress tensors in the fluid equations along with an auxiliary vector in the concentration equation. After a variational approach, the resulting weak model is then augmented using appropriate parameterized Galerkin terms and rewritten as fixed-point problem. Existence, uniqueness and convergence results are obtained under certain regularity assumptions combined with the Lax-Milgram theorem, and the Banach and Brouwer fixed-point theorems. Optimal a priori error estimates are derived and confirmed through some numerical examples that illustrate the performance of the proposed technique.

Key words: Bioconvection, nonlinear partial differential equation, augmented mixed formulation, finite element method, fixed point theory, a priori error analysis.

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q80, 76D05, 76M10, 92C17

1 Introduction

Bioconvective flows, or bioconvection, refers to a spontaneous flow and pattern formation due to the motion of a large number of upswimming micro-organisms as an innate behavioral response to a stimulus like gravity, light, oxygen, food, changes on temperature, or some combination of these. In a fluid of finite depth, upswimming means that cells accumulate near the top surface due to the gathering of micro-organisms, so the upper regions of the suspensions become denser than the lower, and when the density gradient is high enough, micro-organisms fall down; leading to an overturning convection [38].

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By its nature, this phenomenon takes place in several biological processes, including reproduction, infection and the marine life ecosystem [33]. Some direct applications are related to bacterial research, microbiological cultures, separating swimming subpopulations of geotactic micro-organisms (whose movement is gravity-induced) in lab experiments, and controlling population of plankton communities in the oceans, to name a few. In addition, more recently, bioconvective flows have also been considered useful to medical, bioengineering and pharmaceutical applications [6,32]. For instance, it can be used to configure new geometries of bioreactors, to improve the biofuel production and to enhance microfluidics mixing, which are often linked to several pharmaceutical and biotechnological experiments such as analyses of DNA or drugs, screening of patients and combinatorial synthesis.

A fluid dynamical model to describe bioconvection of geotactic microorganisms was introduced in [34] and [37], independently, from a biological and physical point of view. Using the Boussinesq approximation, the resulting model consists of a Navier-Stokes type system for describing the hydrodynamic of the culture fluid assumed to be viscous and incompressible, in terms of the velocity and the pressure, nonlinearly coupled to an advection-diffusion equation for the micro-organisms concentration, which comes from a cell conservation equation.

The mathematical analysis of this model was carried out in [39]. There, the authors prove existence of weak solutions by the Galerkin method, and existence of strong solutions by a semi-group approach along with the method of successive approximations, for both stationary and evolution problems. Also, a positivity property of the concentration is shown there. Later, generalized models in which the effective viscosity depends on the concentration of the organisms are mathematically analyzed in [8], for initial conditions, and in [14], for periodic conditions and assuming that the viscosity is a concentration-dependent continuously differentiable function. In these works, uniqueness results of solutions are further given. Then in [2], the authors complement the results from [39] by addressing the problem of obtaining convergence rates for the error when using spectral Galerkin approximations of the problem with a constant viscosity.

First numerical simulations of bioconvection are developed in [11, 29] in two dimensions. Whilst in [11] the authors integrate the Navier-Stokes equations, they treat the cells as individuals moving points, instead of using the continuum cell conservation. In [29], the problem is solved integrating the incompressible Navier-Stokes equations and the cell conservation equation in a shallow box as a physical domain. To the best of our knowledge, [10] is one of the first finite element analysis for the bioconvection model. There, the problem is considered with concentration-dependent viscosity and the authors firstly improve the existence result from [14], by allowing the viscosity to be a continuous and bounded function. They then state existence and uniqueness results for the continuous and discrete problems, as well as a the convergence associated to the classical primal method based on finite elements; whose solvability requires an inf-sup compatibility condition. Additionally, although the analysis is carried out in two and three dimensions, they test the performance and accuracy of the numerical technique only in the 2d-case, including an example with data obtained from lab experiments. Here the Taylor-Hood finite element of second order is used for approximating the velocity and pressure, whereas piecewise quadratic polynomials are used for the concentration. Other numerical techniques developed for related models and their respective mathematical analysis are [19–22, 25–27, 31, 35, 36, 44] and the references there in, which include gyrotactic, geotactic, oxitactic and chemotactic microorganisms modeling.

As a phenomenon from fluid dynamics, in certain applications some additional physically relevant variables such as the gradient of the fluid velocity or the gradient of the micro-organisms concentration might reveal specific mechanisms of the bioconvection, and hence become of primary interest. Whilst these variables could be obtained via numerical integration of the discrete solutions provided by standard methods, this certainly would lead to a loss of accuracy or deteriorate the expected convergence order. In light of this, the purpose of this work is to contribute with the construction, analysis and implementation of a new numerical technique based on mixed finite elements for simulating bioconvective flows of geotactic micro-organisms, allowing

(a) direct computation of physically relevant variables in the phenomena such as the velocity gradient, the vorticity, the shear stress tensor of the fluid and the micro-organisms concentration gradient,

(b) flexibility regarding the use of finite element subspaces, avoiding any inf-sup compatibility restriction,

(c) high-order approximations, and optimal-order a priori error estimates.

To that end, based on previous mixed methods developed for related problems [3–5, 12, 16, 17], we firstly re-write the original model as a first-order system of equations in which the resulting unknowns become the velocity and concentration (as primal variables) along with the strain tensor, the vorticity tensor, a pseudo-stress tensor and a vectorial unknown depending on the fluid velocity, the microorganism concentration and its gradient (introduced as auxiliary unknowns). After a variational formulation, the problem is then augmented by using redundant parameterized Galerkin terms, which allows to set the problem in standard Hilbert spaces and, in turn, to circumvent any inf-sup compatibility condition between the involved spaces. The analysis is then carried out by a fixed-point approach [15], combining the Lax-Milgram theorem with the classical Banach and Brouwer fixed-point theorems for stating the respective solvability of the continuous problem and the associated Galerkin scheme, under suitable regularity assumptions, a feasible choice of parameters and, in the discrete case, for any family of finite element subspaces. A Strang-type lemma, valid for linear problems, enables us to derive the corresponding Céa estimate and to provide optimal a priori error bounds for the Galerkin solution. In turn, the pressure can be recovered by a post-processed of the discrete solutions, preserving the same rate of convergence. Finally, numerical experiments are presented to illustrate the performance of the technique and confirming the expected orders.

We have organized the contents of this paper as follows. The remainder of this section introduces some standard notations and functional spaces. In Section 2, we introduce the model problem, and the auxiliary variables in terms of which an equivalent first-order set of equations is obtained. Next, in Section 3, we derive the augmented mixed variational formulation and establish its well-posedness. The associated Galerkin scheme is introduced and analyzed in Section 4. In Section 5, we derive the corresponding Céa estimate and, finally, in Section 6 we present a couple of numerical examples illustrating the performance of our augmented fully-mixed finite element method.

Notations

Let us denote by $\Omega \subseteq \mathbb{R}^3$, be a bounded domain with polyhedral boundary $\Gamma := \partial\Omega$, and with outward unit normal $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^{\mathrm{t}}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$, and semi-norm $|\cdot|_{s,\Omega}$. Given a generic scalar functional space A, we let $\mathbf{A}$ and $\mathbb{A}$ be its vectorial and tensor versions, respectively, and we denote by $\|\cdot\|$, with no subscripts, the natural norm of either an element or an operator in any product functional space. As usual, for any vector field $\boldsymbol{v} = (v_i)_{i=1,3}$, we set the gradient, divergence and, tensor product operators, as

$$\nabla \boldsymbol{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,3}, \quad \operatorname{div} \boldsymbol{v} := \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \boldsymbol{v} \otimes \boldsymbol{w} := (v_i w_j)_{i,j=1,3}.$$

Furthermore, given tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,3}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,3}$, we let $\mathbf{div}\,\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,3}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{3} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{3} \tau_{ij}\zeta_{ij}\,, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ stands for the identity tensor in $\mathbb{R}^{3\times 3}$. We recall that the space

$$\mathbb{H}(\mathbf{div};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) \Big\},$$

equipped with the usual norm

$$\|\boldsymbol{\tau}\|^2_{\mathbf{div},\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega},$$

is a Hilbert space. Finally, we employ $\mathbf{0}$ to denote a generic null vector and use c or C, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

2 The bioconvective flows model

In this section, we present the model problem, and define the auxiliary unknowns to be introduced into the respective continuous formulation. From [34, 37, 38], we consider the following system of partial differential equations, describing the three-dimensional hydrodynamics of negatively geotactic micro-organisms in suspension in a viscous and incompressible culture fluid Ω, given by

$$\begin{gathered} -\mathbf{div}\,(\mu(\varphi)\mathbf{e}(\boldsymbol{u})) + (\nabla\boldsymbol{u})\,\boldsymbol{u} + \nabla\,p = \boldsymbol{f} - g\,(1+\gamma\varphi)\,\mathbf{i}_3\,, \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = 0 \qquad \text{in} \quad \Omega, \\ -\kappa\Delta\varphi + \boldsymbol{u}\cdot\nabla\varphi + U\frac{\partial\varphi}{\partial x_3} = 0 \quad \text{in} \quad \Omega, \end{gathered} \tag{2.1}$$

that is, a set of coupled non-linear equations given by a Navier-Stokes type-system and an advection–diffusion equation, in the Boussinesq approximation framework, where the unknowns are the velocity $\boldsymbol{u} = (u_j)_{j=1,3}$, the pressure p and the micro-organism concentration φ of the culture fluid, and in the realistic case in which the micro-organisms concentration might affect the kinematic viscosity $\mu(\,\cdot\,)$. Here, $\mathbf{e}(\boldsymbol{u})$ stands for the symmetric part ot the velocity gradient, defined as $\mathbf{e}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{t}})$, $\boldsymbol{f}$ refers to a volume-distributed external force, g is the gravitational force magnitude, κ and U are constants associated to the diffusion rate and the mean velocity of upward swimming of the microorganisms, respectively, $\mathbf{i}_3 = (0,0,1)^{\mathrm{t}}$ is the vertical unitary vector, and $\gamma := \rho_0/\rho_m - 1 > 0$, is a given constant depending on the micro-organisms density ρ_0 and the culture fluid density ρ_m. In turn, such as in [10] (cf. [14]), we assume that the viscosity $\mu(\,\cdot\,)$ is a Lipschitz continuous and bounded from above and below function; that is, for some constants $L_\mu > 0$ and $\mu_1, \mu_2 > 0$, there hold

$$|\mu(s) - \mu(t)| \,\le\, L_\mu\,|s-t|\,, \qquad \forall\, s,t \ge 0, \tag{2.2}$$

and

$$\mu_1 \,\le\, \mu(s) \,\le\, \mu_2, \qquad \forall s \,\ge 0. \tag{2.3}$$

We complete the system (2.1), with a non-slip condition for the velocity and a zero flux Robin-type condition for the micro-organisms on the boundary, that is

$$\boldsymbol{u} = \mathbf{0}\,, \qquad \text{and} \qquad \kappa\frac{\partial\varphi}{\partial\boldsymbol{\nu}} - \nu_3 U\varphi \,=\, 0 \qquad \text{on} \quad \Gamma, \tag{2.4}$$

as well as the total mass restriction

$$\frac{1}{|\Omega|}\int_\Omega \varphi = \alpha, \tag{2.5}$$

where α is a given positive constant, assuring that no micro-organisms are allowed to leave or enter the physical domain. Note that (2.5) is equivalent to

$$\int_\Omega (\varphi - \alpha) = 0,$$

and consequently, when setting the auxiliary concentration $\varphi_\alpha := \varphi - \alpha$, which satisfies $\displaystyle\int_\Omega \varphi_\alpha = 0$, and by introducing it into (2.1) and (2.4), we get

$$-\mathbf{div}\,(\mu(\varphi_\alpha + \alpha)\mathbf{e}(\boldsymbol{u})) + (\nabla \boldsymbol{u})\,\boldsymbol{u} + \nabla\, p = \boldsymbol{f}_\alpha - g\,(1 + \gamma\varphi_\alpha)\,\mathbf{i}_3 \quad \text{in} \quad \Omega\,,$$

$$\kappa\frac{\partial \varphi_\alpha}{\partial \boldsymbol{\nu}} - \nu_3 U(\varphi_\alpha + \alpha) = 0 \quad \text{on} \quad \Gamma\,,$$

where $\boldsymbol{f}_\alpha := \boldsymbol{f} - g\gamma\alpha\,\mathbf{i}_3$. Note that the rest of equations remains unchanged with φ_α in place of φ. Therefore, to simplify the notation and without confusion, we rename from now on $\varphi := \varphi_\alpha$ and $\boldsymbol{f} := \boldsymbol{f}_\alpha$, so that the original problem (2.1), (2.4) and (2.5), takes the form

$$\begin{gathered} -\mathbf{div}\,(\mu(\varphi + \alpha)\mathbf{e}(\boldsymbol{u})) + (\nabla \boldsymbol{u})\,\boldsymbol{u} + \nabla\, p = \boldsymbol{f} - g\,(1 + \gamma\varphi)\,\mathbf{i}_3\,, \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = 0 \qquad \text{in} \quad \Omega, \\ -\kappa\Delta\varphi + \boldsymbol{u}\cdot\nabla\varphi + U\frac{\partial \varphi}{\partial x_3} = 0 \quad \text{in} \quad \Omega, \quad \text{with} \quad \int_\Omega \varphi = 0\,, \\ \boldsymbol{u} = \mathbf{0} \quad \text{and} \quad \kappa\frac{\partial \varphi}{\partial \boldsymbol{\nu}} - \nu_3 U(\varphi + \alpha) = 0 \quad \text{on} \quad \Gamma\,. \end{gathered} \tag{2.6}$$

From the first equation of (2.6), it is clear that uniqueness of an eventual pressure solution of this problem (see [28] or [40]) is ensured in the space

$$\mathrm{L}^2_0(\Omega) := \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\}.$$

Likewise, from the total mass condition on the auxiliary concentration (second equation of the second row in system (2.6)), we see that an eventual weak solution φ of (2.6) belongs to the space

$$\widetilde{\mathrm{H}}^1(\Omega) := \mathrm{H}^1(\Omega) \cap \mathrm{L}^2_0(\Omega) = \left\{ \psi \in \mathrm{H}^1(\Omega) : \quad \int_\Omega \psi = 0 \right\}, \tag{2.7}$$

which is a closed subspace of $\mathrm{H}^1(\Omega)$, and in which the norm and the seminorm are equivalent (result to be used in Lemma 3.2).

Next, in order to derive our fully-mixed formulation, we firstly need to rewrite (2.6) as a first-order system of equations. To this purpose, inspired by the the approach from [12] (see also [3, 4]), we introduce as additional unknowns the strain and vorticity tensors

$$\boldsymbol{t} := \mathbf{e}(\boldsymbol{u}) \qquad \text{and} \qquad \boldsymbol{\rho} = \frac{1}{2}\left\{ \nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^{\mathrm{t}} \right\} =: \nabla \boldsymbol{u} - \boldsymbol{t}\,, \qquad \text{in} \qquad \Omega \tag{2.8}$$

as well as the pseudo-stress tensor

$$\boldsymbol{\sigma} := \mu(\varphi + \alpha)\boldsymbol{t} - p\mathbb{I} - (\boldsymbol{u} \otimes \boldsymbol{u}) \quad \text{in} \quad \Omega. \tag{2.9}$$

Note that $\mathbf{div}(\boldsymbol{u} \otimes \boldsymbol{u}) = (\nabla \boldsymbol{u})\boldsymbol{u}$ when $\operatorname{div} \boldsymbol{u} = 0$ (incompressibility condition - second equation of first row in (2.6)). Thus, the first equation of (2.6) and the constitutive relation (2.9), gives the equilibrium equation

$$- \mathbf{div}(\boldsymbol{\sigma}) \;=\; \boldsymbol{f} - g\,(1 \,+\, \gamma\,\varphi)\mathbf{i}_3 \quad \text{in} \quad \Omega. \tag{2.10}$$

Again, from the incompressibility condition, we have that $\operatorname{tr}(\nabla \boldsymbol{u}) = 0$ and so $\operatorname{tr}(\boldsymbol{\rho}) = \operatorname{tr}(\boldsymbol{t}) = 0$. In particular, by taking deviatoric part from both sides of (2.9), we find that

$$\boldsymbol{\sigma}^{\mathrm{d}} = \mu(\varphi + \alpha)\boldsymbol{t} - (\boldsymbol{u} \otimes \boldsymbol{u})^{\mathrm{d}}\,, \quad \text{in } \Omega. \tag{2.11}$$

and so the pressure can be eliminated from the system but, by taking trace from both sides of (2.9), we readily deduce that it can be recovered in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ as

$$p = -\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma} + (\boldsymbol{u} \otimes \boldsymbol{u}))\,, \qquad \text{in} \qquad \Omega\,. \tag{2.12}$$

As for the equation modeling the micro-organisms concentration, similarly to [17], we introduce as the new vectorial unknown that we call "pseudo-concentration" gradient

$$\boldsymbol{p} := \kappa\nabla\varphi - \varphi\boldsymbol{u} - U(\varphi + \alpha)\mathbf{i}_3\,, \qquad \text{in} \qquad \Omega, \tag{2.13}$$

so that, from the first equation of second row from 2.6, the incompressibility condition and the Robin condition for the concentration, we get

$$-\operatorname{div} \boldsymbol{p} \,=\, 0 \quad \text{in} \quad \Omega, \qquad \text{and} \quad \boldsymbol{p} \cdot \boldsymbol{\nu} \,=\, 0 \qquad \text{on} \qquad \Gamma\,. \tag{2.14}$$

Finally, gathering together (2.8), (2.10), (2.11), (2.13) and (2.14), we arrive at the following first-order system with unknowns $\boldsymbol{t}$, $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$, $\boldsymbol{u}$, $\boldsymbol{p}$ and φ

$$\begin{gathered}
\boldsymbol{t} + \boldsymbol{\rho} \,=\, \nabla\boldsymbol{u}\,, \quad \boldsymbol{\sigma}^{\mathrm{d}} + (\boldsymbol{u} \otimes \boldsymbol{u})^{\mathrm{d}} \,=\, \mu(\varphi + \alpha)\boldsymbol{t}\,, \quad -\mathbf{div}\,\boldsymbol{\sigma} = \boldsymbol{f} - g\,(1 + \gamma\varphi)\,\mathbf{i}_3\,, \quad \text{in} \quad \Omega \\
\kappa^{-1}\boldsymbol{p} + \kappa^{-1}\varphi\boldsymbol{u} + \kappa^{-1}U(\varphi + \alpha)\mathbf{i}_3 \,=\, \nabla\varphi\,, \quad -\operatorname{div}\boldsymbol{p} \,=\, 0\,, \quad \text{in} \quad \Omega \\
\boldsymbol{u} = \mathbf{0} \quad \text{and} \quad \boldsymbol{p} \cdot \boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma, \\
\int_\Omega \operatorname{tr}(\boldsymbol{\sigma} + (\boldsymbol{u} \otimes \boldsymbol{u})) = 0 \quad \text{and} \quad \int_\Omega \varphi = 0\,.
\end{gathered} \tag{2.15}$$

Note that according to (2.12), the zero mean value restriction of the pressure on the domain is imposed via the first equation in the last row in (2.15). Also, notice that the incompressibility condition of the fluid is implicitly present through the equilibrium relation (2.10) and by stating that $\boldsymbol{t}$ is a trace-free tensor.

3 The continuous formulation

In this section we introduce and analyze the weak formulation of the system described by (2.15). To this end, in Section 3.1 we firstly deduce an augmented variational formulation of (2.15) and then in Section 3.2 we equivalently rewrite it as a fixed-point problem in terms of operators, which arise by decoupling the fluid equations and the concentration equation. Their well-definiteness and solvability are addressed through Sections 3.3 and 3.4.

3.1 The augmented fully-mixed variational formulation

We first recall (see, e.g., [23] or [28]) that there holds

$$\mathbb{H}(\mathbf{div};\Omega) = \mathbb{H}_0(\mathbf{div};\Omega) \oplus \mathrm{R}\mathbb{I}, \tag{3.1}$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) : \int_\Omega \mathrm{tr}\,\boldsymbol{\tau} = 0 \right\}.$$

which means that any $\zeta \in \mathbb{H}(\mathbf{div};\Omega)$, can be uniquely written in terms of its orthogonal projection, namely $\zeta_0 \in \mathbb{H}_0(\mathbf{div},\Omega)$, as

$$\zeta = \zeta_0 + c\,\mathbb{I}\,, \qquad \text{where} \qquad c = \frac{1}{3\,|\Omega|}\int_\Omega \mathrm{tr}\,\zeta\,.$$

In particular, using the first equation in the last row of (2.15), it is easy to see that an eventual solution $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div};\Omega)$ of that system is given by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\mathbb{I},, \qquad \text{with} \qquad \boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div},\Omega)\,, \quad \text{and} \quad c = -\frac{1}{3\,|\Omega|}\int_\Omega \mathrm{tr}\,(\boldsymbol{u}\otimes\boldsymbol{u})\,. \tag{3.2}$$

Then, since $\boldsymbol{\sigma}^{\mathrm{d}} = \boldsymbol{\sigma}_0^{\mathrm{d}}$ and $\mathbf{div}\,\boldsymbol{\sigma}^{\mathrm{d}} = \mathbf{div}\,\boldsymbol{\sigma}_0^{\mathrm{d}}$, it follows that the equations in (2.15) remain unchanged when replacing there $\boldsymbol{\sigma}_0$ in place of $\boldsymbol{\sigma}$. This fact along with (3.2) allows us to reduce the problem by only looking for $\boldsymbol{\sigma}_0$. According to that, and for simplifying the notation, we set from now on $\boldsymbol{\sigma} := \boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div};\Omega)$.
In addition, by their definitions, we introduce the following spaces for the strain tensor $\boldsymbol{t}$ and the vorticity $\boldsymbol{\rho}$, respectively,

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) := \left\{ \boldsymbol{r} \in \mathbb{L}^2(\Omega) : \boldsymbol{r}^{\mathrm{t}} = \boldsymbol{r} \quad \text{and} \quad \mathrm{tr}\,(\boldsymbol{r}) = 0 \right\}, \quad \text{and} \quad \mathbb{L}^2_{\mathrm{skew}}(\Omega) := \left\{ \boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \boldsymbol{\eta}^{\mathrm{t}} = -\boldsymbol{\eta} \right\}.$$

Also, the boundary condition for $\boldsymbol{p}$ on Γ (see third row in (2.15)) suggests the introduction of the functional space

$$\mathbf{H}_\Gamma(\mathrm{div};\Omega) := \left\{ \boldsymbol{q} \in \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{q}\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma \right\}.$$

Now, multiplying the first equation in (2.15) by a test function $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)$, integrating by parts, using the Dirichlet condition for $\boldsymbol{u}$, and the identity $\boldsymbol{t} : \boldsymbol{\tau} = \boldsymbol{t} : \boldsymbol{\tau}^{\mathrm{d}}$ (since $\boldsymbol{t}$ is trace-free), we get

$$\int_\Omega \boldsymbol{t} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\rho} : \boldsymbol{\tau} = 0\,, \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega).$$

Next, testing the second equation from first row in (2.15) with $\boldsymbol{r} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, we obtain

$$\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{r} + \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} : \boldsymbol{r} = \int_\Omega \mu(\varphi+\alpha)\boldsymbol{t} : \boldsymbol{r}\,, \quad \forall\,\boldsymbol{r} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega).$$

In turn, the equilibrium relation associated to $\boldsymbol{\sigma}$ (third equation from first row in (2.15)) is written as

$$-\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} = \int_\Omega \big(\boldsymbol{f} - g\,(1+\gamma\varphi)\,\mathbf{i}_3\big)\cdot\boldsymbol{v}\,, \quad \forall\,\boldsymbol{v} \in \mathbf{L}^2(\Omega),$$

whereas the symmetry of the pseudo-stress tensor is weakly imposed through the identity

$$-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} = 0\,, \quad \forall\,\boldsymbol{\eta} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega).$$

As for the equations associated to the micro-organisms concentration (second row from (2.15)), we firstly multiply the respective constitutive relation by a function $\boldsymbol{q} \in \mathbf{H}_\Gamma(\mathrm{div};\Omega)$ and, after integrating by parts, we find

$$\kappa^{-1}\int_\Omega \boldsymbol{p}\cdot\boldsymbol{q} + \kappa^{-1}\int_\Omega \varphi\boldsymbol{u}\cdot\boldsymbol{q} + \kappa^{-1}\int_\Omega U(\varphi+\alpha)\mathbf{i}_3\cdot\boldsymbol{q} = -\int_\Omega \varphi\,\mathrm{div}\boldsymbol{q}\,, \quad \forall\,\boldsymbol{q}\in\mathbf{H}_\Gamma(\mathrm{div};\Omega)\,,$$

and the equilibrium relation for the concentration is weakly expressed as

$$-\int_\Omega \psi\,\mathrm{div}\,\boldsymbol{p} = 0\,, \quad \forall\,\psi\in\mathrm{L}^2(\Omega).$$

In this way, we arrive at first instance to the mixed formulation: Find $\boldsymbol{t}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\boldsymbol{\sigma}\in\mathbb{H}_0(\mathbf{div};\Omega)$, $\boldsymbol{\rho}\in\mathbb{L}^2_{\mathrm{skew}}(\Omega)$, $\boldsymbol{p}\in\mathbf{H}_\Gamma(\mathrm{div};\Omega)$, and $\boldsymbol{u}$, φ in suitable spaces to be specified below, such that

$$\begin{aligned}
\int_\Omega \boldsymbol{t}:\boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\rho}:\boldsymbol{\tau} &= \ 0,\\
\int_\Omega \mu(\varphi+\alpha)\boldsymbol{t}:\boldsymbol{r} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{r} - \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}}:\boldsymbol{r} &= \ 0,\\
-\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \int_\Omega \boldsymbol{\sigma}:\boldsymbol{\eta} &= \ \int_\Omega (\boldsymbol{f}-g\,(1+\gamma\varphi)\,\mathbf{i}_3)\cdot\boldsymbol{v},\\
\kappa^{-1}\int_\Omega \boldsymbol{p}\cdot\boldsymbol{q} + \int_\Omega \varphi\,\mathrm{div}\boldsymbol{q} + \kappa^{-1}\int_\Omega \varphi\boldsymbol{u}\cdot\boldsymbol{q} &= \ -\kappa^{-1}\int_\Omega U(\varphi+\alpha)\mathbf{i}_3\cdot\boldsymbol{q},\\
-\int_\Omega \psi\,\mathrm{div}\,\boldsymbol{p} &= \ 0,
\end{aligned} \tag{3.3}$$

for all $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega)$, $\boldsymbol{r}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $(\boldsymbol{\eta},\boldsymbol{v})\in\mathbb{L}^2_{\mathrm{skew}}(\Omega)\times\mathbf{L}^2(\Omega)$, $\boldsymbol{q}\in\mathbf{H}_\Gamma(\mathrm{div};\Omega)$, and $\psi\in\mathrm{L}^2(\Omega)$. Note that the third terms on the left-hand side of the second and fourth equations in (3.3) require a suitable regularity for both unknowns $\boldsymbol{u}$ and φ. Indeed, by applying Cauchy-Schwarz and Hölder inequalities, and then the continuous injections $i:\mathrm{H}^1(\Omega)\to\mathrm{L}^4(\Omega)$ and $\boldsymbol{i}:\mathbf{H}^1(\Omega)\to\mathbf{L}^4(\Omega)$ (see e.g. [1] or [41]), we deduce that there exist positive constants $c_1(\Omega):=\|i\|\,\|\boldsymbol{i}\|$ and $c_2(\Omega):=\|\boldsymbol{i}\|^2$, such that

$$\left|\int_\Omega \varphi\boldsymbol{u}\cdot\boldsymbol{q}\right| \le c_1(\Omega)\|\varphi\|_{1,\Omega}\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{q}\|_{0,\Omega}\,, \quad \forall\,\varphi\in\mathrm{H}^1(\Omega),\forall\,\boldsymbol{u}\in\mathbf{H}^1(\Omega),\forall\,\boldsymbol{q}\in\mathbf{L}^2(\Omega), \tag{3.4}$$

and

$$\left|\int_\Omega (\boldsymbol{u}\otimes\boldsymbol{w})^{\mathrm{d}}:\boldsymbol{r}\right| \le c_2(\Omega)\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{w}\|_{1,\Omega}\|\boldsymbol{r}\|_{0,\Omega}\,, \quad \forall\,\boldsymbol{u},\boldsymbol{w}\in\mathbf{H}^1(\Omega),\forall\,\boldsymbol{r}\in\mathbb{L}^2(\Omega). \tag{3.5}$$

In light of the above, and in order to be able to set the variational formulation (3.3) in a framework on standard Hilbert spaces for both the velocity and concentration, we propose to seek $\boldsymbol{u}\in\mathbf{H}^1_0(\Omega)$ and $\varphi\in\widetilde{\mathrm{H}}^1(\Omega)$, and so their respective test spaces. In turn, similarly as in [12, Section 2] (see also [3,17]), we additionally augment (3.3) by incorporating the following redundant Galerkin terms coming from the constitutive and equilibrium equations,

$$\begin{aligned}
\kappa_1\int_\Omega (\mathbf{e}(\boldsymbol{u})-\boldsymbol{t}):\mathbf{e}(\boldsymbol{v}) = 0\,, & \qquad \forall\,\boldsymbol{v}\in\mathbf{H}^1_0(\Omega)\,,\\
\kappa_2\int_\Omega \mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} = -\kappa_2\int_\Omega (\boldsymbol{f}-g\,(1+\gamma\varphi)\,\mathbf{i}_3)\cdot\mathbf{div}\,\boldsymbol{\tau}\,, & \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega),\\
\kappa_3\int_\Omega \Big\{\boldsymbol{\sigma}^{\mathrm{d}} + (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} - \mu(\varphi+\alpha)\boldsymbol{t}\Big\}:\boldsymbol{\tau}^{\mathrm{d}} = 0\,, & \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega),\\
\kappa_4\int_\Omega \big\{\boldsymbol{\rho} - \big(\nabla\boldsymbol{u}-\mathbf{e}(\boldsymbol{u})\big)\big\}:\boldsymbol{\eta} = 0\,, & \qquad \forall\boldsymbol{\eta}\in\mathbb{L}^2_{\mathrm{skew}}(\Omega)\,,
\end{aligned} \tag{3.6}$$

and

$$
\begin{aligned}
\kappa_5 \int_\Omega \big\{ \nabla\varphi - \kappa^{-1}\boldsymbol{p} - \kappa^{-1}\varphi\boldsymbol{u} - \kappa^{-1}U(\varphi+\alpha)\mathbf{i}_3 \big\} \cdot \nabla\psi = 0\,, &\quad \forall\, \psi \in \widetilde{\mathrm{H}}^1(\Omega),\\
\kappa_6 \int_\Omega \operatorname{div}\boldsymbol{p} \operatorname{div}\boldsymbol{q} = 0\,, &\quad \forall\, \boldsymbol{q} \in \mathbf{H}_\Gamma(\operatorname{div};\Omega),
\end{aligned}
\tag{3.7}
$$

where $(\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5,\kappa_6)$ is a vector of positive parameters to be specified later in Section 3.3. Hence, letting

$$
\underline{\boldsymbol{t}} := (\boldsymbol{t},\boldsymbol{\sigma},\boldsymbol{\rho}) \in \mathbb{H} := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega),
$$

where $\mathbb{H}$ is endowed with the natural norm

$$
\|\underline{\boldsymbol{r}}\|_{\mathbb{H}} := \Big\{ \|\boldsymbol{r}\|^2_{0,\Omega} + \|\boldsymbol{\tau}\|^2_{\mathbf{div},\Omega} + \|\boldsymbol{\eta}\|^2_{0,\Omega} \Big\}^{1/2}, \quad \forall\, \underline{\boldsymbol{r}} := (\boldsymbol{r},\boldsymbol{\tau},\boldsymbol{\eta}) \in \mathbb{H}\,,
$$

and adding up (3.3) with (3.6) and (3.7), we arrive at the following augmented fully-mixed formulation for the bioconvective flow problem: Find $(\underline{\boldsymbol{t}},\boldsymbol{u},\boldsymbol{p},\varphi) \in \mathbb{H} \times \mathbf{H}^1_0(\Omega) \times \mathbf{H}_\Gamma(\operatorname{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$ such that

$$
\begin{aligned}
\mathbf{A}_\varphi((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{r}},\boldsymbol{v})) + \mathbf{B}_{\boldsymbol{u}}((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{r}},\boldsymbol{v})) &= F_\varphi(\underline{\boldsymbol{r}},\boldsymbol{v})\,, \quad \forall\, (\underline{\boldsymbol{r}},\boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}^1_0(\Omega),\\
\widetilde{\mathbf{A}}((\boldsymbol{p},\varphi),(\boldsymbol{q},\psi)) + \widetilde{\mathbf{B}}_{\boldsymbol{u}}((\boldsymbol{p},\varphi),(\boldsymbol{q},\psi)) &= \widetilde{F}_\varphi(\boldsymbol{q},\psi)\,, \quad \forall\, (\boldsymbol{q},\psi) \in \mathbf{H}_\Gamma(\operatorname{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega),
\end{aligned}
\tag{3.8}
$$

where, given $\phi \in \widetilde{\mathrm{H}}^1(\Omega)$ and $\boldsymbol{w} \in \mathbf{H}^1_0(\Omega)$, $\mathbf{A}_\phi$, $\mathbf{B}_{\boldsymbol{w}}$, $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}_{\boldsymbol{w}}$ are the bilinear forms defined, respectively, as

$$
\begin{aligned}
\mathbf{A}_\phi((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{r}},\boldsymbol{v})) := &\int_\Omega \mu(\phi+\alpha)\boldsymbol{t} : \big(\boldsymbol{r} - \kappa_3\boldsymbol{\tau}^{\mathrm{d}}\big) + \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \big(\kappa_3\boldsymbol{\tau}^{\mathrm{d}} - \boldsymbol{r}\big) + \int_\Omega \boldsymbol{t} : \boldsymbol{\tau}^{\mathrm{d}}\\
&+ \int_\Omega \big(\boldsymbol{u} + \kappa_2\mathbf{div}\,\boldsymbol{\sigma}\big)\cdot\mathbf{div}\,\boldsymbol{\tau} - \int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} + \int_\Omega \boldsymbol{\rho} : \boldsymbol{\tau} - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta}\\
&+ \kappa_1 \int_\Omega \big(\mathbf{e}(\boldsymbol{u}) - \boldsymbol{t}\big) : \mathbf{e}(\boldsymbol{v}) + \kappa_4 \int_\Omega \big\{\boldsymbol{\rho} - \big(\nabla\boldsymbol{u} - \mathbf{e}(\boldsymbol{u})\big)\big\} : \boldsymbol{\eta}\,,
\end{aligned}
\tag{3.9}
$$

$$
\mathbf{B}_{\boldsymbol{w}}((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{r}},\boldsymbol{v})) := \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{w})^{\mathrm{d}} : \big(\kappa_3\boldsymbol{\tau}^{\mathrm{d}} - \boldsymbol{r}\big)\,,
\tag{3.10}
$$

$$
\begin{aligned}
\widetilde{\mathbf{A}}((\boldsymbol{p},\varphi),(\boldsymbol{q},\psi)) := &\ \kappa^{-1} \int_\Omega \boldsymbol{p}\cdot\big(\boldsymbol{q} - \kappa_5\nabla\psi\big) + \int_\Omega \big(\varphi + \kappa_6\operatorname{div}\boldsymbol{p}\big)\operatorname{div}\boldsymbol{q}\\
&- \int_\Omega \psi\operatorname{div}\boldsymbol{p} + \kappa_5 \int_\Omega \nabla\varphi\cdot\nabla\psi,
\end{aligned}
\tag{3.11}
$$

and

$$
\widetilde{\mathbf{B}}_{\boldsymbol{w}}((\boldsymbol{p},\varphi),(\boldsymbol{q},\psi)) := \kappa^{-1} \int_\Omega \varphi\boldsymbol{w}\cdot\big(\boldsymbol{q} - \kappa_5\nabla\psi\big),
\tag{3.12}
$$

for all $(\underline{\boldsymbol{t}},\boldsymbol{u})$, $(\underline{\boldsymbol{r}},\boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}^1_0(\Omega)$ and for all $(\boldsymbol{p},\varphi)$, $(\boldsymbol{q},\psi) \in \mathbf{H}_\Gamma(\operatorname{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$. In turn, given $\phi \in \widetilde{\mathrm{H}}^1(\Omega)$, F_ϕ and $\widetilde{F}_\phi$ are the bounded linear functionals given by

$$
F_\phi(\underline{\boldsymbol{r}},\boldsymbol{v}) := \int_\Omega \big(\boldsymbol{f} - g\,(1+\gamma\phi)\,\mathbf{i}_3\big)\cdot\big(\boldsymbol{v} - \kappa_2\mathbf{div}\,\boldsymbol{\tau}\big)\,, \qquad \forall\, (\underline{\boldsymbol{r}},\boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}^1_0(\Omega),
\tag{3.13}
$$

and

$$
\widetilde{F}_\phi(\boldsymbol{q},\psi) := -\kappa^{-1} \int_\Omega U(\phi+\alpha)\mathbf{i}_3\cdot\big(\boldsymbol{q} - \kappa_5\nabla\psi\big)\,, \qquad \forall\, (\boldsymbol{q},\psi) \in \mathbf{H}_\Gamma(\operatorname{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega).
\tag{3.14}
$$

3.2 The fixed point approach

Now, we proceed similarly as in [15] (see also [12, 17]) and rewrite (3.8) as an equivalent fixed-point equation in terms of a certain operator $\mathbf{T}$ to be defined below. Firstly, we set $\mathbf{H} := \mathbf{H}_0^1(\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$ and start by introducing the operator $\mathbf{S} : \mathbf{H} \longrightarrow \mathbb{H} \times \mathbf{H}_0^1(\Omega)$ by

$$\mathbf{S}(\boldsymbol{w}, \phi) := \Big(\big(\mathbf{S}_1(\boldsymbol{w}, \phi), \mathbf{S}_2(\boldsymbol{w}, \phi), \mathbf{S}_3(\boldsymbol{w}, \phi)\big)\,,\, \mathbf{S}_4(\boldsymbol{w}, \phi) \Big) = (\underline{\boldsymbol{t}}, \boldsymbol{u})\,, \qquad \forall\, (\boldsymbol{w}, \phi) \in \mathbf{H}, \tag{3.15}$$

where, given $(\boldsymbol{w}, \phi) \in \mathbf{H}$, $(\underline{\boldsymbol{t}}, \boldsymbol{u})$ is the unique solution to the problem: Find $(\underline{\boldsymbol{t}}, \boldsymbol{u}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)$ such that

$$\mathbf{A}_\phi((\underline{\boldsymbol{t}}, \boldsymbol{u}), (\underline{\boldsymbol{r}}, \boldsymbol{v})) \,+\, \mathbf{B}_{\boldsymbol{w}}((\underline{\boldsymbol{t}}, \boldsymbol{u}), (\underline{\boldsymbol{r}}, \boldsymbol{v})) \,=\, F_\phi(\underline{\boldsymbol{r}}, \boldsymbol{v})\,, \qquad \forall\, (\underline{\boldsymbol{r}}, \boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)\,. \tag{3.16}$$

In addition, we also introduce the operator $\widetilde{\mathbf{S}} : \mathbf{H} \longrightarrow \mathbf{H}_\Gamma(\mathrm{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$ defined as

$$\widetilde{\mathbf{S}}(\boldsymbol{w}, \phi) \,:=\, \big(\widetilde{\mathbf{S}}_1(\boldsymbol{w}, \phi), \widetilde{\mathbf{S}}_2(\boldsymbol{w}, \phi)\big) \,=\, (\boldsymbol{p}, \varphi)\,, \qquad \forall\, (\boldsymbol{w}, \phi) \in \mathbf{H}, \tag{3.17}$$

where, given $(\boldsymbol{w}, \phi) \in \mathbf{H}$, $(\boldsymbol{p}, \varphi)$ is the unique solution to the problem: Find $(\boldsymbol{p}, \varphi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$ such that

$$\widetilde{\mathbf{A}}((\boldsymbol{p}, \varphi), (\boldsymbol{q}, \psi)) \,+\, \widetilde{\mathbf{B}}_{\boldsymbol{w}}((\boldsymbol{p}, \varphi), (\boldsymbol{q}, \psi)) \,=\, \widetilde{F}_\phi(\boldsymbol{q}, \psi)\,, \quad \forall\, (\boldsymbol{q}, \psi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega). \tag{3.18}$$

Having introduced the auxiliary mappings $\mathbf{S}$ and $\widetilde{\mathbf{S}}$, we now define the operator $\mathbf{T} : \mathbf{H} \longrightarrow \mathbf{H}$ as

$$\mathbf{T}(\boldsymbol{w}, \phi) \,:=\, \Big(\mathbf{S}_4(\boldsymbol{w}, \phi), \widetilde{\mathbf{S}}_2\big(\mathbf{S}_4(\boldsymbol{w}, \phi), \phi\big)\Big)\,, \quad \forall (\boldsymbol{w}, \phi) \in \mathbf{H}\,, \tag{3.19}$$

and realize that (3.8) can be rewritten as the fixed point problem: Find $(\boldsymbol{u}, \varphi) \in \mathbf{H}$ such that

$$\mathbf{T}(\boldsymbol{u}, \varphi) = (\boldsymbol{u}, \varphi). \tag{3.20}$$

In this way, through the following sections we study the conditions under which the operator $\mathbf{T}$ is well-defined, has a fixed point and when it is unique.

3.3 Well-definiteness of the fixed point operator

In what follows we show that $\mathbf{T}$ is well-defined. Notice that it suffices to prove that the uncoupled problems (3.16) and (3.18) defining $\mathbf{S}$ and $\widetilde{\mathbf{S}}$, respectively, are well-posed. To state the solvability of (3.16), we start studying the stability properties of the forms $\mathbf{A}_\phi$ and $\mathbf{B}_{\boldsymbol{w}}$ and the functional F_ϕ (cf. (3.9), (3.10) and (3.13), respectively. Firstly, given $\phi \in \widetilde{\mathrm{H}}^1(\Omega)$, from the Cauchy-Schwarz inequality we find that there exists a positive constant, denoted by $\|\mathbf{A}_\phi\|$, and depending on μ_2 (cf. (2.3)) and the parameters $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, such that

$$|\mathbf{A}_\phi((\underline{\boldsymbol{t}}, \boldsymbol{u}), (\underline{\boldsymbol{r}}, \boldsymbol{v}))| \,\le\, \|\mathbf{A}_\phi\|\, \|(\underline{\boldsymbol{t}}, \boldsymbol{u})\|\, \|(\underline{\boldsymbol{r}}, \boldsymbol{v})\|\,, \qquad \forall\, (\underline{\boldsymbol{t}}, \boldsymbol{u}),\, (\underline{\boldsymbol{r}}, \boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega). \tag{3.21}$$

Also, given $\boldsymbol{w} \in \mathbf{H}_0^1(\Omega)$, from the estimation (3.5) we have that

$$|\mathbf{B}_{\boldsymbol{w}}((\underline{\boldsymbol{t}}, \boldsymbol{u}), (\underline{\boldsymbol{r}}, \boldsymbol{v}))| \le c_2(\Omega)(1 + \kappa_3^2)^{1/2} \|\boldsymbol{w}\|_{1,\Omega} \|\boldsymbol{u}\|_{1,\Omega} \|(\underline{\boldsymbol{r}}, \boldsymbol{v})\|\,, \qquad \forall\, (\underline{\boldsymbol{t}}, \boldsymbol{u}),\, (\underline{\boldsymbol{r}}, \boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega). \tag{3.22}$$

It then follows from (3.21) and (3.22) that there exists a positive constant, denoted by $\|\mathbf{A}_\phi + \mathbf{B}_{\boldsymbol{w}}\|$, and depending on μ_2, $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, $c_2(\Omega)$, and $\|\boldsymbol{w}\|_{1,\Omega}$, such that

$$|\big(\mathbf{A}_\phi \,+\, \mathbf{B}_{\boldsymbol{w}}\big)((\underline{\boldsymbol{t}}, \boldsymbol{u}), (\underline{\boldsymbol{r}}, \boldsymbol{v}))| \,\le \|\mathbf{A}_\phi + \mathbf{B}_{\boldsymbol{w}}\|\, \|(\underline{\boldsymbol{t}}, \boldsymbol{u})\|\, \|(\underline{\boldsymbol{r}}, \boldsymbol{v})\|\,, \qquad \forall\, (\underline{\boldsymbol{t}}, \boldsymbol{u}),\, (\underline{\boldsymbol{r}}, \boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)\,. \tag{3.23}$$

Regarding the ellipticity of $\mathbf{A}_\phi$ we proceed similarly to [12, Lemma 3.1]. So, we use the bounds for $\mu(\cdot)$ (cf. (2.3)), the Cauchy-Schwarz and Young inequalities (with $\delta_1, \delta_2, \delta_3 > 0$), and subsequently the Korn inequality and the Poincaré inequality (see [42, Théorème 1.2-5]) with constant c_p, to deduce that there exists $\alpha(\Omega) > 0$ satisfying

$$\mathbf{A}_\phi((\underline{\boldsymbol{r}}, \boldsymbol{v}), (\underline{\boldsymbol{r}}, \boldsymbol{v})) \,\geq\, \alpha(\Omega)\, \|(\underline{\boldsymbol{r}}, \boldsymbol{v})\|^2, \qquad \forall\, (\underline{\boldsymbol{r}}, \boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega), \tag{3.24}$$

where

$$\alpha(\Omega) := \min\Big\{\alpha_1(\Omega), \alpha_3(\Omega), c_p\alpha_4(\Omega), \alpha_5(\Omega)\Big\}, \tag{3.25}$$

with

$$\alpha_1(\Omega) := \mu_1 - \frac{\kappa_3\mu_2}{2\delta_1} - \frac{\kappa_1}{2\delta_2}, \quad \alpha_2(\Omega) := \min\left\{\kappa_3\left(1 - \frac{\mu_2\delta_1}{2}\right), \frac{\kappa_2}{2}\right\},$$

$$\alpha_3(\Omega) := \min\left\{c_3(\Omega)\alpha_2(\Omega), \frac{\kappa_2}{2}\right\}, \quad \alpha_4(\Omega) := \frac{\kappa_1}{2}\left(1 - \frac{\delta_2}{2}\right) - \frac{\kappa_4}{4\delta_3}, \quad \text{and} \quad \alpha_5(\Omega) := \kappa_4\left(1 - \frac{\delta_3}{2}\right),$$

and $c_3(\Omega) > 0$ (see [23, Lemma 2.3], for details) is such that

$$c_3(\Omega)\|\boldsymbol{\tau}\|_{0,\Omega}^2 \leq \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2, \qquad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega).$$

In turn, the positivity of $\alpha(\Omega)$ is ensured as long as the constants α_i in (3.25) are positive, which gives the following feasible ranges for the parameters $(\kappa_i)_{1\leq i\leq 4}$,

$$0 \,<\, \kappa_1 < 2\delta_2\Big(\mu_1 - \frac{\mu_2\kappa_3}{2\delta_1}\Big), \quad \kappa_2 \,>\, 0, \quad 0 \,<\, \kappa_3 < \frac{2\delta_1\mu_1}{\mu_2}, \quad \text{and} \quad 0 \,<\, \kappa_4 < 2\delta_3\kappa_3\Big(1 - \frac{\delta_2}{2}\Big) \tag{3.26}$$

with

$$0 \,<\, \delta_1 < \frac{2}{\mu_2}, \quad \text{and} \quad 0 < \delta_2\,,\, \delta_3 \,<\, 2. \tag{3.27}$$

Next, combining (3.22) with (3.24), we have that for all $(\underline{\boldsymbol{r}}, \boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)$ there holds

$$(\mathbf{A}_\phi + \mathbf{B}_{\boldsymbol{w}})((\underline{\boldsymbol{r}}, \boldsymbol{v}), (\underline{\boldsymbol{r}}, \boldsymbol{v})) \geq \Big\{\alpha(\Omega) - c_2(\Omega)(1+\kappa_3^2)^{1/2}\|\boldsymbol{w}\|_{1,\Omega}\Big\}\|(\underline{\boldsymbol{r}}, \boldsymbol{v})\|^2 \geq \frac{\alpha(\Omega)}{2}\|(\underline{\boldsymbol{r}}, \boldsymbol{v})\|^2, \tag{3.28}$$

provided $c_2(\Omega)(1+\kappa_3^2)^{1/2}\|\boldsymbol{w}\|_{1,\Omega} \leq \dfrac{\alpha(\Omega)}{2}$. Therefore, the ellipticity of the form $\mathbf{A}_\phi + \mathbf{B}_{\boldsymbol{w}}$ is ensured with the constant $\dfrac{\alpha(\Omega)}{2} > 0$, independent of $\boldsymbol{w}$, by requiring $\|\boldsymbol{w}\|_{1,\Omega} \leq r_0$, with

$$r_0 \,:=\, \frac{\alpha(\Omega)}{2c_2(\Omega)(1+\kappa_3^2)^{1/2}}. \tag{3.29}$$

Finally, the functional F_ϕ (with $\phi \in \widetilde{\mathrm{H}}^1(\Omega)$, given) is clearly linear in $\mathbb{H} \times \mathbf{H}_0^1(\Omega)$, and using Cauchy-Schwarz inequality, we conclude with $M_{\mathbf{S}} := (1+\kappa_2^2)^{1/2}$, that

$$\|F_\phi\| \,\leq\, M_{\mathbf{S}}\Big\{\|\boldsymbol{f}\|_{0,\Omega} \,+\, \Big(|\Omega|^{1/2} \,+\, \gamma\|\phi\|_{0,\Omega}\Big)\,\|\boldsymbol{g}\|_{\infty,\Omega}\Big\}. \tag{3.30}$$

where $\boldsymbol{g} := g\,\mathbf{i}_3 \,\in\, \mathbf{L}^\infty(\Omega)$. The foregoing analysis essentially gives us conditions for the well-posedness of the uncoupled problem (3.16) or, equivalently, the well definition of the operator $\mathbf{S}$ (cf. (3.15)). This is summarized in the following result.

Lemma 3.1 *Let $r_0 > 0$ given by (3.29) and let $r \in (0, r_0)$. Assume that $\kappa_1 \in \left(0, 2\delta_2\left(\mu_1 - \frac{\kappa_3\mu_2}{2\delta_1}\right)\right)$, $\kappa_2 > 0$, $\kappa_3 \in \left(0, \frac{2\delta_1\mu_1}{\mu_2}\right)$, and $\kappa_4 \in \left(0, 2\delta_3\kappa_1\left(1 - \frac{\delta_2}{2}\right)\right)$, with $\delta_1 \in \left(0, \frac{2}{\mu_2}\right)$, and $\delta_2, \delta_3 \in (0, 2)$. Then, for each $(\boldsymbol{w}, \phi) \in \mathbf{H}$ such that $\|\boldsymbol{w}\|_{1,\Omega} \le r$, there exist a unique solution $(\underline{\boldsymbol{t}}, \boldsymbol{u}) = \mathbf{S}(\boldsymbol{w}, \phi) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega)$ to problem (3.16) and a positive constant $c_{\mathbf{S}} > 0$, independent of $(\boldsymbol{w}, \phi)$, such that*

$$\|\mathbf{S}(\boldsymbol{w}, \phi)\| = \|(\underline{\boldsymbol{t}}, \boldsymbol{u})\| \le c_{\mathbf{S}}\left\{\|\boldsymbol{f}\|_{0,\Omega} + \left(|\Omega|^{1/2} + \gamma\|\phi\|_{0,\Omega}\right)\|\boldsymbol{g}\|_{\infty,\Omega}\right\}. \tag{3.31}$$

Proof. It follows from the estimates (3.23), (3.28) and (3.30) and a straightforward application of the Lax-Milgram Theorem (see e.g. [23, Theorem 1.1]), and the respective continuous dependence result gives the a priori estimate (3.31) with $c_{\mathbf{S}} := \dfrac{2M_{\mathbf{S}}}{\alpha(\Omega)}$. In turn, the ranges for the parameters are stated according to (3.26)-(3.27), guaranteeing the positivity of the ellipticity constant $\alpha(\Omega)$. □

Next, we concentrate in proving that problem (3.18) is well posed or, in other words, that the operator $\widetilde{\mathbf{S}}$ (cf. (3.17)) is well-defined. The following lemma establishes this result.

Lemma 3.2 *Assume that $\kappa_5 \in (0, 2\widetilde{\delta})$, with $\widetilde{\delta} \in (0, 2\kappa)$, and $\kappa_6 > 0$. Then, there exists a positive constant $\widetilde{r}_0$ (see (3.37) below) such that for all $\widetilde{r} \in (0, \widetilde{r}_0)$ and $(\boldsymbol{w}, \phi) \in \mathbf{H}$ with $\|\boldsymbol{w}\|_{1,\Omega} \le r$, the problem (3.18) has a unique solution $(\boldsymbol{p}, \varphi) := \widetilde{\mathbf{S}}(\boldsymbol{w}, \phi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$. Moreover, there exists a constant $c_{\widetilde{\mathbf{S}}} > 0$, independent of $(\boldsymbol{w}, \phi)$, satisfying*

$$\|\widetilde{\mathbf{S}}(\boldsymbol{w}, \phi)\| = \|(\boldsymbol{p}, \varphi)\| \le c_{\widetilde{\mathbf{S}}}\,\kappa^{-1}\,U\left\{\alpha|\Omega|^{1/2} + \|\phi\|_{0,\Omega}\right\}. \tag{3.32}$$

Proof. For a given $\boldsymbol{w} \in \mathbf{H}_0^1(\Omega)$, we firstly observe from (3.11) and (3.12) that $\widetilde{\mathbf{A}} + \widetilde{\mathbf{B}}_{\boldsymbol{w}}$ is clearly a bilinear form. Also, from the Cauchy-Schwarz inequality we have that

$$|\widetilde{\mathbf{A}}((\boldsymbol{p}, \varphi), (\boldsymbol{q}, \psi))| \le \|\widetilde{\mathbf{A}}\|\,\|(\boldsymbol{p}, \varphi)\|\,\|(\boldsymbol{q}, \psi)\|,$$

where $\|\widetilde{\mathbf{A}}\|$ depends on κ, κ_5 and κ_6, and from the estimate (3.4) we get

$$|\widetilde{\mathbf{B}}_{\boldsymbol{w}}((\boldsymbol{p}, \varphi), (\boldsymbol{q}, \psi))| \le \kappa^{-1}(1 + \kappa_5^2)^{1/2} c_1(\Omega)\|\boldsymbol{w}\|_{1,\Omega}\|\varphi\|_{1,\Omega}\|(\boldsymbol{q}, \psi)\|, \tag{3.33}$$

for all $(\boldsymbol{p}, \varphi)$, $(\boldsymbol{q}, \psi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$. Then, by gathering the foregoing estimates, we find that there exists a positive constant, which we denote by $\|\widetilde{\mathbf{A}} + \widetilde{\mathbf{B}}_{\boldsymbol{w}}\|$, only depending on $\kappa, \kappa_5, \kappa_6$ and $c_1(\Omega)$, such that

$$|\widetilde{\mathbf{A}}((\boldsymbol{p}, \varphi), (\boldsymbol{q}, \psi)) + \widetilde{\mathbf{B}}_{\boldsymbol{w}}((\boldsymbol{p}, \varphi), (\boldsymbol{q}, \psi))| \le \|\widetilde{\mathbf{A}} + \widetilde{\mathbf{B}}_{\boldsymbol{w}}\|\,\|(\boldsymbol{p}, \varphi)\|\,\|(\boldsymbol{q}, \psi)\|$$

for all $(\boldsymbol{p}, \varphi)$, $(\boldsymbol{q}, \psi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$. Likewise, from the definition of the bilinear form $\widetilde{\mathbf{A}}$ (cf. (3.11)), we have that

$$\widetilde{\mathbf{A}}((\boldsymbol{q}, \psi), (\boldsymbol{q}, \psi)) = \kappa^{-1}\|\boldsymbol{q}\|_{0,\Omega}^2 - \kappa^{-1}\kappa_5\int_\Omega \boldsymbol{q}\cdot\nabla\psi + \kappa_6\,\|\mathrm{div}\boldsymbol{q}\|_{0,\Omega}^2 + \kappa_5\,|\psi|_{1,\Omega}^2,$$

and hence, using the Cauchy-Schwarz inequality and the Young inequality with $\widetilde{\delta} > 0$, we obtain for all $(\boldsymbol{q}, \psi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$ that

$$\widetilde{\mathbf{A}}((\boldsymbol{q}, \psi), (\boldsymbol{q}, \psi)) \ge \kappa^{-1}\left(1 - \frac{\kappa_5}{2\widetilde{\delta}}\right)\|\boldsymbol{q}\|_{0,\Omega}^2 + \kappa_6\,\|\mathrm{div}\boldsymbol{q}\|_{0,\Omega}^2 + \kappa_5\left(1 - \frac{\kappa^{-1}\widetilde{\delta}}{2}\right)|\psi|_{1,\Omega}^2. \tag{3.34}$$

In this way, recalling that the norm and semi-norm are equivalent in the space $\widetilde{\mathrm{H}}^1(\Omega)$ (cf. 2.7), we apply the generalized Poincaré inequality with constant $\widetilde{c}_p$ to the last term in (3.34) (see [24, Teorema 9.13]), and define the constants

$$\widetilde{\alpha}_1(\Omega) := \min\left\{\kappa^{-1}\left(1 - \frac{\kappa_5}{2\widetilde{\delta}}\right), \kappa_6\right\} \quad \text{and} \quad \widetilde{\alpha}_2(\Omega) := \kappa_5\left(1 - \frac{\kappa^{-1}\widetilde{\delta}}{2}\right),$$

which are positive thanks to the hypotheses on $\widetilde{\delta}, \kappa_5$ and κ_6, to obtain

$$\widetilde{\mathbf{A}}((\boldsymbol{q},\psi),(\boldsymbol{q},\psi)) \,\geq\, \widetilde{\alpha}(\Omega)\|(\boldsymbol{q},\psi)\|^2\,, \qquad \forall (\boldsymbol{q},\psi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega)\times\widetilde{\mathrm{H}}^1(\Omega), \tag{3.35}$$

with $\widetilde{\alpha}(\Omega) := \min\{\widetilde{\alpha}_1(\Omega), \widetilde{c}_p\widetilde{\alpha}_2(\Omega)\}$, which shows that $\widetilde{\mathbf{A}}$ is elliptic. Therefore, combining (3.33) and (3.35), we deduce that for all $(\boldsymbol{q},\psi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega)\times\widetilde{\mathrm{H}}^1(\Omega)$, there holds

$$(\widetilde{\mathbf{A}} + \widetilde{\mathbf{B}}_{\boldsymbol{w}})((\boldsymbol{q},\psi),(\boldsymbol{q},\psi)) \geq \left\{\widetilde{\alpha}(\Omega) - \kappa^{-1}(1+\kappa_5^2)^{1/2}c_1(\Omega)\|\boldsymbol{w}\|_{1,\Omega}\right\}\|(\boldsymbol{q},\psi)\|^2 \geq \frac{\widetilde{\alpha}(\Omega)}{2}\|(\boldsymbol{q},\psi)\|^2, \tag{3.36}$$

whenever $\kappa^{-1}(1+\kappa_5^2)^{1/2}c_1(\Omega)\|\boldsymbol{w}\|_{1,\Omega} \leq \dfrac{\widetilde{\alpha}(\Omega)}{2}$. Thus, the ellipticity of $\widetilde{\mathbf{A}} + \widetilde{\mathbf{B}}_{\boldsymbol{w}}$ with constant $\dfrac{\widetilde{\alpha}(\Omega)}{2}$, independent of $\boldsymbol{w}$, is ensured by requiring $\|\boldsymbol{w}\|_{1,\Omega} \leq \widetilde{r}_0$, with

$$\widetilde{r}_0 := \frac{\widetilde{\alpha}(\Omega)}{2\kappa^{-1}(1+\kappa_5^2)^{1/2}c_1(\Omega)}. \tag{3.37}$$

Next, it is easy to see from (3.14) that the functional $\widetilde{F}_\phi$ is bounded with

$$\|\widetilde{F}_\phi\| \,\leq\, \kappa^{-1}\,U\,(1+\kappa_5^2)^{1/2}\left\{\alpha|\Omega|^{1/2} + \|\phi\|_{0,\Omega}\right\}. \tag{3.38}$$

Summing up, and owing to the hypotheses on κ_5 and κ_6, we have proved that for any sufficiently small $\boldsymbol{w} \in \mathbf{H}_0^1(\Omega)$, the bilinear form $\widetilde{\mathbf{A}} + \widetilde{\mathbf{B}}_{\boldsymbol{w}}$ and the functional $\widetilde{F}_\phi$ satisfy the hypotheses of the Lax-Milgram Theorem, which guarantees the well-posedness of (3.18) and the a priori estimate (3.32) with $c_{\widetilde{\mathbf{S}}} := \dfrac{2}{\widetilde{\alpha}(\Omega)}(1+\kappa_5^2)^{1/2}$. □

At this point, we remark that, for computational purposes, the constants $\alpha(\Omega)$ and $\widetilde{\alpha}(\Omega)$ yielding the ellipticity of $\mathbf{A}_\phi + \mathbf{B}_{\boldsymbol{w}}$ and $\widetilde{\mathbf{A}} + \widetilde{\mathbf{B}}_{\boldsymbol{w}}$, respectively, can be maximized by taking the parameters δ_1, δ_2, δ_3, κ_1, κ_3, κ_4, $\widetilde{\delta}$ and κ_5 as the middle points of their feasible ranges, and by choosing κ_2 and κ_6 so that they maximize the minima defining $\alpha_2(\Omega)$ and $\widetilde{\alpha}_1(\Omega)$, respectively. More precisely, we take

$$\begin{gathered}
\delta_1 = \tfrac{1}{\mu_2}, \quad \delta_2 = \delta_3 = 1, \quad \kappa_3 = \tfrac{\delta_1\mu_1}{\mu_2} = \tfrac{\mu_1}{\mu_2^2}, \quad \kappa_1 = \delta_2\left(\mu_1 - \tfrac{\kappa_3\mu_2}{2\delta_1}\right) = \tfrac{\mu_1}{2}, \\
\kappa_4 = \delta_3\kappa_1\left(1 - \tfrac{\delta_2}{2}\right) = \tfrac{\mu_1}{2}, \quad \kappa_2 = 2\kappa_3\left(\mu_1 - \tfrac{\mu_2\delta_1}{2}\right) = \tfrac{\mu_1}{\mu_2^2}, \quad \widetilde{\delta} = \kappa, \\
\kappa_5 = \widetilde{\delta} = \kappa, \quad \kappa_6 = \kappa^{-1}\left(1 - \tfrac{\kappa_5}{2\widetilde{\delta}}\right) = \tfrac{\kappa^{-1}}{2}
\end{gathered} \tag{3.39}$$

The explicit values of the stabilization parameters κ_i, $i \in \{1,\ldots,6\}$, given above will be employed in Section 6 for the corresponding numerical examples.

3.4 Solvability analysis of the fixed point equation

Having proved the well-posedness of the uncoupled problems (3.16) and (3.18), which ensures that the operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$ and $\mathbf{T}$ are well defined, we now aim to establish the existence of a unique fixed point of the operator $\mathbf{T}$. For this purpose, in what follows we will verify the hypothesis of the Banach fixed-point theorem (e.g. [13, Theorem 3.7-1]). We begin with the following result.

Lemma 3.3 *Suppose that the parameters κ_i, $i \in \{1, \ldots, 6\}$, satisfy the conditions required by Lemmas 3.1 and 3.2. Given $r \in (0, \min\{r_0, \widetilde{r}_0\})$, with r_0 and $\widetilde{r}_0$ given by (3.29) and (3.37), respectively, we let W_r be the closed ball in $\mathbf{H}$ defined by*

$$W_r := \Big\{ (\boldsymbol{w}, \phi) \in \mathbf{H} : \ \|(\boldsymbol{w}, \phi)\| \le r \Big\}, \tag{3.40}$$

and assume that the data satisfy

$$c_{\mathbf{S}} \Big\{ \|\boldsymbol{f}\|_{0,\Omega} + \Big(|\Omega|^{1/2} + \gamma\, r \Big) \|\boldsymbol{g}\|_{\infty,\Omega} \Big\} + c_{\widetilde{\mathbf{S}}}\, \kappa^{-1}\, U \Big\{ \alpha |\Omega|^{1/2} + r \Big\} \le r, \tag{3.41}$$

with $c_{\mathbf{S}}$ and $c_{\widetilde{\mathbf{S}}}$ as in (3.31) and (3.32), respectively. Then $\mathbf{T}(W_r) \subseteq W_r$.

Proof. Given $(\boldsymbol{w}, \phi) \in W_r$, and so $\|\boldsymbol{w}\|_{1,\Omega} \le r_0$, it follows from Lemma 3.1 that there exists a unique $\boldsymbol{u} = \mathbf{S}_4(\boldsymbol{w}, \phi) \in \mathbf{H}_0^1(\Omega)$ solution to problem (3.16) and it satisfies the a priori estimate (3.31). In turn, if the data satisfies (3.41), we have that $\|\mathbf{S}_4(\boldsymbol{w}, \phi)\|_{1,\Omega} \le \widetilde{r}_0$, and according to Lemma 3.2 there exists a unique $\varphi = \widetilde{\mathbf{S}}_2(\mathbf{S}_4(\boldsymbol{w}, \phi), \phi) \in \widetilde{\mathrm{H}}^1(\Omega)$ solution to (3.18) with $\boldsymbol{w} := \mathbf{S}_4(\boldsymbol{w}, \phi)$. As a consequence, from the definition of the operator $\mathbf{T}$ (cf. (3.19)), $\exists!\, (\boldsymbol{u}, \varphi) = (\mathbf{S}_4(\boldsymbol{w}, \phi), \widetilde{\mathbf{S}}_2(\mathbf{S}_4(\boldsymbol{w}, \phi), \phi)) = \mathbf{T}(\boldsymbol{w}, \phi)$ and from (3.31) and (3.32)

$$\begin{aligned} \|(\boldsymbol{u}, \varphi)\| &\le \|\mathbf{S}_4(\boldsymbol{w}, \phi)\|_{1,\Omega} + \|\widetilde{\mathbf{S}}_2\big(\mathbf{S}_4(\boldsymbol{w}, \phi), \phi\big)\|_{1,\Omega}, \\ &\le c_{\mathbf{S}} \Big\{ \|\boldsymbol{f}\|_{0,\Omega} + \big(|\Omega|^{1/2} + \gamma\, \|\phi\|_{0,\Omega} \big) \|\boldsymbol{g}\|_{\infty,\Omega} \Big\} + c_{\widetilde{\mathbf{S}}}\, \kappa^{-1}\, U \Big\{ \alpha |\Omega|^{1/2} + \|\phi\|_{0,\Omega} \Big\}. \end{aligned}$$

The results then follows using that $\|\phi\|_{0,\Omega} \le r$ and the assumption on the data (3.41). □

Next, we establish two lemmas that will be useful to derive conditions under which the operator $\mathbf{T}$ is continuous. To this end, in a similar way to [5, Section 3.3] and [12, Section 3.3], we introduce the following regularity hypotheses on the operator $\mathbf{S}$. From now on, we suppose that $\boldsymbol{f} \in \mathbf{H}^\delta(\Omega)$, for some $\delta \in (1/2, 1)$ and that for each $(\boldsymbol{w}, \phi) \in \mathbf{H}$ with $\|\boldsymbol{w}\| \le r$, $r > 0$ given, there holds

$$\mathbf{S}(\boldsymbol{w}, \phi) \in \Big(\big(\mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{H}^\delta(\Omega)\big) \times \big(\mathbb{H}_0(\mathbf{div}; \Omega) \cap \mathbb{H}^\delta(\Omega)\big) \times \big(\mathbb{L}^2_{\mathrm{skew}}(\Omega) \cap \mathbb{H}^\delta(\Omega)\big) \Big) \times \big(\mathbf{H}_0^1 \cap \mathbf{H}^{1+\delta}(\Omega)\big),$$

and

$$\begin{aligned} &\|\mathbf{S}_1(\boldsymbol{w}, \phi)\|_{\delta,\Omega} + \|\mathbf{S}_2(\boldsymbol{w}, \phi)\|_{\delta,\Omega} + \|\mathbf{S}_3(\boldsymbol{w}, \phi)\|_{\delta,\Omega} + \|\mathbf{S}_4(\boldsymbol{w}, \phi)\|_{1+\delta,\Omega} \\ &\qquad \le \widehat{C}_{\mathbf{S}} \Big\{ \|\boldsymbol{f}\|_{\delta,\Omega} + \Big(|\Omega|^{1/2} + \gamma\, \|\phi\|_{0,\Omega} \Big) \|\boldsymbol{g}\|_{\infty,\Omega} \Big\}, \end{aligned} \tag{3.42}$$

where $\widehat{C}_{\mathbf{S}}$ is a constant independent of $(\boldsymbol{w}, \phi)$. The aforementioned range for δ will become clear in the proof of Lemma 3.4 and Lemma 3.6 below, in which we will require to suitably control an expression involving the norm of $\boldsymbol{t} = \mathbf{S}_1(\boldsymbol{w}, \phi)$ in some $\mathbb{L}^{2p}$-space by the respective norm in the $\mathbb{H}^\delta$-space, so that it then can be bounded by data using the a priori estimate (3.42).

Lemma 3.4 *Let $r \in (0, r_0)$, with r_0 given by (3.29). Then, for all $(\boldsymbol{w}, \phi), (\widetilde{\boldsymbol{w}}, \widetilde{\phi}) \in \mathbf{H}$ such that $\|\boldsymbol{w}\|_{1,\Omega}, \|\widetilde{\boldsymbol{w}}\|_{1,\Omega} \le r$, there exists a positive constant $C_{\mathbf{S}}$, depending on the parameters κ_2, κ_3, the* *constant* $c_2(\Omega)$ *(cf.* (3.5)*), the ellipticity constant* $\alpha(\Omega)$ *of the bilinear form* $\mathbf{A}_\phi$ *(cf.* (3.24)*) and* δ *(cf.* (3.42)*), such that*

$$\begin{aligned}\|\mathbf{S}(\boldsymbol{w},\phi)-\mathbf{S}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \;\le\; C_{\mathbf{S}}\Big\{ & L_\mu\,\|\mathbf{S}_1(\boldsymbol{w},\phi)\|_{\delta,\Omega}\|\phi-\widetilde{\phi}\|_{\mathrm{L}^{3/\delta}(\Omega)}\\ & +\|\mathbf{S}_4(\boldsymbol{w},\phi)\|_{1,\Omega}\|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_{1,\Omega}+\gamma\,\|\boldsymbol{g}\|_{\infty,\Omega}\|\phi-\widetilde{\phi}\|_{0,\Omega}\Big\},\end{aligned} \tag{3.43}$$

where L_μ *and* γ *are given by* (2.2) *and* (2.1)*, respectively.*

Proof. Given $(\boldsymbol{w},\phi)$, $(\widetilde{\boldsymbol{w}},\widetilde{\phi})\in\mathbf{H}$ with $\|\boldsymbol{w}\|_{1,\Omega}$, $\|\widetilde{\boldsymbol{w}}\|_{1,\Omega}\le r$, let $(\underline{\boldsymbol{t}},\boldsymbol{u}):=\mathbf{S}(\boldsymbol{w},\phi)$ and $(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}}):=\mathbf{S}(\widetilde{\boldsymbol{w}},\widetilde{\phi})$ be the corresponding solutions to the problem (3.16), respectively. Firstly, from the bilinearity of the forms $\mathbf{A}_\phi$ and $\mathbf{B}_{\boldsymbol{w}}$, it is observed that

$$\begin{aligned}\big(\mathbf{A}_{\widetilde{\phi}}+\mathbf{B}_{\widetilde{\boldsymbol{w}}}\big)((\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}}),(\underline{\boldsymbol{r}},\boldsymbol{v})) \;=\; & -\big(\mathbf{A}_\phi-\mathbf{A}_{\widetilde{\phi}}\big)((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{r}},\boldsymbol{v}))\\ & -\mathbf{B}_{\boldsymbol{w}-\widetilde{\boldsymbol{w}}}((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{r}},\boldsymbol{v}))+\Big(F_\phi-F_{\widetilde{\phi}}\Big)(\underline{\boldsymbol{r}},\boldsymbol{v}),\qquad\forall\,(\underline{\boldsymbol{r}},\boldsymbol{v})\in\mathbb{H}\times\mathbf{H}^1_0(\Omega).\end{aligned} \tag{3.44}$$

where we can notice that

$$\big(\mathbf{A}_\phi-\mathbf{A}_{\widetilde{\phi}}\big)((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{r}},\boldsymbol{v})) \;=\; \int_\Omega\Big\{\mu(\phi+\alpha)-\mu(\widetilde{\phi}+\alpha)\Big\}\boldsymbol{t}:\{\boldsymbol{r}-\kappa_3\boldsymbol{\tau}^{\mathrm{d}}\}, \tag{3.45}$$

and

$$\Big(F_\phi-F_{\widetilde{\phi}}\Big)(\underline{\boldsymbol{r}},\boldsymbol{v}) \;=\; -\int_\Omega\gamma(\phi-\widetilde{\phi})\boldsymbol{g}\cdot\{\boldsymbol{v}-\kappa_2\,\mathbf{div}\,\boldsymbol{\tau}\}. \tag{3.46}$$

Thus, using the ellipticity of $\mathbf{A}_{\widetilde{\phi}}+\mathbf{B}_{\widetilde{\boldsymbol{w}}}$ (cf. (3.28)) and then the identities (3.44), (3.45) and (3.46) with $(\underline{\boldsymbol{r}},\boldsymbol{v})=(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}})$, we find that

$$\begin{aligned}\frac{\alpha(\Omega)}{2}\|(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}})\|^2 &\le \big(\mathbf{A}_{\widetilde{\phi}}+\mathbf{B}_{\widetilde{\boldsymbol{w}}}\big)((\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}}),(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}})),\\ &= -\big(\mathbf{A}_\phi-\mathbf{A}_{\widetilde{\phi}}\big)((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}}))-\mathbf{B}_{\boldsymbol{w}-\widetilde{\boldsymbol{w}}}((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}}))+\big(F_\phi-F_{\widetilde{\phi}}\big)((\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}})),\\ &= -\int_\Omega\Big\{\mu(\phi+\alpha)-\mu(\widetilde{\phi}+\alpha)\Big\}\boldsymbol{t}:\Big\{(\boldsymbol{t}-\widetilde{\boldsymbol{t}})-\kappa_3(\boldsymbol{\sigma}^{\mathrm{d}}-\widetilde{\boldsymbol{\sigma}}^{\mathrm{d}})\Big\}-\mathbf{B}_{\boldsymbol{w}-\widetilde{\boldsymbol{w}}}((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}}))\\ &\quad -\int_\Omega\gamma(\phi-\widetilde{\phi})\boldsymbol{g}\cdot\{(\boldsymbol{u}-\widetilde{\boldsymbol{u}})-\kappa_2\,\mathbf{div}\,(\boldsymbol{\sigma}-\widetilde{\boldsymbol{\sigma}})\}.\end{aligned}$$

Now, applying Cauchy-Schwarz and Hölder inequalities, the Lipschitz continuity of $\mu(\cdot)$ (cf. (2.2)), and the estimate (3.5), we obtain

$$\begin{aligned}\frac{\alpha(\Omega)}{2}\|(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}})\|^2 \le \Big\{ & L_\mu(1+\kappa_3^2)^{1/2}\|\boldsymbol{t}\|_{\mathbb{L}^{2p}(\Omega)}\|\phi-\widetilde{\phi}\|_{\mathrm{L}^{2q}(\Omega)}\\ & +c_2(\Omega)(1+\kappa_3^2)^{1/2}\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_{1,\Omega}+\gamma(1+\kappa_2^2)^{1/2}\|\boldsymbol{g}\|_{\infty,\Omega}\|\phi-\widetilde{\phi}\|_{0,\Omega}\Big\}\,\|(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\widetilde{\boldsymbol{t}}},\widetilde{\boldsymbol{u}})\|,\end{aligned} \tag{3.47}$$

where $p,q\in[1,+\infty)$ are such that $1/p+1/q=1$. Next, according to the additional regularity assumed in (3.42), and recalling that the Sobolev embedding theorem (cf. [1, Theorem 4.12] or [41, Theorem 1.3.4]) establishes the continuous injection $i_\delta:\mathrm{H}^\delta(\Omega)\to\mathrm{L}^{\delta^*}(\Omega)$ with boundedness constant $C_\delta>0$, where

$$\delta^* := \frac{6}{3-2\delta},$$

we then take p such that $2p = \delta^*$ to deduce that, on the one hand, since $\boldsymbol{t} := \mathbf{S}_1(\boldsymbol{w},\phi)$

$$\|\boldsymbol{t}\|_{\mathbb{L}^{2p}(\Omega)} = \|\mathbf{S}_1(\boldsymbol{w},\phi)\|_{\mathbb{L}^{2p}(\Omega)} \le C_\delta \|\mathbf{S}_1(\boldsymbol{w},\phi)\|_{\delta,\Omega}\,, \tag{3.48}$$

and, on the other hand, the respective conjugate index q is given by

$$2q \;=\; \frac{2p}{p-1} \;=\; \frac{3}{\delta}.$$

Finally, inequalities (3.47) and (3.48) together with the previous identity give (3.43) with constant $C_{\mathbf{S}} := \dfrac{2}{\alpha(\Omega)} \max\Big\{ C_\delta (1+\kappa_3^2)^{1/2},\, c_2(\Omega)(1+\kappa_3^2)^{1/2},\, (1+\kappa_2^2)^{1/2} \Big\}$. □

In turn, the following result establishes the Lipschitz-continuity of the operator $\widetilde{\mathbf{S}}$.

Lemma 3.5 *Let $r \in (0,\widetilde{r}_0)$, with $\widetilde{r}_0$ given by (3.37). Then, for all $(\boldsymbol{w},\phi)$, $(\widetilde{\boldsymbol{w}},\widetilde{\phi}) \in \mathbf{H}$ such that $\|\boldsymbol{w}\|_{1,\Omega}$, $\|\widetilde{\boldsymbol{w}}\|_{1,\Omega} \le r$, there exists a positive constant $C_{\widetilde{\mathbf{S}}}$, depending on the parameter κ_5, the ellipticity constant $\widetilde{\alpha}(\Omega)$ of the bilinear form $\widetilde{\mathbf{A}}$ (cf. (3.35)) and the constant $c_1(\Omega)$ (cf. (3.4)), such that*

$$\|\widetilde{\mathbf{S}}(\boldsymbol{w},\phi) - \widetilde{\mathbf{S}}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \;\le\; \kappa^{-1}\, C_{\widetilde{\mathbf{S}}} \left\{ U\|\phi-\widetilde{\phi}\|_{0,\Omega} + \|\widetilde{\mathbf{S}}_2(\boldsymbol{w},\phi)\|_{1,\Omega}\|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_{1,\Omega} \right\}, \tag{3.49}$$

where κ is given in (2.1).

Proof. Given r and $(\boldsymbol{w},\phi)$, $(\widetilde{\boldsymbol{w}},\widetilde{\phi}) \in \mathbf{H}$ as in the hypothesis, let us denote $(\boldsymbol{p},\varphi) := \widetilde{\mathbf{S}}(\boldsymbol{w},\phi)$ and $(\widetilde{\boldsymbol{p}},\widetilde{\varphi}) := \widetilde{\mathbf{S}}(\widetilde{\boldsymbol{w}},\widetilde{\phi})$, that is, the respective solutions to problem (3.18) in $\mathbf{H}_\Gamma(\mathrm{div};\Omega)\times\widetilde{\mathrm{H}}^1(\Omega)$. Thus, from the bilinearity of $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}_{\boldsymbol{w}}$ for any $\boldsymbol{w}$, we have that

$$(\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\widetilde{\boldsymbol{w}}})((\boldsymbol{p},\varphi)-(\widetilde{\boldsymbol{p}},\widetilde{\varphi}),(\boldsymbol{q},\psi)) = -\widetilde{\mathbf{B}}_{\boldsymbol{w}-\widetilde{\boldsymbol{w}}}((\boldsymbol{p},\varphi),(\boldsymbol{q},\psi)) + \big(\widetilde{F}_\phi - \widetilde{F}_{\widetilde{\phi}}\big)(\boldsymbol{q},\psi),$$

for all $(\boldsymbol{q},\psi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega)\times\widetilde{\mathrm{H}}^1(\Omega)$. Hence, using the ellipticity of $\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\widetilde{\boldsymbol{w}}}$ (cf. (3.36)) and the continuity of $\widetilde{\mathbf{B}}_{\boldsymbol{w}}$ (cf. (3.33)) and the definition of $\widetilde{F}_\phi$ (cf. 3.14), we obtain

$$\begin{aligned}\frac{\widetilde{\alpha}(\Omega)}{2}\|(\boldsymbol{p},\varphi)-(\widetilde{\boldsymbol{p}},\widetilde{\varphi})\|^2 &\le -\widetilde{\mathbf{B}}_{\boldsymbol{w}-\widetilde{\boldsymbol{w}}}((\boldsymbol{p},\varphi),(\boldsymbol{p},\varphi)-(\widetilde{\boldsymbol{p}},\widetilde{\varphi})) + \big(\widetilde{F}_\phi-\widetilde{F}_{\widetilde{\phi}}\big)((\boldsymbol{p},\varphi)-(\widetilde{\boldsymbol{p}},\widetilde{\varphi})),\\ &\le \Big\{\kappa^{-1}(1+\kappa_5^2)^{1/2}\Big(U\|\phi-\widetilde{\phi}\|_{0,\Omega} + c_1(\Omega)\|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_{1,\Omega}\|\varphi\|_{1,\Omega}\Big)\Big\}\,\|(\boldsymbol{p},\varphi)-(\widetilde{\boldsymbol{p}},\widetilde{\varphi})\|\,.\end{aligned}$$

The result then follows with $C_{\widetilde{\mathbf{S}}} := \dfrac{2}{\widetilde{\alpha}(\Omega)}(1+\kappa_5^2)^{1/2}\max\{1,c_1(\Omega)\}$ and recalling that $\varphi = \widetilde{\mathbf{S}}_2(\boldsymbol{w},\phi)$.

□

As a consequence of the previous lemmas, we have the following result.

Lemma 3.6 *Given $r \in (0,\min\{r_0,\widetilde{r}_0\})$, with r_0 and $\widetilde{r}_0$ given by (3.29) and (3.37), respectively, we let W_r be the closed ball in $\mathbf{H}$ defined in (3.40) and assume that the data satisfy (3.41). Then, there holds*

$$\begin{aligned}&\|\mathbf{T}(\boldsymbol{w},\phi)-\mathbf{T}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\|\\ &\quad\le (1+\kappa^{-1})(1+L_\mu)\,C_{\mathbf{T}}\Big\{\|\boldsymbol{f}\|_{\delta,\Omega}+\|\boldsymbol{f}\|_{0,\Omega}+\Big(|\Omega|^{1/2}+\gamma\Big)\|\boldsymbol{g}\|_{\infty,\Omega}+U\Big\}\|(\boldsymbol{w},\phi)-(\widetilde{\boldsymbol{w}},\widetilde{\phi})\|\,,\end{aligned} \tag{3.50}$$

for all $(\boldsymbol{w},\phi)$, $(\widetilde{\boldsymbol{w}},\widetilde{\phi}) \in W_r$, where $C_{\mathbf{T}}$ is a positive constant depending on r, the constants $c_{\mathbf{S}}$, $C_{\mathbf{S}}$, $C_{\widetilde{\mathbf{S}}}$ (cf. (3.31), (3.43), (3.49)) and δ (cf. (3.42)).

Proof. Given $r \in (0, \min\{r_0, \widetilde{r}_0\})$, and $(\boldsymbol{w}, \phi)$, $(\widetilde{\boldsymbol{w}}, \widetilde{\phi}) \in W_r$, from the definition of $\mathbf{T}$ (cf. (3.19)), the Lipschitz-continuity of $\widetilde{\mathbf{S}}$ (cf. (3.49)) and the a priori estimate given for $\widetilde{\mathbf{S}}$ (3.32) we note that

$$\|\mathbf{T}(\boldsymbol{w},\phi) - \mathbf{T}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \le \|\mathbf{S}_4(\boldsymbol{w},\phi) - \mathbf{S}_4(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| + \|\widetilde{\mathbf{S}}_2(\mathbf{S}_4(\boldsymbol{w},\phi),\phi) - \widetilde{\mathbf{S}}_2(\mathbf{S}_4(\widetilde{\boldsymbol{w}},\widetilde{\phi}),\widetilde{\phi})\|$$

$$\le \|\mathbf{S}_4(\boldsymbol{w},\phi) - \mathbf{S}_4(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| + \kappa^{-1} C_{\widetilde{\mathbf{S}}} \left\{ U \|\phi - \widetilde{\phi}\|_{0,\Omega} + \|\widetilde{\mathbf{S}}_2(\mathbf{S}_4(\boldsymbol{w},\phi),\phi)\|_{1,\Omega} \|\mathbf{S}_4(\boldsymbol{w},\phi) - \mathbf{S}_4(\widetilde{\boldsymbol{w}},\widetilde{\phi})\|_{1,\Omega} \right\},$$

$$\le \kappa^{-1} C_{\widetilde{\mathbf{S}}}\, U \,\|\phi - \widetilde{\phi}\|_{0,\Omega} + (1 + \kappa^{-1} C_{\widetilde{\mathbf{S}}}\, r) \|\mathbf{S}_4(\boldsymbol{w},\phi) - \mathbf{S}_4(\widetilde{\boldsymbol{w}},\widetilde{\phi})\|_{1,\Omega}\,,$$

where in the last inequality we have used that the data satisfy (3.41) and so $\|\widetilde{\mathbf{S}}_2(\mathbf{S}_4(\boldsymbol{w},\phi),\phi)\|_{1,\Omega} \le r$. Next, using the Lipschitz-continuity of $\mathbf{S}$ (cf. (3.43)) and then applying the estimates (3.31) and (3.42), and the fact that $\|\phi\|_{1,\Omega} \le r$, we get

$$\|\mathbf{T}(\boldsymbol{w},\phi) - \mathbf{T}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \le \kappa^{-1} C_{\widetilde{\mathbf{S}}}\, U \,\|\phi - \widetilde{\phi}\|_{0,\Omega} + (1 + \kappa^{-1} C_{\widetilde{\mathbf{S}}}\, r) C_{\mathbf{S}} \Big\{ L_\mu \,\|\mathbf{S}_1(\boldsymbol{w},\phi)\|_{\delta,\Omega} \|\phi - \widetilde{\phi}\|_{\mathrm{L}^{3/\delta}(\Omega)}$$

$$+ \|\mathbf{S}_4(\boldsymbol{w},\phi)\|_{1,\Omega} \|\boldsymbol{w} - \widetilde{\boldsymbol{w}}\|_{1,\Omega} + \gamma \,\|\boldsymbol{g}\|_{\infty,\Omega} \|\phi - \widetilde{\phi}\|_{0,\Omega} \Big\}$$

$$\le \kappa^{-1} C_{\widetilde{\mathbf{S}}}\, U \,\|\phi - \widetilde{\phi}\|_{0,\Omega} + (1 + \kappa^{-1} C_{\widetilde{\mathbf{S}}} r) C_{\mathbf{S}} \Big\{ L_\mu \widehat{C}_{\mathbf{S}} \widetilde{C}_\delta \Big[\|\boldsymbol{f}\|_{\delta,\Omega} + \Big(|\Omega|^{1/2} + \gamma\, r\Big) \|\boldsymbol{g}\|_{\infty,\Omega} \Big] \|\phi - \widetilde{\phi}\|_{1,\Omega}$$

$$+ c_{\mathbf{S}} \Big[\|\boldsymbol{f}\|_{0,\Omega} + \Big(|\Omega|^{1/2} + \gamma\, r\Big) \|\boldsymbol{g}\|_{\infty,\Omega} \Big] \|\boldsymbol{w} - \widetilde{\boldsymbol{w}}\|_{1,\Omega} + \gamma \|\boldsymbol{g}\|_{\infty,\Omega} \|\phi - \widetilde{\phi}\|_{0,\Omega} \Big\}\,,$$

where the multiplicative constant $\widetilde{C}_\delta$, appearing in the second term of the last inequality, stands for the boundedness constant of the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{3/\delta}(\Omega)$. In this way, with

$$C(r) := \big(1 + r C_{\widetilde{\mathbf{S}}}\big)(1 + r) C_{\mathbf{S}}\,, \qquad C_{\mathbf{T},1} := \max\{\widehat{C}_{\mathbf{S}} \widetilde{C}_\delta, c_{\mathbf{S}}\} \quad \text{and} \quad C_{\mathbf{T},2} := 3 \max\{\widehat{C}_{\mathbf{S}} \widetilde{C}_\delta, c_{\mathbf{S}}, 1\}\,,$$

after performing some algebraic manipulations, we find that

$$\|\mathbf{T}(\boldsymbol{w},\phi) - \mathbf{T}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \le (1 + \kappa^{-1})(1 + L_\mu) \Big\{ C(r) \Big[C_{\mathbf{T},1} \Big(\|\boldsymbol{f}\|_{\delta,\Omega} + \|\boldsymbol{f}\|_{0,\Omega} \Big)$$

$$+ C_{\mathbf{T},2} \Big(|\Omega|^{1/2} + \gamma \Big) \|\boldsymbol{g}\|_{\infty,\Omega} \Big] + C_{\widetilde{\mathbf{S}}} U \Big\} \|(\boldsymbol{w},\phi) - (\widetilde{\boldsymbol{w}},\widetilde{\phi})\|,$$

and so (3.50) follows with $C_{\mathbf{T}} := \max\{C(r)\, C_{\mathbf{T},1},\, C(r)\, C_{\mathbf{T},2},\, C_{\widetilde{\mathbf{S}}}\}$.

□

We are now in a position to establish sufficient conditions for the existence and uniqueness of a fixed-point for our problem (3.20) (equivalently, the well-posedness of the variational problem (3.8)). Indeed, we have from Lemmas 3.1 and 3.2 that $\mathbf{T}$ is well-defined in any ball W_r, with $r \in (0, \min\{r_0, \widetilde{r}_0\})$, and if the data satisfy (3.41) then $\mathbf{T}(W_r) \subseteq W_r$ (cf. Lema 3.3). Furthermore, Lemma 3.6 guarantees that $\mathbf{T}$ is Lipschitz-continuous. So, if the data is small enough so that

$$(1 + \kappa^{-1})(1 + L_\mu)\, C_{\mathbf{T}} \Big\{ \|\boldsymbol{f}\|_{\delta,\Omega} + \|\boldsymbol{f}\|_{0,\Omega} + \Big(|\Omega|^{1/2} + \gamma\Big) \|\boldsymbol{g}\|_{\infty,\Omega} + U \Big\} < 1\,, \tag{3.51}$$

then $\mathbf{T}$ becomes a contraction. Therefore, the Banach fixed-point Theorem provides the existence of a unique fixed-point of $\mathbf{T}$; that is, a unique solution to the problem (3.20), or equivalently, to the variational problem (3.8). We have then shown the main result of this section, and we state it as follows.

Theorem 3.7 *Let W_r be the closed ball in $\mathbf{H} = \mathbf{H}_0^1(\Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$ defined in* (3.40)*. Suppose that the parameters κ_i, $i \in \{1, \ldots, 6\}$, satisfy the conditions required by Lemmas* 3.4 *and* 3.5*, that the estimate* (3.42) *holds and the data satisfy* (3.41) *and* (3.51)*. Then, the augmented fully-mixed problem* (3.8) *has unique solution $(\underline{\boldsymbol{t}}, \boldsymbol{u}, \boldsymbol{p}, \varphi) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_\Gamma(\mathrm{div}; \Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$, with $(\boldsymbol{u}, \varphi) \in W_r$. Moreover, the following a priori estimates hold*

$$\|(\underline{\boldsymbol{t}}, \boldsymbol{u})\| \leq c_{\mathbf{S}} \Big\{ \|\boldsymbol{f}\|_{0,\Omega} + \Big(|\Omega|^{1/2} + \gamma \|\varphi\|_{0,\Omega} \Big) \|\boldsymbol{g}\|_{\infty,\Omega} \Big\},$$

and

$$\|(\boldsymbol{p}, \varphi)\| \leq c_{\widetilde{\mathbf{S}}} \, \kappa^{-1} U \Big\{ \alpha |\Omega|^{1/2} + \|\varphi\|_{0,\Omega} \Big\}.$$

with $c_{\mathbf{S}}$ and $c_{\widetilde{\mathbf{S}}}$ are given as in Lemmas 3.1 *and* 3.2*, respectively.*

We point out here that in practice micro-organisms are slightly denser than water and so the parameter $\gamma = \rho_0/\rho_m - 1$ is small. Then, the data restrictions (3.41) and (3.51) are equivalent to require the diffusion rate κ to be sufficiently large while the average velocity of upward swimming U and the physical domain Ω to be sufficiently small. Hence, Theorem 3.7 essentially states that our augmented fully-mixed formulation provides unique solutions to the Bioconvection problem for suspensions with viscous culture fluid, large diffusion rate, and slowly upswimming micro-organisms in small containers, similarly to the primal method for bioconvection proposed in [10].

4 The Galerkin Scheme

We here present and analyze the Galerkin scheme of the augmented fully-mixed formulation (3.8). In Section 4.1, after introducing the finite element spaces in which the discretization is based, we set the discrete problem and adapt the same strategy from Section 3.2 to equivalently write it as a fixed-point equation. The respective solvability analysis will be then address in Section 4.2 by adapting the results for the continuous case obtained in Sections 3.3 and 3.4.

4.1 The discrete framework

As usual, given a shape–regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ made up of tetrahedra K of diameter h_K, we define the meshsize $h := \max\{h_k : \; K \in \mathcal{T}_h\}$. Furthermore, for any $k \geq 0$ and for each $K \in \mathcal{T}_h$, let $\mathrm{P}_k(K)$ (resp. $\widetilde{\mathrm{P}}_k(K)$) be the space of polynomial functions on K of degree $\leq k$ (resp. $= k$), and with the same notations from Section 1, we define the local Raviart-Thomas space of order k as

$$\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \widetilde{\mathrm{P}}_k(K)\mathbf{x},$$

where $\mathbf{x}$ is a generic vector in R^3. Thus, we introduce the following finite element spaces for approximating $\boldsymbol{t}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$, respectively,

$$\begin{aligned}
\mathbb{H}_h^{\boldsymbol{t}} &:= \Big\{ \boldsymbol{r}_h \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \boldsymbol{r}_h|_K \in \mathbb{P}_k(K), \quad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbb{H}_h^{\boldsymbol{\sigma}} &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}; \Omega) : \quad \mathbf{c}^{\mathrm{t}} \boldsymbol{\tau}_h\big|_K \in \mathbf{RT}_k(K), \quad \forall\, \mathbf{c} \in \mathrm{R}^3, \quad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbb{H}_h^{\boldsymbol{\rho}} &:= \Big\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega) : \quad \boldsymbol{\eta}_h|_K \in \mathbb{P}_k(K), \quad \forall\, K \in \mathcal{T}_h \Big\},
\end{aligned}$$

and for approximating $\boldsymbol{u}$, $\boldsymbol{p}$ and φ, respectively, we define

$$\mathbf{H}_h^{\boldsymbol{u}} := \Big\{ \boldsymbol{v}_h \in \mathbf{C}(\overline{\Omega}) : \quad \boldsymbol{v}_h|_K \in \mathbf{P}_{k+1}(K) , \quad \forall\, K \in \mathcal{T}_h , \quad \boldsymbol{v}_h = 0 \text{ on } \Gamma \Big\},$$

$$\mathbf{H}_h^{\boldsymbol{p}} := \Big\{ \boldsymbol{q}_h \in \mathbf{H}_\Gamma(\mathrm{div};\Omega) : \quad \boldsymbol{q}_h|_K \in \mathbf{RT}_k(K) , \quad \forall\, K \in \mathcal{T}_h \Big\},$$

$$\mathrm{H}_h^{\varphi} := \left\{ \psi_h \in C(\overline{\Omega}) : \quad \psi_h|_K \in P_{k+1}(K) , \quad \forall\, K \in \mathcal{T}_h , \quad \text{and} \quad \int_\Omega \psi_h = 0 \right\}.$$

That is, trace-free and skew-symmetric tensors, with piecewise polynomials components of degree k, are used for approximating the strain tensor $\boldsymbol{t}$ and the vorticity $\boldsymbol{\rho}$, respectively, Raviart-Thomas elements of degree k for approximating the pseudo-stress $\boldsymbol{\sigma}$ and the pseudo-concentration gradient $\boldsymbol{p}$, whereas the components of the velocity $\boldsymbol{u}$ and the concentration φ are approximating by using the Lagrange space of piecewise polynomials of degree $k+1$ (with zero-mean value for φ).

Then, letting $\mathbb{H}_h := \mathbb{H}_h^{\boldsymbol{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbb{H}_h^{\boldsymbol{\rho}}$ and $\underline{\boldsymbol{t}}_h := (\boldsymbol{t}_h, \boldsymbol{\sigma}_h, \boldsymbol{\rho}_h)$, $\underline{\boldsymbol{r}}_h := (\boldsymbol{r}_h, \boldsymbol{\tau}_h, \boldsymbol{\eta}_h) \in \mathbb{H}_h$, the Galerkin scheme of (3.8) reads: Find $(\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h, \boldsymbol{p}_h, \varphi_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}$ such that

$$\begin{aligned}
\mathbf{A}_{\varphi_h}((\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h), (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h)) + \mathbf{B}_{\boldsymbol{u}_h}((\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h), (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h)) &= F_{\varphi_h}(\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h) , \quad \forall\, (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}, \\
\widetilde{\mathbf{A}}((\boldsymbol{p}_h, \varphi_h), (\boldsymbol{q}_h, \psi_h)) + \widetilde{\mathbf{B}}_{\boldsymbol{u}_h}((\boldsymbol{p}_h, \varphi_h), (\boldsymbol{q}_h, \psi_h)) &= \widetilde{F}_{\varphi_h}(\boldsymbol{q}_h, \psi_h) , \quad \forall\, (\boldsymbol{q}_h, \psi_h) \in \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}.
\end{aligned} \tag{4.1}$$

Similarly to the continuous case, we now rewrite (4.1) as a fixed-point problem in terms of operators arising by decoupling the system. Indeed, adapting the approach from Section (3.2), we firstly define $\mathbf{H}_h := \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\varphi}$ and introduce the operator $\mathbf{S}_h : \mathbf{H}_h \longrightarrow \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}$ as

$$\mathbf{S}_h(\boldsymbol{w}_h, \phi_h) := \Big(\big(\mathbf{S}_{1,h}(\boldsymbol{w}_h, \phi_h), \mathbf{S}_{2,h}(\boldsymbol{w}_h, \phi_h), \mathbf{S}_{3,h}(\boldsymbol{w}_h, \phi_h)\big), \mathbf{S}_{4,h}(\boldsymbol{w}_h, \phi) \Big) = (\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h)$$

for all $(\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h$, where, for $(\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h$ given, $(\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h)$ is the unique solution to the discrete version of the problem (3.16), namely: Find $(\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}$ such that

$$\mathbf{A}_{\phi_h}((\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h), (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h)) + \mathbf{B}_{\boldsymbol{w}_h}((\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h), (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h)) = F_{\phi_h}(\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h) , \quad \forall\, (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}, \tag{4.2}$$

where the bilinear forms $\mathbf{A}_{\phi_h}$ (with ϕ_h in place of ϕ) and $\mathbf{B}_{\boldsymbol{w}_h}$ (with $\boldsymbol{w}_h$ in place of $\boldsymbol{w}$), and the functional F_{ϕ_h} (with ϕ_h instead of ϕ) are defined as in (3.9), (3.10) and (3.13), respectively. Secondly, we define the operator $\widetilde{\mathbf{S}}_h : \mathbf{H}_h \longrightarrow \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}$ as

$$\widetilde{\mathbf{S}}_h(\boldsymbol{w}_h, \phi_h) := \big(\widetilde{\mathbf{S}}_{1,h}(\boldsymbol{w}_h, \phi_h), \widetilde{\mathbf{S}}_{2,h}(\boldsymbol{w}_h, \phi_h)\big) \;=\; (\boldsymbol{p}_h, \varphi_h) , \quad \forall\, (\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h,$$

where, for $(\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h$ given, $(\boldsymbol{p}_h, \varphi_h)$ stands for the unique solution to the discrete version of problem (3.18), that is: Find $(\boldsymbol{p}_h, \varphi_h) \in \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}$ such that

$$\widetilde{\mathbf{A}}((\boldsymbol{p}_h, \varphi_h), (\boldsymbol{q}_h, \psi_h)) + \widetilde{\mathbf{B}}_{\boldsymbol{w}_h}((\boldsymbol{p}_h, \varphi_h), (\boldsymbol{q}_h, \psi_h)) = \widetilde{F}_{\phi_h}(\boldsymbol{q}_h, \psi_h) , \quad \forall\, (\boldsymbol{q}_h, \psi_h) \in \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}, \tag{4.3}$$

where the bilinear forms $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}_{\boldsymbol{w}_h}$ (with $\boldsymbol{w}_h$ in place of $\boldsymbol{w}$), and the functional $\widetilde{F}_{\phi_h}$ (with ϕ_h instead of ϕ) are defined as in (3.11), (3.12), and (3.14), respectively. Hence, by introducing the operator $\mathbf{T}_h : \mathbf{H}_h \longrightarrow \mathbf{H}_h$ as

$$\mathbf{T}_h(\boldsymbol{w}_h, \phi_h) := \Big(\mathbf{S}_{4,h}(\boldsymbol{w}_h, \phi_h), \widetilde{\mathbf{S}}_{2,h}\big(\mathbf{S}_{4,h}(\boldsymbol{w}_h, \phi_h), \phi_h\big) \Big) , \quad \forall (\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h ,$$

we realize that solving (4.1) is equivalent to seeking for a fixed-point of the operator $\mathbf{T}_h$, that is: Find $(\boldsymbol{u}_h, \varphi_h) \in \mathbf{H}_h$ such that

$$\mathbf{T}_h(\boldsymbol{u}_h, \varphi_h) = (\boldsymbol{u}_h, \varphi_h). \tag{4.4}$$

4.2 Solvability analysis

Here we study the solvability of the fixed-point equation (4.4) by adapting the analysis from Sections 3.3 and 3.4. We remark in advance that most of the proofs are almost verbatim from the analogues results at continuous level, and hence we omit the details in those cases. To begin with, using the same arguments from Lemmas 3.1 and 3.2, we firstly state conditions under which the discrete problems (4.2) and (4.3) are well-posed, and therefore the operators $\mathbf{S}_h$ and $\widetilde{\mathbf{S}}_h$ are well-defined.

Lemma 4.1 *Suppose that the parameters κ_i, $i \in \{1, \ldots, 4\}$, satisfy the conditions required by Lemma 3.1. Then, for each $r \in (0, r_0)$, with r_0 given by (3.29), and for each $(\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h$ such that $\|\boldsymbol{w}_h\|_{1,\Omega} \leq r$, the problem (4.2) has a unique solution $(\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h) = \mathbf{S}_h(\boldsymbol{w}_h, \phi_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}$. Moreover, with the same constant $c_{\mathbf{S}} > 0$ from (3.31), which is independent of $(\boldsymbol{w}_h, \phi_h)$, there holds*

$$\|\mathbf{S}_h(\boldsymbol{w}_h, \phi_h)\| = \|(\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h)\| \,\leq\, c_{\mathbf{S}} \left\{ \|\boldsymbol{f}\|_{0,\Omega} \,+\, \left(|\Omega|^{1/2} \,+\, \gamma \|\phi_h\|_{0,\Omega} \right) \|\boldsymbol{g}\|_{\infty,\Omega} \right\}.$$

Lemma 4.2 *Suppose that the parameters κ_i, $i \in \{5, 6\}$, satisfy the conditions required by Lemma 3.2. Then, for each $\widetilde{r} \in (0, \widetilde{r}_0)$, $\widetilde{r}_0$ given by (3.37), and for each $(\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h$ such that $\|\boldsymbol{w}_h\|_{1,\Omega} \leq \widetilde{r}$, the problem (4.3) has a unique solution $(\boldsymbol{p}_h, \varphi_h) \in \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}$. Moreover, with the same constant $c_{\widetilde{\mathbf{S}}} > 0$ from (3.32), which is independent of $(\boldsymbol{w}_h, \phi_h)$, there holds*

$$\|\widetilde{\mathbf{S}}_h(\boldsymbol{w}_h, \phi_h)\| \,=\, \|(\boldsymbol{p}_h, \varphi_h)\| \,\leq\, c_{\widetilde{\mathbf{S}}}\, \kappa^{-1}\, U \left\{ \alpha |\Omega|^{1/2} + \|\phi_h\|_{0,\Omega} \right\}.$$

Now we state the solvability of the fixed-point equation (4.4) by verifying the hypotheses of the Brouwer fixed-point Theorem (cf. [13, Theorem 9.9-2]). On the one hand, as a straightforward combination of Lemmas 4.1 and 4.2, we begin by establishing the discrete version of Lemma 3.3.

Lemma 4.3 *Given $r \in (0, \min\{r_0, \widetilde{r}_0\})$, with r_0 and $\widetilde{r}_0$ given by (3.29) and (3.37), respectively, we let $W_{r,h}$ be the closed ball in $\mathbf{H}_h$ defined by*

$$W_{r,h} := \left\{ (\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h : \quad \|(\boldsymbol{w}_h, \phi_h)\| \leq r \right\}, \tag{4.5}$$

and assume that the data satisfy (3.41). Then $\mathbf{T}_h(W_{r,h}) \subseteq W_{r,h}$.

On the other hand, we focus now on the Lipschitz continuity of the operators $\mathbf{S}_h$ and $\widetilde{\mathbf{S}}_h$. Regarding $\mathbf{S}_h$, the discrete version of Lemma 3.4 is provided next. Here, we particularly notice in advance that the additional regularity assumption (3.42) employed there to suitably bound $\boldsymbol{t}$ in the $\mathbb{L}^{2p}-$norm by some $\mathbb{H}^{\delta}-$norm can not be applied at the present discrete context to bound $\boldsymbol{t}_h$. On the contrary, we will utilize a $\mathrm{L}^4 - \mathrm{L}^4 - \mathrm{L}^2$ argument (Hölder inequality) to bound the respective term in which it is involved and then make use of the fact that $\boldsymbol{t}_h \in \mathbb{H}_h^{\boldsymbol{t}}$, and so their components are piecewise polynomials (see at the beginning of Section (4.1)).

Lemma 4.4 *Let $(\boldsymbol{w}_h, \phi_h)$, $(\widetilde{\boldsymbol{w}}_h, \widetilde{\phi}_h) \in \mathbf{H}_h$ such that $\|\boldsymbol{w}_h\|_{1,\Omega}$, $\|\widetilde{\boldsymbol{w}}_h\|_{1,\Omega} \leq r$, for any $r \in (0, r_0)$ with r_0 given by (3.29). Then, there exists a positive constant $C_{\mathbf{S}_h}$, depending on κ_2, κ_3, $c_2(\Omega)$, and $\alpha(\Omega)$, but independent of h, such that*

$$\begin{aligned} \|\mathbf{S}_h(\boldsymbol{w}_h, \phi_h) - \mathbf{S}_h(\widetilde{\boldsymbol{w}}_h, \widetilde{\phi}_h)\| \,\leq\, C_{\mathbf{S}_h} \Big\{ & L_\mu \,\|\mathbf{S}_{1,h}(\boldsymbol{w}_h, \phi_h)\|_{\mathbb{L}^4(\Omega)} \|\phi_h - \widetilde{\phi}_h\|_{\mathrm{L}^4(\Omega)} \\ & + \|\mathbf{S}_{4,h}(\boldsymbol{w}_h, \phi_h)\|_{1,\Omega} \|\boldsymbol{w}_h - \widetilde{\boldsymbol{w}}_h\|_{1,\Omega} + \gamma\, \|\boldsymbol{g}\|_{\infty,\Omega} \|\phi_h - \widetilde{\phi}_h\|_{0,\Omega} \Big\}. \end{aligned} \tag{4.6}$$

Proof. The proof is almost verbatim to that one of Lemma 3.4. Indeed, it suffices to see that when applying the Hölder inequality with $p = q = 2$, the estimate (3.47) becomes

$$\begin{aligned}\frac{\alpha(\Omega)}{2}\|(\underline{\boldsymbol{t}}_h,\boldsymbol{u}_h)-(\widetilde{\underline{\boldsymbol{t}}}_h,\widetilde{\boldsymbol{u}}_h)\| \le \Big\{ & L_\mu(1+\kappa_3^2)^{1/2}\|\boldsymbol{t}_h\|_{\mathbb{L}^4(\Omega)}\|\phi_h-\widetilde{\phi}_h\|_{\mathrm{L}^4(\Omega)} \\ & + c_2(\Omega)(1+\kappa_3^2)^{1/2}\|\boldsymbol{u}_h\|_{1,\Omega}\|\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h\|_{1,\Omega}+\gamma(1+\kappa_2^2)^{1/2}\|\boldsymbol{g}\|_{\infty,\Omega}\|\phi_h-\widetilde{\phi}_h\|_{0,\Omega}\Big\}\,.\end{aligned} \tag{4.7}$$

Since elements of $\mathbb{H}_h^{\boldsymbol{t}}$ are piecewise polynomials by components we have that $\|\boldsymbol{t}_h\|_{\mathbb{L}^4(\Omega)} < +\infty$, and using the fact that $\mathbf{S}_{1,h}(\boldsymbol{w}_h,\phi_h) = \boldsymbol{t}_h$, the inequality (4.7) immediately yields the estimate (4.6) with $C_{\mathbf{S}_h} := \dfrac{2}{\alpha(\Omega)}\max\Big\{(1+\kappa_3^2)^{1/2},\, c_2(\Omega)(1+\kappa_3^2)^{1/2},\, (1+\kappa_2^2)^{1/2}\Big\}$, which is clearly independent of h. □

Following the same arguments used in the proof Lemma 3.5, we directly have the following result regarding the operator $\widetilde{\mathbf{S}}_h$.

Lemma 4.5 *Let $(\boldsymbol{w}_h,\phi_h)$, $(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h) \in \mathbf{H}_h$ such that $\|\boldsymbol{w}_h\|_{1,\Omega}$, $\|\widetilde{\boldsymbol{w}}_h\|_{1,\Omega} \le \widetilde{r}$, for any $r \in (0,\widetilde{r}_0)$, with r_0 given by (3.37). Then, with the same constant $C_{\widetilde{\mathbf{S}}}$ provided by Lemma 3.5, there holds*

$$\|\widetilde{\mathbf{S}}_h(\boldsymbol{w}_h,\phi_h)-\widetilde{\mathbf{S}}_h(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\| \;\le\; \kappa^{-1}\,C_{\widetilde{\mathbf{S}}}\,\Big\{U\,\|\phi_h-\widetilde{\phi}_h\|_{0,\Omega} \;+\; \|\widetilde{\mathbf{S}}_{2,h}(\boldsymbol{w}_h,\phi_h)\|_{1,\Omega}\|\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h\|_{1,\Omega}\Big\}\,. \tag{4.8}$$

As a result of the previous two lemmas, we can state the Lipschitz-continuity of the operator $\mathbf{T}_h$, which constitutes the discrete version of Lemma 3.6.

Lemma 4.6 *Let $r \in (0,\min\{r_0,\widetilde{r}_0\})$, with r_0 and $\widetilde{r}_0$ given by (3.29) and (3.37), respectively, let $W_{r,h}$ be the closed ball in $\mathbf{H}_h$ defined in (4.5) and assume that the data satisfy (3.41). Then, there exists a constant $C_{\mathbf{T}_h} > 0$, that depends on r and other constants but is independent of h, such that*

$$\begin{aligned}\|\mathbf{T}_h(\boldsymbol{w}_h,\phi_h)-\mathbf{T}_h(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\| \le (1+\kappa^{-1})(1+L_\mu)\,C_{\mathbf{T}_h}\Big\{&\|\mathbf{S}_{1,h}(\boldsymbol{w}_h,\phi_h)\|_{\mathbb{L}^4(\Omega)} \\ &+\|\boldsymbol{f}\|_{0,\Omega} \,+\, \Big(|\Omega|^{1/2}+\gamma\Big)\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, U\Big\}\|(\boldsymbol{w},\phi)-(\widetilde{\boldsymbol{w}},\widetilde{\phi})\|\,,\end{aligned} \tag{4.9}$$

for all $(\boldsymbol{w}_h,\phi_h)$, $(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h) \in W_{r,h}$.

Proof. Proceeding as in the proof of Lemma 3.6, but using (4.6) and (4.8) instead (3.43) and (3.49), respectively, the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^4(\Omega)$ with constant $\widetilde{C}$, and then the a priori estimate provided by Lemma 4.1, we find that

$$\begin{aligned}&\|\mathbf{T}_h(\boldsymbol{w}_h,\phi_h)-\mathbf{T}_h(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\| \\ &\quad\le\; \kappa^{-1}C_{\widetilde{\mathbf{S}}}\,U\,\|\phi_h-\widetilde{\phi}_h\|_{0,\Omega} \;+\; (1+\kappa^{-1}C_{\widetilde{\mathbf{S}}}\,r)C_{\mathbf{S}_h}\,\Big\{L_\mu\widetilde{C}\,\|\mathbf{S}_1(\boldsymbol{w}_h,\phi_h)\|_{\mathbb{L}^4(\Omega)}\|\phi_h-\widetilde{\phi}_h\|_{1,\Omega} \\ &\qquad + c_{\mathbf{S}}\Big[\|\boldsymbol{f}\|_{0,\Omega}+\Big(|\Omega|^{1/2}+\gamma\,r\Big)\|\boldsymbol{g}\|_{\infty,\Omega}\Big]\|\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h\|_{1,\Omega}+\,\gamma\,\|\boldsymbol{g}\|_{\infty,\Omega}\|\phi_h-\widetilde{\phi}_h\|_{0,\Omega}\Big\}\,.\end{aligned}$$

Then, after performing algebraic manipulations and defining

$$\widetilde{C}(r) := \big(1+rC_{\widetilde{\mathbf{S}}}\big)(1+r)C_{\mathbf{S}_h}\,,\qquad \widetilde{C}_{\mathbf{T},1} := \max\{\widetilde{C},c_{\mathbf{S}}\}\quad\text{and}\quad \widetilde{C}_{\mathbf{T},2} := 2\max\{c_{\mathbf{S}},1\}\,,$$

the results follows with $C_{\mathbf{T}} := \max\{\widetilde{C}(r)\,C_{\mathbf{T},1},\, \widetilde{C}(r)\,C_{\mathbf{T},2},\, C_{\widetilde{\mathbf{S}}}\}$, which is independent of h because so the constants $\widetilde{C}(r)$, $C_{\mathbf{T},1}$, $C_{\mathbf{T},2}$ and ,$C_{\widetilde{\mathbf{S}}}$ are. □

The previous lemma provides the continuity required by the Brouwer fixed-point theorem, in the convex and compact set $W_{r,h} \subset \mathbf{H}_h$. Therefore, we have essentially proved the following result.

Theorem 4.7 *Suppose that the parameters* κ_i*,* $i \in \{1, \cdots, 6\}$*, satisfy the conditions required by Lemmas* 3.4 *and* 3.5*. Let* $W_{r,h}$ *be the closed ball in* $\mathbf{H}_h$ *defined in* (4.5) *and assume that the data satisfy* (3.41)*. Then, the Galerkin scheme* (4.1) *has at least one solution* $(\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h, \boldsymbol{p}_h, \varphi_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}$*, with* $(\boldsymbol{u}_h, \varphi_h) \in W_{r,h}$*, and the following a priori estimates hold*

$$\|(\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h)\| \leq c_{\mathbf{S}} \left\{ \|\boldsymbol{f}\|_{0,\Omega} + \left(|\Omega|^{1/2} + \gamma \|\varphi_h\|_{0,\Omega} \right) \|\boldsymbol{g}\|_{\infty,\Omega} \right\},$$

and

$$\|(\boldsymbol{p}_h, \varphi_h)\| \leq c_{\widetilde{\mathbf{S}}}\, \kappa^{-1}\, U \left\{ \alpha |\Omega|^{1/2} + \|\varphi_h\|_{0,\Omega} \right\},$$

with $c_{\mathbf{S}}$ *and* $c_{\widetilde{\mathbf{S}}}$ *as in Lemmas* 3.1 *and* 3.2*, respectively.*

We end this section by remarking that the lack of suitable estimates for $\|\mathbf{S}_{1,h}(\boldsymbol{w}_h, \phi_h)\|_{\mathbb{L}^4(\Omega)}$ (similar to [12, Section 4.2]) stops us of trying to use (4.9) to derive a condition on data so that $\mathbf{T}_h$ becomes a contraction. This is the reason why in the previous theorem we can only guarantee the existence of a discrete solution. In turn, as we commented after Theorem (3.7) for the continuous case, the previous result states that our augmented fully-mixed scheme provides existence of discrete solutions to the bioconvection problem whenever the data satisfy the condition 3.41, that is, for suspensions with viscous culture fluid, large diffusion rate, and slowly upswimming micro-organisms in small containers, similarly to the classical finite element method for bioconvection that was constructed in [10].

5 A priori error analysis

In this section, we undertake the error analysis for the Galerkin scheme (4.1) associated to the problem (3.8). To that end, we will deduce the corresponding Céa estimate as well as the respective theoretical convergence rates according to the approximation properties of the discrete spaces introduced in Section 4.1. To begin with, we let

$$(\underline{\boldsymbol{t}}, \boldsymbol{u}, \boldsymbol{p}, \varphi) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_\Gamma(\mathrm{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega) \quad \text{with} \quad (\boldsymbol{u}, \varphi) \in W_r\,,$$

and

$$(\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h, \boldsymbol{p}_h, \varphi_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi} \quad \text{with} \quad (\boldsymbol{u}_h, \varphi_h) \in W_{r,h}\,,$$

be solutions to the problems (3.8) and (4.1), respectively. Therefore, we have that

$$\begin{aligned} \mathbf{A}_\varphi((\underline{\boldsymbol{t}}, \boldsymbol{u}), (\underline{\boldsymbol{r}}, \boldsymbol{v})) + \mathbf{B}_{\boldsymbol{u}}((\underline{\boldsymbol{t}}, \boldsymbol{u}), (\underline{\boldsymbol{r}}, \boldsymbol{v})) &= F_\varphi(\underline{\boldsymbol{r}}, \boldsymbol{v})\,, \quad \forall\, (\underline{\boldsymbol{r}}, \boldsymbol{v}) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega), \\ \mathbf{A}_{\varphi_h}((\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h), (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h)) + \mathbf{B}_{\boldsymbol{u}_h}((\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h), (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h)) &= F_{\varphi_h}(\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h)\,, \quad \forall\, (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \widetilde{\mathbf{A}}((\boldsymbol{p}, \varphi), (\boldsymbol{q}, \psi)) + \widetilde{\mathbf{B}}_{\boldsymbol{u}}((\boldsymbol{p}, \varphi), (\boldsymbol{q}, \psi)) &= \widetilde{F}_\varphi(\boldsymbol{q}, \psi)\,, \quad \forall\, (\boldsymbol{q}, \psi) \in \mathbf{H}_\Gamma(\mathrm{div};\Omega) \times \widetilde{\mathrm{H}}^1(\Omega), \\ \widetilde{\mathbf{A}}((\boldsymbol{p}_h, \varphi_h), (\boldsymbol{q}_h, \psi_h)) + \widetilde{\mathbf{B}}_{\boldsymbol{u}_h}((\boldsymbol{p}_h, \varphi_h), (\boldsymbol{q}_h, \psi_h)) &= \widetilde{F}_{\varphi_h}(\boldsymbol{q}_h, \psi_h)\,, \quad \forall\, (\boldsymbol{q}_h, \psi_h) \in \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}. \end{aligned} \tag{5.2}$$

Because of the structure of the systems (5.1) and (5.2), in what follows we apply the well-known Strang lemma for elliptic variational problems (see [43, Theorem 11.1]) in order to derive an upper bound for the total error $\|(\underline{\boldsymbol{t}}, \boldsymbol{u}, \boldsymbol{p}, \varphi) - (\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h, \boldsymbol{p}_h, \varphi_h)\|$. We recall this auxiliary result as follows.

Lemma 5.1 *Let V be a Hilbert space, $F \in V'$, and $A : V \times V \to \mathbb{R}$ be a bounded and V-elliptic bilinear form. In addition, let $\{V_h\}_{h>0}$ be a sequence of finite dimensional subspaces of V, and for each $h > 0$ consider a bounded bilinear form $A : V_h \times V_h \to \mathbb{R}$ and a functional $F_h \in V_h'$. Assume that the family $\{A_h\}_{h>0}$ is uniformly elliptic, that is, there exists a constant $\widetilde{\alpha} > 0$, independent of h, such that*

$$A_h(v_h, v_h) \,\geq\, \widetilde{\alpha}\, \|v_h\|_V^2\,, \quad \forall\, v_h \in V_h\,, \quad \forall\, h > 0.$$

In turn, let $u \in V$ and $u_h \in V_h$ such that

$$A(u, v) = F(v)\,, \quad \forall\, v \in V\,, \quad \text{and} \quad A_h(u_h, v_h) = F_h(v_h)\,, \quad \forall\, v_h \in V_h.$$

Then, for each $h > 0$ there holds

$$\begin{aligned} \|u - u_h\|_V \;\leq C_{ST} \Bigg\{ & \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|F(w_h) - F_h(w_h)|}{\|w_h\|_V} \\ & + \inf_{\substack{v_h \in V_h \\ v_h \neq 0}} \left(\|u - v_h\|_V + \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|A(v_h, w_h) - A_h(v_h, w_h)|}{\|w_h\|_V} \right) \Bigg\}, \end{aligned}$$

where $C_{ST} := \widetilde{\alpha}^{-1} \max\{1, \|A\|\}$.

In that follows, we denote as usual

$$\operatorname{dist}\big((\underline{\boldsymbol{t}}, \boldsymbol{u}), \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}\big) := \inf_{(\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}} \|(\underline{\boldsymbol{t}}, \boldsymbol{u}) - (\underline{\boldsymbol{r}}_h, \boldsymbol{v}_h)\|\,,$$

and

$$\operatorname{dist}\big((\boldsymbol{p}, \varphi), \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}\big) := \inf_{(\boldsymbol{q}_h, \psi_h) \in \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}} \|(\boldsymbol{p}, \varphi) - (\boldsymbol{q}_h, \psi_h)\|\,.$$

The following lemma provides a preliminary estimate for the error $\|(\underline{\boldsymbol{t}}, \boldsymbol{u}) - (\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h)\|$ associated to the system (5.1).

Lemma 5.2 *Let $C_{ST} := \dfrac{2}{\alpha(\Omega)} \max\{1, \|\mathbf{A}_\varphi + \mathbf{B}_{\boldsymbol{u}}\|\}$, where $\alpha(\Omega)$ is the constant yielding the ellipticity of $\mathbf{A}_\varphi + \mathbf{B}_{\boldsymbol{u}}$ (cf. (3.28) and Lemma 3.1). Then, there holds*

$$\begin{aligned} \|(\underline{\boldsymbol{t}}, \boldsymbol{u}) - (\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h)\| \;\leq\; C_{ST} \Big\{ & \Big(1 + 2\|\mathbf{A}_\varphi\| + c_2(\Omega)(1 + \kappa_3^2)^{1/2} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}\Big) \operatorname{dist}\big((\underline{\boldsymbol{t}}, \boldsymbol{u}), \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}\big) \\ & + \gamma(1 + \kappa_2^2)^{1/2} \|\boldsymbol{g}\|_{\infty,\Omega} \|\varphi - \varphi_h\|_{0,\Omega} \,+\, L_\mu C_\delta (1 + \kappa_3^2)^{1/2} \|\boldsymbol{t}\|_{\delta,\Omega} \|\varphi - \varphi_h\|_{\mathrm{L}^{3/\delta}(\Omega)} \\ & + c_2(\Omega)(1 + \kappa_3^2)^{1/2} \|\boldsymbol{u}\|_{1,\Omega} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \Big\}\,. \end{aligned} \tag{5.3}$$

Proof. Since $(\boldsymbol{u}, \varphi) \in W_r$ and $(\boldsymbol{u}_h, \varphi_h) \in W_{r,h}$, Lemma 3.1 and Lemma 4.1 guarantee that the bilinear forms $\mathbf{A}_\varphi + \mathbf{B}_{\boldsymbol{u}}$ and $\mathbf{A}_{\varphi_h} + \mathbf{B}_{\boldsymbol{u}_h}$ are $\mathbb{H} \times \mathbf{H}_0^1(\Omega)-$elliptic and $\mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}}-$elliptic ($\forall\, h > 0$), respectively, with the same constant $\frac{\alpha(\Omega)}{2}$ (see (3.28)). Also, F_φ and F_{φ_h} are clearly both linear and bounded functionals. Therefore the system (5.1) satisfies the hypotheses of Strang's lemma and thus, a direct application of the Lemma 5.1 to the specific context (5.1) with

$$A \,:=\, \mathbf{A}_\varphi + \mathbf{B}_{\boldsymbol{u}}\,, \quad \{A_h\}_{h>0} \,:=\, \{\mathbf{A}_{\varphi_h} + \mathbf{B}_{\boldsymbol{u}_h}\}_{h>0}\,, \quad F := F_\varphi\,, \quad \text{and} \quad F_h := F_{\varphi_h}\,,$$

yields

$$\|(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\boldsymbol{t}}_h,\boldsymbol{u}_h)\| \,\le\, C_{ST}\left\{ \Big\|\big(F_\varphi-F_{\varphi_h}\big)|_{\mathbb{H}_h\times\mathbf{H}_h^{\boldsymbol{u}}}\Big\| + \inf_{\substack{(\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h)\,\in\,\mathbb{H}_h\times\mathbf{H}_h^{\boldsymbol{u}}\\ (\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h)\neq\mathbf{0}}} \Bigg(\|(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h)\| \right.$$

$$\left. + \sup_{\substack{(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h)\,\in\,\mathbb{H}_h\times\mathbf{H}_h^{\boldsymbol{u}}\\ (\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h)\neq\mathbf{0}}} \frac{\big|(\mathbf{A}_\varphi-\mathbf{A}_{\varphi_h})((\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h),(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h))+\mathbf{B}_{\boldsymbol{u}-\boldsymbol{u}_h}((\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h),(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h))\big|}{\|(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h)\|}\Bigg)\right\}, \tag{5.4}$$

where $C_{ST}:=\dfrac{2}{\alpha(\Omega)}\max\{1,\|\mathbf{A}_\varphi+\mathbf{B}_{\boldsymbol{u}}\|\}$. Now, from the estimate (3.46), observe that the first term at the right-hand side of (5.4) can be bounded as

$$\Big\|\big(F_\varphi-F_{\varphi_h}\big)|_{\mathbb{H}_h\times\mathbf{H}_h^{\boldsymbol{u}}}\Big\| \,\le\, \gamma\,(1+\kappa_2^2)^{1/2}\,\|\boldsymbol{g}\|_{\infty,\Omega}\|\varphi-\varphi_h\|_{0,\Omega}\,. \tag{5.5}$$

To estimate the supremum in (5.4), on the one hand, we first conveniently add and subtract $(\underline{\boldsymbol{t}},\boldsymbol{u})$ in the first component of the bilinear form $\mathbf{A}_\varphi-\mathbf{A}_{\varphi_h}$ to find

$$\begin{aligned}(\mathbf{A}_\varphi-\mathbf{A}_{\varphi_h})((\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h),(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h)) \;&=\; \mathbf{A}_\varphi((\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h)-(\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h))\\ &+(\mathbf{A}_\varphi-\mathbf{A}_{\varphi_h})((\underline{\boldsymbol{t}},\boldsymbol{u}),(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h))+\mathbf{A}_{\varphi_h}((\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h),(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h))\,.\end{aligned} \tag{5.6}$$

Now, we apply (3.21) to estimate the first and third terms at the right-hand side of (5.6), whereas the the second term is estimated by proceeding similarly to the derivation of (3.47) combined with (3.48), which gives

$$\begin{aligned}\big|(\mathbf{A}_\varphi-\mathbf{A}_{\varphi_h})((\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h),(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h))\big| \;&\le\; 2\|\mathbf{A}_\varphi\|\,\|(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h)\|\,\|(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h)\|\\ &+L_\mu C_\delta(1+\kappa_3^2)^{1/2}\|\underline{\boldsymbol{t}}\|_{\delta,\Omega}\|\varphi-\varphi_h\|_{\mathrm{L}^{3/\delta}(\Omega)}\|(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h)\|.\end{aligned}$$

On the other hand, for estimating the term that involves $\mathbf{B}_{\boldsymbol{u}-\boldsymbol{u}_h}$, we apply (3.22) with $\boldsymbol{w}=\boldsymbol{u}-\boldsymbol{u}_h$,

$$\begin{aligned}\big|\mathbf{B}_{\boldsymbol{u}-\boldsymbol{u}_h}((\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h),(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h))\big| \;&\le\; c_2(\Omega)(1+\kappa_3^2)^{1/2}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\|\boldsymbol{v}_h\|_{1,\Omega}\|(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h)\|\\ &\le\; c_2(\Omega)(1+\kappa_3^2)^{1/2}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\|(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h)\|\,\|(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h)\|\\ &+\; c_2(\Omega)(1+\kappa_3^2)^{1/2}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\|\boldsymbol{u}\|_{1,\Omega}\|(\underline{\boldsymbol{s}}_h,\boldsymbol{z}_h)\|.\end{aligned} \tag{5.7}$$

where the last inequality arises after adding and subtracting $\boldsymbol{u}$ in the term $\|\boldsymbol{v}_h\|_{1,\Omega}$, using triangle inequality and then bounding $\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,\Omega}$ by $\|(\underline{\boldsymbol{t}},\boldsymbol{u})-(\underline{\boldsymbol{r}}_h,\boldsymbol{v}_h)\|$. Finally, by replacing (5.5), (5.6) and (5.7) back into (5.4), we get (5.3). □

Concerning the error $\|(\boldsymbol{p},\varphi)-(\boldsymbol{p}_h,\varphi_h)\|$ associated to the concentration equations (5.2), we have the following result.

Lemma 5.3 *Let* $\widetilde{C}_{ST}:=\dfrac{2}{\widetilde{\alpha}(\Omega)}\max\{1,\|\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\boldsymbol{u}}\|\}$*, where* $\widetilde{\alpha}(\Omega)$ *is the constant yielding the ellipticity of* $\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}_{\boldsymbol{u}}$ *(cf.* (3.36) *and Lemma* 3.2*). Then, there holds*

$$\begin{aligned}\|(\boldsymbol{p},\varphi)-(\boldsymbol{p}_h,\varphi_h)\| \,\le\, \widetilde{C}_{ST}\Big\{&\Big(1+\kappa^{-1}c_1(\Omega)(1+\kappa_5^2)^{1/2}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\Big)\mathrm{dist}\big((\boldsymbol{p},\varphi),\,\mathbf{H}_h^{\boldsymbol{p}}\times\mathrm{H}_h^{\varphi}\big)\\ &+\kappa^{-1}c_1(\Omega)(1+\kappa_5^2)^{1/2}\|\varphi\|_{1,\Omega}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\kappa^{-1}U(1+\kappa_5^2)^{1/2}\|\varphi-\varphi_h\|_{0,\Omega}\Big\}.\end{aligned} \tag{5.8}$$

Proof. It follows from a slight modification of the proof of [16, Lemma 5.3] which makes use of Lemma 5.1. There, the consistency error associated to the functional in the Strang estimate vanishes, but this does not happen in the present case with $\widetilde{F}_\varphi - \widetilde{F}_{\varphi_h}$. We simply bound this term similarly as in the proof of Lemma 3.5. We omit further details. □

We now combine the two previous lemmas to derive an a priori estimate for the total error $\|(\underline{\boldsymbol{t}}, \boldsymbol{u}, \boldsymbol{p}, \varphi) - (\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h, \boldsymbol{p}_h, \varphi_h)\|$. Indeed, by gathering together the estimates (5.3) and (5.8), we get

$$\begin{aligned}
\|(\underline{\boldsymbol{t}}, \boldsymbol{u}, \boldsymbol{p}, \varphi) - (\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h, \boldsymbol{p}_h, \varphi_h)\| \,&\le\, C_{ST} L_\mu C_\delta (1+\kappa_3^2)^{1/2} \|\boldsymbol{t}\|_{\delta,\Omega} \|\varphi - \varphi_h\|_{\mathrm{L}^{3/\delta}(\Omega)} \\
&+ \Big(C_{ST} c_2(\Omega)(1+\kappa_3^2)^{1/2} \|\boldsymbol{u}\|_{1,\Omega} \,+\, \widetilde{C}_{ST} \kappa^{-1} c_1(\Omega)(1+\kappa_5^2)^{1/2} \|\varphi\|_{1,\Omega} \Big) \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \\
&+ \Big(C_{ST} \gamma (1+\kappa_2^2)^{1/2} \|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \widetilde{C}_{ST} \kappa^{-1} U (1+\kappa_5^2)^{1/2} \Big) \|\varphi - \varphi_h\|_{0,\Omega} \\
&+ C_{ST} \Big(1 + 2\|\mathbf{A}_\varphi\| + c_2(\Omega)(1+\kappa_3^2)^{1/2} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \Big) \mathrm{dist}\big((\underline{\boldsymbol{t}}, \boldsymbol{u}),\, \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}} \big) \\
&+ \widetilde{C}_{ST} \Big(1 + \kappa^{-1} c_1(\Omega)(1+\kappa_5^2)^{1/2} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \Big) \mathrm{dist}\big((\boldsymbol{p}, \varphi),\, \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi} \big)\,,
\end{aligned}$$

The first term of the right-hand side of the foregoing inequality is estimated by using (3.42) to bound $\|\boldsymbol{t}\|_{\delta,\Omega}$, and the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{3/\delta}(\Omega)$ to get $\|\varphi - \varphi_h\|_{\mathrm{L}^{3/\delta}(\Omega)} \,\le\, \widetilde{C}_\delta \|\varphi - \varphi_h\|_{1,\Omega}$. In turn, in the second term, we use that $(\boldsymbol{u}, \varphi) \in W_r$ to bound $\|\boldsymbol{u}\|_{1,\Omega}$ and $\|\varphi\|_{1,\Omega}$ by r. In this way, after performing some algebraic manipulations, we can assert that

$$\begin{aligned}
\|(\underline{\boldsymbol{t}}, \boldsymbol{u}, \boldsymbol{p}, \varphi) - (\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h, \boldsymbol{p}_h, \varphi_h)\| \,&\le\, \mathbf{C}(\boldsymbol{f}, \boldsymbol{g}, \kappa, \mu, \gamma, U, r, |\Omega|)\, \|(\underline{\boldsymbol{t}}, \boldsymbol{u}, \boldsymbol{p}, \varphi) - (\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h, \boldsymbol{p}_h, \varphi_h)\| \\
&+ C_{ST} \Big(1 + 2\|\mathbf{A}_\varphi\| + c_2(\Omega)(1+\kappa_3^2)^{1/2} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \Big) \mathrm{dist}\big((\underline{\boldsymbol{t}}, \boldsymbol{u}),\, \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}} \big) \\
&+ \widetilde{C}_{ST} \Big(1 + \kappa^{-1} c_1(\Omega)(1+\kappa_5^2)^{1/2} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \Big) \mathrm{dist}\big((\boldsymbol{p}, \varphi),\, \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi} \big)\,,
\end{aligned} \tag{5.9}$$

where $\mathbf{C}(\boldsymbol{f}, \boldsymbol{g}, \kappa, \mu, \gamma, U, |\Omega|)$ is a constant, depending only on data, r and $|\Omega|$, but is independent of h, defined by

$$\mathbf{C}(\boldsymbol{f}, \boldsymbol{g}, \kappa, \mu, \gamma, U, r, |\Omega|) \,:=\, \max\big\{ \mathbf{C}_1(\boldsymbol{f}, \boldsymbol{g}, \mu, \gamma, r, |\Omega|)\,, \mathbf{C}_2(\kappa, r)\,, \mathbf{C}_3(\boldsymbol{g}, \kappa, \gamma, U) \big\}\,, \tag{5.10}$$

with

$$\mathbf{C}_1(\boldsymbol{f}, \boldsymbol{g}, \mu, \gamma, r, |\Omega|) \,:=\, L_\mu\, C_1 \left\{ \|\boldsymbol{f}\|_{\delta,\Omega} \,+\, \big(|\Omega|^2 + \gamma\, r\big) \|\boldsymbol{g}\|_{\infty,\Omega} \right\}, \quad \mathbf{C}_2(\kappa, r) \,:=\, r\, C_2\, (\kappa^{-1} + 1)\,,$$
$$\text{and} \quad \mathbf{C}_3(\boldsymbol{g}, \kappa, \gamma, U) \,:=\, C_3 \left(\gamma \|\boldsymbol{g}\|_{\infty,\Omega} \,+ \kappa^{-1} U \right),$$

where

$$C_1 := C_{ST}\, C_\delta\, \widetilde{C}_\delta\, \widehat{C}_{\mathbf{S}}\, (1+\kappa_3^2)^{1/2}\,, \quad C_2 := C_{ST}\, c_2(\Omega)\, (1+\kappa_3^2)^{1/2} \,+\, \widetilde{C}_{ST}\, c_1(\Omega)\, (1+\kappa_5^2)^{1/2}\,,$$
$$\text{and} \quad C_3 := C_{ST}\, (1+\kappa_2^2)^{1/2} \,+\, \widetilde{C}_{ST}\, (1+\kappa_5^2)^{1/2}\,.$$

Note that the constants multiplying the distances $\mathrm{dist}\big((\underline{\boldsymbol{t}}, \boldsymbol{u}),\, \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}} \big)$ and $\mathrm{dist}\big((\boldsymbol{p}, \varphi),\, \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi} \big)$ are both controlled by other constants, parameters, and data only because so $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}$ does, according to Theorem 3.7. Consequently, we are in position to establish the following result providing the complete Céa estimate.

Theorem 5.4 *Let* $r \in (0, \min\{r_0, \widetilde{r}_0\})$, *with* r_0 *and* $\widetilde{r}_0$ *given by* (3.29) *and* (3.37), *respectively, and* $(\underline{\boldsymbol{t}}, \boldsymbol{u}, \boldsymbol{p}, \varphi) \in \mathbb{H} \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_\Gamma(\mathrm{div}; \Omega) \times \widetilde{\mathrm{H}}^1(\Omega)$ *and* $(\underline{\boldsymbol{t}}_h, \boldsymbol{u}_h, \boldsymbol{p}_h, \varphi_h) \in \mathbb{H}_h \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbf{H}_h^{\boldsymbol{p}} \times \mathrm{H}_h^{\varphi}$, *with*

$(\boldsymbol{u},\varphi)\in W_r$ *and* $(\boldsymbol{u}_h,\varphi_h)\in W_{r,h}$, *be solutions to the problems* (3.8) *and* (4.1), *respectively. Assume that the data,* r *and* Ω *are such that the constant defined by* (5.10) *satisfies*

$$\mathbf{C}(\boldsymbol{f},\boldsymbol{g},\kappa,\mu,\gamma,U,r,|\Omega|)\,\leq\,\frac{1}{2}\,. \tag{5.11}$$

Then, there exists a positive constant C, *depending only on parameters, data and other constants, all of them independent of* h, *such that*

$$\|(\underline{\boldsymbol{t}},\boldsymbol{u},\boldsymbol{p},\varphi)-(\underline{\boldsymbol{t}}_h,\boldsymbol{u}_h,\boldsymbol{p}_h,\varphi_h)\|\,\leq C\Big\{\mathrm{dist}\big((\underline{\boldsymbol{t}},\boldsymbol{u}),\,\mathbb{H}_h\times\mathbf{H}_h^{\boldsymbol{u}}\big)+\mathrm{dist}\big((\boldsymbol{p},\varphi),\,\mathbf{H}_h^{\boldsymbol{p}}\times\mathrm{H}_h^{\varphi}\big)\Big\}. \tag{5.12}$$

Proof. It follows by using the hypothesis (5.11) into the estimate (5.9) and the fact that $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$ satisfies the a priori estimates provided by Theorem 3.7. □

Finally, we complete our a priori error analysis stating the corresponding convergence rate of our Galerkin scheme (4.1).

Theorem 5.5 *In addition to the hypotheses of theorems* 3.7, 4.7 *and* 5.4, *assume that there exists* $s>0$ *such that* $\boldsymbol{t}\in\mathbb{H}^s(\Omega)$, $\boldsymbol{\sigma}\in\mathbb{H}^s(\Omega)$ *with* $\mathbf{div}\,\boldsymbol{\sigma}\in\mathbf{H}^s(\Omega)$, $\boldsymbol{\rho}\in\mathbb{H}^s(\Omega)$, $\boldsymbol{u}\in\mathbf{H}^{s+1}(\Omega)$, $\boldsymbol{p}\in\mathbf{H}^s(\Omega)$ *with* $\mathrm{div}\,\boldsymbol{p}\in\mathrm{H}^s(\Omega)$, *and* $\varphi\in\mathrm{H}^{s+1}(\Omega)$. *Then, there exists* $C>0$, *independent of* h, *such that*

$$\begin{aligned}\|(\underline{\boldsymbol{t}},\boldsymbol{u},\boldsymbol{p},\varphi)-(\underline{\boldsymbol{t}}_h,\boldsymbol{u}_h,\boldsymbol{p}_h,\varphi_h)\|\,\leq\,&Ch^{\min\{s,k+1\}}\Big\{\|\boldsymbol{t}\|_{s,\Omega}+\|\boldsymbol{\sigma}\|_{s,\Omega}+\|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega}+\|\boldsymbol{\rho}\|_{s,\Omega}\\&+\|\boldsymbol{u}\|_{s+1,\Omega}+\|\boldsymbol{p}\|_{s,\Omega}+\|\mathrm{div}\,\boldsymbol{p}\|_{s,\Omega}+\|\varphi\|_{s+1,\Omega}\Big\}.\end{aligned} \tag{5.13}$$

Proof. It follows directly from the Céa estimate (5.12) and standard approximation properties of the discrete spaces $\mathbb{H}_h^{\boldsymbol{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbb{H}_h^{\boldsymbol{\rho}}$, $\mathbf{H}_h^{\boldsymbol{u}}$, $\mathbf{H}_h^{\boldsymbol{p}}$ and H_h^{φ} (see [7,23], for instance). □

Now, regarding the postprocessing of additional variables, on the one hand, we recall the orthogonal decomposition for the pseudostress tensor provided in (3.1), and then the modified equation for the continuous pressure (2.12) becomes

$$p=-\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}+c\mathbb{I}+(\boldsymbol{u}\otimes\boldsymbol{u})),\quad\text{with}\quad c:=-\frac{1}{3|\Omega|}\int_\Omega\mathrm{tr}(\boldsymbol{u}\otimes\boldsymbol{u}). \tag{5.14}$$

Thus, according to (5.14), we define our discrete approximation of the pressure as

$$p_h=-\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}_h+c_h\mathbb{I}+(\boldsymbol{u}_h\otimes\boldsymbol{u}_h)),\quad\text{with}\quad c_h:=-\frac{1}{3|\Omega|}\int_\Omega\mathrm{tr}(\boldsymbol{u}_h\otimes\boldsymbol{u}_h), \tag{5.15}$$

which yields

$$p-p_h=\frac{1}{3}\mathrm{tr}\Big\{(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})+(\boldsymbol{u}_h\otimes\boldsymbol{u}_h-\boldsymbol{u}\otimes\boldsymbol{u})\Big\}+(c_h-c).$$

On the other hand, such as in [17], it is not difficult to see that that the relation (2.13) gives also the chance to compute the discrete concentration gradient through the formulae

$$\nabla\varphi_h\,=\,\kappa^{-1}\boldsymbol{p}_h+\kappa^{-1}\varphi_h\boldsymbol{u}_h+\kappa^{-1}U(\varphi_h+\alpha)\mathbf{i}_3\,. \tag{5.16}$$

Therefore, similarly to [9, Section 4], we easily deduce that there exist constants $C,\widetilde{C}>0$, independent of h, such that

$$\begin{aligned}\|p-p_h\|_{0,\Omega}&\leq C\Big\{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\Big\},\\ \|\nabla\varphi-\nabla\varphi_h\|_{0,\Omega}\,&\leq\,\widetilde{C}\,\Big\{\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\mathrm{div};\Omega}+\|\varphi-\varphi_h\|_{1,\Omega}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\Big\},\end{aligned} \tag{5.17}$$

and so the convergence rates of the postprocessed variables, in the L^2-norm, coincide with those provided by (5.13) (cf. Theorem 5.5).

6 Numerical results

This section presents a couple of numerical examples to illustrate the performance of our augmented fully-mixed formulation (4.1) and to support the respective convergence theoretical results for the primary and postprocessed variables predicted by Theorem 5.5 and the estimates (5.17), respectively. The fixed-point problem (4.4) has been implemented through a Picard iteration on a **FreeFem++** code (cf. [30]) and the resulting algebraic linear systems have been solved with the direct linear solver UMFPACK (see [18]). As an initial solution, we have simply taken $(\boldsymbol{u}_h^{(0)}, \varphi_h^{(0)}) = (\mathbf{0}, 0)$ to construct, on each step m, the entire solution vector

$$\mathbf{sol}^{(m)} = (\boldsymbol{t}_h^{(m)}, \boldsymbol{\sigma}_h^{(m)}, \boldsymbol{\rho}_h^{(m)}, \boldsymbol{u}_h^{(m)}, \boldsymbol{p}_h^{(m)}, \varphi_h^{(m)}) \quad \text{for all} \quad m \geq 1\,.$$

As stopping criteria, we have prescribed a fixed tolerance tol $= 1\mathrm{E}-8$ to finish the iterative technique when either a maximum number of iterations is reached or the relative error between two consecutive iterations, let us say $\mathbf{sol}^{(m)}$ and $\mathbf{sol}^{(m+1)}$, satisfies

$$\frac{||\mathbf{sol}^{(m+1)} - \mathbf{sol}^{(m)}||_{\ell^2}}{||\mathbf{sol}^{(m+1)}||_{\ell^2}} < \mathtt{tol}\,,$$

where $||\cdot||_{\ell^2}$ stands for the Euclidean ℓ^2-norm in R^N with N denoting the total number of degrees of freedom defined by the finite element family $(\mathbb{H}_h^{\boldsymbol{t}}, \mathbb{H}_h^{\boldsymbol{\sigma}}, \mathbb{H}_h^{\boldsymbol{\rho}}, \mathbf{H}_h^{\boldsymbol{u}}, \mathbf{H}_h^{\boldsymbol{p}}, \mathrm{H}_h^{\varphi})$ specified in Section 4.1.

The individual errors associated to the primary unknowns are denoted and defined by

$$\mathsf{e}(\boldsymbol{t}) := \|\boldsymbol{t} - \boldsymbol{t}_h\|_{0,\Omega}, \quad \mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega}, \quad \mathsf{e}(\boldsymbol{\rho}) := \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\Omega},$$
$$\mathsf{e}(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}, \quad \mathsf{e}(\boldsymbol{p}) := \|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{div};\Omega}, \quad \text{and} \quad \mathsf{e}(\varphi) := \|\varphi - \varphi_h\|_{1,\Omega}\,,$$

and the errors associated to the postprocessed variables (cf. (5.15) and (5.16)) are given, respectively, as

$$\mathsf{e}(p) := \|p - p_h\|_{0,\Omega} \quad \text{and} \quad \mathsf{e}(\nabla\varphi) := \|\nabla\varphi - \nabla\varphi_h\|_{0,\Omega}\,.$$

We also let $\mathsf{e}_{\mathtt{prim}}$ and $\mathsf{e}_{\mathtt{post}}$ be the total errors related to the primary and post-processed variables, respectively, that is,

$$\mathsf{e}_{\mathtt{prim}} := \left\{\mathsf{e}(\boldsymbol{t})^2 + \mathsf{e}(\boldsymbol{\sigma})^2 + \mathsf{e}(\boldsymbol{\rho})^2 + \mathsf{e}(\boldsymbol{u})^2 + \mathsf{e}(\boldsymbol{p})^2 + \mathsf{e}(\varphi)^2\right\}^{1/2} \quad \text{and} \quad \mathsf{e}_{\mathtt{post}} := \left\{\mathsf{e}(p)^2 + \mathsf{e}(\nabla\varphi)^2\right\}^{1/2}.$$

Following the same notation, we denote $\mathtt{r}(\,\cdot\,)$, $\mathtt{r}_{\mathtt{prim}}$ and $\mathtt{r}_{\mathtt{post}}$ as the individual experimental convergence rate associated to each variable, and the total convergence rates of the primary unknowns and post-processed variables, respectively, that is

$$\mathtt{r}(\,\cdot\,) := \frac{\log(\mathsf{e}(\,\cdot\,)/\mathsf{e}'(\,\cdot\,))}{\log(h/h')}, \quad \mathtt{r}_{\mathtt{prim}} := \frac{\log(\mathsf{e}_{\mathtt{prim}}/\mathsf{e}'_{\mathtt{prim}})}{\log(h/h')} \quad \text{and} \quad \mathtt{r}_{\mathtt{post}} := \frac{\log(\mathsf{e}_{\mathtt{post}}/\mathsf{e}'_{\mathtt{post}})}{\log(h/h')}$$

where h and h' denote two consecutive meshsizes with errors e and e'.

Example 1: accuracy assessment in 2D

In our first example we study the accuracy of the method in 2D by manufacturing an exact solution of a corresponding modification of problem (2.6). More precisely, the expressions $\mathbf{i}_3$, $\frac{\partial\varphi}{\partial x_3}$, and ν_3 are replaced in (2.6) by $\mathbf{i}_2 := (0,1)$, $\frac{\partial\varphi}{\partial x_2}$, and ν_2, respectively. Then, we consider the square $\Omega := (-1,1)^2$ and the data

$$\mu(x_1,x_2) = 1+\sin^2(x_1), \quad U = 0.01, \quad \gamma = 0.5, \qquad \kappa = 1, \quad \alpha = 0.5 \quad \text{and} \quad \boldsymbol{g} = (0,1)^t\,. \tag{6.1}$$

It follows that $\mu_1 = 1$ and $\mu_2 = 2$ ((2.3)), and hence the stabilization parameters κ_i, $(i = 1,\cdots,6)$, are chosen as in (3.39) and in accordance to Lemmas 3.1 and 3.2, that is

$$\kappa_1 = \frac{\mu_1}{2}, \quad \kappa_2 = \kappa_3 = \frac{\mu_1}{\mu_2^2}\,, \quad \kappa_4 = \frac{\mu_1}{2}, \quad \kappa_5 = \kappa, \quad \text{and} \quad \kappa_6 = \frac{\kappa^{-1}}{2}\,. \tag{6.2}$$

The terms on the right-hand sides are adjusted in such a way that the exact solutions are given by the smooth functions

$$\boldsymbol{u}(x_1,x_2) \;=\; \left(\begin{array}{c} 2\pi\cos(\pi x_2)\sin(\pi x_2)\sin^2(\pi x_1) \\ -2\pi\cos(\pi x_1)\sin(\pi x_1)\sin^2(\pi x_2) \end{array}\right), \quad p(x_1,x_2) \;=-5x_1\sin(x_2)\,,$$

and

$$\varphi(x_1,x_2) = \vartheta\, exp\left(\frac{U}{\kappa}x_2\right) - \alpha\,, \quad \text{where } \vartheta\in\mathbb{R} \text{ is taken so that } \int_\Omega \varphi = 0.$$

Note that the homogeneous Dirichlet condition for the velocity, the Robin-type boundary condition for the concentration, the incompressibbility condition of the fluid, and the zero-mean value restriction for both the pressure and the concentration are satisfied by the above functions.

Values of errors and corresponding convergence rates associated to the approximations with the finite element families $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ and $\mathbb{P}_1^{disc} - \mathbb{RT}_1 - \mathbb{P}_1^{disc} - \mathbf{P}_2 - \mathbf{RT}_1 - \mathrm{P}_2$ corresponding to approximations of order $k = 0$ and $k = 1$, respectively, are reported in Table 1. There, we observe that the convergence rates are linear (in the case $k = 0$) and quadratic (in the case $k = 1$) with respect to h for all the main unknowns in their respective norms, as well as the post-processed variables in the L^2-norm. Also, it is observed that the errors decay faster when increasing the approximation order from $k = 0$ to $k = 1$. In particular, this behavior can be observed from the values related to the total convergence rates $\mathtt{r_{prim}}$ and $\mathtt{r_{post}}$ for the primary and the variables obtained by post-processing. Our findings are in agreement with the theoretical error bounds predicted from Theorem 5.5 and the estimates (5.17). On the other hand, we mention that 8 and 9 Picard steps were required to reach the prescribed tolerance $\mathtt{tol}$ = 1E-08 in the cases $k = 0$ and $k = 1$, respectively. The approximation of the velocity magnitude, the pressure and concentration are depicted in Figure 1 computed with our fully-mixed method on a mesh with $N = 873843$ degrees of freedom and $k = 0$.

Example 2: accuracy assessment in 3D with concentration-dependent viscosity

In this example we focus on testing the accuracy of our method in the three-dimensional setting and considering the viscosity as a concentration-dependent function. To that end, we define the manufactured exact solution in the cube $\Omega := (0,1)^3$ as

$$\boldsymbol{u}(x_1,x_2,x_3) = \left(\begin{array}{c} 4x_1^2x_2x_3(x_3-1)(x_2-1)(x_2-x_3)(x_1-1)^2 \\ -4x_1x_2^2x_3(x_2-1)^2(x_3-1)(x_1-1)(x_1-x_3) \\ 4x_1x_2x_3^2(x_3-1)^2(x_2-1)(x_1-1)(x_1-x_2) \end{array}\right),$$

$$p(x_1,x_2,x_3) = \cos(\pi x_1)\cos(x_2)\cos(x_3)\,,$$

Fully–mixed $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1\,(k=0)$ scheme

$\mathtt{e}(\boldsymbol{t})$	$\mathtt{r}(\boldsymbol{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{p})$	$\mathtt{r}(\boldsymbol{p})$	$\mathtt{e}(\varphi)$	$\mathtt{r}(\varphi)$
2.4590	–	17.755	–	2.1535	–	4.4929	–	0.1792	–	0.1728	–
1.2280	1.0018	8.9033	0.9958	1.1132	0.9520	2.2405	1.0038	0.0898	0.9968	0.0869	0.9917
0.8925	0.9976	6.4778	0.9943	0.8147	0.9759	1.6285	0.9974	0.0654	0.9912	0.0633	0.9906
0.7010	0.9980	5.0906	0.9958	0.6419	0.9850	1.2792	0.9976	0.0514	0.9953	0.0498	0.9928
0.5607	1.0098	4.0729	1.0085	0.5144	1.0012	1.0232	1.0097	0.0411	1.0112	0.0398	1.0117
0.3924	0.9928	2.8513	0.9919	0.3606	0.9881	0.7162	0.9923	0.0288	0.9892	0.0279	0.9881
0.3567	1.0724	2.5921	1.0715	0.3279	1.0687	0.6510	1.0731	0.0262	1.0637	0.0253	1.0998

$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\nabla\varphi)$	$\mathtt{r}(\nabla\varphi)$	$\mathtt{e}_{\mathtt{prim}}$	$\mathtt{r}_{\mathtt{prim}}$	$\mathtt{e}_{\mathtt{post}}$	$\mathtt{r}_{\mathtt{post}}$	h	N	It
1.5750	–	0.1728	–	18.606	–	1.5845	–	0.0884	18819	8
0.7723	1.0281	0.0869	0.9917	9.3301	0.9958	0.7772	1.0277	0.0442	74499	8
0.5587	1.0122	0.0633	0.9906	6.7884	0.9943	0.5623	1.0119	0.0321	140451	8
0.4378	1.0076	0.0497	0.9995	5.3347	0.9957	0.4406	1.0075	0.0252	227139	8
0.3497	1.0169	0.0398	1.0044	4.2682	1.0085	0.3520	1.0158	0.0202	354483	8
0.2444	0.9966	0.0279	0.9881	2.9881	0.9918	0.2460	0.9964	0.0141	722403	8
0.2222	1.0706	0.0253	1.0998	2.7164	1.0716	0.2236	1.0710	0.0129	873843	8

Fully–mixed $\mathbb{P}_1^{disc} - \mathbb{RT}_1 - \mathbb{P}_1^{disc} - \mathbf{P}_2 - \mathbf{RT}_1 - \mathrm{P}_2\,(k=1)$ scheme

$\mathtt{e}(\boldsymbol{t})$	$\mathtt{r}(\boldsymbol{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{p})$	$\mathtt{r}(\boldsymbol{p})$	$\mathtt{e}(\varphi)$	$\mathtt{r}(\varphi)$
0.3204	–	2.2831	–	0.2628	–	0.5630	–	0.0214	–	0.0205	–
0.1812	1.9853	1.2892	1.9905	0.1523	1.9001	0.3186	1.9830	0.0121	1.9859	0.0117	1.9533
0.0808	1.9898	0.5745	1.9907	0.0695	1.9322	0.1422	1.9868	0.0054	1.9871	0.0052	1.9972
0.0455	1.9988	0.3234	2.0013	0.0395	1.9679	0.0801	1.9990	0.0031	1.9330	0.0030	1.9158
0.0359	2.0168	0.2556	2.0023	0.0313	1.9803	0.0633	2.0033	0.0024	2.1782	0.0023	2.2613
0.0239	2.0105	0.1712	1.9805	0.0208	2.0194	0.0423	1.9919	0.0016	2.0036	0.0015	2.1122

$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\nabla\varphi)$	$\mathtt{r}(\nabla\varphi)$	$\mathtt{e}_{\mathtt{prim}}$	$\mathtt{r}_{\mathtt{prim}}$	$\mathtt{e}_{\mathtt{post}}$	$\mathtt{r}_{\mathtt{post}}$	h	N	It
0.2185	–	0.0205	–	2.3879	–	0.2195	–	0.1178	35139	9
0.1236	1.9843	0.0117	1.9533	1.3490	1.9889	0.1242	1.9841	0.0884	62211	9
0.0550	1.9942	0.0052	1.9972	0.6014	1.9897	0.0552	1.9943	0.0589	139395	9
0.0309	2.0082	0.0030	1.9158	0.3386	2.0007	0.0310	2.0073	0.0442	247299	9
0.0244	2.0100	0.0023	2.2613	0.2676	2.0024	0.0245	2.0123	0.0393	312771	9
0.0163	1.9935	0.0015	2.1122	0.1792	1.9822	0.0164	1.9945	0.0321	466755	9

Table 1: Test 1: Convergence history for the fully-mixed approximation of the Bioconvection problem with $k=0$ (first and second panel) and $k=1$ (third and fourth panel)

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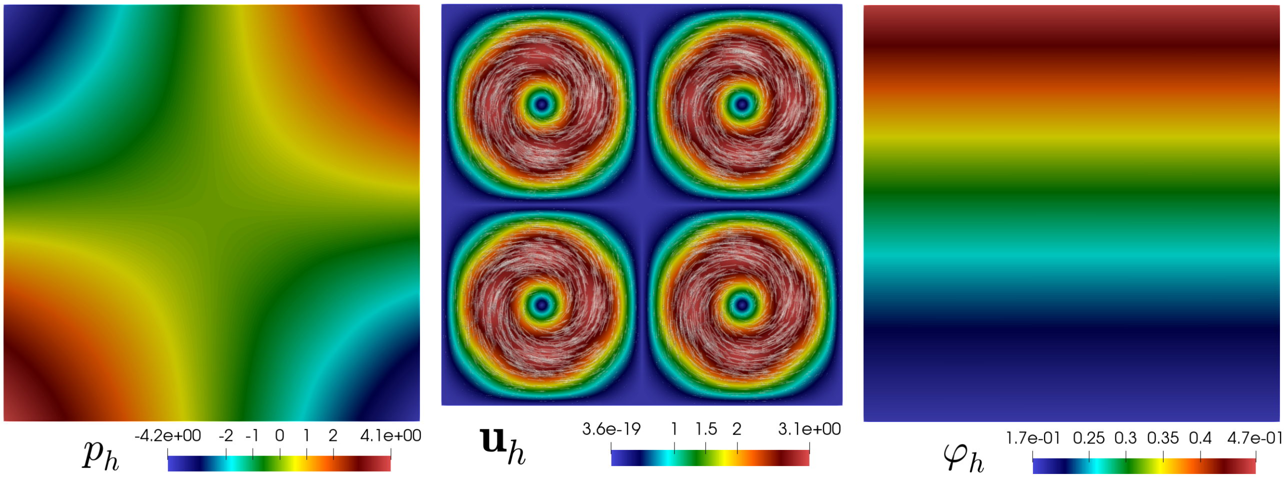

Figure 1: Example 1: Approximated pressure, velocity magnitude, and concentration obtained with the fully-mixed method using $k = 0$ and $N = 873843$ degrees of freedom .

and, similarly as in the first example, the auxiliary exact concentration satisfying the Robin-type boundary condition takes the form

$$\varphi(x_1, x_2, x_3) = \vartheta\, exp\left(\frac{U}{\kappa} x_3\right) - \alpha\,, \quad \text{where } \vartheta \in \mathbb{R} \text{ is taken so that} \quad \int_\Omega \varphi = 0.$$

Next, the viscosity is taken as a concentration-dependent function defined as

$$\mu(\varphi) = 1 + \sin^2(\varphi),$$

satisfying (2.2) and (2.3) with $\mu_1 = 1$ and $\mu_2 = 2$. The rest of data and stabilization parameters are taken as in (6.1) and (6.2).

For this example, we consider the finite element spaces introduced in Section 4.1 with $k = 0$. The convergence history is summarized in Table 2 and it is observed there that the total error decay is of order $\mathcal{O}(h)$ for the primary unknowns and the postprocessed variables as predicted by Theorem 5.5 and the estimates (5.17). In particular, 4 Picard steps were required to achieve the prescribed tolerance `tol` = 1E-08. Next, in Figure 2 we display the streamlines, the component $\boldsymbol{\rho}_{12,h}$ of the vorticity tensor and the concentration profile φ in the first panel, whereas in the second panel are depicted the component $\boldsymbol{t}_{11,h}$ of the shear stress tensor, the component $\boldsymbol{\sigma}_{23,h}$ of the pseudo-stress tensor and the concentration gradient vector field $\nabla\varphi_h$ obtained with $k = 0$ and $N = 1403428$ degrees of freedom.

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Fully–mixed $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbb{P}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_1$ $(k = 0)$ scheme											
$\mathtt{e}(\boldsymbol{t})$	$\mathtt{r}(\boldsymbol{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{p})$	$\mathtt{r}(\boldsymbol{p})$	$\mathtt{e}(\varphi)$	$\mathtt{r}(\varphi)$
0.0817	–	0.6974	–	0.0446	–	0.0916	–	0.4489	–	1.0850	–
0.0676	0.4672	0.4764	0.9398	0.0419	0.1544	0.0647	0.8576	0.4365	0.0692	0.8401	0.6039
0.0550	0.7174	0.3580	0.9941	0.0371	0.4209	0.0452	1.2500	0.3301	0.9713	0.6096	1.1154
0.0389	0.8569	0.2360	1.0274	0.0290	0.6089	0.0252	1.4391	0.1918	1.3385	0.3433	1.4156
0.0297	0.9306	0.1750	1.0400	0.0234	0.7546	0.0161	1.5537	0.1240	1.5178	0.2160	1.6113
0.0240	0.9623	0.1388	1.0375	0.0194	0.8314	0.0113	1.5853	0.0875	1.5611	0.1478	1.6981
0.0200	0.9807	0.1150	1.0355	0.0165	0.8813	0.0085	1.5883	0.0658	1.5629	0.1077	1.7414
0.0151	0.9861	0.0856	1.0239	0.0127	0.9166	0.0054	1.5436	0.0426	1.5095	0.0654	1.7349
0.0121	0.9924	0.0682	1.0169	0.0103	0.9465	0.0039	1.4727	0.0310	1.4282	0.0447	1.7019
0.0101	0.9933	0.0567	1.0105	0.0086	0.9611	0.0030	1.3985	0.0242	1.3510	0.0331	1.6428
0.0093	1.0021	0.0523	1.0152	0.0080	0.9762	0.0027	1.3583	0.0218	1.3048	0.0291	1.6082
	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\nabla\varphi)$	$\mathtt{r}(\nabla\varphi)$	$\mathtt{e}_{\mathtt{prim}}$	$\mathtt{r}_{\mathtt{prim}}$	$\mathtt{e}_{\mathtt{post}}$	$\mathtt{r}_{\mathtt{post}}$	h	N	It
	0.2231	–	1.0680	–	0.7096	–	0.2233	–	0.7071	972	4
	0.1496	0.9845	0.8146	0.6680	0.4874	0.9263	0.1499	0.9837	0.4714	3064	4
	0.1091	1.0995	0.5879	1.1342	0.3669	0.9875	0.1092	1.0099	0.3536	7028	4
	0.0674	1.1858	0.3303	1.4215	0.2423	1.0236	0.0675	1.1864	0.2357	22792	4
	0.0474	1.2239	0.2076	1.6150	0.1798	1.0375	0.0475	1.2247	0.1768	53604	4
	0.0362	1.2111	0.1421	1.6967	0.1426	1.0358	0.0362	1.2120	0.1414	103724	4
	0.0291	1.1919	0.1037	1.7373	0.1182	1.0342	0.0292	1.1926	0.1179	178132	4
	0.0201	1.1502	0.0631	1.7257	0.0880	1.0230	0.0209	1.1508	0.0884	419012	4
	0.0163	1.1107	0.0447	1.5443	0.0702	1.0163	0.0163	1.1110	0.0707	814644	4
	0.0134	1.0808	0.0322	1.7988	0.0583	1.0101	0.0134	1.0812	0.0589	1403428	4
	0.0123	1.0732	0.0284	1.5868	0.0538	1.0149	0.0123	1.0735	0.0129	1782252	4

Table 2: Test 2: Convergence history for the fully-mixed approximation of the three-dimensional Bioconvection problem with concentration-dependent viscosity and using approximation order $k = 0$
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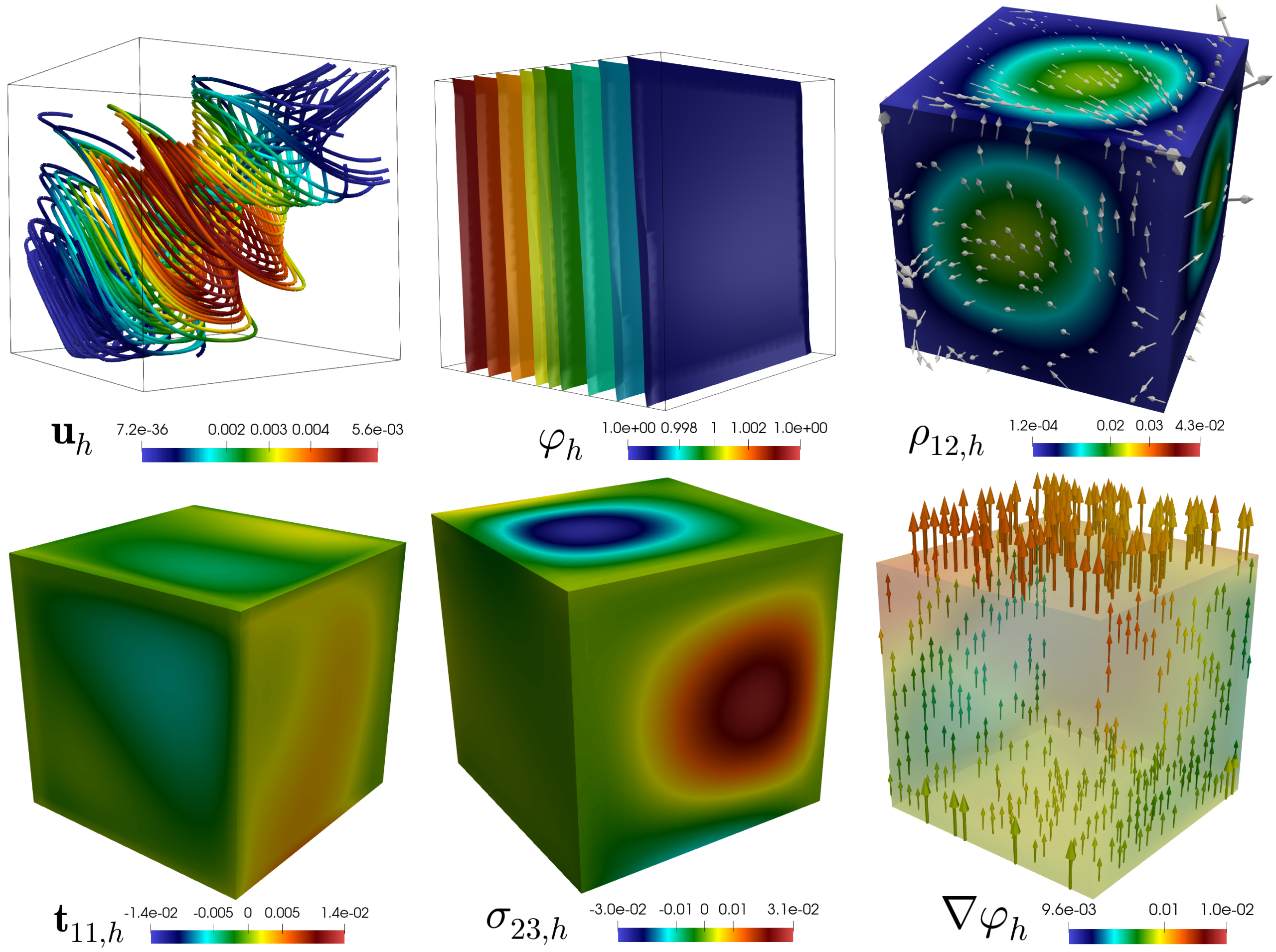

Figure 2: Example 2: Streamlines, concentration profile φ_h, and component $\boldsymbol{\rho}_{12,h}$ of the vorticity tensor (first panel), and the component $\boldsymbol{t}_{11,h}$ of the shear stress tensor, component $\boldsymbol{\sigma}_{23,h}$ of the pseudo-stress tensor and concentration gradient $\nabla\varphi_h$ obtained with the fully-mixed method for the Bioconvection problem using $k = 0$ and $N = 1403428$ degrees of freedom.

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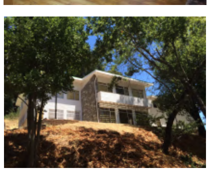

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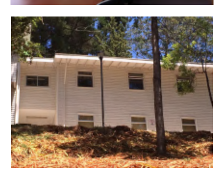
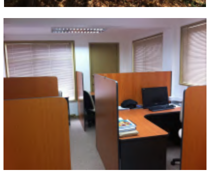

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