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Dedicated to the memory of Francisco Javier Sayas

Abstract

This paper is concerned with new insights on the application of the boundary-field equation approach, which refers basically to the combined use of finite element and boundary integral equation methods, to numerically solve nonlinear exterior transmission problems in 2D and 3D. As a model, we consider a nonlinear second-order elliptic equation in divergence form holding in an annular domain coupled with the Laplace equation in the corresponding unbounded exterior region, together with transmission conditions on the interface and a suitable radiation condition at infinity. We first extend the classical Johnson & Nédélec coupling procedure, which makes use of only one boundary integral equation, to our nonlinear model without assuming any restrictive smoothness requirement on the boundary but only Lipschitz-continuity. Next, we extend the applicability of a recently introduced modification of the Costabel & Han coupling method, which employs the two boundary integral equations arising from the Green representation formula, to our nonlinear model as well. This new boundary-field equation method is based on the introduction of both Cauchy data on the boundary as independent unknowns. Primal and dual-mixed variational formulations are analyzed for each extension described above, and suitable hypotheses on the nonlinear coefficient of the elliptic equation allow to establish well-posedness of the corresponding continuous and discrete schemes by using monotone operator and nonlinear Babuška-Brezzi theories. Finally, a priori error estimates are established.

1 Introduction

The combination of finite element methods (FEM) and boundary element methods (BEM), which is usually employed to solve transmission problems, has been frequently utilized for many years in diverse applications. In this regard, the most popular procedures have been the Johnson & Nédélec and Costabel & Han coupling methods (cf. [6], [10], [30], and [32]), which use the Green representation of the solution in the corresponding region. Moreover, the former, which is based on a single boundary integral equation and the compactness of the double-layer boundary integral operator, was initially applied only to transmission problems involving the Laplace operator. Indeed, its applicability to other elliptic equations, such as the Lamé or Stokes systems, was forbidden due to the lack of the required compactness. This drawback soon motivated the approaches by Costabel and Han in [10]

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and [30], respectively, which, being both based on the incorporation of a second boundary integral equation, differed only in the sign of a common integral identity, thus explaining that, in spite of this minor difference, they were both named as the Costabel & Han method. Actually, and to be more precise, Costabel's approach yielded a symmetric and non-positive definite scheme, whereas the one by Han gave rise to a non-symmetric but elliptic system.

Almost two decades later, the aforementioned limitation of the Johnson & Nédélec coupling method was finally solved in [41] (see also [44], [26] and [42]), where it was proved that the Galerkin scheme for this approach is always stable after all, thus confirming that, on the contrary to the common belief since its creation, the procedure can be applied to other elliptic equations and to arbitrary polygonal/polyhedral regions as well. The analysis from [41] was then generalized in [34] to the coupling of mixed-FEM and BEM for the Poisson equation on Lipschitz-continuous domains, and soon after that, the application to the three dimensional exterior Stokes problem was developed in [24] and [25]. Further contributions along the last three decades dealing with the use of either the Johnson & Nédélec or Costabel & Han coupling procedure to solve 2D and 3D problems, including coupling with mixed-FEM, nonlinear models, fluid-solid interaction, and eddy current problems, among others, can be found, e.g. in [1], [7],[8], [11], [14], [16], [18], [19], [20], [21], [22], [23], [35], [36], and [37].

On the other hand, the combined use of the virtual element method (VEM) - a rather recent technique originally proposed in [2] and [4] - and BEM, has been introduced and analyzed for the first time in [27] for solving a linear exterior transmission problem in 2D and 3D. More recently, the approach in [27] was extended in [28] to the case of acoustic scattering problems. Regarding [27] (similar comments apply to [28]), let us begin by stressing that the philosophy of VEM is based on two main aspects. Firstly, the discrete spaces are defined on meshes formed by polygonal or polyhedral elements, and the corresponding basis functions are not known explicitly (which justify the concept *virtual* utilized), but only the degrees of freedom defining them uniquely on each element are required to implement the method. Secondly, suitable projection operators and stabilizing terms are utilized to define approximated bilinear forms, which provide still consistency and stability of the resulting discrete scheme. The aforementioned degrees of freedom normally have to do with polynomial moments within each element and with traces and normal traces, both polynomial as well, on the boundaries of them. Having said the above, the approach in [27] reduces basically to the usual primal formulation in the corresponding bounded region, coupled by means of the Costabel & Han procedure with the boundary integral equation method in the exterior domain. In this way, and besides the original unknown of the problem, its normal derivative in 2D, and both its normal derivative and its trace in the 3D case, are introduced as auxiliary non-virtual unknowns. Actually, for the correct matching between the degrees of freedom of VEM and the densities of the boundary integral operators, a new variational formulation for the coupling, which is given by a suitable modification of the Costabel & Han technique, and which incorporates precisely both Cauchy data on the boundary as independent unknowns, needs to be introduced in the 3D case. In this regard, and in spite of the very intuitive character of the resulting boundary-field equation method, it is surprising to realize that it had not been employed before. This latter remark motivates our present application of this approach and its dual-mixed version to nonlinear exterior transmission problems.

According to the previous bibliographic discussion, and motivated by the advances from [41], [34], [26], [24], and [25], in the present paper we first extend the applicability of the Johnson & Nédélec coupling method to analyze the weak solvability and Galerkin approximations of nonlinear transmission problems in Lipschitz-continuous domains, that is without requiring the double-layer boundary integral operator to be compact. Next, and bearing in mind eventual forthcoming developments on the coupling of VEM and BEM for nonlinear models, here we also address the same goal as above but, instead of the Johnson & Nédélec technique, using now the primal and dual-mixed versions of the modified Costabel & Han procedure introduced in [27]. The rest of our work is organized as follows. In Section 2 we introduce our nonlinear transmission problem and recall the main definitions and results regarding the boundary integral equation method. Then, in Sections 3 and 4 we consider the primal and dual-mixed approaches, respectively, of the Johnson & Nédélec coupling method as applied to our model from Section 2. Under suitable assumptions on the nonlinearity of the problem, the well-posedness of the continuous and discrete formulations are established and corresponding a priori error estimates and rates of convergence are derived. In Section 5 we basically follow the same structure of the previous two sections, but now employing the modified Costabel & Han coupling procedure from [27]. Finally, we include Appendixes A and B with some details on the optimal choice of certain parameters needed along the analysis, and with the main results that form part of a nonlinear Babuška-Brezzi theory, respectively.

We end this section by stressing that standard notations for spaces and norms will be employed throughout the paper. In particular, given a real number r, a domain $G \subseteq \mathrm{R}^n$, $n \in \{2,3\}$, and a part S of its boundary ∂G, $\mathrm{H}^r(G)$ and $\mathrm{H}^r(S)$ stand for the respective Sobolev spaces of order r, with vector versions given by $\mathbf{H}^r(G) := [\mathrm{H}^r(G)]^n$ and $\mathbf{H}^r(S) := [\mathrm{H}^r(S)]^n$, and whose norms and seminorms are denoted by $\|\cdot\|_{r,G}$ and $|\cdot|_{r,G}$, and $\|\cdot\|_{r,S}$ and $|\cdot|_{r,S}$, respectively (cf. [33]). In addition, we use the convention $\mathrm{L}^2(G) := \mathrm{H}^0(G)$, $\mathbf{L}^2(G) := \mathbf{H}^0(G)$, $\mathrm{L}^2(S) := \mathrm{H}^0(S)$, $\mathbf{L}^2(S) := \mathbf{H}^0(S)$, and recall that for each $r \in (0,1]$, $\mathrm{H}^{-r}(\partial G)$ is the dual of $\mathrm{H}^r(\partial G)$ with respect to the pivot space $\mathrm{L}^2(\partial G)$. Furthermore, denoting by div the usual divergence operator, we also introduce the classical Hilbert space for dual-mixed formulations

$$\mathbf{H}(\mathrm{div};\Omega) := \Big\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^2(\Omega)\Big\}, \tag{1.1}$$

which is endowed with the inner product and induced norm given by

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{\mathrm{div};\Omega} := \int_\Omega \Big\{\boldsymbol{\zeta}\cdot\boldsymbol{\tau} + \mathrm{div}(\boldsymbol{\zeta})\,\mathrm{div}(\boldsymbol{\tau})\Big\} \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div};\Omega)\,,$$

and

$$\|\boldsymbol{\tau}\|^2_{\mathrm{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|^2_{0,\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div};\Omega)\,.$$

respectively. Other definitions and notations are given along the paper when they are needed.

2 The model and the boundary integral equation method

In this section we introduce the model of interest and recall the main results concerning the boundary integral equation method to be employed later on.

2.1 The nonlinear transmission problem

Let Ω_0 be a bounded, simply connected domain in R^n, $n \in \{2,3\}$, with Lipschitz boundary Γ_0, and let Ω be the annular region boundary by Γ_0 and the boundary Γ of a Lipschitz-continuous region $\mathcal{O}$ containing $\bar\Omega_0$. We denote by $\mathcal{O}_e$ the complement of $\bar{\mathcal{O}}$, and let $\boldsymbol{n}$ be the unit outward normal to Γ pointing towards $\mathcal{O}_e$ (see Figure 2.1 below). Then, given $f \in \mathrm{L}^2(\Omega)$, we consider the transmission problem:

$$\begin{array}{rcll} -\mathrm{div}\Big(a\,(\cdot, \|\nabla u\|)\,\nabla u\Big) & = & f & \text{in} \quad \Omega\,, \\ u & = & 0 & \text{on} \quad \Gamma_0\,, \\ u = u_e \quad \text{and} \quad a\,(\cdot,\|\nabla u\|)\,\dfrac{\partial u}{\partial \boldsymbol{n}} & = & \dfrac{\partial u_e}{\partial \boldsymbol{n}} & \text{on} \quad \Gamma\,, \\ \Delta\, u_e & = & 0 & \text{in} \quad \mathcal{O}_e\,, \\ u_e(x) & = & O(\|x\|^{-1}) & \text{as} \quad \|x\| \to +\infty\,, \end{array} \tag{2.1}$$

where $a : \Omega \times \mathrm{R}^+ \to \mathrm{R}$ is a nonlinear scalar function satisfying certain regularity conditions to be specified later on. We remark that in the case $n = 2$, the condition $u_e(x) = O(1)$ as $\|x\| \to +\infty$ would be sufficient. The boundary-field equation methods to be considered in this paper are obtained by using different primal and dual-mixed approaches to transform (2.1) into an equivalent nonlocal boundary problem holding in the bounded domain Ω. This goal is achieved in each case by employing first the boundary integral equation method to solve the Laplace equation in the exterior region Ω_e, which is explained in the following section.

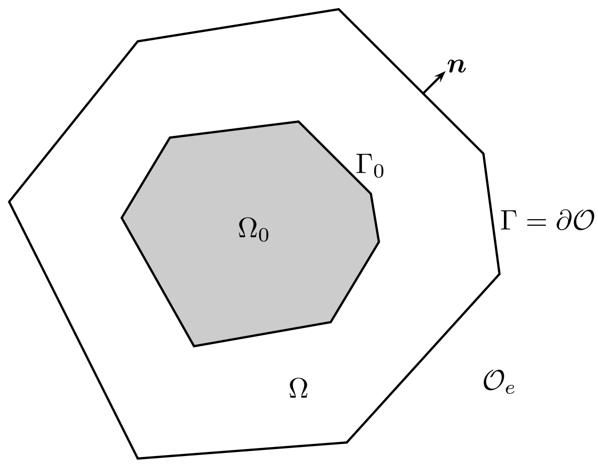

Figure 2.1: 2D geometry of the model problem.

2.2 The boundary integral equation method

We first let γ and $\gamma_{\boldsymbol{n}}$ be the usual trace and normal trace operators on Γ (acting either from Ω or $\mathcal{O}_e$). Then, proceeding in the usual way, we compute the harmonic solution in the exterior region $\mathcal{O}_e$ by means of the Green representation formula

$$u_e(x) = \int_\Gamma \frac{\partial \mathrm{E}(|x-y|)}{\partial \boldsymbol{n}_y} \psi(y)\, \mathrm{d}s_y - \int_\Gamma \mathrm{E}(|x-y|)\lambda(y)\, \mathrm{d}s_y \qquad \forall\, x \in \mathcal{O}_e\,, \tag{2.2}$$

where

$$\mathrm{E}(|x-y|) := \begin{cases} \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{if } n = 3 \\ -\dfrac{1}{2\pi}\log|x-y| & \text{if } n = 2 \end{cases}$$

is the fundamental solution of the Laplacian, and

$$\psi := \gamma(u) = \gamma(u_e) \quad \text{and} \quad \lambda := \gamma_{\boldsymbol{n}}\big(a(\cdot, \|\nabla u\|)\nabla u\big) = \gamma_{\boldsymbol{n}}(\nabla u_e)\,, \tag{2.3}$$

are the Cauchy data on this interface. We stress here that the first and second expression on the right hand side of (2.2) constitute the double $\mathcal{D}$ and single layer $\mathcal{S}$ potentials, respectively, which are defined as

$$(\mathcal{D}\,\varphi)(x) := \int_\Gamma \frac{\partial \mathrm{E}(|x-y|)}{\partial \boldsymbol{n}_y} \varphi(y) \qquad \forall\, x \in \mathrm{R}^n \backslash \Gamma\,, \quad \forall\, \varphi \in \mathrm{H}^{1/2}(\Gamma)\,,$$

and

$$(\mathcal{S}\,\mu)(x) := \int_\Gamma \mathrm{E}(|x-y|)\mu(y)\, \mathrm{d}s_y \qquad \forall\, x \in \mathrm{R}^n \backslash \Gamma\,, \quad \forall\, \mu \in \mathrm{H}^{-1/2}(\Gamma)\,.$$

In what follows, we denote the restrictions $\mathcal{D}_i := \mathcal{D}|_\Omega$, $\mathcal{D}_e := \mathcal{D}|_{\Omega_e}$, $\mathcal{S}_i := \mathcal{S}|_\Omega$, and $\mathcal{S}_e := \mathcal{S}|_{\Omega_e}$. Then, the jump conditions on Γ of these potentials establish that (cf. [31], [40])

$$\begin{aligned} \gamma\big(\mathcal{D}_e(\varphi)\big) \,&=\, \big(\frac{1}{2}\mathrm{I} + K\big)\varphi\,, \qquad \gamma\big(\mathcal{D}_i(\varphi)\big) \,=\, \big(-\,\frac{1}{2}\mathrm{I} + K\big)\varphi\,, \\ \gamma_{\boldsymbol{n}}\big(\nabla\mathcal{D}_e(\varphi)\big) \,&=\, \gamma_{\boldsymbol{n}}\big(\nabla\mathcal{D}_i(\varphi)\big) \,=\, -W\varphi \qquad \forall\,\varphi\in \mathrm{H}^{1/2}(\Gamma)\,, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \gamma_{\boldsymbol{n}}\big(\nabla\mathcal{S}_e(\mu)\big) \,&=\, \big(-\,\frac{1}{2}\mathrm{I} + K^{\mathtt{t}}\big)\mu\,, \qquad \gamma_{\boldsymbol{n}}\big(\nabla\mathcal{S}_i(\mu)\big) \,=\, \big(\frac{1}{2}\mathrm{I} + K^{\mathtt{t}}\big)\mu\,, \\ \gamma\big(\mathcal{S}_e(\mu)\big) \,&=\, \gamma\big(\mathcal{S}_i(\mu)\big) \,=\, V\mu \qquad \forall\,\mu\in \mathrm{H}^{-1/2}(\Gamma)\,, \end{aligned} \tag{2.5}$$

where V, K, $K^{\mathtt{t}}$, and W are the boundary integral operators of the single, double, adjoint of the double, and hypersingular layer potentials (cf. [31]), respectively, whereas I is a generic identity operator. Moreover, from the first two identities in (2.4) and (2.5) we obtain, respectively,

$$\varphi \,=\, \gamma\big(\mathcal{D}_e(\varphi)\big) \,-\, \gamma\big(\mathcal{D}_i(\varphi)\big) \qquad \forall\,\varphi\in \mathrm{H}^{1/2}(\Gamma)\,, \tag{2.6}$$

and

$$\mu \,=\, \gamma_{\boldsymbol{n}}\big(\nabla\mathcal{S}_i(\mu)\big) \,-\, \gamma_{\boldsymbol{n}}\big(\nabla\mathcal{S}_e(\mu)\big) \qquad \forall\,\mu\in \mathrm{H}^{-1/2}(\Gamma)\,. \tag{2.7}$$

In turn, by applying (2.4) and (2.5) in (2.2), we deduce that

$$\gamma(u_e) \,=\, \big(\frac{1}{2}\mathrm{I} + K\big)\psi \,-\, V\lambda \quad \text{on} \quad \Gamma\,, \tag{2.8}$$

and

$$\gamma_{\boldsymbol{n}}(\nabla u_e) \,=\, -\,W\psi \,+\, \big(\frac{1}{2}\mathrm{I} - K^{\mathtt{t}}\big)\lambda \quad \text{on} \quad \Gamma\,. \tag{2.9}$$

We now denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$ with respect to the pivot space $\mathrm{L}^2(\Gamma)$, and introduce the spaces

$$\mathrm{H}^{1/2}_0(\Gamma) := \{\varphi\in \mathrm{H}^{1/2}(\Gamma):\, \langle 1,\varphi\rangle = 0\}$$

and

$$\mathrm{H}^{-1/2}_0(\Gamma) := \{\mu\in \mathrm{H}^{-1/2}(\Gamma):\, \langle \mu,1\rangle = 0\}\,,$$

for which there hold the decompositions

$$\mathrm{H}^{1/2}(\Gamma) \,=\, \mathrm{H}^{1/2}_0(\Gamma)\oplus \mathrm{R} \quad \text{and} \quad \mathrm{H}^{-1/2}(\Gamma) \,=\, \mathrm{H}^{-1/2}_0(\Gamma)\oplus \mathrm{R}\,.$$

Then, the main mapping properties of V, K, $K^{\mathtt{t}}$, and W are given as follows (cf. [31], [40]).

Lemma 2.1. *The linear operators*

$$V:\, \mathrm{H}^{-1/2}(\Gamma) \longrightarrow \mathrm{H}^{1/2}(\Gamma), \qquad K:\, \mathrm{H}^{1/2}(\Gamma) \longrightarrow \mathrm{H}^{1/2}(\Gamma)\,,$$

$$K^{\mathtt{t}}:\, \mathrm{H}^{-1/2}(\Gamma) \longrightarrow \mathrm{H}^{-1/2}(\Gamma), \quad \textit{and} \quad W:\, \mathrm{H}^{1/2}(\Gamma) \longrightarrow \mathrm{H}^{-1/2}(\Gamma)\,,$$

are bounded. In addition, there exist positive constants α_V, α_W such that

$$\langle\mu,\, V\mu\rangle \,\geq\, \alpha_V\,\|\mu\|^2_{-1/2,\Gamma} \qquad \begin{cases} \forall\,\mu\in \mathrm{H}^{-1/2}_0(\Gamma), & \textit{if } n=2\,,\\ \forall\,\mu\in \mathrm{H}^{-1/2}(\Gamma), & \textit{if } n=3\,, \end{cases} \tag{2.10}$$

and

$$\langle W\varphi,\varphi\rangle \,=\, \langle W\varphi_0,\varphi_0\rangle \,\geq\, \alpha_W\,\|\varphi_0\|^2_{1/2,\Gamma} \tag{2.11}$$

for all $\varphi := \varphi_0 + c\in \mathrm{H}^{1/2}(\Gamma)$, with $\varphi_0\in \mathrm{H}^{1/2}_0(\Gamma)$ and $c\in\mathrm{R}$.

The different ways of handling the first two equations of (2.1) with the boundary integral equations provided by (2.8) and (2.9) give rise to the boundary-field equation methods that we introduce and analyze in the following sections. Throughout the rest of the paper, and given any generic normed space X, $[\cdot,\cdot]$ stands for the duality pairing between X' and X.

3 The Johnson & Nédélec coupling: primal approach

In this section we employ the classical Johnson & Nédélec coupling method, in its primal approach, to analyze the respective continuous and discrete formulations of the model introduced in Section 2.

3.1 The continuous formulation

We begin by replacing $\gamma(u_e)$ and ψ in (2.8) by $\gamma(u)$, which yields

$$\big(\frac{1}{2}\mathrm{I} - K\big)\gamma(u) \,+\, V\lambda \,=\, 0 \quad \text{on} \quad \Gamma\,. \tag{3.1}$$

Next, we introduce the space

$$\mathrm{H}^1_{\Gamma_0}(\Omega) \,:=\, \Big\{ v \in \mathrm{H}^1(\Omega): \quad v = 0 \quad \text{on} \quad \Gamma_0 \Big\}\,, \tag{3.2}$$

and let c_p be the positive constant arising from the Poincaré inequality for $\mathrm{H}^1_{\Gamma_0}(\Omega)$, that is such that

$$|v|^2_{1,\Omega} \,\geq\, c_p\,\|v\|^2_{1,\Omega} \qquad \forall\, v \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,. \tag{3.3}$$

Then, multiplying the first equation of (2.1) by a test function $v \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, integrating by parts, and using that $\lambda = \gamma_{\boldsymbol{n}}\big(a\,(\cdot, \|\nabla u\|)\,\nabla u\big)$, we obtain

$$\int_\Omega a\,(\cdot, \|\nabla u\|)\,\nabla u \cdot \nabla v \,-\, \langle \lambda, \gamma(v)\rangle \,=\, \int_\Omega f\,v \qquad \forall\, v \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,. \tag{3.4}$$

In turn, testing (3.1) with $\mu \in \widehat{\mathrm{H}}^{-1/2}(\Gamma) := \mathrm{H}_0^{-1/2}(\Gamma)$ when $n = 2$, or $\widehat{\mathrm{H}}^{-1/2}(\Gamma) := \mathrm{H}^{-1/2}(\Gamma)$ when $n = 3$, and adding the resulting expression to the foregoing identity, we arrive at our first boundary-field equation formulation: Find $(u, \lambda) \in \mathbf{H} \,:=\, \mathrm{H}^1_{\Gamma_0}(\Omega) \times \widehat{\mathrm{H}}^{-1/2}(\Gamma)$ such that

$$[\mathbf{A}(u, \lambda), (v, \mu)] \,=\, \mathbf{F}(v, \mu) \qquad \forall\, (v, \mu) \in \mathbf{H}\,, \tag{3.5}$$

where $\mathbf{A} : \mathbf{H} \longrightarrow \mathbf{H}'$ is the nonlinear operator defined by

$$[\mathbf{A}(w, \xi), (v, \mu)] \,:=\, \int_\Omega a\,(\cdot, \|\nabla w\|)\,\nabla w \cdot \nabla v \,-\, \langle \xi, \gamma(v)\rangle \,+\, \langle \mu, \big(\frac{1}{2}\mathrm{I} - K\big)\gamma(w)\rangle \,+\, \langle \mu, V\xi\rangle \tag{3.6}$$

for all (w, ξ), $(v, \mu) \in \mathbf{H}$, and $\mathbf{F} \in \mathbf{H}'$ is the functional given by

$$\mathbf{F}(v, \mu) \,:=\, \int_\Omega f\,v \qquad \forall\, (v, \mu) \in \mathbf{H}\,. \tag{3.7}$$

3.2 Solvability analysis of the continuous formulation

Here we address the solvability of (3.5). In fact, denoting by $\widetilde{\mathbf{A}}$ the linear part of $\mathbf{A}$, we first observe from (3.6) that for each $(v, \mu) \in \mathbf{H}$ there holds

$$[\widetilde{\mathbf{A}}(v, \mu), (v, \mu)] \,=\, -\,\langle \mu, \gamma(v)\rangle \,+\, \langle \big(\frac{1}{2}\mathrm{I} - K^{\mathtt{t}}\big)\mu, \gamma(v)\rangle \,+\, \langle \mu, V\mu\rangle\,,$$

which, using the first identity in (2.5), and also the one given by (2.7), becomes

$$[\widetilde{\mathbf{A}}(v, \mu), (v, \mu)] \,=\, -\,\langle \gamma_{\boldsymbol{n}}\big(\mathcal{S}_i(\mu)\big) - \gamma_{\boldsymbol{n}}\big(\mathcal{S}_e(\mu)\big), \gamma(v)\rangle \,-\, \langle \gamma_{\boldsymbol{n}}\big(\mathcal{S}_e(\mu)\big), \gamma(v)\rangle \,+\, \langle \mu, V\mu\rangle\,,$$

that is

$$[\widetilde{\mathbf{A}}(v,\mu),(v,\mu)] \,=\, -\,\langle\gamma_{\boldsymbol{n}}\big(\mathcal{S}_i(\mu)\big),\gamma(v)\rangle \,+\, \langle\mu,V\mu\rangle\,.$$

Then, integrating by parts backwardly in Ω, bearing in mind that $v=0$ on Γ_0, $\Delta S_i(\mu)=0$ in Ω, and $\langle\mu,V\mu\rangle \,=\, \|\nabla\mathcal{S}(\mu)\|^2_{0,\mathrm{R}^n}$, and applying the Young inequality with parameter $\delta>0$, we find that

$$\begin{aligned}
[\widetilde{\mathbf{A}}(v,\mu),(v,\mu)] \,&=\, -\int_\Omega \nabla\mathcal{S}_i(\mu)\cdot\nabla v \,+\, \langle\mu,V\mu\rangle\\
&\geq\, -\,\|\nabla\mathcal{S}_i(\mu)\|_{0,\Omega}\,\|\nabla v\|_{0,\Omega} \,+\, \langle\mu,V\mu\rangle\\
&\geq\, -\,\frac{\delta}{2}\,\|\nabla\mathcal{S}(\mu)\|^2_{0,\mathrm{R}^n} \,-\, \frac{1}{2\delta}\,\|\nabla v\|^2_{0,\Omega} \,+\, \langle\mu,V\mu\rangle\\
&=\, \Big(1-\frac{\delta}{2}\Big)\,\langle\mu,V\mu\rangle \,-\, \frac{1}{2\delta}\,\|\nabla v\|^2_{0,\Omega}\,.
\end{aligned} \tag{3.8}$$

In order to combine the foregoing inequality with a suitable estimate for the remaining term defining the operator $\mathbf{A}$, that is its nonlinear part, throughout the rest of the paper we assume the following hypotheses on the nonlinear coefficient a:

(H.1) [CARATHÉODORY AND BOUNDEDNESS CONDITIONS]. The function $a(\cdot,t)$ is measurable in Ω for all $t\in\mathrm{R}^+$, $a(x,\cdot)$ is continuous in R^+ for almost all $x\in\Omega$, and there exists a constant $C>0$ such that

$$|a(x,t)| \,\leq\, C \qquad \forall\, t\in\mathrm{R}^+\,,\quad \forall\, x\in\Omega\ a.e.$$

(H.2) [STRONG MONOTONICITY]. There exists a constant $\alpha_A>0$ such that

$$\int_\Omega \Big\{a(\cdot,\|\mathbf{r}\|)\,\mathbf{r} \,-\, a(\cdot,\|\mathbf{s}\|)\,\mathbf{s}\Big\}\cdot(\mathbf{r}-\mathbf{s}) \,\geq\, \alpha_A\,\|\mathbf{r}-\mathbf{s}\|^2_{0,\Omega}$$

for all $\mathbf{r},\,\mathbf{s}\in\mathbf{L}^2(\Omega)$.

(H.3) [LIPSCHITZ-CONTINUITY]. There exists a constant $\ell_A>0$ such that

$$\left|\int_\Omega \Big\{a(\cdot,\|\mathbf{r}\|)\,\mathbf{r} \,-\, a(\cdot,\|\mathbf{s}\|)\,\mathbf{s}\Big\}\cdot\mathbf{t}\right| \,\leq\, \ell_A\,\|\mathbf{r}-\mathbf{s}\|_{0,\Omega}\,\|\mathbf{t}\|_{0,\Omega}$$

for all $\mathbf{r},\,\mathbf{s},\,\mathbf{t}\in\mathbf{L}^2(\Omega)$.

For explicit examples of nonlinear coefficients a satisfying the above assumptions we refer to [20] and [29] (see, also [13] and [17]). We remark here that **(H.1)** guarantees that the nonlinear part of $\mathbf{A}$ is well-defined, whereas **(H.2)** and **(H.3)** allow to show that $\mathbf{A}$ becomes strongly monotone and Lipschitz-continuous. More precisely, we have the following main result.

Theorem 3.1. *Assume that the constant α_A from* **(H.2)** *is such that $\alpha_A > 1/4$. Then, the variational formulation* (3.5) *has a unique solution $(u,\lambda)\in\mathbf{H}$, and there exists a positive constant C_A, depending only on α_A, c_p* (cf. (3.3))*, and α_V* (cf. (2.10))*, such that*

$$\|(u,\lambda)\|_{\mathbf{H}} \,:=\, \Big\{\|u\|^2_{1,\Omega} \,+\, \|\lambda\|^2_{-1/2,\Gamma}\Big\}^{1/2} \,\leq\, C_A^{-1}\,\|f\|_{0,\Omega}\,. \tag{3.9}$$

Proof. Given (w,ξ), $(v,\mu)\in\mathbf{H}$, we begin by observing from the definition of $\mathbf{A}$ (cf. (3.6)) and the linearity of $\widetilde{\mathbf{A}}$, that

$$\begin{aligned}
[\mathbf{A}(w,\xi)-\mathbf{A}(v,\mu),(w,\xi)-(v,\mu)] \,&=\, \int_\Omega \Big\{a(\cdot,\|\nabla w\|)\,\nabla w \,-\, a(\cdot,\|\nabla v\|)\,\nabla v\Big\}\cdot\nabla(w-v)\\
&\quad +\quad [\widetilde{\mathbf{A}}\big((w,\xi)-(v,\mu)\big),(w,\xi)-(v,\mu)]\,,
\end{aligned}$$

from which, employing the hypothesis **(H.2)** and the estimate (3.8), it follows that

$$[\mathbf{A}(w,\xi)-\mathbf{A}(v,\mu),(w,\xi)-(v,\mu)] \,\geq\, \Big(\alpha_A-\frac{1}{2\delta}\Big)\,\|\nabla(w-v)\|_{0,\Omega}^2 \,+\, \Big(1-\frac{\delta}{2}\Big)\,\langle\xi-\mu,V(\xi-\mu)\rangle$$

for any $\delta>0$. In this way, in order to have positive constants multiplying the expressions on the right hand side of the foregoing equation, we require to choose δ such that

$$\alpha_A \,>\, \frac{1}{2\delta} \qquad \text{and} \qquad 1 \,>\, \frac{\delta}{2}\,,$$

that is $\dfrac{1}{2\alpha_A} \,<\, \delta \,<\, 2$, which is feasible under precisely our assumption that $\alpha_A \,>\, 1/4$. Thus, choosing for instance δ as the midpoint of its range of variability, and applying (3.3) and (2.10), we deduce the existence of a positive constant $C_A \,:=\, \min\Big\{c_p\Big(\alpha_A-\frac{1}{2\delta}\Big),\alpha_V\Big(1-\frac{\delta}{2}\Big)\Big\}$, such that

$$[\mathbf{A}(w,\xi)-\mathbf{A}(v,\mu),(w,\xi)-(v,\mu)] \,\geq\, C_A\Big\{\|w-v\|_{1,\Omega}^2 \,+\, \|\xi-\mu\|_{-1/2,\Gamma}^2\Big\}\,, \tag{3.10}$$

which proves that $\mathbf{A}$ is strongly monotone. In turn, it is straightforward to see from **(H.3)** and the boundedness of $\widetilde{\mathbf{A}}$, that $\mathbf{A}$ is Lipschitz-continuous, that is there exists a constant $L_A>0$, depending on ℓ_A, $\|\gamma\|$, $\|K\|$, and $\|V\|$ (cf. Lemma 2.1), such that

$$\|\mathbf{A}(w,\xi)-\mathbf{A}(v,\mu)\|_{\mathbf{H}'} \,\leq\, L_A\,\|(w,\xi)-(v,\mu)\|_{\mathbf{H}} \qquad \forall\,(w,\xi),\,(v,\mu)\in\mathbf{H}\,. \tag{3.11}$$

Therefore, applying the abstract result from [38, Theorem 3.3.23], we conclude the existence of a unique $(u,\lambda)\in\mathbf{H}$ solution to (3.5). Moreover, employing (3.10) with $(w,\xi)=(u,\lambda)$ and $(v,\mu)=(0,0)$, using that (u,λ) satisfies (3.5), and noting that $\mathbf{A}(0,0)$ is the null functional, we arrive at (3.9) and conclude the proof. □

3.3 The discrete formulation

In this part we introduce the Galerkin formulation associated with (3.5) and analyze its solvability and convergence properties under the same assumption of Theorem 3.1, that is that $\alpha_A \,>\, 1/4$. To this end, we let $\{\mathrm{H}_h^u\}_{h>0}$ and $\{\mathrm{H}_h^\lambda\}_{h>0}$ be families of finite dimensional subspaces of $\mathrm{H}^1_{\Gamma_0}(\Omega)$ and $\widehat{\mathrm{H}}^{-1/2}(\Gamma)$, respectively, and introduce the Galerkin scheme: Find $(u_h,\lambda_h)\in\mathbf{H}_h \,:=\, \mathrm{H}_h^u\times\mathrm{H}_h^\lambda$ such that

$$[\mathbf{A}(u_h,\lambda_h),(v_h,\mu_h)] \,=\, \mathbf{F}(v_h,\mu_h) \qquad \forall\,(v_h,\mu_h)\in\mathbf{H}_h\,. \tag{3.12}$$

Then, it is straightforward to see from (3.10) that $\mathbf{A}|_{\mathbf{H}_h}:\mathbf{H}_h\longrightarrow\mathbf{H}_h'$ is strongly monotone as well. In turn, for all (w_h,ξ_h), $(v_h,\mu_h)\in\mathbf{H}_h$ there holds

$$\begin{aligned}\|\mathbf{A}(w_h,\xi_h)-\mathbf{A}(v_h,\mu_h)\|_{\mathbf{H}_h'} \,&:=\, \sup_{\substack{(z_h,\eta_h)\in\mathbf{H}_h\\ (z_h,\eta_h)\neq\mathbf{0}}}\frac{[\mathbf{A}(w_h,\xi_h)-\mathbf{A}(v_h,\mu_h),(z_h,\eta_h)]}{\|(z_h,\eta_h)\|}\\ \leq\, \sup_{\substack{(z,\eta)\in\mathbf{H}\\ (z,\eta)\neq\mathbf{0}}}\frac{[\mathbf{A}(w_h,\xi_h)-\mathbf{A}(v_h,\mu_h),(z,\eta)]}{\|(z,\eta)\|} \,&=\, \|\mathbf{A}(w_h,\xi_h)-\mathbf{A}(v_h,\mu_h)\|_{\mathbf{H}'}\,,\end{aligned}$$

which, thanks to the estimate (3.11), implies the Lipschitz-continuity of $\mathbf{A}|_{\mathbf{H}_h}:\mathbf{H}_h\longrightarrow\mathbf{H}_h'$. Hence, an application of [38, Theorem 3.3.23] to this discrete setting yields the existence of a unique solution $(u_h,\lambda_h)\in\mathbf{H}_h$ to (3.12). In addition, proceeding exactly as in the proof of Theorem 3.1, we obtain

$$\|(u_h,\lambda_h)\|_{\mathbf{H}} \,\leq\, C_A^{-1}\,\|f\|_{0,\Omega}\,.$$

On the other hand, concerning the corresponding a priori error estimate, we first observe from (3.5) and (3.12) that there holds the Galerkin orthogonality type condition

$$[\mathbf{A}(u,\lambda) - \mathbf{A}(u_h,\lambda_h), (v_h,\mu_h)] \,=\, 0 \qquad \forall\, (v_h,\mu_h) \in \mathbf{H}_h \,. \tag{3.13}$$

Then, given an arbitrary $(w_h,\xi_h) \in \mathbf{H}_h$, we use the strong monotonicity of $\mathbf{A}|_{\mathbf{H}_h}$, the identity (3.13), and the Lipschitz-continuity of $\mathbf{A}$, to find that

$$\begin{aligned} C_A\,\|(u_h,\lambda_h) - (w_h,\xi_h)\|_{\mathbf{H}}^2 \,&\le\, [\mathbf{A}(u_h,\lambda_h) - \mathbf{A}(w_h,\xi_h), (u_h,\lambda_h) - (w_h,\xi_h)] \\ &=\, [\mathbf{A}(u,\lambda) - \mathbf{A}(w_h,\xi_h), (u_h,\lambda_h) - (w_h,\xi_h)] \\ &\le\, L_A\,\|(u,\lambda) - (w_h,\xi_h)\|_{\mathbf{H}}\,\|(u_h,\lambda_h) - (w_h,\xi_h)\|_{\mathbf{H}}\,, \end{aligned}$$

which yields

$$\|(u_h,\lambda_h) - (w_h,\xi_h)\|_{\mathbf{H}} \,\le\, L_A\,C_A^{-1}\,\|(u,\lambda) - (w_h,\xi_h)\|_{\mathbf{H}}\,.$$

Then, applying the triangle inequality and using the foregoing bound, we obtain

$$\|(u,\lambda) - (u_h,\lambda_h)\|_{\mathbf{H}} \,\le\, \Big\{1 + L_A\,C_A^{-1}\Big\}\,\|(u,\lambda) - (w_h,\xi_h)\|_{\mathbf{H}} \qquad \forall\,(w_h,\xi_h) \in \mathbf{H}_h\,,$$

which implies the Cea estimate

$$\|(u,\lambda) - (u_h,\lambda_h)\|_{\mathbf{H}} \,\le\, \Big\{1 + L_A\,C_A^{-1}\Big\}\,\mathrm{dist}\big((u,\lambda), \mathbf{H}_h\big)\,, \tag{3.14}$$

where, as usual, and from now on, dist stands for the distance of a given vector to the specified subspace.

We end this section by stressing that specific families of finite elements subspaces of $\mathrm{H}^1_{\Gamma_0}(\Omega)$ and $\widehat{\mathrm{H}}^{-1/2}(\Gamma)$, and their approximation properties, which together with (3.14) yield the rates of convergence of the Galerkin scheme (3.12), are available in the literature (see, e.g. [12], [39], [40]).

4 The Johnson & Nédélec coupling: dual-mixed approach

Here we employ again the Johnson & Nédélec coupling method, but now in its dual-mixed approach, to analyze the respective continuous and Galerkin schemes of the nonlinear problem introduced in Section 2. In this case, and for sake of simplicity of the analysis, we slightly change the original model (2.1) by considering a homogeneous Neumann boundary condition on Γ_0.

4.1 The continuous formulation

In order to set the dual-mixed formulation, we introduce the auxiliary unknowns

$$\mathbf{t} \,:=\, \nabla u \quad \text{and} \quad \boldsymbol{\sigma} \,:=\, a(\cdot, \|\mathbf{t}\|)\,\mathbf{t} \quad \text{in} \quad \Omega\,, \tag{4.1}$$

so that the transmission problem becomes

$$\begin{array}{rcll} \mathbf{t} \,=\, \nabla u\,, \quad \boldsymbol{\sigma} & = & a(\cdot,\|\mathbf{t}\|)\,\mathbf{t} & \text{in} \quad \Omega\,, \\ \mathrm{div}(\boldsymbol{\sigma}) & = & -f & \text{in} \quad \Omega\,, \\ \gamma_{\boldsymbol{n}}(\boldsymbol{\sigma}) & = & 0 & \text{on} \quad \Gamma_0\,, \\ \gamma(u) = \gamma(u_e) \quad \text{and} \quad \gamma_{\boldsymbol{n}}(\boldsymbol{\sigma}) & = & \gamma_{\boldsymbol{n}}(\nabla u_e) & \text{on} \quad \Gamma\,, \\ \Delta\, u_e & = & 0 & \text{in} \quad \mathcal{O}_e\,, \\ u_e(x) & = & O(\|x\|^{-1}) & \text{as} \quad \|x\| \to +\infty\,. \end{array} \tag{4.2}$$

Then, introducing the subspace of $\mathbf{H}(\mathrm{div};\Omega)$ (cf. (1.1)) given by

$$\mathbf{H}_0(\mathrm{div};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div};\Omega) : \quad \gamma_{\boldsymbol{n}}(\boldsymbol{\tau}) = 0 \quad \text{on} \quad \Gamma_0 \Big\}, \tag{4.3}$$

testing the first equation in the first row of (4.2) against $\boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega)$, integrating by parts, and recalling that $\psi = \gamma(u)$ on Γ, we arrive at

$$\int_\Omega \boldsymbol{\tau} \cdot \mathbf{t} \,+\, \int_\Omega u \,\mathrm{div}(\boldsymbol{\tau}) \,-\, \langle \gamma_{\boldsymbol{n}}(\boldsymbol{\tau}), \psi \rangle \,=\, 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega)\,. \tag{4.4}$$

It is worth mentioning here that, while the Neumann boundary condition on Γ_0 becomes an essential one within the present dual-mixed formulation, the fact that we are assuming it to be homogeneous, which, by the way, motivated the introduction of $\mathbf{H}_0(\mathrm{div};\Omega)$ (cf. (4.3)), avoids the need of incorporating $\gamma(u)|_{\Gamma_0}$ as the Lagrange multiplier taking care of it. Indeed, it is already well-known that if such condition were non-homogeneous, the latter procedure would be a suitable way to deal with it by means of a conforming scheme. On the other hand, testing the second equation in the first row of (4.2) against $\mathbf{s} \in \mathbf{L}^2(\Omega)$, we get

$$\int_\Omega a(\cdot, \|\mathbf{t}\|)\, \mathbf{t} \cdot \mathbf{s} \,-\, \int_\Omega \boldsymbol{\sigma} \cdot \mathbf{s} \,=\, 0 \qquad \forall\, \mathbf{s} \in \mathbf{L}^2(\Omega)\,. \tag{4.5}$$

Moreover, regarding the same equation, and for purposes that will be clarified along the analysis below, we also add the consistent equation

$$\kappa \int_\Omega \Big\{ \boldsymbol{\sigma} \,-\, a(\cdot, \|\mathbf{t}\|)\, \mathbf{t} \Big\} \cdot \boldsymbol{\tau} \,=\, 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega)\,, \tag{4.6}$$

where $\kappa > 0$ is a stabilization parameter to be chosen conveniently. Next, replacing $\gamma_{\boldsymbol{n}}(\nabla u_e)$ and λ in (2.9) by $\gamma_{\boldsymbol{n}}(\boldsymbol{\sigma})$, we obtain

$$W\psi \,+\, \big(\frac{1}{2}\mathrm{I} + K^{\mathrm{t}}\big)\big(\gamma_{\boldsymbol{n}}(\boldsymbol{\sigma})\big) \,=\, 0 \quad \text{on} \quad \Gamma\,,$$

which, tested against $\varphi \in \mathrm{H}^{1/2}(\Gamma)$, leads to

$$\langle W\psi, \varphi \rangle \,+\, \big\langle \big(\frac{1}{2}\mathrm{I} + K^{\mathrm{t}}\big)\big(\gamma_{\boldsymbol{n}}(\boldsymbol{\sigma})\big), \varphi \big\rangle \,=\, 0 \qquad \forall\, \varphi \in \mathrm{H}^{1/2}(\Gamma)\,. \tag{4.7}$$

Finally, from the equation in the second row of (4.2) we have

$$\int_\Omega v \,\mathrm{div}(\boldsymbol{\sigma}) \,=\, -\int_\Omega f\, v \qquad \forall\, v \in \mathrm{L}^2(\Omega)\,. \tag{4.8}$$

In this way, adding the equations (4.4), (4.5), (4.6) and (4.7), keeping (4.8) as it is, performing some algebraic rearrangements, denoting $\vec{\mathbf{t}} := (\mathbf{t}, \boldsymbol{\sigma}, \psi)$, $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, \varphi)$, $\mathbf{H} := \mathbf{L}^2(\Omega) \times \mathbf{H}_0(\mathrm{div};\Omega) \times \mathrm{H}^{1/2}(\Gamma)$, and $\mathbf{Q} := \mathrm{L}^2(\Omega)$, we arrive at the boundary-field equation formulation: Find $(\vec{\mathbf{t}}, u) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} [\mathbf{A}(\vec{\mathbf{t}}), \vec{\mathbf{s}}] \,+\, [\mathbf{B}(\vec{\mathbf{s}}), u] \quad &= \quad \mathbf{F}(\vec{\mathbf{s}}) \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,, \\ [\mathbf{B}(\vec{\mathbf{t}}), v] \quad &= \quad \mathbf{G}(v) \qquad \forall\, v \in \mathbf{Q}\,, \end{aligned} \tag{4.9}$$

where the nonlinear operator $\mathbf{A} : \mathbf{H} \to \mathbf{H}'$ and $\mathbf{B} \in \mathcal{L}\big(\mathbf{H}, \mathbf{Q}'\big)$ are defined for each $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi) \in \mathbf{H}$ as

$$[\mathbf{A}(\vec{\mathbf{r}}), \vec{\mathbf{s}}] \,:=\, \int_\Omega a(\cdot, \|\mathbf{r}\|)\, \mathbf{r} \cdot \mathbf{s} \,-\, \kappa \int_\Omega a(\cdot, \|\mathbf{r}\|)\, \mathbf{r} \cdot \boldsymbol{\tau} \,+\, [\widetilde{\mathbf{A}}(\vec{\mathbf{r}}), \vec{\mathbf{s}}] \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,, \tag{4.10}$$

with $\widetilde{\mathbf{A}} \in \mathcal{L}\big(\mathbf{H}, \mathbf{H}'\big)$ given by

$$\begin{aligned}[\widetilde{\mathbf{A}}(\vec{\mathbf{r}}), \vec{\mathbf{s}}] \,:=\, \kappa \int_\Omega \boldsymbol{\zeta} \cdot \boldsymbol{\tau} \,-\, \int_\Omega \boldsymbol{\zeta} \cdot \mathbf{s} \,+\, \int_\Omega \boldsymbol{\tau} \cdot \mathbf{r} \,-\, \langle \gamma_{\boldsymbol{n}}(\boldsymbol{\tau}), \phi \rangle \\ +\, \langle W\phi, \varphi \rangle \,+\, \Big\langle \big(\frac{1}{2}\mathrm{I} + K^{\mathrm{t}}\big)\big(\gamma_{\boldsymbol{n}}(\boldsymbol{\zeta})\big), \varphi \Big\rangle \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,,\end{aligned} \tag{4.11}$$

and

$$[\mathbf{B}(\vec{\mathbf{r}}), v] \,:=\, \int_\Omega v \,\mathrm{div}(\boldsymbol{\zeta}) \qquad \forall\, v \in \mathbf{Q}\,, \tag{4.12}$$

respectively, whereas $\mathbf{F} \in \mathbf{H}'$ and $\mathbf{G} \in \mathbf{Q}'$ are set as

$$\mathbf{F}(\vec{\mathbf{s}}) \,:=\, 0 \quad \forall\, \vec{\mathbf{s}} \in \mathbf{H} \quad \text{and} \quad \mathbf{G}(v) \,:=\, -\int_\Omega f\, v \quad \forall\, v \in \mathbf{Q}\,. \tag{4.13}$$

4.2 Solvability analysis of the continuous formulation

We aim here to analyze the solvability of (4.9) by applying the abstract result given by Theorem B.1 (see Appendix B), for which we need to state first the main properties of the forms involved. We begin with the inf-sup condition for $\mathbf{B}$.

Lemma 4.1. *There exists $\beta > 0$ such that there holds*

$$\sup_{\substack{\vec{\mathbf{s}} \in \mathbf{H} \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{s}}), v]}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geq\, \beta\, \|v\|_{\mathbf{Q}} \quad \forall\, v \in \mathbf{Q}\,. \tag{4.14}$$

Proof. Given $v \in \mathbf{Q}$, and recalling that $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, \varphi)$, we first observe that

$$\sup_{\substack{\vec{\mathbf{s}} \in \mathbf{H} \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{s}}), v]}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,=\, \sup_{\substack{\boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega v \,\mathrm{div}(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{div};\Omega}}\,. \tag{4.15}$$

Then, we proceed as in [15, Section 2.4.1] and let $z \in \mathrm{H}^1_\Gamma(\Omega) := \Big\{ z \in \mathrm{H}^1(\Omega) : \quad \gamma(z) = 0 \quad \text{on} \quad \Gamma \Big\}$ be the unique weak solution, guaranteed by the Lax-Milgram Lemma, of the boundary value problem

$$\Delta z = v \quad \text{in} \quad \Omega\,, \quad \gamma(z) = 0 \quad \text{on} \quad \Gamma\,, \quad \gamma_{\boldsymbol{n}}(\nabla z) = 0 \quad \text{on} \quad \Gamma_0\,.$$

The corresponding continuous dependence result establishes that $\|z\|_{1,\Omega} \,\leq\, \widetilde{c}_p^{\,-1}\, \|v\|_{0,\Omega}$, where, similarly to (3.3), $\widetilde{c}_p$ is the constant yielding the Poincaré inequality for $\mathrm{H}^1_\Gamma(\Omega)$. Thus, setting $\widetilde{\boldsymbol{\tau}} := \nabla z$, we notice that $\mathrm{div}(\widetilde{\boldsymbol{\tau}}) = v$ and $\gamma_{\boldsymbol{n}}(\widetilde{\boldsymbol{\tau}}) = 0$ on Γ_0, which confirms that $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}_0(\mathrm{div};\Omega)$. Moreover, $\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div};\Omega} \,=\, \big\{ \|\nabla z\|^2_{0,\Omega} + \|v\|^2_{0,\Omega} \big\}^{1/2} \,\leq\, \big\{ 1 + \widetilde{c}_p^{\,-2} \big\}^{1/2} \|v\|_{0,\Omega}$, and hence, bounding below (4.15) with $\boldsymbol{\tau} = \widetilde{\boldsymbol{\tau}}$, we get the required inequality (4.14) with $\beta := \big\{ 1 + \widetilde{c}_p^{\,-2} \big\}^{-1/2}$. $\square$

At this point we remark that for the actual purpose of the above lemma, it would have sufficed to prove the surjectivity of $\mathbf{B}$, which is basically achieved with the fact that $\mathrm{div}(\widetilde{\boldsymbol{\tau}}) = v$. The explicit knowledge of the constant β for this continuous inf-sup condition is not as relevant as it is for the respective discrete one. Next, denoting by $\mathbf{V}$ the null space of $\mathbf{B}$, we easily find that

$$\mathbf{V} \,:=\, \Big\{ \vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi) \in \mathbf{H} : \quad \mathrm{div}(\boldsymbol{\zeta}) = 0 \quad \text{in} \quad \Omega \Big\}\,, \tag{4.16}$$

so that we now aim to look at the behaviour of the linear component $\widetilde{\mathbf{A}}$ of $\mathbf{A}$ with respect to $\mathbf{V}$. More precisely, for each $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi) \in \mathbf{V}$ we obtain from (4.11) and the second identity in the first row of (2.4) that

$$\begin{aligned}
[\widetilde{\mathbf{A}}(\vec{\mathbf{r}}), \vec{\mathbf{r}}] &= \kappa \, \|\boldsymbol{\zeta}\|_{0,\Omega}^2 \, - \, \langle \gamma_{\boldsymbol{n}}(\boldsymbol{\zeta}), \phi \rangle \, + \, \langle W\phi, \phi \rangle \, + \, \big\langle \big(\frac{1}{2}\mathrm{I} + K^{\mathrm{t}}\big)\big(\gamma_{\boldsymbol{n}}(\boldsymbol{\zeta})\big), \phi \big\rangle \\
&= \kappa \, \|\boldsymbol{\zeta}\|_{0,\Omega}^2 \, + \, \langle W\phi, \phi \rangle \, + \, \big\langle \big(-\frac{1}{2}\mathrm{I} + K^{\mathrm{t}}\big)\big(\gamma_{\boldsymbol{n}}(\boldsymbol{\zeta})\big), \phi \big\rangle \\
&= \kappa \, \|\boldsymbol{\zeta}\|_{0,\Omega}^2 \, + \, \langle W\phi, \phi \rangle \, + \, \big\langle \gamma_{\boldsymbol{n}}(\boldsymbol{\zeta}), \big(-\frac{1}{2}\mathrm{I} + K\big)\phi \big\rangle \\
&= \kappa \, \|\boldsymbol{\zeta}\|_{0,\Omega}^2 \, + \, \langle W\phi, \phi \rangle \, + \, \big\langle \gamma_{\boldsymbol{n}}(\boldsymbol{\zeta}), \gamma\big(\mathcal{D}_i(\phi)\big) \big\rangle \, .
\end{aligned}$$

Then, integrating by parts backwardly the last term in the foregoing equation, and bearing in mind that $\mathrm{div}(\boldsymbol{\zeta}) = 0$ in Ω and $\gamma_{\boldsymbol{n}}(\boldsymbol{\zeta}) = 0$ on Γ_0, we arrive at

$$[\widetilde{\mathbf{A}}(\vec{\mathbf{r}}), \vec{\mathbf{r}}] = \kappa \, \|\boldsymbol{\zeta}\|_{0,\Omega}^2 \, + \, \langle W\phi, \phi \rangle \, + \, \int_\Omega \nabla \mathcal{D}_i(\phi) \cdot \boldsymbol{\zeta} \, ,$$

from which, applying Cauchy-Schwarz's and Young's inequalities, the latter with parameter $\delta > 0$, and employing that $\|\nabla \mathcal{D}_i(\phi)\|_{0,\Omega}^2 \, \le \, \|\nabla \mathcal{D}(\phi)\|_{0,\mathrm{R}^n}^2 = \langle W\phi, \phi \rangle$, we find that

$$[\widetilde{\mathbf{A}}(\vec{\mathbf{r}}), \vec{\mathbf{r}}] \, \ge \, \left(\kappa - \frac{1}{2\delta} \right) \, \|\boldsymbol{\zeta}\|_{0,\Omega}^2 \, + \, \left(1 - \frac{\delta}{2} \right) \, \langle W\phi, \phi \rangle \quad \forall \, \vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi) \in \mathbf{V} \, . \tag{4.17}$$

Note here that the constants multiplying $\|\boldsymbol{\zeta}\|_{0,\Omega}^2$ and $\langle W\phi, \phi \rangle$ become positive if $\frac{1}{2\kappa} < \delta < 2$, which constitutes a feasible range for δ if κ is chosen such that $\kappa > \frac{1}{4}$. Having established the above, we now look at the whole nonlinear operator $\mathbf{A}$ in order to derive conditions under which it becomes strongly monotone on $\mathbf{V}$. In fact, using the assumptions (**H.2**) and (**H.3**), and applying again Young's inequality, but now with a parameter $\varepsilon > 0$, we deduce from (4.10) that for each $\vec{\mathbf{q}} \in \mathbf{H}$, and for all $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi)$, $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, \varphi) \in \mathbf{V}$ there holds

$$\begin{aligned}
[\mathbf{A}(\vec{\mathbf{r}} + \vec{\mathbf{q}}) - \mathbf{A}(\vec{\mathbf{s}} + \vec{\mathbf{q}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}] \, &= \, [\mathbf{A}(\vec{\mathbf{r}} + \vec{\mathbf{q}}) - \mathbf{A}(\vec{\mathbf{s}} + \vec{\mathbf{q}}), (\vec{\mathbf{r}} + \vec{\mathbf{q}}) - (\vec{\mathbf{s}} + \vec{\mathbf{q}})] \\
&\ge \, \alpha_A \, \|\mathbf{r} - \mathbf{s}\|_{0,\Omega}^2 \, - \, \kappa \, \ell_A \, \|\mathbf{r} - \mathbf{s}\|_{0,\Omega} \, \|\boldsymbol{\zeta} - \boldsymbol{\tau}\|_{0,\Omega} \, + \, [\widetilde{\mathbf{A}}(\vec{\mathbf{r}} - \vec{\mathbf{s}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}] \\
&\ge \, \left(\alpha_A \, - \, \frac{\kappa \, \ell_A}{2\varepsilon} \right) \, \|\mathbf{r} - \mathbf{s}\|_{0,\Omega}^2 \, - \, \frac{\kappa \, \ell_A \, \varepsilon}{2} \, \|\boldsymbol{\zeta} - \boldsymbol{\tau}\|_{0,\Omega}^2 \, + \, [\widetilde{\mathbf{A}}(\vec{\mathbf{r}} - \vec{\mathbf{s}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}] \, .
\end{aligned} \tag{4.18}$$

Thus, bounding $[\widetilde{\mathbf{A}}(\vec{\mathbf{r}} - \vec{\mathbf{s}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}]$ according to (4.17), replacing the resulting estimate back into (4.18), and noting that $\|\boldsymbol{\zeta}\|_{0,\Omega} = \|\boldsymbol{\zeta}\|_{\mathrm{div};\Omega}$ for all $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi) \in \mathbf{V}$, we obtain

$$[\mathbf{A}(\vec{\mathbf{r}} + \vec{\mathbf{q}}) - \mathbf{A}(\vec{\mathbf{s}} + \vec{\mathbf{q}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}] \, \ge \, C_1 \, \|\mathbf{r} - \mathbf{s}\|_{0,\Omega}^2 \, + \, C_2 \, \|\boldsymbol{\zeta} - \boldsymbol{\tau}\|_{\mathrm{div};\Omega}^2 \, + \, C_3 \, \langle W(\phi - \varphi), \phi - \varphi \rangle \, , \tag{4.19}$$

for each $\vec{\mathbf{q}} \in \mathbf{H}$, and for all $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi)$, $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, \varphi) \in \mathbf{V}$, where

$$C_1 \, := \, \left(\alpha_A - \frac{\kappa \, \ell_A}{2\varepsilon} \right) , \quad C_2 \, := \, \left(\left(\kappa - \frac{1}{2\delta} \right) - \frac{\kappa \, \ell_A \, \varepsilon}{2} \right) , \qquad \text{and} \quad C_3 \, := \, \left(1 - \frac{\delta}{2} \right) . \tag{4.20}$$

Hence, in order for the above constants to be positive, we need, in addition to the already stated conditions $\frac{1}{2\kappa} < \delta < 2$ and $\kappa > \frac{1}{4}$, that $\alpha_A > \frac{\kappa \, \ell_A}{2\varepsilon}$ and $\left(\kappa - \frac{1}{2\delta} \right) > \frac{\kappa \, \ell_A \, \varepsilon}{2}$, which turns into the constraint $\frac{\kappa \, \ell_A}{2 \, \alpha_A} < \varepsilon < \frac{2}{\ell_A} \left(1 - \frac{1}{2\kappa\delta} \right)$. Certainly, the latter is a feasible range for ε if the difference between the upper and lower ends of the interval is positive, which is equivalent to requiring that $\alpha_A > \frac{\ell_A^2}{4} \, g(\kappa, \delta)$, where

$$g(\kappa, \delta) \, := \, \frac{\kappa}{\left(1 - \frac{1}{2\kappa\delta}\right)} \qquad \forall \, \kappa \in \left(\tfrac{1}{4}, +\infty \right) , \quad \forall \, \delta \in \left(\tfrac{1}{2\kappa}, 2 \right) . \tag{4.21}$$

In Appendix A below we show that the infimum of g within the above specified region for κ and δ is given by 1, which is attained for $\kappa = \frac{1}{2}$ and $\delta = \frac{1}{\kappa} = 2$. In this way, we conclude that a sufficient condition for the constants in (4.20) to be positive is that $\alpha_A > \frac{\ell_A^2}{4}$. Indeed, this inequality is equivalent to stating $\frac{2\alpha_A}{\ell_A^2} > \frac{1}{2}$, so that choosing $\kappa \in \left(\frac{1}{2}, \frac{2\alpha_A}{\ell_A^2}\right)$ and $\delta = \frac{1}{\kappa}$, we satisfy the conditions required for these parameters. In addition, it follows in this case that $\frac{\ell_A^2}{4}\, g(\kappa,\delta) = \frac{\ell_A^2\,\kappa}{2} < \alpha_A$, which confirms the feasibility of the range for ε, that is $\varepsilon \in \left(\frac{\kappa\,\ell_A}{2\alpha_A}, \frac{1}{\ell_A}\right)$. Hence, choosing κ, δ, and ε as indicated, the constants from (4.20) become

$$C_1 := \frac{\alpha_A}{\varepsilon}\left(\varepsilon - \frac{\kappa\,\ell_A}{2\alpha_A}\right), \quad C_2 := \frac{\kappa\ell_A}{2}\left(\frac{1}{\ell_A} - \varepsilon\right), \qquad \text{and} \quad C_3 := \frac{1}{\kappa}\left(\kappa - \frac{1}{2}\right),$$

which are clearly all positive. Certainly, the values of these constants are maximized if κ and ε are chosen as the midpoints of their respective ranges. In any case, from the previous analysis and (4.19) we conclude that there exists a positive constant $C_A := \min\Big\{C_1, C_2, C_3\Big\}$, depending on ℓ_A and α_A, in particular on the difference $4\alpha_A - \ell_A^2$, such that

$$[\mathbf{A}(\vec{\mathbf{r}}+\vec{\mathbf{q}}) - \mathbf{A}(\vec{\mathbf{s}}+\vec{\mathbf{q}}), \vec{\mathbf{r}}-\vec{\mathbf{s}}] \,\geq\, C_A\Big\{\|\mathbf{r}-\mathbf{s}\|_{0,\Omega}^2 \,+\, \|\boldsymbol{\zeta}-\boldsymbol{\tau}\|_{\mathrm{div};\Omega}^2 \,+\, \langle W(\phi-\varphi), \phi-\varphi\rangle\Big\}, \tag{4.22}$$

for each $\vec{\mathbf{q}} \in \mathbf{H}$, and for all $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi)$, $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, \varphi) \in \mathbf{V}$.

Unfortunately, this estimate does not guarantee the strong monotonicity of $\mathbf{A}$ on $\mathbf{V}$ since the ellipticity property of W given by (2.11) (cf. Lemma 2.1) holds for $\mathrm{H}_0^{1/2}(\Gamma)$ but not for $\mathrm{H}^{1/2}(\Gamma)$, and therefore the last term in (4.22) does not yield the required norm for $\phi - \varphi$. However, in what follows we circumvent this difficulty by observing that there is actually a lack of uniqueness for the nonlinear problem (4.9). To confirm this, we now consider the homogeneous linear problem arising from (4.9) after replacing $\mathbf{A}$ by $\widetilde{\mathbf{A}}$, and $\mathbf{F}$ and $\mathbf{G}$ by the respective null functionals, that is:

$$\begin{aligned} [\widetilde{\mathbf{A}}(\vec{\mathbf{t}}), \vec{\mathbf{s}}] \,+\, [\mathbf{B}(\vec{\mathbf{s}}), u] &= 0 \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,, \\ [\mathbf{B}(\vec{\mathbf{t}}), v] &= 0 \qquad \forall\, v \in \mathbf{Q}\,. \end{aligned} \tag{4.23}$$

Then, it is easy to see that if $(\vec{\mathbf{t}}, u) \in \mathbf{H}\times\mathbf{Q}$ is a solution of (4.23) with $\vec{\mathbf{t}} = (\mathbf{0}, \boldsymbol{\sigma}, \psi)$, necessarily there holds $\boldsymbol{\sigma} = \mathbf{0}$, $\psi = c$, and $u = c$, with $c \in \mathrm{R}$. In fact, it readily follows from the second equation of (4.23) that $\mathrm{div}(\boldsymbol{\sigma}) = 0$, so that taking $\vec{\mathbf{s}} = \vec{\mathbf{t}}$ in the first equation, and using (4.17) and (2.11), we deduce for $\boldsymbol{\sigma}$ and ψ. In turn, knowing this, the first equation becomes

$$0 = -\langle \gamma_{\boldsymbol{n}}(\boldsymbol{\tau}), c\rangle + \int_\Omega u\,\mathrm{div}(\boldsymbol{\tau}) = \int_\Omega (u-c)\,\mathrm{div}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega)\,,$$

which, thanks to the inf-sup condition (4.14), yields $u = c$. Conversely, it is straightforward to check that for each $c \in \mathrm{R}$, the pair $(\vec{\mathbf{t}}, u) := \big((\mathbf{0}, \mathbf{0}, c), c\big) \in \mathbf{H}\times\mathbf{Q}$ solves (4.23).

According to the above discussion, we conclude that given any solution $(\vec{\mathbf{t}}, u) \in \mathbf{H}\times\mathbf{Q}$ of (4.9), $(\vec{\mathbf{t}}, u) + \big((\mathbf{0}, \mathbf{0}, c), c\big)$ also becomes a solution for any $c \in \mathrm{R}$. Therefore, in order to avoid this non-uniqueness, we could look either for $u \in \mathrm{L}_0^2(\Omega) := \big\{v \in \mathrm{L}^2(\Omega) : \int_\Omega v = 0\big\}$ instead of $u \in \mathrm{L}^2(\Omega)$, or for $\varphi \in \mathrm{H}_0^{1/2}(\Gamma)$ instead of $\varphi \in \mathrm{H}^{1/2}(\Gamma)$. For simplicity of the analysis, and particularly due to the $\mathrm{H}_0^{1/2}(\Gamma)$-ellipticity of W (cf. (2.11)), we prefer the second option. Thus, instead of $\mathbf{H}$, we now introduce the space $\mathbf{H}_0 := \mathbf{L}^2(\Omega)\times\mathbf{H}_0(\mathrm{div};\Omega)\times\mathrm{H}_0^{1/2}(\Gamma)$, and observe that testing the first equation of (4.9) against $\vec{\mathbf{s}} \in \mathbf{H}$ is equivalent to doing it against $\vec{\mathbf{s}} \in \mathbf{H}_0$, which follows from the fact that $W(1) = 0$, identity that is implicit in (2.11), and $K(1) = -\frac{1}{2}$ (cf. [31], [40]). In this way, from now on we consider, in replacement of (4.9), the formulation: Find $(\vec{\mathbf{t}}, u) \in \mathbf{H}_0 \times \mathbf{Q}$ such that

$$\begin{aligned} [\mathbf{A}(\vec{\mathbf{t}}), \vec{\mathbf{s}}] + [\mathbf{B}(\vec{\mathbf{s}}), u] &= \mathbf{F}(\vec{\mathbf{s}}) \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{H}_0\,, \\ [\mathbf{B}(\vec{\mathbf{t}}), v] &= \mathbf{G}(v) \qquad \forall\, v \in \mathbf{Q}\,. \end{aligned} \tag{4.24}$$

Then, instead of $\mathbf{V}$ (cf. (4.16)), the null space of $\mathbf{B}$ now becomes

$$\mathbf{V}_0 := \Big\{ \vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi) \in \mathbf{H}_0 : \quad \mathrm{div}(\boldsymbol{\zeta}) = 0 \quad \text{in} \quad \Omega \Big\}\,, \tag{4.25}$$

and hence, thanks to (4.22) and (2.11), and defining $\widetilde{\alpha} := C_A \min\big\{1, \alpha_W\big\}$, we deduce that

$$[\mathbf{A}(\vec{\mathbf{r}}+\vec{\mathbf{q}}) - \mathbf{A}(\vec{\mathbf{s}}+\vec{\mathbf{q}}), \vec{\mathbf{r}}-\vec{\mathbf{s}}] \,\geq\, \widetilde{\alpha} \Big\{ \|\mathbf{r}-\mathbf{s}\|_{0,\Omega}^2 + \|\boldsymbol{\zeta}-\boldsymbol{\tau}\|_{\mathrm{div};\Omega}^2 + \|\phi-\varphi\|_{1/2,\Gamma}^2 \Big\}\,, \tag{4.26}$$

for each $\vec{\mathbf{q}} \in \mathbf{H}$, and for all $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi)$, $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, \varphi) \in \mathbf{V}_0$, which proves the strong monotonicity of $\mathbf{A}$ on $\mathbf{V}_0$. In turn, starting from the definition of the nonlinear operator $\mathbf{A}$ (cf. (4.10)), and employing hypothesis (**H.3**), Cauchy-Schwarz's inequality, the boundedness of the normal trace $\gamma_{\boldsymbol{n}}$ (cf. [15, Theorem 1.7]), and the corresponding properties of the boundary integral operators W and $K^{\mathbf{t}}$ (cf. Lemma 2.1), we deduce the existence of a positive constant $\widetilde{L}_A$, depending on ℓ_A, κ, and the norms of W, $K^{\mathbf{t}}$, and $\gamma_{\boldsymbol{n}}$, such that

$$\|\mathbf{A}(\vec{\mathbf{r}}) - \mathbf{A}(\vec{\mathbf{s}})\|_{\mathbf{H}'} \,\leq\, \widetilde{L}_A \, \|\vec{\mathbf{r}} - \vec{\mathbf{s}}\|_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{r}}, \vec{\mathbf{s}} \in \mathbf{H}\,, \tag{4.27}$$

which shows the Lipschitz-continuity of $\mathbf{A}$.

We are now in position to establish the main result concerning (4.24).

Theorem 4.1. *In addition to the hypotheses* (**H.1**), (**H.2**), *and* (**H.3**), *assume that* $\alpha_A > \dfrac{\ell_A^2}{4}$. *Then, problem* (4.24) *has a unique solution* $(\vec{\mathbf{t}}, u) \in \mathbf{H}_0 \times \mathbf{Q}$. *Moreover, there hold*

$$\|\vec{\mathbf{t}}\|_{\mathbf{H}} \,\leq\, \frac{1}{\beta} \left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}} \right) \|f\|_{0,\Omega} \quad \textit{and} \quad \|u\|_{\mathbf{Q}} \,\leq\, \frac{\widetilde{L}_A}{\beta^2} \left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}} \right) \|f\|_{0,\Omega}\,. \tag{4.28}$$

Proof. In virtue of Lemma 4.1 (which is easily seen to be valid for $\mathbf{H}_0$ as well, instead of $\mathbf{H}$), (4.26), and (4.27), the proof reduces to a straightforward application of Theorem B.1. In particular, the a priori estimates in (4.28) follow from (B.5) and (B.6) after observing that in this case $\mathbf{F} = \mathbf{0}$, $\|\mathbf{G}\|_{\mathbf{Q}'} = \|f\|_{0,\Omega}$, and $\mathbf{A}(\mathbf{0}) = \mathbf{0}$. ☐

4.3 The discrete formulation

In what follows we introduce a discrete formulation associated with (4.24) and discuss its solvability and convergence properties. For this purpose, we now let $\big\{\mathrm{H}_h^{\mathbf{t}}\big\}_{h>0}$, $\big\{\mathrm{H}_h^{\boldsymbol{\sigma}}\big\}_{h>0}$, $\big\{\mathrm{H}_h^{\psi}\big\}_{h>0}$, and $\big\{\mathrm{H}_h^{u}\big\}_{h>0}$, be families of finite dimensional subspaces of $\mathbf{L}^2(\Omega)$, $\mathbf{H}_0(\mathrm{div};\Omega)$, $\mathrm{H}_0^{1/2}(\Gamma)$, and $\mathrm{L}^2(\Omega)$, respectively, set $\mathbf{H}_{0,h} := \mathrm{H}_h^{\mathbf{t}} \times \mathrm{H}_h^{\boldsymbol{\sigma}} \times \mathrm{H}_h^{\psi}$, $\mathbf{Q}_h := \mathrm{H}_h^{u}$, and consider the Galerkin scheme: Find $(\vec{\mathbf{t}}_h, u_h) := \big((\mathbf{t}_h, \boldsymbol{\sigma}_h, \psi_h), u_h\big) \in \mathbf{H}_{0,h} \times \mathbf{Q}_h$ such that

$$\begin{aligned} [\mathbf{A}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}_h] + [\mathbf{B}(\vec{\mathbf{s}}_h), u_h] &= \mathbf{F}(\vec{\mathbf{s}}_h) \qquad \forall\, \vec{\mathbf{s}}_h := (\mathbf{s}_h, \boldsymbol{\tau}_h, \varphi_h) \in \mathbf{H}_{0,h}\,, \\ [\mathbf{B}(\vec{\mathbf{t}}_h), v_h] &= \mathbf{G}(v_h) \qquad \forall\, v_h \in \mathbf{Q}_h\,. \end{aligned} \tag{4.29}$$

In this case, and aiming to apply later on Theorem B.2, the solvability analysis of (4.29) requires to identify first the null space of the discrete restriction $\mathbf{B}|_{\mathbf{H}_{0,h}} : \mathbf{H}_{0,h} \to \mathbf{Q}_h'$, which is defined as

$$\mathbf{V}_{0,h} := \left\{ \vec{\mathbf{r}}_h := (\mathbf{r}_h, \boldsymbol{\zeta}_h, \phi_h) \in \mathbf{H}_{0,h} : \quad [\mathbf{B}(\vec{\mathbf{r}}_h), v_h] = 0 \quad \forall\, v_h \in \mathbf{Q}_h \right\},$$

that is

$$\mathbf{V}_{0,h} := \left\{ \vec{\mathbf{r}}_h := (\mathbf{r}_h, \boldsymbol{\zeta}_h, \phi_h) \in \mathbf{H}_{0,h} : \quad \int_\Omega v_h \operatorname{div}(\boldsymbol{\zeta}_h) = 0 \quad \forall\, v_h \in \mathbf{Q}_h \right\}. \tag{4.30}$$

Having this in mind, we assume the following hypothesis on $\mathrm{H}_h^{\boldsymbol{\sigma}}$ and $\mathbf{Q}_h = \mathrm{H}_h^u$:

($\mathbf{H.4}$) $\operatorname{div}\big(\mathrm{H}_h^{\boldsymbol{\sigma}}\big) \subseteq \mathrm{H}_h^u$,

thanks to which (4.30) becomes

$$\mathbf{V}_{0,h} = \left\{ \vec{\mathbf{r}}_h := (\mathbf{r}_h, \boldsymbol{\zeta}_h, \phi_h) \in \mathbf{H}_{0,h} : \quad \operatorname{div}(\boldsymbol{\zeta}_h) = 0 \quad \text{in} \quad \Omega \right\},$$

and hence $\mathbf{V}_{0,h}$ is clearly contained in $\mathbf{V}_0$ (cf. (4.25)). In this way, it follows straightforwardly from (4.26) that the discrete restriction $\mathbf{A}|_{\mathbf{H}_{0,h}} : \mathbf{H}_{0,h} \to \mathbf{H}_{0,h}'$ is strongly monotone on $\mathbf{V}_{0,h}$ with the same constant $\widetilde{\alpha}$ from (4.26). Thus, in order to complete the hypotheses of Theorem B.2, we need to satisfy the corresponding discrete inf-sup condition (B.12), for which we incorporate the following additional assumption:

($\mathbf{H.5}$) There exists a positive constant $\beta_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{H}_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega v_h \operatorname{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\operatorname{div};\Omega}} \;\geq\; \beta_{\mathtt{d}}\, \|v_h\|_{0,\Omega} \qquad \forall\, v_h \in \mathbf{Q}_h\,.$$

Then, proceeding analogously to the first part of the proof of Lemma 4.1, and employing now ($\mathbf{H.5}$), we find that for each $v_h \in \mathbf{Q}_h$ there holds

$$\sup_{\substack{\vec{\mathbf{s}}_h \in \mathbf{H}_{0,h} \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{s}}_h), v_h]}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \;=\; \sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{H}_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega v_h \operatorname{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\operatorname{div};\Omega}} \;\geq\; \beta_{\mathtt{d}}\, \|v_h\|_{0,\Omega}\,,$$

which provides the missing hypothesis. For specific examples of finite element subspaces $\mathrm{H}_h^{\boldsymbol{\sigma}}$ and H_h^u verifying ($\mathbf{H.4}$) and ($\mathbf{H.5}$), we refer to [3], [5], and [15], among several other suitable references.

Consequently, we are now able to establish the following result providing the well posedness of (4.29) and the associated Cea estimate.

Theorem 4.2. *In addition to the hypotheses of Theorem* 4.1*, assume that* ($\mathbf{H.4}$) *and* ($\mathbf{H.5}$) *are satisfied. Then, problem* (4.29) *has a unique solution* $(\vec{\mathbf{t}}_h, u_h) \in \mathbf{H}_{0,h} \times \mathbf{Q}_h$*, and there hold*

$$\|\vec{\mathbf{t}}_h\|_{\mathbf{H}} \;\leq\; \frac{1}{\beta_{\mathtt{d}}} \left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}} \right) \|f\|_{0,\Omega} \quad \textit{and} \quad \|u_h\|_{\mathbf{Q}} \;\leq\; \frac{\widetilde{L}_A}{\beta_{\mathtt{d}}^2} \left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}} \right) \|f\|_{0,\Omega}\,.$$

Moreover, recalling that $(\vec{\mathbf{t}}, u) \in \mathbf{H}_0 \times \mathbf{Q}$ *stands for the unique solution of* (4.24)*, the corresponding a priori error estimates reduce to*

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} \;\leq\; \left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}} \right) \left(1 + \frac{1}{\beta_{\mathtt{d}}} \right) \operatorname{dist}\big(\vec{\mathbf{t}}, \mathbf{H}_{0,h}\big)$$

and

$$\|u - u_h\|_{0,\Omega} \,\le\, \left(1 + \frac{1}{\beta_{\mathtt{d}}}\right) \mathrm{dist}(u, \mathbf{Q}_h) \,+\, \frac{\widetilde{L}_A}{\beta_{\mathtt{d}}} \left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}}\right) \left(1 + \frac{1}{\beta_{\mathtt{d}}}\right) \mathrm{dist}\big(\vec{\mathbf{t}}, \mathbf{H}_{0,h}\big)\,.$$

Proof. According to the previous discussion, the proof follows from a direct application of Theorem B.2 and the simplified Cea estimates (B.22) and (B.23), observing again that $\mathbf{F} = \mathbf{0}$, $\|\mathbf{G}\|_{\mathbf{Q}'} = \|f\|_{0,\Omega}$, and $\mathbf{A}(\mathbf{0}) = \mathbf{0}$, noting from (4.12) that $\|\mathbf{B}\| \le 1$, and using the fact that $\mathbf{V}_{0,h} \subseteq \mathbf{V}_0$. $\square$

In addition to the already stated remark on $\mathrm{H}_h^{\boldsymbol{\sigma}}$ and H_h^u, and similarly as we did at the end of Section 3, we emphasize here that specific families of the required finite element subspaces, and their approximation properties, can be found at several places in the literature.

5 The modified Costabel & Han coupling

In this section we apply the primal and dual-mixed approaches of the new coupling procedure introduced in [27] in the context of VEM and BEM (see also [28]), to analyze, via FEM or mixed-FEM instead of VEM, the respective continuous and discrete formulations of the model introduced in Section 2. The main difference of the procedures to be employed here with respect to the original Costabel & Han coupling method (see [10], [30]), lies on the simultaneous use of both Cauchy data on the boundary as independent unknowns, namely

$$\psi \,:=\, \gamma(u) = \gamma(u_e) \in \mathrm{H}_0^{1/2}(\Gamma) \quad \text{and} \quad \lambda \,:=\, \gamma_{\boldsymbol{n}}\big(a\,(\cdot, \|\nabla u\|)\,\nabla u\big) = \gamma_{\boldsymbol{n}}(\nabla u_e) \in \widehat{\mathrm{H}}^{-1/2}(\Gamma)\,, \tag{5.1}$$

with $\widehat{\mathrm{H}}^{-1/2}(\Gamma)$ defined in Section 3.1.

5.1 The primal approach

In this case, we proceed similarly to the derivation of (3.4), but now adding and subtracting the expression $\langle \lambda, \varphi\rangle$ with arbitrary $\varphi \in \mathrm{H}_0^{1/2}(\Gamma)$, and imposing weakly the relation $\psi = \gamma(u)$ with a test function $\mu \in \widehat{\mathrm{H}}^{-1/2}(\Gamma)$. As a consequence of it, we obtain

$$\int_\Omega a\,(\cdot, \|\nabla u\|)\,\nabla u \cdot \nabla v \,-\, \langle \lambda, \gamma(v) - \varphi\rangle \,-\, \langle \lambda, \varphi\rangle \,+\, \langle \mu, \gamma(u) - \psi\rangle \,=\, \int_\Omega f\, v \tag{5.2}$$

for all $(v, \varphi, \mu) \in \mathbf{X} \,:=\, \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}_0^{1/2}(\Gamma) \times \widehat{\mathrm{H}}^{-1/2}(\Gamma)$. On the other hand, we rewrite the identities (2.8) and (2.9) as

$$\big(\frac{1}{2}\mathrm{I} - K\big)\psi \,+\, V\lambda \,=\, 0 \quad \text{on} \quad \Gamma\,, \tag{5.3}$$

and

$$\lambda \,=\, -\,W\psi \,+\, \big(\frac{1}{2}\mathrm{I} - K^{\mathtt{t}}\big)\lambda \quad \text{on} \quad \Gamma\,. \tag{5.4}$$

Thus, replacing λ in the third term on the left hand side of (5.2) by the expression provided by (5.4), and incorporating (5.3) into the fourth one, we arrive at the variational formulation: Find $\vec{u} := (u, \psi, \lambda) \in \mathbf{X}$ such that

$$[\mathbf{A}(\vec{u}), \vec{v}] \,=\, \mathbf{F}(\vec{v}) \qquad \forall\, \vec{v} := (v, \varphi, \mu) \in \mathbf{X}\,, \tag{5.5}$$

where the nonlinear operator $\mathbf{A} : \mathbf{X} \to \mathbf{X}'$ is defined for each $\vec{w} := (w, \phi, \xi) \in \mathbf{X}$ as

$$[\mathbf{A}(\vec{w}), \vec{v}] \,:=\, \int_\Omega a\,(\cdot, \|\nabla w\|)\,\nabla w \cdot \nabla v \,+\, [\widetilde{\mathbf{A}}(\vec{w}), \vec{v}] \qquad \forall\, \vec{v} := (v, \varphi, \mu) \in \mathbf{X}\,, \tag{5.6}$$

with $\widetilde{\mathbf{A}} \in \mathcal{L}(\mathbf{X}, \mathbf{X}')$ given by

$$\begin{aligned}[\widetilde{\mathbf{A}}(\vec{w}), \vec{v}] \,:=\, &\langle W\phi, \varphi\rangle \,+\, \langle \mu, V\xi\rangle \,-\, \langle \xi, \gamma(v) - \varphi\rangle \,+\, \langle \mu, \gamma(w) - \phi\rangle \\ &- \langle \xi, (\frac{1}{2}\mathrm{I} - K)\varphi\rangle \,+\, \langle \mu, (\frac{1}{2}\mathrm{I} - K)\phi\rangle \qquad \forall\, \vec{v} := (v, \varphi, \mu) \in \mathbf{X}\,,\end{aligned} \tag{5.7}$$

and $\mathbf{F} \in \mathbf{X}'$ is set as

$$\mathbf{F}(\vec{v}) \,:=\, \int_\Omega f\, v \qquad \forall\, \vec{v} := (v, \varphi, \mu) \in \mathbf{X}\,. \tag{5.8}$$

We stress here that the transmission conditions (5.1) (see also (2.3)), which are employed in the derivation of the present continuous formulation, are recovered precisely from (5.5) and the Green representation formula (2.2), similarly as done in [27]. We omit details and refer to the last paragraph of [27, Section 4.2].

Next, concerning the solvability analysis of (5.5), we begin by observing, in virtue of the definition of $\widetilde{\mathbf{A}}$ (cf. (5.7)) and the ellipticity properties of V and W (cf. (2.10) and (2.11) in Lemma 2.1), that for each $\vec{v} := (v, \varphi, \mu) \in \mathbf{X}$ we obtain

$$[\widetilde{\mathbf{A}}(\vec{v}), \vec{v}] \,=\, \langle W\varphi, \varphi\rangle \,+\, \langle \mu, V\mu\rangle \,\geq\, \min\big\{\alpha_W, \alpha_V\big\} \Big\{\|\varphi\|^2_{1/2,\Gamma} + \|\mu\|^2_{-1/2,\Gamma}\Big\}\,. \tag{5.9}$$

Hence, proceeding similarly as in the proof of Theorem 3.1 (cf. Section 3.2), we have from (5.6) that for each $\vec{w} := (w, \phi, \xi)\,,\ \vec{v} := (v, \varphi, \mu) \in \mathbf{X}$ there holds

$$[\mathbf{A}(\vec{w}) - \mathbf{A}(\vec{v}), \vec{w} - \vec{v}] = \int_\Omega \Big\{a(\cdot, \|\nabla w\|)\, \nabla w - a(\cdot, \|\nabla v\|)\, \nabla v\Big\} \cdot \nabla(w - v) \,+\, [\widetilde{\mathbf{A}}(\vec{w} - \vec{v}), \vec{w} - \vec{v}]\,,$$

from which, invoking the hypothesis **(H.2)**, and employing (3.3) and (5.9), we find that

$$[\mathbf{A}(\vec{w}) - \mathbf{A}(\vec{v}), \vec{w} - \vec{v}] \,\geq\, C_A \Big\{\|w - v\|^2_{1,\Omega} \,+\, \|\phi - \varphi\|^2_{1/2,\Gamma} + \|\xi - \mu\|^2_{-1/2,\Gamma}\Big\} = C_A\, \|\vec{w} - \vec{v}\|^2_{\mathbf{X}}\,,$$

with $C_A := \min\big\{c_p\, \alpha_A, \alpha_W, \alpha_V\big\}$, which proves the strong monotonicity of $\mathbf{A}$. In addition, appealing now to **(H.3)** and the boundedness of $\widetilde{\mathbf{A}}$, whose norm depends on $\|W\|$, $\|V\|$, $\|\gamma\|$, and $\|K\|$, we deduce that $\mathbf{A}$ is Lipschitz-continuous with a constant L_A depending on ℓ_A and $\|\widetilde{\mathbf{A}}\|$. In this way, applying again the abstract result from [38, Theorem 3.3.23], and noting that $\|\mathbf{F}\|_{\mathbf{X}'} \,\leq\, \|f\|_{0,\Omega}$, we can establish the following main result.

Theorem 5.1. *The variational formulation* (5.5) *has a unique solution* $\vec{u} := (u, \psi, \lambda) \in \mathbf{X}$*, and there holds*

$$\|\vec{u}\|_{\mathbf{X}} \,\leq\, C_A^{-1}\, \|f\|_{0,\Omega}\,. \tag{5.10}$$

On the other hand, in order to set the discrete formulation of the present coupling method, we first let $\{\mathrm{H}_h^u\}_{h>0}$, $\{\mathrm{H}_h^\psi\}_{h>0}$, and $\{\mathrm{H}_h^\lambda\}_{h>0}$ be families of finite dimensional subspaces of $\mathrm{H}^1_{\Gamma_0}(\Omega)$, $\mathrm{H}_0^{1/2}(\Gamma)$, and $\widehat{\mathrm{H}}^{-1/2}(\Gamma)$, respectively. Then, the Galerkin scheme associated with (5.5) reads: Find $\vec{u}_h := (u_h, \psi_h, \lambda_h) \in \mathbf{X}_h := \mathrm{H}_h^u \times \mathrm{H}_h^\psi \times \mathrm{H}_h^\lambda$ such that

$$[\mathbf{A}(\vec{u}_h), \vec{v}_h] \,=\, \mathbf{F}(\vec{v}_h) \qquad \forall\, \vec{v}_h := (v_h, \varphi_h, \mu_h) \in \mathbf{X}_h\,. \tag{5.11}$$

Since the strong monotonicity and Lipschitz-continuity properties of $\mathbf{A}$ are certainly transferred to $\mathbf{A}|_{\mathbf{X}_h}$, analogously as explained in Section 3.3 for the primal approach of the Johnson & Nédélec coupling procedure, we conclude the well-posedness of (5.11) by applying again [38, Theorem 3.3.23], but now to the present discrete setting. In particular, the corresponding a priori estimate becomes the analogue of (5.10), that is

$$\|\vec{u}_h\|_{\mathbf{X}} \,\leq\, C_A^{-1}\, \|f\|_{0,\Omega}\,. \tag{5.12}$$

Moreover, following the same arguments from the second half of Section 3.3, we are able to establish the respective Cea estimate, which has the same structure of (3.14), namely

$$\|\vec{u}-\vec{u}_h\|_{\mathbf{X}} \,\le\, \Big\{1+L_A\,C_A^{-1}\Big\}\,\mathrm{dist}(\vec{u},\mathbf{X}_h)\,. \tag{5.13}$$

We conclude this section by observing that an analogue remark to the one stated at the end of Section 3, referring now to specific families of finite element subspaces of $\mathrm{H}^1_{\Gamma_0}(\Omega)$, $\mathrm{H}_0^{1/2}(\Gamma)$, and $\widehat{\mathrm{H}}^{-1/2}(\Gamma)$, is valid here. Additionally, we highlight that, as compared with the method from Section 3, the present primal version of the modified Costabel & Han method has the advantages, on one hand, of not requiring any restriction on the strong monotonicity constant α_A, and on the other hand, of yielding direct approximations, without any further computation or differentiation, of both Cauchy data. Nevertheless, assuming that the restriction on α_A is satisfied, the former method, involving only one boundary integral equation and two boundary integral operators, is certainly of less complexity and hence computationally cheaper.

5.2 The dual-mixed approach

We now address the dual-mixed version of the modified Costabel & Han coupling method as applied to the nonlinear model introduced in Section 2. Similarly as we did for the same approach of the Johnson & Nédélec coupling in Section 4, and for sake of simplicity of the discussion to be presented below, we consider again a homogeneous Neumann boundary condition on Γ_0. Then, after introducing $\mathbf{t} := \nabla u$ and $\boldsymbol{\sigma} := a(\cdot,\|\mathbf{t}\|)\,\mathbf{t}$ as auxiliary unknowns (cf. (4.1), Section 4.1), we proceed similarly to the derivation of (4.4), but now adding and subtracting the expression $\langle\xi,\psi\rangle$ with arbitrary $\xi\in\widehat{\mathrm{H}}^{-1/2}(\Gamma)$, and imposing weakly the relation $\gamma_{\boldsymbol{n}}(\sigma)=\lambda$ with a test function $\varphi\in\mathrm{H}_0^{1/2}(\Gamma)$. The resulting equation of the above arrangements reads

$$\int_\Omega \boldsymbol{\tau}\cdot\mathbf{t} \,+\, \int_\Omega u\,\mathrm{div}(\boldsymbol{\tau}) \,-\, \langle\gamma_{\boldsymbol{n}}(\boldsymbol{\tau})-\xi,\psi\rangle \,-\, \langle\xi,\psi\rangle \,+\, \langle\gamma_{\boldsymbol{n}}(\boldsymbol{\sigma})-\lambda,\varphi\rangle \,=\, 0 \tag{5.14}$$

for all $(\boldsymbol{\tau},\varphi,\xi)\in\mathbf{H}_0(\mathrm{div};\Omega)\times\mathrm{H}_0^{1/2}(\Gamma)\times\widehat{\mathrm{H}}^{-1/2}(\Gamma)$. In turn, the identities (2.8) and (2.9) (or, equivalently, (5.3) and (5.4)), are now rewritten as

$$\psi \,=\, \big(\frac{1}{2}\mathrm{I}+K\big)\psi \,-\, V\lambda \,=\, 0 \quad \text{on} \quad \Gamma\,, \tag{5.15}$$

and

$$W\psi \,+\, \big(\frac{1}{2}\mathrm{I}+K^{\mathbf{t}}\big)\lambda \,=\, 0 \quad \text{on} \quad \Gamma\,. \tag{5.16}$$

Therefore, replacing ψ in the fourth term on the left hand side of (5.14) by the expression provided by (5.15), and incorporating (5.16) into the fifth one, we obtain

$$\begin{aligned}\int_\Omega \boldsymbol{\tau}\cdot\mathbf{t} \,+\, \int_\Omega u\,\mathrm{div}(\boldsymbol{\tau}) \,-\, \langle\gamma_{\boldsymbol{n}}(\boldsymbol{\tau})-\xi,\psi\rangle \,+\, \langle\gamma_{\boldsymbol{n}}(\boldsymbol{\sigma})-\lambda,\varphi\rangle\\ +\,\langle\xi,V\lambda\rangle \,-\, \langle\xi,\big(\frac{1}{2}\mathrm{I}+K\big)\psi\rangle \,+\, \langle W\psi,\varphi\rangle \,+\, \langle\lambda,\big(\frac{1}{2}\mathrm{I}+K\big)\varphi\rangle \,=\, 0\end{aligned} \tag{5.17}$$

for all $(\boldsymbol{\tau},\varphi,\xi)\in\mathbf{H}_0(\mathrm{div};\Omega)\times\mathrm{H}_0^{1/2}(\Gamma)\times\widehat{\mathrm{H}}^{-1/2}(\Gamma)$. Moreover, adding (4.5) and (4.6) to (5.17), keeping (4.8) as it is, performing some algebraic rearrangements, and denoting $\vec{\mathbf{t}} := (\mathbf{t},\boldsymbol{\sigma},\psi,\lambda)$, $\vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau},\varphi,\xi)$, $\mathbf{H} := \mathbf{L}^2(\Omega)\times\mathbf{H}_0(\mathrm{div};\Omega)\times\mathrm{H}_0^{1/2}(\Gamma)\times\widehat{\mathrm{H}}^{-1/2}(\Gamma)$, and $\mathbf{Q} := \mathrm{L}^2(\Omega)$, we arrive now at the variational formulation: Find $(\vec{\mathbf{t}},u)\in\mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned}[\mathbf{A}(\vec{\mathbf{t}}),\vec{\mathbf{s}}] \,+\, [\mathbf{B}(\vec{\mathbf{s}}),u] \;&=\; \mathbf{F}(\vec{\mathbf{s}}) \qquad \forall\,\vec{\mathbf{s}}\in\mathbf{H}\,,\\ [\mathbf{B}(\vec{\mathbf{t}}),v] \;&=\; \mathbf{G}(v) \qquad \forall\,v\in\mathbf{Q}\,,\end{aligned} \tag{5.18}$$

where the nonlinear operator $\mathbf{A} : \mathbf{H} \to \mathbf{H}'$ and $\mathbf{B} \in \mathcal{L}\big(\mathbf{H}, \mathbf{Q}'\big)$ are defined for each $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi, \boldsymbol{\eta}) \in \mathbf{H}$ as

$$[\mathbf{A}(\vec{\mathbf{r}}), \vec{\mathbf{s}}] \,:=\, \int_\Omega a(\cdot, \|\mathbf{r}\|)\, \mathbf{r} \cdot \mathbf{s} \,-\, \kappa \int_\Omega a(\cdot, \|\mathbf{r}\|)\, \mathbf{r} \cdot \boldsymbol{\tau} \,+\, [\widetilde{\mathbf{A}}(\vec{\mathbf{r}}), \vec{\mathbf{s}}] \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,, \tag{5.19}$$

with $\widetilde{\mathbf{A}} \in \mathcal{L}\big(\mathbf{H}, \mathbf{H}'\big)$ given by

$$\begin{aligned} [\widetilde{\mathbf{A}}(\vec{\mathbf{r}}), \vec{\mathbf{s}}] \,:=\, & \kappa \int_\Omega \boldsymbol{\zeta} \cdot \boldsymbol{\tau} \,-\, \int_\Omega \boldsymbol{\zeta} \cdot \mathbf{s} \,+\, \int_\Omega \boldsymbol{\tau} \cdot \mathbf{r} \,-\, \langle \gamma_{\boldsymbol{n}}(\boldsymbol{\tau}) - \xi, \phi \rangle \,+\, \langle \gamma_{\boldsymbol{n}}(\boldsymbol{\zeta}) - \boldsymbol{\eta}, \varphi \rangle \\ & + \langle W\phi, \varphi \rangle \,+\, \langle \xi, V\boldsymbol{\eta} \rangle \,-\, \langle \xi, \big(\tfrac{1}{2}\mathrm{I} + K\big)\phi \rangle \,+\, \langle \boldsymbol{\eta}, \big(\tfrac{1}{2}\mathrm{I} + K\big)\varphi \rangle \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,, \end{aligned} \tag{5.20}$$

and

$$[\mathbf{B}(\vec{\mathbf{r}}), v] \,:=\, \int_\Omega v \,\mathrm{div}(\boldsymbol{\zeta}) \qquad \forall\, v \in \mathbf{Q}\,, \tag{5.21}$$

respectively, whereas $\mathbf{F} \in \mathbf{H}'$ and $\mathbf{G} \in \mathbf{Q}'$ are set as

$$\mathbf{F}(\vec{\mathbf{s}}) \,:=\, 0 \quad \forall\, \vec{\mathbf{s}} \in \mathbf{H} \quad \text{and} \quad \mathbf{G}(v) \,:=\, -\int_\Omega f\, v \quad \forall\, v \in \mathbf{Q}\,. \tag{5.22}$$

We now discuss the solvability of (5.18) by applying, analogously to the analysis in Section 4, the abstract result given by Theorem B.1. For this purpose, we first observe that, irrespective of the fact that the product space $\mathbf{H}$ now includes additionally $\widehat{\mathrm{H}}^{-1/2}(\Gamma)$ as a fourth component, the continuous inf-sup condition for the present operator $\mathbf{B}$ (cf. (5.21)) is proved exactly as we did in Section 4.2 (cf. Lemma 4.1) since the only spaces involved in it are the second component of $\mathbf{H}$ and $\mathbf{Q}$. In this way, with the same constant β from that lemma, there holds

$$\sup_{\substack{\vec{\mathbf{s}} \in \mathbf{H} \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{s}}), v]}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geq\, \beta\, \|v\|_{\mathbf{Q}} \quad \forall\, v \in \mathbf{Q}\,. \tag{5.23}$$

Moreover, the null space $\mathbf{V}$ of $\mathbf{B}$ remains basically unchanged with respect to (4.16), namely

$$\mathbf{V} \,:=\, \Big\{ \vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi, \boldsymbol{\eta}) \in \mathbf{H} : \quad \mathrm{div}(\boldsymbol{\zeta}) = 0 \quad \text{in} \quad \Omega \Big\}\,.$$

It follows from (5.20) and the ellipticity properties of W and V (cf. Lemma 2.1) that for each $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi, \boldsymbol{\eta}) \in \mathbf{V}$ there holds

$$[\widetilde{\mathbf{A}}(\vec{\mathbf{r}}), \vec{\mathbf{r}}] \,\geq\, \kappa \|\boldsymbol{\zeta}\|^2_{0,\Omega} \,+\, \alpha_W\, \|\phi\|^2_{1/2,\Gamma} \,+\, \alpha_V\, \|\boldsymbol{\eta}\|^2_{-1/2,\Gamma}\,. \tag{5.24}$$

Therefore, employing once again the hypotheses (**H.2**) and (**H.3**), and applying Young's inequality with a parameter $\delta > 0$, we deduce from (5.19) and (5.24) that for each $\vec{\mathbf{q}} \in \mathbf{H}$, and for all $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\zeta}, \phi, \boldsymbol{\eta})$, $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, \varphi, \xi) \in \mathbf{V}$ there holds

$$\begin{aligned} [\mathbf{A}(\vec{\mathbf{r}} + \vec{\mathbf{q}}) - \mathbf{A}(\vec{\mathbf{s}} + \vec{\mathbf{q}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}] \,&=\, [\mathbf{A}(\vec{\mathbf{r}} + \vec{\mathbf{q}}) - \mathbf{A}(\vec{\mathbf{s}} + \vec{\mathbf{q}}), (\vec{\mathbf{r}} + \vec{\mathbf{q}}) - (\vec{\mathbf{s}} + \vec{\mathbf{q}})] \\ &\geq\, \left(\alpha_A \,-\, \frac{\kappa\, \ell_A}{2\delta}\right) \|\mathbf{r} - \mathbf{s}\|^2_{0,\Omega} \,+\, \kappa \left(1 \,-\, \frac{\ell_A\, \delta}{2}\right) \|\boldsymbol{\zeta} - \boldsymbol{\tau}\|^2_{\mathrm{div};\Omega} \\ &+\quad \alpha_W\, \|\phi - \varphi\|^2_{1/2,\Gamma} \,+\, \alpha_V\, \|\boldsymbol{\eta} - \xi\|^2_{-1/2,\Gamma}\,, \end{aligned}$$

from which we conclude the strong monotonicity of $\mathbf{A}$ on $\mathbf{V}$ if δ and κ are chosen such that $\delta \in \big(0, \frac{2}{\ell_A}\big)$ and $\kappa \in \big(0, \frac{2\,\delta\,\alpha_A}{\ell_A}\big)$. In particular, taking the midpoints of each range, that is $\delta = \frac{1}{\ell_A}$ and $\kappa = \frac{\delta\,\alpha_A}{\ell_A}$, which maximize the respective constants, we arrive finally at

$$[\mathbf{A}(\vec{\mathbf{r}} + \vec{\mathbf{q}}) - \mathbf{A}(\vec{\mathbf{s}} + \vec{\mathbf{q}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}] \,\geq\, \widetilde{\alpha} \Big\{ \|\mathbf{r} - \mathbf{s}\|^2_{0,\Omega} + \|\boldsymbol{\zeta} - \boldsymbol{\tau}\|^2_{\mathrm{div};\Omega} + \|\phi - \varphi\|^2_{1/2,\Gamma} + \|\boldsymbol{\eta} - \xi\|^2_{-1/2,\Gamma} \Big\}\,, \tag{5.25}$$

with $\widetilde{\alpha} := \min\left\{\frac{\alpha_A}{2}, \frac{\alpha_A}{2\,\ell_A^2}, \alpha_W, \alpha_V\right\}$. We stress here that, differently from the analysis in Section 4.2 (cf. (4.17), (4.19), (4.20)), and thanks to the restriction-free positiveness condition (5.24) satisfied by $\widetilde{\mathbf{A}}$, the inequality (5.25) has been derived without assuming any relation between α_A and ℓ_A. On the other hand, the Lipschitz-continuity of $\mathbf{A}$ is established analogously as done in that same section, using (H.3), Cauchy-Schwarz's inequality, and the boundedness of the linear operators involved, so that with a positive constant $\widetilde{L}_A$, depending on ℓ_A and the norms of W, V, K, and $\gamma_{\boldsymbol{n}}$, there holds

$$\|\mathbf{A}(\vec{\mathbf{r}}) - \mathbf{A}(\vec{\mathbf{s}})\|_{\mathbf{H}'} \,\le\, \widetilde{L}_A\,\|\vec{\mathbf{r}} - \vec{\mathbf{s}}\|_{\mathbf{H}} \qquad \forall\,\vec{\mathbf{r}},\,\vec{\mathbf{s}} \in \mathbf{H}\,.$$

Consequently, bearing in mind (5.23) and the previous discussion on the properties of $\mathbf{A}$, we can state the following main theorem concerning the solvability of (5.18), whose proof, similarly to the one of Theorem 4.1, follows from a direct application of Theorem B.1.

Theorem 5.2. *Assume the hypotheses* (**H.1**), (**H.2**)*, and* (**H.3**). *Then, problem* (5.18) *has a unique solution* $(\vec{\mathbf{t}}, u) \in \mathbf{H}\times\mathbf{Q}$. *Moreover, there hold*

$$\|\vec{\mathbf{t}}\|_{\mathbf{H}} \,\le\, \frac{1}{\beta}\left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}}\right)\|f\|_{0,\Omega} \quad \textit{and} \quad \|u\|_{\mathbf{Q}} \,\le\, \frac{\widetilde{L}_A}{\beta^2}\left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}}\right)\|f\|_{0,\Omega}\,.$$

Furthermore, regarding the Galerkin scheme associated with (5.18), we now let let $\big\{\mathrm{H}_h^{\mathbf{t}}\big\}_{h>0}$, $\big\{\mathrm{H}_h^{\boldsymbol{\sigma}}\big\}_{h>0}$, $\big\{\mathrm{H}_h^{\psi}\big\}_{h>0}$, $\big\{\mathrm{H}_h^{\lambda}\big\}_{h>0}$, and $\big\{\mathrm{H}_h^{u}\big\}_{h>0}$, be families of finite dimensional subspaces of $\mathbf{L}^2(\Omega)$, $\mathbf{H}_0(\mathrm{div};\Omega)$, $\mathrm{H}_0^{1/2}(\Gamma)$, $\widehat{\mathrm{H}}^{-1/2}(\Gamma)$, and $\mathrm{L}^2(\Omega)$, respectively, set $\mathbf{H}_h := \mathrm{H}_h^{\mathbf{t}}\times\mathrm{H}_h^{\boldsymbol{\sigma}}\times\mathrm{H}_h^{\psi}\times\mathrm{H}_h^{\lambda}$, $\mathbf{Q}_h := \mathrm{H}_h^{u}$, and consider the discrete formulation: Find $(\vec{\mathbf{t}}_h, u_h) := \big((\mathbf{t}_h, \boldsymbol{\sigma}_h, \psi_h, \lambda_h), u_h\big) \in \mathbf{H}_h\times\mathbf{Q}_h$ such that

$$\begin{aligned} [\mathbf{A}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}_h] \,+\, [\mathbf{B}(\vec{\mathbf{s}}_h), u_h] \quad &= \quad \mathbf{F}(\vec{\mathbf{s}}_h) \qquad \forall\,\vec{\mathbf{s}}_h := (\mathbf{s}_h, \boldsymbol{\tau}_h, \varphi_h, \xi_h) \in \mathbf{H}_h\,, \\ [\mathbf{B}(\vec{\mathbf{t}}_h), v_h] \quad &= \quad \mathbf{G}(v_h) \qquad \forall\,v_h \in \mathbf{Q}_h\,. \end{aligned} \tag{5.26}$$

The rest of the discussion proceeds as we did in Section 4.3 for the dual mixed approach of the Johnson & Nédélec coupling procedure, by assuming additionally the hypotheses (**H.4**) and (**H.5**). We omit further details and just summarize the main results in the following theorem, which is the analogue of Theorem 4.2.

Theorem 5.3. *In addition to the hypotheses of Theorem* 4.1*, assume that* (**H.4**) *and* (**H.5**) *are satisfied. Then, problem* (5.26) *has a unique solution* $(\vec{\mathbf{t}}_h, u_h) \in \mathbf{H}_h\times\mathbf{Q}_h$*, and there hold*

$$\|\vec{\mathbf{t}}_h\|_{\mathbf{H}} \,\le\, \frac{1}{\beta_{\mathtt{d}}}\left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}}\right)\|f\|_{0,\Omega} \quad \textit{and} \quad \|u_h\|_{\mathbf{Q}} \,\le\, \frac{\widetilde{L}_A}{\beta_{\mathtt{d}}^2}\left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}}\right)\|f\|_{0,\Omega}\,.$$

Moreover, the corresponding a priori error estimates reduce to

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} \,\le\, \left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}}\right)\left(1 + \frac{1}{\beta_{\mathtt{d}}}\right)\mathrm{dist}\big(\vec{\mathbf{t}}, \mathbf{H}_h\big)$$

and

$$\|u - u_h\|_{0,\Omega} \,\le\, \left(1 + \frac{1}{\beta_{\mathtt{d}}}\right)\mathrm{dist}(u, \mathbf{Q}_h) \,+\, \frac{\widetilde{L}_A}{\beta_{\mathtt{d}}}\left(1 + \frac{\widetilde{L}_A}{\widetilde{\alpha}}\right)\left(1 + \frac{1}{\beta_{\mathtt{d}}}\right)\mathrm{dist}\big(\vec{\mathbf{t}}, \mathbf{H}_h\big)\,.$$

We conclude this section by highlighting that the very same remark stated at the end of Section 5.1 applies here when comparing the advantages and complexity of the present dual-mixed approach of the modified Costabel & Han coupling procedure with those of the method from Section 4.

A Optimal choices of parameters κ and δ

In this brief appendix we confirm the optimal choices of κ and δ given in Section 4.2. In fact, let us consider minimizing the function g (cf. (4.21)) in its range of definition, that is in the region $\mathcal{R} := \Big\{(\kappa,\delta) \in \mathrm{R}^2 : \quad \kappa \in \big(\frac{1}{4}, +\infty\big)\,, \quad \delta \in \big(\frac{1}{2\kappa}, 2\big)\Big\}$. Then, we observe that

$$
\frac{\partial g}{\partial \kappa} = \frac{1 - \frac{1}{\kappa\delta}}{\left(1 - \frac{1}{2\kappa\delta}\right)^2} \begin{cases} > 0 & \text{if} \quad \kappa > \dfrac{1}{\delta}, \\ = 0 & \text{if} \quad \kappa = \dfrac{1}{\delta}, \\ < 0 & \text{if} \quad \kappa < \dfrac{1}{\delta}, \end{cases}
$$

$$
\frac{\partial g}{\partial \delta} = \frac{-\frac{1}{2\delta^2}}{\left(1 - \frac{1}{2\kappa\delta}\right)^2} < 0 \qquad \forall\, (\kappa,\delta) \,\in\, \mathcal{R}\,.
$$

It follows that for given δ , the function $g(\cdot,\delta)$ is decreasing in the interval $(\frac{1}{4}, \frac{1}{\delta})$ and increasing in $(\frac{1}{\delta}, +\infty)$, which means that it attains its minimum at $\kappa = \frac{1}{\delta}$, whereas given κ, the function $g(\kappa,\cdot)$ is always decreasing (see Figure 2 below for a graphic illustration of the region $\mathcal{R}$).

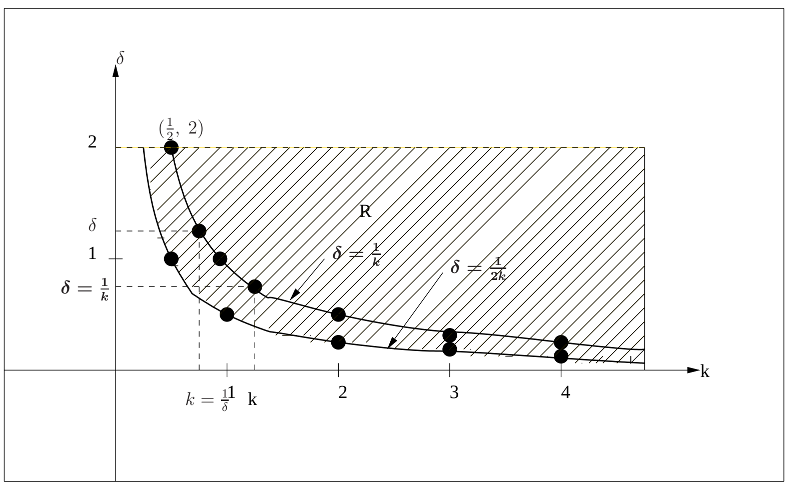

Figure A.1: $\mathcal{R}$, the feasible region for the choice of parameters κ and δ.

Therefore, $g(\kappa,\delta)$ attains its infimum at $\kappa = \frac{1}{2}$ and $\delta = \frac{1}{\kappa} = 2$, whose value is 1. In this way, we confirm the already announced sufficient condition for a well-posedness of the dual-mixed formulation (4.24) (cf. Section 4.2), namely $\alpha_A > \frac{\ell_A^2}{4}$.

B A nonlinear Babuška-Brezzi theory

In this appendix we address a nonlinear version of the classical Babuška-Brezzi theory for saddle point problems in a Hilbertian framework. While the results to be presented in what follows can be obtained as particular cases of more general abstract theories (see, e.g. [43, Proposition 2.3], [9, Theorem 3.1]), particular proofs for them as such are, up to our knowledge, not available in the literature. This is the reason why, for sake of clearness of the discussion in some sections of this work, we provide them below. Our problem of interest is precisely the one for which the following theorem establishes sufficient conditions under which it is uniquely solvable.

Theorem B.1. *Let* H *and* Q *be real Hilbert spaces, and let* $\mathrm{A}:\mathrm{H}\to\mathrm{H}'$ *be a nonlinear operator, and* $\mathrm{B}\in\mathcal{L}\big(\mathrm{H},\mathrm{Q}'\big)$. *In addition, let* V *be the null space of* B, *and assume that*

a) *the family of operators* $\mathrm{A}(\cdot+\widetilde{\tau}):\mathrm{V}\to\mathrm{V}'$, *with* $\widetilde{\tau}\in\mathrm{H}$, *is uniformly strongly monotone, that is there exists* $\alpha>0$ *such that*

$$[\mathrm{A}(\tau+\widetilde{\tau})-\mathrm{A}(\zeta+\widetilde{\tau}),\tau-\zeta]\,\geq\,\alpha\,\|\tau-\zeta\|_{\mathrm{H}}^2\qquad\forall\,\widetilde{\tau}\in\mathrm{H},\quad\forall\,\tau,\,\zeta\in\mathrm{V}\,,\tag{B.1}$$

b) $\mathrm{A}:\mathrm{H}\to\mathrm{H}'$ *is Lipschitz-continuous, that is, there exists* $L_{\mathrm{A}}>0$ *such that*

$$\|\mathrm{A}(\tau)-\mathrm{A}(\zeta)\|_{\mathrm{H}'}\,\leq\,L_{\mathrm{A}}\,\|\tau-\zeta\|_{\mathrm{H}}\qquad\forall\,\tau,\,\zeta\in\mathrm{H}\,,\tag{B.2}$$

c) *there exists* $\beta>0$ *such that*

$$\sup_{\substack{\tau\in\mathrm{H}\\ \tau\neq 0}}\frac{[\mathrm{B}(\tau),v]}{\|\tau\|_{\mathrm{H}}}\,\geq\,\beta\,\|v\|_{\mathrm{Q}}\qquad\forall\,v\in\mathrm{Q}\,.\tag{B.3}$$

Then, for each pair $(\mathrm{F},\mathrm{G})\in\mathrm{H}'\times\mathrm{Q}'$ *there exists a unique* $(\sigma,u)\in\mathrm{H}\times\mathrm{Q}$ *such that*

$$\begin{aligned}[][\mathrm{A}(\sigma),\tau]\,+\,[\mathrm{B}(\tau),u]\,&=\,\mathrm{F}(\tau)\qquad\forall\,\tau\in\mathrm{H}\,,\\ [\mathrm{B}(\sigma),v]\,&=\,\mathrm{G}(v)\qquad\forall\,v\in\mathrm{Q}\,,\end{aligned}\tag{B.4}$$

and there hold

$$\|\sigma\|_{\mathrm{H}}\,\leq\,\frac{1}{\alpha}\|\mathrm{F}\|_{\mathrm{H}'}\,+\,\frac{1}{\beta}\left(1+\frac{L_A}{\alpha}\right)\|\mathrm{G}\|_{\mathrm{Q}'}\,+\,\frac{1}{\alpha}\,\|\mathrm{A}(\mathbf{0})\|_{\mathrm{H}'}\,,\tag{B.5}$$

and

$$\|u\|_{\mathrm{Q}}\,\leq\,\frac{1}{\beta}\left(1+\frac{L_A}{\alpha}\right)\|\mathrm{F}\|_{\mathrm{H}'}\,+\,\frac{L_A}{\beta^2}\left(1+\frac{L_A}{\alpha}\right)\|\mathrm{G}\|_{\mathrm{Q}'}\,+\,\frac{1}{\beta}\left(1+\frac{L_A}{\alpha}\right)\|\mathrm{A}(\mathbf{0})\|_{\mathrm{H}'}\,.\tag{B.6}$$

Proof. Let us first introduce the operator $\underline{\mathrm{B}}\in\mathcal{L}\big(\mathrm{Q},\mathrm{H}'\big)$ defined for each $v\in\mathrm{Q}$ by $[\underline{\mathrm{B}}(v),\tau]:=[\mathrm{B}(\tau),v]$ for all $\tau\in\mathrm{H}$, so that (B.4) can be rewritten as the operator equation system:

$$\begin{aligned}\mathrm{A}(\sigma)\,+\,\underline{\mathrm{B}}(u)\,&=\,\mathrm{F}\,,\\ \mathrm{B}(\sigma)\,&=\,\mathrm{G}\,.\end{aligned}\tag{B.7}$$

Furthermore, denoting by $\mathcal{R}_{\mathrm{H}}:\mathrm{H}'\to\mathrm{H}$ and $\mathcal{R}_{\mathrm{Q}}:\mathrm{Q}'\to\mathrm{Q}$ the respective Riesz operators, and defining $\widetilde{\mathrm{B}}:=\mathcal{R}_{\mathrm{Q}}\,\mathrm{B}:\mathrm{H}\to\mathrm{Q}$, it readily follows that its Hilbert-adjoint is given by $\widetilde{\mathrm{B}}^*=\mathcal{R}_{\mathrm{H}}\,\underline{\mathrm{B}}:\mathrm{Q}\to\mathrm{H}$. In turn, it is easy to see that the inf-sup condition (B.3) can be rewritten as $\|\underline{\mathrm{B}}(v)\|_{\mathrm{H}'}\,\geq\,\beta\,\|v\|_{\mathrm{Q}}$ for all $v\in\mathrm{Q}$, which, according to the above identity, is the same that $\|\widetilde{\mathrm{B}}^*(v)\|_{\mathrm{H}}\,\geq\,\beta\,\|v\|_{\mathrm{Q}}$ for all $v\in\mathrm{Q}$. Moreover, noting that the null space of B and $\widetilde{\mathrm{B}}$ coincide, the foregoing inequality also says that $\widetilde{\mathrm{B}}^*:\mathrm{Q}\to\mathrm{V}^{\perp}$ is an isomorphism, which, thanks to a classical result on linear operators (see, e.g. [15, Lemma 2.1]), is equivalent to saying that $\widetilde{\mathrm{B}}:\mathrm{V}^{\perp}\to\mathrm{Q}$ is an isomorphism as well, and that $\|\widetilde{\mathrm{B}}(\tau)\|_{\mathrm{Q}}\,\geq\,\beta\,\|\tau\|_{\mathrm{H}}$ for all $\tau\in\mathrm{V}^{\perp}$. Thus, going back to B and $\underline{\mathrm{B}}$, we conclude from the above discussion that $\mathrm{B}:\mathrm{V}^{\perp}\to\mathrm{Q}'$ and $\underline{\mathrm{B}}:\mathrm{Q}\to\mathcal{R}_{\mathrm{H}}^{-1}(\mathrm{V}^{\perp})$ are isomorphisms and that there hold

$$\|\mathrm{B}(\tau)\|_{\mathrm{Q}'}\,\geq\,\beta\,\|\tau\|_{\mathrm{H}}\quad\forall\,\tau\in\mathrm{V}^{\perp}\qquad\text{and}\qquad\|\underline{\mathrm{B}}(v)\|_{\mathrm{H}'}\,\geq\,\beta\,\|v\|_{\mathrm{Q}}\quad\forall\,v\in\mathrm{Q}\,.\tag{B.8}$$

Now, given $(\mathrm{F}, \mathrm{G}) \in \mathrm{H}' \times \mathrm{Q}'$, we proceed to construct $(\sigma, u) \in \mathrm{H} \times \mathrm{Q}$ solving (B.7). In fact, we first let σ_g be the unique element in $\mathrm{V}^\perp$ satisfying $\mathrm{B}(\sigma_g) = \mathrm{G}$ and $\|\sigma_g\|_{\mathrm{H}} \le \dfrac{1}{\beta}\|\mathrm{G}\|_{\mathrm{Q}'}$ (cf. first inequality in (B.8)). Next, thanks to the assumptions (B.1) and (B.2), a straightforward application of [38, Theorem 3.3.23] implies that $\mathrm{A}(\cdot + \sigma_g) : \mathrm{V} \to \mathrm{V}'$ is bijective, and hence there exists a unique $\sigma_0 \in \mathrm{V}$ such that $\mathrm{A}(\sigma_0 + \sigma_g) = \mathrm{F}|_{\mathrm{V}} \in \mathrm{V}'$, that is $[\mathrm{F} - \mathrm{A}(\sigma_0 + \sigma_g), \tau] = 0$ for all $\tau \in \mathrm{V}$. This means that $\mathrm{F} - \mathrm{A}(\sigma_0 + \sigma_g) \in \mathcal{R}_{\mathrm{H}}^{-1}(\mathrm{V}^\perp)$, and therefore there exists a unique $u \in \mathrm{Q}$ such that $\underline{\mathrm{B}}(u) = \mathrm{F} - \mathrm{A}(\sigma_0 + \sigma_g)$ and $\|u\|_{\mathrm{Q}} \le \dfrac{1}{\beta}\|\mathrm{F} - \mathrm{A}(\sigma_0 + \sigma_g)\|_{\mathrm{H}'}$ (cf. second inequality in (B.8)). In this way, defining $\boldsymbol{\sigma} := \boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_g \in \mathrm{H}$, we have $\mathrm{A}(\sigma) + \underline{\mathrm{B}}(u) = \mathrm{F}$ and $\mathrm{B}(\sigma) = \mathrm{B}(\sigma_g) = \mathrm{G}$, which proves that (σ, u) solves (B.7). Concerning the boundedness of (σ, u), we begin by applying (B.1) to $\widetilde{\tau} = \sigma_g$, $\tau = \sigma_0$, and $\zeta = \mathbf{0}$, which yields

$$\alpha\,\|\sigma_0\|_{\mathrm{H}}^2 \le [\mathrm{A}(\sigma_0 + \sigma_g) - \mathrm{A}(\sigma_g), \sigma_0] = [\mathrm{F} - \mathrm{A}(\sigma_g), \sigma_0]\,.$$

Then, adding and subtracting $\mathrm{A}(\mathbf{0})$ in the foregoing inequality, and applying (B.2), we deduce that

$$\|\sigma_0\|_{\mathrm{H}} \le \frac{1}{\alpha}\Big\{\|\mathrm{F}\|_{\mathrm{H}'} + \|\mathrm{A}(\mathbf{0})\|_{\mathrm{H}'} + L_{\mathrm{A}}\,\|\sigma_g\|_{\mathrm{H}}\Big\}\,. \tag{B.9}$$

On the other hand, proceeding similarly with the bound for $\|u\|_{\mathrm{Q}}$, we find that

$$\|u\|_{\mathrm{Q}} \le \frac{1}{\beta}\Big\{\|\mathrm{F}\|_{\mathrm{H}'} + \|\mathrm{A}(\mathbf{0})\|_{\mathrm{H}'} + L_{\mathrm{A}}\,\|\sigma_0\|_{\mathrm{H}} + L_{\mathrm{A}}\,\|\sigma_g\|_{\mathrm{H}}\Big\}\,. \tag{B.10}$$

Consequently, recalling that $\sigma = \sigma_0 + \sigma_g$, the required estimates (B.5) and (B.6) are obtained directly from (B.9), (B.10), and the already stated bound for $\|\sigma_g\|_{\mathrm{H}}$. It remains to show that (σ, u) is the unique solution of (B.4) (equivalently (B.7)). For this purpose, we now let $(\widetilde{\sigma}, \widetilde{u}) \in \mathrm{H} \times \mathrm{Q}$ be another solution of (B.7). It follows that $\sigma - \widetilde{\sigma} \in \mathrm{V}$ and $\mathrm{A}(\sigma) - \mathrm{A}(\widetilde{\sigma}) = \underline{\mathrm{B}}(\widetilde{u} - u)$, so that from the latter we see that $[\mathrm{A}(\sigma) - \mathrm{A}(\widetilde{\sigma}), \tau] = 0$ for all $\tau \in \mathrm{V}$. Then, applying (B.1) to $\widetilde{\tau} = \widetilde{\sigma}$, $\tau = \sigma - \widetilde{\sigma}$, and $\zeta = \mathbf{0}$, we get

$$\alpha\,\|\sigma - \widetilde{\sigma}\|_{\mathrm{H}}^2 \le [\mathrm{A}(\sigma) - \mathrm{A}(\widetilde{\sigma}), \sigma - \widetilde{\sigma}] = 0\,,$$

from which it is clear that $\sigma = \widetilde{\sigma}$, and thus $\underline{\mathrm{B}}(\widetilde{u} - u) = \mathbf{0}$. Finally, the bijectivity of $\underline{\mathrm{B}} : \mathrm{Q} \to \mathcal{R}_{\mathrm{H}}^{-1}(\mathrm{V}^\perp)$ gives $u = \widetilde{u}$, which completes the uniqueness and ends the proof. □

The discrete version of Theorem B.1, including the associated Cea estimates, is stated as follows.

Theorem B.2. *In addition to the hypotheses of Theorem* B.1, *let* $\{\mathrm{H}_h\}_{h>0}$ *and* $\{\mathrm{Q}_h\}_{h>0}$ *be families of finite dimensional subspaces of* H *and* Q, *respectively, and let* V_h *be the null space of* $\mathrm{B}|_{\mathrm{H}_h} : \mathrm{H}_h \to \mathrm{Q}_h'$, *that is*

$$\mathrm{V}_h := \Big\{\tau_h \in \mathrm{H}_h : \quad [\mathrm{B}(\tau_h), v_h] = 0 \quad \forall\, v_h \in \mathrm{Q}_h\Big\}\,.$$

Assume that

a) *the family of operators* $\mathrm{A}(\cdot + \widetilde{\tau}_h) : \mathrm{V}_h \to \mathrm{V}_h'$, *with* $\widetilde{\tau}_h \in \mathrm{H}_h$, *is uniformly strongly monotone, that is there exists* $\alpha_{\mathtt{d}} > 0$, *independent of* h, *such that*

$$[\mathrm{A}(\tau_h + \widetilde{\tau}_h) - \mathrm{A}(\zeta_h + \widetilde{\tau}_h), \tau_h - \zeta_h] \ge \alpha_{\mathtt{d}}\,\|\tau_h - \zeta_h\|_{\mathrm{H}}^2 \qquad \forall\, \widetilde{\tau}_h \in \mathrm{H}_h, \quad \forall\, \tau_h, \zeta_h \in \mathrm{V}_h\,, \tag{B.11}$$

b) *there exists* $\beta_{\mathtt{d}} > 0$ *such that*

$$\sup_{\substack{\tau_h \in \mathrm{H}_h \\ \tau_h \ne 0}} \frac{[\mathrm{B}(\tau_h), v_h]}{\|\tau_h\|_{\mathrm{H}}} \ge \beta_{\mathtt{d}}\,\|v_h\|_{\mathrm{Q}} \qquad \forall\, v_h \in \mathrm{Q}_h\,. \tag{B.12}$$

Then, for each pair $(\mathrm{F},\mathrm{G}) \in \mathrm{H}' \times \mathrm{Q}'$ *there exists a unique* $(\sigma_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ *such that*

$$\begin{aligned} [\mathrm{A}(\sigma_h),\tau_h] \,+\, [\mathrm{B}(\tau_h),u_h] &= \mathrm{F}(\tau_h) \qquad \forall\, \tau_h \in \mathrm{H}_h\,, \\ [\mathrm{B}(\sigma_h),v_h] &= \mathrm{G}(v_h) \qquad \forall\, v_h \in \mathrm{Q}_h\,, \end{aligned} \tag{B.13}$$

and there hold

$$\|\sigma_h\|_{\mathrm{H}} \,\le\, \frac{1}{\alpha_{\mathtt{d}}}\|\mathrm{F}\|_{\mathrm{H}'} \,+\, \frac{1}{\beta_{\mathtt{d}}}\left(1+\frac{L_A}{\alpha_{\mathtt{d}}}\right)\|\mathrm{G}\|_{\mathrm{Q}'} \,+\, \frac{1}{\alpha_{\mathtt{d}}}\,\|\mathrm{A}(\mathbf{0})\|_{\mathrm{H}'}\,, \tag{B.14}$$

and

$$\|u_h\|_{\mathrm{Q}} \,\le\, \frac{1}{\beta_{\mathtt{d}}}\left(1+\frac{L_A}{\alpha_{\mathtt{d}}}\right)\|\mathrm{F}\|_{\mathrm{H}'} \,+\, \frac{L_A}{\beta_{\mathtt{d}}^2}\left(1+\frac{L_A}{\alpha_{\mathtt{d}}}\right)\|\mathrm{G}\|_{\mathrm{Q}'} \,+\, \frac{1}{\beta_{\mathtt{d}}}\left(1+\frac{L_A}{\alpha_{\mathtt{d}}}\right)\|\mathrm{A}(\mathbf{0})\|_{\mathrm{H}'}\,. \tag{B.15}$$

Moreover, bearing in mind that $(\sigma,u) \in \mathrm{H}\times\mathrm{Q}$ *is the unique solution of* (B.4)*, the respective Cea estimates become*

$$\|\sigma-\sigma_h\|_{\mathbf{H}} \,\le\, \frac{\|\mathrm{B}\|}{\alpha_{\mathtt{d}}}\mathrm{dist}(u,\mathrm{Q}_h) \,+\, \left(1+\frac{L_A}{\alpha_{\mathtt{d}}}\right)\left(1+\frac{\|\mathrm{B}\|}{\beta_{\mathtt{d}}}\right)\,\mathrm{dist}\big(\sigma,\mathrm{H}_h\big) \tag{B.16}$$

and

$$\|u-u_h\|_{0,\Omega} \,\le\, \left(1+\frac{\|\mathrm{B}\|}{\beta_{\mathtt{d}}}+\frac{L_A\,\|\mathrm{B}\|}{\alpha_{\mathtt{d}}\,\beta_{\mathtt{d}}}\right)\mathrm{dist}(u,\mathrm{Q}_h) \,+\, \frac{L_A}{\beta_{\mathtt{d}}}\left(1+\frac{L_A}{\alpha_{\mathtt{d}}}\right)\left(1+\frac{\|\mathrm{B}\|}{\beta_{\mathtt{d}}}\right)\mathrm{dist}\big(\sigma,\mathrm{H}_h\big)\,. \tag{B.17}$$

Proof. We first observe that the Lipschitz-continuity of A, which is guaranteed by (B.2) (hypothesis b) from Theorem B.1), implies the same property for $\mathrm{A}|_{\mathrm{H}_h} : \mathrm{H}_h \to \mathrm{H}'_h$ and with the same constant L_A. Thus, in virtue of this fact and the present hypotheses a) and b), we deduce that the existence of a unique $(\sigma_h,u_h) \in \mathrm{H}_h\times\mathrm{Q}_h$ solution of (B.13), and the a priori estimates (B.14) and (B.15), follow from a straightforward application of Theorem B.1 to the present discrete context. Hence, it only remains to show the Cea estimates (B.16) and (B.17), for which we proceed very similarly to the proof of the general linear case of the Babuška-Brezzi theory (see, e.g. [15, Theorem 2.6]), even using some of the estimates provided there. In fact, we first define a kind of translation of V_h given by

$$\mathrm{V}_h^g := \Big\{\tau_h \in \mathrm{H}_h : \quad [\mathrm{B}(\tau_h),v_h] = G(v_h) \quad \forall\, v_h \in \mathrm{Q}_h\Big\}\,,$$

and notice that σ_h belongs to V_h^g, whence it readily follows that $\sigma_h - \tau_h^g \in \mathrm{V}_h$ for all $\tau_h^g \in \mathrm{V}_h^g$. Then, applying the strong monotonicity assumption (B.11) to $\widetilde{\tau}_h = \tau_h^g \in \mathrm{H}_h$, $\tau_h = \sigma_h - \tau_h^g \in \mathrm{V}_h$, and $\zeta_h = 0 \in \mathrm{V}_h$, we find that

$$\alpha_{\mathtt{d}}\,\|\sigma_h-\tau_h^g\|_{\mathrm{H}}^2 \,\le\, [\mathrm{A}(\sigma_h)-\mathrm{A}(\tau_h^g),\sigma_h-\tau_h^g]\,,$$

from which, adding and subtracting $[\mathrm{A}(\sigma),\sigma_h-\tau_h^g]$ on the right hand side, and using the first equations of (B.4) and (B.13), as well as the fact that $[\mathrm{B}(\sigma_h-\tau_h^g),v_h] = 0$ for all $v_h \in \mathrm{Q}_h$, we arrive at

$$\alpha_{\mathtt{d}}\,\|\sigma_h-\tau_h^g\|_{\mathrm{H}}^2 \,\le\, [\mathrm{B}(\sigma_h-\tau_h^g),u-v_h] \,+\, [\mathrm{A}(\sigma)-\mathrm{A}(\tau_h^g),\sigma_h-\tau_h^g] \quad \forall\, v_h \in \mathrm{Q}_h\,. \tag{B.18}$$

This is a key point of the argumentation in the sense that, if we had V_h contained in V, then the first term on the right hand side of (B.18) would also vanish, and, on the contrary to what we show next, the a priori estimate for $\|\sigma-\sigma_h\|_{\mathrm{H}}$ would not depend on $\mathrm{dist}(u,\mathrm{Q}_h)$ but only on $\mathrm{dist}(\sigma,\mathrm{H}_h)$. After finishing the present proof, we go back to this issue. Now, without assuming a priori any relation between V_h and V, we simply apply the boundedness of B and the Lipschitz-continuity of A (cf. (B.2)), to derive from (B.18) that

$$\|\sigma_h-\tau_h^g\|_{\mathrm{H}} \,\le\, \frac{\|\mathrm{B}\|}{\alpha_{\mathtt{d}}}\,\|u-v_h\|_{\mathrm{Q}} \,+\, \frac{L_A}{\alpha_{\mathtt{d}}}\,\|\sigma-\tau_h^g\|_{\mathrm{H}} \qquad \forall\, v_h \in \mathrm{Q}_h\,, \quad \forall\, \tau_h^g \in \mathrm{V}_h^g\,, \tag{B.19}$$

so that, employing additionally the triangle inequality to bound $\|\sigma - \sigma_h\|_{\mathrm{H}}$, we conclude that

$$\|\sigma - \sigma_h\|_{\mathrm{H}} \,\le\, \frac{\|\mathrm{B}\|}{\alpha_{\mathtt{d}}}\,\mathrm{dist}\big(u, \mathrm{Q}_h\big) \,+\, \left(1 + \frac{L_A}{\alpha_{\mathtt{d}}}\right)\,\mathrm{dist}\big(\sigma, \mathrm{V}_h^g\big)\,. \tag{B.20}$$

Next, we recall from the proof of [15, Theorem 2.6] that there holds

$$\mathrm{dist}\big(\sigma, \mathrm{V}_h^g\big) \,\le\, \left(1 + \frac{\|\mathrm{B}\|}{\beta_{\mathtt{d}}}\right)\,\mathrm{dist}\big(\sigma, \mathrm{H}_h\big)\,,$$

which, replaced back into (B.20), yields (B.16). In turn, the preliminary estimate for $\|u - u_h\|_{\mathrm{Q}}$ is obtained almost exactly as done in the proof of [15, Theorem 2.6], except that besides applying the triangle inequality and the discrete inf-sup condition (B.12), the Lipschitz-continuity of A needs to be employed as well. In this way, we easily get

$$\|u - u_h\|_{\mathrm{Q}} \,\le\, \left(1 + \frac{\|\mathrm{B}\|}{\beta_{\mathtt{d}}}\right)\,\mathrm{dist}\big(u, \mathrm{Q}_h\big) \,+\, \frac{L_A}{\beta_{\mathtt{d}}}\,\|\sigma - \sigma_h\|_{\mathrm{H}}\,, \tag{B.21}$$

which, together with (B.16), imply (B.17) and complete the proof. $\square$

We end this appendix by stressing, as commented in the proof of the foregoing theorem, that in the case that V_h is contained in V, the a priori estimate for $\|\sigma - \sigma_h\|_{\mathrm{H}}$ does not depend on $\mathrm{dist}(u, \mathrm{Q}_h)$, and hence the Cea estimates (B.16) and (B.17) simplify to

$$\|\sigma - \sigma_h\|_{\mathbf{H}} \,\le\, \left(1 + \frac{L_A}{\alpha_{\mathtt{d}}}\right)\left(1 + \frac{\|\mathrm{B}\|}{\beta_{\mathtt{d}}}\right)\,\mathrm{dist}\big(\sigma, \mathrm{H}_h\big) \tag{B.22}$$

and

$$\|u - u_h\|_{0,\Omega} \le \left(1 + \frac{\|\mathrm{B}\|}{\beta_{\mathtt{d}}}\right)\mathrm{dist}(u, \mathrm{Q}_h) + \frac{L_A}{\beta_{\mathtt{d}}}\left(1 + \frac{L_A}{\alpha_{\mathtt{d}}}\right)\left(1 + \frac{\|\mathrm{B}\|}{\beta_{\mathtt{d}}}\right)\mathrm{dist}\big(\sigma, \mathrm{H}_h\big). \tag{B.23}$$

References

[1] M. Aurada, M. Feischl, T. Führer, M. Karkulik, J.M. Melenk and D. Praetorius, *Classical FEM-BEM coupling methods: nonlinearities, well-posedness, and adaptivity.* Comput. Mech. 51 (2013), no. 4, 399-419.

[2] L. Beirão da Veiga, F. Brezzi, A. Cangiani, L.D Marini, G. Manzini and A. Russo, *Basic principles of virtual elements methods.* Math. Models Methods Appl. Sci. 23 (2013), no. 1, 199–214.

[3] D. Boffi, F. Brezzi and M. Fortin, Mixed Finite Element Methods and Applications. Springer Series in Computational Mathematics, 44. Springer, Heidelberg, 2013.

[4] F. Brezzi, R.S. Falk and L.D. Marini, *Basic principles of mixed virtual element methods.* ESAIM Math. Model. Numer. Anal. 48 (2014), no. 4, 1227–1240.

[5] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Springer Series in Computational Mathematics, 15. Springer-Verlag, New York, 1991.

[6] F. Brezzi and C. Johnson, *On the coupling of boundary integral and finite element methods.* Calcolo 16 (1979), no. 2, 189–201.

[7] U. Brink, C. Carstensen and E. Stein, *Symmetric coupling of boundary elements and Raviart-Thomas-type mixed finite elements in elastostatics*. Numer. Math. 75 (1996), no. 2, 153–174.

[8] C. Carstensen and S. Funken, *Coupling of mixed finite elements and boundary elements*. IMA J. Numer. Anal. 20 (2000), no. 3, 461–480.

[9] S. Caucao, M. Discacciati, G.N. Gatica and R. Oyarzúa, *A conforming mixed finite element method for the Navier–Stokes/Darcy–Forchheimer coupled problem*. ESAIM Math. Model. Numer. Anal., https://doi.org/10.1051/m2an/2020009, to appear.

[10] M. Costabel, *Symmetric methods for the coupling of finite elements and boundary elements*. In: Boundary Elements IX (C.A. Brebbia, G. Kuhn, W.L. Wendland eds.), Springer, Berlin, pp. 411-420, (1987).

[11] M. Costabel and E.P. Stephan, *Coupling of finite and boundary element methods for an elastoplastic interface problem*. SIAM J. Numer. Anal. 27 (1990), no. 5, 1212–1226.

[12] A. Ern and J.-L Guermond, Theory and Practice of Finite Elements. Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.

[13] A.I. Garralda-Guillem, G.N. Gatica, M. Ruiz Galán and A. Márquez, *A posteriori error analysis of twofold saddle point variational formulations for nonlinear boundary value problems*. IMA J. Numer. Anal. 34 (2014), no. 1, 326–361.

[14] G.N. Gatica, *On the Coupling of Boundary Integral and Finite Element Methods for Non-linear Boundary Value Problems*. Ph.D. Dissertation, University of Delaware, Delaware, USA, 1989.

[15] G.N. Gatica, A Simple Introduction to the Mixed Finite Element Method: Theory and Applications. SpringerBriefs in Mathematics. Springer, Cham, 2014.

[16] G.N. Gatica and N. Heuer, *A dual-dual formulation for the coupling of mixed-FEM and BEM in hyperelasticity*. SIAM J. Numer. Anal. 38 (2000), no. 2, 380–400.

[17] G.N. Gatica, N. Heuer and S. Meddahi, *On the numerical analysis of nonlinear twofold saddle point problems*. IMA J. Numer. Anal. 23 (2003), no. 2, 301–330.

[18] G.N. Gatica and G.C. Hsiao, *The coupling of boundary element and finite element methods for a nonlinear exterior boundary value problem*. Z. Anal. Anwendungen 8 (1989), no. 4, 377–387.

[19] G.N. Gatica and G.C. Hsiao, *On a class of variational formulations for some nonlinear interface problems*. Rend. Mat. Appl. (7) 10 (1990), no. 4, 681–715 (1991).

[20] G.N. Gatica and G.C. Hsiao, *On the coupled BEM and FEM for a nonlinear exterior Dirichlet problem in* $\mathbb{R}^2$. Numer. Math. 61 (1992), no. 2, 171–214.

[21] G.N. Gatica and G.C. Hsiao, *The coupling of boundary integral and finite element methods for non-monotone nonlinear problems*. Numer. Funct. Anal. Optim. 13 (1992), no. 5-6, 431–447.

[22] G.N. Gatica and G.C. Hsiao, *The uncoupling of boundary integral and finite element methods for nonlinear boundary value problems*. J. Math. Anal. Appl. 189 (1995), no. 2, 442–461.

[23] G.N. Gatica and G.C. Hsiao, Boundary-Field Equation Methods for a Class of Nonlinear Problems. Pitman Research Notes in Mathematics Series, vol. 331, (1995).

[24] G.N. Gatica, G.C. Hsiao, S. Meddahi and F.J. Sayas, *On the dual-mixed formulation for an exterior Stokes problem*. ZAMM Z. Angew. Math. Mech. 93 (2013), no. 6-7, 437–445.

[25] G.N. Gatica, G.C. Hsiao, S. Meddahi and F.J. Sayas, *New developments on the coupling of mixed-FEM and BEM for the three-dimensional exterior Stokes problem.* Int. J. Numer. Anal. Model. 13 (2016), no. 3, 457–492.

[26] G.N. Gatica, G.C. Hsiao and F.J. Sayas, *Relaxing the hypotheses of the Bielak-MacCamy BEM-FEM coupling.* Numer. Math. 120 (2012), no. 3, 465–487.

[27] G.N. Gatica and S. Meddahi, *On the coupling of VEM and BEM in two and three dimensions.* SIAM J. Numer. Anal. 57 (2019), no. 6, 2493–2518.

[28] G.N. Gatica and S. Meddahi, *Coupling of virtual element and boundary element methods for the solution of acoustic scattering problems.* J. Numer. Math. to appear, DOI: https://doi.org/10.1515/jnma-2019-0068.

[29] G.N. Gatica and W.L. Wendland, *Coupling of mixed finite elements and boundary elements for linear and nonlinear elliptic problems.* Appl. Anal. 63 (1996), no. 1-2, 39–75.

[30] H. Han, *A new class of variational formulations for the coupling of finite and boundary element methods.* J. Comput. Math. 8 (1990), no. 3, 223–232.

[31] G.C. Hsiao and W.L. Wendland, Boundary Integral Equations. Applied Mathematical Sciences, 164. Springer-Verlag, Berlin, 2008.

[32] C. Johnson and J.C. Nédélec, *On the coupling of boundary integral and finite element methods.* Math. Comp. 35 (1980), no. 152, 1063–1079.

[33] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge, 2000.

[34] S. Meddahi, F.J. Sayas and V. Selgás, *Nonsymmetric coupling of BEM and mixed FEM on polyhedral interfaces.* Math. Comp. 80 (2011), no. 273, 43–68.

[35] S. Meddahi and F.J. Sayas, *Analysis of a new BEM-FEM coupling for two-dimensional fluid-solid interaction.* Numer. Methods Partial Differential Equations 21 (2005), no. 6, 1017–1042.

[36] S. Meddahi and V. Selgás, *A mixed-FEM and BEM coupling for a three-dimensional eddy current problem.* M2AN Math. Model. Numer. Anal. 37 (2003), no. 2, 291–318.

[37] S. Meddahi, J. Valdés, O. Menéndez and P. Pérez, *On the coupling of boundary integral and mixed finite element methods.* J. Comput. Appl. Math. 69 (1996), no. 1, 113–124.

[38] J. Nečas, Introduction to the Theory of Nonlinear Elliptic Equations. Reprint of the 1983 edition. A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1986.

[39] A. Quarteroni and A. Valli, Numerical Approximation of Partial Differential Equations. Springer Series in Computational Mathematics, 23. Springer-Verlag, Berlin, 1994.

[40] S.A. Sauter and C. Schwab, Boundary Element Methods. Springer Series in Computational Mathematics, 39. Springer-Verlag, Berlin, 2011.

[41] F.J. Sayas, *The validity of Johnson-Nédélec's BEM-FEM coupling on polygonal interfaces.* SIAM J. Numer. Anal. 47 (2009), no. 5, 3451–3463.

[42] F.J. Sayas, *The validity of Johnson-Nédélec's BEM-FEM coupling on polygonal interfaces.* SIAM Rev. 55 (2013), no. 1, 131–146.

[43] B. Scheurer, *Existence et approximation de points selles pour certains problèmes non linéaires.* RAIRO Anal. Numér. 11 (1977), no. 4, 369–400.

[44] O. Steinbach, *A note on the stable one-equation coupling of finite and boundary elements.* SIAM J. Numer. Anal. 49 (2011), no. 4, 1521–1531.

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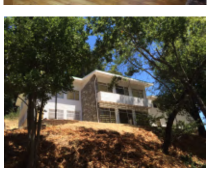

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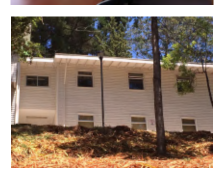
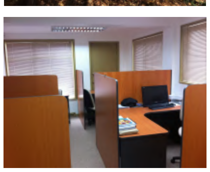

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