UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en
Ingeniería Matemática (CI^{2}MA)

A virtual marriage a la mode: some recent results on the coupling of VEM and BEM

Gabriel N. Gatica, Antonio Marquez,
Salim Meddahi

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A virtual marriage à la mode: some recent results on the coupling of VEM and BEM

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Abstract The aim of this chapter is to present in a unified way some recent results on the combined use of the virtual element method (VEM) and the boundary element method (BEM) to numerically solve linear transmission problems in 2D and 3D. As models we consider an elliptic equation in divergence form holding in an annular domain coupled with the Laplace equation in the corresponding unbounded exterior region, and an acoustic scattering problem determined by a bounded obstacle and a time harmonic incident wave, so that the scattered field, and hence the total wave as well, satisfies the homogeneous Helmholtz equation. Both sets of corresponding equations are complemented with proper transmission conditions at the respective interfaces, and suitable radiation conditions at infinity. We employ the usual primal formulation and the associated VEM approach in the respective bounded regions, and combine it, by means of either the Costabel & Han approach or a recent modification of it, with the boundary integral equation method in the exterior domain, thus yielding two possible VEM/BEM schemes. The first method is valid only in 2D and considers the main variable and its normal derivative as unknowns, whereas the second one, which includes additionally the trace of the former as a third unknown, is applicable in both dimensions. The well-posedness of the continuous and discrete formulations is established and a priori error estimates together with corresponding rates of convergence are derived. Finally, several numerical examples in 2D illustrating the performance of the proposed discrete schemes are reported.

Gabriel N. Gatica
CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, e-mail: ggatica@ci2ma.udec.cl

Antonio Márquez
Departamento de Construcción e Ingeniería de Fabricación, Universidad de Oviedo, Oviedo, España, e-mail: amarquez@uniovi.es

Salim Meddahi
Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, Calvo Sotelo s/n, Oviedo, España, e-mail: salim@uniovi.es

1 Introduction

The concept "marriage à la mode" was originally employed in [10], one of the seminal papers on the subject back in the 70', to refer to the combined use (also named coupling) of the finite element (FEM) and boundary element (BEM) methods and the advantages of performing this "marriage". The mathematical fundamentals of this novel idea was provided either around the same time or short after in [9], [13] and [28], and the first resulting method, which uses a single boundary integral equation arising from the Green representation formula of the solution, is known nowadays as the Johnson & Nédélec approach. Until a couple of decades after, the applicability of this technique, being based on the compactness of a boundary integral operator involved and the Fredholm theory, was restricted mainly to transmission problems involving the Laplace operator. For other problems of interest, such as the Lamé system, the compactness property does not hold and hence the method could not be employed to this model.

The aforementioned limitation motivated the coupling procedures by Costabel and Han in [19] and [26], respectively, which were both based on the incorporation of a second boundary integral equation, namely the one that is obtained after applying the normal derivative (or traction in the case of elasticity) to the Green formula. In this way, the former technique yielded a symmetric and non-positive definite scheme, whereas the latter, on the contrary, gave rise to a non-symmetric but elliptic system. However, one simply refers to either one of them as the Costabel & Han method since they only differ in the sign of a common integral identity. In turn, the historical drawback of the Johnson & Nédélec coupling method, was surprinsingly solved in [33] (see also [34] and [35]), where it was shown that actually all Galerkin schemes for this approach are stable, thus extending its use to other elliptic equations and to arbitrary polygonal/polyhedral domains.

On the other hand, the virtual element method (VEM) has become during the last decade a very promising technique to numerically solve diverse linear and nonlinear boundary value problems in continuum mechanics. Among its many applications, we can mention linear elasticity, plate bending problems, the Steklov eigenvalue problem, acoustic vibration, and diverse models in fluid mechanics. In particular, the latter includes stream function-based, divergence free, and non-conforming virtual element methods for the classical velocity-pressure formulation of the Stokes equation, primal virtual element approaches for the Darcy, Brinkman, and Navier-Stokes models, and dual-mixed variational formulations yielding mixed virtual element schemes for the Stokes equation, the linear and nonlinear Brinkman problems, the nonlinear Stokes equation arising from quasi-Newtonian Stokes flows, and the Navier-Stokes equations, as well. A representative, though not exhaustive, list of works concerning theoretical and applied aspects of VEM, besides certainly the other chapters of the present proceedings, includes [2], [3], [4], [5], [7], [8], [11], [12], [14], [15], [16], [25], [31], [36] and the references therein.

In the same direction as above, and aiming to continue extending the applicability of VEM, we have recently introduced and analyzed in [23] and [24], up to our knowledge for the first time, the combined use of VEM and BEM for solving transmission problems in 2D and 3D. The own advantages of each method, properly discussed in those references, are certainly transferred to the combined use of them.

The main purpose of this chapter is precisely to present a unified treatment of the main tools and results from [23] and [24]. While specific models are considered there, the main motivation of these works and hence of the present one, is to settle the main basis allowing to analyze later on any other particular model of interest that is solved via the coupling of VEM and BEM. In particular, this might be the case for unbounded domains with a bounded complex heterogeneous region for which the corresponding partitions are constructed in a much easier way by using nonconvex elements. Other recent contributions on the coupling of VEM and BEM, mainly referring to computational and applied aspects of it, are presented in [1] and [20].

Our models from [23] and [24] are described in what follows. To this end, we let Ω_0 and $\mathcal{O}$ be two simply connected and bounded polygonal/polyhedral domains in $\mathbb{R}^d$, $d = 2, 3$, with boundaries $\Gamma_0 := \partial\Omega_0$ and $\Gamma := \partial\mathcal{O}$, respectively, such that $\Omega_0 \subseteq \mathcal{O}$. In addition, we introduce the annular region $\Omega := \mathcal{O} \setminus \overline{\Omega}_0$ and the exterior domain $\mathcal{O}_e := \mathbb{R}^d \setminus \overline{\mathcal{O}}$ (see Figure 1 below), and denote by $\boldsymbol{n}$ the unit outward normal to Γ pointing towards $\mathcal{O}_e$.

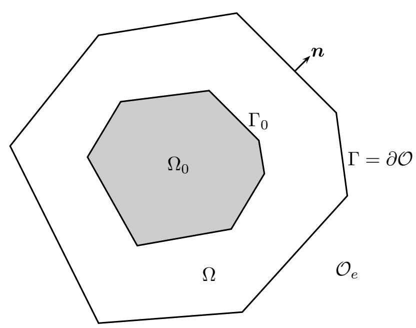

Fig. 1 2D geometry of the model problems.

The first model consists of an elliptic equation in divergence form holding in Ω coupled with the Laplace equation in the unbounded exterior region $\mathcal{O}_e$, together with transmission conditions on the interface Γ and a suitable radiation condition at infinity, that is we look for $u : \Omega \longrightarrow \mathbb{R}$ and $u_e : \mathcal{O}_e \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} -\operatorname{div}(\kappa\nabla u) &= f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma_0, \\ u = u_e \quad \text{on} \quad \Gamma, &\quad \kappa\frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial u_e}{\partial \boldsymbol{n}} \quad \text{on} \quad \Gamma, \\ \Delta u_e = 0 \quad \text{in} \quad \mathcal{O}_e, &\quad u_e(x) = O(\frac{1}{|x|}) \quad \text{as} \quad |x| \longrightarrow \infty, \end{aligned} \tag{1}$$

where $f \in \mathrm{L}^2(\Omega)$ and $\kappa \in \mathrm{L}^\infty(\Omega)$ are given functions. Additionally, we assume that there exists a constant $\underline{\kappa} > 0$ such that

$$\underline{\kappa} \leq \kappa(x) \leq \overline{\kappa} := \|\kappa\|_{L^\infty(\Omega)} \qquad \forall\, x \in \Omega\,.$$

Throughout the rest of the paper we call (1) the Poisson model.

In turn, in order to define the second model we let $\theta : \mathbb{R}^d \to \mathbb{C}$ be a complex-valued piecewise constant function satisfying $\mathrm{Re}\big(\theta(\boldsymbol{x})\big) > 0$ and $\mathrm{Im}\big(\theta(\boldsymbol{x})\big) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^d$, and such that $1 - \theta(\boldsymbol{x})$ has a compact support contained in O. Also, we let $\kappa > 0$ be a given constant, and let w be a function satisfying the Helmholtz equation $\Delta w + \kappa^2 w = 0$ in $\mathbb{R}^d$. Then, we consider an obstacle occupying O with refractive index given by θ, and assume that w acts as a time harmonic incident wave, so that the scattered field u^s, and hence the total wave $u := w + u^s$ as well, satisfy the homogeneous Helmholtz equation in O_e. In this way, the resulting coupled problem, which is complemented with suitable transmission conditions on Γ and the Sommerfeld radiation condition at infinity, reduces to find $u : O \longrightarrow \mathbb{C}$ and $u^s : O_e \longrightarrow \mathbb{C}$ such that

$$\begin{gathered}
\Delta u + \kappa^2 \theta(\boldsymbol{x})\, u \;=\; 0 \quad \text{in} \quad O\,, \\
u \;=\; u^s + w \quad \text{on} \quad \Gamma, \quad \frac{\partial u}{\partial \boldsymbol{n}} \;=\; \frac{\partial u^s}{\partial \boldsymbol{n}} + \frac{\partial w}{\partial \boldsymbol{n}} \quad \text{on} \quad \Gamma\,, \\
\Delta u^s + \kappa^2\, u^s \;=\; 0 \quad \text{in} \quad O_e\,, \\
\frac{\partial u^s}{\partial r} - \imath\kappa u^s = o(r^{\frac{1-d}{2}}) \quad \text{when} \quad r := |\boldsymbol{x}| \to \infty\,.
\end{gathered} \tag{2}$$

The above is named from now on the Helmholtz model.

The rest of the chapter is organized as follows. In section 2 we first describe the basic aspects of the boundary integral equation method (BIEM), and then introduce and analyze the Costabel & Han and modified Costabel & Han coupling methods. The discrete VEM/BEM schemes for the Costabel & Han approach as applied to both models from Section 1 are studied in Section 3 for the 2D case. Next, in Section 4 we explain the necessity of introducing the modified Costabel & Han coupling procedure in the 3D case and introduce and analyze its applicability to the VEM/BEM scheme for the Poisson model. Finally, in Section 5 we illustrate the performance of our discrete methods with several numerical results in 2D.

We end this section with some notations to be employed throughout the rest of the paper. In particular, given a real number $r \geq 0$ and a polyhedron $\mathcal{G} \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$, we denote by $\|\cdot\|_{r,\mathcal{G}}$ and $|\cdot|_{r,\mathcal{G}}$, respectively, the norm and seminorm of the usual Sobolev space $\mathrm{H}^r(\mathcal{G})$ (cf. [29]). Also, we use the convention $\mathrm{L}^2(\mathcal{G}) := \mathrm{H}^0(\mathcal{G})$, and for all $t \in (0, 1]$ we let $\mathrm{H}^{-t}(\partial\mathcal{G})$ be the dual of $\mathrm{H}^t(\partial\mathcal{G})$ with respect to the pivot space $\mathrm{L}^2(\partial\mathcal{G})$. In addition, we set $\mathcal{P}_{-1} = \{0\}$, and for a nonnegative integer m, $\mathcal{P}_m$ is the space of polynomials of degree $\leq m$. Then, given a set $D \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$, $\mathcal{P}_m(D)$ stands for the restriction of $\mathcal{P}_m$ to D.

2 The coupling procedures

Here we introduce and analyze the continuous formulations of the two coupling procedures that we utilize for the combination of VEM and BEM. Both approaches require the basic aspects of the boundary integral equation method (BIEM) as applied to the Laplace and Helmholtz equations, which is addressed in the following section.

2.1 BIEM for Laplace and Helmholtz

We begin by letting γ and $\gamma_{\boldsymbol{n}}$ be the trace and normal trace operators, respectively, on Γ, acting either from O (equivalently from Ω) or from O_e. Then, the harmonic solution u_e in the exterior domain O_e (cf. third row of (1)) can be represented by the Green formula

$$u_e(\boldsymbol{x}) \;=\; \int_\Gamma \frac{\partial \mathbf{E}(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\,\gamma u(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}} \;-\; \int_\Gamma \mathbf{E}(|\boldsymbol{x}-\boldsymbol{y}|)\lambda(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}} \qquad \forall\, \boldsymbol{x}\in O_e\,, \quad (3)$$

where

$$\mathbf{E}(|\boldsymbol{x}-\boldsymbol{y}|) := \begin{cases} \dfrac{1}{4\pi}\dfrac{1}{|\boldsymbol{x}-\boldsymbol{y}|} & \text{if } d=3\,,\\[2mm] -\dfrac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{y}| & \text{if } d=2\,, \end{cases}$$

is the fundamental solution of the Laplace operator, and, according to the transmission conditions at the second row of (1), $\gamma u = \gamma u_e$ and $\lambda := \gamma_{\boldsymbol{n}}(\kappa\nabla u) = \gamma_{\boldsymbol{n}}(\nabla u_e)$ are the Cauchy data on the interface Γ. Then, applying γ and $\gamma_{\boldsymbol{n}}$ from O_e to (3) and its gradient, respectively, and employing the jump conditions on Γ of the two potentials in the right hand side of (3), we arrive at (cf. [27], [32])

$$\gamma u_e \;=\; \big(\frac{\mathrm{id}}{2} + K\big)\gamma u \;-\; V\lambda \quad \text{on} \quad \Gamma\,, \qquad (4)$$

and

$$\gamma_{\boldsymbol{n}}(\nabla u_e) \;=\; -\,W\gamma u \;+\; \big(\frac{\mathrm{id}}{2} - K^{\mathtt{t}}\big)\lambda \quad \text{on} \quad \Gamma\,, \qquad (5)$$

where V, K, $K^{\mathtt{t}}$ are the boundary integral operators representing the single, double and adjoint of the double layer, respectively, id is a generic identity operator, and W is the hypersingular operator. In this way, replacing γu_e and $\gamma_{\boldsymbol{n}}(\nabla u_e)$ by γu and λ, respectively, (4) and (5) become

$$0 \;=\; \big(\frac{\mathrm{id}}{2} - K\big)\gamma u \;+\; V\lambda \quad \text{on} \quad \Gamma\,, \qquad (6)$$

and

$$\lambda \;=\; -\,W\gamma u \;+\; \big(\frac{\mathrm{id}}{2} - K^{\mathtt{t}}\big)\lambda \quad \text{on} \quad \Gamma\,. \qquad (7)$$

In turn, denoting by $H_0^{(1)}$ the Hankel function of order 0 and first type, it can be proved that the solution u^s of the homogeneous Helmholtz equation in O_e (cf. third row of (2)) admits the integral representation

$$u^s(\boldsymbol{x}) = \int_\Gamma \frac{\partial \mathbf{E}_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\gamma u^s(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}} - \int_\Gamma \mathbf{E}_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\lambda(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}} \qquad \forall \boldsymbol{x}\in O_e\,, \quad (8)$$

where

$$\mathbf{E}_\kappa(|\boldsymbol{x}-\boldsymbol{y}|) := \begin{cases} \dfrac{\imath}{4} H_0^{(1)}(\kappa|\boldsymbol{x}-\boldsymbol{y}|) & \text{if } d=2,\\ \dfrac{e^{\imath\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} & \text{if } d=3, \end{cases}$$

is the fundamental solution of the Helmholtz equation with wave number κ, and $\lambda := \gamma_{\boldsymbol{n}}(\nabla u^s) = \dfrac{\partial u^s}{\partial \boldsymbol{n}}$. Next, proceeding similarly to the derivation of (6) and (7), which means now applying γ and $\gamma_{\boldsymbol{n}}$ to (8), and taking into account the corresponding jump properties of the potentials involved (see again [27], [32]), we arrive at

$$0 = (\frac{\mathrm{id}}{2} - K_\kappa)\gamma u^s + V_\kappa \lambda, \tag{9}$$

and

$$\lambda = -W_\kappa \gamma u^s + (\frac{\mathrm{id}}{2} - K_\kappa^{\mathtt{t}})\lambda, \tag{10}$$

where V_κ, K_κ, $K_\kappa^{\mathtt{t}}$, and W_κ are the boundary integral operators representing the single, double, adjoint of the double, and hypersingular layer potentials, respectively.

We end this section with some useful properties of the boundary integral operators involved in (6) - (7) and (9) - (10). Indeed, V, K, $K^{\mathtt{t}}$, and W are formally defined at almost every point $\boldsymbol{x} \in \Gamma$ by

$$\begin{aligned} V\lambda(\boldsymbol{x}) &:= \int_\Gamma \mathbf{E}(|\boldsymbol{x}-\boldsymbol{y}|)\lambda(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}}\,,\\ K\varphi(\boldsymbol{x}) &:= \int_\Gamma \frac{\partial \mathbf{E}(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\,\varphi(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}}\,,\\ K^{\mathtt{t}}\lambda(\boldsymbol{x}) &:= \int_\Gamma \frac{\partial \mathbf{E}(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\,\lambda(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}}\,,\\ W\varphi(\boldsymbol{x}) &:= -\frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\int_\Gamma \frac{\partial \mathbf{E}(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\,\varphi(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}}\,, \end{aligned} \tag{11}$$

for suitable functions λ and φ, whereas V_κ, K_κ, $K_\kappa^{\mathtt{t}}$, and W_κ are defined analogously to (11) by replacing $\mathbf{E}$ by $\mathbf{E}_\kappa$. Moreover, the main mapping properties of these operators are collected in the following lemma (cf. [32]).

Lemma 1 *The operators*

$$V,\, V_\kappa :\, H^{-1/2+\sigma}(\Gamma) \longrightarrow H^{1/2+\sigma}(\Gamma), \qquad K,\, K_\kappa :\, H^{1/2+\sigma}(\Gamma) \longrightarrow H^{1/2+\sigma}(\Gamma)$$

$$K^{\mathtt{t}},\, K_\kappa^{\mathtt{t}} :\, H^{-1/2+\sigma}(\Gamma) \longrightarrow H^{-1/2+\sigma}(\Gamma), \qquad W,\, W_\kappa :\, H^{1/2+\sigma}(\Gamma) \longrightarrow H^{-1/2+\sigma}(\Gamma),$$

are continuous for all $\sigma \in [-1/2,\ 1/2]$.

Furthermore, we now let $\langle \cdot, \cdot \rangle$ be both the inner product in $\mathrm{L}^2(\Gamma)$ and the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$ with respect to the pivot space $\mathrm{L}^2(\Gamma)$, and introduce the subspaces

$$\mathrm{H}_0^{1/2}(\Gamma) := \{\varphi \in \mathrm{H}^{1/2}(\Gamma) :\ \langle 1, \varphi \rangle = 0\}$$

and

$$\mathrm{H}_0^{-1/2}(\Gamma) := \{\mu \in \mathrm{H}^{-1/2}(\Gamma) : \langle \mu, 1 \rangle = 0\} .$$

Then, we have the following lemma providing ellipticity-type properties of the operators V and W (cf. [29, 32]).

Lemma 2 *There exist positive constants α_V, C_V, and α_W such that*

$$\langle \bar{\mu},\, V\mu \rangle \;\geq\; \alpha_V \;\|\mu\|_{-1/2,\Gamma}^2 \qquad \begin{cases} \forall\, \mu \in \mathrm{H}_0^{-1/2}(\Gamma), & \text{if } d = 2\,, \\ \forall\, \mu \in \mathrm{H}^{-1/2}(\Gamma), & \text{if } d = 3\,, \end{cases} \tag{12}$$

$$\langle \bar{\mu}, V\mu \rangle_\Gamma + \left|\langle \bar{\mu}, 1 \rangle\right|^2 \geq C_V \;\|\mu\|_{-1/2,\Gamma}^2 \quad \forall \mu \in \mathrm{H}^{-1/2}(\Gamma) \quad \text{if } d = 2\,, \tag{13}$$

and

$$\langle W\varphi,\, \bar{\varphi} \rangle \;\geq\; \alpha_W \;\|\varphi\|_{1/2,\Gamma}^2 \quad \forall\, \varphi \in \mathrm{H}_0^{1/2}(\Gamma)\,. \tag{14}$$

We end this section by stressing that the operators associated to the Helmholtz equation, that is V_κ, K_κ, $K_\kappa^{\mathtt{t}}$, and W_κ, may be regarded as compact perturbations of those corresponding to the Laplacian, that is V, K, $K^{\mathtt{t}}$, and W. In fact, we have the following lemma (cf. [32]).

Lemma 3 *The operators*

$$V_\kappa - V :\; H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \qquad K_\kappa - K :\; H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$$

$$K_\kappa^{\mathtt{t}} - K^{\mathtt{t}} :\; H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \qquad W_\kappa - W :\; H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma),$$

are compact.

2.2 The Costabel & Han coupling

Our first coupling method, which makes use of the pairs of boundary integral equations (6) - (7) and (9) - (10) to reformulate problems (1) and (2) in the bounded domains Ω and $\mathcal{O}$, respectively, is due to Costabel and Han (cf. [19] and [26]). More precisely, the reformulation of (1) reads: Find $u : \Omega \to \mathbb{R}$ and $\lambda : \Gamma \to \mathbb{R}$ such that

$$\begin{aligned} -\operatorname{div}(\kappa \nabla u) \;=\; f \quad \text{in} \quad \Omega, \quad u \;=\; 0 \quad \text{on} \quad \Gamma_0\,, \\ \kappa \frac{\partial u}{\partial \boldsymbol{n}} \;=\; \lambda \quad \text{on} \quad \Gamma, \\ 0 \;=\; \big(\frac{\mathrm{id}}{2} - K\big)\gamma u \;+\; V\lambda \quad \text{on} \quad \Gamma, \\ \lambda \;=\; -W\gamma u \;+\; \big(\frac{\mathrm{id}}{2} - K^{\mathtt{t}}\big)\lambda \quad \text{on} \quad \Gamma, \end{aligned} \tag{15}$$

whereas the one of (2) becomes: Find $u : \mathcal{O} \to \mathbb{C}$ and $\lambda : \Gamma \to \mathbb{C}$ such that

$$
\begin{gathered}
\Delta u + \kappa^2\theta(\boldsymbol{x})\, u = 0 \quad \text{in} \quad \mathcal{O}\,, \\
\gamma u = \gamma u^s + \gamma w \quad \text{on} \quad \Gamma\,, \quad \frac{\partial u}{\partial \boldsymbol{n}} = \lambda + \frac{\partial w}{\partial \boldsymbol{n}} \quad \text{on} \quad \Gamma\,, \\
0 = (\frac{\mathrm{id}}{2} - K_\kappa)\gamma u^s + V_\kappa \lambda \quad \text{on} \quad \Gamma\,, \\
\lambda = -W_\kappa \gamma u^s + (\frac{\mathrm{id}}{2} - K_\kappa^{\mathtt{t}})\lambda \quad \text{on} \quad \Gamma\,.
\end{gathered} \tag{16}
$$

Then, introducing the spaces

$$
X \,:=\, \Big\{ v \in \mathrm{H}^1(\Omega) :\; v|_{\Gamma_0} = 0 \Big\} \quad \text{and} \quad \mathbf{X} \,:=\, X \times \mathrm{H}_0^{-1/2}(\Gamma)\,,
$$

multiplying the first equation of the first row of (15) by $v \in X$, integrating the resulting expression by parts, replacing $\lambda = \kappa \dfrac{\partial u}{\partial \boldsymbol{n}}$ on Γ by the right hand side of the fourth row of (15), and finally testing the third row of (15) against $\mu \in \mathrm{H}_0^{-1/2}(\Gamma)$, we arrive at the variational formulation: Find $(u, \lambda) \in \mathbf{X}$ such that

$$
\begin{aligned}
\int_\Omega \kappa \nabla u \cdot \nabla v + \big\langle W\gamma u, \gamma v \big\rangle - \big\langle \lambda, (\frac{\mathrm{id}}{2} - K)\gamma v) \big\rangle &= \int_\Omega f v \qquad \forall v \in X\,, \\
\big\langle \mu, V\lambda \big\rangle + \big\langle \mu, (\frac{\mathrm{id}}{2} - K)\gamma u) \big\rangle &= 0 \qquad\quad \forall \mu \in \mathrm{H}_0^{-1/2}(\Gamma)\,.
\end{aligned} \tag{17}
$$

Equivalently, (17) can be rewritten as: Find $(u, \lambda) \in \mathbf{X}$ such that

$$
\mathbf{A}\big((u, \lambda), (v, \mu)\big) \;=\; \mathbf{F}(v, \mu) \qquad \forall\, (v, \mu) \in \mathbf{X}\,, \tag{18}
$$

where

$$
\begin{aligned}
\mathbf{A}\big((z, \xi), (v, \mu)\big) \;:=\; & a(z, v) + \big\langle W\gamma z, \gamma v \big\rangle + \big\langle \mu, V\xi \big\rangle \\
& + \big\langle \mu, (\frac{\mathrm{id}}{2} - K)\gamma z) \big\rangle - \big\langle \xi, (\frac{\mathrm{id}}{2} - K)\gamma v) \big\rangle\,,
\end{aligned} \tag{19}
$$

with

$$
a(z, v) \;:=\; \int_\Omega \kappa \nabla z \cdot \nabla v\,, \tag{20}
$$

and

$$
\mathbf{F}(v, \mu) \;:=\; \int_\Omega f v\,, \tag{21}
$$

for all (z, ξ), $(v, \mu) \in \mathbf{X}$.

Proceeding similarly to the derivation of (17), we readily find that the variational formulation of (16) reads: Find $(u, \lambda) \in \mathbb{X} := \mathrm{H}^1(\mathcal{O}) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$
\mathbb{A}_\kappa\big((u, \lambda), (v, \mu)\big) = \mathbb{F}(v, \mu) \qquad \forall (v, \mu) \in \mathbb{X}\,, \tag{22}
$$

where

$$
\begin{aligned}
\mathbb{A}_\kappa\big((z, \xi), (v, \mu)\big) := \; & a_\kappa(z, v) + \big\langle W_\kappa \gamma z, \gamma v \big\rangle + \big\langle \mu, V_\kappa \xi \big\rangle \\
& + \big\langle \mu, (\frac{\mathrm{id}}{2} - K_\kappa)\gamma z \big\rangle - \big\langle \xi, (\frac{\mathrm{id}}{2} - K_\kappa)\gamma v \big\rangle\,,
\end{aligned} \tag{23}
$$

with

$$a_\kappa(z,v) := \int_O \nabla z \cdot \nabla v - \kappa^2 \int_O \theta z v\,, \tag{24}$$

and

$$\mathbb{F}(v,\mu) := \left\langle \frac{\partial w}{\partial \boldsymbol{n}} + W_\kappa \gamma w, \gamma v \right\rangle + \left\langle \mu, (\frac{\mathrm{id}}{2} - K_\kappa)\gamma w \right\rangle,$$

for all (z,ξ), $(v,\mu) \in \boldsymbol{X}$.

2.3 The modified Costabel & Han coupling

We now consider the modified Costabel & Han coupling method that was introduced for the first time in [23, Section 4.2]. More precisely, in addition to $\lambda = \kappa \dfrac{\partial u}{\partial \boldsymbol{n}}$ (cf. (15)) or $\lambda = \gamma_{\boldsymbol{n}}(\nabla u^s)$ (cf. (16)), this approach introduces the trace $\psi := \gamma u$ or $\psi := \gamma u^s$ as a boundary unknown as well of the formulation. As a consequence, instead of (3) and (8), the harmonic function u_e and the scattered field u^s are computed as

$$u_e(\boldsymbol{x}) \;=\; \int_\Gamma \frac{\partial \mathbf{E}(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\, \psi(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}} \;-\; \int_\Gamma \mathbf{E}(|\boldsymbol{x}-\boldsymbol{y}|)\lambda(\boldsymbol{y})\,\mathrm{d}s_{\boldsymbol{y}} \qquad \forall\, \boldsymbol{x} \in O_e\,,$$

and

$$u^s(\boldsymbol{x}) = \int_\Gamma \frac{\partial E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_y}\psi(\boldsymbol{y})\,\mathrm{d}s_y - \int_\Gamma E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\lambda(\boldsymbol{y})\,\mathrm{d}s_y \qquad \forall \boldsymbol{x} \in O_e\,,$$

respectively, whence the corresponding pairs of identities (6) - (7) and (9) - (10) become

$$\begin{aligned} 0 \;&=\; \big(\frac{\mathrm{id}}{2} - K\big)\psi \;+\; V\lambda \quad \text{on} \quad \Gamma,\\ \lambda \;&=\; -W\psi \;+\; \big(\frac{\mathrm{id}}{2} - K^{\mathrm{t}}\big)\lambda \quad \text{on} \quad \Gamma, \end{aligned}$$

and

$$\begin{aligned} 0 &= (\frac{\mathrm{id}}{2} - K_\kappa)\psi + V_\kappa \lambda \quad \text{on} \quad \Gamma,\\ \lambda &= -W_\kappa \psi + (\frac{\mathrm{id}}{2} - K_\kappa^{\mathrm{t}})\lambda \quad \text{on} \quad \Gamma\,. \end{aligned}$$

Then, proceeding as for the derivation of (18), but additionally adding and subtracting the expression $\langle \lambda, \varphi\rangle$ with arbitrary $\varphi \in \mathrm{H}_0^{1/2}(\Gamma)$, and imposing weakly the relation $\psi = \gamma u$ in $\mathrm{H}^{1/2}(\Gamma)$, the modified Costabel & Han formulation of (1) reduces to: Find $(u,\psi,\lambda) \in \widetilde{\mathbf{X}} := X \times \mathrm{H}_0^{1/2}(\Gamma) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\widetilde{\mathbf{A}}\big((u,\psi,\lambda),(v,\varphi,\mu)\big) \;=\; \widetilde{\mathbf{F}}(v,\varphi,\mu) \qquad \forall\,(v,\varphi,\mu) \in \widetilde{\mathbf{X}}, \tag{25}$$

where

$$\begin{aligned} \widetilde{\mathbf{A}}\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;&=\; \mathbf{a}\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;+\; \big\langle W\phi, \varphi \big\rangle \\ &+\; \big\langle \mu, V\xi \big\rangle \;-\; \big\langle \xi, \big(\frac{\mathrm{id}}{2} - K\big)\varphi \big\rangle \;+\; \big\langle \mu, \big(\frac{\mathrm{id}}{2} - K\big)\phi \big\rangle, \end{aligned} \tag{26}$$

with

$$\mathbf{a}\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;=\; a(z,v) \;-\; \big\langle \xi, \gamma v - \varphi \big\rangle \;+\; \big\langle \mu, \gamma z - \phi \big\rangle\,, \tag{27}$$

a being defined by (20), and

$$\widetilde{\mathbf{F}}(v,\varphi,\mu) := \int_\Omega f\,v\,, \tag{28}$$

for all $(z,\phi,\xi),\ (v,\varphi,\mu) \in \widetilde{\mathbf{X}}$.

Analogously, proceeding similarly to the derivation of (25), but now imposing weakly the relation $\psi = \gamma u^s$ in $\mathrm{H}^{1/2}(\Gamma)$, the modified Costabel & Han formulation of (2) reads: Find $(u,\psi,\lambda) \in \widetilde{\mathbb{X}} := \mathrm{H}^1(\mathcal{O}) \times \mathrm{H}^{1/2}_0(\Gamma) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\widetilde{\mathbb{A}}_\kappa\big((u,\psi,\lambda),(v,\varphi,\mu)\big) \;=\; \widetilde{\mathbb{F}}(v,\varphi,\mu) \qquad \forall\,(v,\varphi,\mu) \in \widetilde{\mathbb{X}}\,, \tag{29}$$

where

$$\begin{aligned}\widetilde{\mathbb{A}}_\kappa\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;&:=\; \mathbf{a}_\kappa\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;+\; \big\langle W_\kappa\phi, \varphi \big\rangle \\ &+\; \big\langle \mu, V_\kappa\xi \big\rangle \;+\; \big\langle \mu, \big(\frac{\mathrm{id}}{2} - K_\kappa\big)\phi \big\rangle \;-\; \big\langle \xi, \big(\frac{\mathrm{id}}{2} - K_\kappa\big)\varphi \big\rangle\end{aligned} \tag{30}$$

with

$$\mathbf{a}_\kappa\big((z,\phi,\xi),(v,\varphi,\mu)\big) \;:=\; a_\kappa(z,v) \;-\; \big\langle \xi, \gamma v - \varphi \big\rangle \;+\; \big\langle \mu, \gamma z - \phi \big\rangle\,,$$

a_κ given by (24), and

$$\widetilde{\mathbb{F}}(v,\varphi,\mu) := \langle \frac{\partial w}{\partial \boldsymbol{n}}, \gamma v \rangle \;+\; \langle \mu, \gamma w \rangle\,,$$

for all $(z,\phi,\xi),\ (v,\varphi,\mu) \in \widetilde{\mathbb{X}}$.

2.4 Solvability analysis

In this section we address the solvability of the Costabel & Han and modified Costabel & Han coupling procedures as applied to the Poisson (cf. (18), (25)) and Helmholtz (cf. (22), (29)) models.

We begin the analysis with the formulations (18) and (25). Indeed, bearing in mind the definitions of the bilinear forms $\mathbf{A}$ (cf. (19)) and $\widetilde{\mathbf{A}}$ (cf. (26)), we easily deduce from Lemmas 1 and 2 that there exist positive constants $\|\mathbf{A}\|$, $\|\widetilde{\mathbf{A}}\|$, α, and $\widetilde{\alpha}$, such that

$$\mathbf{A}(z,\xi),(v,\mu) \le \|\mathbf{A}\|\,\|(z,\xi)\|\,\|(v,\mu)\| \quad \forall\,(z,\xi),\ (v,\mu) \in \boldsymbol{X}\,,$$

$$\widetilde{\mathbf{A}}(z,\phi,\xi),(v,\varphi,\mu) \le \|\widetilde{\mathbf{A}}\|\,\|(z,\phi,\xi)\|\,\|(v,\varphi\mu)\| \quad \forall\,(z,\phi,\xi),\ (v,\varphi,\mu) \in \widetilde{\mathbf{X}}\,,$$

$$\mathbf{A}(v,\mu),(v,\mu) \ge \alpha\,\|(v,\mu)\|^2 \quad \forall\,(v,\mu) \in \boldsymbol{X}\,,$$

and

$$\widetilde{\mathbf{A}}(v,\varphi,\mu),(v,\mu) \ge \alpha\,\|(v,\varphi,\mu)\|^2 \quad \forall\,(v,\varphi,\mu) \in \widetilde{\mathbf{X}}\,,$$

where

$$\|(v,\mu)\|^2 := \|v\|_{1,\Omega}^2 + \|\mu\|_{-1/2,\Gamma}^2$$

and

$$\|(v,\varphi,\mu)\|^2 := \|v\|_{1,\Omega}^2 + \|\varphi\|_{1/2,\Gamma}^2 + \|\mu\|_{-1/2,\Gamma}^2 \,.$$

In this way, since the boundedness of the linear functionals $\mathbf{F}$ (cf. (21)) and $\widetilde{\mathbf{F}}$ (cf. (28)) follow from a simple application of the Cauchy-Schwarz inequality, we conclude the well-posedness of problems (18) and (25) as a straightforward consequence of the foregoing estimates and the Lax-Milgram lemma.

Next, we deal with the solvability analysis of (22) and (29). To this end, we now introduce the compact perturbations (fact to be confirmed later on) of the bilinear forms $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ that are obtained from (23) and (30), respectively, by taking $\kappa = 0$, by replacing V_κ, K_κ, K_κ^{t}, and W_κ by V, K, K^{t}, and W, respectively, and by adding suitable one-dimensional terms, that is

$$\begin{aligned}\mathbb{A}_0\big((z,\xi),(v,\mu)\big) &:= a(z,v) + \left(\int_\Gamma z\right)\left(\int_\Gamma v\right) + \big\langle W\gamma z, \gamma v\big\rangle + \big\langle \mu, V\xi\big\rangle \\ &+ \big\langle \xi, 1\big\rangle\big\langle \mu, 1\big\rangle + \big\langle \mu, (\frac{\mathrm{id}}{2} - K)\gamma z\big\rangle - \big\langle \xi, (\frac{\mathrm{id}}{2} - K)\gamma v\big\rangle\end{aligned} \tag{31}$$

for all (z,ξ), $(v,\mu) \in \mathbb{X}$, and

$$\begin{aligned}\widetilde{\mathbb{A}}_0\big((z,\phi,\xi),(v,\varphi,\mu)\big) &= \mathbf{a}\big((z,\phi,\xi),(v,\varphi,\mu)\big) + \left(\int_\Gamma z\right)\left(\int_\Gamma v\right) + \big\langle W\phi, \varphi\big\rangle \\ &+ \big\langle \mu, V\xi\big\rangle + \big\langle \xi, 1\big\rangle\big\langle \mu, 1\big\rangle + \big\langle \mu, (\frac{\mathrm{id}}{2} - K)\phi\big\rangle - \big\langle \xi, (\frac{\mathrm{id}}{2} - K)\varphi\big\rangle\end{aligned} \tag{32}$$

for all (z,ϕ,ξ), $(v,\varphi,\mu) \in \widetilde{\mathbb{X}}$. Then, bearing in mind now (31), (32), and the definitions of $\mathbb{A}_\kappa$ (cf. (23)) and $\widetilde{\mathbb{A}}_\kappa$ (cf. (30)), we easily deduce thanks to Lemma 1 that all these bilinear forms are bounded. Equivalently, there exist positive constants denoted $\|\mathbb{A}_\kappa\|$, $\|\mathbb{A}_0\|$, $\|\widetilde{\mathbb{A}}_\kappa\|$, and $\|\widetilde{\mathbb{A}}_0\| > 0$, such that for each $* \in \{\kappa, 0\}$ there hold

$$\big|\mathbb{A}_*\big((z,\xi),(v,\mu)\big)\big| \le \|\mathbb{A}_*\| \, \|(z,\xi)\| \, \|(v,\mu)\| \tag{33}$$

for all (z,ξ), $(v,\mu) \in \boldsymbol{X}$, and

$$\Big|\widetilde{\mathbb{A}}_*\big((z,\phi,\xi),(v,\varphi,\mu)\big)\Big| \le \|\widetilde{\mathbb{A}}_*\| \, \|(z,\phi,\xi)\| \, \|(v,\varphi,\mu)\| \tag{34}$$

for all (z,ϕ,ξ), $(v,\varphi,\mu) \in \widetilde{\boldsymbol{X}}$. In addition, if follows from Lemma 2 that $\mathbb{A}_0$ and $\widetilde{\mathbb{A}}_0$ are both elliptic, which means that there exist positive constants α_0, $\widetilde{\alpha}_0$, such that

$$\mathrm{Re}\Big(\mathbb{A}_0\big((v,\mu),(\bar{v},\bar{\mu})\big)\Big) \ge \alpha_0 \, \|(v,\mu)\|^2 \qquad \forall (v,\mu) \in \boldsymbol{X}\,, \tag{35}$$

and

$$\mathrm{Re}\Big(\widetilde{\mathbb{A}}_0\big((v,\varphi,\mu),(\bar{v},\bar{\varphi},\bar{\mu})\big)\Big) \ge \widetilde{\alpha}_0 \, \|(v,\varphi,\mu)\|^2 \qquad \forall (v,\varphi,\mu) \in \widetilde{\boldsymbol{X}}\,. \tag{36}$$

Regarding the ellipticity of $\widetilde{\mathbb{A}}_0$ given by the foregoing equation, we stress here that, due to the inequalities (12) and (13), the expression $\langle \xi, 1\rangle\langle \mu, 1\rangle$ is needed in the definition of $\widetilde{\mathbb{A}}_0$ (cf. (32)) only for the 2D analysis, and hence it is omitted for the 3D one.

Next, we let $\mathbb{X}'$ and $\widetilde{\mathbb{X}}'$ be the duals of $\mathbb{X}$ and $\widetilde{\mathbb{X}}$ pivotal to $\mathrm{L}^2(\mathcal{O}) \times \mathrm{L}^2(\Gamma)$ and $\mathrm{L}^2(\mathcal{O}) \times \mathrm{L}^2(\Gamma) \times \mathrm{L}^2(\Gamma)$, respectively, which yields $\mathbb{X} \subset \mathrm{L}^2(\mathcal{O}) \times \mathrm{L}^2(\Gamma) \subset \mathbb{X}'$ and $\widetilde{\mathbb{X}} \subset \mathrm{L}^2(\mathcal{O}) \times \mathrm{L}^2(\Gamma) \times \mathrm{L}^2(\Gamma) \subset \widetilde{\mathbb{X}}'$ with dense inclusions. Thus, we denote by $[\cdot,\cdot]$ the corresponding duality pairings, and let $\mathcal{A}_\kappa : \boldsymbol{X} \to \boldsymbol{X}'$, $\mathcal{A}_0 : \boldsymbol{X} \to \boldsymbol{X}'$, $\widetilde{\mathcal{A}}_\kappa : \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{X}}'$, and $\widetilde{\mathcal{A}}_0 : \widetilde{\boldsymbol{X}} \to \widetilde{\boldsymbol{X}}'$ be the linear operators induced by $\mathbb{A}_\kappa$, $\mathbb{A}_0$, $\widetilde{\mathbb{A}}_\kappa$, and $\widetilde{\mathbb{A}}_0$, respectively, that is, for each $* \in \{\kappa, 0\}$

$$\big[\mathcal{A}_*(z,\xi),(v,\mu)\big] \;:=\; \mathbb{A}_*\big((z,\xi),(v,\mu)\big)$$

for all (z,ξ), $(v,\mu) \in \boldsymbol{X}$, and

$$\big[\widetilde{\mathcal{A}}_*(z,\phi,\xi),(v,\varphi,\mu)\big] \;:=\; \widetilde{\mathbb{A}}_*\big((z,\phi,\xi),(v,\varphi,\mu)\big)$$

for all (z,ϕ,ξ), $(v,\varphi,\mu) \in \widetilde{\boldsymbol{X}}$. It is clear from (33) and (34) that $\mathcal{A}_\kappa$, $\mathcal{A}_0$, $\widetilde{\mathcal{A}}_\kappa$, and $\widetilde{\mathcal{A}}_0$ are all bounded. In addition, (35) and (36) guarantee that $\mathcal{A}_0$ and $\widetilde{\mathcal{A}}_0$ are isomorphisms. Furthermore, we easily deduce from Lemma 3 and the compactness of the canonical injection from $\mathrm{H}^1(\mathcal{O})$ into $\mathrm{L}^2(\mathcal{O})$, that $\mathcal{A}_\kappa - \mathcal{A}_0 : \mathbb{X} \mapsto \mathbb{X}'$ and $\widetilde{\mathcal{A}}_\kappa - \widetilde{\mathcal{A}}_0 : \widetilde{\mathbb{X}} \mapsto \widetilde{\mathbb{X}}'$ are compact, whence $\mathcal{A}_\kappa$ and $\widetilde{\mathcal{A}}_\kappa$ are Fredholm operators of index zero.

Furthermore, we recall from [24, Theorem 2.1] the following result.

Theorem 1 *A function $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying* (2) *with $w = 0$ vanishes identically everywhere.*

We are now in position to establish the conditions under which problems (18) and (29) are uniquely solvable.

Theorem 2 *Assume that κ^2 is not an eigenvalue of the Laplacian in $\mathcal{O}$ with a Dirichlet boundary condition on Γ. Then, problems* (18) *and* (29) *are well posed.*

Proof The proof is adapted from [30, Theorem 3.2]. According to our previous analysis, the Fredholm alternative is applicable and therefore the proof reduces to show uniqueness of solution for (18) and (29). In what follows we restrict ourselves to (18), the proof for (29) being analogous. To this end, given a solution $(u_0,\lambda_0) \in \mathrm{H}^1(\mathcal{O}) \times \mathrm{H}^{-1/2}(\Gamma)$ of (18) with $w = 0$, we introduce the function

$$\widetilde{u}(\boldsymbol{x}) := \begin{cases} u_0(\boldsymbol{x}) & \forall\, \boldsymbol{x} \in \mathcal{O}, \\ \displaystyle\int_\Gamma \frac{\partial E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial \boldsymbol{n}_y} u^s(\boldsymbol{y})\, \mathrm{d}\sigma_y - \int_\Gamma E_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\lambda_0\, \mathrm{d}\sigma_y & \forall\, \boldsymbol{x} \in \mathcal{O}_e\,. \end{cases}$$

It is easy to verify that u_0 solves the equation

$$\Delta u_0 + \kappa^2 \theta(\boldsymbol{x})\, u_0 = 0 \quad \text{in} \quad \mathcal{O}\,, \tag{37}$$

and that $q := \widetilde{u}|_{\mathcal{O}_e}$ is a radiating solution of the Helmholtz equation with wave number κ, that is

$$
\begin{aligned}
\Delta q + \kappa^2 q &= 0 \quad \text{in} \quad O_e\,, \\
\frac{\partial q}{\partial r} - \imath\kappa q &= o(r^{\frac{1-d}{2}}) \qquad r := |\boldsymbol{x}| \to \infty\,.
\end{aligned} \tag{38}
$$

Furthermore, using the jump relations of the acoustic potential layers we obtain the identities

$$
\gamma q = (\frac{\mathrm{id}}{2} + K_\kappa)\gamma u_0 - V_\kappa \lambda_0 \quad \text{on} \quad \Gamma, \tag{39}
$$

$$
\lambda_0 = -W_\kappa \gamma u_0 + (\frac{\mathrm{id}}{2} - K_\kappa^{\mathtt{t}})\lambda_0 \quad \text{on} \quad \Gamma, \tag{40}
$$

from which, comparing in particular (4) and (39), we deduce that

$$
\gamma q = \gamma u_0. \tag{41}
$$

In turn, subtracting equations (5) and (40) yields

$$
(\frac{\mathrm{id}}{2} - K_\kappa^{\mathtt{t}})\left(\frac{\partial u_0}{\partial \boldsymbol{n}} - \lambda_0\right) = 0\,, \tag{42}
$$

and using that, under our hypothesis on k, operator $\frac{\mathrm{id}}{2} - K_\kappa^{\mathtt{t}}$ is injective (cf. [18]), we deduce from (42) the identity

$$
\frac{\partial q}{\partial \boldsymbol{n}} = \frac{\partial u_0}{\partial \boldsymbol{n}} \quad \text{on} \quad \Gamma\,. \tag{43}
$$

Finally, equations (37), (38), (41) and (43) show that $\widetilde{u} \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a solution of (2) with $w = 0$, and therefore Theorem 1 ensures that such a function $\widetilde{u}$ should vanish identically in $\mathbb{R}^d$, which ends the proof. □

Finally, as a consequence of Theorem 2 and the Fredholm alternative we conclude that the operators $\mathcal{A}_\kappa : \mathbb{X} \to \mathbb{X}'$ and $\widetilde{\mathcal{A}}_\kappa : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}'$ are bijective.

3 The Costabel & Han VEM/BEM schemes in 2D

In this section we introduce and analyze the discrete VEM/BEM schemes for the Costabel & Han coupling procedure as applied to the Poisson and Helmholtz models in the 2D case. Similar analyses hold for the modified Costabel & Han approach, and hence they are omitted. We begin with some fundamental notations and results on VEM in 2D.

3.1 Preliminaries

From now on we assume that there exist polygonal partitions $\cup_{i=1}^{I}\overline{\Omega}_i = \overline{\Omega}$ and $\cup_{i=1}^{I}\overline{O}_i = \overline{O}$, and an integer $k \geq 1$, such that $f|_{\Omega_i} \in \mathrm{H}^k(\Omega_i)$, $\kappa|_{\Omega_i} \in \mathrm{W}^{k+1,\infty}(\Omega_i)$, and $\theta|_{O_i} \in \mathbb{C}$ for all $i \in \{1, \ldots, I\}$. Then we let $\{\mathcal{F}_h\}_h$ be a family of partitions of $\overline{\Omega}$ (resp. of $\overline{O}$), constituted of connected polygons $F \in \mathcal{F}_h$ of diameter $h_F \leq h$, and assume that the meshes $\{\mathcal{F}_h\}_h$ are aligned with each Ω_i (resp. O_i), $i \in \{1, \ldots, I\}$. For each $F \in \mathcal{F}_h$ the boundary ∂F is subdivided into straight segments e, which are referred to in what follows as edges. In particular, we introduce the set

$$\mathcal{E}_h \ := \ \left\{\text{edges of } \mathcal{F}_h \, : \ e \subseteq \Gamma\right\}.$$

In addition, we assume that the family $\left\{\mathcal{F}_h\right\}_h$ of meshes satisfy the following conditions: There exists $\rho \in (0, 1)$ such that

(A1) each F of $\{\mathcal{F}_h\}_h$ is star-shaped with respect to a disk D_F of radius ρh_F,
(A2) for each F of $\{\mathcal{F}_h\}_h$ and for all edges $e \subseteq \partial F$ it holds $|e| \geq \rho h_F$.

Then, for each F of $\{\mathcal{F}_h\}_h$, we introduce the projection operator $\Pi_k^{\nabla,F} \, : \, \mathrm{H}^1(F) \to \mathcal{P}_k(F)$, which, given $v \in \mathrm{H}^1(F)$, is uniquely characterized by (see [6])

$$\int_F \nabla(\Pi_k^{\nabla,F} v) \cdot \nabla p + \left(\int_{\partial F} \Pi_k^{\nabla,F} v\right)\left(\int_{\partial F} p\right) = \int_F \nabla v \cdot \nabla p + \left(\int_{\partial F} v\right)\left(\int_{\partial F} p\right) \quad (44)$$

for all $p \in \mathcal{P}_k(F)$. Moreover, we let Π_k^F be the $\mathrm{L}^2(F)$–orthogonal projection onto $\mathcal{P}_k(F)$ with vectorial counterpart $\boldsymbol{\Pi}_k^F \, : \, \mathrm{L}^2(F)^2 \to \mathcal{P}_k(F)^2$, and following [2] we introduce, for $k \geq 1$, the local virtual element space

$$\begin{aligned} X_h^k(F) \ := \ \Big\{ v \in \mathrm{H}^1(F) : \quad & v|_e \in \mathcal{P}_k(e), \quad \forall e \subseteq \partial F, \\ & \Delta v \in \mathcal{P}_k(F), \quad \Pi_k^F v - \Pi_k^{\nabla,F} v \ \in \ \mathcal{P}_{k-2}(F) \Big\}. \end{aligned} \quad (45)$$

It can be shown (cf. [2]) that the degrees of freedom of $X_h^k(F)$ consist of:

i) the values at the vertices of F, and additionally for $k \geq 2$
ii) the moments of order $\leq k-2$ on the edges of F, and
iii) the moments of order $\leq k-2$ on F.

We are then allowed to introduce the global virtual element space as

$$X_h^k \ := \ \left\{ v \in X(\text{resp. } \mathrm{H}^1(\Omega)) : \quad v|_F \in X_h^k(F) \quad \forall F \in \mathcal{F}_h \right\}. \quad (46)$$

On the other hand, for any integer $k \geq 0$, we denote by $\mathcal{P}_k(\mathcal{F}_h)$ the space of piecewise polynomials of degree $\leq k$ with respect to $\mathcal{F}_h$, and let $\Pi_k^{\mathcal{F}}$ be the global $\mathrm{L}^2(\Omega)$-orthogonal (resp. $\mathrm{L}^2(O)$-orthogonal) projection onto $\mathcal{P}_k(\mathcal{F}_h)$, which is assembled cellwise, i.e.

$$(\Pi_k^{\mathcal{F}} v)|_F \ := \ \Pi_k^F(v|_F) \quad \forall\, F \in \mathcal{F}_h\,, \quad \forall\, v \in \mathrm{L}^2(\Omega) \text{ (resp. } v \in \mathrm{L}^2(O)). \quad (47)$$

Similarly, for any $\boldsymbol{q} \in \mathrm{L}^2(\Omega)^2$ (resp. $\boldsymbol{q} \in \mathrm{L}^2(O)^2$), $\boldsymbol{\Pi}_k^{\mathcal{F}} \boldsymbol{q}$ is defined by $(\boldsymbol{\Pi}_k^{\mathcal{F}} \boldsymbol{q})|_F = \boldsymbol{\Pi}_k^F(\boldsymbol{q}|_F)$ for all $F \in \mathcal{F}_h$. It is important to notice that $\mathcal{P}_k(F) \subseteq X_h^k(F)$ and that the projectors $\Pi_k^{\nabla,F} v$ and $\Pi_k^F v$ are computable for all $v \in X_h^k(F)$. Furthermore, it is also easy to check that $\mathbf{\Pi}_{k-1}^F \nabla v$ is explicitly known for all $v \in X_h^k(F)$ (cf. [6]).

Hereafter, given any positive functions A_h and B_h of the mesh parameter h, the notation $A_h \lesssim B_h$ means that $A_h \leq C B_h$ with $C > 0$ independent of h, whereas $A_h \simeq B_h$ means that $A_h \lesssim B_h$ and $B_h \lesssim A_h$. Then, under the conditions on $\mathcal{F}_h$, the technique of averaged Taylor polynomials introduced in [22] permits to prove the following error estimates,

$$\|v - \Pi_k^F v\|_{0,F} + h_F |v - \Pi_k^F v|_{1,F} \lesssim h_F^{\ell+1} |v|_{\ell+1,F} \quad \forall\, \ell \in \{0, 1, ..., k\}, \tag{48}$$

$$\|v - \Pi_k^{\nabla,F} v\|_{0,F} + h_F \|v - \Pi_k^{\nabla,F} v\|_{1,F} \lesssim h_F^{\ell+1} |v|_{\ell+1,F} \quad \forall\, \ell \in \{1, 2, ..., k\} \tag{49}$$

for all $v \in \mathrm{H}^{\ell+1}(F)$. In turn, the local interpolation operator $I_k^F : \mathrm{H}^2(F) \to X_h^k(F)$ is defined for each $v \in \mathrm{H}^2(F)$ by imposing that $v - I_k^F v$ has vanishing degrees of freedom, which satisfies (cf. [11, Lemma 2.23])

$$\|v - I_k^F v\|_{0,F} + h_F \left|v - I_k^F v\right|_{1,F} \lesssim h_F^{\ell+1} |v|_{\ell+1,F} \quad \forall \ell \in \{1, 2, ..., k\} \tag{50}$$

for all $v \in \mathrm{H}^{\ell+1}(F)$. In addition, we denote by $I_k^{\mathcal{F}}$ the global virtual element interpolation operator, i.e., for each $v \in C^0(\overline{\Omega})$ (resp. $v \in C^0(\overline{O})$), we set locally

$$(I_k^{\mathcal{F}} v)|_F \;=\; I_k^F(v|_F) \qquad \forall\, F \in \mathcal{F}_h\,. \tag{51}$$

On the other hand, in order to approximate the unknown $\lambda \in \mathrm{H}_0^{-1/2}(\Gamma)$, we introduce the non-virtual (but explicit) subspace

$$\Lambda_h^{k-1} := \left\{\mu \in \mathrm{L}^2(\Gamma): \quad \mu|_e \in P_{k-1}(e), \quad \forall e \in \mathcal{E}_h\,, \quad \int_\Gamma \mu = 0\right\}, \tag{52}$$

and let $\Pi_{k-1}^{\mathcal{E}} : \mathrm{L}^2(\Gamma) \to \Lambda_h^{k-1}$ be the $\mathrm{L}^2(\Gamma)$-orthogonal projection. In addition, we let $\{\Gamma_1, ..., \Gamma_J\}$ be the set of segments constituting Γ, and for any $t \geq 0$ we consider the broken Sobolev space $\mathrm{H}_b^t(\Gamma) := \prod_{j=1}^J \mathrm{H}^t(\Gamma_j)$ endowed with the graph norm

$$\|\varphi\|_{t,b,\Gamma}^2 := \sum_{j=1}^J \|\varphi\|_{t,\Gamma_j}^2 \quad \forall\, \varphi \in \mathrm{H}_b^t(\Gamma)\,.$$

Next, we recall from [32] the approximation property of the operator $\Pi_{k-1}^{\mathcal{E}}$.

Lemma 4 *Assume that* $\mu \in \mathrm{H}^{-1/2}(\Gamma) \cap \mathrm{H}_b^r(\Gamma)$ *for some* $r \geq 0$. *Then,*

$$\left\|\mu - \Pi_{k-1}^{\mathcal{E}} \mu\right\|_{-t,\Gamma} \;\lesssim\; h^{\min\{r,k\}+t}\; \|\mu\|_{r,b,\Gamma} \qquad \forall\, t \in \{0, 1/2\}.$$

3.2 The Costabel & Han VEM/BEM scheme for Poisson

In this section we introduce and analyze the VEM/BEM scheme for the continuous formulation (18) in the 2D case.

3.2.1 The discrete setting

For all $F \in \mathcal{F}_h$ we let S_h^F be the symmetric bilinear form defined on $\mathrm{H}^1(F) \times \mathrm{H}^1(F)$ by

$$S_h^F(v,w) := h_F^{-1} \sum_{e \subseteq \partial F} \int_e \pi_k^e v \, \pi_k^e w \qquad \forall\, v,\, w \in \mathrm{H}^1(F), \tag{53}$$

where π_k^e is the $\mathrm{L}^2(e)$-projection onto $\mathcal{P}_k(e)$. It is shown in [11, Lemma 3.2] that

$$S_h^F(v,v) \simeq a^F(v,v) \qquad \forall\, v \in X_h^k(F) \quad \text{such that} \quad \Pi_k^{\nabla,F} v = 0, \tag{54}$$

where a^F is the local version of a, that is

$$a^F(v,w) := \int_F \kappa \, \nabla v \cdot \nabla w \qquad \forall\, v,\, w \in \mathrm{H}^1(F). \tag{55}$$

It is important to notice that S_h^F is computable on $X_h^k(F) \times X_h^k(F)$ since $\pi_k^e v = v \in \mathcal{P}_k(e)$ for all $v \in X_h^k(F)$, and that, by symmetry, there holds

$$S_h^F(v,w) \le S_h^F(v,v)^{1/2} S_h^F(w,w)^{1/2} \lesssim a^F(v,v)^{1/2} a^F(w,w)^{1/2}$$

for all $v,\, w \in X_h^k(F)$ satisfying $\Pi_k^{\nabla,F} v = \Pi_k^{\nabla,F} w = 0$. Next, for each $F \in \mathcal{F}_h$ we introduce

$$a_h^F(v,w) := \int_F \kappa \, \mathbf{\Pi}_{k-1}^F \nabla v \cdot \mathbf{\Pi}_{k-1}^F \nabla w + S_h^F(v - \Pi_k^{\nabla,F} v, w - \Pi_k^{\nabla,F} w), \tag{56}$$

and let a_h be the global extension of it, that is

$$a_h(v,w) = \sum_{F \in \mathcal{F}_h} a_h^F(v,w) \qquad \forall\, v,\, w \in X_h^k. \tag{57}$$

We now stress, as shown in [6], that the first term defining a_h^F is also computable on $X_h^k(F) \times X_h^k(F)$ even if κ is not a polynomial function. Indeed, using the fact that $\mathbf{\Pi}_{k-1}^F$ is self-adjoint and integrating by parts, we find that there holds

$$\begin{aligned}
\int_F \kappa \, \mathbf{\Pi}_{k-1}^F \nabla v \cdot \mathbf{\Pi}_{k-1}^F \nabla w &= \int_F \mathbf{\Pi}_{k-1}^F \big(\kappa \, \mathbf{\Pi}_{k-1}^F \nabla v\big) \cdot \nabla w \\
&= -\int_F \operatorname{div}\big(\mathbf{\Pi}_{k-1}^F \big(\kappa \, \mathbf{\Pi}_{k-1}^F \nabla v\big)\big) w + \int_{\partial F} \mathbf{\Pi}_{k-1}^F \big(\kappa \, \mathbf{\Pi}_{k-1}^F \nabla v\big) \cdot \boldsymbol{n}_{\partial F} \, w
\end{aligned}$$

for all $v,\, w \in X_h^k(F)$. Then, we notice that the first term on the right hand side of the foregoing identity is computable thanks to the moments of w on F of order $\le k-2$, whereas the second one is computable as well since each factor of it is a known polynomial.

We now let $\mathbf{X}_h := X_h^k \times \Lambda_h^{k-1}$ and introduce the discrete version of problem (18): Find $(u_h, \lambda_h) \in \mathbf{X}_h$ such that

$$\mathbf{A}_h\big((u_h,\lambda_h),(v_h,\mu_h)\big) = \mathbf{F}_h(v_h,\mu_h) \qquad \forall\, (v_h,\mu_h) \in \mathbf{X}_h, \tag{58}$$

where

$$\mathbf{A}_h\big((z_h,\xi_h),(v_h,\mu_h)\big) \;:=\; a_h(z_h,v_h) + \big\langle W\gamma z_h, \gamma v_h\big\rangle + \big\langle \mu_h, V\xi_h\big\rangle$$
$$+ \big\langle \mu_h, (\frac{\mathrm{id}}{2} - K)\gamma z_h)\big\rangle - \big\langle \xi_h, (\frac{\mathrm{id}}{2} - K)\gamma v_h)\big\rangle\,, \tag{59}$$

for all $z_h, \xi_h)$, $(v_h, \mu_h) \in \mathbf{X}_h$, and

$$\mathbf{F}_h(v_h,\mu_h) \;:=\; \int_\Omega (\Pi^{\mathcal{F}}_{k-1} f)\, v_h \qquad \forall\,(v_h,\mu_h) \in \mathbf{X}_h\,. \tag{60}$$

We stress that, due to the degrees of freedom of the virtual element subspace X_h^k (cf. (46)), and thanks to the non-virtual character of the finite element subspace Λ_h^{k-1} (cf. (52)), all the terms in (59) involving the boundary integral operators are computable

3.2.2 Solvability and a priori error analyses

We begin with the boundedness property of $\mathbf{A}_h$.

Lemma 5 *There hold*

$$|a_h^F(z,v)| \;\lesssim\; \|z\|_{1,F}\,\|v\|_{1,F} \qquad \forall\, F \in \mathcal{F}_h\,, \quad \forall\, z,\; v \in \mathrm{H}^1(F)\,, \tag{61}$$

and

$$|\mathbf{A}_h\big((z,\eta),(v,\mu)\big)| \;\lesssim\; \|(z,\eta)\|\;\|(v,\mu)\| \qquad \forall\,(z,\eta),\,(v,\mu) \;\in\; \mathbf{X}_h\,. \tag{62}$$

Proof The local estimate (61) is basically a consequence of the Cauchy-Schwarz inequality and the fact that (see [6])

$$S_h^F(z - \Pi_k^{\nabla,F} z, v - \Pi_k^{\nabla,F} v) \;\lesssim\; |z - \Pi_k^{\nabla,F} z|_{1,F}\,|v - \Pi_k^{\nabla,F} v|_{1,F} \;\lesssim\; |z|_{1,F}\,|v|_{1,F}\,, \tag{63}$$

whereas (62) follows from (61) and the mapping properties provided by Lemma 1.□

The following lemma recalls from [6] some useful estimates between a^F and a_h^F, which involve the local operators Π_k^F and I_k^F.

Lemma 6 *For each $F \in \mathcal{F}_h$ there hold*

$$|a^F(\Pi_k^F z, v_h) - a_h^F(\Pi_k^F z, v_h)| \;\lesssim\; h_F^k\;\|z\|_{k+1,F}\;\|v_h\|_{1,F} \tag{64}$$

for all $(z, v_h) \;\in\; \mathrm{H}^{k+1}(F) \times X_h^k(F)$,

$$|a^F(v_h, I_k^F z) - a_h^F(v_h, I_k^F z)| \;\lesssim\; h_F\;\|v_h\|_{1,F}\;\|z\|_{2,F} \tag{65}$$

for all $(z, v_h) \;\in\; \mathrm{H}^2(F) \times X_h^k(F)$, *and*

$$|a^F(\Pi_k^F z, I_k^F v) - a_h^F(\Pi_k^F z, I_k^F v)| \;\lesssim\; h_F^{k+1}\;\|z\|_{k+1,F}\;\|v\|_{2,F} \tag{66}$$

for all $(z,v) \;\in\; \mathrm{H}^{k+1}(F) \times \mathrm{H}^2(F)$.

Proof For the proof of (64) we refer to [6, Lemma 5.5], whereas (65) can be proved as explained in [6, Remark 5.1]. In turn, (66) follows by combining the proofs of (64) and (65). □

The $\mathbf{X}_h$-ellipticity of the bilinear form $\mathbf{A}_h$ is established next.

Lemma 7 *There holds*

$$\mathbf{A}_h\big((v,\mu),(v,\mu)\big) \;\gtrsim\; \|(v,\mu)\|^2 \qquad \forall\,(v,\mu)\,\in\,\mathbf{X}_h\,. \tag{67}$$

Proof We begin by observing, thanks to (13) and (14), that for all $(v,\mu)\in\mathbf{X}_h$ we obtain

$$\mathbf{A}_h\big((v,\mu),(v,\mu)\big) \;=\; a_h(v,v) + \big\langle W\gamma v,\gamma v\big\rangle + \big\langle \mu, V\mu\big\rangle \;\geq\; a_h(v,v) \;+\; \alpha_V\;\|\mu\|^2_{-1/2,\Gamma}\,. \tag{68}$$

On the other hand, according to the definition of a_h^F (cf. (56)), noting that certainly there holds $\Pi_k^{\nabla,F}\big(v-\Pi_k^{\nabla,F}v\big) \;=\; 0$, and then employing (54) and the fact that

$$|v-\Pi_k^{\nabla,F}v|_{1,F} \;=\; \|\nabla v - \nabla\Pi_k^{\nabla,F}v\|_{0,F} \;\geq\; \|\nabla v - \mathbf{\Pi}_{k-1}^F\nabla v\|_{0,F}\,,$$

we deduce that

$$\begin{aligned} a_h^F(v,v) \;&\gtrsim\; \big\|\mathbf{\Pi}_{k-1}^F\nabla v\big\|_{0,F}^2 \;+\; a^F(v-\Pi_k^{\nabla,F}v,\, v-\Pi_k^{\nabla,F}v)\\ &\gtrsim\; \Big\{\big\|\mathbf{\Pi}_{k-1}^F\nabla v\big\|_{0,F}^2 \;+\; |v-\Pi_k^{\nabla,F}v|_{1,F}^2\Big\}\\ &\gtrsim\; \Big\{\big\|\mathbf{\Pi}_{k-1}^F\nabla v\big\|_{0,F}^2 \;+\; \big\|\nabla v-\mathbf{\Pi}_{k-1}^F\nabla v\big\|_{0,F}^2\Big\} \;\gtrsim\; |v|_{1,F}^2\,. \end{aligned} \tag{69}$$

In this way, the proof follows from the definition of a_h (cf. (57)), (68), and (69). □

As a consequence of Lemmas 5 and 7, a straightforward application again of the Lax-Milgram lemma shows that (58) admits a unique solution $(u_h,\lambda_h)\in\mathbf{X}_h$. Moreover, we have the following a priori error estimate.

Theorem 3 *Under the assumption that $u\in X\cap\prod_{i=1}^I \mathrm{H}^2(\Omega_i)$, there holds*

$$\begin{aligned} \|(u,\lambda)-(u_h,\lambda_h)\| \;&\lesssim\; \Big\|(u,\lambda)-(I_k^{\mathcal{F}}u,\Pi_{k-1}^{\mathcal{E}}\lambda)\Big\|\\ &+\;\sup_{w_h\in X_h^k}\frac{|a(u,w_h)-a_h(I_k^{\mathcal{F}}u,w_h)|}{\|w_h\|_{1,\Omega}} \;+\; \Big\|f-\Pi_{k-1}^{\mathcal{F}}f\Big\|_{0,\Omega}\,. \end{aligned} \tag{70}$$

Proof From the definitions of $\mathbf{F}$ and $\mathbf{F}_h$ (cf. (18) and (60)) we have that

$$\sup_{\substack{(v_h,\mu_h)\in\mathbf{X}_h\\(v_h,\mu_h)\neq\mathbf{0}}}\frac{\big|\mathbf{F}(v_h,\mu_h)-\mathbf{F}_h(v_h,\mu_h)\big|}{\|(v_h,\mu_h)\|} \;\leq\; \Big\|f-\Pi_{k-1}^{\mathcal{F}}f\Big\|_{0,\Omega}\,.$$

In turn, according to the definitions of $\mathbf{A}$ and $\mathbf{A}_h$ (cf. (19) and (59)) it readily follows that

$$\mathbf{A}((v_h,\mu_h),(w_h,\xi_h)) - \mathbf{A}_h((v_h,\mu_h),(w_h,\xi_h)) \;=\; a(v_h,w_h)-a_h(v_h,w_h)$$

for all $(v_h, \mu_h), (w_h, \xi_h) \in \mathbf{X}_h$. In addition, adding and subtracting u to the first component of a, and using the boundedness of this bilinear form, we obtain

$$\left|a(v_h, w_h) - a_h(v_h, w_h)\right| \lesssim \left\{\|u - v_h\| \, \|w_h\|_{1,\Omega} + \left|a(u, w_h) - a_h(v_h, w_h)\right|\right\}$$

for all $v_h, w_h \in X_h^k$. Hence, bearing in mind the foregoing estimates, a straightforward application of the first Strang Lemma (cf. [17, Theorem 4.1.1]) to the context given by (18) and (58) gives

$$\begin{aligned} \|(u,\lambda) - (u_h, \lambda_h)\| \lesssim \inf_{(v_h,\mu_h)\in \mathbf{X}_h} \Bigg\{ \|(u,\lambda) - (v_h,\mu_h)\| \\ + \sup_{\substack{w_h\in X_h^k\\ w_h\neq \mathbf{0}}} \frac{\left|a(u, w_h) - a_h(v_h, w_h)\right|}{\|w_h\|_{1,\Omega}} \Bigg\} + \left\| f - \Pi_{k-1}^{\mathcal{F}} f \right\|_{0,\Omega} . \end{aligned} \tag{71}$$

Next, since $X \cap \prod_{i=1}^{I} \mathrm{H}^2(\Omega_i) \subseteq C^0(\overline{\Omega})$ and $\mathrm{H}_b^{1/2}(\Gamma) \subseteq \mathrm{L}^2(\Gamma)$, we deduce by hypotheses that $u \in C^0(\overline{\Omega})$ and $\lambda = \kappa \nabla u \cdot \boldsymbol{n} \in \mathrm{L}^2(\Gamma)$, which implies that $I_k^{\mathcal{F}} u$ and $\Pi_{k-1}^{\mathcal{E}} \lambda$ are meaningful. In this way, taking in particular $(v_h, \mu_h) = (I_k^{\mathcal{F}} u, \Pi_{k-1}^{\mathcal{E}} \lambda) \in \mathbf{X}_h$ in (71) we arrive at (70) and conclude the proof. □

It remains to bound the supremum in (70), for which we begin by noticing that for each $w_h \in X_h^k$ there holds

$$a(u, w_h) - a_h(I_k^{\mathcal{F}} u, w_h) = \sum_{F\in\mathcal{F}_h} \left\{ a^F(u, w_h) - a_h^F(I_k^F u, w_h) \right\}, \tag{72}$$

where each term of the sum in (72) can be decomposed as

$$\begin{aligned} a^F(u, w_h) - a_h^F(I_k^F u, w_h) = a^F(u - \Pi_k^F u, w_h) \\ + a^F(\Pi_k^F u, w_h) - a_h^F(\Pi_k^F u, w_h) + a_h^F(\Pi_k^F u - I_k^F u, w_h) . \end{aligned} \tag{73}$$

Then, employing the boundedness of a^F (cf. (55)) and a_h^F (cf. (61)), we obtain

$$|a^F(u - \Pi_k^F u, w_h)| \lesssim \left\|u - \Pi_k^F u\right\|_{1,F} \|w_h\|_{1,F}$$

and

$$|a_h^F(\Pi_k^F u - I_k^F u, w_h)| \lesssim \left\{ \left\|u - I_k^F u\right\|_{1,F} + \left\|u - \Pi_k^F u\right\|_{1,F} \right\} \|w_h\|_{1,F} ,$$

respectively, which, replaced back in (73) and then in (72), yields

$$\sup_{w_h \in X_h^k} \frac{|a(u,w_h) - a_h(I_k^{\mathcal{F}} u, w_h)|}{\|w_h\|_{1,\Omega}} \lesssim \left\| u - I_k^{\mathcal{F}} u \right\|_{1,\Omega} + \left(\sum_{F \in \mathcal{F}_h} \left\| u - \Pi_k^F u \right\|_{1,F}^2 \right)^{1/2}$$
$$+ \sup_{w_h \in X_h^k} \frac{\displaystyle\sum_{F \in \mathcal{F}_h} \left| a^F(\Pi_k^F u, w_h) - a_h^F(\Pi_k^F u, w_h) \right|}{\|w_h\|_{1,\Omega}} . \tag{74}$$

Note that we used here that $\left\| u - I_k^{\mathcal{F}} u \right\|_{1,\Omega}^2 = \sum_{F \in \mathcal{F}_h} \left\| u - I_k^F u \right\|_{1,F}^2$. Hence, employing (74) in (70), we find that

$$\|(u,\lambda) - (u_h,\lambda_h)\| \lesssim \left\{ \left\| (u,\lambda) - (I_k^{\mathcal{F}} u, \Pi_{k-1}^{\mathcal{E}} \lambda) \right\| + \left(\sum_{F \in \mathcal{F}_h} \left\| u - \Pi_k^F u \right\|_{1,F}^2 \right)^{1/2} \right.$$
$$\left. + \sup_{w_h \in X_h^k} \frac{\displaystyle\sum_{F \in \mathcal{F}_h} \left| a^F(\Pi_k^F u, w_h) - a_h^F(\Pi_k^F u, w_h) \right|}{\|w_h\|_{1,\Omega}} + \left\| f - \Pi_{k-1}^{\mathcal{F}} f \right\|_{0,\Omega} \right\} . \tag{75}$$

The foregoing a priori error estimate together with the regularity assumptions on u and f, and the approximation properties of the projection and interpolation operators involved, allow us to establish the rates of convergence of our VEM/BEM scheme (58). More precisely, we have the following result.

Theorem 4 *Assuming that* $u \in X \cap \prod_{i=1}^I \mathrm{H}^{k+1}(\Omega_i)$ *and* $f \in \prod_{i=1}^I \mathrm{H}^k(\Omega_i)$, *there holds*

$$\begin{aligned} \|(u,\lambda) - (u_h,\lambda_h)\| &:= \|u - u_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \\ &\lesssim h^k \sum_{i=1}^I \left\{ \|u\|_{k+1,\Omega_i} + \|f\|_{k,\Omega_i} \right\} . \end{aligned} \tag{76}$$

Proof We first notice from (64) (cf. Lemma 6) that

$$\left| a^F(\Pi_k^F u, w_h) - a_h^F(\Pi_k^F u, w_h) \right| \lesssim h_F^k \|u\|_{k+1,F} \|w_h\|_{1,F} \qquad \forall\, F \in \mathcal{F}_h ,$$

which implies

$$\sup_{w_h \in X_h^k} \frac{\displaystyle\sum_{F \in \mathcal{F}_h} \left| a^F(\Pi_k^F u, w_h) - a_h^F(\Pi_k^F u, w_h) \right|}{\|w_h\|_{1,\Omega}} \lesssim h^k \sum_{i=1}^I \|u\|_{k+1,\Omega_i} . \tag{77}$$

Then, by applying (48) and (50), we readily deduce that

$$\begin{aligned} &\left\| u - I_k^{\mathcal{F}} u \right\|_{1,\Omega} + \left(\sum_{F \in \mathcal{F}_h} \left\| u - \Pi_k^F u \right\|_{1,F}^2 \right)^{1/2} + \left\| f - \Pi_{k-1}^{\mathcal{F}} f \right\|_{0,\Omega} \\ &\qquad \lesssim h^k \sum_{i=1}^I \left\{ \|u\|_{k+1,\Omega_i} + \|f\|_{k,\Omega_i} \right\} . \end{aligned} \tag{78}$$

In turn, by hypothesis $\lambda = \kappa\nabla u \cdot \boldsymbol{n}$ satisfies $\lambda|_{\Gamma_j} \in \mathrm{H}^{k-1/2}(\Gamma_j)$ on each straight segment Γ_j, $j \in \{1, ..., J\}$, constituting Γ, and therefore Lemma 4 and the trace theorem yield

$$\left\|\lambda - \Pi_{k-1}^{\mathcal{E}}\lambda\right\|_{-1/2,\Gamma} \lesssim h^k \sum_{j=1}^{J} \|\lambda\|_{k-1/2,\Gamma_j} \lesssim h^k \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i} . \tag{79}$$

Finally, replacing (77), (78), and (79) in (75) we obtain (76) and conclude the proof.□

We end this section by stressing that rates of convergence for u in the $\mathrm{L}^2(\Omega)$-norm, and for a computable approximation $\widehat{u}$ of u in a broken $\mathrm{H}^1(\Omega)$-norm, can also be derived. In fact, under a suitable regularity hypothesis on the solution of the dual problem to (18), and assuming that $u \in X \cap \prod_{i=1}^{I} \mathrm{H}^{k+1}(\Omega_i)$ and $f \in \prod_{i=1}^{I} \mathrm{H}^{k}(\Omega_i)$, there holds (cf. [23, Theorem 3.7])

$$\|u - u_h\|_{0,\Omega} \lesssim h^{k+1} \sum_{i=1}^{I} \left\{ \|u\|_{k+1,\Omega_i} + \|f\|_{k,\Omega_i} \right\}.$$

In turn, introducing the fully computable approximation of u given by $\widehat{u} := \Pi_k^{\mathcal{F}} u_h$, defining the broken $\mathrm{H}^1(\Omega)$-seminorm

$$|u - \widehat{u}|_{1,b,\Omega} := \left\{ \sum_{F\in\mathcal{F}_h} |u - \widehat{u}_h|_{1,F}^2 \right\}^{1/2} ,$$

and assuming that $u \in X \cap \prod_{i=1}^{I} \mathrm{H}^{k+1}(\Omega_i)$ and $f \in \prod_{i=1}^{I} \mathrm{H}^{k}(\Omega_i)$, there holds (cf. [23, Theorem 3.8])

$$\|u - \widehat{u}_h\|_{0,\Omega} + h\,|u - \widehat{u}|_{1,b,\Omega} \lesssim h^{k+1} \sum_{i=1}^{I} \left\{ \|u\|_{k+1,\Omega_i} + \|f\|_{k,\Omega_i} \right\}.$$

3.3 The Costabel & Han VEM/BEM scheme for Helmholtz

In what follows we introduce and analyze the VEM/BEM scheme for the continuous formulation (22) in the 2D case.

3.3.1 The discrete setting

We also make use of the symmetric bilinear form S_h^F (cf. (53)) for each $F \in \mathcal{F}_h$, and notice now from [11, Lemma 3.2] that

$$S_h^F(v, \bar{v}) \simeq a_0^F(v, \bar{v}) \quad \forall v \in X_h^k(F) \quad \text{such that } \Pi_k^{\nabla,F} v = 0,$$

where a_0^F is the local version of a_0, that is

$$a_0^F(z,v) := \int_F \nabla z \cdot \nabla v \qquad \forall\, z,\ v \in \mathrm{H}^1(F)\,. \tag{80}$$

In addition, by symmetry there holds

$$S_h^F(z,v) \ \le\ S_h^F(z,z)^{1/2}\, S_h^F(v,v)^{1/2} \ \lesssim\ a_0^F(z,z)^{1/2}\, a_0^F(v,v)^{1/2} \tag{81}$$

for all $z,\ v \in X_h^k(F)$. Next, for each $F \in \mathcal{F}_h$ we introduce

$$a_{0,h}^F(z,v) \ :=\ a_0^F(\Pi_k^{\nabla,F} z, \Pi_k^{\nabla,F} v) + S_h^F(z - \Pi_k^{\nabla,F} z, v - \Pi_k^{\nabla,F} v) \quad \forall\, z,\ v \in X_h^k(F)\,, \tag{82}$$

and

$$a_{\kappa,h}^F(z,v) := a_{0,h}^F(z,v) - \kappa^2 \theta_F \int_F (\Pi_{k-1}^F z)(\Pi_{k-1}^F v) \quad \forall\, z,\ v \in X_h^k(F)\,, \tag{83}$$

where $\theta_F = \theta|_F \in \mathbb{C}$. We also let $a_{0,h}$ and $a_{\kappa,h}$ be the corresponding global extensions of $a_{0,h}^F$ and $a_{\kappa,h}^F$, respectively, that is

$$a_{0,h}(z,v) := \sum_{F \in \mathcal{F}_h} a_{0,h}^F(z,v)$$

and

$$a_{\kappa,h}(z,v) := \sum_{F \in \mathcal{F}_h} a_{\kappa,h}^F(z,v) \quad \forall\, z,\ v \in X_h^k. \tag{84}$$

Then, denoting $\boldsymbol{X}_h^k := X_h^k \times \Lambda_h^{k-1}$, the discrete version of problem (22) reduces to: Find $(u_h, \lambda_h) \in \boldsymbol{X}_h^k$ such that

$$\mathbb{A}_{\kappa,h}\big((u_h,\lambda_h),(v_h,\mu_h)\big) = \mathbb{F}(v_h,\mu_h) \qquad \forall (v_h,\mu_h) \in \boldsymbol{X}_h^k\,, \tag{85}$$

where

$$\begin{aligned}\mathbb{A}_{\kappa,h}\big((z_h,\xi_h),(v_h,\mu_h)\big) &:= a_{\kappa,h}(z_h,v_h) + \big\langle W_\kappa \gamma z_h, \gamma v_h \big\rangle + \big\langle \mu_h, V_\kappa \xi_h \big\rangle \\ &+ \ \big\langle \mu_h, (\frac{\mathrm{id}}{2} - K_\kappa)\gamma z_h \big\rangle - \big\langle \xi_h, (\frac{\mathrm{id}}{2} - K_\kappa)\gamma v_h \big\rangle\end{aligned}$$

for all $(z_h,\xi_h),\,(v_h,\mu_h) \in \boldsymbol{X}_h^k$.

3.3.2 Solvability and a priori error analyses

For the solvability of (85), we now introduce the perturbation of the bilinear form $\mathbb{A}_{\kappa,h}$ given by

$$\begin{aligned}\mathbb{A}_{0,h}\big((z_h,\xi_h),(v_h,\mu_h)\big) &:= a_{0,h}(z_h,v_h) + \left\{\int_\Gamma z_h\right\}\left\{\int_\Gamma v_h\right\} + \big\langle W\gamma z_h, \gamma v_h\big\rangle \\ + \ \big\langle \mu_h, V\xi_h\big\rangle + \big\langle \xi_h, 1\big\rangle\big\langle \mu_h, 1\big\rangle &+ \big\langle \mu_h, (\frac{\mathrm{id}}{2} - K)\gamma z_h\big\rangle - \big\langle \xi_h, (\frac{\mathrm{id}}{2} - K)\gamma v_h\big\rangle\end{aligned} \tag{86}$$

for all $(z_h, \xi_h), (v_h, \mu_h) \in \boldsymbol{X}_h^k$. Next, the boundedness of $\mathbb{A}_{\kappa,h}$ and $\mathbb{A}_{0,h}$, and the ellipticity of $\mathbb{A}_{0,h}$, are provided by the following two lemmas.

Lemma 8 *There exist positive constants M_κ and M_0, independent of h, such that for each $* \in \{\kappa, 0\}$ there hold*

$$|\mathbb{A}_{*,h}\big((z_h, \xi_h), (v_h, \mu_h)\big)| \;\le\; M_* \,\|(z_h, \xi_h)\|\,\|(v_h, \mu_h)\|$$

for all (z_h, ξ_h), $(v_h, \mu_h) \in \boldsymbol{X}_h^k$.

Proof Starting from the corresponding definitions (cf. (59) and (86)), it suffices to employ the mapping properties of the boundary integral operators (cf. Lemma 1), and then notice from (80), (81) and [6], that for each $F \in \mathcal{F}_h$ there holds

$$\begin{aligned} &S_h^F(z_h - \Pi_k^{\nabla,F} z_h, v_h - \Pi_k^{\nabla,F} v_h) \\ &\qquad \lesssim\; |z_h - \Pi_k^{\nabla,F} z_h|_{1,F}\, |v_h - \Pi_k^{\nabla,F} v_h|_{1,F} \;\lesssim\; |z_h|_{1,F}\, |v_h|_{1,F} \end{aligned}$$

for all $z_h,\ v_h \in X_h^k(F)$. □

Lemma 9 *There exist a positive constant β_0, independent of h, such that*

$$Re\Big\{\mathbb{A}_{0,h}\big((v_h, \mu_h), (\bar v_h, \bar\mu_h)\big)\Big\} \;\ge\; \beta_0\, \|(v_h, \mu_h)\|^2 \qquad \forall\, (v_h, \mu_h) \in \boldsymbol{X}_h^k\,.$$

Proof Bearing in mind the definition (86), and proceeding as in the deduction of (35), we first apply the positivity properties of the boundary integral operators (cf. Lemma 2). In this way, noticing from (82), (80) and (54) that for each $F \in \mathcal{F}_h$ there holds

$$\begin{aligned} a_{0,h}^F(v, \bar v) \;&=\; |\Pi_k^{\nabla,F} v|_{1,F}^2 + S_h^F(v - \Pi_k^{\nabla,F} v, \bar v - \Pi_k^{\nabla,F} \bar v) \\ &\gtrsim\; |\Pi_k^{\nabla,F} v|_{1,F}^2 \;+\; |v - \Pi_k^{\nabla,F} v|_{1,F}^2 \;\gtrsim\; |v|_{1,F}^2 \qquad \forall\, v \in X_h^k(F)\,, \end{aligned}$$

we arrive at the required inequality and conclude the proof. □

Furthermore, thanks to Lemmas 8 and 9, and the boundedness estimate (33), we can apply the Lax-Milgram lemma to introduce the Galerkin projection-type operator $\mathcal{R}_h : \boldsymbol{X} \to \boldsymbol{X}_h^k$, which, given $(z, \xi) \in \boldsymbol{X}$, is uniquely characterized by

$$\mathbb{A}_{0,h}\big(\mathcal{R}_h(z, \xi), (v_h, \mu_h)\big) \;=\; \mathbb{A}_0\big((z, \xi), (v_h, \mu_h)\big) \qquad \forall\, (v_h, \mu_h) \in \boldsymbol{X}_h^k\,. \tag{87}$$

Moreover, it readily follows from the aforementioned classical lemma that $\mathcal{R}_h$ is uniformly bounded in h with $\|\mathcal{R}_h\| \le \|\mathbb{A}_0\|/\beta_0$. The approximation property of this operator is established next. As usual, given a finite dimensional subspace X_h of a normed space X, we set for each $x \in X$, $\operatorname{dist}(x, X_h) := \inf\limits_{x_h \in X_h} \|x - x_h\|$.

Theorem 5 *There exists a positive constant C, independent of h, such that*

$$\|\mathcal{R}_h(z, \xi) - (z, \xi)\| \;\le\; C\left\{\operatorname{dist}\big((z, \xi), \boldsymbol{X}_h^k\big) \;+\; \Big(\sum_{F \in \mathcal{F}_h} |z - \Pi_k^{\nabla,F} z|_{1,F}^2\Big)^{1/2}\right\} \tag{88}$$

for all $(z, \xi) \in \boldsymbol{X}$.

Proof Given $(z,\xi) \in \boldsymbol{X}$ and $(z_h,\xi_h) \in \boldsymbol{X}_h^k$, we first observe by triangle inequality that

$$\|\mathcal{R}_h(z,\xi) - (z,\xi)\| \;\leq\; \|(v_h,\mu_h)\| \;+\; \|(z,\xi) - (z_h,\xi_h)\|\,, \tag{89}$$

with $(v_h,\mu_h) := \mathcal{R}_h(z,\xi) - (z_h,\xi_h) \in \boldsymbol{X}_h^k$, so that in what follows we focus on estimating $\|(v_h,\mu_h)\|$. In fact, applying the ellipticity property (67), the identity (87), the boundedness of $\mathbb{A}_0$ (cf. (33)), and the fact that the difference between $\mathbb{A}_0$ and $\mathbb{A}_{0,h}$ (cf. (31), (86)) reduces to $a_0 - a_{0,h}$, we obtain

$$\begin{aligned}
\beta_0\,\|(v_h,\mu_h)\|^2 \;&\leq\; \mathrm{Re}\Big\{\mathbb{A}_{0,h}\big((v_h,\mu_h),(\bar v_h,\bar\mu_h)\big)\Big\} \\
&=\; \mathrm{Re}\Big\{\mathbb{A}_0\big((z,\xi),(\bar v_h,\bar\mu_h)\big) - \mathbb{A}_{0,h}\big((z_h,\xi_h),(\bar v_h,\bar\mu_h)\big)\Big\} \\
&\leq \Big|\mathbb{A}_0\big((z,\xi)-(z_h,\xi_h),(\bar v_h,\bar\mu_h)\big)\Big| \\
&\quad + \Big|\mathbb{A}_0\big((z_h,\xi_h),(\bar v_h,\bar\mu_h)\big) - \mathbb{A}_{0,h}\big((z_h,\xi_h),(\bar v_h,\bar\mu_h)\big)\Big| \\
&\leq\; \|\mathbb{A}_0\|\,\|(z,\xi)-(z_h,\xi_h)\|\,\|(v_h,\mu_h)\| \\
&\quad + \sum_{F\in\mathcal{F}_h}\Big|a_0^F(z_h,\bar v_h) - a_{0,h}^F(z_h,\bar v_h)\Big|\,.
\end{aligned} \tag{90}$$

Then, subtracting and adding $\Pi_k^{\nabla,F} z$ in the first component of the expression $a_{0,h}^F(z_h,\bar v_h)$, using that $a_{0,h}^F(\Pi_k^{\nabla,F} z, v_h) \;=\; a_0^F(\Pi_k^{\nabla,F} z, \Pi_k^{\nabla,F} v_h) \;=\; a_0^F(\Pi_k^{\nabla,F} z, v_h)$ (which follows from (82) and after taking $(v,p) = (v_h,1)$ and $(v,p) = (v_h,\Pi_k^{\nabla,F} z)$ in (44)), and employing the triangle inequality and the boundedness of a_0^F and $a_{0,h}^F$, the latter being consequence of (81), we find that

$$\begin{aligned}
\Big|a_0^F(z_h,\bar v_h) - a_{0,h}^F(z_h,\bar v_h)\Big| \;&\leq\; \Big|a_0^F(z_h - \Pi_k^{\nabla,F} z,\bar v_h)\Big| + \Big|a_{0,h}^F(z_h - \Pi_k^{\nabla,F} z,\bar v_h)\Big| \\
&\lesssim\; |z_h - \Pi_k^{\nabla,F} z|_{1,F}\,|v_h|_{1,F} \;\lesssim\; \Big\{|z_h - z|_{1,F} \;+\; |z - \Pi_k^{\nabla,F} z|_{1,F}\Big\}\,|v_h|_{1,F}\,.
\end{aligned}$$

In this way, summing up over $F \in \mathcal{F}_h$, it follows that

$$\begin{aligned}
&\sum_{F\in\mathcal{F}_h}\Big|a_0^F(z_h,\bar v_h) - a_{0,h}^F(z_h,\bar v_h)\Big| \\
&\qquad\lesssim\; \Big\{|z_h - z|_{1,\mathcal{O}} \;+\; \Big(\sum_{F\in\mathcal{F}_h}|z - \Pi_k^{\nabla,F} z|_{1,F}^2\Big)^{1/2}\Big\}\,|v_h|_{1,\mathcal{O}} \\
&\qquad\lesssim\; \Big\{\|(z,\xi)-(z_h,\xi_h)\| \;+\; \Big(\sum_{F\in\mathcal{F}_h}|z - \Pi_k^{\nabla,F} z|_{1,F}^2\Big)^{1/2}\Big\}\,\|(v_h,\mu_h)\|\,,
\end{aligned}$$

which, combined with (90), yields

$$\|(v_h,\mu_h)\| \;\lesssim\; \|(z,\xi)-(z_h,\xi_h)\| \;+\; \Big(\sum_{F\in\mathcal{T}_h}|z - \Pi_k^{\nabla,F} z|_{1,F}^2\Big)^{1/2}\,.$$

Finally, replacing the foregoing inequality back into (89) and taking infimum with respect to $(z_h,\xi_h) \in \boldsymbol{X}_h^k$, we arrive at (88) and finish the proof. □

Having proved Theorem 5, we now employ classical density arguments, the aproximation properties provided by (49), (50), and Lemma 4, and the uniform boundedness of $\mathcal{R}_h$, to deduce that

$$\lim_{h\to 0} \|\mathcal{R}_h(z,\xi) - (z,\xi)\| \;=\; 0 \qquad \forall\,(z,\xi)\in \boldsymbol{X}\,, \tag{91}$$

or, equivalently, that $\mathcal{R}_h$ converges pointwise to the identity operator in $\boldsymbol{X}$.

The unique solvability and associated stability estimate of the VEM/BEM scheme (85) follows from the discrete inf-sup condition for $\mathbb{A}_{\kappa,h}$, which is established next. For later use, we now let $\langle\cdot,\cdot\rangle_{\boldsymbol{X}}$ be the inner product of $\boldsymbol{X}$.

Theorem 6 *Assume that κ^2 is not an eigenvalue of the Laplacian in O with a Dirichlet boundary condition on Γ. Then, there exist positive constants h_0 and α_κ, independent of h, such that for each $h \;\le\; h_0$ there holds*

$$\sup_{\substack{(z_h,\xi_h)\in \boldsymbol{X}_h^k\\ (z_h,\xi_h)\neq \mathbf{0}}} \frac{\big|\mathbb{A}_{\kappa,h}\big((z_h,\xi_h),(v_h,\mu_h)\big)\big|}{\|(z_h,\xi_h)\|} \;\ge\; \alpha_\kappa\,\|(v_h,\mu_h)\| \qquad \forall\,(v_h,\mu_h)\in \boldsymbol{X}_h^k\,. \tag{92}$$

Proof We first employ the bijectivity of $\mathcal{A}_\kappa : \boldsymbol{X}\to\boldsymbol{X}'$ to deduce the existence of a bounded operator $\Theta : \boldsymbol{X}\to\boldsymbol{X}$ such that, given $(z,\xi)\in\boldsymbol{X}$, $\Theta(z,\xi)\in\boldsymbol{X}$ is uniquely characterized by the identity

$$\mathbb{A}_\kappa\big(\Theta(z,\xi),(v,\mu)\big) \;=\; \langle (z,\xi),(v,\mu)\rangle_{\boldsymbol{X}} \qquad \forall\,(v,\mu)\in\boldsymbol{X}\,,$$

which implies, in particular, that

$$\mathbb{A}_\kappa\big(\Theta(z,\xi),(z,\xi)\big) \;=\; \|(z,\xi)\|^2 \qquad \forall\,(z,\xi)\in\boldsymbol{X}\,. \tag{93}$$

Then, given $(v_h,\mu_h)\in\boldsymbol{X}_h^k$, we set $(z_h^+,\xi_h^+) := \mathcal{R}_h\Theta(v_h,\mu_h)\in\boldsymbol{X}_h^k$, and observe that certainly

$$\sup_{\substack{(z_h,\xi_h)\in \boldsymbol{X}_h^k\\ (z_h,\xi_h)\neq \mathbf{0}}} \frac{\big|\mathbb{A}_{\kappa,h}\big((z_h,\xi_h),(v_h,\mu_h)\big)\big|}{\|(z_h,\xi_h)\|} \;\ge\; \frac{\big|\mathbb{A}_{\kappa,h}\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big)\big|}{\|(z_h^+,\xi_h^+)\|}\,. \tag{94}$$

In turn, adding and subtracting the bilinear forms $\mathbb{A}_\kappa$, $\mathbb{A}_0$, and $\mathbb{A}_{0,h}$, so that

$$\mathbb{A}_{\kappa,h} \;=\; \mathbb{A}_{0,h} + (\mathbb{A}_\kappa - \mathbb{A}_0) + (\mathbb{A}_0 - \mathbb{A}_{0,h}) + (\mathbb{A}_{\kappa,h} - \mathbb{A}_\kappa)\,,$$

and noticing from the definitions of $\mathbb{A}_\kappa$, $\mathbb{A}_0$, $\mathbb{A}_{\kappa,h}$, and $\mathbb{A}_{0,h}$ (cf. (19), (31), (59), and (86)), that

$$\big(\mathbb{A}_0 - \mathbb{A}_{0,h}\big)\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big) \;=\; \int_O \nabla z_h^+\cdot\nabla v_h \;-\; a_{0,h}(z_h^+,v_h)$$

and

$$
\begin{aligned}
\big(\mathbb{A}_{\kappa,h} - \mathbb{A}_{\kappa}\big)\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big) \;&=\; a_{\kappa,h}(z_h^+,v_h) \;-\; a_{\kappa}(z_h^+,v_h)\\
&=\; a_{0,h}(z_h^+,v_h) - \int_O \nabla z_h^+\cdot\nabla v_h\\
&+ \kappa^2 \sum_{F\in\mathcal{F}_h} \theta_F \int_F \Big\{ z_h^+ v_h - \big(\Pi_{k-1}^F z_h^+\big)\,\big(\Pi_{k-1}^F v_h\big)\Big\}\,,
\end{aligned}
\tag{95}
$$

we readily arrive at

$$
\begin{aligned}
\mathbb{A}_{\kappa,h}\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big) \;&=\; \mathbb{A}_{0,h}\big(\mathcal{R}_h\Theta(v_h,\mu_h),(v_h,\mu_h)\big)\\
&+ \big[\mathcal{C}\mathcal{R}_h\Theta(v_h,\mu_h),(v_h,\mu_h)\big]\\
&+ \kappa^2 \sum_{F\in\mathcal{F}_h} \theta_F \int_E \Big\{ z_h^+ v_h - \big(\Pi_{k-1}^F z_h^+\big)\,\big(\Pi_{k-1}^F v_h\big)\Big\}\,,
\end{aligned}
\tag{96}
$$

where $\mathcal{C} := \mathcal{A}_\kappa - \mathcal{A}_0 : \boldsymbol{X} \to \boldsymbol{X}'$ is a compact operator. Hence, starting from (96), denoting by I the identity operator from $\boldsymbol{X}$ into itself, letting θ_M be the maximum value of $|\theta_F|$, $F\in\mathcal{F}_h$, and employing the characterization of $\mathcal{R}_h$ (cf. (87)), the orthogonality condition satisfied by Π_{k-1}^F, the identity (93), the approximation properties of Π_{k-1}^F (cf. (48)), and the fact that $\mathcal{R}_h$ is uniformly bounded, we find that

$$
\begin{aligned}
\mathbb{A}_{\kappa,h}\big((z_h^+,\xi_h^+),(v_h,\mu_h)\big) \;&=\; \mathbb{A}_{0}\big(\Theta(v_h,\mu_h),(v_h,\mu_h)\big) \;+\; \big[\mathcal{C}\mathcal{R}_h\Theta(v_h,\mu_h),(v_h,\mu_h)\big]\\
&+\; \kappa^2 \sum_{F\in\mathcal{F}_h} \theta_F \int_F \Big\{ z_h^+ v_h - \big(\Pi_{k-1}^F z_h^+\big)\,\big(\Pi_{k-1}^F v_h\big)\Big\}\\
&=\; \mathbb{A}_{\kappa}\big(\Theta(v_h,\mu_h),(v_h,\mu_h)\big) \;+\; \big[\mathcal{C}\big(\mathcal{R}_h - \mathrm{I}\big)\Theta(v_h,\mu_h),(v_h,\mu_h)\big]\\
&+\; \kappa^2 \sum_{F\in\mathcal{F}_h} \theta_F \int_F \Big\{ z_h^+ - \Pi_{k-1}^F z_h^+\Big\}\Big\{ v_h - \Pi_{k-1}^F v_h\Big\}\\
&\ge\; \Big\{1 - \|\mathcal{C}\big(\mathcal{R}_h - \mathrm{I}\big)\|\|\Theta\|\Big\}\,\|(v_h,\mu_h)\|^2\\
&-\; \kappa^2\theta_M \sum_{F\in\mathcal{F}_h} \big\|z_h^+ - \Pi_{k-1}^F z_h^+\big\|_{0,F}\,\big\|v_h - \Pi_{k-1}^F v_h\big\|_{0,F}\\
&\ge\; \Big\{1 - \|\mathcal{C}\big(\mathcal{R}_h - \mathrm{I}\big)\|\|\Theta\|\Big\}\,\|(v_h,\mu_h)\|^2 - Ch^2\|(z_h^+,\xi_h^+)\|\,\|(v_h,\mu_h)\|\\
&\ge\; \Big\{1 - \|\mathcal{C}\big(\mathcal{R}_h - \mathrm{I}\big)\|\|\Theta\| - Ch^2\Big\}\,\|(z_h^+,\xi_h^+)\|\,\|(v_h,\mu_h)\|\,,
\end{aligned}
$$

where C is a positive constant depending on κ and θ_M, but independent of h, and the last inequality uses that $\|(v_h,\mu_h)\| \gtrsim \|(z_h^+,\xi_h^+)\|$. Finally, the compactness of $\mathcal{C}$ and the pointwise convergence of $\mathcal{R}_h - \mathrm{I}$ to zero (cf. (91)) guarantee that $\lim\limits_{h\to 0}\|\mathcal{C}(\mathcal{R}_h - \mathrm{I})\| = 0$, which, together with the foregoing estimate and (94), yield (92) for a sufficiently small h_0. □

Under the same assumptions of Theorem 6, and as a straightforward consequence of (92), we conclude that, given $\mathbb{F}\in\boldsymbol{X}'$ and $h\le h_0$, the VEM/BEM scheme (85) has a unique solution $(u_h,\lambda_h)\in\boldsymbol{X}_h^k$.

We now turn to provide a priori error bounds and associated rates of convergence for the solution of the VEM/BEM scheme (85). For this purpose, we first define the discrete analogue of Θ (though with respect to the second component of the bilinear form involved), namely the operator $\Theta_h : \boldsymbol{X}_h^k \to \boldsymbol{X}_h^k$ that, given $(v_h, \mu_h) \in \boldsymbol{X}_h^k$, is uniquely characterized by the equation

$$\mathbb{A}_{\kappa,h}\big((z_h, \xi_h), \Theta_h(v_h, \mu_h)\big) \;=\; \langle (z_h, \xi_h), (v_h, \mu_h)\rangle_{\boldsymbol{X}} \qquad \forall\, (z_h, \xi_h) \in \boldsymbol{X}_h^k\,,$$

so that, in particular,

$$\mathbb{A}_{\kappa,h}\big((v_h, \mu_h), \Theta_h(v_h, \mu_h)\big) \;=\; \|(v_h, \mu_h)\|^2 \qquad \forall\, (v_h, \mu_h) \in \boldsymbol{X}_h^k\,. \tag{97}$$

Note that the above identity and the discrete inf-sup condition (92) yield

$$\|\Theta_h\| \;\leq\; \frac{1}{\alpha_\kappa}\,. \tag{98}$$

Hence, we have the following Cea-type estimate, which makes use of $\Pi_{k-1}^{\mathcal{F}}$ (cf. (47)), the global $\mathrm{L}^2(\mathcal{O})$-orthogonal projection onto $\mathcal{P}_{k-1}(\mathcal{F}_h)$.

Theorem 7 *Assume that κ^2 is not an eigenvalue of the Laplacian in $\mathcal{O}$ with a Dirichlet boundary condition on Γ, and let $h_0 > 0$ be the constant whose existence is guaranteed by Theorem 6. Then, there exists a constant $C > 0$, independent of h, such that for each $h \leq h_0$ there holds*

$$\begin{aligned} &\|(u,\lambda) - (u_h, \lambda_h)\| \\ &\quad\leq C\left\{\mathrm{dist}\big((u,\lambda), \boldsymbol{X}_h^k\big) + \Big(\sum_{F\in\mathcal{F}_h}\big\|u - \Pi_k^{\nabla,F} u\big\|_{1,F}^2\Big)^{1/2} + \big\|u - \Pi_{k-1}^{\mathcal{F}} u\big\|_{0,\mathcal{O}}\right\}. \end{aligned} \tag{99}$$

Proof We begin by observing, thanks to the triangle inequality, that

$$\|(u,\lambda) - (u_h, \lambda_h)\| \;\leq\; \|(u,\lambda) - (v_h, \mu_h)\| \;+\; \|(z_h, \xi_h)\| \qquad \forall\, (v_h, \mu_h) \in \boldsymbol{X}_h^k\,, \tag{100}$$

where $(z_h, \xi_h) := (u_h, \lambda_h) - (v_h, \mu_h)$. Then, setting $(z_h^+, \xi_h^+) := \Theta_h(z_h, \xi_h) \in \boldsymbol{X}_h^k$, employing the identity (97) and the fact that $\mathbb{A}_\kappa\big((u,\lambda), \cdot\big)$ and $\mathbb{A}_{\kappa,h}\big((u_h,\lambda_h), \cdot\big)$ coincide on $\boldsymbol{X}_h^k$ (which follows from (18) and (58)), adding and subtracting (v_h, μ_h) in the first component of $\mathbb{A}_\kappa$, using the uniform boundedness of Θ_h (cf. (98)) and the identity provided by the first row of (95), and then adding and subtracting u in the first component of a_κ, we obtain

$$\begin{aligned} \|(z_h, \xi_h)\|^2 \;&=\; \mathbb{A}_{\kappa,h}\big((u_h, \lambda_h), \Theta_h(z_h, \xi_h)\big) \;-\; \mathbb{A}_{\kappa,h}\big((v_h, \mu_h), \Theta_h(z_h, \xi_h)\big) \\ &=\; \mathbb{A}_\kappa\big((u,\lambda) - (v_h, \mu_h), \Theta_h(z_h, \xi_h)\big) + \big(\mathbb{A}_\kappa - \mathbb{A}_{\kappa,h}\big)\big((v_h, \mu_h), (z_h^+, \xi_h^+)\big) \\ &\leq\; \|\mathbb{A}_\kappa\|\,\alpha_\kappa^{-1}\,\|(u,\lambda) - (v_h, \mu_h)\|\,\|(z_h, \xi_h)\| \;+\; \big|a_\kappa(v_h, z_h^+) - a_{\kappa,h}(v_h, z_h^+)\big| \\ &\leq\; \Big(\|\mathbb{A}_\kappa\| + \|a_\kappa\|\Big)\,\alpha_\kappa^{-1}\,\|(u,\lambda) - (v_h, \mu_h)\|\,\|(z_h, \xi_h)\| \\ &\qquad + \big|a_\kappa(u, z_h^+) - a_{\kappa,h}(v_h, z_h^+)\big|\,. \end{aligned} \tag{101}$$

In this way, we now focus on estimating the last term on the right hand side of the foregoing equation. Indeed, according to the definitions of a_κ and $a_{\kappa,h}$ (cf. (24) and (83) - (84)), we first obtain

$$\begin{aligned}\left|a_\kappa(u,z_h^+) - a_{\kappa,h}(v_h,z_h^+)\right| \,&\le\, \sum_{F\in\mathcal{F}_h}\left|a_\kappa^F(u,z_h^+) - a_{\kappa,h}^F(v_h,z_h^+)\right|\\ \le\, \sum_{F\in\mathcal{F}_h}&\left|a_0^F(u,z_h^+) - a_{0,h}^F(v_h,z_h^+)\right| \\ +\,\kappa^2\sum_{F\in\mathcal{F}_h}&|\theta_F|\,\left|\int_F\left\{u z_h^+ - \big(\Pi_{k-1}^F v_h\big)\big(\Pi_{k-1}^F z_h^+\big)\right\}\right|\,.\end{aligned} \tag{102}$$

Next, adding and subtracting $\Pi_k^{\nabla,F}u$ in the first component of $a_0^F(u,z_h^+)$, recalling that there holds $a_0^F(\Pi_k^{\nabla,F}u,z_h^+) \;=\; a_{0,h}^F(\Pi_k^{\nabla,F}u,z_h^+)$ (cf. proof of Theorem 5), and thanks to the uniform boundedness of $a_{0,h}^F$, we find that

$$\begin{aligned}\left|a_0^F(u,z_h^+) - a_{0,h}^F(v_h,z_h^+)\right| \,&=\, \left|a_0^F(u-\Pi_k^{\nabla,F}u,z_h^+) \,+\, a_0^F(\Pi_k^{\nabla,F}u,z_h^+) - a_{0,h}^F(v_h,z_h^+)\right|\\ &=\, \left|a_0^F(u-\Pi_k^{\nabla,F}u,z_h^+) \,+\, a_{0,h}^F(\Pi_k^{\nabla,F}u - v_h,z_h^+)\right|\\ &\lesssim\, \left\{\left\|u-\Pi_k^{\nabla,F}u\right\|_{1,F} \,+\, \left\|\Pi_k^{\nabla,F}u - v_h\right\|_{1,F}\right\}\|z_h^+\|_{1,F}\\ &\lesssim\, \left\{\left\|u-\Pi_k^{\nabla,F}u\right\|_{1,F} \,+\, \|u-v_h\|_{1,F}\right\}\|z_h^+\|_{1,F}\,.\end{aligned} \tag{103}$$

In turn, the orthogonality condition satisfied by Π_{k-1}^F and the triangle inequality yield

$$\begin{aligned}\left|\int_F\left\{u z_h^+ - \big(\Pi_{k-1}^F v_h\big)\big(\Pi_{k-1}^F z_h^+\big)\right\}\right| \,&=\, \left|\int_F\left\{u - \big(\Pi_{k-1}^F v_h\big)\right\}z_h^+\right|\\ &\le\, \left\{\left\|u-v_h\right\|_{0,F} \,+\, \left\|u-\Pi_{k-1}^F u\right\|_{0,F}\right\}\|z_h^+\|_{0,F}\,.\end{aligned} \tag{104}$$

Hence, plugging (103) and (104) in (102), and applying the Cauchy-Schwarz inequality, we deduce the existence of a positive constant C_1, depending on κ and θ_M, but independent of h, such that

$$\begin{aligned}\left|a_\kappa(u,z_h^+) - a_{\kappa,h}(v_h,z_h^+)\right| \,&\le\, C_1\left\{\Big(\sum_{F\in\mathcal{F}_h}\left\|u-\Pi_k^{\nabla,F}u\right\|_{1,F}^2\Big)^{1/2}\right.\\ &\left.+\left\|u-v_h\right\|_{1,\mathcal{O}} \,+\, \left\|u-\Pi_{k-1}^{\mathcal{F}}u\right\|_{0,\mathcal{O}}\right\}\|z_h^+\|_{1,\mathcal{O}}\,.\end{aligned} \tag{105}$$

Thus, replacing (105) back into (101), and bounding $\|z_h^+\|_{1,\mathcal{O}}$ by $\alpha_\kappa^{-1}\,\|(z_h,\xi_h)\|$, we conclude that

$$\begin{aligned}\|(z_h,\xi_h)\| \,&\le\, C_2\left\{\left\|(u,\lambda)-(v_h,\mu_h)\right\|\right.\\ &\left.+\Big(\sum_{F\in\mathcal{F}_h}\left\|u-\Pi_k^{\nabla,F}u\right\|_{1,F}^2\Big)^{1/2} + \left\|u-\Pi_{k-1}^{\mathcal{F}}u\right\|_{0,\mathcal{O}}\right\},\end{aligned} \tag{106}$$

where C_2 is a positive constant depending on $\|\mathbb{A}_\kappa\|$, $\|a_\kappa\|$, α_κ, and C_1, but independent of h. Finally, combining (100) and (106), and then taking infimum over $(v_h, \mu_h) \in \boldsymbol{X}_h^k$, we arrive at (99). □

The rates of convergence of our discrete scheme are provided next. To this end, we recall from (51) that $I_k^{\mathcal{F}}$ stands for the global virtual element interpolation operator. Then, we have the following result.

Theorem 8 *Assume that κ^2 is not an eigenvalue of the Laplacian in $\mathcal{O}$ with a Dirichlet boundary condition on Γ, and that both u and the datum w belong to $\mathrm{H}^1(\mathcal{O}) \cap \prod_{i=1}^{I} \mathrm{H}^{k+1}(\mathcal{O}_i)$. In addition, let $h_0 > 0$ be the constant whose existence is guaranteed by Theorem 6. Then, there exists a constant $C_0 > 0$, independent of h, such that for each $h \le h_0$ there holds*

$$\|(u,\lambda) - (u_h,\lambda_h)\| \;\le\; C_0\, h^k \sum_{i=1}^{I} \|u\|_{k+1,\mathcal{O}_i}\,. \tag{107}$$

Proof We begin by observing that $\mathrm{H}^1(\mathcal{O}) \cap \prod_{i=1}^{I} \mathrm{H}^{k+1}(\mathcal{O}_i) \subseteq C^0(\overline{\mathcal{O}})$, which implies that $I_k^{\mathcal{F}} u$ is meaningful. Furthermore, we have that $\lambda = \gamma_{\boldsymbol{n}}\big(\nabla(u-w)\big) \in \mathrm{H}^{-1/2}(\Gamma) \cap \mathrm{H}_b^{k-1/2}(\Gamma) \subseteq \mathrm{L}^2(\Gamma)$, whence $\Pi_{k-1}^{\mathcal{E}}\lambda$ is meaningful as well, and hence

$$\mathrm{dist}\big((u,\lambda), \boldsymbol{X}_h^k\big) \;\le\; \|u - I_k^{\mathcal{T}} u\|_{1,\Omega} \;+\; \|\lambda - \Pi_{k-1}^{\mathcal{E}}\lambda\|_{-1/2,\Gamma}\,.$$

In this way, replacing the foregoing estimate back into (99), applying the approximation properties of $I_k^{\mathcal{F}}$ (cf. (50)), $\Pi_{k-1}^{\mathcal{E}}$ (cf. Lemma 4), $\Pi_k^{\nabla,F}$ (cf. (49)), and $\Pi_{k-1}^{\mathcal{F}}$ (cf. (48)), and employing the trace inequality

$$\|\lambda\|_{k-1/2,b,\Gamma} \;\lesssim\; \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i}\,,$$

we are led to (107), thus concluding the proof. □

4 The modified Costabel & Han VEM/BEM schemes in 3D

In spite of the plural sense of its title, in this section we introduce and analyze the discrete VEM/BEM scheme for the modified Costabel & Han coupling procedure as applied to the Poisson model only, in the 3D case. The corresponding analysis for the Helmholtz model arises from a suitable combination of the tools to be employed in what follows with those utilized in Section 3.3. We refer to [24, Section 6] for details.

We begin by stressing that the Costabel & Han coupling procedure, that is the variational formulation (18), is not applicable to a VEM/BEM coupling in three dimensions. In fact, as it will become clear below from definitions (108) and (109), the restriction of a VEM function v_h to the boundary of a given element in 3D is not a polynomial function but a virtual function as well. As a consequence, the term

$\left\langle \mu_h, (\frac{\mathrm{id}}{2} - K)\gamma v_h) \right\rangle$ of the bilinear form $\mathbf{A}_h$ (cf. (59)) defining (58), is not computable. Moreover, it is easy to show that, replacing this term by $\left\langle \mu_h, (\frac{\mathrm{id}}{2} - K)\Pi_k^{\mathcal{F}} \gamma v_h) \right\rangle$, implies a dramatic loss of accuracy because, as Γ is a polyhedral Lipschitz boundary, the boundary integral operator K does not yield any further regularity. Summarizing, the fact that the original Costabel & Han coupling method is only applicable to a VEM/BEM scheme in 2D has motivated the introduction of the modified version that we employ in this section.

Similarly as in Section 3.1, we previously need to collect some fundamental notations and results on VEM in 3D.

4.1 Preliminaries

We let $\{\mathcal{T}_h\}_h$ be a family of decompositions of $\overline{\Omega}$ into polyhedral elements E of diameter $h_E \leq h$, and assume again that the meshes $\{\mathcal{T}_h\}_h$ are aligned with each of the subdomaines Ω_i, $i = 1, \ldots, I$ mentioned at the beginning of Section 3.1. In turn, the boundary ∂E of each $E \in \mathcal{T}_h$ is subdivided into planar faces denoted by F, and we let $\mathcal{F}_h$ be the set of faces of $\mathcal{T}_h$ that are contained in Γ. In addition, we assume that the family $\{\mathcal{T}_h\}_h$ of meshes satisfy the following conditions: There exists $\rho \in (0, 1)$ such that

(B1) each E of $\{\mathcal{T}_h\}_h$ is star-shaped with respect to a ball B_E of radius ρh_E,
(B2) for each E of $\{\mathcal{T}_h\}_h$, the diameters h_F of all its faces $F \subseteq \partial E$ satisfy $h_F \geq \rho h_E$,
(B3) the faces of each $E \in \{\mathcal{T}_h\}_h$, viewed as 2-dimensional elements, satisfy the properties **(A1)** and **(A2)** (cf. Section 3.1) with the same ρ.

Next, given an integer $k \geq 1$ and $E \in \mathcal{T}_h$, and bearing in mind the definition (45), we set

$$X_h^k(\partial E) := \left\{ v \in C^0(\partial E) : \quad v|_F \in X_h^k(F) \quad \forall\, F \subseteq \partial E \right\}, \tag{108}$$

and introduce the local virtual element space

$$\begin{aligned} W_h^k(E) := \Big\{ v \in \mathrm{H}^1(E) : \ v|_{\partial E} \in X_h^k(\partial E),\ \Delta v \in \mathcal{P}_k(E), \\ \Pi_k^E v - \Pi_k^{\nabla,E} v \in \mathcal{P}_{k-2}(E) \Big\}, \end{aligned} \tag{109}$$

where, analogously to the 2D case (cf. Section 3.1), Π_k^E is now the $\mathrm{L}^2(E)$–orthogonal projection onto $\mathcal{P}_k(E)$, and the projection operator $\Pi_k^{\nabla,E} : \mathrm{H}^1(E) \to \mathcal{P}_k(E)$ is defined as in (44) after replacing F with E. In addition, the degrees of freedom of $W_h^k(E)$ consist of:

i) the values at the vertices of E,
ii) the moments of order $\leq k - 2$ on the edges of E,
iii) the moments of order $\leq k - 2$ on the faces of E, and
iv) the moments of order $\leq k - 2$ on E.

We can then define the global virtual element space as

$$W_h^k \;:=\; \Big\{ v \in X : \quad v|_E \in W_h^k(E) \quad \forall E \in \mathcal{T}_h \Big\} \,. \tag{110}$$

In addition, and coherently with the notations of Section 3.1, given any integer $k \geq 0$, we let Π_k^E and $\Pi_k^{\mathcal{T}}$ be the L^2–orthogonal projections onto $\mathcal{P}_k(E)$ and $\mathcal{P}_k(\mathcal{T}_h)$, respectively, and denote by $\boldsymbol{\Pi}_k^E$ and $\boldsymbol{\Pi}_k^{\mathcal{T}}$ their corresponding vectorial counterparts. Here again, we stress that $\mathcal{P}_k(E) \subseteq X_h^k(E)$ and that $\Pi_k^{\nabla,E} v$, $\Pi_k^E v$ and $\boldsymbol{\Pi}_{k-1}^E \nabla v$ are all computable for each $v \in X_h^k(E)$ (cf. [2]). In turn, we let $I_k^E : \mathrm{H}^2(E) \to W_h^k(E)$ be the local interpolation operator, which is uniquely determined by the degrees of freedom of $W_h^k(E)$, and whose corresponding global operator is denoted $I_k^{\mathcal{T}} : \mathrm{H}^2(\Omega) \to W_h^k$. The error estimates satisfied by the operators Π_k^E, $\Pi_k^{\nabla,E}$ and I_k^E are given by analogue versions of (48), (49) and (50), respectively, in which F is replaced with E.

Furthermore, we also introduce the simplicial submesh $\mathfrak{F}_h$ of Γ obtained by subdividing each face $F \in \mathcal{F}_h$ into the set of triangles that arise after joining each vertex of F with the midpoint of the disc with respect to which F is star-shaped. Since we are assuming that the meshes satisfy conditions **(A1)** and **(A2)** (cf. Section 3.1), the triangles $T \in \mathfrak{F}_h$ have a shape ratio that is uniformly bounded with respect to h. According to the above, and in order to approximate the non-virtual boundary unknowns of our scheme (cf. Section 4.2 below), we now introduce the piecewise polynomial spaces

$$\Lambda_h^{k-1} \;:=\; \Big\{ \mu_h \in \mathrm{L}^2(\Gamma) : \quad \mu_h|_T \in P_{k-1}(T) \quad \forall T \in \mathfrak{F}_h \Big\} \tag{111}$$

and

$$\Psi_h^k \;:=\; \Big\{ \varphi_h \in C^0(\Gamma) : \quad \varphi_h|_T \in \mathcal{P}_k(T) \quad \forall T \in \mathfrak{F}_h \Big\} \,\cap\, \mathrm{H}_0^{1/2}(\Gamma) \,. \tag{112}$$

Thus, we let $\Pi_{k-1}^{\mathfrak{F}}$ be the $\mathrm{L}^2(\Gamma)$-orthogonal projection onto Λ_h^{k-1}, and let $\mathcal{L}_k^{\mathfrak{F}} : C^0(\Gamma) \to \Psi_h^k$ be the corresponding global Lagrange interpolation operator of order k. Then, denoting by $\{\Gamma_1, ..., \Gamma_J\}$ the open polygons, contained in different hyperplanes of $\mathbb{R}^3$, such that $\Gamma = \cup_{j=1}^J \overline{\Gamma}_j$, we now recall from [32] the following approximation properties of $\Pi_{k-1}^{\mathfrak{F}}$ and $\mathcal{L}_k^{\mathfrak{F}}$.

Lemma 10 *Assume that $\mu \in H_0^{-1/2}(\Gamma) \,\cap\, H_b^r(\Gamma)$ for some $r \geq 0$. Then*

$$\Big\| \mu - \Pi_{k-1}^{\mathfrak{F}} \mu \Big\|_{-t,\Gamma} \;\lesssim\; h^{\min\{r,k\}+t} \;\|\mu\|_{r,b,\Gamma} \qquad \forall\, t \in \{0, 1/2\} \,.$$

Proof See [32, Theorem 4.3.20]. □

Lemma 11 *Assume that $\varphi \in H_b^{r+1/2}(\Gamma) \,\cap\, H^1(\Gamma)$ for some $r > 1/2$. Then*

$$\Big\| \varphi - \mathcal{L}_k^{\mathfrak{F}} \varphi \Big\|_{t,\Gamma} \;\lesssim\; h^{\min\{r+1/2,k+1\}-t} \;\|\varphi\|_{r+1/2,b,\Gamma} \qquad \forall\, t \in \{0, 1/2, 1\} \,.$$

Proof See [32, Proposition 4.1.50]. □

4.2 The discrete setting

According to the finite dimensional subspaces defined in (110), (111), and (112), we now propose the following discrete formulation for (25): Find $(u_h, \psi_h, \lambda_h) \in \widetilde{\mathbf{X}}_h := W_h^k \times \Psi_h^k \times \Lambda_h^{k-1}$ such that

$$\widetilde{\mathbf{A}}_h\big((u_h, \psi_h, \lambda_h), (v_h, \varphi_h, \mu_h)\big) = \widetilde{\mathbf{F}}_h(v_h, \varphi_h, \mu_h) \quad \forall\, (v_h, \varphi_h, \mu_h) \in \widetilde{\mathbf{X}}_h\,, \tag{113}$$

where

$$\begin{aligned} \widetilde{\mathbf{A}}_h\big((z_h, \phi_h, \xi_h), (v_h, \varphi_h, \mu_h)\big) &= \mathbf{a}_h\big((z_h, \phi_h, \xi_h), (v_h, \varphi_h, \mu_h)\big) + \big\langle W\phi_h, \varphi_h \big\rangle \\ &+ \big\langle \mu_h, V\xi_h \big\rangle - \big\langle \xi_h, \big(\frac{\mathrm{id}}{2} - K\big)\varphi_h \big\rangle + \big\langle \mu_h, \big(\frac{\mathrm{id}}{2} - K\big)\phi_h \big\rangle\,, \end{aligned} \tag{114}$$

$$\begin{aligned} \mathbf{a}_h\big((z_h, \phi_h, \xi_h), (v_h, \varphi_h, \mu_h)\big) &= a_h(z_h, v_h) - \sum_{F \in \mathcal{F}_h} \int_F \xi_h\, \Pi_{k-1}^F(\gamma v_h - \varphi_h) \\ &+ \sum_{F \in \mathcal{F}_h} \int_F \mu_h\, \Pi_{k-1}^F(\gamma z_h - \phi_h)\,, \end{aligned} \tag{115}$$

and

$$\widetilde{\mathbf{F}}_h(v_h, \varphi_h, \mu_h) := \int_\Omega \Pi_{k-1}^{\mathcal{T}} f\, v_h \tag{116}$$

for all (z_h, ϕ_h, ξ_h), $(v_h, \varphi_h, \mu_h) \in \widetilde{\mathbf{X}}_h$. We notice here that the bilinear form a_h forming part of the definition of $\mathbf{a}_h$ (cf. (115)) is defined as in Section 3. Namely, denoting by $\mathcal{E}(E)$ and $\mathcal{F}(E)$ the sets of edges and faces, respectively, of a given $E \in \mathcal{T}_h$, we introduce

$$S_h^E(v, z) := \sum_{e \in \mathcal{E}(E)} \int_e \Pi_k^e v\, \Pi_k^e z + h_E^{-1} \sum_{F \in \mathcal{F}(E)} \int_F \Pi_{k-2}^F v\, \Pi_{k-2}^F z$$

for all $v,\ z \in W_h^k(E)$, set

$$a_h^E(v, z) := \int_E \kappa\, \mathbf{\Pi}_{k-1}^E \nabla v \cdot \mathbf{\Pi}_{k-1}^E \nabla z + S_h^E(v - \Pi_k^{\nabla, E} v, z - \Pi_k^{\nabla, E} z) \tag{117}$$

for all $v,\ z \in \mathrm{H}^1(E)$, and define

$$a_h(v, z) := \sum_{E \in \mathcal{T}_h} a_h^E(v, z)$$

for all $v,\ z \in W_h^k$. Furthermore, we stress that the boundary terms in (115) are certainly induced by the corresponding boundary terms in (26). More precisely, the fact that the discrete version of the second term on the right hand side of (26), that is $\big\langle \xi_h, \gamma v_h - \varphi_h \big\rangle$, is not computable since the virtual trace γv_h is not known, suggests to replace $\gamma v_h - \varphi_h$ by a suitable projection such as $\Pi_{k-1}^{\mathfrak{F}}(\gamma v_h - \varphi_h)$, thus yielding the new term $\sum_{F\in\mathcal{F}_h}\int_F \xi_h\,\Pi^F_{k-1}(\gamma v_h-\varphi_h)$. An analogue reason explains the replacement of $\langle\mu_h,\gamma z_h-\phi_h\rangle$ by $\sum_{F\in\mathcal{F}_h}\int_F \mu_h\,\Pi^F_{k-1}(\gamma z_h-\phi_h)$. In turn, the use here of the global orthogonal projector $\Pi^{\mathfrak{F}}_{k-1}:\mathrm{L}^2(\Gamma)\to\Lambda^{k-1}_h$, equivalently the local projections Π^F_{k-1} on each face F of $\mathcal{F}_h$, is strongly motivated by the fact that λ_h and μ_h live in the subspace Λ^{k-1}_h, which allows to apply later on the corresponding orthogonality condition, a key property for the derivation of the a priori error estimate and the associated rates of convergence (see below Theorem 9, estimate (124), and Theorem 10, all in Section 4.3).

Finally, we highlight that the discrete problem (113) is meaningful since $S^E_h(\cdot,\cdot)$ is computable on $W^k_h(E)\times W^k_h(E)$. Moreover, it can be shown that $S^E_h(v,z)$ scales like $a^E(v,z):=\int_E\kappa\nabla v\cdot\nabla z$ on the kernel of $\Pi^{\nabla,E}_k$ in $W^k_h(E)$. In other words, the three-dimensional counterpart of (54) holds true (cf. [11, Section 5.5]), which implies, in particular, that we have the corresponding 3D versions of (63) and (69) as well.

4.3 Solvability and a priori error analyses

We begin by introducing further notations to be employed later on. In fact, for any $s\geq 0$ we define the broken Sobolev spaces

$$\mathrm{H}^s(\mathcal{T}_h)\;:=\;\prod_{E\in\mathcal{T}_h}\mathrm{H}^s(K),\qquad \mathrm{H}^s(\mathcal{F}_h)\;:=\;\prod_{F\in\mathcal{F}_h}\mathrm{H}^s(F),$$

which are endowed with the Hilbertian norms and corresponding seminorms, given respectively, by

$$\|v\|^2_{s,\mathcal{T}_h}\;:=\;\sum_{E\in\mathcal{T}_h}\|v\|^2_{s,E}\,,\qquad \|\varphi\|^2_{s,\mathcal{F}_h}\;:=\;\sum_{F\in\mathcal{F}_h}\|\varphi\|^2_{s,F}\,.$$

and

$$|v|^2_{s,\mathcal{T}_h}\;:=\;\sum_{E\in\mathcal{T}_h}|v|^2_{s,E}\,,\qquad |\varphi|^2_{s,\mathcal{F}_h}\;:=\;\sum_{F\in\mathcal{F}_h}|\varphi|^2_{s,F}\,,$$

for all $v\in\mathrm{H}^s(\mathcal{T}_h)$ and for all $\varphi\in\mathrm{H}^s(\mathcal{F}_h)$. In addition, we set as usual $\mathrm{H}^0(\mathcal{T}_h)=\mathrm{L}^2(\mathcal{T}_h)$ and $\mathrm{H}^0(\mathcal{F}_h)=\mathrm{L}^2(\mathcal{F}_h)$.

Now, concerning the solvability of (113), we first notice that the boundedness of $\widetilde{\mathbf{A}}_h$ follows exactly as proved for the 2D case (cf. Section 3.3). Thus, we continue the analysis with the $\widetilde{\mathbf{X}}_h$-ellipticity of $\widetilde{\mathbf{A}}_h$ with respect to the usual product norm of $\widetilde{\mathbf{X}}$.

Lemma 12 *There holds*

$$\widetilde{\mathbf{A}}_h\big(v_h,\varphi_h,\mu_h\big),\big(v_h,\varphi_h,\mu_h\big)\;\gtrsim\;\|(v_h,\varphi_h,\mu_h)\|^2$$

for all $\big(v_h,\varphi_h,\mu_h\big)\in\widetilde{\mathbf{X}}_h$.

Proof Given $(v_h,\varphi_h,\mu_h)\in\widetilde{\mathbf{X}}_h$, it follows from (114) and (115) that

$$\widetilde{\mathbf{A}}_h\big(v_h,\varphi_h,\mu_h\big),\big(v_h,\varphi_h,\mu_h\big) \;=\; a_h(v_h,v_h) \,+\, \big\langle W\varphi_h,\varphi_h\big\rangle \,+\, \big\langle \mu_h, V\mu_h\big\rangle\,,$$

and hence the 3D version of (69) and Lemma 1 finish the proof. □

As a consequence of the previous analysis and the Lax-Milgram lemma, we conclude that (113) has a unique solution $(u_h,\psi_h,\lambda_h) \in \mathbb{X}_h$. Next, in order to establish the corresponding a priori error estimate, we follow the same notations from Section 3.1 and for each planar face $F \in \mathcal{F}_h$ we let Π_k^F be the $\mathrm{L}^2(F)$-orthogonal projection onto $\mathcal{P}_k(F)$ with vectorial counterpart $\mathbf{\Pi}_k^F$. In addition, $\Pi_k^{\mathcal{F}}$ and $\mathbf{\Pi}_k^{\mathcal{F}}$ stand for their global extensions to $\mathrm{L}^2(\Gamma)$ and $\mathrm{L}^2(\Gamma)^2$, respectively, which are assembled cellwise. The approximation properties of Π_k^F (and hence of $\mathbf{\Pi}_k^F$, $\Pi_k^{\mathcal{F}}$ and $\mathbf{\Pi}_k^{\mathcal{F}}$) are exactly those given by (or derived from) (48).

The 3D analogue of Theorem 3 is given by the following result.

Theorem 9 *Under the assumption that $u \in X \cap \prod_{i=1}^I \mathrm{H}^2(\Omega_i)$, there holds*

$$\begin{aligned}
&\|(u,\psi,\lambda)-(u_h,\psi_h,\lambda_h)\| \\
&\quad\lesssim \big\|f-\Pi_{k-1}^{\mathcal{T}} f\big\|_{0,\Omega} \,+\, \|(u,\psi,\lambda)-(I_k^{\mathcal{T}}u,\mathcal{L}_k^{\mathfrak{F}}\psi,\Pi_{k-1}^{\mathfrak{F}}\lambda)\| \\
&\quad+ \sup_{\substack{(w_h,\phi_h,\xi_h)\in\widetilde{\mathbf{X}}_h\\(w_h,\phi_h,\xi_h)\neq\mathbf{0}}} \frac{\big|\mathbf{a}\big((u,\psi,\lambda),(w_h,\phi_h,\xi_h)\big)-\mathbf{a}_h\big((I_k^{\mathcal{T}}u,\mathcal{L}_k^{\mathfrak{F}}\psi,\Pi_{k-1}^{\mathfrak{F}}\lambda),(w_h,\phi_h,\xi_h)\big)\big|}{\|(w_h,\phi_h,\xi_h)\|}\,.
\end{aligned} \tag{118}$$

Proof We follow basically the same sequence of arguments provided in the proof of Theorem 3. Indeed, according to the definitions of $\widetilde{\mathbf{F}}$ (cf. (28)), $\widetilde{\mathbf{F}}_h$ (cf. (116)), $\widetilde{\mathbf{A}}$ (cf. (26) - (27)) and $\widetilde{\mathbf{A}}_h$ (cf. (114) - (115)), which yields, in particular

$$\big(\widetilde{\mathbf{A}}-\widetilde{\mathbf{A}}_h\big)\big((v_h,\varphi_h,\mu_h),(w_h,\phi_h,\xi_h)\big) \;=\; \big(\mathbf{a}-\mathbf{a}_h\big)\big((v_h,\varphi_h,\mu_h),(w_h,\phi_h,\xi_h)\big)$$

for all (v_h,φ_h,μ_h), $(w_h,\phi_h,\xi_h) \in \widetilde{\mathbf{X}}_h$, and using the boundedness of $\mathbf{a}$, we find that a direct application of the first Strang Lemma (cf. [17, Theorem 4.1.1]) to the context given now by (25) and (113), gives

$$\begin{aligned}
&\|(u,\psi,\lambda)-(u_h,\psi_h,\lambda_h)\| \\
&\quad\lesssim \big\|f-\Pi_{k-1}^{\mathcal{T}} f\big\|_{0,\Omega} + \inf_{(v_h,\varphi_h,\mu_h)\in\widetilde{\mathbf{X}}_h} \Bigg\{ \|(u,\psi,\lambda)-(v_h,\varphi_h,\mu_h)\| \\
&\quad+ \sup_{\substack{(w_h,\phi_h,\xi_h)\in\widetilde{\mathbf{X}}_h\\(w_h,\phi_h,\xi_h)\neq\mathbf{0}}} \frac{\big|\mathbf{a}\big((u,\psi,\lambda),(w_h,\phi_h,\xi_h)\big)-\mathbf{a}_h\big((v_h,\varphi_h,\mu_h),(w_h,\phi_h,\xi_h)\big)\big|}{\|(w_h,\phi_h,\xi_h)\|} \Bigg\}\,.
\end{aligned} \tag{119}$$

Thanks to the hypothesis we have that both u and $\psi = \gamma u$ are continuous, and hence $I_k^{\mathcal{T}} u$ and $\mathcal{L}_k^{\mathfrak{F}}\psi$ are meaningful. In addition, the fact that $u \in \prod_{i=1}^I \mathrm{H}^2(\Omega_i)$ implies that $\lambda = \kappa\nabla u\cdot\boldsymbol{n} \in \mathrm{H}_b^{1/2}(\Gamma) \subseteq \mathrm{L}^2(\Gamma)$, and hence $\Pi_{k-1}^{\mathfrak{F}}\lambda$ is meaningful as well. In this way, taking in particular $(v_h,\varphi_h,\mu_h) = (I_k^{\mathcal{T}}u,\mathcal{L}_k^{\mathfrak{F}}\psi,\Pi_{k-1}^{\mathfrak{F}}\lambda) \in \widetilde{\mathbf{X}}_h$ in (119) we arrive at (118) and conclude the proof. □

Analogously to the analysis for the 2D case, we now aim to estimate the supremum in (118), for which we first notice from the definitions of $\mathbf{a}$ (cf. (27)) and $\mathbf{a}_h$ (cf. (115)), and using that $\psi = \gamma u$, that

$$\begin{aligned} &\mathbf{a}\big((u,\psi,\lambda),(w_h,\phi_h,\xi_h)\big) - \mathbf{a}_h\big((I_k^{\mathcal{T}} u, \mathcal{L}_k^{\mathfrak{F}}\psi, \Pi_{k-1}^{\mathfrak{F}}\lambda),(w_h,\phi_h,\xi_h)\big) \\ &\quad = a(u,w_h) - a_h(I_k^{\mathcal{T}} u, w_h) - \langle \lambda, \gamma w_h - \phi_h\rangle \\ &\quad + \int_\Gamma \Pi_{k-1}^{\mathfrak{F}}\lambda\, \Pi_{k-1}^{\mathcal{F}}(\gamma w_h - \phi_h) - \int_\Gamma \xi_h\, \Pi_{k-1}^{\mathcal{F}}(\gamma I_k^{\mathcal{T}} u - \mathcal{L}_k^{\mathfrak{F}}\psi) \end{aligned} \tag{120}$$

for all $(w_h,\phi_h,\xi_h) \in \mathbb{X}_h$. Then, in what follows we proceed to estimate the right hand side of (120) by splitting it into the three expressions determined by the first and second terms, the third and fourth terms, and the fifth term, respectively.

Firstly, recalling that κ has been assumed to be piecewise constant, and noting that certainly $\nabla\Pi_k^E u \in \mathcal{P}_{k-1}(E)^3$, we deduce, according to the definition of a_h^E (cf. (117)), that

$$a_h^E(\Pi_k^E u, w_h) = a^E(\Pi_k^E u, w_h) \qquad \forall\, E \in \mathcal{T}_h\,, \quad \forall\, w_h \in W_h^k(E)\,,$$

and therefore, adding and subtracting $\Pi_k^E u$ in the first components of a^E and a_h^E, we readily find that

$$a(u,w_h) - a_h(I_k^{\mathcal{T}} u, w_h) = \sum_{E\in\mathcal{T}_h} \Big\{ a^E(u - \Pi_k^E u, w_h) + a_h^E(\Pi_k^E u - I_k^E u, w_h)\Big\}$$

for all $w_h \in W_h^k$. In this way, thanks to the foregoing identity and the boundedness of a^E and a_h^E, the latter being proved similarly to the proof of Lemma 5, and then adding and subtracting u in the expression resulting from bounding a_h^E, we arrive at

$$\big|a(u,w_h) - a_h(I_k^{\mathcal{T}} u, w_h)\big| \lesssim \Big\{ |u - I_k^{\mathcal{T}} u|_{1,\Omega} + |u - \Pi_k^{\mathcal{T}} u|_{1,\mathcal{T}_h}\Big\}\, |w_h|_{1,\Omega}\,. \tag{121}$$

Secondly, noting that $\Pi_{k-1}^{\mathcal{F}}(\gamma w_h - \phi_h) \in \Lambda_h^{k-1}$ (cf. (111)), and employing the orthogonality condition satisfied by $\Pi_{k-1}^{\mathfrak{F}}$, as well as the symmetry of $\Pi_{k-1}^{\mathcal{F}}$, we obtain

$$\int_\Gamma \Pi_{k-1}^{\mathfrak{F}}\lambda\, \Pi_{k-1}^{\mathcal{F}}(\gamma w_h - \phi_h) = \int_\Gamma \lambda\, \Pi_{k-1}^{\mathcal{F}}(\gamma w_h - \phi_h) = \int_\Gamma \Pi_{k-1}^{\mathcal{F}}\lambda\,(\gamma w_h - \phi_h)\,,$$

which yields

$$-\langle \lambda, \gamma w_h - \phi_h\rangle + \int_\Gamma \Pi_{k-1}^{\mathfrak{F}}\lambda\, \Pi_{k-1}^{\mathcal{F}}(\gamma w_h - \phi_h) = \langle \Pi_{k-1}^{\mathcal{F}}\lambda - \lambda, \gamma w_h - \phi_h\rangle\,,$$

and hence, according to the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, and using the trace theorem, we obtain

$$
\begin{aligned}
& \Big| \int_\Gamma \Pi^{\mathfrak{F}}_{k-1}\lambda \, \Pi^{\mathcal{F}}_{k-1}(\gamma w_h - \phi_h) \, - \, \langle \lambda, \gamma w_h - \phi_h \rangle \Big| \\
& \qquad \lesssim \, \|\lambda - \Pi^{\mathcal{F}}_{k-1}\lambda\|_{-1/2,\Gamma} \left\{ \|w_h\|_{1,\Omega} + \|\phi_h\|_{1/2,\Gamma} \right\} .
\end{aligned} \tag{122}
$$

Finally, adding and subtracting $\gamma u = \psi$, we readily get

$$
- \int_\Gamma \xi_h \, \Pi^{\mathcal{F}}_{k-1}(\gamma I^{\mathcal{T}}_k u - \mathcal{L}^{\mathfrak{F}}_k \psi) \; = \; \int_\Gamma \xi_h \, \Pi^{\mathcal{F}}_{k-1}\big(\gamma(u - I^{\mathcal{T}}_k u) - (\psi - \mathcal{L}^{\mathfrak{F}}_k \psi)\big) ,
$$

from which, applying the Cauchy-Schwarz inequality in $\mathrm{L}^2(\Gamma)$ and the inverse inequality satisfied by Λ^{k-1}_h (cf. (111)), we find that

$$
\begin{aligned}
& \Big| \int_\Gamma \xi_h \, \Pi^{\mathcal{F}}_{k-1}(\gamma I^{\mathcal{T}}_k u - \mathcal{L}^{\mathfrak{F}}_k \psi) \Big| \\
& \qquad \lesssim \, h^{-1/2} \left\{ \|\gamma(u - I^{\mathcal{T}}_k u)\|_{0,\Gamma} + \|\psi - \mathcal{L}^{\mathfrak{F}}_k \psi\|_{0,\Gamma} \right\} \|\xi_h\|_{-1/2,\Gamma} .
\end{aligned} \tag{123}
$$

Consequently, using (121), (122), and (123) to bound (120), and then replacing the resulting estimate into (118), we arrive at the a priori error estimate

$$
\begin{aligned}
\|(u,\psi,\lambda) - (u_h,\psi_h,\lambda_h)\| \; & \lesssim \; \big\|f - \Pi^{\mathcal{T}}_{k-1} f\big\|_{0,\Omega} \, + \, |u - I^{\mathcal{T}}_k u|_{1,\Omega} \, + \, \|\psi - \mathcal{L}^{\mathfrak{F}}_k \psi\|_{1/2,\Gamma} \\
& + \, \|\lambda - \Pi^{\mathfrak{F}}_{k-1}\lambda\|_{-1/2,\Gamma} \, + \, |u - \Pi^{\mathcal{T}}_k u|_{1,\mathcal{T}_h} \, + \, \|\lambda - \Pi^{\mathcal{F}}_{k-1}\lambda\|_{-1/2,\Gamma} \\
& + \, h^{-1/2} \left\{ \|\gamma(u - I^{\mathcal{T}}_k u)\|_{0,\Gamma} + \|\psi - \mathcal{L}^{\mathfrak{F}}_k \psi\|_{0,\Gamma} \right\} .
\end{aligned} \tag{124}
$$

The foregoing inequality constitutes the key estimate to derive the rates of convergence of the present 3D VEM/BEM scheme. In this regard, and in order to bound one of the terms involved, we also need the scaled trace inequality (cf. [21, Lemma 1.49]), which says that for each $E \in \mathcal{T}_h$ there holds

$$
\|v\|^2_{0,\partial E} \; \lesssim \; \left\{ h_E^{-1} \, \|v\|^2_{0,E} \, + \, h_E \, |v|^2_{1,E} \right\} \qquad \forall \, v \in \mathrm{H}^1(E) . \tag{125}
$$

Indeed, we end this section with the following main result.

Theorem 10 *Assuming that* $u \in X \cap \prod_{i=1}^I \mathrm{H}^{k+1}(\Omega_i)$ *and* $f \in \prod_{i=1}^I \mathrm{H}^k(\Omega_i)$*, there holds*

$$
\|(u,\psi,\lambda) - (u_h,\psi_h,\lambda_h)\| \; \lesssim \; h^k \, \sum_{i=1}^{I} \left\{ \, \|u\|_{k+1,\Omega_i} \, + \, \|f\|_{k,\Omega_i} \, \right\} . \tag{126}
$$

Proof We first observe, thanks to the regularity assumption on u, that $\psi = \gamma u \in \mathrm{H}^{k+1/2}_b(\Gamma)$ and $\lambda = \kappa \nabla u \cdot \boldsymbol{n} \in \mathrm{H}^{k-1/2}_b(\Gamma)$. Throughout the rest of the proof we identify the terms on the right hand side of (124) according to the order they have been written there, from left to right and from up to down. Thus, applying the 3D versions of (48) (to the first and fifth terms), (50) (to the second term), and Lemma 4 (to the sixth term), and using by the trace inequality that $\|\lambda\|_{k-1/2,b,\Gamma} \, \leq \, c \, \sum_{i=1}^I \|u\|_{k+1,\Omega_i}$, we obtain

$$
\begin{aligned}
&\|f - \Pi_{k-1}^{\mathcal{T}} f\|_{0,\Omega} \,+\, |u - I_k^{\mathcal{T}} u|_{1,\Omega} \,+\, |u - \Pi_k^{\mathcal{T}} u|_{1,\mathcal{T}_h} \,+\, \|\lambda - \Pi_{k-1}^{\mathcal{F}} \lambda\|_{-1/2,\Gamma} \\
&\qquad \lesssim\, h^k \sum_{i=1}^{I} \Big\{ \|f\|_{k,\Omega_i} \,+\, \|u\|_{k+1,\Omega_i} \Big\}\,.
\end{aligned}
\tag{127}
$$

In turn, invoking Lemmas 11 and 10 to bound the third and fourth terms, respectively, and employing also by trace theorem that $\|\psi\|_{k+1/2,b,\Gamma} \,\leq\, c\, \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i}$, we find that

$$
\begin{aligned}
&\|\psi - \mathcal{L}_k^{\mathfrak{F}} \psi\|_{1/2,\Gamma} \,+\, \|\lambda - \Pi_{k-1}^{\mathfrak{F}} \lambda\|_{-1/2,\Gamma} \\
&\qquad \lesssim\, h^k \Big\{ \|\psi\|_{k+1/2,b,\Gamma} + \|\lambda\|_{k-1/2,b,\Gamma} \Big\} \,\lesssim\, h^k \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i}\,.
\end{aligned}
\tag{128}
$$

Furthermore, a straightforward application of Lemma 11 to the eighth term, gives

$$
\|\psi - \mathcal{L}_k^{\mathfrak{F}} \psi\|_{0,\Gamma} \,\lesssim\, h^{k+1/2}\, \|\psi\|_{k+1/2,b,\Gamma}\,,
$$

and hence

$$
h^{-1/2}\, \|\psi - \mathcal{L}_k^{\mathfrak{F}} \psi\|_{0,\Gamma} \,\lesssim\, h^k \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i}\,. \tag{129}
$$

Next, employing the scaled trace inequality (125) and the 3D version of (50), we obtain that for each face F of an element $E \in \mathcal{T}_h$ there holds

$$
\begin{aligned}
&h_F^{-1}\, \big\|\gamma\big(u - I_k^E u\big)\big\|_{0,F}^2 \,\leq\, h_F^{-1} \big\|\gamma\big(u - I_k^E u\big)\big\|_{0,\partial E}^2 \\
&\quad \lesssim\, h_E^{-2}\, \big\|u - I_k^E u\big\|_{0,E}^2 \,+\, \big|u - I_k^E u\big|_{1,E} \,\lesssim\, h_E^{2k}\, \|u\|_{k+1,E}^2\,,
\end{aligned}
$$

from which we arrive at

$$
h^{-1/2}\, \big\|\gamma\big(u - I_k^{\mathcal{T}} u\big)\big\|_{0,\Gamma} \,\lesssim\, h^k \sum_{i=1}^{I} \|u\|_{k+1,\Omega_i}\,. \tag{130}
$$

Finally, utilizing (127), (128), (129), and (130) in (124), we get (126) and conclude the proof. □

5 Numerical results

In this section we show that the numerical rates of convergence delivered by the VEM/BEM schemes (58), (85), and the 2D version of (113) are in accordance with the theoretical ones. For simplicity, we restrict our tests to two-dimensional model problems and to the lowest polynomial degree $k = 1$.

In what follows h and N stand for the meshsize and the total number of degrees of freedom, respectively, of each partition of $\overline{\Omega}$. In addition, the individual and global errors are defined by

$$\mathtt{e}(u) := \|u - \widehat{u}\|_{0,\Omega} + |u - \widehat{u}|_{1,b,\Omega},\ \mathtt{e}(\lambda) := \|\lambda - \lambda_h\|_{-1/2,\Gamma},\ \mathtt{e}(\psi) := \|\psi - \psi_h\|_{1/2,\Gamma},$$

with corresponding rates of convergence

$$\mathtt{r}(\star) := \frac{\log(\mathtt{e}(\star)/\mathtt{e}'(\star))}{\log(h/h')} \quad \forall \star \in \{u, \lambda, \psi\},$$

where h and h' denote two consecutive meshsizes with errors $\mathtt{e}$ and $\mathtt{e}'$. Note here that the exact error for u, that is $\|u-u_h\|_{1,\Omega}$, is not computable since u_h, being virtual, is not known explicitly in Ω. This is the reason why $\mathtt{e}(u)$ is defined above with $\widehat{u}$ (cf. the end of Subsection 3.2) instead of u_h.

5.1 Convergence tests for the Poisson model

We first investigate the performance of the discrete schemes (58) and (113) when applied to problem (1). We point out that the VEM/BEM discretization method (113) has been introduced and analyzed in the 3D case only. However, it is not difficult to see that it is also applicable to two-dimensional problems.

h	N	$\mathtt{e}(u)$	$\mathtt{r}(u)$	$\mathtt{e}(\lambda)$	$\mathtt{r}(\lambda)$
1/64	3521	2.914E−01	–	1.066E−01	–
1/128	13185	1.458E−01	0.999	5.327E−02	1.001
1/256	50945	7.292E−02	1.000	2.663E−02	1.000
1/512	200193	3.646E−02	1.000	1.332E−02	1.000

h	N	$\mathtt{e}(u)$	$\mathtt{r}(u)$	$\mathtt{e}(\psi)$	$\mathtt{r}(\psi)$	$\mathtt{e}(\lambda)$	$\mathtt{r}(\lambda)$
1/64	3777	2.914E−01	–	7.522E−03	–	3.104E−01	–
1/128	13697	1.458E−01	0.999	2.659E−03	1.500	5.338E−02	1.001
1/256	51969	7.292E−02	1.000	9.454E−04	1.492	2.669E−02	1.000
1/512	202241	3.646E−02	1.000	3.387E−04	1.481	1.335E−02	1.000

Table 1 Convergence history of the VEM/BEM schemes (58) and (113) for Poisson, Example 1

We choose $\kappa = 1$ so that the harmonic function $u(\boldsymbol{x}) = \dfrac{x_1 + x_2}{|\boldsymbol{x}|^2}$, $\boldsymbol{x} := (x_1, x_2)$, is a solution of problem (1) with a nonhomogeneous Dirichlet boundary condition on Γ_0. We consider two different geometry settings. In the first example, we take $\Omega_0 = (-0.25, 0.25)^2$ and $\mathcal{O} = (-0.5, 0.5)^2$, and use a sequence of meshes constructed out of square elements. In turn, in the second example, we select $\Omega_0 = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 < 0.25^2\}$ and $\mathcal{O} = (-0.5, 0.5)^2$, and employ a sequence of Voronoi meshes initially generated from random seeds and subsequently smoothed using Lloyd's relaxation algorithm.

The convergence history of both schemes are reported in Tables 1 and 2 for Examples 1 and 2, respectively. There we can see that the rates of convergence predicted by Theorems 4 and 10, that is $O(h)$ for $k = 1$, are confirmed for each one of the unknowns in both examples. The higher rate provided by the unknown ψ for the scheme (113) must be simply understood as a proper super convergence

h	N	$\mathtt{e}(u)$	$\mathtt{r}(u)$	$\mathtt{e}(\lambda)$	$\mathtt{r}(\lambda)$
1/64	7197	3.488E−01	–	1.053E−01	–
1/128	27529	1.749E−01	0.997	5.294E−02	0.993
1/256	107683	8.743E−02	1.001	2.656E−02	0.996
1/384	240512	5.825E−02	1.002	1.774E−02	0.996

h	N	$\mathtt{e}(u)$	$\mathtt{r}(u)$	$\mathtt{e}(\psi)$	$\mathtt{r}(\psi)$	$\mathtt{e}(\lambda)$	$\mathtt{r}(\lambda)$
1/64	7459	3.488E−01	–	1.744E−02	–	1.053E−01	–
1/128	28047	1.749E−01	0.997	6.559E−03	1.539	5.295E−02	0.994
1/256	108713	8.743E−02	1.001	2.472E−03	1.402	2.655E−02	0.997
1/384	242054	5.825E−02	1.002	1.349E−03	1.493	1.772E−02	0.997

Table 2 Convergence history of the VEM/BEM schemes (58) and (113) for Poisson, Example 2

behavior of this particular exact solution u. In addition, we observe that, except for the additional direct approximation ψ_h of the trace of u provided by (113), both VEM/BEM schemes behave very similarly since, at each partition, they yield basically the same errors for each common unknown. Certainly, the advantage of (113) with respect to (58) is that the former is applicable in 2D and 3D, whereas the latter is restricted to 2D. In turn, the advantage of (58) with respect to (113), which is obviously valid only in 2D, is that the former, having one less boundary unknown, is a bit cheaper than (113) in terms of the total number of degrees of freedom. This is illustrated in the present examples by the second columns of Tables 1 and 2.

5.2 Convergence tests for the Helmholtz model

We finally report numerical results carried out with the method based on the scheme (85) (cf. Subsection 3.3) and with an adaptation of scheme (113) for problem (2).

h	N	$\mathtt{e}(u)$	$\mathtt{r}(u)$	$\mathtt{e}(\lambda)$	$\mathtt{r}(\lambda)$
1/64	3951	2.920E−01	–	7.550E−01	–
1/128	15336	1.464E−01	0.966	3.584E−01	1.021
1/192	34133	9.715E−02	1.050	2.394E−01	1.035
1/256	60430	7.259E−02	0.962	1.757E−01	1.020

Table 3 Convergence history of the VEM/BEM scheme (85) for Helmholtz with $\kappa = 2$.

We consider problem (2) with $\theta = 1$, $\Omega_0 = \left\{\boldsymbol{x} \in \mathbb{R}^2 : \frac{x_1^2}{0.5^2} + \frac{x_2^2}{0.7^2} < 1\right\}$, $\mathcal{O} = \left\{\boldsymbol{x} \in \mathbb{R}^2 : \frac{x_1^2}{1.1^2} + \frac{x_2^2}{1.5^2} < 1\right\}$, and use a sequence of of successively refined Voronoi meshes of the domain $\Omega = \mathcal{O} \setminus \Omega_0$. We select the incident wave w in such a way that the exact solution is given in the following closed form

$$u(\boldsymbol{x}) = \begin{cases} (1+\imath)\,\dfrac{x_1 + x_2}{|\boldsymbol{x}|^2} & \text{in } \Omega \\ H_0^{(1)}(\kappa|\boldsymbol{x}|) & \text{in } \mathcal{O}_e. \end{cases}$$

h	N	$\mathtt{e}(u)$	$\mathtt{r}(u)$	$\mathtt{e}(\lambda)$	$\mathtt{r}(\lambda)$	$\mathtt{e}(\psi)$	$\mathtt{r}(\psi)$
1/64	4113	2.882E−01	–	1.280E−01	–	5.682E−03	–
1/128	15666	1.447E−01	0.965	6.467E−02	0.962	2.735E−03	0.997
1/192	34620	9.602E−02	1.051	4.345E−02	1.019	1.800E−03	1.072
1/256	61089	7.178E−02	0.960	3.240E−02	0.969	1.352E−03	0.945

Table 4 Convergence history of the VEM/BEM scheme (113) adapted for Helmholtz with $\kappa = 1$.

We observe from Tables 3 and 4 that the expected linear convergence rate of the error is attained in each variable for both schemes.

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References

1. D. Adak, A. Pramod, E.T. Ooi and S. Natarajan, *A combined virtual element method and the scaled boundary finite element method for linear elastic fracture mechanics*. Eng. Anal. Bound. Elem. 113 (2020), 9–16.
2. B. Ahmad, A. Alsaedi, F. Brezzi, L.D. Marini and A. Russo, *Equivalent projectors for virtual element methods*. Comput. Math. Appl. 66 (2013), no. 3, 376–391.
3. P.F. Antonietti, L. Beirão da Veiga, D. Mora and M. Verani, *A stream virtual element formulation of the Stokes problem on polygonal meshes*. SIAM J. Numer. Anal. 52 (2014), no. 1, 386–404.
4. L. Beirão da Veiga, F. Brezzi, A. Cangiani, L.D Marini, G. Manzini and A. Russo, *Basic principles of virtual elements methods*. Math. Models Methods Appl. Sci. 23 (2013), no. 1, 199–214.
5. L. Beirão da Veiga, F. Brezzi and L.D. Marini, *Virtual elements for linear elasticity problems*. SIAM J. Numer. Anal. 51 (2013), no. 2, 794–812.
6. L. Beirão da Veiga, F. Brezzi, L.D. Marini and A. Russo, *Virtual element method for general second-order elliptic problems on polygonal meshes*. Math. Models Methods Appl. Sci. 26 (2016), no. 4, 729–750.
7. L. Beirão da Veiga, C. Lovadina and G. Vacca, *Divergence free virtual elements for the Stokes problem on polygonal meshes*. ESAIM Math. Model. Numer. Anal. 51 (2017), no. 2, 509–535.
8. L. Beirão da Veiga, C. Lovadina and G. Vacca, *Virtual elements for the Navier-Stokes problem on polygonal meshes*. SIAM J. Numer. Anal. 56 (2018), no. 3, 1210–1242.
9. P. Bettess, D.W. Kelly and O.C. Zienkiewicz, *The coupling of the finite element method and boundary solution procedures*. Internat. J. Numer. Methods Engrg. 11 (1977), no. 2, 355–375.
10. P. Bettess, D.W. Kelly and O.C. Zienkiewicz, *Marriage à la mode. The best of both worlds (Finite elements and boundary integrals)*. Energy Methods in Finite Element Analysis. Glowinski R.; Rodin E.Y.; Zienkiewicz O.C. (eds.). Chichester: J. Wiley (1979).
11. S.C. Brenner, Q. Guan and L.Y. Sung, *Some estimates for virtual element methods*. Comput. Methods Appl. Math. 17 (2017), no. 4, 553–574.
12. F. Brezzi, R.S. Falk and L.D. Marini, *Basic principles of mixed virtual element methods*. ESAIM Math. Model. Numer. Anal. 48 (2014), no. 4, 1227–1240.

13. F. Brezzi and C. Johnson, *On the coupling of boundary integral and finite element methods*. Calcolo 16 (1979), no. 2, 189–201.
14. E. Cáceres and G.N. Gatica, *A mixed virtual element method for the pseudostress-velocity formulation of the Stokes problem*. IMA J. Numer. Anal. 37 (2017), no. 1, 296–331.
15. E. Cáceres, G.N. Gatica and F.A. Sequeira, *A mixed virtual element method for the Brinkman problem*. Math. Models Methods Appl. Sci. 27 (2017), no. 4, 707–743.
16. E. Cáceres, G.N. Gatica and F.A. Sequeira, *A mixed virtual element method for quasi-Newtonian Stokes flows*. SIAM J. Numer. Anal. 56 (2018), no. 1, 317–343.
17. P. Ciarlet, The Finite Element Method for Elliptic Problems. North-Holland, 1978.
18. D. Colton, D. and Kress, R., Inverse Acoustic and Electromagnetic Scattering Theory. Second edition. Applied Mathematical Sciences, 93. Springer-Verlag, Berlin, 1998.
19. M. Costabel, *Symmetric methods for the coupling of finite elements and boundary elements*. In: Boundary Elements IX (C.A. Brebbia, G. Kuhn, W.L. Wendland eds.), Springer, Berlin, pp. 411-420, (1987).
20. L. Desiderio, S. Falletta and L. Scuderi, *A virtual element method coupled with a boundary integral non reflecting condition for 2D exterior Helmholtz problems*. Comput. Math. Appl. 84 (2021), 296–313.
21. D.A. Di Pietro and A. Ern, Mathematical Aspects of Discontinuous Galerkin Methods. Mathématiques & Applications (Berlin) [Mathematics & Applications], 69. Springer, Heidelberg, 2012.
22. T. Dupont and R. Scott, *Polynomial approximation of functions in Sobolev spaces*, Math. Comp. 34 (1980), no. 150, 441–463.
23. G.N. Gatica and S. Meddahi, *On the coupling of* VEM *and* BEM *in two and three dimensions*. SIAM J. Numer. Anal. 57 (2019), no. 6, 2493–2518.
24. G.N. Gatica and S. Meddahi, *Coupling of virtual element and boundary element methods for the solution of acoustic scattering problems*. J. Numer. Math. 28 (2020), no. 4, 223–245.
25. G.N. Gatica, M. Munar and F.A. Sequeira, *A mixed virtual element method for the Navier-Stokes equations*. Math. Models Methods Appl. Sci. 28 (2018), no. 14, 2719–2762.
26. H. Han, *A new class of variational formulations for the coupling of finite and boundary element methods*. J. Comput. Math. 8 (1990), no. 3, 223–232.
27. G.C. Hsiao and W.. Wendland, Boundary Integral Equations. Applied Mathematical Sciences, vol. 164, Springer-Verlag, Berlin Heidelberg, 2008.
28. C. Johnson and J.C. Nédélec, *On the coupling of boundary integral and finite element methods*. Math. Comp. 35 (1980), no. 152, 1063–1079.
29. W. McLean, Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, 2000.
30. S. Meddahi, A. Márquez and V. Selgas, *Computing acoustic waves in an inhomogeneous medium of the plane by a coupling of spectral and finite elements*. SIAM J. Numer. Anal. 41 (2003), no. 5, 1729–1750.
31. D. Mora, G. Rivera and R. Rodríguez, *A virtual element method for the Steklov eigenvalue problem*. Math. Models Methods Appl. Sci. 25 (2015), no. 8, 1421–1445.
32. S.A. Sauter and C. Schwab, Boundary Element Methods. Springer Series in Computational Mathematics, 39. Springer-Verlag, Berlin, 2011.
33. F.J. Sayas, *The validity of Johnson-Nédélec's BEM-FEM coupling on polygonal interfaces*. SIAM J. Numer. Anal. 47 (2009), no. 5, 3451–3463.
34. F.J. Sayas, *The validity of Johnson-Nédélec's BEM-FEM coupling on polygonal interfaces*. SIAM Rev. 55 (2013), no. 1, 131–146.
35. O. Steinbach, *A note on the stable one-equation coupling of finite and boundary elements*. SIAM J. Numer. Anal. 49 (2011), no. 4, 1521–1531.
36. G. Vacca, *An H^1-conforming virtual element for Darcy and Brinkman equations*. Math. Models Methods Appl. Sci. 28 (2018), no. 1, 159–194.

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