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GLOBAL EXISTENCE IN A FOOD CHAIN MODEL CONSISTING OF TWO COMPETITIVE PREYS, ONE PREDATOR AND CHEMOTAXIS

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Abstract. We consider a mathematical model for the spatio-temporal evolution of three biological species in a food chain model consisting of two competitive preys and one predator with intra-specific competition. Besides diffusing, the predator species moves toward higher concentrations of a chemical substance which is produced by the prey, which move away from a substance produced by the predators. The resulting reaction-diffusion system consists of three parabolic equations along with three elliptic equation describing the chemical. First the local existence of a nonnegative solutions is proved, then we provide uniform estimates in Lebesgue spaces which lead to boundedness and the global well-posedness for the system. Finally we report and discuss some numerical simulations.

1. Introduction

1.1. **Scope.** We consider a reaction-diffusion system describing three interacting species in a food chain model, where each species secretes a chemical substance. The governing model, which is based on the treatments in [23, 36], is a strongly coupled nonlinear system of six partial differential equations (PDEs) with chemotactic terms, namely three parabolic equations describing the evolution of the densities u_i coupled with three elliptic equations for the concentrations y_i, $i = 1, 2, 3$:

$$\begin{aligned} \partial_t u_1 - D_1 \Delta u_1 - \chi_1 \operatorname{div}(u_1 \nabla y_3) &= F_1(\boldsymbol{u}) \\ \partial_t u_2 - D_2 \Delta u_2 - \chi_2 \operatorname{div}(u_2 \nabla y_3) &= F_2(\boldsymbol{u}) \\ \partial_t u_3 - D_3 \Delta u_3 + \chi_3 \operatorname{div}(u_3 \nabla (y_1 + y_2)) &= F_3(\boldsymbol{u}), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T] \\ -\mathcal{D}_i \Delta y_i + \theta_i y_i &= \delta_i u_i, \quad i = 1, 2, 3, \end{aligned} \tag{1.1}$$

where $u_i(\boldsymbol{x}, t)$, $i = 1, 2, 3$ is the density of species i at position $\boldsymbol{x}$ at time t. At the lowest level of the food chain we find the prey ($i = 1, 2$). Species 3, the predator, preys upon species 1 and 2. Moreover, $y_i(\boldsymbol{x}, t)$ denotes the concentration of the chemical substance secreted by species i at position $\boldsymbol{x}$ at time t, and $\boldsymbol{y}(\boldsymbol{x}, t) := (y_1(\boldsymbol{x}, t), y_2(\boldsymbol{x}, t), y_3(\boldsymbol{x}, t))^{\mathrm{T}}$.

The chemotactic movement of the species is due to chemical substances secreted by the other species. Its orientation is determined by the sign of the chemotactic coefficients χ_i (see [10]). In this work, we consider that the prey (species 1 and 2) moves in the direction of *decreasing* concentration

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of the chemical secreted by species 3 (trying to avoid that species), while the predator (species 3) moves in the direction of *increasing* concentration of the chemical secreted by species 1 and 2. Notice that the equations for the chemical substances are elliptic, rather than parabolic. This is justified in cases where the diffusion of the chemical substances occurs in a much faster time scale than the movement of individuals. This property is reasonable in a variety of ecological settings. The constants D_1 D_2 and D_3 are the diffusion coefficients of the prey (species 1 and 2) and predator (species 3) respectively, the coefficients $\mathcal{D}_i$, θ_i and δ_i are positive constants for $i = 1, 2, 3$.

The functional responses F_i, $i = 1, 2, 3$ are chosen as Holling type II (cf., e.g., [21, p. 38]),

$$\begin{aligned} F_1(\boldsymbol{u}) &\coloneqq r_1 u_1 \left(1 - \frac{u_1}{k_1}\right) - \sigma_1 u_1 u_2 - \frac{M_1 u_1}{A_1 + u_1} u_3, \\ F_2(\boldsymbol{u}) &\coloneqq r_2 u_2 \left(1 - \frac{u_2}{k_2}\right) - \sigma_2 u_1 u_2 - \frac{M_2 u_2}{A_2 + u_2} u_3, \\ F_3(\boldsymbol{u}) &\coloneqq \gamma_1 \frac{M_1 u_1}{A_1 + u_1} u_3 + \gamma_2 \frac{M_2 u_2}{A_2 + u_2} u_3 - L u_3 - H u_3^2, \end{aligned} \tag{1.2}$$

where r_1 and r_2 are biotic potentials, k_1 and k_2 are environmental carrying capacities of the two prey species, σ_1 and σ_2 are coefficients of inter-specific competition between two prey species, M_1 and M_2 are predation coefficients, γ_1 and γ_2 are conversion factors, A_1 and A_2 are half-saturation constants, L is the natural death rate of the predator, and H is the intra-specific competition among predator. We assume Neumann boundary conditions

$$\frac{\partial u_i}{\partial \boldsymbol{\nu}} = \frac{\partial y_i}{\partial \boldsymbol{\nu}} = 0, \qquad i = 1, 2, 3, \tag{1.3}$$

and the initial condition

$$u_i(\boldsymbol{x}, 0) = u_{i,0}(\boldsymbol{x}), \quad i = 1, 2, 3. \tag{1.4}$$

It is the purpose of this work to prove the existence and uniqueness of global classical and weak solutions of the initial-boundary value problem (1.1)-(1.4). First, the local existence of a negative solution is proven by using the Banach fixed point theorem and the properties of the heat semigroup. In addition, we show that the solution of the problem satisfies the L^α-integrability property. Then, using the local existence of the solution and the L^α-integrability, existence of a global solution is proven. In order to prove the existence of weak solutions, first we shown a stability result for the classical solutions. Then, we consider, for $k \in \mathbb{N}$ a classical solution $(\boldsymbol{u}^k, \boldsymbol{y}^k)$, and we prove some k-independent estimates. Therefore we can invoke the Aubin-Lions Lemma to guarante the existence of the limit function, which is a weak solution of our problem. Uniqueness follows from a stability property. Finally, we report some numerical examples.

1.2. **Related work.** Lotka-Volterra models have played an important role in the analysis of the interspecies relations in biology and ecology. After the pioneering work of Lotka [22] and Volterra [31] numerous works have contributed to the development of this analysis (cf. [5, 13, 15, 35]). This has been accompanied by the study of a natural phenomenon that arises in biology, called chemotaxis, which can be described by the Keller-Segel model [19]. We refer to [6, 16, 17] for an extensive analysis and results for some general formulations of the classical chemotaxis models also known as Keller-Segel models.

In [18] the authors studied a Keller–Segel-type chemotaxis model, where the chemotactic sensitivity equals some nonlinear function of the particle density. They determine the critical blow-up exponent and assuming some growth conditions for the chemotactic sensitivity function, establish an a priori estimate for the solution of the problem considered and conclude the global existence and boundedness of the solution. Furthermore, they prove the existence of solutions that become unbounded in finite or infinite time. The analysis of the parabolic-elliptic system arising in chemotaxis involving a logistic term was studied in [30]. The existence of global bounded classical solutions is proved under the assumption that either the space dimension does not exceed two, or that the logistic damping effect is strong enough. Also, the corresponding stationary problem is studied and some regularity properties are given. In [33] the chemotaxis system with logistic source was studied. The author introduced the concept of a very weak solution and global existence is proved.

In [26] the competitive exclusion phenomena was investigated. Parameter regimes are identified for which one of the species dies out asymptotically, whereas the other reaches its carrying capacity in the large time limit. The global existence and large-time behavior is addressed in [7] for weakly competitive species case and for the partially strong competition setting. In [27] the multiscale invasion of tumor cells through the surrounding tissue matrix model was studied. The resulting system, featuring three partial and three ordinary differential equations including a temporal delay, involves chemotactic and haptotactic cross-diffusion. The authors prove global existence, along with some basic boundedness properties, of weak solutions.

In [3] a system of PDEs describing the dynamics of ant foraging was analyzed. The system is made of convection–diffusion–reaction equations, and the coupling is driven by chemotaxis mechanisms. The authors established well-posedness for the model, and investigated the regularity issue for a large class of integrable data. In [4] the authors proposed and analyzed a reaction-diffusion model for predator-prey interaction, featuring both prey and predator taxis mediated by nonlocal sensing. They proved uniform estimates in Lebesgue spaces which lead to boundedness and the global well-posedness for the system, also numerical experiments are presented and discussed. See also [28] for a related model. In [9] a reaction–diffusion system is formulated to describe three interacting species within the Hastings–Powell (HP) food chain structure with chemotaxis produced by three chemicals. A convergent finite volume scheme was constructed and the existence of a discrete solution of the FV scheme is proved together with the convergence to the corresponding weak solution of the model.

1.3. **Outline of the paper.** The remainder of the paper is organized as follows. In Section 2 we present some preliminary material, including relevant notation and assumptions on the data, and collect some tools that will be used frequently. Next, Section 3 is devoted to the construction of the global classical solution. First, in Section 3.1, we prove the existence of the maximal time $T_{\max}$ for which there exists a unique classical solution, then in Section 3.2 we guarantee that any solution of the system (1.1) satisfies the L^α integrability property. This section ends with the proof of the existence and uniqueness of the global classical solution. Section 4 is concerned with the construction of a weak solution of our system. We start with the definition of a weak solution of (1.1), and present a stability result for classical solutions. Then we define a sequence of global classical solution of (1.1) and some a priori estimates, allowing us to prove the existence and uniqueness of the weak solution to (1.1). In Section 5, we adapt the finite volume method used in [9] to the present context. Finally, in Section 6 we provide numerical examples to illustrate the behaviour of the solutions.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with piecewise smooth boundary $\partial\Omega$. We employ standard Lebesgue and Sobolev spaces $W^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$ and $L^p(\Omega)$ (with their usual norms [1]) for all $m \in \mathbb{N}$ and $p \in [1, \infty]$. If X is a Banach space, $a < b$ and $p \in [1, \infty]$, then $L^p(a, b; X)$ denotes the space of all measurable functions $u : (a,b) \longrightarrow X$ such that $\|u(\cdot)\|_X \in L^p(a,b)$. Next, for $T>0$ we define $\Omega_T := \Omega \times (0,T]$. Furthermore, we define

$$\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} := \begin{pmatrix} -y_3 \\ -y_3 \\ y_1 + y_2 \end{pmatrix} = \boldsymbol{B}\boldsymbol{y}, \quad \text{where} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1^{\mathrm{T}} \\ \boldsymbol{b}_2^{\mathrm{T}} \\ \boldsymbol{b}_3^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then the system (1.1) can then be written as

$$\partial_t u_i - D_i \Delta u_i + \chi_i \operatorname{div}\big(u_i \nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{y})\big) = F_i(\boldsymbol{u}), \tag{2.1}$$

$$-\mathcal{D}_i \Delta y_i + \theta_i y_i = \delta_i u_i, \quad i = 1,2,3, \quad (\boldsymbol{x},t) \in \Omega_T. \tag{2.2}$$

Our basic requirements are as follows. The functional responses F_i are locally Lipschitz continuous, with L_i denoting a Lipschitz constant for F_i for all $i = 1,2,3$; $D_i > 0$, $\mathcal{D}_i > 0$, $\theta_i \geq 0$, $|\chi_i| \leq a$, and $\delta_i \geq 0$ for $i = 1,2,3$, where a is a constant. Finally, we assume that the constants arising in (1.2) satisfy $\gamma_1 M_1 + \gamma_2 M_2 - L > 0$.

Next, we collect some tools that will frequently be used in this work.

Lemma 2.1. *Assume that u_1, u_2 and u_3 are nonnegative functions. Then there exists a constant $C > 0$ such that*

$$\sum_{i=1}^{3} \|F_i\|_{L^\infty(\Omega)} \leq C. \tag{2.3}$$

Proof. Due to the nonnegativity of the functions u_i and (1.2) we get

$$\big|F_i(\boldsymbol{u})\big| \leq r_i u_i \left(1 - \frac{u_i}{k_i}\right) \leq \frac{1}{4} r_i k_i, \quad i = 1,2,$$

$$\big|F_3(\boldsymbol{u})\big| \leq (\gamma_1 M_1 + \gamma_2 M_2 - L) u_3 - H u_3^2 \leq \frac{(\gamma_1 M_1 + \gamma_2 M_2 - L)^2}{4H}.$$

Taking the supremum in each of the previous inequalities and summing the results yields (2.3). $\square$

We shall need the following consequence of the Gagliardo-Nirenberg interpolation inequality in two dimensions (see e.g. [24, Eq. 6.34]). Namely, there exists a constant $C = C(\Omega, \alpha)$ such that

$$\int_\Omega \xi^{\alpha+1} \, \mathrm{d}\boldsymbol{x} \leq C(\Omega, \alpha) \|\xi\|_{L^1(\Omega)} \int_\Omega \big|\nabla \xi^{\alpha/2}\big|^2 \, \mathrm{d}\boldsymbol{x} \tag{2.4}$$

and elliptic regularity in the L^p sense (cf. [12]): the linear equation

$$-\Delta v + v = u \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega,$$

admits a unique solution v satisfying

$$\|v\|_{W^{2,p}(\Omega)} \leq C \|u\|_{L^p(\Omega)}. \tag{2.5}$$

For the proof of the L^α integrability property, we shall require the two following lemmas (see [3, Appendix A]).

Lemma 2.2 (ODE comparison)**.** *Assume Y and X are non-negative absolutely continuous functions on $[0,T]$ and such that for every $t > 0$:*

$$Y'(t) + aY^\alpha(t) \geq b + \delta + c(1 + t^{-\gamma}) \sup_{\tau(t) \leq s \leq t} Y^{\alpha_0}(s),$$

$$X'(t) + aX^\alpha(t) \leq b + c(1 + t^{-\gamma}) \sup_{\tau(t) \leq s \leq t} X^{\alpha_0}(s)$$

for some continuous mapping $t \mapsto \tau(t) \in [0,t]$ *and constants* $b \geq 0$, $c \geq 0$, $a > 0$, $\delta > 0$, $\alpha > \alpha_0 \geq 0$, *and* $\gamma \geq 0$. *If* $Y(0) > X(0)$, *then* $Y \geq X$ *in* $[0,T]$. *In particular, if* $\gamma = 0$,

$$\sup_{t\in[0,T]} X(t) \leq \max\big\{X(0), C\big\},$$

where the constant $C > 0$ *depends on all parameters but* $\tau(\cdot)$, δ *and* T.

Lemma 2.3. *Assume* X *is an absolutely continuous function on* $[0,T]$ *and such that*

$$X'(t) + aX^{\alpha}(t) \leq b + c(1 + t^{-\gamma}) \sup_{t/2\leq s\leq t} X^{\alpha_0}(s)$$

with $b \geq 0$, $c \geq 0$, $a > 0$, $\alpha > \alpha_0 \geq 0$, *and* $\gamma \geq 0$. *Then*

$$X(t) \leq C(1 + t^{-\beta}), \quad \beta = \max\left\{\frac{1}{\alpha - 1}, \frac{\gamma}{\alpha - \alpha_0}\right\},$$

where the constant $C > 0$ *depends on all parameters but it is independent of* T.

For $p \in (1,\infty)$, let $A := A_p$ denote the sectorial operator defined by

$$A_p u := -\Delta u, \quad \text{for} \quad u \in D(A_p) := \left\{\psi \in W^{2,p}(\Omega) : \frac{\partial \psi}{\partial \boldsymbol{\nu}} = 0\right\}. \tag{2.6}$$

Then we define the operators $\exp(-tA)$ by

$$\big(\exp(-tA)f\big)(\boldsymbol{x}) = \int_\Omega G(\boldsymbol{x}, \boldsymbol{y}, t) f(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y},$$

where G represents the Green function and the family $(\exp(-tA))_{t\geq 0}$ denotes the Neumann heat semigroup. We use the following property of the Neumann heat semigroup to prove the existence global classical solution:

$$\big\|\exp(-tA)w\big\|_{L^p(\Omega)} \leq Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|w\|_{L^q(\Omega)}. \tag{2.7}$$

We refer to [32, Lemma 1.3] for other properties of Neumann heat semigroup.

Furthermore, the fact that the spectrum of A is a p-independent countable set of positive real numbers, namely $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, entails the following consequence. The operator $A + 1$ possesses fractional powers $(A + 1)^{\beta}$, $\beta \geq 0$, whose domains have the embedding properties (see [14, Theorem 1.6.1])

$$D((A_p + 1)^{\beta}) \hookrightarrow C^{\delta}(\overline{\Omega}) \qquad \text{if} \quad 2\beta - \frac{n}{p} > \delta \geq 0. \tag{2.8}$$

Moreover, it can easily be seen ([18, Lemma 2.1]) that for $t > 0$ the operator $(A+1)^{\beta}\exp(-tA)\operatorname{div}(\cdot)$ possesses a unique extension from $C_0^{\infty}(\Omega)$ to $L^p(\Omega)$ that satisfies the following lemma.

Lemma 2.4. *Let* $\beta \geq 0$ *and* $p \in (1,\infty)$. *Then for all* $\epsilon > 0$ *there exists* $c(\epsilon) > 0$ *such that for all* $w \in C_0^{\infty}$ *there exists* $\mu > 0$ *such that*

$$\begin{aligned}\big\|(A + 1)^{\beta}\exp(-tA)\operatorname{div}(w)\big\|_{L^p(\Omega)} &\leq c(\epsilon)t^{-\beta-\epsilon-1/2}\exp(-\mu t)\|w\|_{L^p(\Omega)} \\ &\leq c(\epsilon)t^{-\beta-\epsilon-1/2}\|w\|_{L^p(\Omega)} \quad \text{for all } t > 0.\end{aligned}$$

3. Global Classical Solutions

The goal of this section is to guarantee the global existence of solution for system (1.1). In order to achieve this, first we show local existence of a nonnegative solution, next we prove some a priori estimates and finally we establish global existence. The local existence proof is valid for $n \geq 2$.

3.1. **Local existence.** Here we prove local existence of a nonnegative solution. The proof is based on the Banach fixed-point theorem.

Lemma 3.1. *Suppose that $u_{i,0} \in C^0(\overline{\Omega})$ for all $i = 1, 2, 3$ are nonnegative. Then there exists $T_{\max} \in (0, \infty]$ and a unique classical solution $(\boldsymbol{u}, \boldsymbol{y})$ of (2.1) which is nonnegative. Each u_i, y_i belongs to $C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ and we have the following extensibility criterion:*

$$T_{\max} = \infty \quad \text{or} \quad \lim_{t \nearrow T_{\max}} \sum_{i=1}^{3} \|u_i\|_{L^\infty(\Omega)} = \infty. \tag{3.1}$$

Proof. We claim that for all $R > 0$ there exists $T = T(R) > 0$ such that if in addition to the above assumptions, $\|u_{i,0}\|_{L^\infty(\Omega)} \leq R$ for all $i = 1, 2, 3$, then the statement of the theorem holds. Let $L_i(R) > 0$ denote a Lipschitz constant for F_i on $(-R, R)$. For a small $T \in (0, 1)$, we introduce the Banach space

$$X := \left[C^0\big([0, T]; C^0(\bar{\Omega})\big)\right]^3$$

along with its closed subset

$$S := \big\{(u_1, u_2, u_3)^{\mathrm{T}} \in X \ : \ \|u_i\|_{L^\infty((0,T); L^\infty(\Omega))} \leq 2R, \, i = 1, 2, 3\big\},$$

where $R = \max_{i=1,2,3} \|u_{i,0}\|_\infty$. For $\boldsymbol{u} := (u_1, u_2, u_3)^{\mathrm{T}} \in S$ and $t \in [0, T]$, we introduce a mapping Φ on S by

$$\Phi(\boldsymbol{u}) := \big(\Phi_1(u_1), \Phi_2(u_2), \Phi_3(u_3)\big)^{\mathrm{T}},$$

where, for all $i = 1, 2, 3$, $\Phi_i(u_i)$ is defined by

$$\begin{aligned} \Phi_i(u_i) := \exp(-D_i t A) u_{i,0} - \chi_i \int_0^t \exp\big(-D_i(t-s)A\big) \operatorname{div}\big(u_i \nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{y})\big) \, \mathrm{d}s \\ + \int_0^t \exp\big(-D_i(t-s)A\big) F_i\big(\boldsymbol{u}(s)\big) \, \mathrm{d}s. \end{aligned}$$

Let $y_i \in \cap_{1<p<\infty} L^\infty((0, T); W^{2,p}(\Omega))$ denote the (weak) solution of

$$-\mathcal{D}_i \Delta y_i + \theta_i y_i = \delta_i u_i \quad \text{on } \Omega, \quad \frac{\partial y_i}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

Then, for all $i = 1, 2, 3$ we have

$$\|\Phi_i(u_i)\|_{L^\infty(\Omega)} \leq I_1 + I_2 + I_3, \tag{3.3}$$

where we define

$$I_1 := \big\|\exp(-D_i t A) u_{i,0}\big\|_{L^\infty(\Omega)}, \quad I_2 := \chi_i \int_0^t \big\|\exp\big(-D_i(t-s)A\big) \operatorname{div}\big(u_i(s) \nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{y}(s))\big)\big\|_{L^\infty(\Omega)} \, \mathrm{d}s,$$

$$I_3 := \int_0^t \big\|\exp\big(-D_i(t-s)A\big) F_i(\boldsymbol{u}(s))\big\|_{L^\infty(\Omega)} \, \mathrm{d}s.$$

It is clear that for all $t \in (0, T)$,

$$I_1 \leq \|u_{i,0}\|_{L^\infty(\Omega)} \leq R. \tag{3.4}$$

Using (2.3), we get

$$I_3 \leq \int_0^t \big\|F_i(\boldsymbol{u}(s))\big\|_{L^\infty(\Omega)} \, \mathrm{d}s \leq \|F_i\|_{L^\infty((-R,R))} \cdot T \quad \text{for all } t \in (0, T). \tag{3.5}$$

Now, in order to control the second member of (3.3), we fix $p \in (1,\infty)$ with $p > n$. Let $\beta \in (\frac{n}{2p}, \frac{1}{2})$ and $\varepsilon \in (0, \frac{1}{2} - \beta)$. Then, by Lemma 2.4

$$\begin{aligned} I_2 &\le C \int_0^t \big\|(A+1)^\beta \exp\big(-D_i(t-s)A\big) \operatorname{div}\big(u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s))\big)\big\|_{L^p(\Omega)} \,\mathrm{d}s \\ &\le C \int_0^t (t-s)^{-(\beta+\varepsilon+1/2)} \big\|u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s))\big\|_{L^p(\Omega)} \,\mathrm{d}s \\ &\le C \int_0^t (t-s)^{-(\beta+\varepsilon+1/2)} \big\|u_i(s)\big\|_{L^\infty(\Omega)} \big\|\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s)\big\|_{W^{1,p}(\Omega)} \,\mathrm{d}s \\ &\le C \int_0^t (t-s)^{-(\beta+\varepsilon+1/2)} \big\|u_i(s)\big\|_{L^\infty(\Omega)} \big\|\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s)\big\|_{W^{2,p}(\Omega)} \,\mathrm{d}s \\ &\le C(R) \int_0^t (t-s)^{-(\beta+\varepsilon+1/2)} \,\mathrm{d}s \le C(R) T^{-(\beta+\varepsilon)+1/2} \quad \text{for all } t \in (0,T), \end{aligned} \tag{3.6}$$

were have used that $T < 1$, elliptic regularity (cf. 2.5) for (3.2), and that

$$\big\|\exp(\tau A) \operatorname{div} w\big\|_{L^p\Omega} \le c(\varepsilon) t^{-(\beta+\varepsilon+1/2)} \|w\|_{L^p} \quad \text{for all } w \in L^p$$

(see Lemma 2.4). From (3.4), (3.5), (3.6) and $1/2 - \beta - \epsilon > 0$, it follows that if we choose T sufficiently small, then Φ maps S into itself.

Now, let $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in S$, then for all $i = 1,2,3$ we estimate

$$\big\|\Phi_i(u_i)(t) - \Phi_i(\tilde{u}_i)(t)\big\|_{L^\infty(\Omega)} \le J_1 + J_2,$$

where we define

$$\begin{aligned} J_1 &:= \chi_i \int_0^t \big\|\exp\big(-D_i(t-s)A\big) \operatorname{div}\big(u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s)) - \tilde{u}_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}}(s))\big)\big\|_{L^\infty(\Omega)} \,\mathrm{d}s, \\ J_2 &:= \int_0^t \big\|\exp\big(-D_i(t-s)A\big)\big(F_i(\boldsymbol{u}) - F_i(\tilde{\boldsymbol{u}})\big)\big\|_{L^\infty(\Omega)} \,\mathrm{d}s. \end{aligned}$$

Since the functional responses F_i are locally Lipschitz continuous, we get

$$J_2 \le \int_0^t \big\|F_i(\boldsymbol{u}) - F_i(\tilde{\boldsymbol{u}})\big\|_{L^\infty(\Omega)} \,\mathrm{d}s \le L_i(R) \int_0^t \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_X \,\mathrm{d}s \le T \cdot L_i(R) \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_X.$$

Using the properties of the operator $A+1$ we find that

$$\begin{aligned} J_1 &\le C \int_0^t \big\|(A+1)^\beta \exp\big(-D_i(t-s)A\big) \operatorname{div}\big(u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s)) - \tilde{u}_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}}(s))\big)\big\|_{L^p(\Omega)} \,\mathrm{d}s \\ &\le C \int_0^t (t-s)^{-(\beta+\varepsilon+1/2)} \big\|\big(u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s)) - \tilde{u}_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}}(s))\big)\big\|_{L^p(\Omega)} \,\mathrm{d}s \\ &\le C \int_0^t (t-s)^{-(\beta+\varepsilon+1/2)} \Big(\big\|\big(u_i(s)\big(\nabla(\boldsymbol{b}_i^{\mathrm{T}}(\boldsymbol{y}(s) - \tilde{\boldsymbol{y}}(s)))\big)bigr\|_{L^p(\Omega)} \\ &\qquad + \big\|\nabla\big(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}}(s)\big)\big(u_i(s) - \tilde{u}_i(s)\big)\big\|_{L^p(\Omega)}\Big) \,\mathrm{d}s. \end{aligned}$$

Now, using elliptic regularity (cf. 2.5) and keeping in mind that Equation (3.2) is linear we get

$$\big\|\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}}(s)\big\|_{W^{1,p}(\Omega)} \le C\big\|\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}}(s)\big\|_{W^{2,p}(\Omega)} \le C\big\|\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{u}}(s)\big\|_{L^p(\Omega)} \le C\big\|\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{u}}(s)\big\|_{L^\infty(\Omega)}$$

and

$$\big\|\boldsymbol{b}_i^{\mathrm{T}}(\boldsymbol{y}(s) - \tilde{\boldsymbol{y}}(s))\big\|_{W^{1,p}(\Omega)} \le C\big\|\boldsymbol{b}_i^{\mathrm{T}}(\boldsymbol{y}(s) - \tilde{\boldsymbol{y}}(s))\big\|_{W^{2,p}(\Omega)} \le C\big\|\boldsymbol{b}_i^{\mathrm{T}}(\boldsymbol{u}(s) - \tilde{\boldsymbol{u}}(s))\big\|_{L^p(\Omega)}$$

$$\leq C\big\|\boldsymbol{b}_i^{\mathrm{T}}(\boldsymbol{u}(s)-\tilde{\boldsymbol{u}}(s))\big\|_{L^\infty(\Omega)}.$$

Thus

$$\begin{aligned} J_1 \leq C\int_0^t (t-s)^{-(\beta+\varepsilon+1/2)}\Big(&\|u_i(s)\|_{L^\infty(\Omega)}\big\|\boldsymbol{b}_i^{\mathrm{T}}(\boldsymbol{u}(s)-\tilde{\boldsymbol{u}}(s))\big\|_{L^\infty(\Omega)} \\ &+\big\|u_i(s)-\tilde{u}_i(s)\big\|_{L^\infty(\Omega)}\big\|\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{u}}(s)\big\|_{L^\infty(\Omega)}\Big)\,\mathrm{d}s \\ \leq C(R)T^{-(\beta+\varepsilon+1/2)}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_X. \end{aligned}$$

Therefore collecting the previous inequalities we get

$$\begin{gathered}\big\|\Phi_i(u_i)(t)-\Phi_i(\tilde{u}_i)(t)\big\|_{L^\infty(\Omega)} \leq C(R)T^{-(\beta+\varepsilon+1/2)}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_X + T\cdot L_i(R)\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_X \\ \text{for all } t\in(0,T),\end{gathered}$$

which shows that if T is chosen sufficiently small, then Φ acts as a contraction on S. Accordingly, the Banach fixed point theorem asserts the existence of some $\boldsymbol{u}\in S$ such that $\Phi(\boldsymbol{u})=\boldsymbol{u}$, along with the existence of y_1, y_2 and y_3 as obtained from (3.2).

Since the above choice of T depends only on $\|u_{i,0}\|_{L^\infty}$, the existence of maximal time $T_{\max}$, that satisfies (3.1) can be ensured by [25, Proposition 16.1]. Relying on this, the inclusions $u_1,u_2,u_3\in C^{2,1}(\overline{\Omega}\times(0,T_{\max})$ result from straightforward regularity arguments including standard semigroup techniques and parabolic Schauder estimates ([20, Theorem IV.5.3]). Again by standard regularity arguments, we are able to establish the regularity of y_1,y_2 and y_3.

An application of the strong maximum principle applied to (2.1) implies the claim concerning the positivity of u_1,u_2 and u_3. Hence u_1,u_2 and u_3 are positive in $\overline{\Omega}\times(0,T_{\max})$ and the strong elliptic maximum principle applied to (2.2) yields positivity also of y_1,y_2 and y_3.

Let us finally prove uniqueness of solutions in the indicated class, without loss of generality, we assume that $D_i=1$ for all $i=1,2,3$. Assume that $T>0$ and two classical solutions of the system (1.1) $(\boldsymbol{u},\boldsymbol{y})$ and $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{y}})$ in $\Omega\times(0,T)$ are given. We fix $T_0\in(0,T)$, and define $w_i=u_i-\tilde{u}_i$ and $z=y_i-\tilde{y}_i$. The system for these differences is given by

$$\begin{aligned} \partial_t w_i - D_i\Delta w_i + \chi_i \operatorname{div}\big(w_i\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}})\big) + \chi_i\operatorname{div}\big(u_i\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{z})\big) &= F_i(\boldsymbol{u})-F_i(\tilde{\boldsymbol{u}}), \\ -\mathcal{D}_i\Delta z_i + \theta_i z_i &= \delta_i w_i, \quad i=1,2,3, \quad (\boldsymbol{x},t)\in\Omega_T. \end{aligned} \tag{3.7}$$

Multiplying (3.7) by w_i and integrating the result in space, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w_i|^2\,\mathrm{d}\boldsymbol{x} + D_i\int_\Omega |\nabla w_i|^2\,\mathrm{d}\boldsymbol{x} \\ &= \chi_i\int_\Omega \big(u_i\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}) - \tilde{u}_i\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}})\big)\nabla w_i\,\mathrm{d}\boldsymbol{x} + \int_\Omega \big(F_i(\boldsymbol{u})-F_i(\tilde{\boldsymbol{u}})\big)w_i\,\mathrm{d}\boldsymbol{x} \\ &= \chi_i\int_\Omega u_i\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{z})\nabla w_i\,\mathrm{d}\boldsymbol{x} + \chi_i\int_\Omega w_i\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}})\nabla w_i\,\mathrm{d}\boldsymbol{x} + \int_\Omega\big(F_i(\boldsymbol{u})-F_i(\tilde{\boldsymbol{u}})\big)w_i\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{3.8}$$

for all $t \in (0,T)$. By the Hölder, Gagliardo-Nirenberg and Young inequalities,

$$\begin{aligned}\chi_i \int_\Omega w_i \nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}})\nabla w_i \,\mathrm{d}\boldsymbol{x} &\leq |\chi_i| \|\nabla w_i\|_{L^2(\Omega)} \|\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}})\|_{L^p(\Omega)} \|w_i\|_{L^{2p/(p-2)}(\Omega)} \\ &\leq C\|\nabla w_i\|_{L^2(\Omega)} \|\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}})\|_{L^p(\Omega)} \|\nabla w_i\|_{L^2(\Omega)}^{n/p} \|w_i\|_{L^2(\Omega)}^{(p-n)/p} \\ &\leq C\|\nabla w_i\|_{L^2(\Omega)} \|\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}})\|_{L^p(\Omega)} \|\nabla w_i\|_{L^2(\Omega)}^{n/p} \|w_i\|_{L^2(\Omega)}^{(p-n)/p} \\ &\leq C\|\nabla w_i\|_{L^2(\Omega)}^{(p+n)/p} \|\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}})\|_{L^p(\Omega)} \|w_i\|_{L^2(\Omega)}^{(p-n)/p} \\ &\leq \frac{1}{2}\|\nabla w_i\|_{L^2(\Omega)}^2 + C\|w_i\|_{L^2(\Omega)}^2,\end{aligned} \tag{3.9}$$

where we have used that $\|\nabla(\boldsymbol{b}_i^{\mathrm{T}}\tilde{\boldsymbol{y}})\|_{L^p(\Omega)} \leq C$ for $t \in (0,T_0)$, and that $p > n \geq 2$. Furthermore, in view of the boundedness of u_i and $\tilde{u}_i$ in $\Omega \times (0,T_0)$ and the Lipschitz continuity of F_i, we obtain

$$\chi_i \int_\Omega u_i \nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{z})\nabla w_i \,\mathrm{d}\boldsymbol{x} \leq \frac{1}{2}\|\nabla w_i\|_{L^2(\Omega)}^2 + C\|\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{z})\|_{L^2(\Omega)}^2$$

and

$$\int_\Omega \big(F_i(\boldsymbol{u}) - F_i(\tilde{\boldsymbol{u}})\big) w_i \,\mathrm{d}\boldsymbol{x} \leq C\|w_i\|_{L^2(\Omega)}^2.$$

We conclude upon (3.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w_i|^2 \,\mathrm{d}\boldsymbol{x} \leq C\|w_i\|_{L^2(\Omega)}^2 \qquad \text{for all } t \in (0,T_0).$$

The Gronwall inequality clearly implies uniqueness in $\Omega \times (0,T_0)$ and hence the uniqueness in $\Omega \times (0,T)$ because $T_0 \in (0,T)$ was arbitrary. □

3.2. **Global classical solutions.** In this subsection, we prove existence and uniqueness of a global solution to the system (1.1). That is, we prove $T_{\max} = \infty$ which implies that $u_1, u_2, u_3, y_1, y_2, y_3$ belong to $C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))$. First, we prove L^1-integrability.

Lemma 3.2. *Let $(\boldsymbol{u}, \boldsymbol{y})$ be sufficiently smooth non-negative solutions of the system* (1.1) *with the boundary conditions* (1.3)*. Then there exists a constant $\mathcal{M}$ depending on γ_1, γ_2, $|\Omega|$, and $\|u_{i,0}\|_{L^1(\Omega)}$ but not on t, such that for all $t > 0$,*

$$\int_\Omega (u_1 + u_2 + u_3) \,\mathrm{d}\boldsymbol{x} \leq \mathcal{M}. \tag{3.10}$$

Proof. Integrating the first, second and third equations of (1.1) and using the Neumann boundary conditions we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\gamma_1 u_1 + \gamma_2 u_2 + u_3) \,\mathrm{d}\boldsymbol{x} \leq r_1\gamma_1 \int_\Omega u_1 \left(1 - \frac{u_1}{k}\right) \,\mathrm{d}\boldsymbol{x} + r_2\gamma_2 \int_\Omega u_1 \left(1 - \frac{u_1}{k}\right) \,\mathrm{d}\boldsymbol{x} - L\int_\Omega u_3 \,\mathrm{d}\boldsymbol{x}$$

for all $t \in (0, T_{\max})$. From the inequality

$$r_i\left(u_i - \frac{u_i}{k_i}\right) \leq \frac{k_i(r_i+1)^2}{4r_i} - u_i$$

we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\gamma_1 u_1 + \gamma_2 u_2 + u_3) \,\mathrm{d}\boldsymbol{x} \leq{}& \gamma_1 \frac{k_1(r_1+1)^2}{4r_1}\int_\Omega \mathrm{d}\boldsymbol{x} - \gamma_1 \int_\Omega u_1 \,\mathrm{d}\boldsymbol{x} + \gamma_2 \frac{k_2(r_2+1)^2}{4r_2}\int_\Omega \mathrm{d}\boldsymbol{x} \\ &- \gamma_2 \int_\Omega u_2 \,\mathrm{d}\boldsymbol{x} - L\int_\Omega u_3 \,\mathrm{d}\boldsymbol{x}\end{aligned}$$

$$\leq C|\Omega| - \int_\Omega (\gamma_1 u_1 + \gamma_2 u_2 + u_3)\,\mathrm{d}\boldsymbol{x}.$$

Taking $A(t) = \gamma_1 u_1 + \gamma_2 u_2 + u_3$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) + A(t) \leq C|\Omega|,$$

which implies

$$A(t) \leq \exp(-t)A(0) + \bigl(1 - \exp(-t)\bigr)C|\Omega|.$$

The conclusion of the lemma readily follows. □

Our main ingredient for the proof of global existence is the following a priori estimate which asserts that if $u_{i,0} \in L^\alpha(\Omega)$, then u_i is uniformly bounded in L^α for some $\alpha > 1$. We adapt the proof shown in [4, Proposition 3.2] to our context.

Lemma 3.3. *Let $(\boldsymbol{u}, \boldsymbol{y})$ be sufficiently smooth non-negative solutions of the system (1.1) with the boundary conditions (1.3) and integrable initial data, and let $t > 0$ be arbitrary. Then, for any $\alpha \in (1, \infty)$, we have the estimate.*

$$\sum_{i=1}^{3} \|u_i\|_\alpha \leq C(\alpha, \mathcal{M})(1 + t^{1-\alpha}). \tag{3.11}$$

Moreover, if $u_{i,0} \in L^\alpha(\Omega)$ for $i = 1, 2, 3$, then actually

$$\sum_{i=1}^{3} \|u_i\|_\alpha \leq C\bigl(\alpha, \mathcal{M}, \|u_{1,0}\|_\alpha, \|u_{2,0}\|_\alpha, \|u_{3,0}\|_\alpha\bigr). \tag{3.12}$$

Proof. For simplicity we put $\|\cdot\|_{L^\alpha(\Omega)} = \|\cdot\|_\alpha$. Multiplying the first equation in (1.1) by $u_1^{\alpha-1}$ and integrating by parts we obtain

$$\frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1\|_\alpha^\alpha + D_1(\alpha - 1)\int_\Omega u_1^{\alpha-2}|\nabla u_1|^2\,\mathrm{d}\boldsymbol{x} + \frac{\chi_1(\alpha-1)}{\alpha}\int_\Omega \nabla y_3 \nabla u_1^\alpha\,\mathrm{d}\boldsymbol{x} = \int_\Omega F_1(\boldsymbol{u})u_1^{\alpha-1}\,\mathrm{d}\boldsymbol{x}.$$

Now, multiplying the equation $-\mathcal{D}_3\Delta y_3 + \theta_3 y_3 = \delta_3 u_3$ by u_1^α we obtain

$$-\int_\Omega \nabla y_3 \nabla u_1^\alpha\,\mathrm{d}\boldsymbol{x} \leq \frac{\theta_3}{\mathcal{D}_3}\int_\Omega y_3 u_1^\alpha\,\mathrm{d}\boldsymbol{x}.$$

Then, using the equality

$$\int_\Omega w^{\alpha-2}|\nabla w|^2\,\mathrm{d}\boldsymbol{x} = \frac{4}{\alpha^2}\int |\nabla w^{\alpha/2}|^2\,\mathrm{d}\boldsymbol{x}$$

and

$$\int_\Omega F_1(\boldsymbol{u})u_1^{\alpha-1}\,\mathrm{d}\boldsymbol{x} \leq r_1 \int_\Omega u_1^\alpha\,\mathrm{d}\boldsymbol{x},$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_1\|_\alpha^\alpha + 4\frac{D_1(\alpha-1)}{\alpha}\bigl\|\nabla u_1^{\alpha/2}\bigr\|_2^2 \leq \frac{(\alpha-1)\chi_1\delta_3}{\mathcal{D}_3}\int_\Omega y_3 u_1^\alpha\,\mathrm{d}\boldsymbol{x} + r_1\alpha\int_\Omega u_1^\alpha\,\mathrm{d}\boldsymbol{x}. \tag{3.13}$$

In order to estimate the right-hand side of the previous inequality, we take $\epsilon > 0$ to be specified later. We use the following consequence of the Young inequality:

$$y_3 u_1^\alpha \leq \epsilon u_1^{\alpha+1} + \epsilon^{-\alpha} y_3^{\alpha+1}, \tag{3.14}$$

and also the inequality

$$\int_\Omega u_1^\alpha \,\mathrm{d}\boldsymbol{x} \le \|u_1\|_1^{1/\alpha^2}\|u_1\|_{\alpha+1}^{(\alpha^2-1)/\alpha} \le C_1'(\mathcal{M},\epsilon,\alpha) + \epsilon\|u_1\|_{\alpha+1}^{\alpha+1}.$$

Therefore, for some constant $C_1' = C_1'(\mathcal{M},\epsilon,\alpha,\delta_3,\chi_1,\mathcal{D}_3)$ we have

$$\frac{(\alpha-1)\chi_1\delta_3}{\mathcal{D}_3}\int_\Omega y_3 u_1^\alpha \,\mathrm{d}\boldsymbol{x} + r_1\alpha\int_\Omega u_1^\alpha \,\mathrm{d}\boldsymbol{x} \le C_1' + C_1'\epsilon\|u_1\|_{\alpha+1}^{\alpha+1} + C_1'\|y_3\|_{\alpha+1}^{\alpha+1}.$$

The last inequality together with (3.13) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_1\|_\alpha^\alpha + 4\frac{D_1(\alpha-1)}{\alpha}\|\nabla u_1^{\alpha/2}\|_2^2 \le C_1' + C_1'\|u_1\|_{\alpha+1}^{\alpha+1} + C_1'\|y_3\|_{\alpha+1}^{\alpha+1}.$$

From the Gagliardo-Nirenberg-Sobolev (GNS) inequality (2.4) and for sufficiently small ϵ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_1\|_\alpha^\alpha + C_1'\|u_1\|_{\alpha+1}^{\alpha+1} \le C_1' + C_1'\|u_1\|_\alpha^\alpha + C_1'\|y_3\|_{\alpha+1}^{\alpha+1} \tag{3.15}$$

for some C_1' depending on α and $\mathcal{M}$.

Now we deal with the last term on the right-hand side of (3.15). First we multiply the six equations in (1.1) by $y_3^{\alpha-1}$ to get

$$\int_\Omega |\nabla y_3^{\alpha/2}|^2 \,\mathrm{d}\boldsymbol{x} \le \frac{\delta_3}{\mathcal{D}_3}\int_\Omega u_3 y_3^{\alpha-1} \,\mathrm{d}\boldsymbol{x} \le C_1''\int_\Omega u_3^\alpha \,\mathrm{d}\boldsymbol{x} + C_1''\int_\Omega y_3^\alpha \,\mathrm{d}\boldsymbol{x}. \tag{3.16}$$

Then from (2.4) and (3.16) we deduce that

$$\|y_3\|_{\alpha+1}^{\alpha+1} \le C_1''\|u_3\|_\alpha^\alpha + C_1''\|y_3\|_\alpha^\alpha \le C_1'' + C_1''\|u_3\|_\alpha^\alpha + C_1''\epsilon\|y_3\|_{\alpha+1}^{\alpha+1}.$$

Taking ϵ sufficiently small yields

$$\|y_3\|_{\alpha+1}^{\alpha+1} \le C_1'' + C_1''\|u_3\|_\alpha^\alpha. \tag{3.17}$$

In view of (3.17), the estimates (3.15) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_1\|_\alpha^\alpha + C_1\|u_1\|_{\alpha+1}^{\alpha+1} \le C_1 + C_1\|u_3\|_\alpha^\alpha,$$

for some C_1 depending on the $\alpha, \mathcal{M}$, the GNS constant and the parameters of the system.

Reasoning in the same way for the second equation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|_\alpha^\alpha + C_1\|u_2\|_{\alpha+1}^{\alpha+1} \le C_1 + C_1\|u_3\|_\alpha^\alpha. \tag{3.18}$$

Very similar computations yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_3\|_\beta^\beta + 4\frac{D_3(\beta-1)}{\beta}\|\nabla u_3^{\beta/2}\|_2^2 \le (\beta-1)\chi_3\left(\frac{\delta_1}{\mathcal{D}_1}\int_\Omega u_1 u_3^\beta \,\mathrm{d}x + \frac{\delta_2}{\mathcal{D}_2}\int_\Omega u_2 u_3^\beta \,\mathrm{d}\boldsymbol{x}\right)\\ + (\gamma_1 M_1 + \gamma_2 M_2 - L)\int_\Omega u_3^\beta \,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.19}$$

Therefore, using (3.14) we find that for $\epsilon_1 > 0$ (to be specified later) there exists a positive constant $C_2 = C_2(\mathcal{M},\epsilon_1,\beta,\delta_1,\delta_2\chi_3,\mathcal{D}_1,\mathcal{D}_2)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_3\|_\beta^\beta + 4\frac{D_3(\beta-1)}{\beta}\|\nabla u_3^{\beta/2}\|_2^2 \le C_2 + C_2\epsilon_1\|u_3\|_{\beta+1}^{\beta+1} + C_2\|u_1\|_{\beta+1}^{\beta+1} + C_2\|u_2\|_{\beta+1}^{\beta+1}.$$

Again, from the GNS inequality and choosing ϵ_1 sufficiently small we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_3\|_\beta^\beta + C_2\|u_3\|_{\beta+1}^{\beta+1} \le C_2 + C_2\|u_1\|_{\beta+1}^{\beta+1} + C_2\|u_2\|_{\beta+1}^{\beta+1}. \tag{3.20}$$

In light of (3.18), (3.19) and (3.20) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u_1\|_\alpha^\alpha + \|u_2\|_\alpha^\alpha + \|u_3\|_\beta^\beta\big) + C\big(\|u_1\|_{\alpha+1}^{\alpha+1} + \|u_2\|_{\alpha+1}^{\alpha+1} + \|u_3\|_{\beta+1}^{\beta+1}\big) \le C + CU_1 + CU_2 + CU_3$$

for some constant C depending on α, β and $\mathcal{M}$, and where we define

$$U_1 := \|u_1\|_{\beta+1}^{\beta+1}, \quad U_2 := \|u_2\|_{\beta+1}^{\beta+1}, \quad U_3 := \|u_3\|_\alpha^\alpha.$$

In order to conveniently bound the terms on the right-hand side using the left-hand side, we should take $\beta < \alpha < \beta + 1$. Now, due to the interpolation inequalities

$$\|u_i\|_{\beta+1} \le \|u_i\|_1^{1-\theta_1}\|u_i\|_{\alpha+1}^{\theta_1}, \quad \theta_1 = \frac{\beta(\alpha+1)}{\alpha(\beta+1)} \in (0,1), \quad i = 1,2,$$
$$\|u_3\|_\alpha \le \|u_3\|_1^{1-\theta_2}\|u_i\|_{\beta+1}^{\theta_2}, \quad \theta_2 = \frac{(\beta+1)(\alpha-1)}{\alpha\beta} \in (0,1),$$

and (3.10) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u_1\|_\alpha^\alpha + \|u_2\|_\alpha^\alpha + \|u_3\|_\beta^\beta\big) + C\big(\|u_1\|_{\alpha+1}^{\alpha+1} + \|u_2\|_{\alpha+1}^{\alpha+1} + \|u_3\|_{\beta+1}^{\beta+1}\big) \le C + C\big(S_1 + S_2 + S_3\big)$$

for some constant C depending on α, β and $\mathcal{M}$, where we define

$$S_1 := \|u_1\|_{\alpha+1}^{\theta_1(\beta+1)}, \quad S_2 := \|u_2\|_{\alpha+1}^{\theta_1(\beta+1)}, \quad S_3 := \|u_3\|_{\beta+1}^{\theta_2\alpha}.$$

We observe that $\theta_1(\beta+1) < \alpha+1$ and $\theta_2\alpha < \beta+1$, so using Young's inequality with a sufficiently small ϵ allows the terms on the right-hand side to be absorbed into the left-hand side. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u_1\|_\alpha^\alpha + \|u_2\|_\alpha^\alpha + \|u_3\|_\beta^\beta\big) + C\big(\|u_1\|_{\alpha+1}^{\alpha+1} + \|u_2\|_{\alpha+1}^{\alpha+1} + \|u_3\|_{\beta+1}^{\beta+1}\big) \le C$$

for some constant C depending on α, β and $\mathcal{M}$. Applying the interpolation inequality

$$\|w\|_\alpha \le \|w\|_1^{1-\xi}\|w\|_{\alpha+1}^{\xi}, \qquad \xi = \frac{\alpha^2-1}{\alpha^2} \in (0,1),$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u_1\|_\alpha^\alpha + \|u_2\|_\alpha^\alpha + \|u_3\|_\beta^\beta\big) + C\Big((\|u_1\|_\alpha^\alpha)^{\frac{\alpha}{\alpha-1}} + (\|u_2\|_\alpha^\alpha)^{\frac{\alpha}{\alpha-1}} + (\|u_3\|_\beta^\beta)^{\frac{\beta}{\beta-1}}\Big) \le C$$

and so, from

$$\big(\|u_3\|_\beta^\beta\big)^{\frac{\alpha}{\alpha-1}} \le \big(\|u_3\|_\beta^\beta\big)^{\frac{\beta}{\beta-1}}$$

and the convexity of $x \mapsto a^x$, we find, setting $Z(t) := \|u_1\|_\alpha^\alpha + \|u_2\|_\alpha^\alpha + \|u_3\|_\beta^\beta$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(t) + CZ(t)^{\frac{\alpha}{\alpha-1}} \le C.$$

In light of the ODE comparison (2.3) we obtain that $Z(t) \le C(1+t^{1-\alpha})$. Furthermore, by invoking (2.2) we obtain the estimate (3.12). □

The following lemma contains a general statement on extensibility and regularity of solutions known to be bounded in $L^\infty((0,T_{\max});L^p(\Omega))$ for some $p > 1$. It will be used to prove global existence and boundedness. We adapt the methods used in the proof of [6, Lemma 3.2] and [34, Lemma 2.6] to our context for its proof.

Lemma 3.4. *Let $p > 1$ such that $u_{i,0} \in L^p(\Omega)$. Then $T_{\max} = \infty$ and*

$$\sup \left(\sum_{i=1}^{3} \|u_i(\cdot, t)\|_{L^\infty(\Omega)} \right) < \infty.$$

Proof. For each $T \in (0, T_{\max})$ we have

$$M(T) := \sup_{t \in (0,T)} \left(\sum_{i=1}^{3} \|u_i\|_{L^\infty(\Omega)} \right) < \infty.$$

To estimate $M(T)$ adequately, we fix an arbitrary $t \in (0, T)$. Let $t_0 := \max\{t - 1, 0\}$. By the variation of constants formula, $\|u_i(\cdot, t)\|_{L^\infty(\Omega)} \le K_1 + K_2 + K_3$, where we define

$$K_1 := \big\|\exp\big(-D_i(t - t_0)A\big)u_i(\cdot, t_0)\big\|_{L^\infty(\Omega)}, \quad K_2 := \int_{t_0}^{t} \big\|\exp\big(-D_i(t - s)A\big)F_i(\boldsymbol{u}(s))\big\|_{L^\infty(\Omega)} \,\mathrm{d}s,$$

$$K_3 := \chi_i \int_{t_0}^{t} \big\|\exp\big(-D_i(t - s)A\big)\operatorname{div}\big(u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s))\big)\big\|_{L^\infty(\Omega)} \,\mathrm{d}s.$$

Here, if $t \le 1$ then $t_0 = 0$, and we may we use the comparison principle to see that

$$K_1 \le \|u_{i,0}\|_{L^\infty(\Omega)}, \tag{3.21}$$

whereas in the case $t > 1$, we invoke (2.7) and (3.10) to estimate

$$K_1 \le C(t - t_0)^{-1}\big\|u_i(\cdot, t_0)\big\|_{L^1(\Omega)} \le C_1\mathcal{M}.$$

Again using the parabolic maximum principle and (2.3), we get

$$K_2 \le \int_{t_0}^{t} \big\|F_i\big(\boldsymbol{u}(s)\big)\big\|_{L^\infty(\Omega)} \,\mathrm{d}s \le \|F_i\|_{L^\infty(\Omega)}.$$

Now, let $p > 2$, $\beta \in (\frac{1}{p}, \frac{1}{2})$, and $\varepsilon \in (0, \frac{1}{2} - \beta)$. Then

$$\begin{aligned} K_3 &\le C_2 \int_{t_0}^{t} \big\|(A + 1)^\beta \exp\big(-D_i(t - s)A\big)\operatorname{div}\big(u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s))\big)\big\|_{L^p(\Omega)} \,\mathrm{d}s \\ &\le C_2 \int_{t_0}^{t} (t - s)^{-(\beta+\varepsilon+1/2)}\big\|u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s))\big\|_{L^p(\Omega)} \,\mathrm{d}s \qquad \forall t \in (0, T_{\max}). \end{aligned}$$

By Lemma 3.3, $\|u_i(t)\|_{L^\alpha(\Omega)} \le C'$ holds for any $\alpha > 1$, $t > 0$ and $i = 1, 2, 3$ with $C' > 0$ depending on $\alpha, \|u_{1,0}\|_{L^\alpha(\Omega)}, \|u_{2,0}\|_{L^\alpha(\Omega)}, \|u_{3,0}\|_{L^\alpha(\Omega)}$ and $\mathcal{M}$ but not on t. Thus elliptic regularity theory applied to (2.2) tells us that also $\|\nabla y_i(t)\|_{L^\alpha(\Omega)} \le C_3$ holds for any $\alpha > 1$ and $t > 0$. In particular, $\|u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s))\|_{L^p(\Omega)} \le C_3$ for any $s \in (0, T_{\max})$ with $C_3 > 0$ depending on p, $\|u_{1,0}\|_{L^p(\Omega)}$, $\|u_{2,0}\|_{L^p(\Omega)}$, $\|u_{3,0}\|_{L^p(\Omega)}$, and $\mathcal{M}$ but not on t. Consequently, using Lemma 2.4, we arrive at

$$\begin{aligned} K_3 &\le C_2 \int_{t_0}^{t} (t - s)^{-(\beta+\varepsilon+1/2)}\big\|u_i(s)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}(s))\big\|_{L^p(\Omega)} \,\mathrm{d}s \\ &\le C_3 \int_{t_0}^{t} (t - s)^{-(\beta+\varepsilon+1/2)} \,\mathrm{d}s =: C_4 \quad \text{for all } t \in (0, T), \end{aligned} \tag{3.22}$$

where C_4 is independent of t. Collecting (3.21)–(3.22) we conclude that

$$\big\|u_i(\cdot, t)\big\|_{L^\infty(\Omega)} \le C_5 \qquad \text{for all } t \in (0, T).$$

Herein the constant $C_5 > 0$ is independent of t. Thus, we obtain $T_{\max} = \infty$ in view of (3.1). □

The last lemma entails the main result of this section, namely the existence and uniqueness of the global classical solution to the system (1.1).

Theorem 3.1. *Let $u_{i,0} \in C^0(\overline{\Omega}) \cap L^p(\Omega)$ be nonnegative for some $p > 2$ with $i = 1, 2, 3$. Then* (1.1) *possesses a unique global classical solution $(\boldsymbol{u}, \boldsymbol{y})$ for which both u_i and y_i are nonnegative and each $u_1, u_2, u_3, y_1, y_2, y_3$ belongs to $C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$.*

4. Weak Solutions

In this section, we prove a well-posedness result for weak solutions to system (1.1), as a limit of classical solutions with smoothed initial data.

Definition 4.1. *A* weak solution *of* (1.1) *in the time interval* $(0, T)$ *is a set of non-negative functions $(\boldsymbol{u}, \boldsymbol{y})$ such that for all $i = 1, 2, 3$*

$$u_i, y_i \in L^2\big(0, T; H^1(\Omega)\big), \qquad \partial_t u_i \in L^2\big(0, T; \big(H^1(\Omega)\big)^*\big),$$

and for all test function $\xi \in C^\infty\big([0, T) \times \Omega\big)$, u_i and y_i satisfy the following identities for $i = 1, 2, 3$:

$$\begin{aligned} -\int_0^T u_i \partial_t \xi \,\mathrm{d}t + \int_{\Omega_T} \big(D_i \nabla u_i \cdot \nabla \xi - \chi_i u_i \nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{y}) \cdot \nabla \xi\big) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t & \\ - \int_\Omega u_{i,0}(\boldsymbol{x}) \xi(\boldsymbol{x}, 0) \,\mathrm{d}t &= \int_{\Omega_T} F_i \cdot \xi \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t, \\ \mathcal{D}_i \int_{\Omega_T} \nabla y_i \cdot \nabla \xi \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t + \theta_i \int_{\Omega_T} y_i \cdot \xi \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t &= \int_{\Omega_T} u_i \cdot \xi \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t. \end{aligned}$$

The main result of this section is the following.

Theorem 4.1. *Fix an arbitrary $T > 0$. Then for all nonnegative $u_{i,0} \in L^4(\Omega)$ there exists a unique weak solution to the system* (1.1) *in the sense of Definition 4.1.*

We postpone the proof of the Theorem to the end of the section, and prove now the auxiliary results needed. The first is a stability result for the classical solutions of (1.1) obtained in Theorem 3.1.

Lemma 4.1. *Let $p > 2$ and $u_{i,0}^a, u_{i,0}^b \in C^0(\overline{\Omega}) \cap L^p(\Omega)$ be two sets of nonnegative initial data with $i = 1, 2, 3$. Then, the respective classical solutions $(\boldsymbol{u}^a, \boldsymbol{y}^a)$ and $(\boldsymbol{u}^b, \boldsymbol{y}^b)$ obtained in Theorem 3.1 are stable in the sense that there exists a constant $C > 0$ depending only on the L^p norms of the initial data, on Ω, and on the constants appearing in* (1.1) *such that*

$$\sum_{i=1}^3 \big\|u_i^a(t) - u_i^b(t)\big\|_{L^2(\Omega)} \leq \left(\sum_{i=1}^3 \big\|u_{i,0}^a - u_{i,0}^b\big\|_{L^2(\Omega)}\right) \exp(Ct). \tag{4.1}$$

Proof. Let $\overline{u}_i := u_i^a - u_i^b$, for $i = 1, 2, 3$, and similarly for y_i. The equations for $\overline{u}_i$ read

$$\partial_t \overline{u}_i - D_i \Delta \overline{u}_i - \chi_i \operatorname{div}(\overline{u}_i \nabla y_3^a) - \chi_i \operatorname{div}(u_i^b \nabla \overline{y}_3) = F_i(\boldsymbol{u}^a) - F_i(\boldsymbol{u}^b), \tag{4.2}$$

if $i = 1, 2$, and for $i = 3$,

$$\partial_t \overline{u}_3 - D_3 \Delta \overline{u}_3 - \chi_3 \operatorname{div}\big(\overline{u}_3 \nabla(y_1^a + y_2^a)\big) - \chi_3 \operatorname{div}\big(u_3^b \nabla(\overline{y}_1 + \overline{y}_2)\big) = F_3(\boldsymbol{u}^a) - F_3(\boldsymbol{u}^b).$$

Multiplying (4.2) by $\overline{u}_i$ and integrating in Ω, we find

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\overline{u}_i|^2 \,\mathrm{d}\boldsymbol{x} + D_i \int_\Omega |\nabla \overline{u}_i|^2 \,\mathrm{d}\boldsymbol{x} = -T_1 - T_2 + T_3, \tag{4.3}$$

where we define

$$T_1 \coloneqq \chi_i \int_\Omega \overline{u}_i \nabla y_3^a \cdot \nabla \overline{u}_i \,\mathrm{d}\boldsymbol{x}, \qquad T_2 \coloneqq \chi_i \int_\Omega u_i^b \nabla \overline{y}_3 \cdot \nabla \overline{u}_i \,\mathrm{d}\boldsymbol{x}, \qquad T_3 \coloneqq \int_\Omega \big(F_i(\boldsymbol{u}^a) - F_i(\boldsymbol{u}^b)\big)\overline{u}_i \,\mathrm{d}\boldsymbol{x}.$$

First note that since $n = 2$, we find from Sobolev embedding, elliptic regularity and the estimate (3.12) that

$$\|\nabla y_3^a\|_{L^\infty(\Omega)} \le C\|y_3^a\|_{W^{2,p}} \le C\|u_3^a\|_{L^p(\Omega)} \le C\big(\|u_{i,0}^a\|_{L^p(\Omega)}\big). \tag{4.4}$$

Thus we can write, using an appropriate Young inequality,

$$T_1 \le \chi_i \|\nabla y_3^a\|_{L^\infty(\Omega)} \int_\Omega |\overline{u}_i \nabla \overline{u}_i| \,\mathrm{d}\boldsymbol{x} \le C\big(\|u_{i,0}^a\|_{L^p(\Omega)}\big) \int_\Omega |\overline{u}_i|^2 \,\mathrm{d}\boldsymbol{x} + \frac{D_i}{2} \int_\Omega |\nabla \overline{u}_i|^2 \,\mathrm{d}\boldsymbol{x},$$

where $C(\|u_{i,0}^a\|_{L^p(\Omega)})$ depends also on the constants appearing in (1.1). For the second term in the right-hand side of (4.3), we find

$$\begin{aligned} T_2 &\le \frac{D_i}{2} \int_\Omega |\nabla \overline{u}_i|^2 \,\mathrm{d}\boldsymbol{x} + C\|u_i^b\|_{L^\infty(\Omega)}^2 \int_\Omega |\nabla \overline{y}_3|^2 \,\mathrm{d}\boldsymbol{x} \\ &\le \frac{D_i}{2} \int_\Omega |\nabla \overline{u}_i|^2 \,\mathrm{d}\boldsymbol{x} + C(\|u_{i,0}^b\|_{L^p(\Omega)})\|\overline{u}_3\|_{L^2(\Omega)}^2, \end{aligned}$$

where we used Lemma 3.4 and elliptic regularity. Finally, from the locally Lipschitz property of F_i, and the L^∞ estimate of Lemma 3.4, we find that

$$T_3 \le L \sum_{i=1}^3 \|\overline{u}_i\|_{L^2(\Omega)}^2$$

for some constant $L > 0$. Combining these estimates in (4.3) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}_i\|_{L^2(\Omega)}^2 \le C \sum_{j=1}^3 \|\overline{u}_j\|_{L^2(\Omega)}^2 \quad \text{for } i = 1, 2.$$

For the third equation of (1.1), similar calculations yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}_3\|_2^2 \le C \sum_{j=1}^3 \|\overline{u}_j\|_{L^2(\Omega)}^2.$$

Therefore, with $\zeta(t) = \sum_{j=1}^3 \|\overline{u}_j\|_2^2$, we find $\zeta'(t) \le C\zeta(t)$ and so $\zeta(t) \le \zeta(0)\exp(Ct)$, which is (4.1). $\square$

Now we take a sequence of smoothed initial data $u_{i,0}^k \in C^0(\overline{\Omega}) \cap L^4(\Omega)$ such that $u_{i,0}^k \to u_{i,0}$ in $L^4(\Omega)$. We consider, for $k \in \mathbb{N}$, the classical solution $(\boldsymbol{u}^k, \boldsymbol{y}^k) \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))$ of the system

$$\begin{aligned} \partial_t u_1^k - D_1 \Delta u_1^k - \chi_1 \operatorname{div}\big(u_1^k \nabla (y_3^k)\big) &= F_1(\boldsymbol{u}^k), \\ \partial_t u_2^k - D_2 \Delta u_2^k - \chi_2 \operatorname{div}\big(u_2^k \nabla (y_3^k)\big) &= F_2(\boldsymbol{u}^k) \\ \partial_t u_3^k - D_3 \Delta u_3^k + \chi_3 \operatorname{div}\big(u_3^k \nabla (y_1^k + y_2^k)\big) &= F_3(\boldsymbol{u}^k) \\ -\mathcal{D}_i \Delta y_i^k + \theta_i y_i^k = \delta_i u_i^k&, \quad i = 1, 2, 3, \quad (\boldsymbol{x}, t) \in \Omega_T, \end{aligned} \tag{4.5}$$

which is given by Theorem 3.1. The next lemma provides the remaining estimates needed to obtain a weak solution.

Lemma 4.2. *Let $(\boldsymbol{u}^k, \boldsymbol{y}^k)$ be the sequence of classical solutions of the system* (1.1) *described above. Fix an arbitrary $T > 0$. Then there exist constants $C_1, \dots, C_5 > 0$ not depending on k such that for $i = 1, 2, 3$,*

$$\|u_i^k\|_{L^\infty(0,T;L^2(\Omega))} + \|y_i^k\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1, \tag{4.6}$$

$$\left\|F_i(u^k)\right\|_{L^2(\Omega_T)} \leq C_2, \tag{4.7}$$

$$\|\nabla u_i^k\|_{L^2(\Omega_T)} + \|\nabla y_i^k\|_{L^2(\Omega_T)} \leq C_3, \tag{4.8}$$

$$\left\|u_i^k \nabla (b_i^T \boldsymbol{y}^k)\right\|_{L^2(\Omega_T)} \leq C_4, \tag{4.9}$$

$$\|\partial_t u_i^k\|_{L^2(0,T;(H^1(\Omega))^*)} \leq C_5. \tag{4.10}$$

Proof. Multiplying the first equation in (4.5) by u_1^k and integrating by parts yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1^k\|^2_{L^2(\Omega)} + D_1\|\nabla u_1^k\|^2_{L^2(\Omega)} + \chi_1 \int_\Omega u_1^k \nabla y_3^k \cdot \nabla u_1^k \,\mathrm{d}\boldsymbol{x} \leq r_1 \|u_1^k\|^2_{L^2(\Omega)}. \tag{4.11}$$

We have

$$\chi_1 \int_\Omega |u_1^k||\nabla y_3^k||\nabla u_1^k| \,\mathrm{d}\boldsymbol{x} \leq \|\nabla y_3^k\|_{L^\infty(\Omega)} \chi_1 \int_\Omega |u_1^k||\nabla u_1^k| \,\mathrm{d}\boldsymbol{x}$$

and, by (4.4), $\|\nabla y_3^k\|_{L^\infty(\Omega)} \leq C(\|u_{i,0}^k\|_{L^4(\Omega)})$. Since $u_{i,0}^k \to u_{i,0}$ in $L^4(\Omega)$, $\|u_{i,0}^k\|_{L^4(\Omega)}$ is bounded uniformly in k. Therefore, $\|\nabla y_3^k\|_{L^\infty(\Omega)} \leq C$ and we get

$$\|\nabla y_3^k\|_{L^\infty(\Omega)} \chi_1 \int_\Omega |u_1^k||\nabla u_1^k| \,\mathrm{d}\boldsymbol{x} \leq C \int_\Omega |u_1^k||\nabla u_1^k| \,\mathrm{d}\boldsymbol{x} \leq \frac{D_1}{2} \int_\Omega |\nabla u_1^k|^2 \,\mathrm{d}\boldsymbol{x} + C \int_\Omega |u_1^k|^2 \,\mathrm{d}\boldsymbol{x}$$

for some appropriate constant C. Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1^k\|^2_{L^2(\Omega)} + \frac{D_1}{2}\|\nabla u_1^k\|^2_{L^2(\Omega)} \leq C\|u_1^k\|^2_{L^2(\Omega)}. \tag{4.12}$$

In view of Gronwall's inequality it follows from (4.11) that

$$\sup_{(0,T)} \|u_1^k\|_{L^2(\Omega)} \leq C_1'. \tag{4.13}$$

Now, we apply elliptic regularity theory and (4.13) to find

$$\sup_{(0,T)} \|y_1^k\|_{L^2(\Omega)} \leq C_1', \tag{4.14}$$

for some constant $C_1' > 0$. The treatment of the second species u_2^k is exactly the same, and we obtain

$$\sup_{(0,T)} \|u_2^k\|_{L^2(\Omega)} \leq C_2' \text{ and } \sup_{(0,T)} \|y_2^k\|_{L^2(\Omega)} \leq C_2', \tag{4.15}$$

for some constant $C_2' > 0$.

For the third species, the procedure is still the same, except that we need to bound $\|\nabla(y_1^k + y_2^k)\|_{L^\infty(\Omega)}$ instead of $\|\nabla y_3^k\|_{L^\infty(\Omega)}$. But in a similar way, we find that

$$\|\nabla(y_1^k + y_2^k)\|_{L^\infty(\Omega)} \leq C_3',$$

where C_3' depends on $\|u_{i,0}^k\|_{L^4(\Omega)}$, which is uniformly bounded in k. We thus get

$$\sup_{(0,T)} \|u_3^k\|_{L^2(\Omega)} \leq C_3' \text{ and } \sup_{(0,T)} \|y_3^k\|_{L^2(\Omega)} \leq C_3' \tag{4.16}$$

for some constant $C_3' > 0$. Then (4.6) follows from (4.13) to (4.16).

The quadratic growth of F_i and the $L^\infty(0,T;L^4(\Omega))$ estimates on the solutions u_i^k, ensure the $L^2(\Omega_T)$ bound for F_i, (4.7).

Now from (4.12) and (4.6) we obtain (4.8).

To obtain (4.9), we observe that

$$\int_0^T \left\|u_i^k \nabla(b_i^T \boldsymbol{y}^k)\right\|_{L^2(\Omega)} \,\mathrm{d}s \le \sup_{(0,T)} \left\|\nabla(b_i^T \boldsymbol{y}^k)\right\|_{L^\infty(\Omega)} \int_0^T \|u_i^k\|_{L^2(\Omega)} \,\mathrm{d}s.$$

By (4.4) and (3.12), we have that

$$\sup_{(0,T)} \|\nabla y_i^k\|_{L^\infty(\Omega)} \le C\big(\|u_{i,0}^k\|_{L^4(\Omega)}\big) \le C \quad \text{uniformly in } k.$$

Therefore, using (4.6), we get

$$\int_0^T \left\|u_i^k \nabla(b_i^T \boldsymbol{y}^k)\right\|_{L^2(\Omega)} \,\mathrm{d}s \le C_4',$$

which is (4.9).

Finally, we deduce from (4.7), (4.8) and (4.9) that for all $\phi \in L^2(0,T;H^1(\Omega))$,

$$\begin{aligned}
\left|\int_0^T \int_\Omega \partial_t u_i^k \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\right| &\le \left| -D_i \int_0^T \int_\Omega \nabla u_i^k \cdot \nabla\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \chi_i \int_0^T \int_\Omega u_i^k \nabla(b_i^T \boldsymbol{y}^k)\cdot\nabla\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \right.\\
&\qquad \left. + \int_0^T \int_\Omega F_i(\boldsymbol{u}^k)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \right| \\
&\le D_i \|\nabla u_i^k\|_{L^2(\Omega_T)} \|\nabla\phi\|_{L^2(\Omega_T)} + |\chi_i| \|u_i^k \nabla(b_i^T \boldsymbol{y}^k)\|_{L^2(\Omega_T)} \|\nabla\phi\|_{L^2(\Omega_T)} \\
&\quad + \|F_i(u^k)\|_{L^2(\Omega_T)} \|\phi\|_{L^2(\Omega_T)} \\
&\le C_5 \|\phi\|_{L^2(0,T;H^1(\Omega))},
\end{aligned}$$

for some constant C_5 independent of k. From this we deduce the bound (4.10). This completes the proof of Lemma 4.2. ☐

We are now ready to prove the well-posedness result of Theorem 4.1.

Proof of Theorem 4.1. For $k,l \in \mathbb{N}$, consider the Cauchy differences

$$Z_{k,l}(t) = \sum_{i=1}^3 \left\|u_i^k(t) - u_i^l(t)\right\|_{L^2(\Omega)}$$

constructed from the smooth solutions in Lemma 4.2. From the stability result in Lemma 4.1, we see that

$$Z_{k,l}(t) \le Z_{k,l}(0)\exp(Ct),$$

with C independent of the indices k,l. Therefore, the sequences $(u_i^k)_{k\in\mathbb{N}}$ are Cauchy sequences in $L^\infty(0,T;L^2(\Omega))$. As a consequence, there exist $u_i \in L^\infty(0,T;L^2(\Omega))$ with

$$u_i^k \to u_i \quad \text{in } L^\infty(0,T;L^2(\Omega)).$$

From the equations for y_i^k we easily deduce (for instance with $i=1$) that

$$\left\|y_1^k(t) - y_1^l(t)\right\|_{H^1}^2 \le C\|u_3^k - u_3^l\|_{L^2}^2 \to 0 \quad \text{as } k,l \to \infty,$$

and so $(y_i^k)_k$ are Cauchy sequences in $L^\infty(0,T;H^1(\Omega))$. Therefore we have

$$y_i^k \to y_i \quad \text{in } L^\infty(0,T;H^1(\Omega)), \text{ as } k \to \infty.$$

From the estimates (4.8) and (4.10) we deduce also that

$$\begin{aligned} u_i^k &\rightharpoonup u_i && \text{weakly in } L^2(0,T;H^1(\Omega)),\\ \partial_t u_i^k &\rightharpoonup \partial_t u_i && \text{weakly in } L^2\big(0,T;(H^1(\Omega))^*\big). \end{aligned}$$

As a consequence of the previous estimates we find, in addition, that

$$\begin{aligned} u_i^k\nabla(b_i^T\boldsymbol{y}^k) &\to u_i\nabla(b_i^T\boldsymbol{y}) && \text{in } L^1((0,T)\times\Omega),\\ F_i(u_i^k) &\to F_i(u_i) && \text{in } L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

The above convergences, along with a time continuity property in $L^2((0,T)\times\Omega)$ (which is a consequence of the Aubin-Lions lemma [29, Theorem 2.1]), ensure that for each $\xi \in C^\infty\big([0,T)\times\Omega\big)$ we can pass to the limit on each term of

$$\begin{aligned} -\int_0^T u_i^k\partial_t\xi \,\mathrm{d}t + \int_{\Omega_T}\Big(D_i\nabla u_i^k - \chi_i u_i^k\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}^k)\Big)\nabla\xi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &\\ - \int_\Omega u_{i,0}(\boldsymbol{x})\xi(\boldsymbol{x},0)\,\mathrm{d}t &= \int_{\Omega_T} F_i\,\xi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\\ \mathcal{D}_i\int_{\Omega_T}\nabla y_i^k\cdot\nabla\xi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \theta_i\int_{\Omega_T} y_i^k\xi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \int_{\Omega_T} u_i^k\,\xi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \end{aligned}$$

to obtain a weak solution according to Definition 4.1. The uniqueness follows from the fact that the stability property in Lemma 4.1 holds, by approximation, for weak solutions as well. This completes the proof of Theorem 4.1. □

5. Finite Volume Scheme

In this section, we construct approximate solutions of problem (1.1). For this purpose, we introduce a notion of admissible finite volume mesh (see e.g. [11]).

5.1. **Admissible mesh.** Let $\Omega\subset\mathbb{R}^n$, $n=2$ denote an open bounded polygonal connected domain with boundary $\partial\Omega$. An admissible FV mesh of Ω is given by a family $\mathcal{T}_h$ of control volumes (open and convex polygonal subsets of Ω), a family $\mathcal{E}\subset\bar{\Omega}$ of hyperplanes of $\mathbb{R}^d$ (edges in two-dimensional case or sides in three-dimensional) and a family of points $\mathcal{P}=(\boldsymbol{x}_K)_{K\in\mathcal{T}_h}$ that satisfy

$$\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h}\bar{K},\qquad \mathcal{E} = \bigcup_{K\in\mathcal{T}_h}\mathcal{E}_K,\qquad \partial K = \bigcup_{L\in\mathcal{N}(K)}\bar{\sigma}.$$

Let $K,L\in\mathcal{T}_h$ with $K\neq L$. If $\bar{K}\cap\bar{L}=\bar{\sigma}$ for some $\sigma\in\mathcal{E}$, then $\sigma=K|L$ (common edge). We introduce the set of interior (respectively boundary) edges denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), that is $\mathcal{E}_{\mathrm{int}}=\{\sigma\in\mathcal{E}\ :\ \sigma\not\subset\partial\Omega\}$ (resp. $\mathcal{E}_{\mathrm{ext}}=\{\sigma\in\mathcal{E}\ :\ \sigma\subset\partial\Omega\}$). The set of neighbours of K is given by $\mathcal{N}(K)=\{L\in\mathcal{T}_h\ :\ \exists\sigma\in\mathcal{E},\bar{\sigma}=\bar{K}\cap\bar{L}\}$. The family $\mathcal{P}$ is such that for all $K\in\mathcal{T}_h$, $\boldsymbol{x}_K\in\bar{K}$, and, if $\sigma=K|L$, it is assumed that $\boldsymbol{x}_K\neq\boldsymbol{x}_L$, and that the segment $\overline{\boldsymbol{x}_K\boldsymbol{x}_L}$ is orthogonal to $\sigma=K|L$ Let $d_{K|L}$ denote the Euclidean distance between $\boldsymbol{x}_k$ and $\boldsymbol{x}_L$ and by $d_{K,\sigma}$ the distance from $\boldsymbol{x}_K$ to σ. The transmissibility through $\sigma\in\mathcal{E}_{\mathrm{int}}$ is defined by $\tau_{K|L}=m(K|L)/d_{K|L}=m(\sigma)/d_\sigma$ and for $\sigma\in\mathcal{E}_{\mathrm{ext}}$ by $\tau_{K,\sigma}=m(\sigma)/d_{K,\sigma}$. We require local regularity restrictions on the family of meshes $\mathcal{T}_h$; namely

$$\begin{aligned} \exists\gamma>0\ \forall h\ \forall K\in\mathcal{T}_h\ \forall L\in\mathcal{N}(K):&\quad \mathrm{diam}(K)+\mathrm{diam}(L)\le\gamma d_{K,L}\\ \exists\gamma>0\ \forall h\ \forall K\in\mathcal{T}_h\ \forall L\in\mathcal{N}(K):&\quad m(K|L)d_{K,L}\le\gamma m(K). \end{aligned}$$

A discrete function on the mesh $\mathcal{T}_h$ is a set $(u_K)_{K\in\mathcal{T}_h}$. Whenever convenient, we identify it with the piecewise constant function $u_h \in \Omega$ such that $u_h|_K = u_K$. Finally, the discrete gradient $\nabla_h u_h$ of a constant per control volume function u_h is defined on $\bar{K}\cap\bar{L}$ by

$$\nabla_{K,L} u_{i,h} := \frac{u_L - u_K}{d_{K|L}} \boldsymbol{n}_{K|L}.$$

5.2. **Description of the finite volume (FV) scheme.** We adapt the finite volume scheme given in [9] to our context, recalling that in [9] the convergence to the weak solution of FV scheme has been proven. To discretize (1.1) we choose an admissible discretization of Ω_T consisting of an admissible mesh $\mathcal{T}_h$ of Ω along with a time step $\Delta t_h > 0$; both Δt_h and the size $\max_{K\in\mathcal{T}} \operatorname{diam}(K)$ tend to zero as $h \to 0$. We define $N_h > 0$ as the smallest integer such that $(N_h + 1)\Delta t_h \geq T$, and set $t_n = n\Delta t_h$ for $n \in \{0,\dots,N_h\}$. Whenever Δt_h is fixed, we will drop the subscript h in the notation.

To formulate the resulting scheme, we introduce the terms

$$\mathcal{A}^{n+1}_{i,K,L} := \min\bigl\{(u^{n+1}_{i,K})^+, (u^{n+1}_{i,L})^+\bigr\}, \quad F^{n+1}_{i,K} := F_i\bigl((u^{n+1}_{1,K})^+, (u^{n+1}_{2,K})^+, (u^{n+1}_{3,K})^+\bigr), \quad i = 1,2,3.$$

The computation starts from the initial cell averages

$$u^0_{i,K} := \frac{1}{m(K)} \int_K u_{i,0}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad i = 1,2,3.$$

We state the FV scheme for (1.1) as follows: for all $K \in \mathcal{T}_h$ and $n \in \{0,1,\dots,N_h\}$, find $(u^{n+1}_{i,K})_{K\in\mathcal{T}_h}$, $i = 1,2,3$, such that

$$\begin{aligned} -\mathcal{D}_i \sum_{L\in\mathcal{N}(K)} \tau_{K|L}\bigl(y^{n+1}_{i,L} - y^{n+1}_{i,K}\bigr) + \theta_i m(K) y^{n+1}_{i,K} &= \delta_i m(K) u^n_{i,K}, \quad i = 1,2,3, \\ m(K)\frac{u^{n+1}_{i,K} - u^n_{i,K}}{\Delta t} - D_i \sum_{L\in\mathcal{N}(K)} \tau_{K|L}\bigl(u^{n+1}_{i,L} - u^{n+1}_{i,K}\bigr) & \\ +\chi_i \sum_{L\in\mathcal{N}(K)} \tau_{K|L} \mathcal{A}^{n+1}_{i,K,L} \boldsymbol{b}_i^{\mathrm{T}}\bigl(\boldsymbol{y}^{n+1}_L - \boldsymbol{y}^{n+1}_K\bigr) &= m(K) F^{n+1}_{i,K}, \quad i = 1,2,3. \end{aligned} \tag{5.1}$$

As usual, homogeneous Neumann boundary conditions are taken into account implicitly. Indeed, the parts of ∂K that lie in $\partial\Omega$ do not contribute to the sums over $L \in \mathcal{N}(K)$ terms, which means that the flux is zero is imposed on the external edge of the mesh.

The sets of values $(u^{n+1}_{1,K}, u^{n+1}_{2,K}, u^{n+1}_{3,K})_{K\in\mathcal{T}_h, n\in\{0,1,\dots,N_h\}}$ and $(y^{n+1}_{1,K}, y^{n+1}_{2,K}, y^{n+1}_{3,K})_{K\in\mathcal{T}_h, n\in\{0,1,\dots,N_h\}}$ satisfying (5.1) will be called a discrete solution. We associate a discrete solution of the scheme at $t = t_{n+1}$ with the triples $\boldsymbol{u}^{n+1}_h = (u^{n+1}_{1,h}, u^{n+1}_{2,h}, u^{n+1}_{3,h})^{\mathrm{T}}$ and $\boldsymbol{y}^{n+1}_h = (y^{n+1}_{1,h}, y^{n+1}_{2,h}, y^{n+1}_{3,h})^{\mathrm{T}}$ of the piecewise constant on Ω functions given by

$$u^{n+1}_{y,h}|_K = u^{n+1}_{i,K}, \quad y^{n+1}_{i,h}|_K = y^{n+1}_{i,K}, \quad \text{for all } K \in \mathcal{T}_h, \text{ all } n \in \{0,1,\dots,N_h - 1\} \text{ and all } i = 1,2,3.$$

Furthermore, we define the piecewise constant function

$$\boldsymbol{u}_h(\boldsymbol{x},t) = \bigl(u_{1,h}(\boldsymbol{x},t), u_{2,h}(\boldsymbol{x},t), u_{3,h}(\boldsymbol{x},t)\bigr)^{\mathrm{T}} := \sum_{\substack{K\in\mathcal{T}_h \\ n\in\{0,1,\dots,N_h\}}} \boldsymbol{u}^{n+1}_h \mathbb{1}_{(t_n,t_{n+1}]\times K}.$$

Herein, the expression $\mathbb{1}_{(t_n,t_{n+1}]\times K}$ denotes the characteristic function of set $(t_n, t_{n+1}]\times K$, in similar way we define the piecewise constant function $\boldsymbol{y}_h(\boldsymbol{x},t)$.

6. Numerical Examples

We present in this section some numerical results obtained by the finite volume scheme (5.1). To obtain a numerical test case, we reduce the number of parameters in the model (1.1), (1.2). To this end we non-dimensionalize the system following [2]. We choose

$$U_1 = \frac{u_1}{k_1}, \quad U_2 = \frac{M_1 A_1 u_2}{k_1 A_2 M_2}, \quad U_3 = \frac{\gamma_1 K_1}{u_3}, \quad \tau = r_1 t.$$

Making the substitution and simplifying, we obtain

$$\begin{aligned} F_1(\boldsymbol{U}) &= U_1(1-U_1) - c_1 U_1 U_2 - \frac{a_1 U_1}{b_1 + U_1} U_3, \\ F_2(\boldsymbol{U}) &= r U_1 (k - U_1) - c_2 U_1 U_2 - \frac{a_2 U_2}{b_2 + U_2} U_2, \\ F_3(\boldsymbol{U}) &= \frac{a_1 U_1}{b_1 + U_1} U_3 + \frac{a_3 U_2}{b_2 + U_2} U_3 - d U_3 - f U_3^2. \end{aligned}$$

On the domain $\Omega \coloneqq [0,50] \times [0,50]$ we define a uniform Cartesian grid

$$\mathcal{T}_h = \big\{ K_{ij} \subseteq \Omega \ : \ K_{ij} = ((i-1)N_x, iN_x) \times ((j-1)N_x, jN_x) \big\}$$

with $N_x \times N_y$ control volumes. For the simulations, we choose $N_x = N_y = 400$, and $\Delta t = 0.01$, the corresponding diffusion coefficients are chosen as $D_1 = D_2 = D_3 = 1$, and the sensitivity chemotactic parameters are chosen as in [4], namely $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 10$ and $\theta_1 = \theta_2 = \theta_3 = 0.1$. The initial distribution and the others parameters are specified in each example.

6.1. **Example 1: pursuit and evasion.** For this numerical test, we suppress the terms on the right-hand side of (1.1) in order to ignore the population dynamics and emphasize the effect due to chemotaxis terms. To this end, we consider the chemotactic coefficients $\chi_1 = 1$, $\chi_2 = 2$, and $\chi_3 = 5$. (These can be interpreted in the sense that the action of prey species 2 to evade the predator is greater than that of prey species 1.) Moreover, we set $\delta_1 = \delta_2 = \delta_3 = 100$ (which means that the chemical sensitivity of the predator and prey are large compared to their diffusion rates). The initial conditions are displayed in the top row of Figure 1. In the second to fourth row of Figure 1, we display the numerical approximation at three different simulation times $T = 0.54$, $T = 1.12$ and $T = 1.84$. We can see that the predator begins to chase the prey species while the two group of prey species closer to the predator immediately take evasive action. We can appreciate that there is a tendency for the predator to choose prey species 1 because its evasion action is less than for prey species 2.

6.2. **Example 2: full dynamics.** In this example we choose the following parameters in the reaction terms:

$$\begin{gathered} r = 1.5, \quad k = 0.6, \quad a_1 = 1.9, \quad a_2 = 0.57, \quad a_3 = 0.55, \quad b_1 = 0.5, \\ b_2 = 0.8, \quad c_1 = 0.1, \quad c_2 = 0.9, \quad d = 0.6, \quad f = 0.2. \end{gathered}$$

According to [2], for the chosen parameters, the solution of the ODE system is a constant equilibrium point $(u_1^*, u_2^*, u_3^*) = (0.1632, 0.4153, 0.2776)$. In order to observe the full dynamics of model (1.1), (1.2) we consider the chemotactic coefficients $\chi_1 = 1$, $\chi_2 = 2$, $\chi_3 = 5$, $\delta_1 = 100$, $\delta_2 = 90$, and $\delta_3 = 40$. The initial conditions are displayed in the top row of Figure 2. The initial data represent a uniform distribution plus a perturbation around the constant equilibrium point (u_1^*, u_2^*, u_3^*). The "random" initial datum has been chosen to test whether small perturbations would give rise to large-scale regular structures akin to the well-known mechanism of pattern formation,

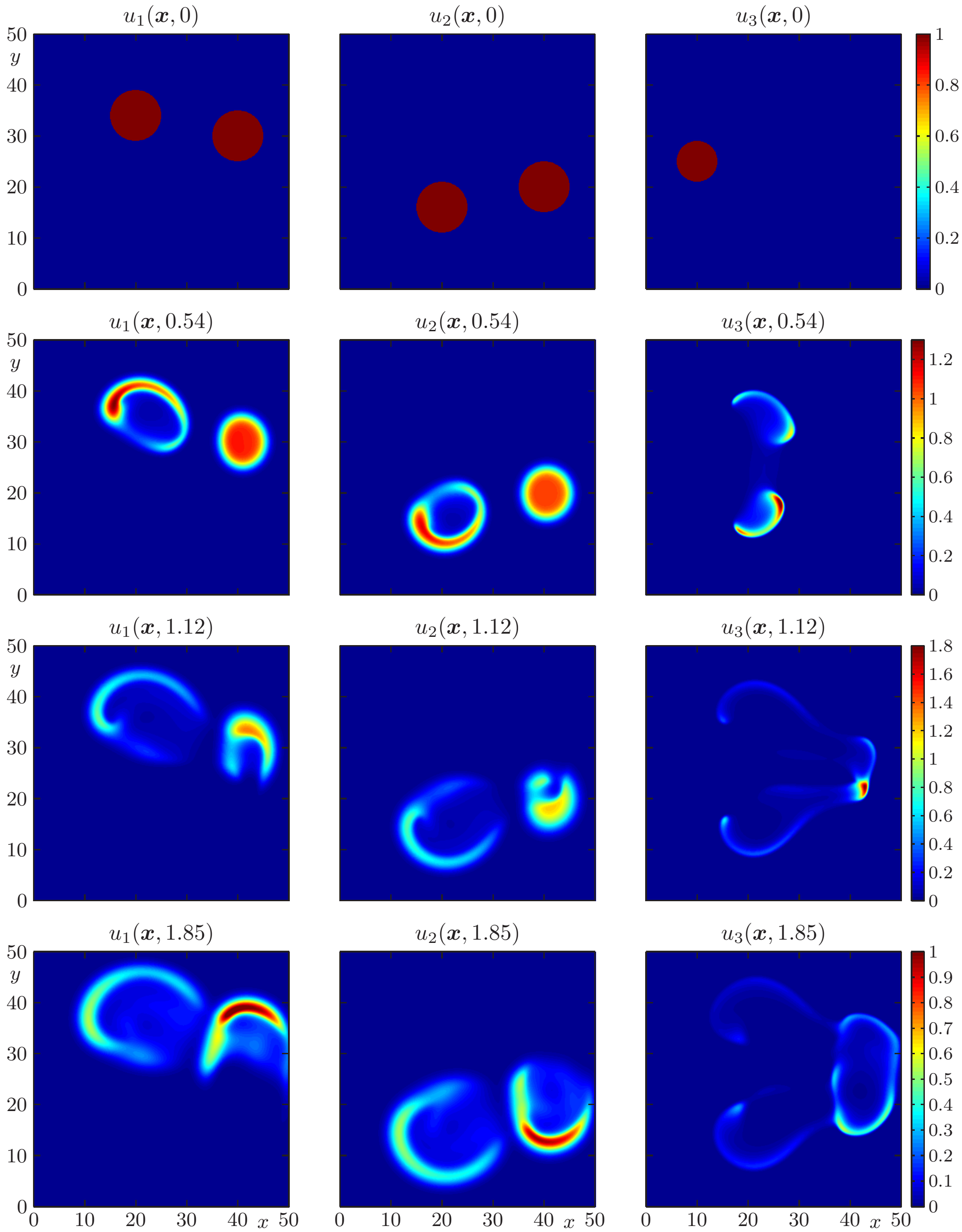

FIGURE 1. Example 1: Numerical solution of (1.1) without reaction terms and with chemotactic coefficients given by $\chi_1 = 1,\ \chi_2 = 2,\ \chi_3 = 5,\ \delta_1 = \delta_2 = \delta_3 = 100$.

or rather, the small fluctuations in the initial datum would simply be smoothed out. In Fig 2 we display numerical approximation at simulation times $T = 34$, $T = 141$ and $T = 300$. It turns out that each species aggregates in a kind of groups structure which forming zones of different densities. This structure varies with time (not shown here), which lends further support to the conjecture that

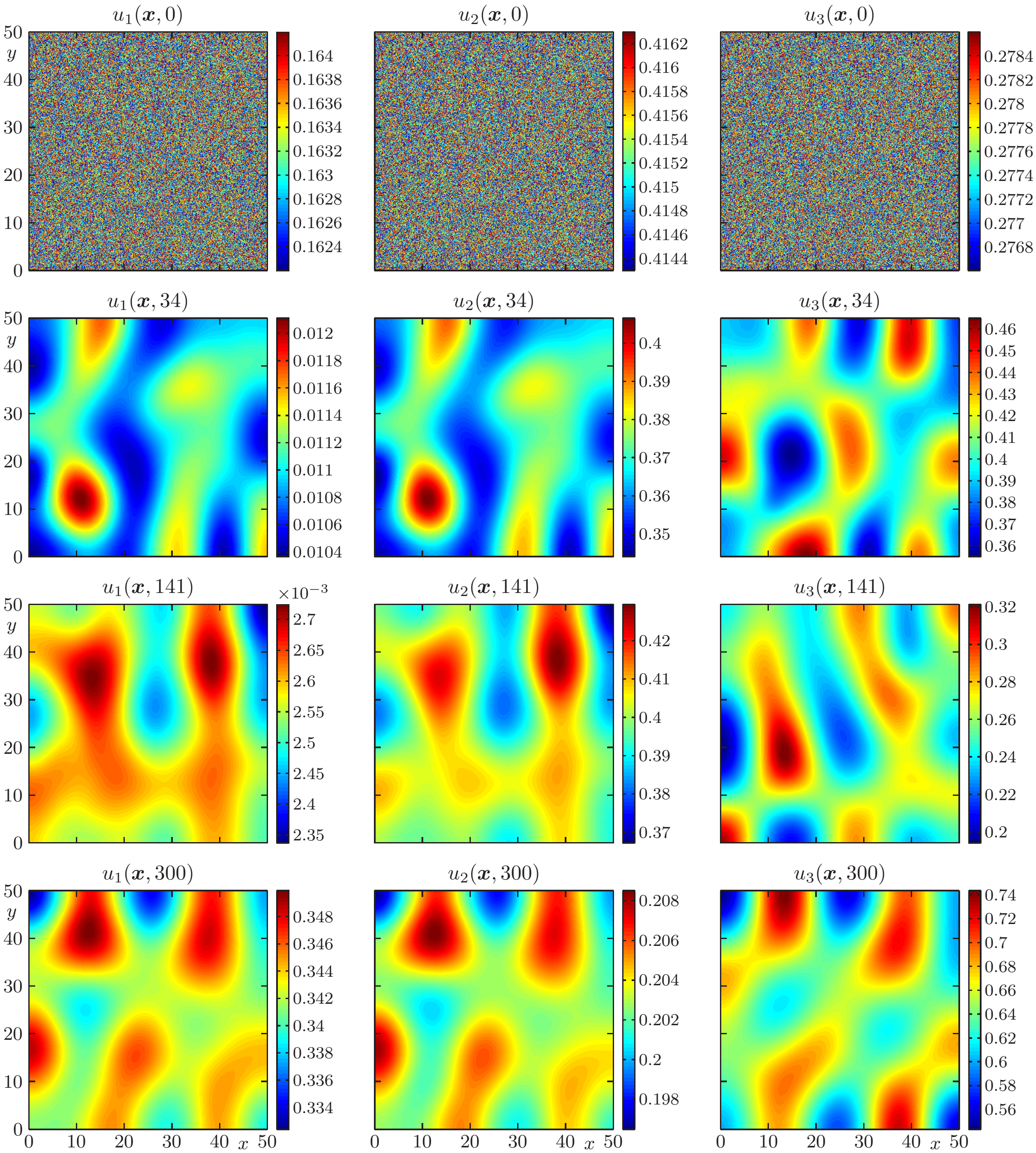

FIGURE 2. Example 2: Numerical solution of (1.1) with chemotactic coefficients given by $\chi_1 = 1$, $\chi_2 = 2$, $\chi_3 = 5$, $\delta_1 = 100$, $\delta_2 = 90$ and $\delta_3 = 40$

this model (at least with the parameters chosen) exhibits spatial–temporal oscillatory behavior. On the other hand, in Fig 3 we display the quantities

$$\mathcal{I}(u_i, t^n) := \sum_{K\in\mathcal{T}_h} m(K)u_{i,K}^n = \int_\Omega u_i(x,t^n)\,\mathrm{d}\boldsymbol{x},$$

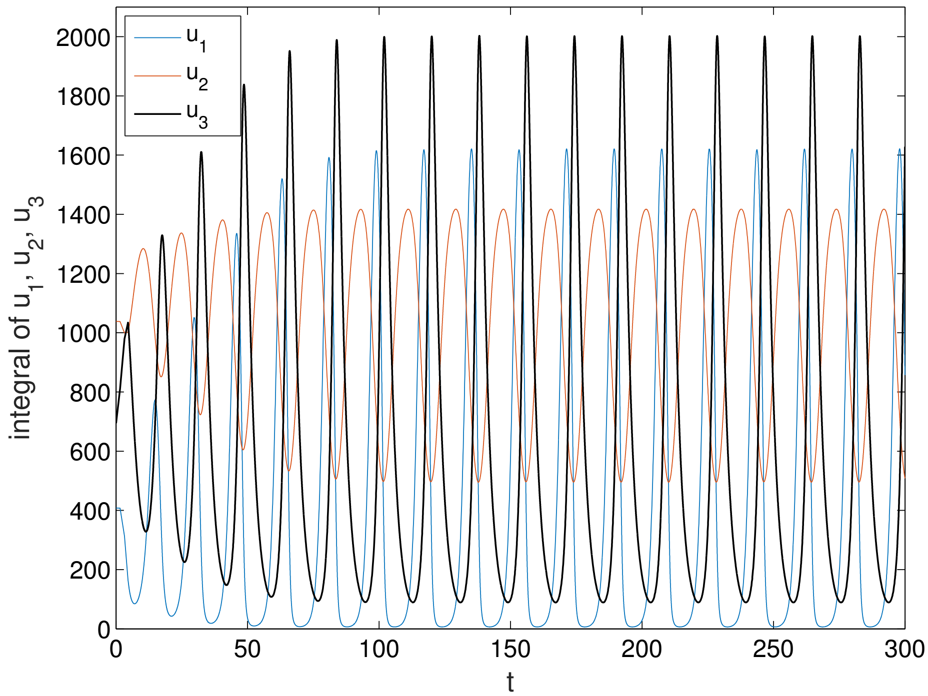

FIGURE 3. Example 2: $\mathcal{I}(u_i,)$ for $i = 1, 2, 3$ and $0 \leq t \leq 300$.

which represents the approximate total number in Ω of individuals of compartment u_i. We can observe that for the parameters chosen, unlike the model without diffusion and chemotaxis terms, the values of $\mathcal{I}(u_i, t^n)$ generate a dynamic of oscillations which is maintained over time.

CONCLUSIONS

We have proposed and studied a reaction-diffusion system consisting of three parabolic equations which describe the dynamic of a food chain model with two competitive preys and one predator, and three elliptic equations describing the chemotaxis produced by three chemicals. For this purpose we have proven the existence and uniqueness of global classical and weak solutions of the initial-boundary value of the proposed problem. The local existence of a negative solution was proved by using the Banach xed point theorem and the properties of the heat semigroup. Next, we showed that the solution of the problem satises the L^α-integrability property, with this, existence of a global solution is proven. In order to prove the existence of weak solutions, we have dened for $k \in \mathbb{N}$ a sequence of the classical solution $(\boldsymbol{u}^k, \boldsymbol{y}^k)$, and then we proved some k-independent estimates. Therefore by using the Aubin-Lions Lemma, we guaranteed the existence of the limit function, which is a weak solution of our problem. Uniqueness follows from a stability property. Numerical tests showed the rapid movement of the species due to the choice of the chemotactic coefficients, In Example 1, when the reaction term is not considered, we can see that the predator begins to chase the prey species which immediately take evasive action. In Example 2, we have observed that small perturbations would give rise to large-scale spatial–temporal oscillatory behavior akin to the well-known mechanism of pattern formation. Additionally, we have seen that the approximate total number in ω of individuals of species u_i at time t, $\mathcal{I}(u_i, t)$ generates a dynamic of oscillations which is maintained over time.

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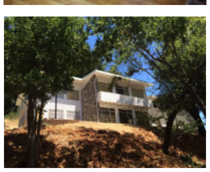

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