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Analysis of an unfitted mixed finite element method for a class of quasi-Newtonian Stokes flow

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Abstract

We propose and analyze an unfitted method for a dual-dual mixed formulation of a class of Stokes models with variable viscosity depending on the velocity gradient, in which the pseudoestress, the velocity and its gradient are the main unknowns. On a fluid domain Ω with curved boundary Γ we consider a Dirichlet boundary condition and employ an approach previously applied to the Stokes equations with constant viscosity, which consists of approximating Ω by a polyhedral computational subdomain Ω_h, not necessarily fitting Ω, where a Galerkin method is applied to compute solution. Furthermore, to approximate the Dirichlet data on the computational boundary Γ_h, we make use of a transferring technique based on integrating the discrete velocity gradient. Then the associated Galerkin scheme can be defined by employing Raviart–Thomas of order $k \geq 0$ for the pseudostress, and discontinuous polynomials of degree k for the velocity and its gradient. For the *a priori* error analysis we provide suitable assumptions on the mesh near the boundary Γ ensuring that the associated Galerkin scheme is well-posed and optimally convergent with $\mathcal{O}(h^{k+1})$. Next, for the case when Γ_h is taken as a piecewise linear interpolation of Γ, we develop a reliable and quasi-efficient residual-based *a posteriori* error estimator. Numerical experiments verify our analysis and illustrate the performance of the associated *a posteriori* error indicator.

Key words: curved domains, high-order, nonlinear Stokes flow, mixed methods, error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 76D07

1 Introduction

Although finite element methods are well-known in the numerical treatment of partial differential equations (PDEs) in regard to *a priori* error estimates to guarantee convergence, there is evidence demonstrating a loss of accuracy of these methods when applied to PDEs on domains Ω with curved boundary Γ (see, e.g., [3]). In practice, the real domain Ω is approximated by a convenient computational domain Ω_h. Often Ω_h does not exactly match the boundary of Ω. As a consequence, the discrete space in which one looks for the finite element solution is no longer a subspace of the continuous space. This approximation therefore introduces a "variational crime". Strang [36] was the first who studied this

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fact and established how to estimate the consistency error term introduced by the crime. This term is typically of low order and dominates the error estimates. An approach to remedy this drawback is to match or "fit" the mesh to Ω using a suitable local interpolation of the boundary Γ [26]. However, the generation of high-order meshes is non-trivial for the latter case and can be expensive if remeshing is required, e.g., deforming domain problems. It is then preferable to use unfitted methods which reduce the computational cost of mesh generation by, for instance, immersing Ω in a background mesh and setting Ω_h to be the union of all the elements of the mesh that lie inside Ω. However, it is not easy to construct a higher-order accurate unfitted method, primarily because the boundary data is imposed "away" from the true boundary.

Let us briefly discuss the literature on unfitted methods. A common approach to construct these methods consists of using a Nitsche-like strategy [29] to weakly impose the data on the computational boundary $\Gamma_h := \partial\Omega_h$. In this direction, the cut finite element method (CutFEM) has been successfully applied to the Poisson and Stokes problems [5, 6], among others. Alternatively, an unfitted method based on shifting the location of the boundary data was presented by Main and Scovazzi [28]. They proposed a suitable Taylor expansion of the solution allowing for an accurate approximation of the boundary data on Γ_h. This approach is closely related to the one developed in [14], and later analyzed in [13], using hybridizable discontinuous Galerkin (HDG) methods and a transferring (or shifting) technique proposed for one-dimensional problems [12]. More precisely, if u is an unknown of the PDE and g is a datum satisfying $u = g$ on Γ, the method proposes to rewrite u on Γ_h by integrating $\sigma := \nabla u$ along a family of segments connecting Γ and Γ_h. Proceeding as for u and integrating the extrapolation of the discrete approximation of σ, a suitable approximation of g on Γ_h is obtained. Then, the problem is solved in Ω_h and its solution is extended by local extrapolations to the region $\Omega \setminus \overline{\Omega}_h$. In [13] it has been shown that for a piecewise $\mathcal{C}^2$ boundary Γ, the method keeps high order accuracy if the distance $d(\Gamma, \Gamma_h)$ between Γ and Γ_h is of order of h^s, where h denotes the meshsize and $s \geq 1$. Moreover, mixed formulations based on this transferring technique have been recently proposed for difussive problems by [30] and for linear Stokes flow by [31].

In this paper, we consider a higher-order accurate unfitted mixed method for a class of nonlinear Stokes models arising in quasi-Newtonian fluids with Dirichlet data. On the domain Ω_h we combine the above transferring technique with the dual-dual mixed formulation of these models, where the pseudoestress, the velocity and its gradient are the main unknowns [22]. Assuming similar hypotheses as in [30, 31], we show well-posedness and optimal *a priori* error estimates of the Galerkin scheme through a fixed point argument and standard results on nonlinear monotone operators. In particular, under the choice of finite element subspaces given by Raviart–Thomas of order $k \geq 0$ for the pseudostress, and discontinuous polynomials of degree k for the velocity and its gradient, we show an overall convergence of order of h^{k+1} when $d(\Gamma, \Gamma_h)$ is of order of h^s, with $s \geq 1$. One of the main differences between our work and [22] is that we introduce a boundary-value correction in the dual-dual mixed scheme while [22] presents results on polyhedral domains for which a transferring technique is not required.

The second contribution of this paper is a reliable and quasi-efficient residual-based *a posteriori* error estimator for nonlinear Stokes flow. We first show reliability estimates for higher-order accurate approximations of the dual-dual mixed formulation of our model by using a postprocessed velocity with enhanced accuracy, and standard arguments (inf-sup conditions on the involved finite element spaces, Helmholtz decompositions and local approximation properties of Clément and Raviart–Thomas interpolation operators), provided Γ_h is constructed by a piecewise linear interpolation of Γ for which $d(\Gamma, \Gamma_h)$ is of order of h^2. A similar approach was previously used in the *a posteriori* analysis of the standard pseudostress-velocity formulation of the linear Stokes problem in [21], and later extended to domains with curved boundaries in [31]. We furthermore prove that our estimator is efficient up to a fully computable and residual term involving the curved boundary.

The rest of this paper is structured as follows. The governing equations and corresponding dual-dual mixed formulation are presented in Section 2. In Section 3 we introduce the Galerkin scheme and derive hypotheses on the curved and computational boundaries providing the well-posedness of the problem. The corresponding *a priori* error estimates are discussed in Sections 4. In Section 5 we derive a reliable and quasi-efficient residual-based *a posteriori* error estimator. Numerical examples to validate our theory are reported in Section 6 and conclusions are drawn in Section 7.

2 The continuous problem

2.1 Notation

Let $\Omega \subseteq \mathbb{R}^n$ denote a bounded and open region with Lipschitz continuous boundary Γ which is not necessarily polygonal ($n=2$) or polyhedral ($n=3$), and denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ. In what follows we use standard notation for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $\mathrm{H}^{1/2}(\Gamma)$ is the trace space of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vector and tensor counterparts of the generic scalar functional space M. When no confusions arises, $\|\cdot\|$ with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space, and $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$. In turn, given $\boldsymbol{\sigma} := (\sigma_{ij}), \boldsymbol{\tau} := (\tau_{ij}) \in \mathbb{R}^{n\times n}$, we let $\mathbf{div}\,\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and write as usual

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji}), \quad \mathrm{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\sigma} : \boldsymbol{\tau} := \sum_{i,j=1}^{n} \sigma_{ij}\tau_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}\,(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{n\times n}$. For tensor-valued functions we also require the Hilbert space

$$\mathbb{H}(\mathbf{div}\,;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) \Big\},$$

equipped with the norm

$$\|\cdot\|_{\mathbf{div}\,;\Omega} := \Big(\|\cdot\|_{0,\Omega}^2 + \|\mathbf{div}\,(\cdot)\|_{0,\Omega}^2 \Big)^{1/2}.$$

Finally, by $\mathbf{0}$ we will refer to the generic null vector (including the null functional and operator), and we will denote by C, with or without subscripts, bar, tildes, or hats, generic constants independent of the discretization parameters, but might depend on the polynomial degree, the shape-regularity of the triangulation and the domain.

2.2 Governing equations

We are interested in approximating, by a mixed finite element method, the nonlinear Stokes equations describing a steady quasi-Newtonian Stokes flow occupying the region Ω, under the action of external forces, given by

$$\begin{gathered} \boldsymbol{\sigma} = 2\mu(|\nabla \mathbf{u}|)\nabla \mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega, \quad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega, \\ \mathrm{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma, \end{gathered} \tag{2.1}$$

where the unknowns are the velocity $\mathbf{u}$, the pressure p, and the pseudostress tensor $\boldsymbol{\sigma}$. Furthermore, $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is a given volume force, $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ is a prescribed velocity on Γ, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ describes a given nonlinear kinematic viscosity.

As suggested by the third equation in (2.1), the compatibility condition $\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} = 0$ will be assumed throughout this work. Moreover, in order to ensure the uniqueness of solution, we shall consider (2.1) with $p \in \mathrm{L}_0^2(\Omega) := \{q \in \mathrm{L}^2(\Omega) : \int_\Omega q = 0\}$.

It is worth recalling that (2.1) covers a large family of nonlinear viscosity functions (see, e.g., [1, 25, 27, 33]). For instance, the Carreau law for viscoplastic flows (see, e.g., [27]) given by $\mu(s) := \mu_0 + \mu_1(1+s^2)^{(\beta-2)/2}$ for all $s \in \mathbb{R}^+$, with $\mu_0 \geq 0$, $\mu_1 > 0$, and $\beta \geq 1$; and the Ladyzhenskaya law (or power Law) for fluids with large stresses [25], which reads $\mu(s) := \mu_0 + \mu_1 s^{\beta-2}$ for all $s \in \mathbb{R}^+$, with $\mu_0 \geq 0$, $\mu_1 > 0$, and $\beta > 1$.

We end this section with the assumptions made in [22] concerning the viscosity in (2.1). For each $\mathbf{r} := (r_{ij}) \in \mathbb{R}^{n\times n}$ and for all $i,j \in \{1,\ldots,d\}$, we let $\mu_{ij} : \mathbb{R}^{n\times n} \to \mathbb{R}$ be the mapping given by $\mu_{ij}(\mathbf{r}) := \mu(|\mathbf{r}|)r_{ij}$. We assume that μ is of class $\mathscr{C}^1$ and that there exist constants $\gamma_0, \alpha_0 > 0$ such that for all $\mathbf{r} := (r_{ij})$, $\mathbf{s} := (s_{ij}) \in \mathbb{R}^{n\times n}$, there hold

$$|\mu_{ij}(\mathbf{r})| \leq \gamma_0|\mathbf{r}|, \qquad \left|\frac{\partial}{\partial r_{kl}}\mu_{ij}(\mathbf{r})\right| \leq \gamma_0 \qquad \forall\, i,j,k,l \in \{1,\ldots,n\}, \tag{2.2}$$

and

$$\sum_{i,j,k,l=1}^{n} \frac{\partial}{\partial r_{kl}}\mu_{ij}(\mathbf{r})s_{ij}s_{kl} \geq \alpha_0|\mathbf{s}|^2. \tag{2.3}$$

It is in particular possible to check that the Carreau law satisfies (2.2) and (2.3) for all $\mu_0 > 0$ and for all $\beta \in [1,2]$.

2.3 The dual-dual mixed formulation

In this section we recall the dual-dual mixed formulation of (2.1) which has been studied in [22]. Let us first observe that the incompressibility condition together with the constitutive equation in (2.1) are equivalent to the pair of equations given by

$$\boldsymbol{\sigma} = 2\mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega, \quad \text{and} \quad p + \frac{1}{n}\mathrm{tr}\,(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad \Omega, \tag{2.4}$$

from which p can be eliminated from (2.1) and recovered afterwards by a post-processing technique that will be specified in Section 4.2. In this way, by introducing the auxiliary unknown $\mathbf{t} := \nabla\mathbf{u}$ to make the nonlinear viscosity easier to handle, we can rewrite (2.1) equivalently as

$$\begin{aligned} \boldsymbol{\sigma}^{\mathrm{d}} = 2\mu(|\mathbf{t}|)\mathbf{t} \quad \text{in} \quad \Omega, \quad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{t} = \nabla\mathbf{u} \quad \text{in} \quad \Omega, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma. \end{aligned} \tag{2.5}$$

Motivated by (2.4) and by the fact that $p \in \mathrm{L}_0^2(\Omega)$, in the sequel we consider the decomposition $\mathbb{H}(\mathbf{div}\,;\Omega) = \mathbb{H}_0(\mathbf{div}\,;\Omega) \oplus \mathrm{P}_0(\Omega)\mathbb{I}$, where $\mathbb{H}_0(\mathbf{div}\,;\Omega) := \{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega) : \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}) = 0\}$ and $\mathrm{P}_0(\Omega)$ is the space of constant polynomials defined on Ω. Furthermore, due to the impressibility condition, we let $\mathbb{L}^2_{\mathrm{tr}}(\Omega) := \{\mathbf{s} \in \mathbb{L}^2(\Omega) : \mathrm{tr}\,(\mathbf{s}) = 0 \quad \text{in} \quad \Omega\}$. Then, it is not difficult to obtain the following variational formulation of (2.5): Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}\,;\Omega) \times \mathbf{L}^2(\Omega)$ such that

$$2\int_\Omega \mu(|\mathbf{t}|)\mathbf{t} : \mathbf{s} - \int_\Omega \mathbf{s} : \boldsymbol{\sigma}^{\mathrm{d}} = 0 \qquad \forall\,\mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega), \tag{2.6a}$$

$$-\int_\Omega \mathbf{t} : \boldsymbol{\tau}^{\mathrm{d}} - \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} = -\langle\boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega), \tag{2.6b}$$

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} = \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v}\in\mathbf{L}^2(\Omega), \tag{2.6c}$$

whose well-posedness was established in [22, Section 2.2] via the abstract theory for twofold saddle point formulations (see, e.g., [18, Theorem 1] and [19, Theorem 2.4]).

3 The Galerkin scheme

In this section we introduce and analize the Galerkin approximation of (2.6). For simplicity, however, we restrict ourselves to the problem in two dimensions. The extension to three dimensions requires some technicalities that will be discussed in Section 7.

3.1 Preliminary results

In what follows we suppose that Γ is piecewise $\mathscr{C}^2$ and that Ω can be approximated by a family $\{\Omega_h\}_{h>0}$ of polygonal subdomains of Ω. We allow, in principle, any construction of Ω_h (not necessarily fitting the curved boundary Γ) and write $\Gamma_h := \partial\Omega_h$. In addition, the index h will refer to the size of a given triangulation $\mathcal{T}_h$ of $\overline{\Omega}_h$.

Let us discuss the link between the problem (2.6) and Ω_h. Applying integration by parts in (2.6) and using suitable test functions, one can recover the original system of equations given by (2.5), and hence, the solution of (2.6) satisfies in a distributional sense,

$$\boldsymbol{\sigma}^{\mathrm{d}} = 2\mu(|\mathbf{t}|)\mathbf{t} \quad \text{in} \quad \Omega_h, \quad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega_h, \quad \mathbf{t} = \nabla\mathbf{u} \quad \text{in} \quad \Omega_h. \tag{3.1}$$

Let now $\tilde{\mathbf{g}}$ be the trace of $\mathbf{u}$ on Γ_h. Proceeding as in [13] (see also [14]), we will employ a transferring technique to provide a more convenient expression for $\tilde{\mathbf{g}}$. Let $\boldsymbol{x}\in\Gamma_h$ and $\bar{\boldsymbol{x}}\in\Gamma$ be a point associated to $\bar{\boldsymbol{x}}$, i.e., $\bar{\boldsymbol{x}}(\boldsymbol{x})$. The precise construction of $\bar{\boldsymbol{x}}$ will be explained in Section 6. We denote by $\rho(\boldsymbol{x})$ the segment (often called transferring path) starting at $\boldsymbol{x}$ and ending at $\bar{\boldsymbol{x}}$, with unit tangent vector $\boldsymbol{m}(\boldsymbol{x})$ and length $|\rho(\boldsymbol{x})|$. Then, integrating the equation $\mathbf{t} = \nabla\mathbf{u}$ along $\rho(\boldsymbol{x})$, we obtain

$$\tilde{\mathbf{g}}(\boldsymbol{x}) := \bar{\mathbf{g}}(\boldsymbol{x}) - \int_0^{|\rho(\boldsymbol{x})|} \mathbf{t}(\boldsymbol{x} + \varepsilon\boldsymbol{m}(\boldsymbol{x}))\boldsymbol{m}(\boldsymbol{x})\,d\varepsilon, \tag{3.2}$$

where $\bar{\mathbf{g}}(\boldsymbol{x}) := \mathbf{g}(\bar{\boldsymbol{x}}(\boldsymbol{x}))$ and $\mathbf{g}$ is the boundary condition prescribed on Γ. In Section 3.3 we will comment on the main considerations to construct the transferring paths.

From (3.1) and (3.2) we find that the solution of (2.6) satisfies

$$2\int_{\Omega_h} \mu(|\mathbf{t}|)\mathbf{t} : \mathbf{s} - \int_{\Omega_h} \mathbf{s} : \boldsymbol{\sigma}^{\mathrm{d}} = 0 \qquad \forall\,\mathbf{s}\in\mathbb{L}^2(\Omega_h), \tag{3.3a}$$

$$-\int_{\Omega_h} \mathbf{t} : \boldsymbol{\tau} - \int_{\Omega_h} \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} = -\langle \boldsymbol{\tau}\boldsymbol{\nu}_{\Gamma_h}, \tilde{\mathbf{g}}\rangle_{\Gamma_h} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}\,;\Omega_h), \tag{3.3b}$$

$$-\int_{\Omega_h} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} = \int_{\Omega_h} \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v}\in\mathbf{L}^2(\Omega_h), \tag{3.3c}$$

and

$$\int_{\Omega_h} \mathrm{tr}\,(\boldsymbol{\sigma}) = -\int_{\Omega_h^c} \mathrm{tr}\,(\boldsymbol{\sigma}), \tag{3.3d}$$

where $\Omega_h^c := \Omega \setminus \overline{\Omega}_h$ and $\boldsymbol{\nu}_{\Gamma_h}$ denotes the outward unit normal vector on Ω_h.

Let now $\boldsymbol{\sigma}_0 \in \mathbb{H}(\mathbf{div}\,;\Omega_h)$ be given by

$$\boldsymbol{\sigma}_0 := \boldsymbol{\sigma} - \omega_{\boldsymbol{\sigma}}\mathbb{I}, \quad \text{with} \quad \omega_{\boldsymbol{\sigma}} := -\frac{1}{2|\Omega_h|}\int_{\Omega_h^c} \operatorname{tr}(\boldsymbol{\sigma}) \in \mathbb{R}. \tag{3.4}$$

Observe that $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}\,;\Omega_h)$ if and only if (3.3d) holds, and since $\boldsymbol{\sigma}_0^{\mathrm{d}} = \boldsymbol{\sigma}^{\mathrm{d}}$ and $\operatorname{div}\boldsymbol{\sigma}_0 = \operatorname{div}\boldsymbol{\sigma}$, equations (3.3a) and (3.3c) remain unchanged when $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$. Moreover, from (3.1) we have $\operatorname{tr}(\mathbf{t}) = 0$ in Ω_h, which implies that $\mathbf{t} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_h)$, and $\int_{\Gamma_h} \tilde{\mathbf{g}}\cdot\boldsymbol{\nu}_{\Gamma_h} = 0$. Consequently, the system (3.3) can be rewritten equivalently as

$$[\mathbf{A}_{1,h}(\mathbf{t}),\mathbf{s}] + [\mathbf{B}_{1,h}(\mathbf{s}),\boldsymbol{\sigma}_0] = 0 \qquad \forall\,\mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_h), \tag{3.5a}$$

$$[\mathbf{B}_{1,h}(\mathbf{t}),\boldsymbol{\tau}] + [\mathbf{B}_h(\boldsymbol{\tau}),\mathbf{u}] = [\mathbf{G}_h^{\mathbf{t}},\boldsymbol{\tau}] \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega_h), \tag{3.5b}$$

$$[\mathbf{B}_h(\boldsymbol{\sigma}_0),\mathbf{v}] = [\mathbf{F}_h,\mathbf{v}] \qquad \forall\,\mathbf{v} \in \mathbf{L}^2(\Omega_h), \tag{3.5c}$$

where the nonlinear operator $\mathbf{A}_{1,h} : \mathbb{L}^2_{\mathrm{tr}}(\Omega_h) \to \big[\mathbb{L}^2_{\mathrm{tr}}(\Omega_h)\big]'$, and the linear and bounded operators $\mathbf{B}_{1,h} : \mathbb{L}^2_{\mathrm{tr}}(\Omega_h) \to [\mathbb{H}_0(\mathbf{div}\,;\Omega_h)]'$ and $\mathbf{B}_h : \mathbb{H}_0(\mathbf{div}\,;\Omega_h) \to \big[\mathbf{L}^2(\Omega_h)\big]'$, are defined by

$$\begin{gathered}[\mathbf{A}_{1,h}(\mathbf{r}),\mathbf{s}] := 2\int_{\Omega_h} \mu(|\mathbf{r}|)\mathbf{r} : \mathbf{s}, \quad [\mathbf{B}_{1,h}(\mathbf{r}),\boldsymbol{\tau}] := -\int_{\Omega_h} \mathbf{r} : \boldsymbol{\tau}^{\mathrm{d}}, \\ \text{and} \quad [\mathbf{B}_h(\boldsymbol{\tau}),\mathbf{v}] := -\int_{\Omega_h} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau},\end{gathered} \tag{3.6}$$

and the linear functionals $\mathbf{F}_h \in \big[\mathbf{L}^2(\Omega_h)\big]'$ and $\mathbf{G}_h^{\mathbf{t}} \in [\mathbb{H}(\mathbf{div}\,;\Omega_h)]'$ by

$$[\mathbf{F}_h,\mathbf{v}] := \int_{\Omega_h} \mathbf{f}\cdot\mathbf{v} \quad \text{and} \quad [\mathbf{G}_h^{\mathbf{t}},\boldsymbol{\tau}] := -\langle \boldsymbol{\tau}\boldsymbol{\nu}_{\Gamma_h}, \tilde{\mathbf{g}}\rangle_{\Gamma_h}. \tag{3.7}$$

Above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators and functionals.

We end this section by specifying the boundedness properties of the operators and functionals appearing in (3.5). Using the Cauchy–Schwarz inequality, it follows that

$$\big|[\mathbf{B}_{1,h}(\mathbf{r}),\boldsymbol{\tau}]\big| \le \|\mathbf{r}\|_{0,\Omega_h}\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\Omega_h} \qquad \forall\,\mathbf{r} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_h), \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega_h), \tag{3.8}$$

$$\big|[\mathbf{B}_h(\boldsymbol{\tau}),\mathbf{v}]\big| \le \|\boldsymbol{\tau}\|_{\mathbf{div}\,;\Omega_h}\|\mathbf{v}\|_{0,\Omega_h} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega_h), \quad \forall\,\mathbf{v} \in \mathbf{L}^2(\Omega_h), \tag{3.9}$$

$$\big|[\mathbf{F}_h,\mathbf{v}]\big| \le \|\mathbf{f}\|_{0,\Omega}\|\mathbf{v}\|_{0,\Omega_h} \qquad \forall\,\mathbf{v} \in \mathbf{L}^2(\Omega_h), \tag{3.10}$$

and by the boundedness of the normal trace operator in $\mathbb{H}(\mathbf{div}\,;\Omega_h)$, there holds

$$\big|[\mathbf{G}_h^{\mathbf{t}},\boldsymbol{\tau}]\big| \le \|\tilde{\mathbf{g}}\|_{1/2,\Gamma_h}\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\Omega_h} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega_h).$$

Finally, we recall from [22, Lemma 2.1] that $\mathbf{A}_{1,h}$ is strongly monotone and Lipschitz continuous, that is, with the constants α_0 and γ_0 given by (2.2) and (2.3), respectively, we have

$$[\mathbf{A}_{1,h}(\mathbf{r}) - \mathbf{A}_{1,h}(\mathbf{s}),\mathbf{r}-\mathbf{s}] \ge 2\alpha_0\|\mathbf{r}-\mathbf{s}\|^2_{0,\Omega_h}, \tag{3.11}$$

and

$$\|\mathbf{A}_{1,h}(\mathbf{r}) - \mathbf{A}_{1,h}(\mathbf{s})\| \le 2\gamma_0\|\mathbf{r}-\mathbf{s}\|_{0,\Omega_h}, \tag{3.12}$$

for all $\mathbf{r},\mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_h)$.

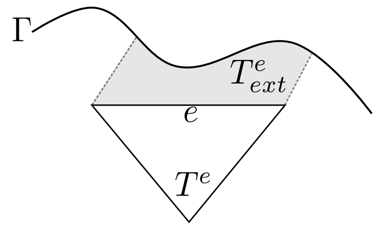

Figure 3.1: A two-dimensional example of T^e_{ext} is shown in gray.

3.2 Statement of the Galerkin scheme

In this section we propose a Galerkin approximation of the problem (3.5). We start by introducing some useful notation and definitions. Hereafter, $\mathcal{T}_h$ stands for a shape-regular triangulation of $\overline{\Omega}_h$ made of triangles T of diameter h_T, i.e., $h := \max\{h_T : T \in \mathcal{T}_h\}$. Moreover, given an integer $l \geq 0$ and a subset S of $\mathbb{R}^2$, we let $\mathrm{P}_l(S)$ (resp. $\tilde{\mathrm{P}}_l(S)$) denote the space of polynomials of degree at most l on S (resp. of degree equal to l on S). Then, for each integer $k \geq 0$ and for each $T \in \mathcal{T}_h$, we define the local Raviart–Thomas space of order k as $\mathbf{RT}_k(T) := [\mathrm{P}_k(T)]^2 \oplus \tilde{\mathrm{P}}_k(T)\boldsymbol{x}$, where $\boldsymbol{x} := (x_1, x_2)^{\mathrm{t}}$ is a generic vector of $\mathbb{R}^2$.

For each $T \in \mathcal{T}_h$, we denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h$. Then we write $\mathcal{E}_h = \mathcal{E}_h(\Omega_h) \cup \mathcal{E}_h(\Gamma_h)$, where $\mathcal{E}_h(\Omega_h) := \{e \in \mathcal{E}_h : e \subseteq \Omega_h\}$ and $\mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h : e \subseteq \Gamma_h\}$. Given $e \in \mathcal{E}_h(\Gamma)$, we associate an element T^e of $\mathcal{T}_h$ having e as an edge. In addition, for every $e \in \mathcal{E}_h(\Gamma)$, we let T^e_{ext} be the region delimited by e, the transferring paths associated to the vertices of e, and Γ (see an illustration in Figure 3.1). Finally, we let $\mathcal{T}_h^c$ denote the partition of Ω_h^c into all sets T^e_{ext}. Clearly, the latter makes sense if the paths from the vertices do not intersect each other and do not intersect the interior of the domain Ω_h, and if for a vertex $\boldsymbol{x} \in \Gamma$, $\bar{\boldsymbol{x}}(\boldsymbol{x})$ is uniquely defined (cf. Section 6), which will be assumed from now on.

We now define the finite element subspaces:

$$\mathbb{X}_{1,h}(\Omega_h) := \Big\{\mathbf{s}_h \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_h) : \quad \mathbf{s}_h|_T \in [\mathrm{P}_k(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h\Big\}, \tag{3.13}$$

$$\mathbb{M}_{1,h}(\Omega_h) := \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}\,;\Omega_h) : \quad \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau}_h|_T \in \mathbf{RT}_k(T) \quad \forall\, \mathbf{c} \in \mathbb{R}^2, \quad \forall\, T \in \mathcal{T}_h\Big\}, \tag{3.14}$$

$$\mathbf{M}_h(\Omega_h) := \Big\{\mathbf{v}_h \in \mathbf{L}^2(\Omega_h) : \quad \mathbf{v}_h|_T \in [\mathrm{P}_k(T)]^2 \quad \forall\, T \in \mathcal{T}_h\Big\}. \tag{3.15}$$

Then, letting $\mathbb{M}^0_{1,h}(\Omega_h) := \mathbb{M}_{1,h}(\Omega_h) \cap \mathbb{H}_0(\mathbf{div}\,;\Omega_h)$, the Galerkin scheme associated to the problem (3.5) reads: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_{0,h}, \mathbf{u}_h) \in \mathbb{X}_{1,h}(\Omega_h) \times \mathbb{M}^0_{1,h}(\Omega_h) \times \mathbf{M}_h(\Omega_h)$ such that

$$[\mathbf{A}_{1,h}(\mathbf{t}_h), \mathbf{s}_h] + [\mathbf{B}_{1,h}(\mathbf{s}_h), \boldsymbol{\sigma}_{0,h}] = 0 \qquad \forall\, \mathbf{s}_h \in \mathbb{X}_{1,h}(\Omega_h), \tag{3.16a}$$

$$[(\mathbf{B}_{1,h} + \mathbf{D}_h)(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathbf{B}_h(\boldsymbol{\tau}_h), \mathbf{u}_h] = [\mathbf{G}_h, \boldsymbol{\tau}_h] \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{M}^0_{1,h}(\Omega_h), \tag{3.16b}$$

$$[\mathbf{B}_h(\boldsymbol{\sigma}_{0,h}), \mathbf{v}_h] = [\mathbf{F}_h, \mathbf{v}_h] \qquad \forall\, \mathbf{v}_h \in \mathbf{M}_h(\Omega_h), \tag{3.16c}$$

where $\mathbf{A}_{1,h}$, $\mathbf{B}_{1,h}$, $\mathbf{B}_h$ and $\mathbf{F}_h$ are defined in (3.6) and (3.7), and

$$[\mathbf{D}_h(\mathbf{r}_h), \boldsymbol{\tau}_h] := -\sum_{e \in \mathcal{E}_h(\Gamma_h)} \int_e \left(\int_0^{|\rho(\boldsymbol{x})|} \boldsymbol{E}_h(\mathbf{r}_h)(\boldsymbol{x} + \varepsilon \boldsymbol{m}(\boldsymbol{x}))\boldsymbol{m}(\boldsymbol{x})\, d\varepsilon\right) \cdot (\boldsymbol{\tau}_h \boldsymbol{\nu}_e)(\boldsymbol{x})\, dS_{\boldsymbol{x}}, \tag{3.17}$$

and

$$[\mathbf{G}_h, \boldsymbol{\tau}_h] := - \sum_{e\in\mathcal{E}_h(\Gamma_h)} \int_e \bar{\mathbf{g}}(\boldsymbol{x}) \cdot (\boldsymbol{\tau}_h \boldsymbol{\nu}_e)(\boldsymbol{x})\, dS_{\boldsymbol{x}}, \tag{3.18}$$

for all $\mathbf{r}_h \in \mathbb{X}_{1,h}(\Omega_h)$ and for all $\boldsymbol{\tau}_h \in \mathbb{M}^0_{1,h}(\Omega_h)$, where $\boldsymbol{\nu}_e := (\boldsymbol{\nu}_{\Gamma_h})|_e$, $\bar{\mathbf{g}}(\boldsymbol{x}) = \mathbf{g}(\bar{\boldsymbol{x}}(\boldsymbol{x}))$, and $\boldsymbol{E}_h$ is the extrapolation operator defined for each integer $\ell \geq 0$ and for all $\boldsymbol{p} \in \prod_{T\in\mathcal{T}_h} [\mathrm{P}_\ell(T)]^2$ as

$$\boldsymbol{E}_h(\boldsymbol{p})(\boldsymbol{y}) := \begin{cases} \boldsymbol{p}(\boldsymbol{y}) & \forall\, \boldsymbol{y} \in T, \quad \forall\, T \in \mathcal{T}_h, \\ \boldsymbol{p}|_{T^e}(\boldsymbol{y}) & \forall\, \boldsymbol{y} \in T^e_{ext}, \quad \forall\, e \in \mathcal{E}_h(\Gamma_h). \end{cases} \tag{3.19}$$

Note that (3.16) can be seen in general (with only few exceptions, including the case of a polygonal domain $\Omega = \Omega_h$) as a perturbation of the standard approximation of the problem (2.6) due to the presence of the following approximation of $\tilde{\mathbf{g}}$ (cf. (3.2)):

$$\tilde{\mathbf{g}}_h(\boldsymbol{x}) := \bar{\mathbf{g}}(\boldsymbol{x}) - \int_0^{|\rho(\boldsymbol{x})|} \boldsymbol{E}_h(\mathbf{t}_h)(\boldsymbol{x} + \varepsilon \boldsymbol{m}(\boldsymbol{x}))\boldsymbol{m}(\boldsymbol{x})\, d\varepsilon \tag{3.20}$$

for all $e \in \mathcal{E}_h(\Gamma_h)$ and for each $\boldsymbol{x} \in e$. Therefore, the theory for twofold saddle point formulations cannot be straightforwardly applied here to ensure the well-posedness of (3.16). Nevertheless, this drawback can be overcome if we use a fixed point approach. More precisely, in what follows we reformulate (3.16) as a fixed point problem and show that its associated operator results to be a contraction mapping provided $\|\mathbf{D}_h\|$ is sufficiently small.

3.3 Well-posedness

We begin by introducing the aforementioned fixed point operator. In fact, we let

$$\mathcal{J}_h : \mathbb{X}_{1,h}(\Omega_h) \to \mathbb{X}_{1,h}(\Omega_h), \qquad \mathbf{r}_h \mapsto \mathcal{J}_h(\mathbf{r}_h) = \mathbf{t}_h,$$

where $\mathbf{t}_h \in \mathbb{X}_{1,h}(\Omega_h)$ is the first component of the solution of problem: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_{0,h}, \mathbf{u}_h) \in \mathbb{X}_{1,h}(\Omega_h) \times \mathbb{M}^0_{1,h}(\Omega_h) \times \mathbf{M}_h(\Omega_h)$, such that

$$[\mathbf{A}_{1,h}(\mathbf{t}_h), \mathbf{s}_h] + [\mathbf{B}_{1,h}(\mathbf{s}_h), \boldsymbol{\sigma}_{0,h}] = 0 \qquad \forall\, \mathbf{s}_h \in \mathbb{X}_{1,h}(\Omega_h), \tag{3.21a}$$

$$[\mathbf{B}_{1,h}(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathbf{B}_h(\boldsymbol{\tau}_h), \mathbf{u}_h] = [\mathbf{G}_h - \mathbf{D}_h(\mathbf{r}_h), \boldsymbol{\tau}_h] \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{M}^0_{1,h}(\Omega_h), \tag{3.21b}$$

$$[\mathbf{B}_h(\boldsymbol{\sigma}_{0,h}), \mathbf{v}_h] = [\mathbf{F}_h, \mathbf{v}_h] \qquad \forall\, \mathbf{v}_h \in \mathbf{M}_h(\Omega_h), \tag{3.21c}$$

and realize that proving the unique solvability of problem (3.16) is equivalent to proving the unique solvability of the fixed point problem: Find $\mathbf{t}_h \in \mathbb{X}_{1,h}(\Omega_h)$, such that

$$\mathcal{J}_h(\mathbf{t}_h) = \mathbf{t}_h. \tag{3.22}$$

Then, in what follows we apply the classical Banach fixed point theorem to prove existence and uniqueness of solution of problem (3.22). Before doing that, we must study the well-definiteness of the operator $\mathcal{J}_h$. To that end, we recall the following result taken from [18, Theorem 3] (see also [19, Theorem 2.2] or [20, Theorem 3.1]).

Theorem 3.1. *Let X_1, M_1 and M be Hilbert spaces, and let X_1', M_1', and M' be their respective dual spaces. Let $A_1 : X_1 \to X_1'$ be a nonlinear operator, and $B : M_1 \to M'$ and $B_1 : X_1 \to M_1'$ be linear and bounded operators. In turn, let $X_{1,h}$, $M_{1,h}$ and M_h be finite dimensional subspaces of X_1, M_1,* *and M, respectively. In addition, let $\widetilde{M}_{1,h} := \{\boldsymbol{\tau}_h \in M_{1,h} : \ [B(\boldsymbol{\tau}_h), \mathbf{v}_h] = 0 \quad \forall\, \mathbf{v}_h \in M_h\}$, define $V_{1,h} := \{\mathbf{s}_h \in X_{1,h} : \ [B_1(\mathbf{s}_h), \boldsymbol{\tau}_h] = 0 \quad \forall\, \boldsymbol{\tau}_h \in \widetilde{M}_{1,h}\}$, and let $\Pi_{1,h} : X'_{1,h} \to V'_{1,h}$ be the canonical imbedding. Finally, let $A_{1,h} := i_h^* A_1 : X_1 \to X'_{1,h}$, where $i_h : X_{1,h} \to X_1$ is the canonical injection with adjoint $i_h^* : X'_1 \to X'_{1,h}$, and assume that*

(i) The nonlinear operator $A_{1,h} : X_1 \to X'_{1,h}$ is Lipschitz–continuous with a Lipschitz constant $\gamma_h^* > 0$, and for any $\tilde{\mathbf{t}}_h \in X_{1,h}$, the nonlinear operator $\Pi_{1,h} A_{1,h}(\,\cdot\, + \tilde{\mathbf{t}}_h) : V_{1,h} \to V'_{1,h}$ is strongly monotone with a monotonicity constant $\alpha_h^* > 0$ independent of $\tilde{\mathbf{t}}_h$.

(ii) There exists $\beta_h^* > 0$ such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[B(\boldsymbol{\tau}_h), \mathbf{v}_h]}{\|\mathbf{v}_h\|_{X_1}} \geq \beta_h^* \|\mathbf{v}_h\|_M \qquad \forall\, \mathbf{v}_h \in M_h.$$

(iii) There exists $\beta_{1,h}^* > 0$ such that

$$\sup_{\substack{\mathbf{s}_h \in X_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[B_1(\mathbf{s}_h), \tau_h]}{\|\mathbf{s}_h\|_{X_1}} \geq \beta_{1,h}^* \|\boldsymbol{\tau}_h\|_{M_1} \qquad \forall\, \boldsymbol{\tau}_h \in \widetilde{M}_{1,h}.$$

Then, for each $(H, G, F) \in X'_1 \times M'_1 \times M'$ there exists a unique $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_{1,h} \times M_{1,h} \times M_h$, satisfying

$$[A_1(\mathbf{t}_h), \mathbf{s}_h] + [B_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] = [H, \mathbf{s}_h] \qquad \forall\, \mathbf{s}_h \in X_{1,h}, \tag{3.23a}$$

$$[B_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [B(\boldsymbol{\tau}_h), \mathbf{u}_h] = [G, \boldsymbol{\tau}_h] \qquad \forall\, \boldsymbol{\tau}_h \in M_{1,h}, \tag{3.23b}$$

$$[B(\boldsymbol{\sigma}_h), \mathbf{v}_h] = [F, \mathbf{v}_h] \qquad \forall\, \mathbf{v}_h \in M_h. \tag{3.23c}$$

Moreover, there exists $C_h > 0$, depending only on α_h^*, β_h^*, γ_h^* $\beta_{1,h}^*$ and $\|B_1\|$, such that

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leq C_h \Big(\|H|_{X_{1,h}}\| + \|F|_{M_h}\| + \|G|_{M_{1,h}}\| + \|A_{1,h}(\mathbf{0})\| \Big). \tag{3.24}$$

Proof. See [18, Theorem 3], [19, Theorem 2.2] or [20, Theorem 3.1]. □

According to the definition of the fixed point operator, it becomes clear that to prove the well-definitenes of $\mathcal{J}_h$ it suffices to prove the well-posedness of problem (3.21) by means of Theorem 3.1. We begin by noticing that the strong monotonicity and Lipschitz continuity of the nonlinear operator $\mathbf{A}_{1,h}$ also hold at the discrete level (see [22, Secton 2.4]). These imply the first hypothesis of Theorem 3.1 with monotonicity constant α_h^* and Lipschitz constant γ_h^* independent of h, because they actually coincide with those provided by (3.11) and (3.12), respectively. Furthermore, the boundedness of the linear operators $\mathbf{B}_{1,h}$ and $\mathbf{B}_h$, as well of the functional $\mathbf{F}_h$, are inherited from the continuous case with the same boundedness constants as in (3.8), (3.9) and (3.10), respectively. Next, to obtain the boundedness of the functional $\mathbf{G}_h - \mathbf{D}_h(\mathbf{r}_h)$ we proceed exactly as in [31, Lemma 3.1] and provide first an estimate for the linear operator $\mathbf{D}_h$. To that end, we need to introduce further notations and definitions. Given $e \in \mathcal{E}_h(\Gamma_h)$, we denote by $h_e^\perp$ the distance between the vertex of T^e, opposite to e, and the plane determined by e, and set $H_e := \max_{\boldsymbol{x} \in e} |\rho(\boldsymbol{x})|$ and $r_e := H_e / h_e^\perp$. Given an integer $\ell \geq 0$, we also set

$$C_{ext}^e := r_e^{-1/2} \sup_{\substack{\mathbf{r} \in [\mathrm{P}_\ell(T^e)]^{2\times 2} \\ \mathbf{r} \neq \mathbf{0}}} \frac{|||\boldsymbol{E}_h(\mathbf{r})|||_e}{\|\mathbf{r}\|_{0,T^e}} \quad \text{and} \quad C_{eq}^e := \sup_{\substack{\mathbf{p} \in [\mathrm{P}_\ell(\partial T^e)]^2 \\ \mathbf{p} \neq \mathbf{0}}} \frac{\|\mathbf{p}\|_{0,\partial T^e}}{\|\mathbf{p}\|_{-1/2,\partial T^e}}, \tag{3.25}$$

where $[\mathrm{P}_\ell(\partial T^e)]^2 := \prod_{\tilde{e}\subseteq\partial T^e}[\mathrm{P}_\ell(\tilde{e})]^2$ and

$$|\!|\!|\mathbf{w}|\!|\!|_e := \left(\int_e \int_0^{|\rho(\boldsymbol{x})|} |\mathbf{w}(\boldsymbol{x}+\varepsilon\boldsymbol{m}(\boldsymbol{x}))|^2 \,\mathrm{d}\varepsilon\,\mathrm{dS}_{\boldsymbol{x}}\right)^{1/2} \qquad \forall\,\mathbf{w}\in\mathbb{L}^2(T^e_{ext}).$$

We recall from [30] that the constant C^e_{ext} is independent of the meshsize h, but depend on the shape-regularity constant and the polynomial degree, and similarly for C^e_{eq} (see, e.g., [15, Lemma 3.2]). In this way, applying the Cauchy–Schwarz inequality and using the fact that $h^\perp_e \leq h_{T^e}$, we obtain, after some algebraic manipulations,

$$|[\mathbf{D}_h(\mathbf{r}_h),\boldsymbol{\tau}_h]| \leq \max_{e\in\mathcal{E}_h(\Gamma_h)}\left\{r_e C^e_{ext} C^e_{eq}\right\}\|\mathbf{r}_h\|_{0;\Omega_h}\|\boldsymbol{\tau}_h\|_{\mathbf{div}\,;\Omega_h}, \quad \forall\,\mathbf{r}_h\in\mathbb{L}^2_{\mathrm{tr}}(\Omega_h),\ \boldsymbol{\tau}_h\in\mathbb{H}_0(\mathbf{div}\,;\Omega_h), \tag{3.26}$$

which implies

$$\|\mathbf{D}_h\| \leq \max_{e\in\mathcal{E}_h(\Gamma_h)}\left\{r_e C^e_{ext} C^e_{eq}\right\}. \tag{3.27}$$

Above we do not know, a priori, how r_e varies with h. To control this quantity, we assume that

$$R := \max_{e\in\mathcal{E}_h(\Gamma_h)} r_e \leq C, \tag{3.28}$$

where $C > 0$ is independent of h. At the end of this section we will see that (3.28) makes sense.

Now, to bound $\mathbf{G}_h$ we assume that the mapping $\bar{\boldsymbol{x}}:\Gamma_h\to\Gamma$ (cf. Section 3.1) is continuous (this will be ensured by construction in Section 6) and apply the boundedness of the normal trace operator in $\mathbb{H}(\mathbf{div}\,;\Omega_h)$, to immediately find

$$|[\mathbf{G}_h,\boldsymbol{\tau}_h]| \leq \|\bar{\mathbf{g}}\|_{1/2,\Gamma_h}\|\boldsymbol{\tau}_h\|_{\mathbf{div}\,;\Omega_h} \qquad \forall\,\boldsymbol{\tau}_h\in\mathbb{H}(\mathbf{div}\,;\Omega_h), \tag{3.29}$$

which implies

$$\|\mathbf{G}_h\| \leq \|\bar{\mathbf{g}}\|_{1/2,\Gamma_h}. \tag{3.30}$$

From (3.26) and (3.29) it readily follows that the functional $\mathbf{G}_h - \mathbf{D}_h(\mathbf{r}_h)$ is continuous.

Finally, it is well-known that the following inf-sup conditions hold (see [22, Section 2.4]):

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{M}^0_{1,h}(\Omega_h)\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{[\mathbf{B}_h(\boldsymbol{\tau}_h),\mathbf{v}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}\,;\Omega_h}} \geq \beta\|\mathbf{v}_h\|_{0,\Omega_h} \qquad \forall\,\mathbf{v}_h\in\mathbf{M}_h(\Omega_h), \tag{3.31}$$

and

$$\sup_{\substack{\mathbf{s}_h\in\mathbb{X}_{1,h}(\Omega_h)\\ \mathbf{s}_h\neq\mathbf{0}}} \frac{[\mathbf{B}_{1,h}(\mathbf{s}_h),\boldsymbol{\tau}_h]}{\|\mathbf{s}_h\|_{0,\Omega_h}} \geq \beta_1\|\boldsymbol{\tau}_h\|_{\mathbf{div}\,;\Omega_h} \qquad \forall\,\boldsymbol{\tau}_h\in\mathbb{V}_h(\Omega_h), \tag{3.32}$$

where $\beta,\beta_1 > 0$ depend on $|\Omega_h|$, and

$$\begin{aligned}\mathbb{V}_h(\Omega_h) &:= \Big\{\boldsymbol{\tau}_h\in\mathbb{M}^0_{1,h}(\Omega_h): \quad [\mathbf{B}_h(\boldsymbol{\tau}_h),\mathbf{v}_h] = 0 \quad \forall\,\mathbf{v}_h\in\mathbf{M}_h(\Omega_h)\Big\}\\ &= \Big\{\boldsymbol{\tau}_h\in\mathbb{M}^0_{1,h}(\Omega_h): \quad \mathbf{div}\,\boldsymbol{\tau}_h = \mathbf{0} \quad \text{in} \quad \Omega_h\Big\}.\end{aligned} \tag{3.33}$$

From the above discussion, the well-posedness of problem (3.21), or equivalently the well-definiteness of $\mathcal{J}_h$, is a straightforward consequence of Theorem 3.1.

Now we turn to prove the Lipschitz continuity of $\mathcal{J}_h$.

Theorem 3.2. *There holds*

$$\|\mathcal{J}_h(\mathbf{r}_h) - \mathcal{J}_h(\mathbf{r}_h^*)\|_{0,\Omega_h} \le \frac{\gamma_0}{\alpha_0\beta_1}\|\mathbf{D}_h\|\,\|\mathbf{r}_h - \mathbf{r}_h^*\|_{0,\Omega_h} \qquad \forall\, \mathbf{r}_h, \mathbf{r}_h^* \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_h). \tag{3.34}$$

Proof. Let $\mathbf{r}_h, \mathbf{r}_h^*, \mathbf{t}_h, \mathbf{t}_h^* \in \mathbb{X}_{1,h}(\Omega_h)$ be such that $\mathcal{J}_h(\mathbf{r}_h) = \mathbf{t}_h$ and $\mathcal{J}_h(\mathbf{r}_h^*) = \mathbf{t}_h^*$. By definition of $\mathcal{J}_h$, it follows that there exist $\boldsymbol{\sigma}_{0,h}, \boldsymbol{\sigma}_{0,h}^* \in \mathbb{M}^0_{1,h}(\Omega_h)$ and $\mathbf{u}_h, \mathbf{u}_h^* \in \mathbf{M}_h(\Omega_h)$ such that $(\mathbf{t}_h, \boldsymbol{\sigma}_{0,h}, \mathbf{u}_h)$ and $(\mathbf{t}_h^*, \boldsymbol{\sigma}_{0,h}^*, \mathbf{u}_h^*)$ satisfy the problem (3.21), where on the right-hand side the operator $\mathbf{D}_h$ is evaluated at $\mathbf{r}_h$ and $\mathbf{r}_h^*$, respectively. We can therefore write

$$[\mathbf{A}_{1,h}(\mathbf{t}_h) - \mathbf{A}_{1,h}(\mathbf{t}_h^*), \mathbf{s}_h] + [\mathbf{B}_{1,h}(\mathbf{s}_h), \boldsymbol{\sigma}_{0,h} - \boldsymbol{\sigma}_{0,h}^*] = 0, \tag{3.35a}$$

$$[\mathbf{B}_{1,h}(\mathbf{t}_h - \mathbf{t}_h^*), \boldsymbol{\tau}_h] + [\mathbf{B}_h(\boldsymbol{\tau}_h), \mathbf{u}_h - \mathbf{u}_h^*] = [\mathbf{D}_h(\mathbf{r}_h^* - \mathbf{r}_h), \boldsymbol{\tau}_h], \tag{3.35b}$$

$$[\mathbf{B}_h(\boldsymbol{\sigma}_{0,h} - \boldsymbol{\sigma}_{0,h}^*), \mathbf{v}_h] = 0, \tag{3.35c}$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{X}_{1,h}(\Omega_h) \times \mathbb{M}^0_{1,h}(\Omega_h) \times \mathbf{M}_h(\Omega_h)$.

From (3.35c) we have $(\boldsymbol{\sigma}_{0,h} - \boldsymbol{\sigma}_{0,h}^*) \in \mathbb{V}_h(\Omega_h)$ (cf. (3.33)). Then, using the inf-sup condition in (3.32), equation (3.35a) and the Lipschitz continuity of $\mathbf{A}_{1,h}$ (cf. (3.12)), we obtain

$$\begin{aligned}
\beta_1\|\boldsymbol{\sigma}_{0,h} - \boldsymbol{\sigma}_{0,h}^*\|_{\mathbf{div};\Omega_h} &\le \sup_{\substack{\mathbf{s}_h \in \mathbb{X}_{1,h}(\Omega_h)\\ \mathbf{s}_h \neq \mathbf{0}}} \frac{\left|[\mathbf{B}_{1,h}(\mathbf{s}_h), \boldsymbol{\sigma}_{0,h} - \boldsymbol{\sigma}_{0,h}^*]\right|}{\|\mathbf{s}_h\|_{0,\Omega_h}} \\
&= \sup_{\substack{\mathbf{s}_h \in \mathbb{X}_{1,h}(\Omega_h)\\ \mathbf{s}_h \neq \mathbf{0}}} \frac{\left|[\mathbf{A}_{1,h}(\mathbf{t}_h) - \mathbf{A}_{1,h}(\mathbf{t}_h^*), \mathbf{s}_h]\right|}{\|\mathbf{s}_h\|_{0,\Omega_h}} \\
&\le 2\gamma_0\|\mathbf{t}_h - \mathbf{t}_h^*\|_{0,\Omega_h}.
\end{aligned} \tag{3.36}$$

On the other hand, using the strong monotonocity of $\mathbf{A}_{h,1}$ (cf. (3.11)), system (3.35) and the boundedness of $\mathbf{D}_h$, there holds

$$\begin{aligned}
2\alpha_0\|\mathbf{t}_h - \mathbf{t}_h^*\|^2_{0,\Omega_h} &\le \left[\mathbf{A}_{1,h}(\mathbf{t}_h) - \mathbf{A}_{1,h}(\mathbf{t}_h^*), \mathbf{t}_h - \mathbf{t}_h^*\right] = [\mathbf{D}_h(\mathbf{r}_h - \mathbf{r}_h^*), \boldsymbol{\sigma}_{0,h} - \boldsymbol{\sigma}_{0,h}^*] \\
&\le \|\mathbf{D}_h\|\|\mathbf{r}_h - \mathbf{r}_h^*\|_{0,\Omega_h}\|\boldsymbol{\sigma}_{0,h} - \boldsymbol{\sigma}_{0,h}^*\|_{\mathbf{div};\Omega_h}.
\end{aligned}$$

Combined with (3.36), this yields

$$\|\mathcal{J}_h(\mathbf{r}_h) - \mathcal{J}_h(\mathbf{r}_h^*)\|_{0,\Omega_h} = \|\mathbf{t}_h - \mathbf{t}_h^*\|_{0,\Omega_h} \le \frac{\gamma_0}{\alpha_0\beta_1}\|\mathbf{D}_h\|\,\|\mathbf{r}_h - \mathbf{r}_h^*\|_{0,\Omega_h},$$

which completes the proof. □

It is quite clear from Theorem 3.2 and estimate (3.27) that provided

$$\left(\frac{\gamma_0}{\alpha_0\beta_1}\right) \max_{e \in \mathcal{E}_h(\Gamma_h)} \left\{r_e C^e_{ext} C^e_{eq}\right\} < 1, \tag{3.37}$$

$\mathcal{J}_h$ becomes a contraction mapping.

We are now in a position of stating the well-posedness of the problem (3.16).

Theorem 3.3. *Assume that*

$$C_{wp} \max_{e\in\mathcal{E}_h(\Gamma_h)} \left\{r_e C^e_{ext} C^e_{eq}\right\} \le \frac{1}{2}, \tag{3.38}$$

with

$$C_{wp} := \frac{\gamma_0}{\alpha_0\beta_1} \max\left\{1, \frac{\gamma_0}{\alpha_0\beta_1}\right\}. \tag{3.39}$$

Then, there exists a unique $(\mathbf{t}_h, \boldsymbol{\sigma}_{0,h}, \mathbf{u}_h) \in \mathbb{X}_{1,h}(\Omega_h) \times \mathbb{M}^0_{1,h}(\Omega_h) \times \mathbf{M}_h(\Omega_h)$ *satisfying* (3.16)*. Moreover, there exists* $C > 0$*, independent of* h*, such that*

$$\|\mathbf{t}_h\|_{0,\Omega_h} + \|\boldsymbol{\sigma}_{0,h}\|_{\mathbf{div};\Omega_h} + \|\mathbf{u}_h\|_{0,\Omega_h} \le C\Big(\|\bar{\mathbf{g}}\|_{1/2,\Gamma_h} + \|\mathbf{f}\|_{0,\Omega_h}\Big). \tag{3.40}$$

Proof. Since assumption (3.38) implies (3.37), the unique solvability of the problem (3.16) follows from Theorem 3.2 and the Banach's fixed point theorem.

All that remains is to prove (3.40). To that end, we let $[\mathbb{V}_h(\Omega_h)]^\perp$ be the orthogonal complement of the space $\mathbb{V}_h(\Omega_h)$ given by (3.33). It follows that there exist $\tilde{\boldsymbol{\sigma}}_{0,h} \in \mathbb{V}_h(\Omega_h)$ and $\boldsymbol{\sigma}^\perp_{0,h} \in [\mathbb{V}_h(\Omega_h)]^\perp$ such that $\boldsymbol{\sigma}_{0,h} = \tilde{\boldsymbol{\sigma}}_{0,h} + \boldsymbol{\sigma}^\perp_{0,h}$. Then, considering the monotonicity and Lipschitz constants in (3.11) and (3.12), respectively, and the inf-sup conditions given by (3.31) and (3.32), we can proceed as in the proof of [20, Theorem 3.1] to obtain

$$\|\mathbf{u}_h\|_{0,\Omega_h} \le \frac{1}{\beta}\|\mathbf{G}_h - (\mathbf{B}_{1,h} + \mathbf{D}_h)(\mathbf{t}_h)\|, \tag{3.41}$$

$$\|\mathbf{t}_h\|_{0,\Omega_h} \le \frac{1}{2\alpha_0}\left(\|\boldsymbol{\sigma}^\perp_{0,h}\|_{\mathbf{div};\Omega_h} + \|\tilde{\boldsymbol{\sigma}}_{0,h}\|_{\mathbf{div};\Omega_h} + \|\mathbf{A}_{1,h}(\mathbf{0})\|\right), \tag{3.42}$$

$$\|\tilde{\boldsymbol{\sigma}}_{0,h}\|_{\mathbf{div};\Omega_h} \le \frac{2\gamma_0^2}{\alpha_0\beta_1^2}\left(\|\mathbf{G}_h - \mathbf{D}_h(\mathbf{t}_h)\| + \frac{1}{2\alpha_0}\left(\|\boldsymbol{\sigma}^\perp_{0,h}\|_{\mathbf{div};\Omega_h} + \|\mathbf{A}_{1,h}(\mathbf{0})\|\right)\right), \tag{3.43}$$

and

$$\|\boldsymbol{\sigma}^\perp_{0,h}\|_{\mathbf{div};\Omega_h} \le \frac{1}{\beta}\|\mathbf{F}_h\|. \tag{3.44}$$

Next, combining (3.43) and (3.44), using the boundedness of $\mathbf{D}_h$, noting that $\mathbf{A}_{1,h}(\mathbf{0})$ becomes the null operator, and finally using the estimate (3.42), we find

$$\|\tilde{\boldsymbol{\sigma}}_{0,h}\|_{\mathbf{div};\Omega_h} \le \frac{2\gamma_0^2}{\alpha_0\beta_1^2}\left(\|\mathbf{G}_h\| + \frac{1}{2\alpha_0}\|\mathbf{D}_h\|\|\tilde{\boldsymbol{\sigma}}_{0,h}\|_{\mathbf{div};\Omega_h} + \frac{1}{2\alpha_0\beta}\Big(1 + \|\mathbf{D}_h\|\Big)\|\mathbf{F}_h\|\right).$$

This, together with estimate (3.27) and assumption (3.38), gives

$$\|\tilde{\boldsymbol{\sigma}}_{0,h}\|_{\mathbf{div};\Omega_h} \le \frac{4\gamma_0^2}{\alpha_0\beta_1^2}\|\mathbf{G}_h\| + \frac{1}{\beta}\left(1 + \frac{2\gamma_0^2}{\alpha_0^2\beta_1^2}\right)\|\mathbf{F}_h\|. \tag{3.45}$$

Therefore, from (3.10), (3.30), (3.44) and (3.45), and by the triangle inequality, we get

$$\|\boldsymbol{\sigma}_{0,h}\|_{\mathbf{div};\Omega_h} \le \frac{4\gamma_0^2}{\alpha_0\beta_1^2}\|\bar{\mathbf{g}}\|_{1/2,\Gamma_h} + \frac{2}{\beta}\left(1 + \frac{\gamma_0^2}{\alpha_0^2\beta_1^2}\right)\|\mathbf{f}\|_{0,\Omega_h}.$$

Moreover, from (3.27), (3.41), (3.42), (3.44) and (3.45), it is immediate to see that

$$\|\mathbf{t}_h\|_{0,\Omega_h} \le \frac{2\gamma_0^2}{\alpha_0\beta_1^2}\|\bar{\mathbf{g}}\|_{1/2,\Gamma_h} + \frac{1}{\alpha_0\beta}\left(1 + \frac{\gamma_0^2}{\alpha_0^2\beta_1^2}\right)\|\mathbf{f}\|_{0,\Omega_h},$$

and

$$\|\mathbf{u}_h\|_{0,\Omega_h} \leq \frac{1}{\beta}\left(1+\frac{2\gamma_0^2}{\alpha_0\beta_1^2}\left(1+\frac{1}{2C_{wp}}\right)\right)\|\bar{\mathbf{g}}\|_{1/2,\Gamma_h} + \frac{1}{\alpha_0\beta^2}\left(1+\frac{\gamma_0^2}{\alpha_0^2\beta_1^2}\right)\left(1+\frac{1}{2C_{wp}}\right)\|\mathbf{f}\|_{0,\Omega_h},$$

where C_{wp} is defined in (3.39). This completes the proof. □

We end this section by noting that the closeness between Γ_h and Γ is critical in verifying the assumptions (3.28) and (3.38) providing the above results. Therefore, we consider two situations where we think that our method can be applied. The first is the scenario where Γ_h is constructed through a picewise linear interpolation of Γ. In this case $d(\Gamma,\Gamma_h)=\mathcal{O}(h^2)$, $R=\mathcal{O}(h)$ and (3.28) and (3.38) are clearly satisfied for h small enough. Alternatively, Ω can be immersed in a background mesh in order to set Ω_h as the union of all elements inside Ω. Now R is of order 1 and (3.28) holds, but we fail to ensure (3.38) since $d(\Gamma,\Gamma_h)=\mathcal{O}(h)$ only. Nevertheless, as we will see in Section 6, our numerical results suggest that (3.38) can be relaxed to fit also the latter case.

4 *A priori* error analysis

We now derive the *a priori* error estimates for the Galerkin scheme (3.16). To this end, we proceed as in [20] and employ a suitable Strang-type estimate for twofold saddle point formulations. Furthermore, by a postprocessing procedure, we provide the corresponding estimates for the pseudostress $\boldsymbol{\sigma}$ and the pressure p.

We will assume throughout the rest of this section that (3.28) and (3.38) hold true without stating them in the results.

4.1 Estimates on Ω_h

To alleviate the notation, hereafter we denote by $\vec{\mathbf{t}}_0$ and $\vec{\mathbf{t}}_{0,h}$ the solutions of the problems (3.5) and (3.16), respectively. In addition, let us consider the spaces

$$\mathbb{X}_0(\Omega_h) := \mathbb{L}^2_{\mathrm{tr}}(\Omega_h)\times\mathbb{H}_0(\mathbf{div}\,;\Omega_h)\times\mathbf{L}^2(\Omega_h),$$

and

$$\mathbb{X}_{0,h}(\Omega_h) := \mathbb{X}_{1,h}(\Omega_h)\times\mathbb{M}^0_{1,h}(\Omega_h)\times\mathbf{M}_h(\Omega_h),$$

and let $\mathbb{P}_h : \mathbb{X}_0(\Omega_h)\to[\mathbb{X}_0(\Omega_h)]'$ be the nonlinear operator obtained after adding the three equations on the left-hand side of the problem (3.5), that is,

$$[\mathbb{P}_h(\vec{\mathbf{r}}),\vec{\mathbf{s}}] := [\mathbf{A}_{1,h}(\mathbf{r}),\mathbf{s}] + [\mathbf{B}_{1,h}(\mathbf{s}),\boldsymbol{\rho}] + [\mathbf{B}_{1,h}(\mathbf{r}),\boldsymbol{\tau}] + [\mathbf{B}_h(\boldsymbol{\tau}),\mathbf{w}] + [\mathbf{B}_h(\boldsymbol{\rho}),\mathbf{v}] \tag{4.1}$$

for all $\vec{\mathbf{r}} := (\mathbf{r},\boldsymbol{\rho},\mathbf{w}), \vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau},\mathbf{v}) \in \mathbb{X}_0(\Omega_h)$. Then we can rewrite the problems (3.5) and (3.16) equivalently as

$$[\mathbb{P}_h(\vec{\mathbf{t}}_0),\vec{\mathbf{s}}] = [\mathbb{F}_h^{\vec{\mathbf{t}}_0},\vec{\mathbf{s}}] \qquad \forall\,\vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0(\Omega_h), \tag{4.2}$$

and

$$[\mathbb{P}_h(\vec{\mathbf{t}}_{0,h}),\vec{\mathbf{s}}_h] = [\mathbb{G}_h^{\vec{\mathbf{t}}_{0,h}},\vec{\mathbf{s}}_h] \qquad \forall\,\vec{\mathbf{s}}_h := (\mathbf{s}_h,\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{X}_{0,h}(\Omega_h). \tag{4.3}$$

where

$$[\mathbb{F}_h^{\vec{\mathbf{t}}_0},\vec{\mathbf{s}}] := [\mathbf{G}_h^{\mathbf{t}},\boldsymbol{\tau}] + [\mathbf{F}_h,\mathbf{v}], \tag{4.4}$$

and

$$[\mathbb{G}_h^{\vec{\mathbf{t}}_{0,h}}, \vec{\mathbf{s}}_h] := [\mathbf{G}_h - \mathbf{D}_h(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathbf{F}_h, \mathbf{v}_h]. \tag{4.5}$$

We now turn our attention to the Strang–type estimate for our Galerkin scheme (3.16), or equivalently for the problem (4.3). Our general strategy consists of mimicking the proofs of the results in [20].

We start by recalling the following lemma.

Lemma 4.1. *Let X_1 be a Hilbert space and let X_1' be its dual. Let $A_1 : X_1 \to X_1'$ be a nonlinear operator. Assume that $A_1 : X_1 \to X_1'$ is Lipschitz continuous with constant $\gamma^* > 0$ and strongly monotone with constant $\alpha^* > 0$. Suppose further that $A_1 : X_1 \to X_1'$ has a hemi-continuous first order Gâteaux derivative $\mathcal{D}A_1 : X_1 \to \mathcal{L}(X_1, X_1')$, that is, for any $\mathbf{s}, \mathbf{r} \in X_1$, the mapping $\mathbb{R} \ni \varepsilon \mapsto \mathcal{D}A_1(\mathbf{s} + \varepsilon\mathbf{r})(\mathbf{r}, \cdot) \in X_1'$ is continuous. Then, for any $\mathbf{q} \in X_1$, $\mathcal{D}A_1(\mathbf{q})$ is a bounded and X_1-elliptic form, with boundedness and ellipticity constants given by γ^* and α^*, respectively.*

Proof. See [20, Lemma 3.1]. □

Recall that in Section 2.2 we have assumed that μ is of class $\mathscr{C}^1$. In this case, it is not difficult to see that $\mathbf{A}_{1,h}$ (cf. (3.6)) verifies the hypotheses of Lemma 4.1, and hence, $\mathcal{D}\mathbf{A}_{1,h}(\mathbf{q})$ is a uniformly bounded and uniformly elliptic bilinear form on $\mathbb{L}^2_{\mathrm{tr}}(\Omega_h) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega_h)$ for all $\mathbf{q} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_h)$. Moreover, due to the linearity of the operators $\mathbf{B}_h$ and $\mathbf{B}_{1,h}$ defining $\mathbb{P}_h$, we immediately conclude, after replacing $[\mathbf{A}_{1,h}(\mathbf{t}), \mathbf{s}]$ in (4.1) by $\mathcal{D}\mathbf{A}_{1,h}(\mathbf{q})(\mathbf{t}, \mathbf{s})$, that for any $\vec{\mathbf{q}} := (\mathbf{q}, \boldsymbol{\tau}, \mathbf{v}) \in \mathbb{X}_0(\Omega_h)$, the first order Gâteaux derivative of $\mathbb{P}_h$ is given by

$$\begin{aligned}\mathcal{D}\mathbb{P}_h(\vec{\mathbf{q}})(\vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h) := \mathcal{D}\mathbf{A}_{1,h}(\mathbf{q})(\mathbf{r}_h, \mathbf{s}_h) + [\mathbf{B}_{1,h}(\mathbf{s}_h), \boldsymbol{\rho}_h] + [\mathbf{B}_{1,h}(\mathbf{r}_h), \boldsymbol{\tau}_h] \\ + [\mathbf{B}_h(\boldsymbol{\tau}_h), \mathbf{w}_h] + [\mathbf{B}_h(\boldsymbol{\rho}_h), \mathbf{v}_h]\end{aligned}$$

for all $\vec{\mathbf{r}}_h := (\mathbf{r}_h, \boldsymbol{\rho}_h, \mathbf{w}_h), \vec{\mathbf{s}}_h := (\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{X}_{0,h}(\Omega_h)$.

Next, due to the inf-sup conditions for $\mathbf{B}_h$ and $\mathbf{B}_{1,h}$ provided by (3.31) and (3.32), respectively, and by the already mentioned properties of $\mathcal{D}\mathbf{A}_{1,h}(\mathbf{q})$, it follows that the hypotheses of the linear version of Theorem 3.1 (see [20, Theorem 3.2] for details) are satisfied by the problem obtained after replacing $[\mathbb{P}_h(\vec{\mathbf{t}}_{0,h}), \vec{\mathbf{s}}_h]$ in (4.3) by $\mathcal{D}\mathbb{P}_h(\vec{\mathbf{q}})(\vec{\mathbf{t}}_{0,h}, \vec{\mathbf{s}}_h)$. Therefore, the corresponding continuous dependence result implies that the following global inf-sup condition holds:

$$C_{glob} \|\vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} \leq \sup_{\substack{\vec{\mathbf{s}}_h \in \mathbb{X}_{0,h}(\Omega_h) \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\mathcal{D}\mathbb{P}_h(\vec{\mathbf{q}})(\vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathbb{X}_0(\Omega_h)}} \qquad \forall\, \vec{\mathbf{r}}_h \in \mathbb{X}_{0,h}(\Omega_h), \tag{4.6}$$

with $C_{glob} > 0$ independent of h.

We have then the following intermediate result.

Lemma 4.2. *There exists a constant $C_S > 0$, independent of h, such that*

$$\begin{aligned}&\|\vec{\mathbf{t}}_0 - \vec{\mathbf{t}}_{0,h}\|_{\mathbb{X}_0(\Omega_h)} \\ &\quad \leq C_S \left(\mathbb{T}_0^{\vec{\mathbf{t}}_0} + \inf_{\vec{\mathbf{r}}_h \in \mathbb{X}_{0,h}(\Omega_h)} \|\vec{\mathbf{t}}_0 - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} + \inf_{\mathbf{r}_h \in \mathbb{X}_h(\Omega_h)} \sum_{e \in \mathcal{E}_h(\Gamma_h)} \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h)|\!\|_e \right),\end{aligned} \tag{4.7}$$

where

$$\mathbb{T}_0^{\vec{\mathbf{t}}_0} := \sup_{\substack{\vec{\mathbf{s}}_h \in \mathbb{X}_{0,h}(\Omega_h) \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\left|[\mathbb{F}_h^{\vec{\mathbf{t}}_0}, \vec{\mathbf{s}}_h] - [\mathbb{G}_h^{\vec{\mathbf{t}}_{0,h}}, \vec{\mathbf{s}}_h]\right|}{\|\vec{\mathbf{s}}_h\|_{\mathbb{X}_0(\Omega_h)}}. \tag{4.8}$$

Proof. Note that if we prove the existence of a positive constant C, independent of h, such that for all $\vec{\mathbf{r}}_h \in \mathbb{X}_{0,h}(\Omega_h)$,

$$\|\vec{\mathbf{t}}_{0,h} - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} \leq C \left(\mathbb{T}_0^{\vec{\mathbf{t}}_0} + \|\vec{\mathbf{t}}_0 - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} + \inf_{\mathbf{r}_h \in \mathbb{X}_h(\Omega_h)} \sum_{e \in \mathcal{E}_h(\Gamma_h)} \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h)|\!\|_e \right), \tag{4.9}$$

then one could simply use the triangle inequality to obtain (4.7). Therefore, in the sequel we focus on proving (4.9) and proceed as in [20, Theorem 3.3] (see also [8, Lemma 5.1]).

We begin by noting that the hemi-continuity of $\mathcal{D}\mathbf{A}_{1,h}$ yields the same property for $\mathcal{D}\mathbb{P}_h$. Then, given $\vec{\mathbf{r}} := (\mathbf{r}_h, \boldsymbol{\rho}_h, \mathbf{w}_h), \vec{\mathbf{s}}_h := (\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{X}_{0,h}(\Omega_h)$, there exists $\varepsilon_0 \in (0,1)$, satisfying

$$\begin{aligned} [\mathbb{P}_h(\vec{\mathbf{t}}_{0,h}), \vec{\mathbf{s}}_h] - [\mathbb{P}_h(\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h] &= \int_0^1 \frac{d}{d\varepsilon} \Big([\mathbb{P}_h(\varepsilon \vec{\mathbf{t}}_{0,h} + (1-\varepsilon)\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h] \Big)\, d\varepsilon \\ &= \mathcal{D}\mathbb{P}_h(\varepsilon_0 \vec{\mathbf{t}}_{0,h} + (1-\varepsilon_0)\vec{\mathbf{r}}_h)(\vec{\mathbf{t}}_{0,h} - \vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h). \end{aligned} \tag{4.10}$$

Applying now the global inf-sup condition in (4.6) to $\|\vec{\mathbf{t}}_{0,h} - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)}$, taking $\vec{\mathbf{q}} := \varepsilon_0 \vec{\mathbf{t}}_{0,h} + (1-\varepsilon_0)\vec{\mathbf{r}}_h$, and using (4.10), we find

$$C_{glob} \|\vec{\mathbf{t}}_{0,h} - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} \leq \sup_{\substack{\vec{\mathbf{s}}_h \in \mathbb{X}_{0,h}(\Omega_h) \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{[\mathbb{P}_h(\vec{\mathbf{t}}_{0,h}), \vec{\mathbf{s}}_h] - [\mathbb{P}_h(\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h]}{\|\vec{\mathbf{s}}_h\|_{\mathbb{X}_0(\Omega_h)}}. \tag{4.11}$$

Furthermore, by (4.2) and (4.3), adding and subtracting convenient terms, it follows that

$$[\mathbb{P}_h(\vec{\mathbf{t}}_{0,h}), \vec{\mathbf{s}}_h] - [\mathbb{P}_h(\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h] = [\mathbb{G}_h^{\vec{\mathbf{t}}_{0,h}}, \vec{\mathbf{s}}_h] - [\mathbb{F}_h^{\vec{\mathbf{t}}_0}, \vec{\mathbf{s}}_h] + [\mathbb{F}_h^{\vec{\mathbf{t}}_0}, \vec{\mathbf{s}}_h] - [\mathbb{G}_h^{\vec{\mathbf{r}}_h}, \vec{\mathbf{s}}_h].$$

Combined with (4.11) this yields

$$\|\vec{\mathbf{t}}_0 - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} \leq \frac{1}{C_{glob}} \Big(\mathbb{T}_0^{\vec{\mathbf{t}}_0} + \mathbb{T}_1^{\vec{\mathbf{t}}_0} \Big), \tag{4.12}$$

where $\mathbb{T}_0^{\vec{\mathbf{t}}_0}$ is defined in (4.8) and

$$\mathbb{T}_1^{\vec{\mathbf{t}}_0} := \sup_{\substack{\vec{\mathbf{s}}_h \in \mathbb{X}_{0,h}(\Omega_h) \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\big| [\mathbb{F}_h^{\vec{\mathbf{t}}_0}, \vec{\mathbf{s}}_h] - [\mathbb{G}_h^{\vec{\mathbf{r}}_h}, \vec{\mathbf{s}}_h] \big|}{\|\vec{\mathbf{s}}_h\|_{\mathbb{X}_0(\Omega_h)}}.$$

All that remains is to estimate $\mathbb{T}_1^{\vec{\mathbf{t}}_0}$. The approach of [30, Section 2.4] is useful for this purpose. Let $\boldsymbol{\zeta}$ be defined for each $\boldsymbol{x} \in e$ and for any $\mathbf{r}_h \in \mathbb{X}_{1,h}(\Omega_h)$ (cf. (3.13)) as

$$\boldsymbol{\zeta}(\boldsymbol{x}) := \int_0^{|\rho(\boldsymbol{x})|} (\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h))\,(\boldsymbol{x} + \varepsilon \boldsymbol{m}(\boldsymbol{x}))\boldsymbol{m}(\boldsymbol{x})\, d\varepsilon,$$

and note, by definition of the functionals $\mathbb{F}_h^{\vec{\mathbf{t}}_0}$ and $\mathbb{G}_h^{\vec{\mathbf{r}}_h}$ (cf. (4.4) and (4.5), respectively), that

$$[\mathbb{F}_h^{\vec{\mathbf{t}}_0}, \vec{\mathbf{s}}_h] - [\mathbb{G}_h^{\vec{\mathbf{r}}_h}, \vec{\mathbf{s}}_h] = -\int_{\Gamma_h} \boldsymbol{\zeta}(\boldsymbol{x}) \cdot (\boldsymbol{\tau}_h \boldsymbol{\nu}_{\Gamma_h})(\boldsymbol{x})\, dS_{\boldsymbol{x}}.$$

Then, applying the Cauchy–Schwarz inequality, and using the constant C_{eq}^e of (3.25), we have

$$\left|[\mathbb{F}_h^{\vec{\mathbf{t}}_0}, \vec{\mathbf{s}}_h] - [\mathbb{G}_h^{\vec{\mathbf{r}}_h}, \vec{\mathbf{s}}_h]\right| \leq \sum_{e \in \mathcal{E}_h(\Gamma_h)} C_{eq}^e h_{T^e}^{-1/2} \|\boldsymbol{\zeta}\|_{0,e} \|\boldsymbol{\tau}_h\|_{\mathbf{div}\,;T^e},$$

where T^e denotes an element of $\mathcal{T}_h$ having e as an edge. To estimate $\|\boldsymbol{\zeta}\|_{0,e}$ above, we can apply the Cauchy–Schwarz inequality and use the fact that $h_e^\perp \leq h_{T^e}$. This yields

$$\|\boldsymbol{\zeta}\|_{0,e}^2 \leq H_e \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h)|\!\|_e^2 \leq r_e h_{T^e} \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h)|\!\|_e^2,$$

providing

$$\mathbb{T}_1^{\vec{\mathbf{t}}_0} \leq \sum_{e \in \mathcal{E}_h(\Gamma_h)} (r_e)^{1/2} C_{eq}^e \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h)|\!\|_e. \tag{4.13}$$

The result (4.9) follows by combining (4.12) and (4.13). □

Having proved Lemma 4.2, it is clear that in order to derive the Strang-type estimate of our Galerkin scheme (3.16), we only need to bound $\mathbb{T}_0^{\vec{\mathbf{t}}_0}$. The main result of this section is the following.

Theorem 4.3. *Assume that*

$$C_S \max_{e \in \mathcal{E}_h(\Gamma_h)} \left\{ r_e C_{ext}^e C_{eq}^e \right\} \leq \frac{1}{2}, \tag{4.14}$$

where C_S is the constant appearing in (4.7)*. Then, there exists $C > 0$, independent of h, such that*

$$\|\vec{\mathbf{t}}_0 - \vec{\mathbf{t}}_{0,h}\|_{\mathbb{X}_0(\Omega_h)} \leq C \left(\inf_{\vec{\mathbf{r}}_h \in \mathbb{X}_{0,h}(\Omega_h)} \|\vec{\mathbf{t}}_0 - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} + \inf_{\mathbf{r}_h \in \mathbb{X}_{1,h}(\Omega_h)} \sum_{e \in \mathcal{E}_h(\Gamma_h)} \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h)|\!\|_e \right). \tag{4.15}$$

Proof. First, applying similar arguments as in the proof of (4.13), it is immediate to see that

$$\mathbb{T}_0^{\vec{\mathbf{t}}_0} \leq \sum_{e \in \mathcal{E}_h(\Gamma_h)} (r_e)^{1/2} C_{eq}^e \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{t}_h)|\!\|_e.$$

Let $\mathbf{r}_h \in \mathbb{X}_{1,h}(\Omega_h)$. Adding and subtracting $\boldsymbol{E}_h(\mathbf{r}_h)$, using the constant C_{ext}^e (cf. (3.25)) to bound the norm $\|\!|\cdot|\!\|_e$, and assumption (4.14), there holds

$$\begin{aligned} \mathbb{T}_0^{\vec{\mathbf{t}}_0} &\leq \sum_{e \in \mathcal{E}_h(\Gamma_h)} (r_e)^{1/2} C_{eq}^e \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{t}_h)|\!\|_e \\ &\leq \sum_{e \in \mathcal{E}_h(\Gamma_h)} (r_e)^{1/2} C_{eq}^e \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h)|\!\|_e + \frac{1}{2C_{\mathrm{S}}} \|\mathbf{t}_h - \mathbf{r}_h\|_{0,\Omega_h}, \end{aligned}$$

from which, adding and subtracting $\mathbf{t}_0$, we have

$$\mathbb{T}_0^{\vec{\mathbf{t}}_0} \leq \sum_{e \in \mathcal{E}_h(\Gamma_h)} (r_e)^{1/2} C_{eq}^e \|\!|\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h)|\!\|_e + \frac{1}{2C_S} \Big(\|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega_h} + \|\mathbf{t} - \mathbf{r}_h\|_{0,\Omega_h} \Big). \tag{4.16}$$

The proof ends by combining (4.16) and (4.7), and observing that $\mathbf{r}_h$ is arbitrary. □

4.2 Estimates by postprocessing

Besides the variables approximated by the Galerkin scheme (3.16), with our approach we can also approximate the pressure p and the pseudostress $\boldsymbol{\sigma}$. To that end, we proceed as in [31] and present here postprocessing formulae according to definition (3.4) and the second equation of (2.4). More precisely, given $(\mathbf{t}_h, \boldsymbol{\sigma}_{0,h}, \mathbf{u}_h) \in \mathbb{X}_{1,h}(\Omega_h) \times \mathbb{M}^0_{1,h}(\Omega_h) \times \mathbf{M}_h(\Omega_h)$ the unique solution of (3.16), we propose to approximate $\boldsymbol{\sigma}$ and p by

$$\boldsymbol{\sigma}_h := \boldsymbol{\sigma}_{0,h} + \omega_{\boldsymbol{\sigma}_h}\mathbb{I}, \tag{4.17}$$

and

$$p_h := -\frac{1}{2}\mathrm{tr}\,(\boldsymbol{\sigma}_h), \tag{4.18}$$

where

$$\omega_{\boldsymbol{\sigma}_h} := -\frac{1}{2|\Omega_h|}\int_{\Omega_h^c} \mathrm{tr}\,\left(\boldsymbol{\sigma}_{0,h} - \left(\frac{1}{2|\Omega|}\int_{\Omega_h^c}\mathrm{tr}\,(\boldsymbol{\sigma}_{0,h})\right)\mathbb{I}\right). \tag{4.19}$$

Above we use the extrapolation operator (cf. (3.19)) whenever an evaluation of $\boldsymbol{\sigma}_{0,h}$ on Ω_h^c is required.

It is not difficult to check that the following identities hold:

$$-\,2|\Omega_h|\omega_{\boldsymbol{\sigma}_h} = \int_{\Omega_h^c}\mathrm{tr}\,(\boldsymbol{\sigma}_h) \quad \text{and} \quad 2|\Omega_h|\omega_{\boldsymbol{\sigma}_h} = \int_{\Omega_h}\mathrm{tr}\,(\boldsymbol{\sigma}_h). \tag{4.20}$$

We furthermore note that the normal component of $\boldsymbol{\sigma}_h$ is in general discontinuous across the transferring paths from the vertices of Γ_h to their corresponding points in Γ. Therefore, for the subsequent analysis we consider that $\boldsymbol{\sigma}_h$ on Ω_h^c belongs in the broken Sobolev space

$$\mathbb{H}(\mathbf{div}\,;\mathcal{T}_h^c) := \prod_{e\in\mathcal{E}_h(\Gamma_h)}\mathbb{H}(\mathbf{div}\,;T^e_{ext}), \tag{4.21}$$

equipped with the norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\mathcal{T}_h^c} := \left(\sum_{e\in\mathcal{E}_h(\Gamma_h)}\|\boldsymbol{\tau}\|^2_{\mathbf{div}\,;T^e_{ext}}\right)^{1/2} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}\,;\mathcal{T}_h^c).$$

The following result will be of paramount importance to derive the error estimation of our postprocessing technique.

Lemma 4.4. *Suppose that there exists an integer $\ell \geq 0$ such that $\mathbf{t}\in\mathbb{H}^{\ell+1}(\Omega)$ and $\boldsymbol{\sigma}\in\mathbb{H}^{\ell+1}(\Omega)$, with $\mathrm{div}\,\boldsymbol{\sigma}\in\mathbf{H}^{\ell+1}(\Omega)$. Then, there exist positive constants C_1 and C_2, independent of h, such that for any $\mathbf{r}_h\in\mathbb{X}_{1,h}(\Omega_h)$ and for any $\boldsymbol{\zeta}_h\in\mathbb{M}_{1,h}(\Omega_h)$,*

$$\|\mathbf{t} - \boldsymbol{E}_h(\mathbf{r}_h)\|_{0,\Omega_h^c} \leq C_1\left(\|\mathbf{t}-\mathbf{r}_h\|_{0,\Omega_h} + h^{\ell+1}\|\mathbf{t}\|_{\ell+1,\Omega}\right), \tag{4.22}$$

and

$$\|\boldsymbol{\sigma} - \boldsymbol{E}_h(\boldsymbol{\zeta}_h)\|_{\mathbf{div}\,;\mathcal{T}_h^c} \leq C_2\left(\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{\mathbf{div}\,;\Omega_h} + h^{\ell+1}\|\boldsymbol{\sigma}\|_{\ell+1,\Omega} + h^{\ell+1}\|\mathbf{div}\,\boldsymbol{\sigma}\|_{\ell+1,\Omega}\right). \tag{4.23}$$

Furthermore, if $d(\Gamma,\Gamma_h) = \mathcal{O}(h^s)$, with $s\geq 1$, then there exists a constant $C_3>0$, independent of h, such that

$$\sum_{e\in\mathcal{E}_h(\Gamma_h)}|T^e_{ext}|\,\|\boldsymbol{\sigma} - \boldsymbol{E}_h(\boldsymbol{\zeta}_h)\|_{0,T^e_{ext}} \leq C_3\left(h^{s+1}\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{0,\Omega_h} + h^{\ell+s+2}\|\boldsymbol{\sigma}\|_{\ell+1,\Omega}\right). \tag{4.24}$$

Proof. We refer to [30, Lemma 3.5] for the proofs of (4.22) and (4.23).

On the other hand, it is not difficult to see that the proof of (4.24) is similar to that of (4.22), except that now higher order terms appear from the fact that $|T^e_{ext}| = \mathcal{O}(h^{s+1})$ when $d(\Gamma, \Gamma_h) = \mathcal{O}(h^s)$. $\square$

Before proceeding any further, let us specify the family of transferring paths $\{\rho(\boldsymbol{x})\}_{\boldsymbol{x}\in\Gamma_h}$ connecting Γ_h and Γ. Given $e \in \mathcal{E}_h(\Gamma_h)$, we let $\{\boldsymbol{p}_1, \boldsymbol{p}_2\}$ be the set of all vertices of e. To each of them, we assign a unique point in the boundary Γ, denoted by $\tilde{\boldsymbol{p}}_1$ and $\tilde{\boldsymbol{p}}_2$, respectively. In Section 6 we will guaranty that $\tilde{\boldsymbol{p}}_i$ $(i = 1, 2)$ can be obtained in such a way that $\rho(\boldsymbol{p}_i)$ satisfies the assumptions made in Section 3.2. Let $\widehat{\boldsymbol{m}}^{\boldsymbol{p}_i} := \tilde{\boldsymbol{p}}_i - \boldsymbol{p}_i$. We then set $\boldsymbol{m}_i := \widehat{\boldsymbol{m}}^{\boldsymbol{p}_i}/|\widehat{\boldsymbol{m}}^{\boldsymbol{p}_i}|$ if $\widehat{\boldsymbol{m}}^{\boldsymbol{p}_i} \neq 0$ and $\boldsymbol{m}^{\boldsymbol{p}_i} := \boldsymbol{\nu}_e$, otherwise. Given $e \in \mathcal{E}_h(\Gamma_h)$, the transferring path $\rho(\boldsymbol{x})$ is determined as a convex combination of $\rho(\boldsymbol{p}_1)$ and $\rho(\boldsymbol{p}_2)$. More precisely, for any $\theta \in [0, 1]$, we fix a point $\boldsymbol{x}(\theta) := \boldsymbol{p}_1 + \theta(\boldsymbol{p}_2 - \boldsymbol{p}_1)$ on e, and let $\boldsymbol{m}(\theta) := \boldsymbol{m}^{\boldsymbol{p}_1} + \theta(\boldsymbol{m}^{\boldsymbol{p}_2} - \boldsymbol{m}^{\boldsymbol{p}_2})$ be the tangent vector to the transferring path associated to $\boldsymbol{x}(\theta)$. Finally, by setting $c(\theta) := |\widehat{\boldsymbol{m}}(\theta)|$ if $\widehat{\boldsymbol{m}}(\theta) \neq \boldsymbol{0}$ and $c(\theta) = 1$, otherwise, we let $\boldsymbol{m}(\theta) := \widehat{\boldsymbol{m}}(\theta)/c(\theta)$.

Then we recall from [30, Lemma 3.4] the following useful result.

Lemma 4.5. *Let $\mathbf{r} \in \mathbb{L}^2(T^e_{ext})$ and consider the following conditions:*

(i) $\boldsymbol{m}^{\boldsymbol{p}_1} \cdot \boldsymbol{m}^{\boldsymbol{p}_2} \geq 0$,

(ii) there exists a constant δ_e, independent of h, such that $\boldsymbol{m}(\theta) \cdot \boldsymbol{\nu}_e \geq \delta_e > 0$ for all $\theta \in [0, 1]$; and

(iii) $\boldsymbol{m}^{\boldsymbol{p}_1} \cdot (\boldsymbol{m}^{\boldsymbol{p}_2})^\perp \geq 0$, where $(\boldsymbol{m}^{\boldsymbol{p}_2})^\perp$ is the image of $\boldsymbol{m}^{\boldsymbol{p}_2}$ under a $\pi/2$ counterclockwise rotation about the origin.

If (i) holds, then there exists $C^e_1 > 0$, independent of h, such that $\|\mathbf{r}\|_{0,T^e_{ext}} \leq C^e_1 |||\mathbf{r}|||_e$. Moreover, if (ii) and (iii) are satisfied, then $|||\mathbf{r}|||_e \leq C^e_2 \|\mathbf{r}\|_{0,T^e_{ext}}$, where $C^e_2 > 0$ is also independent of h.

We are now in a position to present the error estimate of our postprocessing technique.

Lemma 4.6. *Assume that the hypotheses of Lemma 4.4 hold. Suppose further that*

$$\max_{e\in\mathcal{E}_h(\Gamma_h)} \left\{ |T^e_{ext}| (r_e)^{1/2} C^e_{ext} C^e_1 \right\} \leq \frac{1}{2} |\Omega_h|^{1/2}. \tag{4.25}$$

Then, there exists a constant $C > 0$, independent of h, such

$$\begin{aligned} &\|p - p_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega_h} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\mathcal{T}^c_h} \\ &\quad \leq C \inf_{\vec{\mathbf{r}}_h \in \mathbb{X}_{0,h}(\Omega_h)} \|\vec{\mathbf{t}}_0 - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} + C h^{\ell+1} \Big(\|\mathbf{t}\|_{\ell+1,\Omega} + \|\boldsymbol{\sigma}\|_{\ell+1,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{\ell+1,\Omega} \Big). \end{aligned} \tag{4.26}$$

Proof. Adding and subtracting convenient terms, using the constant C^e_{ext} of (3.25) and applying the estimate (4.23), it follows that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\mathcal{T}^c_h} \leq \widehat{C}_1 \Big(\|\boldsymbol{\sigma} - \boldsymbol{\zeta}_h\|_{\mathbf{div};\Omega_h} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega_h} + h^{\ell+1}\|\boldsymbol{\sigma}\|_{\ell+1,\Omega} + h^{\ell+1}\|\mathbf{div}\,\boldsymbol{\sigma}\|_{\ell+1,\Omega} \Big) \tag{4.27}$$

for all $\boldsymbol{\zeta}_h \in \mathbb{M}_{1,h}(\Omega_h)$. Furthermore, by (3.4), (4.17) and (4.20),

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega_h} &\leq \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{0,h}\|_{0,\Omega_h} + 2^{1/2}|\Omega_h|^{1/2}|\omega_{\boldsymbol{\sigma}} - \omega_{\boldsymbol{\sigma}_h}| \\ &= \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{0,h}\|_{0,\Omega_h} + \frac{1}{2^{1/2}|\Omega_h|^{1/2}} \left| \int_{\Omega^c_h} \operatorname{tr}\Big(\boldsymbol{\sigma} - \boldsymbol{E}_h(\boldsymbol{\sigma}_h)\Big) \right| \\ &\leq \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{0,h}\|_{0,\Omega_h} + \frac{1}{|\Omega_h|^{1/2}} \sum_{e\in\mathcal{E}_h(\Gamma_h)} |T^e_{ext}|\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T^e_{ext}}. \end{aligned}$$

Then, using the identity $\mathbf{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)=\mathbf{div}\,(\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_{0,h})$, we find

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega_h}\le\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_{0,h}\|_{\mathbf{div};\Omega_h}+\frac{1}{|\Omega_h|^{1/2}}\sum_{e\in\mathcal{E}_h(\Gamma_h)}|T^e_{ext}|\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T^e_{ext}}. \tag{4.28}$$

Similar to the steps in the proof of (4.23) (see [30, Lemma 3.5]) we can apply Lemma 4.5 and use the extrapolation constant C^e_{ext} of (3.25) to obtain

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T^e_{ext}}\le\|\boldsymbol{\sigma}-\boldsymbol{E}_h(\boldsymbol{\zeta}_h)\|_{0,T^e_{ext}}+(r_e)^{1/2}C^e_{ext}C^e_1\|\boldsymbol{\zeta}_h-\boldsymbol{\sigma}_h\|_{0,T^e}.$$

Multiplying this by $|T^e_{ext}|$, summing over all edges in $\mathcal{E}_h(\Gamma_h)$, applying the estimate (4.24), adding and subtracting $\boldsymbol{\sigma}$, and finally using (4.25), we have

$$\begin{aligned}&\sum_{e\in\mathcal{E}_h(\Gamma_h)}|T^e_{ext}|\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T^e_{ext}}\\&\quad\le C_3h^{\ell+s+2}\|\boldsymbol{\sigma}\|_{\ell+1,\Omega}+\left(C_3h^{s+1}+\frac{1}{2}|\Omega_h|^{1/2}\right)\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{0,\Omega_h}+\frac{1}{2}|\Omega_h|^{1/2}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega_h},\end{aligned} \tag{4.29}$$

where $s\ge1$. Substituting (4.29) into (4.28) yields

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega_h}\le\widehat{C}_2\left(\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_{0,h}\|_{\mathbf{div};\Omega_h}+\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{\mathbf{div};\Omega_h}+h^{\ell+s+2}\|\boldsymbol{\sigma}\|_{\ell+1,\Omega}\right). \tag{4.30}$$

Let now $\boldsymbol{\zeta}_{0,h}:=\boldsymbol{\zeta}_h-\omega_{\boldsymbol{\zeta}_h}\mathbb{I}$, where $\omega_{\boldsymbol{\zeta}_h}\in\mathbb{R}$ is chosen such that $\int_{\Omega_h}\mathrm{tr}\,(\boldsymbol{\zeta}_h)=0$. Then, by the same computations as before,

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{\mathbf{div};\Omega_h}&\le\|\boldsymbol{\sigma}_0-\boldsymbol{\zeta}_{0,h}\|_{\mathbf{div};\Omega_h}+\frac{1}{|\Omega_h|^{1/2}}\sum_{e\in\mathcal{E}_h(\Gamma_h)}|T^e_{ext}|\,\|\boldsymbol{\sigma}-\boldsymbol{E}_h(\boldsymbol{\zeta}_h)\|_{0,T^e_{ext}}\\&\le\|\boldsymbol{\sigma}_0-\boldsymbol{\zeta}_{0,h}\|_{\mathbf{div};\Omega_h}+\frac{1}{|\Omega_h|^{1/2}}\left(\widehat{C}_3h^{\ell+s+2}\|\boldsymbol{\sigma}\|_{\ell+1,\Omega}+\sum_{e\in\mathcal{E}_h(\Gamma_h)}|T^e_{ext}|(r_e)^{1/2}C^e_{ext}C^e_1\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{0,T^e}\right),\end{aligned}$$

which, thanks to assumption (4.25), implies that

$$\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{\mathbf{div};\Omega_h}\le\widehat{C}_4\Big(\|\boldsymbol{\sigma}_0-\boldsymbol{\zeta}_{0,h}\|_{\mathbf{div};\Omega_h}+h^{\ell+s+2}\|\boldsymbol{\sigma}\|_{\ell+1,\Omega}\Big). \tag{4.31}$$

Combining (4.27), (4.30) and (4.31), applying (4.15) with $\vec{\mathbf{r}}_h:=(\mathbf{r}_h,\boldsymbol{\zeta}_{0,h},\mathbf{w}_h)\in\mathbb{X}_{0,h}(\Omega_h)$, and noting, by the second equation of (2.4) and by the definition of p_h (cf. (4.18)), that $p-p_h=-\frac{1}{2}\mathrm{tr}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)$, we readily obtain

$$\begin{aligned}&\|p-p_h\|_{0,\Omega}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega_h}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div};\mathcal{T}^c_h}\\&\quad\le\widehat{C}_5\inf_{\vec{\mathbf{r}}_h\in\mathbb{X}_{0,h}(\Omega_h)}\|\vec{\mathbf{t}}_0-\vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)}+\widehat{C}_5h^{\ell+1}\Big(\|\boldsymbol{\sigma}\|_{\ell+1,\Omega}+\|\mathbf{div}\,\boldsymbol{\sigma}\|_{\ell+1,\Omega}\Big)\\&\qquad+\widehat{C}_5\inf_{\mathbf{r}_h\in\mathbb{X}_{1,h}(\Omega_h)}\sum_{e\in\mathcal{E}_h(\Gamma_h)}|||\mathbf{t}-\boldsymbol{E}_h(\mathbf{r}_h)|||_e\end{aligned} \tag{4.32}$$

To bound the last term on the right-hand side of (4.32), we may apply the equivalence of our norms over T^e_{ext} and the estimate given by (4.22). This yields (4.26). □

4.3 Further estimates on Ω_h^c and rates of convergence

We are now interested in approximating $\mathbf{t}$ and $\mathbf{u}$ on Ω_h^c. For simplicity of notation, these approximations will be also denoted by $\mathbf{t}_h$ and $\mathbf{u}_h$.

First, we set $\mathbf{t}_h$ on Ω_h^c to be $\boldsymbol{E}_h(\mathbf{t}_h)$ in the sense given by (3.19). We then have the following result.

Lemma 4.7. *Suppose that there exists an integer $\ell \geq 0$ such that $\mathbf{t} \in \mathbf{H}^{\ell+1}(\Omega)$. Then, there exists $C > 0$, independent of h, such that*

$$\|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega_h^c} \leq C \left(\inf_{\vec{\mathbf{r}}_h \in \mathbb{X}_0(\Omega_h)} \|\vec{\mathbf{t}}_0 - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} + h^{\ell+1} \|\mathbf{t}\|_{\ell+1,\Omega} \right). \tag{4.33}$$

Proof. With minor modifications the proof follows from [30, Lemma 3.6]. □

Let us now specify $\mathbf{u}_h$ on Ω_h^c. To this end, we proceed as in [13, Section 2.1.3]. We start by noting that for each $e \in \mathcal{E}_h(\Gamma_h)$ and any $\boldsymbol{y} \in T_{ext}^e$, there exists a transferring path $\rho(\boldsymbol{x})$, connecting $\boldsymbol{x} \in \Gamma_h$ and $\bar{\boldsymbol{x}} \in \Gamma$, such that $\boldsymbol{y} = \boldsymbol{x} + (\varepsilon/|\rho(\mathbf{x})|)(\bar{\boldsymbol{x}} - \boldsymbol{x})$ for some $\varepsilon \in [\,0, |\rho(\boldsymbol{x})|\,]$. Therefore, by similar arguments as in (3.2), we can write

$$\mathbf{u}_h(\boldsymbol{y}) := \mathbf{u}(\bar{\boldsymbol{y}}) - \int_0^{|\bar{\boldsymbol{y}}-\boldsymbol{y}|} \mathbf{t}_h(\boldsymbol{y} + \varepsilon \boldsymbol{k}(\boldsymbol{y}))\boldsymbol{k}(\boldsymbol{y})\, d\varepsilon,$$

where $\bar{\boldsymbol{y}} := \bar{\boldsymbol{x}}$ and $\boldsymbol{k}(\boldsymbol{y}) := (\bar{\boldsymbol{y}} - \boldsymbol{y})/|\bar{\boldsymbol{y}} - \boldsymbol{y}|$. Then, the following result establishes the corresponding error estimate. The proof is omitted since it is essentially the same as in [30, Lemma 3.7].

Lemma 4.8. *Assume that the hypotheses of Lemma 4.7 hold. Then, there exists a constant $C > 0$, independent of h, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_h^c} \leq CRh \left(\inf_{\vec{\mathbf{r}}_h \in \mathbb{X}_0(\Omega_h)} \|\vec{\mathbf{t}}_0 - \vec{\mathbf{r}}_h\|_{\mathbb{X}_0(\Omega_h)} + h^{\ell+1} \|\mathbf{t}\|_{\ell+1,\Omega} \right), \tag{4.34}$$

where R is defined in (3.28).

From the above discussion, the following theorem provides the theoretical rate of convergence of the Galerkin scheme (3.16) and the main unknowns under suitable regularity assumptions on the exact solution.

Theorem 4.9. *Additionally to the hypotheses of Theorem 3.3, Theorem 4.3 and Lemma 4.6, assume that there exists $s \in (0, k+1]$ such that $\mathbf{t} \in \mathbb{H}^s(\Omega)$, $\mathbf{u} \in \mathbf{H}^s(\Omega)$ and $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$, with $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$. Then, there exist positive constants C_i, $i \in \{1,2,3\}$, independent of h, such that*

$$\|\vec{\mathbf{t}}_0 - \vec{\mathbf{t}}_{0,h}\|_{\mathbb{X}_0(\Omega_h)} \leq C_1 h^s \Big(\|\mathbf{t}\|_{s,\Omega} + \|\mathbf{u}\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} \Big),$$

$$\|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega_h^c} \leq C_2 h^s \Big(\|\mathbf{t}\|_{s,\Omega} + \|\mathbf{u}\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} \Big),$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_h^c} \leq C_3 R h^{s+1} \Big(\|\mathbf{t}\|_{s,\Omega} + \|\mathbf{u}\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} \Big).$$

Furthermore, for the postprocessed solutions $\boldsymbol{\sigma}_h$ and p_h, given by (4.17) *and* (4.18)*, respectively, it follows that there exists a constant $C_4 > 0$, independent of h, such that*

$$\|p - p_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega_h} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\mathcal{T}_h^c}$$
$$\leq C_4 h^s \Big(\|\mathbf{t}\|_{s,\Omega} + \|\mathbf{u}\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} \Big).$$

Proof. The result is a straightforward application of Theorem 4.3, Lemma 4.6, Lemma 4.7, Lemma 4.8, and the usual interpolation estimates. □

5 A residual-based *a posteriori* error estimator

In this section we develop a reliable and quasi-efficient residual-based *a posteriori* error estimator for the Galerkin scheme (3.16). For simplicity, we restrict ourselves to the problem in two dimensions, yet retaining all the characteristic features of the general case. Furthermore, by choosing a suitable norm of the total error, we consider Γ_h to be constructed through a piecewise linear interpolation of the boundary Γ, yielding $d(\Gamma,\Gamma_h) = \mathcal{O}(h^2)$. Despite this, we recall that the *a priori* error analysis in previous sections also holds under the case $d(\Gamma,\Gamma_h) = \mathcal{O}(h)$, provided (3.28), (3.37), (4.14) and (4.25) are satisfied. However, the *a posteriori* error estimator is not trivial for the latter case and is a topic of ongoing work.

We start by introducing some useful notation and preliminary results. In what follows, h_e stands for the length of a given edge $e \in \mathcal{E}_h$. For each $T \in \mathcal{T}_h$, we let $\mathcal{E}_{h,T}$ the set of all edges of T, and denote $\mathcal{E}_{h,T}(\Gamma_h) := \{e \subseteq \partial T : e \in \mathcal{E}_h(\Gamma_h)\}$ and $\mathcal{E}_{h,T}(\Omega_h) := \{e \subseteq \partial T : e \in \mathcal{E}_h(\Omega_h)\}$. For every $e \in \mathcal{E}_h(\Gamma_h)$, we let Γ_e be the intersection between Γ and the closure of the region T_{ext}. Furthermore, for every $e \in \mathcal{E}_h(\Gamma_h)$ we fix a unit normal vector $\boldsymbol{\nu}_e := (\nu_1^e, \nu_2^e)^{\mathrm{t}}$ to the edge e, and let $\mathbf{S}_e := (-\nu_2^e, \nu_1^e)^{\mathrm{t}}$ be the unit tangential vector along e. We define $\boldsymbol{\nu}_{\Gamma_e}$ and $\mathbf{S}_{\Gamma_e}$ along Γ_e similarly. Moreover, for every $e \in \mathcal{E}_h(\Gamma_h)$ (resp. Γ_e), we take $\boldsymbol{\nu}_e$ (resp. $\boldsymbol{\nu}_{\Gamma_e}$) as the vector pointing in the outward direction of Γ_h (resp. Γ) from Ω_h (resp. Ω). However, when no confusion arises we will simply write $\boldsymbol{\nu}$ and $\mathbf{S}$ instead of $\boldsymbol{\nu}_e$ and $\mathbf{S}_e$ (or, $\boldsymbol{\nu}_{\Gamma_e}$ and $\mathbf{S}_{\Gamma_e}$), respectively. Given $e \in \mathcal{E}_h(\Omega_h)$, $\mathbf{v} \in \mathbf{L}^2(\Omega_h)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_h)$, such that $\mathbf{v}|_T \in [\mathcal{C}(T)]^2$ and $\boldsymbol{\tau}|_T \in [\mathcal{C}(T)]^{2\times 2}$ for all $T \in \mathcal{T}_h$, we let $[\![\mathbf{v}]\!]$ and $[\![\boldsymbol{\tau}\mathbf{S}]\!]$ be the corresponding jumps across e, that is, $[\![\mathbf{v}]\!] := (\mathbf{v}|_{T^+})|_e - (\mathbf{v}|_{T^-})|_e$ and $[\![\boldsymbol{\tau}\mathbf{S}]\!] := \{(\boldsymbol{\tau}|_{T^+})|_e - (\boldsymbol{\tau}|_{T^-})|_e\}\,\mathbf{S}$, where T^+ and T^- are two triangles of $\mathcal{T}_h$ having e as a common edge. Finally, given vector and tensor-valued fields $\mathbf{v} := (v_i)_{1\le i\le 2}$ and $\boldsymbol{\tau} := (\tau_{ij})_{1\le i,j\le 2}$, respectively, we set

$$\underline{\mathbf{curl}}(\mathbf{v}) := \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix} \quad \text{and} \quad \mathrm{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \dfrac{\partial \tau_{12}}{\partial x_1} - \dfrac{\partial \tau_{11}}{\partial x_2} \\ \dfrac{\partial \tau_{22}}{\partial x_1} - \dfrac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

In what follows we assume that the hypotheses of Theorem 3.3, Theorem 4.3 and Lemma 4.6 hold true without explicitly stating them in the results. Let $(\mathbf{t}_h, \boldsymbol{\sigma}_{0,h}, \mathbf{u}_h) \in \mathbb{X}_{1,h} \times \mathbb{M}_{1,h}^0(\Omega_h) \times \mathbf{M}_h(\Omega_h)$ be the unique solution of problem (3.16) and $\boldsymbol{\sigma}_h$ be given by (4.17). For the subsequent analysis we further consider a postprocessing $\mathbf{u}_h^*$ of the fluid velocity $\mathbf{u}$. More precisely, we take $\mathbf{u}_h^* \in \prod_{T\in\mathcal{T}_h} \mathbf{P}_{k+1}(T)$ satisfying, for all $T \in \mathcal{T}_h$,

$$\begin{aligned} \int_T \nabla \mathbf{u}_h^* : \nabla \mathbf{q} &= \int_T \mathbf{t}_h : \nabla \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbf{P}_{k+1}(T), \\ \int_T \mathbf{u}_h^* &= \int_T \mathbf{u}_h. \end{aligned} \tag{5.1}$$

Similar to [11, Theorem 5.2] we note that if $\mathbf{t} \in \mathbb{H}^s(\Omega)$ and $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$, with $s \ge 1$, then $\mathbf{u}_h^*$ converges to $\mathbf{u}$ with $\mathcal{O}(h^{s+1})$. To alleviate the notation, we will also write $\mathbf{u}_h^*$ to denote its extrapolation to the region Ω_h^c. It follows from the previous analysis that $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h^*)$ is an optimal convergent approximation of $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X}_0(\Omega) := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}\,; \Omega) \times \mathbf{L}^2(\Omega)$.

We now introduce the global *a posteriori* error estimator

$$\Theta := \left(\sum_{T\in\mathcal{T}_h} \Theta_T^2 \right)^{1/2}, \tag{5.2}$$

where Θ_T is the local error indicator defined for each $T \in \mathcal{T}_h$ as follows:

$$\begin{aligned}
\Theta_T^2 &:= \Psi_T^2 + \|\mathbf{u}_h - \mathbf{u}_h^*\|_{0,T}^2 + h_T^2\|\mathrm{curl}\,(\mathbf{t}_h)\|_{0,T}^2 + \|\mathbf{t}_h - \nabla \mathbf{u}_h^*\|_{0,T}^2 \\
&\quad + \sum_{e\in\mathcal{E}_{h,T}(\Omega_h)} \Big(h_e \|[\![\mathbf{t}_h \mathbf{S}_e]\!]\|_{0,e}^2 + h_e^{-1}\|[\![\mathbf{u}_h^*]\!]\|_{0,e}^2 \Big) + \|\boldsymbol{\sigma}_h^{\mathrm{d}} - 2\mu(|\mathbf{t}_h|)\mathbf{t}_h\|_{0,T}^2 \\
&\quad + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_{h,T}(\Gamma_h)} \Big(\|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T_{ext}^e}^2 + \|\boldsymbol{\sigma}_h^{\mathrm{d}} - 2\mu(|\mathbf{t}_h|)\mathbf{t}_h\|_{0,T_{ext}^e}^2 \Big).
\end{aligned} \tag{5.3}$$

Above, Ψ_T is the fully computable term defined by

$$\Psi_T^2 := \sum_{e\in\mathcal{E}_{h,T}(\Gamma_h)} \left(h_e^{-1}\|\mathbf{g} - \mathbf{u}_h^*\|_{0,\Gamma_e}^2 + h_{T^e} \left\| \frac{d\mathbf{g}}{d\mathbf{S}_{\Gamma_e}} - \mathbf{t}_h \mathbf{S}_{\Gamma_e} \right\|_{0,\Gamma_e}^2 \right). \tag{5.4}$$

The residual character of each term defining Θ is a consequence of the strong problem (2.5) and the regularity of the weak formulation at the continuous level. It is important to remark that the second term of Ψ_T requires $(d\mathbf{g}/d\mathbf{S}_{\Gamma_e})|_{\Gamma_e} \in \mathrm{L}^2(\Gamma_e)$ for each $\Gamma_e \subseteq \Gamma$, which will be overcome below by simply assuming that $\mathbf{g} \in \mathbf{H}^1(\Gamma)$. Furthermore, if $\mu(\mathbf{t}_h)$ is a polynomial, as in the examples given in Section 2.2, then we can use Lemma 4.5 and the constant C_{ext}^e of (3.25) to bound $\|\boldsymbol{\sigma}_h^{\mathrm{d}} - 2\mu(|\mathbf{t}_h|)\mathbf{t}_h\|_{0,T_{ext}^e}$ by $\|\boldsymbol{\sigma}_h^{\mathrm{d}} - 2\mu(|\mathbf{t}_h|)\mathbf{t}_h\|_{0,T^e}$. In the latter case, the last term in the definition of Θ_T is not required.

In what follows we prove the main properties of Θ, namely, its reliability and quasi-efficiency.

5.1 Reliability of the *a posteriori* error estimator

In this section we focus on the proof of the following result.

Theorem 5.1. *There exists a constant $C_{rel} > 0$, independent of h, such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{X}_0(\Omega)} \le C_{rel}\Theta. \tag{5.5}$$

We emphasize that the norm appearing in (5.5) makes sense because $\boldsymbol{\sigma}_h \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$. In fact, since the extrapolation of any function of $\mathbb{H}(\mathbf{div}\,;\Omega_h)$ is a function of $\mathbb{H}(\mathbf{div}\,;\Omega)$ as Γ_h is given by a piecewise linear interpolation of the boundary Γ, it suffices to note, thanks to the identities in (4.20), that $\int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}_h) = 0$.

The proof of Theorem 5.1 is presented in several steps.

5.2 Preliminary results

We start by providing a preliminary upper bound for the total error, as done in [22] (see also [7]).

Lemma 5.2. *There exists $C > 0$, independent of h, such that*

$$\begin{aligned}
&\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{X}_0(\Omega)} \\
&\qquad \le C \Big(\|\mathbf{u}_h^* - \mathbf{u}_h\|_{0,\Omega} + \|\boldsymbol{\sigma}_h^{\mathrm{d}} - 2\mu(|\mathbf{t}_h|)\mathbf{t}_h\|_{0,\Omega} + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega} + \|\mathcal{R}\| \Big),
\end{aligned} \tag{5.6}$$

where $\mathbf{u}_h^*$ *is the postprocessed velocity provided by* (5.1)*, and* $\mathcal{R} : \mathbb{H}_0(\mathbf{div}\,;\Omega) \to \mathbb{R}$ *is the linear and bounded functional defined by*

$$\mathcal{R}(\boldsymbol{\tau}) := \int_\Omega \mathbf{t}_h : \boldsymbol{\tau} + \int_\Omega \mathbf{u}_h^* \cdot \mathbf{div}\,\boldsymbol{\tau} - \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega). \tag{5.7}$$

Proof. First observe, by similar arguments as in (4.6), that for any $\vec{\mathbf{q}} \in \mathbb{X}_0(\Omega) := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}\,;\Omega) \times \mathbf{L}^2(\Omega)$, the following global inf-sup condition holds:

$$\widehat{C}_{glob} \|\vec{\mathbf{r}}\|_{\mathbb{X}_0(\Omega)} \le \sup_{\substack{\vec{\mathbf{s}} \in \mathbb{X}_0(\Omega) \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{\mathcal{D}\mathbb{P}(\vec{\mathbf{q}})(\vec{\mathbf{r}}, \vec{\mathbf{s}})}{\|\vec{\mathbf{s}}\|_{\mathbb{X}_0(\Omega)}} \qquad \forall\, \vec{\mathbf{r}} \in \mathbb{X}_0(\Omega), \tag{5.8}$$

where $\mathbb{P} : \mathbb{X}_0(\Omega) \to [\mathbb{X}_0(\Omega)]'$ is the nonlinear operator induced by the left-hand side of the continuous problem (2.6). Moreover, since $\vec{\mathbf{t}} := (\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}), \vec{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_0(\Omega)$, an application of the mean value theorem yields the existence of a convex combination of $\vec{\mathbf{t}}$ and $\vec{\mathbf{t}}_h$, say $\vec{\mathbf{q}} \in \mathbb{X}_0(\Omega)$, such that

$$\mathcal{D}\mathbb{P}(\vec{\mathbf{q}})(\vec{\mathbf{t}} - \vec{\mathbf{t}}_h, \vec{\mathbf{s}}) = [\mathbb{P}(\vec{\mathbf{t}}), \vec{\mathbf{s}}] - [\mathbb{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}] \qquad \forall\, \vec{\mathbf{s}} \in \mathbb{X}_0(\Omega). \tag{5.9}$$

Next, adding and subtracting $(\mathbf{0}, \mathbf{0}, \mathbf{u}_h^*)$ to the total error, applying (5.8), using the identity (5.9), problem (2.6), and noting that $\mathbf{t}_h : \boldsymbol{\tau}^{\mathrm{d}} = \mathbf{t}_h : \boldsymbol{\tau}$ since $\mathrm{tr}\,(\mathbf{t}_h) = 0$, we readily obtain

$$\begin{aligned}
&\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{X}_0(\Omega)} \\
&\quad \le \|\mathbf{u}_h^* - \mathbf{u}_h\|_{0,\Omega} + \|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h^*)\|_{\mathbb{X}_0(\Omega)} \\
&\quad \le \|\mathbf{u}_h^* - \mathbf{u}_h\|_{0,\Omega} + \frac{1}{\widehat{C}_{glob}} \sup_{\substack{\vec{\mathbf{s}} \in \mathbb{X}_0(\Omega) \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{\big|[\mathbb{P}(\vec{\mathbf{t}}), \vec{\mathbf{s}}] - [\mathbb{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}]\big|}{\|\vec{\mathbf{s}}\|_{\mathbb{X}_0(\Omega)}} \\
&\quad \le \|\mathbf{u}_h^* - \mathbf{u}_h\|_{0,\Omega} + \frac{1}{\widehat{C}_{glob}} \Big(\|\boldsymbol{\sigma}_h^{\mathrm{d}} - 2\mu(|\mathbf{t}_h|)\mathbf{t}_h\|_{0,\Omega} + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega} + \|\mathcal{R}\| \Big),
\end{aligned} \tag{5.10}$$

where $\mathcal{R}$ is defined in (5.7). Since $\widehat{C}_{glob}$ is independent of h, the result follows. □

It is now clear that in order to show (5.5), we need a suitable upper bound for $\|\mathcal{R}\|$. In doing so, we can write

$$\|\mathcal{R}\| = \sup_{\substack{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\mathcal{R}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\Omega}} + \sup_{\substack{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\mathcal{R}(\boldsymbol{\tau} - \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\Omega}}, \tag{5.11}$$

with $\boldsymbol{\tau}_h$ a suitably chosen function that will be defined later by employing a Helmholtz decomposition and the Clément and Raviart–Thomas interpolation operators. This approach has been widely used in *a posteriori* error estimators for mixed methods, see for instance [7, 9, 21, 22, 23]. However, the case of curved domains Ω requires further technicalities that have been recently addressed in [31] with the help of suitable triangles that are added to the triangulation $\mathcal{T}_h$ of $\overline{\Omega}_h$, even though, as we will see in Section 6, they are not needed to compute our *a posteriori* error estimator. Let us now discuss how this can be done.

We associate with each $e \in \mathcal{E}_h(\Gamma_h)$ a triangle $T_\star^e$, with diameter $h_{T_\star^e}$, satisfying

- $T_\star^e$ has e as a boundary edge, $\Gamma_e \subseteq T_\star^e$, $|\Gamma_e| \simeq h_e$ and $h_{T_\star^e} \simeq h_{T^e}$.
- If $F := \overline{T_\star^{e_i}} \cap \overline{T_\star^{e_j}}$, with $e_i, e_j \in \mathcal{E}_h(\Gamma_h)$, $i \neq j$, then F is either a common vertex or a common edge of $T_\star^{e_i}$ and $T_\star^{e_j}$.

These hypotheses are expected to be satisfied on sufficiently fine meshes since Γ_h is constructed through a piecewise linear interpolation of the boundary Γ.

Let $\Omega_h^\star$ be the polygon whose triangulation $\mathcal{T}_h^\star$ consists of $\mathcal{T}_h \cup \{T_\star^e : e \in \mathcal{E}_h(\Gamma_h)\}$, and assume, for the sake of convenience, that $\mathcal{T}_h^\star$ is shape-regular. Let $\mathbb{M}_{1,h}(\Omega_h^\star)$ be the analogue of the space defined in (3.14), and $\mathbf{\Pi}_h^k : \mathbb{H}^1(\Omega_h^\star) \to \mathbb{M}_{1,h}(\Omega_h^\star)$ be the Raviart–Thomas interpolation operator. The following local approximation properties are well-known to hold [4, 32]:

- There exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau} \in \mathbb{H}^\ell(\Omega_h^\star)$, with $1 \leq \ell \leq k+1$, there holds
$$\|\boldsymbol{\tau} - \mathbf{\Pi}_h^k(\boldsymbol{\tau})\|_{0,T} \leq C h_T^\ell |\boldsymbol{\tau}|_{\ell,T} \qquad \forall\, T \in \mathcal{T}_h^\star. \tag{5.12}$$

- There exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau} \in \mathbb{H}^1(\Omega_h^\star)$, with $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{H}^\ell(\Omega_h^\star)$ and $0 \leq \ell \leq k+1$, there holds
$$\left\|\mathbf{div}\,(\boldsymbol{\tau} - \mathbf{\Pi}_h^k(\boldsymbol{\tau}))\right\|_{0,T} \leq C h_T^\ell |\mathbf{div}\,\boldsymbol{\tau}|_{\ell,T} \qquad \forall\, T \in \mathcal{T}_h^\star. \tag{5.13}$$

- There exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau} \in \mathbb{H}^1(\Omega_h^\star)$, there holds
$$\|(\boldsymbol{\tau} - \mathbf{\Pi}_h^k(\boldsymbol{\tau}))\boldsymbol{\nu}_e\|_{0,e} \leq C h_e^{1/2} \|\boldsymbol{\tau}\|_{1,T^e} \qquad \forall\, \text{edge } e \text{ of } \mathcal{T}_h^\star. \tag{5.14}$$

Similarly, due to the above hypotheses on the triangles $T_\star^e$, the following approximation properties on curved segments hold [31]:

- There exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau} \in \mathbb{H}^1(\Omega_h^\star)$, there holds
$$\|(\boldsymbol{\tau} - \mathbf{\Pi}_h^k(\boldsymbol{\tau}))\boldsymbol{\nu}_{\Gamma_e}\|_{0,\Gamma_e} \leq C h_{T^e}^{1/2} \|\boldsymbol{\tau}\|_{1,T_\star^e} \qquad \forall\, e \in \mathcal{E}_h(\Gamma_h). \tag{5.15}$$

- There exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau} \in \mathbb{H}^1(\Omega_h^\star)$, there holds
$$\|\boldsymbol{\tau} - \mathbf{\Pi}_h^k(\boldsymbol{\tau})\|_{0,\Gamma_e} \leq C h_{T^e}^{1/2} \|\boldsymbol{\tau}\|_{1,T_\star^e} \qquad \forall\, e \in \mathcal{E}_h(\Gamma_h). \tag{5.16}$$

- There exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau} \in \mathbb{H}^1(\Omega_h^\star)$, there holds
$$\|\boldsymbol{\tau}\boldsymbol{\nu}_{\Gamma_e}\|_{0,\Gamma_e} \leq C h_{T^e}^{-1/2} \|\boldsymbol{\tau}\|_{1,T_\star^e} \qquad \forall\, e \in \mathcal{E}_h(\Gamma_h). \tag{5.17}$$

In addition, we let $\mathcal{I}_h : \mathrm{H}^1(\Omega_h^\star) \to \left\{ v \in \mathcal{C}(\,\overline{\Omega_h^\star}\,) : v|_T \in \mathrm{P}_1(T)\, \forall\, T \in \mathcal{T}_h^\star \right\}$ be the Clément interpolation operator. We recall from [10] that there exist constants $C_1, C_2 > 0$, independent of h, such that for each $v \in \mathrm{H}^1(\Omega)$ there hold
$$\|v - \mathcal{I}_h(v)\|_{0,T} \leq C_1 h_T |v|_{1,\Delta(T)} \qquad \forall\, T \in \mathcal{T}_h^\star, \tag{5.18}$$
and
$$\|v - \mathcal{I}_h(v)\|_{0,e} \leq C_2 h_e^{1/2} |v|_{1,\Delta(e)} \qquad \forall\, \text{edge } e \text{ of } \mathcal{T}_h^\star, \tag{5.19}$$
where $\Delta(T)$ and $\Delta(e)$ are the union of all the elements intersecting with T and e, respectively.

We end this section by recalling that for each $v \in \mathrm{H}^1(\Omega)$ there exists an extension $\mathscr{E}(v) \in \mathrm{H}^1(\mathbb{R}^2)$, satisfying $\mathscr{E}(v)|_\Omega = v$ and $\|\mathscr{E}(v)\|_{1,\mathbb{R}^2} \leq C\|v\|_{1,\Omega}$, with $C > 0$ independent of v. For its proof, we refer the reader to [35].

5.3 Upper bound for $\|\mathcal{R}\|$

In order to introduce a suitable $\boldsymbol{\tau}_h$ for the estimation of $\|\mathcal{R}\|$ in (5.11), we proceed as in [21, Section 4.1] and provide a stable Helmholtz decomposition of $\mathbb{H}_0(\mathbf{div}\,;\Omega)$. More precisely, for each $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$ there exist $\boldsymbol{\zeta} \in \mathbb{H}^1(\Omega)$ and $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)^{\mathrm{t}} \in \mathbf{H}^1(\Omega)$, with $\int_\Omega \varphi_1 = \int_\Omega \varphi_2 = 0$, such that

$$\boldsymbol{\tau} = \boldsymbol{\zeta} + \underline{\mathbf{curl}}(\boldsymbol{\varphi}) \quad \text{in} \quad \Omega, \quad \text{and} \quad \|\boldsymbol{\zeta}\|_{1,\Omega} + \|\boldsymbol{\varphi}\|_{1,\Omega} \le C\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\Omega}, \tag{5.20}$$

where C is a positive constant independent of $\boldsymbol{\tau}$, $\boldsymbol{\zeta}$ and $\boldsymbol{\varphi}$.

For simplicity of notation, given $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$ and its Helmholtz decomposition (5.20), we let $\boldsymbol{\zeta}_h := \boldsymbol{\Pi}_h^k(\boldsymbol{\mathscr{E}}(\boldsymbol{\zeta})|_{\Omega_h^\star})$ and $\boldsymbol{\varphi}_h := \boldsymbol{\mathcal{I}}_h(\boldsymbol{\mathscr{E}}(\boldsymbol{\varphi})|_{\Omega_h^\star})$, where $\boldsymbol{\mathscr{E}}$ and $\boldsymbol{\mathcal{I}}$ are defined componentwise by the extension operator $\mathscr{E}$ and the Clément interpolant $\mathcal{I}_h$, respectively. We then set the discrete Helmholtz decomposition as $\boldsymbol{\tau}_h := \boldsymbol{\zeta}_h + \underline{\mathbf{curl}}(\boldsymbol{\varphi}_h) + \omega_{\boldsymbol{\tau}_h}\mathbb{I}$, where $\omega_{\boldsymbol{\tau}_h} \in \mathbb{R}$ is chosen so that $\int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) = 0$.

From the above discussion, by definition of $\mathcal{R}$ (cf. (5.7)) and by the compatibility condition $\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} = 0$, we can write

$$\mathcal{R}(\boldsymbol{\tau} - \boldsymbol{\tau}_h) = \mathcal{R}(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h) + \mathcal{R}(\underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)). \tag{5.21}$$

We will bound each term on the right-hand side of (5.21) separately.

The upper bound for $|\mathcal{R}(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h)|$ follows by similar arguments as in [31, Lemma 5.5] (see also [21, Lemma 4.4]). Indeed, for each $\boldsymbol{\xi} \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$ we can decompose the integrals over Ω, within $\mathcal{R}(\boldsymbol{\xi})$, into integrals over Ω_h and its complement. In this way, integrating by parts on each element of $\mathcal{T}_h$ and $\mathcal{T}_h^c$, and using the fact that for any sufficiently smooth tensor field $\boldsymbol{\xi}$ and for each $e \in \mathcal{E}_h(\Gamma_h)$, $\big\{(\boldsymbol{\xi}|_{T^e})\,|_e - \big(\boldsymbol{\xi}|_{T^e_{ext}}\big)\,|_e\big\}\,\boldsymbol{\nu}_e = 0$, it is not difficult to see that

$$\begin{aligned}\mathcal{R}(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h) = &\sum_{T\in\mathcal{T}_h} \int_T (\mathbf{t}_h - \nabla\mathbf{u}_h^*) : (\boldsymbol{\zeta} - \boldsymbol{\zeta}_h) + \sum_{e\in\mathcal{E}_h(\Gamma_h)} \int_{T^e_{ext}} (\mathbf{t}_h - \nabla\mathbf{u}_h^*) : (\boldsymbol{\zeta} - \boldsymbol{\zeta}_h)\\ &+ \sum_{e\in\mathcal{E}_h(\Gamma_h)} \int_{\Gamma_e} (\mathbf{u}_h^* - \mathbf{g})\cdot(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h)\boldsymbol{\nu}_{\Gamma_e} + \sum_{e\in\mathcal{E}_h(\Omega_h)} \int_e [\![\mathbf{u}_h^*]\!]\cdot(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h)\boldsymbol{\nu}_e.\end{aligned} \tag{5.22}$$

Applying the Cauchy–Schwarz inequality to each term above, using the approximation properties (5.12), (5.14) and (5.15), and the constant C^e_{ext} of (3.25), we obtain, after some algebraic manipulations,

$$\begin{aligned}|\mathcal{R}(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h)| \le C\Bigg(&\sum_{T\in\mathcal{T}_h} h_T^2\|\mathbf{t}_h - \nabla\mathbf{u}_h^*\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(\Gamma_h)} (C^e_{ext})^2 r_e h_{T^e}^2 \|\mathbf{t}_h - \nabla\mathbf{u}_h^*\|_{0,T^e}^2\\ &+ \sum_{e\in\mathcal{E}_h(\Gamma_h)} h_{T^e}\|\mathbf{g} - \mathbf{u}_h^*\|_{0,\Gamma_e}^2 + \sum_{e\in\mathcal{E}_h(\Omega_h)} h_{T^e}\|[\![\mathbf{u}_h^*]\!]\|_{0,e}^2\Bigg)^{1/2}\Bigg(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{\zeta}\|_{1,T}^2 + \sum_{e\in\mathcal{E}_h(\Gamma_h)}\mathbb{B}_e^2\Bigg)^{1/2},\end{aligned}$$

where $\mathbb{B}_e^2 := \|\boldsymbol{\zeta}\|_{1,T^e_{ext}}^2 + \|\boldsymbol{\mathscr{E}}(\boldsymbol{\zeta})\|_{1,T^e_\star}^2 + \|\boldsymbol{\zeta}\|_{1,\mathcal{K}(e)}^2$ and $\mathcal{K}(e) := \{T' : e \subseteq \partial T'\}$. Therefore, the stability property of the extension operator $\boldsymbol{\mathscr{E}}$ yields

$$\begin{aligned}|\mathcal{R}(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h)| \le \widehat{C}\Bigg(&\sum_{T\in\mathcal{T}_h} h_T^2\|\mathbf{t}_h - \nabla\mathbf{u}_h^*\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(\Gamma_h)} h_{T^e}\|\mathbf{g} - \mathbf{u}_h^*\|_{0,\Gamma_e}^2\\ &+ \sum_{e\in\mathcal{E}_h(\Omega_h)} h_{T^e}\|[\![\mathbf{u}_h^*]\!]\|_{0,e}^2\Bigg)^{1/2}\|\boldsymbol{\zeta}\|_{1,\Omega}.\end{aligned} \tag{5.23}$$

We now bound $|\mathcal{R}(\underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)|$ following similar steps as in [21, Lemma 4.3]. First, using the identity $\underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\boldsymbol{\nu} = \frac{d}{d\mathbf{S}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)$, assuming that $\frac{d\mathbf{g}}{d\mathbf{S}} \in \mathrm{L}^2(\Gamma)$, and integrating by parts on Γ, it follows that

$$\langle \underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma = -\sum_{e\in\mathcal{E}_h(\Gamma_h)} \int_{\Gamma_e} \frac{d\mathbf{g}}{d\mathbf{S}_{\Gamma_e}} \cdot (\boldsymbol{\varphi}-\boldsymbol{\varphi}_h). \tag{5.24}$$

We can then rewrite $\mathcal{R}(\underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)$, using (5.24), applying [24, Theorem 2.11] to integrate on each element of $\mathcal{T}_h$ and $\mathcal{T}_h^c$, as

$$\begin{aligned}\mathcal{R}(\underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)) &= \sum_{T\in\mathcal{T}_h}\int_T \mathrm{curl}\,(\mathbf{t}_h)\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) + \sum_{e\in\mathcal{E}_h(\Gamma_h)}\int_{T^e_{ext}} \mathrm{curl}\,(\mathbf{t}_h)\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\\ &\quad - \sum_{e\in\mathcal{E}_h(\Omega_h)}\int_e [\![\mathbf{t}_h\mathbf{S}_e]\!]\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) + \sum_{e\in\mathcal{E}_h(\Gamma_h)}\int_{\Gamma_e}\left(\mathbf{t}_h\mathbf{S}_{\Gamma_e} - \frac{d\mathbf{g}}{d\mathbf{S}_{\Gamma_e}}\right)\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h).\end{aligned} \tag{5.25}$$

Next, using similar arguments as before, the approximation properties of $\boldsymbol{\mathcal{I}}_h$ (cf. (5.18) and (5.19)), and the fact that the number of triangles in $\Delta(T)$ and $\Delta(e)$ are bounded (due to the shape-regularity of the mesh), we readily obtain

$$\begin{aligned}|\mathcal{R}(\underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)| &\le C\Bigg(\sum_{T\in\mathcal{T}_h} h_T^2\|\mathrm{curl}\,(\mathbf{t}_h)\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(\Omega_h)} h_e\|[\![\mathbf{t}_h\mathbf{S}_e]\!]\|_{0,e}^2\\ &\quad + \sum_{e\in\mathcal{E}_h(\Gamma_h)} h_{T^e}\left\|\frac{d\mathbf{g}}{d\mathbf{S}_{\Gamma_e}} - \mathbf{t}_h\mathbf{S}_{\Gamma_e}\right\|_{0,\Gamma_e}^2\Bigg)^{1/2}\|\boldsymbol{\varphi}\|_{1,\Omega}.\end{aligned} \tag{5.26}$$

Finally, in order to bound the first term on the right-hand side of (5.11), it suffices to note, by the estimate (5.17), that $\|\boldsymbol{\tau}_h\boldsymbol{\nu}_{\Gamma_e}\|_{0,\Gamma_e} \le C h_{T^e}^{-1/2}\|\boldsymbol{\tau}_h\|_{1,T^e_\star}$ for all $e\in\mathcal{E}_h(\Gamma_h)$. In fact, this yields

$$\begin{aligned}|\mathcal{R}(\boldsymbol{\tau}_h)| &\le C\Bigg(\sum_{T\in\mathcal{T}_h}\|\mathbf{t}_h - \nabla\mathbf{u}_h^*\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(\Gamma_h)} h_{T^e}^{-1}\|\mathbf{g}-\mathbf{u}_h^*\|_{0,\Gamma_e}^2\\ &\quad + \sum_{e\in\mathcal{E}_h(\Omega_h)} h_{T^e}^{-1}\|[\![\mathbf{u}_h^*]\!]\|_{0,e}^2\Bigg)^{1/2}\Big(\|\boldsymbol{\zeta}\|_{1,\Omega} + \|\boldsymbol{\varphi}\|_{1,\Omega}\Big).\end{aligned} \tag{5.27}$$

The reliability estimate (5.5) now follows directly from (5.11), the estimates given by (5.23), (5.26) and (5.27), and the stability of the Helmholtz decomposition (5.20).

5.4 Quasi-efficiency of the *a posteriori* error estimator

The main result of this section reads as follows.

Theorem 5.3. *There exists a constant $C_{eff} > 0$, independent of h, such that*

$$C_{eff}\Theta \le \|(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbb{X}_0(\Omega)} + \Psi, \tag{5.28}$$

where $\Psi := \Big(\sum_{T\in\mathcal{T}_h}\Psi_T^2\Big)^{1/2}$ *and* Ψ_T *is given by* (5.4).

We note that in the above inequality the estimator is quasi-efficient in the sense that it is efficient up to the fully computable and residual term Ψ. If this quantity is of the same order of the total error, we can ensure that Θ is an optimal convergent estimator. In [31] it was presented numerical evidence showing that Ψ is optimally convergent as an unfitted mixed method for the Stokes equations with constant viscosity is considered. In the case of nonlinear viscosity, as we will see in Section 6, the estimator term Ψ has the same behavior.

To obtain (5.28), we will found upper bounds for each estimator term in the definition of Θ_T in (5.3), separately. In doing so, we frequently use of the original system of equations given by (2.1), which can be recovered from the continuous twofold saddle point formulation (2.6) by applying integration by parts in the corresponding equations and using suitable test functions. In particular, we have that $\boldsymbol{\sigma}^{\mathrm{d}} = \mu(|\mathbf{t}|)\mathbf{t}$ and $\mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f}$ hold. Then, it is immediate to see that the estimates for the last four terms appearing in the definition of Θ_T (cf. (5.3)) follow by applying the triangle inequality and the Lipschitz continuity of the nonlinear operator $\mathbf{A}_{1,h}$ (cf. (3.12)). We have the following lemma.

Lemma 5.4. *For all $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h(\Gamma_h)$, there hold*

$$\begin{aligned}
\|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T} &\leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}\,;T}, \\
\|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T^e_{ext}} &\leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}\,;T^e_{ext}}, \\
\|\boldsymbol{\sigma}_h^{\mathrm{d}} - 2\mu(|\mathbf{t}_h|)\mathbf{t}_h\|_{0,T} &\leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} + 2\gamma_0\|\mathbf{t} - \mathbf{t}_h\|_{0,T}, \\
\|\boldsymbol{\sigma}_h^{\mathrm{d}} - 2\mu(|\mathbf{t}_h|)\mathbf{t}_h\|_{0,T^e_{ext}} &\leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T^e_{ext}} + 2\gamma_0\|\mathbf{t} - \mathbf{t}_h\|_{0,T^e_{ext}},
\end{aligned}$$

where γ_0 is the constant given in (2.2).

The following three lemmas provide upper bounds for the remaining five estimator terms in the definition of Θ_T.

Lemma 5.5. *There exits positive constants C_1 and C_2, independent of h, such that for all $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h(\Omega_h)$,*

$$h_T^2\|\mathrm{curl}\,(\mathbf{t}_h)\|_{0,T}^2 \leq C_1\|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2, \tag{5.29}$$

$$h_e\|[\![\mathbf{t}_h\mathbf{S}_e]\!]\|_{0,e}^2 \leq C_2\|\mathbf{t} - \mathbf{t}_h\|_{0,\mathcal{K}(e)}^2, \tag{5.30}$$

where $\mathcal{K}(e) := \{T' \in \mathcal{T}_h : e \subseteq \partial T'\}$.

Proof. Since $\mathrm{curl}\,(\mathbf{t}) = \mathrm{curl}\,(\nabla\mathbf{u}) = 0$, the result can be readily deduced. In fact, applying Lemma 4.9 and Lemma 4.10 of [21] to $\boldsymbol{\rho} := \mathbf{t}$ and $\boldsymbol{\rho}_h := \mathbf{t}_h$, we obtain (5.29) and (5.30), respectively. □

Lemma 5.6. *For each $T \in \mathcal{T}_h$, there exits a constant $C_3 > 0$, independent of h, such that*

$$\|\mathbf{t}_h - \nabla\mathbf{u}_h^*\|_{0,T}^2 \leq C_3\|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2. \tag{5.31}$$

Furthermore, for each $e \in \mathcal{E}_h(\Omega_h)$, there exists a constant $C_4 > 0$, independent of h, such that

$$h_e^{-1}\|[\![\mathbf{u}_h^*]\!]\|_{0,e}^2 \leq C_4\sum_{T\in\mathcal{K}(e)}\|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2. \tag{5.32}$$

Proof. To obtain (5.31), it suffices to apply the result stated in Lemma 3.7 of [16].

It remains to prove (5.32), for which we proceed as in [31, Section 5.2]. First, adding and subtracting convenient terms, it follows that

$$h_e^{-1}\|[\![\mathbf{u}_h^*]\!]\|_{0,e}^2 \leq 2h_e^{-1}\|\boldsymbol{\mathcal{P}}_h([\![\mathbf{u}_h^*]\!])\|_{0,e}^2 + 2h_e^{-1}\|(\boldsymbol{I} - \boldsymbol{\mathcal{P}}_h([\![\mathbf{u}_h^*]\!])\|_{0,e}^2, \tag{5.33}$$

where $\boldsymbol{I}$ is the identity operator and $\boldsymbol{\mathcal{P}}_h$ denotes the $\mathbf{L}^2$-orthogonal projection onto the space of piecewise constant functions on each $e \in \mathcal{E}_h$ (cf. Section 3.2). To estimate the first term on the right-hand side of (5.33), we can proceed as in [31, Lemma 5.4] and use equation (3.16b), the system (5.1) providing $\mathbf{u}_h^*$, and the equations defining the Raviart–Thomas interpolation operator $\boldsymbol{\Pi}_h^k$ (see, e.g., [4]), to obtain

$$h_e^{-1}\|\boldsymbol{\mathcal{P}}_h([\![\mathbf{u}_h^*]\!])\|_{0,e}^2 \leq C \sum_{T\in\mathcal{K}(e)} \|\mathbf{t}_h - \nabla\mathbf{u}_h^*\|_{0,T}^2. \tag{5.34}$$

Furthermore, proceeding exactly as in [16, Lemma 3.5], we find

$$h_e^{-1}\|(\boldsymbol{I} - \boldsymbol{\mathcal{P}}_h([\![\mathbf{u}_h^*]\!])\|_{0,e}^2 \leq \widehat{C} \sum_{T\in\mathcal{K}(e)} \|\nabla(\mathbf{u} - \mathbf{u}_h^*)\|_{0,T}^2. \tag{5.35}$$

The result (5.32) follows from (5.33), (5.34), (5.35), by using the identity $\mathbf{t} = \nabla\mathbf{u}$ and the estimate (5.31). □

Lemma 5.7. *There exists a constant $C_5 > 0$, independent of h, such that for all $T \in \mathcal{T}_h$,*

$$\|\mathbf{u}_h - \mathbf{u}_h^*\|_{0,T}^2 \leq C_5\Big(\|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2\Big).$$

Proof. Due to the second equation of (5.1) and the identity $\mathbf{t} = \nabla\mathbf{u}$, the result follows by mimicking the steps in the proof of [31, Lemma 5.10]. We omit the mathematical details. □

We end this section by noting that the quasi-efficiency estimate (5.28) is a straightforward consequence of Lemma 5.4, Lemma 5.5, Lemma 5.6 and Lemma 5.7.

6 Numerical results

In this section we present numerical examples in two dimensions illustrating the good performance of our discrete scheme (3.16), validating the reliability and quasi-efficiency of the a posteriori error estimator Θ in (5.2), and showing the behavior of the associated adaptive algorithm. The nonlinear systems resulting from (3.16) were solved using Newton's method with a tolerance of 1e-6 and taking as initial guess the solution of the associated linear problem with $\mu = 1$. All simulations were implemented using MATLAB. At each Newton iteration, we used UMFPACK [17] as a direct solver.

The error estimates presented in this work are independent of the construction of basis functions. We chose hierarchical basis for the local Raviart–Thomas space of order k, as presented in [2], and the Dubiner basis (see, e.g., [34]) for the local polynomial space of degree less or equal to k.

In what follows, we denote by N the total number of elements of the mesh $\mathcal{T}_h$ of $\overline{\Omega}_h$. The global error and effectivity index associated to the global estimator Θ are given by

$$\mathrm{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) := \Big\{[\mathrm{e}(\mathbf{t})]^2 + [\mathrm{e}(\boldsymbol{\sigma})]^2 + [\mathrm{e}(\mathbf{u})]^2\Big\}^{1/2} \quad \text{and} \quad \mathrm{eff}(\Theta) := \Theta/\mathrm{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}),$$

where

$$\mathrm{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}, \quad \mathrm{e}(\boldsymbol{\sigma}) := \Big\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega_h}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\mathcal{T}_h^c}^2\Big\}^{1/2}, \quad \mathrm{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega},$$

and $\boldsymbol{\sigma}_h$ is the postprocessed solution given by (4.17). We furthermore consider the errors

$$\mathrm{e}(p) := \|p - p_h\|_{0,\Omega} \quad \text{and} \quad \mathrm{e}^*(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h^*\|_{0,\Omega},$$

where p_h and $\mathbf{u}_h^*$ are the approximations provided by (4.18) and (5.1), respectively. Moreover, using the fact that $h \simeq N^{-1/2}$, the experimental rate of convergence of any of the above quantities will be computed as

$$\text{rate} := -2[\log(\mathrm{e}/\mathrm{e}')/\log(N/N')],$$

where N and N' denote the total number of elements associated to two consecutive triangulations with errors e and e'.

The examples to be considered in this section are described next. In all them we set the nonlinear viscosity to

$$\mu(s) := \mu_0 + \mu_1(1 + s^2)^{(\beta-2)/2} \qquad \forall\, s > 0,$$

with $\mu_0 = 1$, $\mu_1 = 0.5$ and $\beta = 1.5$. In Examples 1 and 2 we explore the performance of the Galerkin scheme (3.16) under different constructions of the computational domain. Furthermore, Example 2 is used to corroborate the reliability and quasi-efficiency of the *a posteriori* error estimator Θ under a quasi-uniform refinement, whereas the simulations in Examples 3 demonstrate the behavior of the adaptive algorithm associated to Θ, which reads:

1. Start with a coarse mesh $\mathcal{T}_h$ of $\overline{\Omega}_h$.
2. Solve the Newton iterative method associated to (3.16) on the current mesh $\mathcal{T}_h$.
3. Compute Θ_T for each $T \in \mathcal{T}_h$.
4. Check the stopping criterion and decide whether to finish or go to next step.
5. Use red-green-blue algorithm to refine each $T' \in \mathcal{T}_h$ satisfying: $\Theta_{T'} \geq 0.5\,(\max_{T\in\mathcal{T}_h} \Theta_T)$.
6. Project every new vertex $\boldsymbol{x}$ of Γ_h onto the closest point $\bar{\boldsymbol{x}}$ of Γ.
7. Define the resulting mesh as the current mesh $\mathcal{T}_h$, and go to step 2.

Note that the above procedure is the usual adaptive refinement strategy from [37], except that the $6th$ step has been added to expect assumptions (3.28), (3.37), (4.14) and (4.25) to hold. Without this modification the region Ω_h^c remains unchanged.

Example 1: Accuracy assessment with $d(\Gamma, \Gamma_h) = \mathcal{O}(h)$

In our first test we choose the domain Ω as the circle centered at $(0,0)$ with a radius of 0.75, and the data $\mathbf{f}$ and $\mathbf{g}$ such that the solution of problem (2.1) is given by $\mathbf{u} := (u_1, u_2)^{\mathrm{t}}$, where $u_1(x_1,x_2) := x_1\cos(x_1x_2)$ and $u_2(x_1,x_2) := -x_2\cos(x_1x_2)$, and $p(x_1,x_2) := x_1^4x_2^4 - p_0$, where $p_0 \in \mathbb{R}$ is chosen so that $p \in \mathrm{L}_0^2(\Omega)$, and $\mathbf{t}$ and $\boldsymbol{\sigma}$ are defined as in (2.1) and (2.5), respectively.

In Table 6.1 we report the convergence history obtained for this example under a sequence of uniform triangulations of a background mesh of Ω, see Figure 6.1. In this case we set Ω_h as the union of all elements inside Ω. Note that $d(\Gamma,\Gamma_h) = \mathcal{O}(h)$ for which assumptions (3.38) and (4.25) hold for h small enough, while (3.38) and (4.14) are not guaranteed theoretically. To compute the transferring paths from the vertices $\boldsymbol{x}_v$ of Γ_h, we use the algorithm in [14, Section 2.4.1] which uniquely determinate a point $\bar{\boldsymbol{x}}(\boldsymbol{x}_v)$ in Γ as the closest point to $\boldsymbol{x}_v$, as shown in the right panel of Figure 6.1. All conditions in

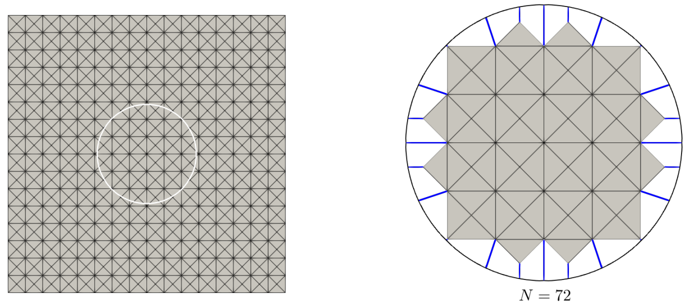

Figure 6.1: Example 1: *Left*, a background mesh of Ω and *right*, corresponding computational domain, shown in gray, and transferring paths from the vertices of Γ_h, shown in blue.

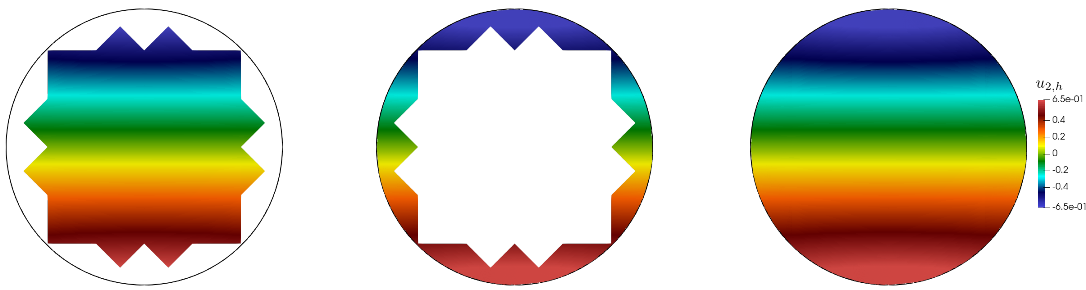

Figure 6.2: Example 1: Approximate velocity component $u_{2,h}$ obtained with $N = 72$ and $k = 3$.

Section 3.2 on the paths $\rho(\boldsymbol{x}_v)$ hold for this approach. If $\boldsymbol{x} \in \Gamma_h$ and $\boldsymbol{x} \neq \boldsymbol{x}_v$, we compute the path $\rho(\boldsymbol{x})$ using the procedure in Section 4.2.

From Table 6.1 we observe that, even when $\mathrm{e}(\mathbf{t})$, $\mathrm{e}(\boldsymbol{\sigma})$ and $\mathrm{e}(p)$ deteriorate for $N = 360$ and $k = 3$, an overall rate of convergence of $\mathcal{O}(h^{k+1})$ is obtained for $k = 1, 2, 3$ as predicted by Theorem 4.9 under the assumption that (3.28), (3.37), (4.14) and (4.25) hold. We furthermore observe that $\mathrm{e}^{\star}(\mathbf{u})$ converges with one order higher than the method, as expected from Section 5. In Figure 6.2 we display the approximation of the second component of the velocity for $N = 72$ and $k = 3$.

Example 2: Accuracy assessment with $d(\Gamma, \Gamma_h) = \mathcal{O}(h^2)$

This second example is aimed at evaluating the accuracy of the method, as well as the properties of the *a posteriori* error estimator through the effectivity index $\mathrm{eff}(\Theta)$, under a quasi-uniform refinement strategy. For this, we consider the same the domain Ω and data as in Example 1 and let Ω_h be the region enclosed by a piecewise linear interpolation of Γ. For every $e \in \mathcal{E}_h(\Gamma_h)$ the transferring paths associated to the interior points of e are chosen so that they perpendicular to this edge. All assumptions in this work can be now verified for h small enough since $d(\Gamma, \Gamma_h) = \mathcal{O}(h^2)$.

In Table 6.2, we present the convergence history obtained for this example. As expected, the method converges with $\mathcal{O}(h^{k+1})$ for $k = 1, 2, 3$. In Figure 6.3, we report the history of convergence of the estimator term Ψ in Theorem 5.3. From these results, we conclude that Ψ converges with the same order as the method, as anticipated in Section 5.4. Finally, the effectivity index $\mathrm{eff}(\Theta)$ is depicted in

		e(**u**)		e($\boldsymbol{\sigma}$)		e(**t**)		e*(**u**)		e(p)	
$k=1$											
N	iter	error	rate	error	rate	error	rate	error	rate	error	rate
72	2	1.66e-03	0.52	1.16e-01	–	1.75e-02	–	1.56e-03	–	3.90e-02	–
360	2	1.95e-04	2.67	2.37e-02	1.98	3.24e-03	2.10	1.64e-04	2.80	8.86e-03	1.84
1560	2	3.33e-05	2.41	3.45e-03	2.63	4.91e-04	2.57	1.34e-05	3.42	1.13e-03	2.80
7000	2	8.54e-06	1.81	9.11e-04	1.77	1.24e-04	1.83	2.99e-06	1.99	2.41e-04	2.06
28504	2	2.10e-06	2.00	1.59e-04	2.48	2.31e-05	2.40	3.26e-07	3.16	3.91e-05	2.59
$k=2$											
		e(**u**)		e($\boldsymbol{\sigma}$)		e(**t**)		e*(**u**)		e(p)	
N	iter	error	rate	error	rate	error	rate	error	rate	error	rate
72	2	4.08e-04	–	2.60e-02	–	3.72e-03	–	3.92e-04	–	1.45e-02	–
360	2	1.24e-04	1.48	1.25e-02	0.92	2.14e-03	0.69	1.23e-04	1.44	7.80e-03	0.77
1560	2	8.61e-07	6.78	1.17e-04	6.37	1.99e-05	6.38	6.04e-07	7.25	5.76e-05	6.69
7000	2	9.82e-08	2.89	1.60e-05	2.64	2.45e-06	2.79	5.92e-08	3.09	6.92e-06	2.82
28504	2	1.05e-08	3.19	1.23e-06	3.66	1.88e-07	3.66	3.03e-09	4.23	5.11e-07	3.71
$k=3$											
		e(**u**)		e($\boldsymbol{\sigma}$)		e(**t**)		e*(**u**)		e(p)	
N	iter	error	rate	error	rate	error	rate	error	rate	error	rate
72	2	8.31e-05	–	3.45e-03	–	7.33e-04	–	8.13e-05	–	1.59e-03	–
360	3	4.52e-05	0.76	6.71e-03	-0.83	1.04e-03	-0.43	4.40e-05	0.76	4.31e-03	-1.24
1560	2	7.80e-08	8.68	1.65e-05	8.20	2.56e-06	8.19	7.72e-08	8.66	1.04e-05	8.22
7000	2	4.23e-10	6.95	2.47e-07	5.60	1.24e-08	7.10	2.06e-10	7.89	5.10e-08	7.09
28504	2	2.37e-11	4.11	9.53e-09	4.64	4.79e-10	4.63	5.16e-12	5.25	2.08e-09	4.56

Table 6.1: Example 1: Convergence history of the errors under a uniform refinement strategy.

Figure 6.4. It is clear that eff(Θ) increases as k does. This behavior was also observed in [31] and it is in agreement with the estimates (5.5) and (5.28) since the constants C_{rel} and C_{eff} depend on the polynomial degree. Moreover, for each value of k, the effectivity index remains bounded, thus verifying not only the reliability of the *a posteriori* error estimator Θ, but also suggesting its efficiency.

Example 3: Adaptive refinement strategy

In our final example we choose Ω as the kidney-shaped domain whose boundary satisfies

$$\left(2\left[(x_1+0.5)^2+x_2^2\right]^2-x_1-0.5\right)^2-\left[(x_1+0.5)^2+x_2^2\right]+0.1=0.$$

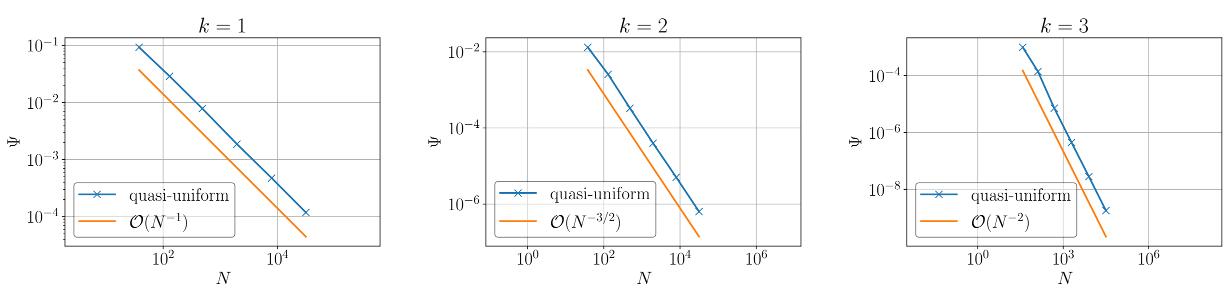

Figure 6.3: Example 2: Log-log plots of N vs. Ψ for a quasi-uniform refinement strategy.

		e($\mathbf{u}$)		e($\boldsymbol{\sigma}$)		e($\mathbf{t}$)		e*($\mathbf{u}$)		e(p)	
$k=1$											
N	iter	error	rate	error	rate	error	rate	error	rate	error	rate
38	2	1.47e-03	–	7.74e-02	–	1.02e-02	–	3.17e-04	–	1.54e-02	–
130	2	4.43e-04	1.95	2.30e-02	1.97	3.10e-03	1.94	5.40e-05	2.88	3.98e-03	2.20
486	2	1.24e-04	1.93	5.90e-03	2.06	8.61e-04	1.94	7.09e-06	3.08	1.05e-03	2.03
1961	2	2.88e-05	2.09	1.39e-03	2.07	2.11e-04	2.02	8.58e-07	3.03	2.67e-04	1.96
7950	2	7.55e-06	1.91	3.51e-04	1.97	5.35e-05	1.96	1.09e-07	2.95	6.59e-05	2.00
31566	2	1.90e-06	2.00	8.80e-05	2.00	1.34e-05	2.01	1.37e-08	3.01	1.64e-05	2.01
$k=2$											
N	iter	error	rate	error	rate	error	rate	error	rate	error	rate
38	2	2.35e-04	–	6.39e-03	–	1.19e-03	–	1.95e-05	–	1.77e-03	–
130	2	4.01e-05	2.87	1.16e-03	2.78	2.03e-04	2.87	1.98e-06	3.72	2.75e-04	3.03
486	2	5.35e-06	3.06	1.41e-04	3.20	2.65e-05	3.08	1.25e-07	4.19	3.51e-05	3.12
1961	2	6.20e-07	3.09	1.73e-05	3.01	3.20e-06	3.03	7.39e-09	4.06	4.26e-06	3.02
7950	2	8.14e-08	2.90	2.17e-06	2.96	4.05e-07	2.95	4.65e-10	3.95	5.37e-07	2.96
31566	2	1.02e-08	3.01	2.74e-07	3.00	5.09e-08	3.01	2.95e-11	4.00	6.73e-08	3.01
$k=3$											
N	iter	error	rate	error	rate	error	rate	error	rate	error	rate
38	2	1.98e-05	–	2.64e-04	–	6.06e-05	–	6.80e-07	–	9.04e-05	–
130	2	2.12e-06	3.63	2.92e-05	3.58	7.46e-06	3.40	7.45e-08	3.60	9.87e-06	3.60
486	2	1.31e-07	4.22	1.75e-06	4.27	4.01e-07	4.43	1.65e-09	5.77	5.61e-07	4.35
1961	2	7.80e-09	4.05	1.02e-07	4.08	2.58e-08	3.94	5.53e-11	4.87	3.41e-08	4.01
7950	2	5.00e-10	3.92	6.70e-09	3.89	1.60e-09	3.97	1.73e-12	4.95	2.16e-09	3.95
31566	2	3.15e-11	4.01	4.21e-10	4.01	1.02e-10	4.00	5.55e-14	4.99	1.37e-10	4.00

Table 6.2: Example 2: Convergence history of the errors under a quasi-uniform refinement strategy.

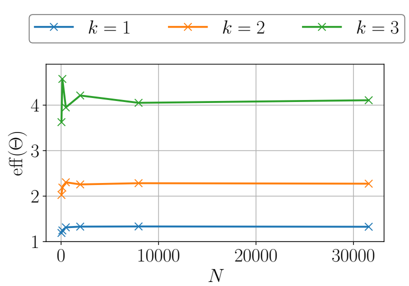

Figure 6.4: Example 2: N vs. eff(Θ) for a quasi-uniform refinement strategy.

We consider two manufactured solutions. In all of them $p(x_1,x_2) := \sin(x_1^2+x_2^2) - p_0$, where $p_0 \in \mathbb{R}$ is chosen in such a way $p \in \mathrm{L}_0^2(\Omega)$. In practice, we compute p_0 using an extremely fine polygonal mesh approximating Ω. Furthermore, the fluid velocity is given by two options,

$$\mathbf{u}(x_1,x_2) := \begin{pmatrix} 4(x_1+x_2)^3 + \frac{x_2}{\sqrt{(x_1+0.19)^2+x_2^2}} \\ -4(x_1+x_2)^3 - \frac{(x_1-0.55)}{\sqrt{(x_1-0.55)^2+x_2^2}} \end{pmatrix} \quad \text{and} \quad \mathbf{u}(x_1,x_2) := \begin{pmatrix} 6(x_1+x_2)^5 + \frac{1}{x_2+0.7} \\ -6(x_1+x_2)^5 \end{pmatrix},$$

and $\mathbf{t}$ and $\boldsymbol{\sigma}$ are defined as in (2.1) and (2.5), respectively. Note that one velocity exhibits a singularity at $(0.55,0)$, and one exhibits singularities along the line $\{(x_1,x_2)^{\mathrm{t}} \in \mathbb{R}^2 : x_2 = -0.7\}$ so that for both cases, high gradients of the velocity are likely to occur near the boundary Γ.

For this example the region of Ω^c intersecting with Ω_h has positive measure because Γ_h is constructed through a piecewise linear interpolation of Γ, and therefore, the validity of the error estimates in this work are not entirely verifiable as $\boldsymbol{\sigma}_h$ is given by (4.17). In fact, we fail to ensure that $\int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}_h) = 0$ for the latter case. To remedy this, we replace the constant $\omega_{\boldsymbol{\sigma}_h}$ in (4.17) by

$$\begin{aligned}\omega_{\boldsymbol{\sigma}_h} :=& -\frac{1}{2|\Omega_h|}\int_{\Omega_h^c} \mathrm{tr}\left(\boldsymbol{\sigma}_{0,h} - \frac{1}{2|\Omega|}\left(\int_{\Omega_h^c}\mathrm{tr}\,(\boldsymbol{\sigma}_{0,h}) - \int_{\Omega^c\cap\Omega_h}\mathrm{tr}\,(\boldsymbol{\sigma}_{0,h})\right)\mathbb{I}\right) \\ &+ \frac{1}{2|\Omega_h|}\int_{\Omega^c\cap\Omega_h} \mathrm{tr}\left(\boldsymbol{\sigma}_{0,h} - \frac{1}{2|\Omega|}\left(\int_{\Omega_h^c}\mathrm{tr}\,(\boldsymbol{\sigma}_{0,h}) - \int_{\Omega^c\cap\Omega_h}\mathrm{tr}\,(\boldsymbol{\sigma}_{0,h})\right)\mathbb{I}\right),\end{aligned}$$

where $\boldsymbol{\sigma}_{0,h} \in \mathbb{M}_{1,h}^0(\Omega_h)$. This is considered in our simulations. Furthermore, noting that $\int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) = 0$ from the system (2.6) and assuming $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}\,;\Omega_h\cup\Omega)$, we replace $\omega_{\boldsymbol{\sigma}}$ in (3.4) by

$$\omega_{\boldsymbol{\sigma}} := -\frac{1}{2|\Omega_h|}\left(\int_{\Omega_h^c}\mathrm{tr}\,(\boldsymbol{\sigma}) - \int_{\Omega^c\cap\Omega_h}\mathrm{tr}\,(\boldsymbol{\sigma})\right).$$

The proofs of the corresponding error estimates proceed now as in the case $\Omega_h \subseteq \Omega$.

In Figure 6.5, we report the decay of the total error with respect to N for quasi-uniform and adaptive refinement strategies. It is clear that the errors using the adaptive refinement are considerably smaller than when using quasi-uniform refinement. Moreover, the adaptive procedure reduces the magnitude of $\mathrm{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$ with optimal convergence of $\mathcal{O}(h^{k+1})$ for $k=1,2,3$. Some adapted meshes are depicted in Figure 6.6, from which is evident that the *a posteriori* error estimator Θ detects the singularities.

7 Concluding remarks

We have introduced a higher-order accurate unfitted mixed method for a class of nonlinear Stokes models with Dirichlet boundary condition. For this, we have extended the boundary-valued correction developed in [12, 14] to the dual-dual mixed formulation of our models. Therefore, the approximation to the solution is first computed on a convenient polyhedron Ω_h approximating the real domain Ω and then extended by local extrapolations to the region $\Omega \setminus \overline{\Omega}_h$. We have proven *a priori* error bounds under the assumption that (3.28), (3.37), (4.14) and (4.25) hold. Furthermore, we have provided *a posteriori* error estimates on Ω_h to be the region enclosed by a piecewise linear interpolation of the real boundary, for which all assumptions in this work hold. These estimates have been verified by numerical experiments in 2D. In particular, an adaptive algorithm associated to the proposed *a posteriori* error estimator has been improved the accuracy of the approximation under complex situations, such as

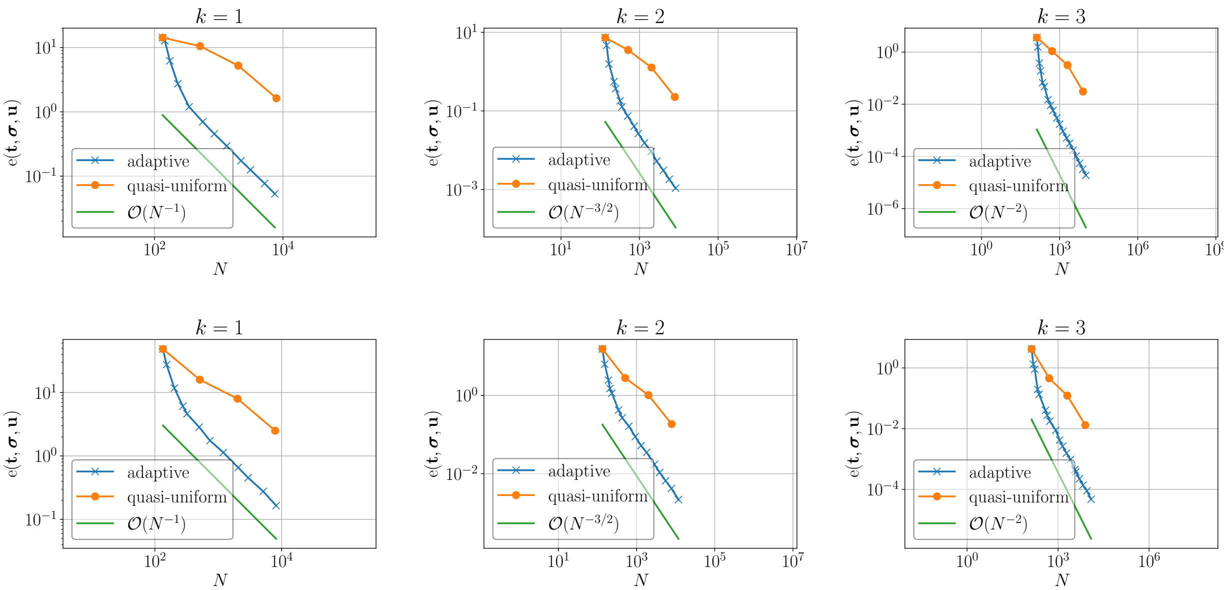

Figure 6.5: Example 3: Log-log plots of N vs. $\mathrm{e}(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})$ for both refinement strategies.

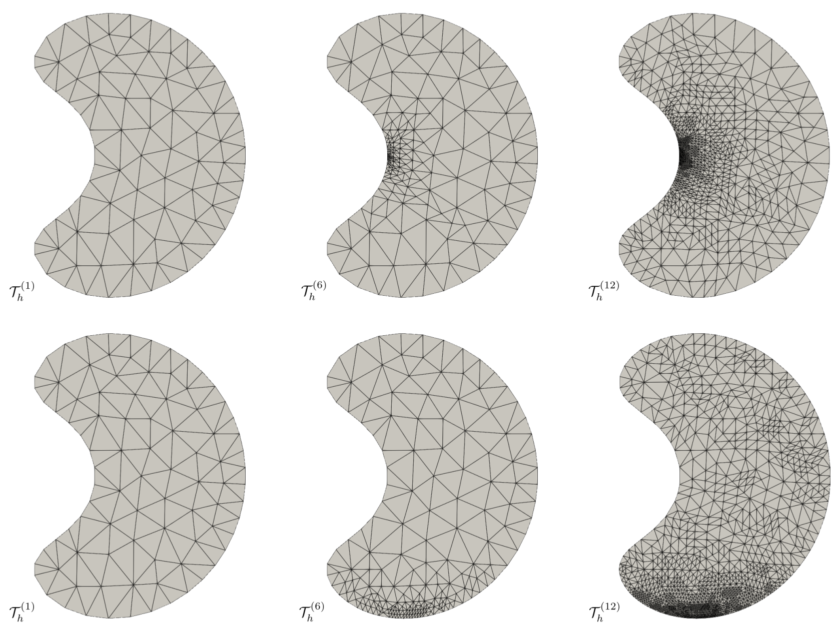

Figure 6.6: Example 3: Initial mesh and two adapted meshes obtained with the adaptive algorithm and $k = 2$ for the two solutions under consideration, one with high gradients near the nonconvex part of Ω (*top*), and one with high gradients near the convex part of Ω (*bottom*).

high gradients or singularities of the solution. The extension of the analysis to three dimension relies on the validity of the norm equivalence in Lemma 4.5, which has been successfully addressed in [31].

On the other hand, further research is needed to obtain an *a posteriori* error estimator for unfitted mixed methods where Ω_h arises from background meshes as in Example 1 of Section 6. This is ongoing work.

Acknowledgments

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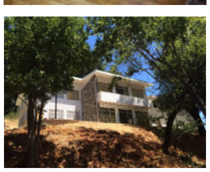

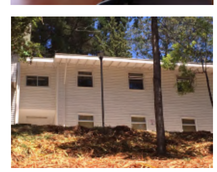

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