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A pseudostress-based mixed-primal finite element method for stress-assisted diffusion problems in Banach spaces*

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Abstract

In this paper we consider the system of partial differential equations describing the stress-assisted diffusion of a solute into an elastic material, and introduce and analyze a Banach spaces-based variational approach yielding a new mixed-primal finite element method for its numerical solution. The elasticity model involved, which is initially defined according to the constitutive relation given by Hooke's law, and whose momentum equation holds with a concentration-depending source term, is reformulated by using the non-symmetric pseudostress tensor and the displacement as the only unknowns of the associated mixed scheme, in addition to assuming a Dirichlet boundary condition for the latter. In turn, the diffusion equation, whose diffusivity function and source term depend on the pseudostress and the displacement of the solid, respectively, is set in primal form in terms of the concentration unknown and a Dirichlet boundary condition for it as well. The resulting coupled formulation is rewritten as an equivalent fixed point operator equation, so that its unique solvability is established by employing the classical Banach theorem along with the corresponding Babuška-Brezzi theory and the Lax-Milgram theorem. The aforementioned dependence of the diffusion coefficient and the subsequent treatment of this term in the continuous analysis, suggest to better look for the solid unknowns in suitable Lebesgue spaces. The discrete analysis is performed similarly, and the Brouwer theorem yields existence of a Galerkin solution. A priori error estimates are derived, and rates of convergence for specific finite element subspaces satisfying the required discrete inf-sup conditions, are established in 2D. Finally, several numerical examples illustrating the performance of the method and confirming the theoretical convergence, are reported.

Key words: linear elasticity, stress-assisted diffusion, fixed point theory, finite element methods

Mathematics subject classifications (2000): 65N30, 65N12, 76R05, 76D07, 65N15, 35Q79

1 Introduction

The so-called stress-assisted diffusion models, which refer to diffusion processes in deformable solids, are present in diverse applications, which include, among others, diffusion of boron and arsenic in silicon [22], voiding of aluminum conductor lines in integrated circuits [26], sorption in polymers [23], damage of electrodes in lithium ion batteries [3], and anisotropy of cardiac dynamics [7]. The usual assumptions in most of these models are, on one hand, that the solid follows an elastic regime, and on

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the other hand, that the diffusion obeys a Fickean law enriched with further contributions arising from local effects by exerted stresses. Mathematically, this second hypothesis means that the respective diffusion coefficient is a continuous function depending precisely on the stress, which acts then as a coupling variable.

While many contributions on the modelling of stress-assisted (and even strain-assisted) diffusion problems are available in the literature, the same can not be said of the corresponding mathematical and numerical analyses of them, which are rather scarce. Indeed, for the first of the latter issues we can mention the recent works [21], [25], and [12], which deal with a general local-global well-posedness theory for static and transient problems via a primal formulation, homogenization of concentration - electric potential systems, and multiscale analysis of the deterioration of binder in electrodes, respectively. In turn, concerning the second of those issues, and up to our knowledge, we can only refer to [16] and [17], where mixed-primal and fully-mixed finite element methods have been introduced and analyzed to numerically solve the stationary problem describing the diffusion of a solute into an elastic material. This diffusion-deformation model is represented by the linear elasticity equations along with a diffusion equation whose function of diffusivity depends on the Cauchy stress of the solid. Further interactions between them are given by the corresponding source terms, which depend on the concentration and the displacement, respectively. In other words, the diffusing species affects the behavior of the solid, whereas the displacement of the latter influences the solute concentration, both through the corresponding external forces, thus yielding a two-way coupled system.

Regarding further details on [16] and [17], we first notice that the approach in [16] follows the usual methodology for the dual-mixed formulation of the linear elasticity problem (cf. [6], [14]), so that the symmetry of the Cauchy stress is imposed weakly through the incorporation of the tensor of solid rotations as the corresponding Lagrange multiplier. In contrast, a primal formulation is employed for the diffusion equation. The well-posedness of the resulting coupled variational formulation is addressed by means of a fixed-point strategy and by applying the Lax-Milgram lemma, the Babuška-Brezzi theory, Sobolev embedding theorems, and suitable regularity estimates. In this way, the Schauder and Banach fixed-point theorems allow to establish existence and uniqueness of continuous solution, respectively. An analogue reasoning is applied to analyze the associated Galerkin scheme and an augmented version of it (for the elasticity equations only), thus deriving existence of discrete solutions, as well as corresponding a priori error estimates and rates of convergence, by employing the Brouwer theorem and a Strang-type lemma.

In turn, while keeping the same dual-mixed scheme for the elasticity equations, an augmented mixed formulation instead of the primal one from [16] is utilized in [17] for the diffusion equation. In addition, similarly to previous works (see, e.g. [19]), the concentration gradient and the diffusive flux are introduced as further unknowns for a more suitable treatment of the nonlinearity arising from the stress-dependent diffusivity. The rest of the continuous and discrete analyses in [17] follows by applying basically the same theoretical tools utilized in [16]. In particular, we highlight that two families of finite element subspaces yielding stable Galerkin schemes are proposed, namely either PEERS or Arnold-Falk-Winther elements for elasticity, and Raviart-Thomas and piecewise polynomials for the mixed formulation of the diffusion equation. We end our discussion on [16] and [17] by pointing out that a significant drawback of their approaches is given by the use of a regularity result for the uncoupled elasticity problem (cf. [16, Theorem 2.4]), which is valid only for convex domains in 2D. In this regard, we remark that the need of this result arises from the handling of the stress-dependent diffusion term when trying to prove a Lipschitz-continuity property of one of the components of the continuous fixed-point operator.

According to the above discussion, and in order to overcome the aforementioned drawback, we have recently realized that the required Lipschitz-continuity property can be established, without any regularity nor convexity assumptions for the linear elasticity problem, by previously restating the whole coupled variational formulation in terms of suitable Lebesgue and Sobolev-type Banach spaces. Moreover, the continuous and discrete analyses can be carried out in this case without employing any augmentation procedure, thus simplifying the computational complexity of the resulting discrete scheme. The purpose of the present work is precisely to introduce and analyze, at the continuous and discrete levels, this new Banach spaces-based formulation for the stress-assisted diffusion problem studied in [16] and [17]. In doing so, we will resort to some of the results provided in our recent related works [18] and [20]. Moreover, because of greater interest in applications, we consider the nearly incompressible case in linear elasticity, and for sake of further simplicity of its analysis, we adopt a pseudostress-based approach instead of the usual stress-based one.

The rest of the paper is organized as follows. Required notations and basic definitions are collected at the end of this introductory section. In Section 2 we introduce the stress-assisted diffusion model and reformulate the elasticity problem in terms of the non-symmetric pseudostress tensor. The continuous formulation is derived in Section 3, and its solvability is studied by means of a fixed-point strategy that arises after decoupling the model into the elasticity and diffusion problems. In turn, the well-posedness of each one of the latter is deduced by applying the Babuška-Brezzi theory in Banach spaces and the classical Lax-Milgram theorem, respectively, whereas the unique solvability of the whole coupled model is concluded thanks to the Banach fixed-point theorem. In Section 4 we consider arbitrary finite element subspaces, assume that they satisfy suitable stability conditions, and employ the discrete version of the fixed-point strategy introduced in Section 3 to analyze the solvability of the associated Galerkin scheme. In this way, and along with the corresponding versions of the theoretical tools employed in Section 3, a straightforward application of Brouwer's theorem allows us to conclude the existence of discrete solution. An a priori error estimate in the form of Cea's estimate is also derived here. Next, in Section 5 we restrict ourselves to the 2D case and introduce specific finite element subspaces satisfying the theoretical hypotheses that were assumed in Section 4. Actually, the latter refer only to a couple of discrete inf-sup conditions for the elasticity equation since any finite element subspace will work for the diffusion model. The lack of a required boundedness property for a particular projector involved stops us of extending the analysis from Section 5 to the 3D case. Finally, several numerical results illustrating the performance of the method and confirming the theoretical rates of convergence provided in Section 5, are reported in Section 6.

Preliminary notations

Throughout the paper, Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star shaped with respect to a ball, and whose outward normal at $\Gamma := \partial\Omega$ is denoted by $\boldsymbol{\nu}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$ and Sobolev spaces $\mathrm{W}^{l,t}(\Omega)$ and $\mathrm{W}^{l,t}_0(\Omega)$, with $l \geq 0$ and $t \in [1,+\infty)$, whose corresponding norms, either for the scalar and vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{l,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and if $t = 2$ we write $\mathrm{H}^l(\Omega)$ instead of $\mathrm{W}^{l,2}(\Omega)$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$, respectively. In addition, letting t, $t' \in (1,+\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we denote by $\mathrm{W}^{1/t',t}(\Gamma)$ the trace space of $\mathrm{W}^{1,t}(\Omega)$, and let $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the dual of $\mathrm{W}^{1/t',t}(\Gamma)$ endowed with the norms $\|\cdot\|_{-1/t',t';\Gamma}$ and $\|\cdot\|_{1/t',t;\Gamma}$, respectively. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$ will be employed for the norm of any element or operator whenever there is no confusion about the spaces to which they belong. Furthermore, as usual, $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathrm{R}^n$. Also, for any vector field $\boldsymbol{v} = (v_i)_{i=1,n}$ we set the gradient and divergence operators, respectively, as

$$\nabla\boldsymbol{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n} \quad \text{and} \quad \mathrm{div}(\boldsymbol{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}\,.$$

Additionally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product operators, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} = (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{n} \boldsymbol{\tau}_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}\,, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}\,.$$

2 The model problem

The stress-assisted diffusion problem studied in [16] and [17], which models the diffusion of a solute into an elastic material occupying the domain Ω, is described by the following system of partial differential equations:

$$\begin{aligned} \boldsymbol{\rho} \,=\, \mathcal{C}\big(\mathbf{e}(\boldsymbol{u})\big) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\rho}) \,=\, \boldsymbol{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \boldsymbol{u} = \boldsymbol{u}_D \quad \text{on} \quad \Gamma\,, \\ \widetilde{\boldsymbol{\sigma}} = \widetilde{\vartheta}(\mathbf{e}(\boldsymbol{u}))\nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) = g(\boldsymbol{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\rho}$ is the Cauchy solid stress, $\boldsymbol{u}$ is the displacement field, $\mathbf{e}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^{\mathtt{t}})$ is the infinitesimal strain tensor (symmetrised gradient of displacements), and $\mathcal{C}$ stands for the linear operator defining the Hooke law (cf. [14, eq. (2.36)]), that is

$$\mathcal{C}\big(\mathbf{e}(\boldsymbol{u})\big) \;:=\; \lambda\,\mathrm{tr}\big(\mathbf{e}(\boldsymbol{u})\big)\,\mathbb{I} \,+\, 2\mu\,\mathbf{e}(\boldsymbol{u})\,, \tag{2.2}$$

with the Lamé constants λ, $\mu > 0$ (dilation and shear moduli) characterizing the properties of the material. In turn, ϕ represents the local concentration of species, $\widetilde{\boldsymbol{\sigma}}$ is the diffusive flux, and $\widetilde{\vartheta} : \mathbb{R} \to \mathbb{R}$ is a tensorial diffusivity function. Finally, $\boldsymbol{f} : \mathrm{R} \to \mathbf{R}$ is a vector field of body loads (which depends on the species concentration), $g : \mathbf{R} \to \mathrm{R}$ denotes an additional source term depending on the solid displacement $\boldsymbol{u}$, and $\boldsymbol{u}_D$ is the Dirichlet datum for $\boldsymbol{u}$, which belongs to a suitable trace space to be identified later on. Specific requirements on $\boldsymbol{f}$ and g will be given below. We note that system (2.1) describes the constitutive relations inherent to linear elastic materials, conservation of linear momentum, the constitutive description of diffusive fluxes, and the mass transport of the diffusive substance, respectively. It also assumes that diffusive time scales are much lower than those of the elastic wave propagation, justifying the static character of the system (cf. [21]).

On the other hand, in this work we are particularly interested in the nearly incompressible case, which reduces to assume from now on that λ is sufficiently large. In addition, in order to avoid the weak imposition of the symmetry of $\boldsymbol{\rho}$, we now reformulate (2.1) in terms of the non-symmetric pseudostress tensor $\boldsymbol{\sigma}$ introduced in [15]. More precisely, according to the analysis provided in [15, Section 2.1], we know that the first row of (2.1) is equivalent to

$$\boldsymbol{\sigma} \,=\, \widehat{\mathcal{C}}\big(\nabla\boldsymbol{u}\big) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) = \boldsymbol{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \boldsymbol{u} = \boldsymbol{u}_D \quad \text{on} \quad \Gamma\,, \tag{2.3}$$

where

$$\widehat{\mathcal{C}}\big(\nabla\boldsymbol{u}\big) \;:=\; (\lambda+\mu)\,\mathrm{tr}\big(\nabla\boldsymbol{u}\big)\mathbb{I} \,+\, \mu\nabla\boldsymbol{u}\,. \tag{2.4}$$

Hence, bearing in mind (2.4) and applying matrix trace to the first equation of (2.3), we can express $\mathrm{tr}(\nabla\boldsymbol{u})$ in terms of $\mathrm{tr}(\boldsymbol{\sigma})$ (cf. [15, eq. (2.3)]), so that the former is eliminated and (2.3) is rewritten, equivalently, as

$$\nabla\boldsymbol{u} \,=\, \widehat{\mathcal{C}}^{-1}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) \,=\, \boldsymbol{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \boldsymbol{u} \,=\, \boldsymbol{u}_D \quad \text{on} \quad \Gamma\,, \tag{2.5}$$

where

$$\widehat{\mathcal{C}}^{-1}(\boldsymbol{\sigma}) \;:=\; \frac{1}{\mu}\,\boldsymbol{\sigma}^{\mathrm{d}} \,+\, \frac{1}{n\big(n\lambda + (n+1)\mu\big)}\;\mathrm{tr}(\boldsymbol{\sigma})\,\mathbb{I}\,. \tag{2.6}$$

We point out here that the original Cauchy stress tensor $\boldsymbol{\rho}$ can be expressed in terms of the pseudostress $\boldsymbol{\sigma}$ through the formula

$$\boldsymbol{\rho} \,=\, \boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{t}} \,-\, \frac{\lambda + 2\mu}{n\lambda + (n+1)\mu}\,\mathrm{tr}(\boldsymbol{\sigma})\,\mathbb{I}\,. \tag{2.7}$$

In turn, using (2.7) and the first equation of (2.1), from which we get $\mathbf{e}(\boldsymbol{u}) \,=\, \mathcal{C}^{-1}(\boldsymbol{\rho})$, where (cf. [14, Section 2.4.3])

$$\mathcal{C}^{-1}(\boldsymbol{\rho}) \,:=\, \frac{1}{2\mu}\,\boldsymbol{\rho} \,-\, \frac{\lambda}{2\mu(n\lambda + 2\mu)}\,\mathrm{tr}(\boldsymbol{\rho})\,\mathbb{I}\,,$$

we can recast the strain-dependent diffusivity $\widetilde{\vartheta}(\mathbf{e}(\boldsymbol{u}))$ as a pseudostress-dependent diffusivity $\vartheta(\boldsymbol{\sigma})$. In this way, we finally obtain that the model (2.1) can be restated as

$$\begin{aligned} \nabla\boldsymbol{u} \,&=\, \widehat{\mathcal{C}}^{-1}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) \,=\, \boldsymbol{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \boldsymbol{u} = \boldsymbol{u}_D \quad \text{on} \quad \Gamma\,, \\ \widetilde{\boldsymbol{\sigma}} \,&=\, \vartheta(\boldsymbol{\sigma})\,\nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \,=\, g(\boldsymbol{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{2.8}$$

Throughout this work, we suppose that ϑ is of class C^1 and uniformly positive definite, meaning the latter that there exists $\vartheta_0 > 0$ such that

$$\vartheta(\boldsymbol{\tau})\boldsymbol{w}\cdot\boldsymbol{w} \,\geq\, \vartheta_0\,|\boldsymbol{w}|^2 \quad \forall\,\boldsymbol{w} \in \mathbf{R}\,, \quad \forall\,\boldsymbol{\tau} \in \mathbb{R}\,. \tag{2.9}$$

We also require uniform boundedness and Lipschitz continuity of ϑ, that is that there exist positive constants ϑ_1, ϑ_2 and L_ϑ, such that

$$\vartheta_1 \,\leq\, |\vartheta(\boldsymbol{\tau})| \,\leq\, \vartheta_2 \quad \text{and} \quad |\vartheta(\boldsymbol{\tau}) - \vartheta(\boldsymbol{\zeta})| \,\leq\, L_\vartheta\,|\boldsymbol{\tau} - \boldsymbol{\zeta}| \quad \forall\,\boldsymbol{\tau},\boldsymbol{\zeta} \in \mathbb{R}\,. \tag{2.10}$$

Similar hypotheses are assumed on the source functions $\boldsymbol{f}$ and g, which means that there exist positive constants f_1, f_2, L_f, g_1, g_2 and L_g, such that

$$f_1 \,\leq\, |\boldsymbol{f}(s)| \,\leq\, f_2\,, \quad |\boldsymbol{f}(s) - \boldsymbol{f}(t)| \,\leq\, L_f\,|s - t| \quad \forall\,s,t \in \mathrm{R}\,, \tag{2.11}$$

$$g_1 \,\leq\, |g(\boldsymbol{w})| \,\leq\, g_2\,, \quad \text{and} \quad |g(\boldsymbol{v}) - g(\boldsymbol{w})| \,\leq\, L_g\,|\boldsymbol{v} - \boldsymbol{w}| \quad \forall\,\boldsymbol{v},\boldsymbol{w} \in \mathbf{R}\,. \tag{2.12}$$

3 The continuous formulation

In this section we introduce a suitable Banach spaces-based variational formulation for (2.8), and then analyze its solvability by means of a fixed-point strategy.

3.1 The mixed-primal formulation

We begin by noticing, as suggested by the Dirichlet boundary condition satisfied by the concentration ϕ, that the appropriate trial and test space reduces in this case to

$$\mathrm{H}^1_0(\Omega) = \Big\{\psi \in \mathrm{H}^1(\Omega) : \quad \psi = 0 \quad \text{on} \quad \Gamma\Big\}\,.$$

Thus, performing the usual integration by parts procedure in $\mathrm{H}^1(\Omega)$, the primal formulation for the diffusion equation becomes: find $\phi \in \mathrm{H}^1_0(\Omega)$ such that

$$A_{\boldsymbol{\sigma}}(\phi,\psi) \,=\, G_{\boldsymbol{u}}(\psi) \qquad \forall\,\psi \in \mathrm{H}^1_0(\Omega)\,, \tag{3.1}$$

where, given $\boldsymbol{\zeta}$ and $\boldsymbol{w}$ lying, respectively, in the same spaces where $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ will be sought,

$$A_{\boldsymbol{\zeta}}(\phi,\psi) \,:=\, \int_\Omega \vartheta(\boldsymbol{\zeta})\nabla\phi\cdot\nabla\psi \qquad \forall\,\phi,\,\psi \in \mathrm{H}^1_0(\Omega)\,, \tag{3.2}$$

and

$$G_{\boldsymbol{w}}(\psi) := \int_\Omega g(\boldsymbol{w})\,\psi \qquad \forall\,\psi \in \mathrm{H}_0^1(\Omega)\,. \tag{3.3}$$

Next, before proceeding with the elasticity equations, we remark that in order to study the continuity property of the diffusivity function ϑ within the definition of the bilinear form A (cf. (3.2)), which will be required for the solvability analysis of the fixed-point operator equation to be proposed afterwards, we need to be able to control the expression

$$\int_\Omega (\vartheta(\boldsymbol{\tau}) - \vartheta(\boldsymbol{\zeta}))\,\nabla\phi\cdot\nabla\psi\,, \tag{3.4}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$ are generic tensors belonging to the same space in which we will seek the unknown $\boldsymbol{\sigma}$. In this regard, and employing the Lipschitz-continuity property of ϑ (cf. (2.10)), straightforward applications of the Cauchy-Schwarz and Hölder inequalities yield

$$\left|\int_\Omega (\vartheta(\boldsymbol{\tau}) - \vartheta(\boldsymbol{\zeta}))\,\nabla\phi\cdot\nabla\psi\right| \,\le\, L_\vartheta\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,2p;\Omega}\,\|\nabla\phi\|_{0,2q;\Omega}\,\|\nabla\psi\|_{0,\Omega}\,, \tag{3.5}$$

where p, $q \in (1,+\infty)$ are conjugate to each other, which makes sense for $\boldsymbol{\tau}$, $\boldsymbol{\zeta} \in \mathbb{L}^{2p}(\Omega)$ and $\nabla\psi \in \mathbf{L}^{2q}(\Omega)$. In this way, the above leads us to initially look for $\boldsymbol{\sigma}$ in the space $\mathbb{L}^r(\Omega)$, with $r := 2p$. The specific choice of r will be discussed later on, so that meanwhile we consider a generic r and let $s \in (1,+\infty)$ be its respective conjugate. In turn, a suitable bounding of the expression $\|\nabla\phi\|_{0,2q;\Omega}$ in (3.5) for a particular ϕ will also be explained subsequently by means of a regularity argument.

Having set the above preliminary choice for the space to which $\boldsymbol{\sigma}$ belongs, it follows now from (2.6) and the first equation of (2.8) that $\boldsymbol{u}$ should be initially sought in $\mathbf{W}^{1,r}(\Omega)$. Thus, in order to derive the variational formulation of the elasticity equations, we need to invoke a suitable integration by parts formula. Indeed, we first introduce for each $t \in (1,+\infty)$ the Banach space

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^t(\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}\,, \tag{3.6}$$

which is endowed with the natural norm defined as

$$\|\boldsymbol{\tau}\|_{t,\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\,. \tag{3.7}$$

Then, given t, $t' \in (1,+\infty)$ conjugate to each other, there holds (cf. [11, Corollary B. 57])

$$\langle \boldsymbol{\tau\nu}, \boldsymbol{v}\rangle_\Gamma \,=\, \int_\Omega \Big\{\boldsymbol{\tau} : \nabla\boldsymbol{v} \,+\, \boldsymbol{v}\cdot\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{H}^t\,(\mathbf{div}_t;\Omega)\times\mathbf{W}^{1,t'}(\Omega)\,, \tag{3.8}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathbf{W}^{-1/t,t}(\Gamma)$ and $\mathbf{W}^{1/t,t'}(\Gamma)$. Moreover, thanks to the surjectivity of the trace operator $\gamma_{0,t'} : \mathbf{W}^{1,t'}(\Omega) \longrightarrow \mathbf{W}^{1/t,t'}(\Gamma)$, a straightforward application of the open mapping theorem and (3.8) yield the existence of a constant $C_{t'} > 0$ such that

$$\|\boldsymbol{\tau\nu}\|_{-1/t,t;\Gamma} \,\le\, C_{t'}\,\|\boldsymbol{\tau}\|_{t,\mathbf{div}_t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}^t\,(\mathbf{div}_t;\Omega)\;. \tag{3.9}$$

Now, applying (3.8) with $t = s$ and $t' = r$ to $\boldsymbol{u} \in \mathbf{W}^{1,r}(\Omega)$ and $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$, and using the Dirichlet boundary condition satisfied by $\boldsymbol{u}$, for which we assume from now on that $\boldsymbol{u}_D \in \mathbf{W}^{1/s,r}(\Gamma)$, we find that

$$\int_\Omega \boldsymbol{\tau} : \nabla\boldsymbol{u} \,=\, -\int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \langle\boldsymbol{\tau\nu},\boldsymbol{u}_D\rangle_\Gamma\,, \tag{3.10}$$

so that, according to (2.6), the testing of the first equation of (2.8) against $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$ gives

$$\frac{1}{\mu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{n(n\lambda+(n+1)\mu)}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma})\,\mathrm{tr}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \langle\boldsymbol{\tau\nu},\boldsymbol{u}_D\rangle_\Gamma\;. \tag{3.11}$$

It follows from the third term on the left hand side of (3.11) that actually it suffices to look for $\boldsymbol{u}$ in $\mathbf{L}^r(\Omega)$. Furthermore, testing the second equation of (2.8), also named equilibrium equation, against $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$, we obtain

$$\int_\Omega \boldsymbol{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) \,=\, -\int_\Omega \boldsymbol{f}(\phi) \cdot \boldsymbol{v}\,, \tag{3.12}$$

which makes sense for $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^r(\Omega)$, and hence $\boldsymbol{\sigma}$ is sought from now in $\mathbb{H}^r(\mathbf{div}_r;\Omega)$. To be more precise about the latter, we notice that for each $t \in (1,+\infty)$ there holds the decomposition

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) \,=\, \mathbb{H}^t_0(\mathbf{div}_t;\Omega) \oplus \mathrm{R}\mathbb{I}\,,$$

where

$$\mathbb{H}^t_0(\mathbf{div}_t;\Omega) \,:=\, \left\{ \boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \right\}.$$

Equivalently, each $\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)$ can be decomposed, uniquely, as

$$\boldsymbol{\tau} \,=\, \boldsymbol{\tau}_0 + d\,\mathbb{I}, \quad \text{with} \quad \boldsymbol{\tau}_0 \in \mathbb{H}^t_0(\mathbf{div}_t;\Omega) \quad \text{and} \quad d := \frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \in \mathrm{R}\,. \tag{3.13}$$

In this way, taking $\boldsymbol{\tau} = \mathbb{I}$ in (3.11) we get

$$\frac{1}{n\lambda + (n+1)\mu}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, \int_\Gamma \boldsymbol{u}_D \cdot \nu\,,$$

from which, along with an application of (3.13) to $t = r$ and $\boldsymbol{\tau} = \boldsymbol{\sigma} \in \mathbb{H}^r(\mathbf{div}_r;\Omega)$, we deduce that

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\,\mathbb{I}\,, \quad \text{with} \quad \boldsymbol{\sigma}_0 \in \mathbb{H}^r_0(\mathbf{div}_r;\Omega) \quad \text{and} \quad c := \frac{n\lambda + (n+1)\mu}{n|\Omega|}\int_\Gamma \boldsymbol{u}_D \cdot \boldsymbol{\nu} \in \mathrm{R}\,. \tag{3.14}$$

The above shows that, in order to attain the full explicit knowledge of the unknown $\boldsymbol{\sigma}$, it only remains to find its $\mathbb{H}^r_0(\mathbf{div}_r;\Omega)$-component $\boldsymbol{\sigma}_0$. Therefore, replacing $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\,\mathbb{I}$ back into (3.11), redenoting $\boldsymbol{\sigma}_0$ simply by $\boldsymbol{\sigma}$, replacing $\vartheta(\boldsymbol{\sigma})$ by $\vartheta(\boldsymbol{\sigma} + c\mathbb{I})$ in the diffusion equation, noting that the testing of the resulting (3.11) against $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}^s_0(\mathbf{div}_s;\Omega)$, and placing this new equation jointly with (3.12), we arrive at the following mixed variational formulation of the first row of (2.8): Find $(\boldsymbol{\sigma},\boldsymbol{u}) \in X_2 \times M_1$ such that

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b_1(\boldsymbol{\tau},\boldsymbol{u}) \;&=\; G(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in X_1,\\ b_2(\boldsymbol{\sigma},\boldsymbol{v}) \;&=\; F_\phi(\boldsymbol{v}) \qquad \forall\,\boldsymbol{v} \in M_2, \end{aligned} \tag{3.15}$$

where

$$X_2 := \mathbb{H}^r_0(\mathbf{div}_r;\Omega)\,, \quad M_1 := \mathbf{L}^r(\Omega)\,, \quad X_1 := \mathbb{H}^s_0(\mathbf{div}_s;\Omega) \quad \text{and} \quad M_2 := \mathbf{L}^s(\Omega)\,, \tag{3.16}$$

and the bilinear forms $a : X_2 \times X_1 \to \mathrm{R}$ and $b_i : X_i \times M_i \to \mathrm{R}$, $i \in \{1,2\}$, and the functionals $F_\phi \in M_2'$ and $G \in X_1'$, are defined, respectively, as

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) := \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{n\big(n\lambda + (n+1)\mu\big)}\int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\,\mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau}) \in X_2 \times X_1\,, \tag{3.17}$$

$$b_i(\boldsymbol{\tau},\boldsymbol{v}) \,:=\, \int_\Omega \boldsymbol{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in X_i \times M_i\,, \tag{3.18}$$

$$G(\boldsymbol{\tau}) \,:=\, \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{u}_D \rangle_\Gamma, \qquad \forall\,\boldsymbol{\tau} \in X_1, \tag{3.19}$$

$$F_\phi(\boldsymbol{v}) \,:=\, -\int_\Omega \boldsymbol{f}(\phi) \cdot \boldsymbol{v} \qquad \forall\,\boldsymbol{v} \in M_2. \tag{3.20}$$

In this way, the mixed-primal formulation of (2.8) reduces to (3.15) and (3.1), that is: Find $(\boldsymbol{\sigma}, \boldsymbol{u}, \phi) \in X_2 \times M_1 \times \mathrm{H}_0^1(\Omega)$ such that

$$
\begin{aligned}
a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b_1(\boldsymbol{\tau}, \boldsymbol{u}) &= G(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in X_1 \,, \\
b_2(\boldsymbol{\sigma}, \boldsymbol{v}) &= F_\phi(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in M_2 \,, \\
A_{\boldsymbol{\sigma}}(\phi, \psi) &= G_{\boldsymbol{u}}(\psi) \qquad \forall\, \psi \in \mathrm{H}_0^1(\Omega) \,.
\end{aligned} \tag{3.21}
$$

3.2 Fixed-point approach

In this section we follow a similar approach to those employed in previous works, e.g. in [2], [9], [16], and [20], and make use of the decoupled variational formulations (3.15) and (3.1) to introduce a fixed-point strategy for the solvability analysis of (3.21). Indeed, we first let $\mathrm{S} : \mathrm{H}_0^1(\Omega) \to X_2 \times M_1$ be the operator defined for each $\varphi \in \mathrm{H}_0^1(\Omega)$ as $\mathrm{S}(\varphi) := (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}})$, where $(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}}) \in X_2 \times M_1$ is the unique solution (to be confirmed below) of (3.15) with φ instead of ϕ, that is

$$
\begin{aligned}
a(\widetilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}) \,+\, b_1(\boldsymbol{\tau}, \widetilde{\boldsymbol{u}}) &= G(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in X_1, \\
b_2(\widetilde{\boldsymbol{\sigma}}, \boldsymbol{v}) &= F_\varphi(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in M_2 \,.
\end{aligned} \tag{3.22}
$$

In turn, we let $\widetilde{\mathrm{S}} : X_2 \times M_1 \to \mathrm{H}_0^1(\Omega)$ be the operator defined for each $(\boldsymbol{\zeta}, \boldsymbol{w}) \in X_2 \times M_1$ as $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) := \widetilde{\phi}$, where $\widetilde{\phi} \in \mathrm{H}_0^1(\Omega)$ is the unique solution (to be confirmed below as well) of (3.1) with $(\boldsymbol{\zeta}, \boldsymbol{w})$ instead $(\boldsymbol{\sigma}, \boldsymbol{u})$, that is

$$
A_{\boldsymbol{\zeta}}(\widetilde{\phi}, \psi) \,=\, G_{\boldsymbol{w}}(\psi) \qquad \forall\, \psi \in \mathrm{H}_0^1(\Omega) \,. \tag{3.23}
$$

Thus, we define the operator $T : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ as

$$
T(\varphi) \,:=\, \widetilde{\mathrm{S}}\big(\mathrm{S}(\varphi)\big) \qquad \forall\, \varphi \in \mathrm{H}_0^1(\Omega) \,, \tag{3.24}
$$

and notice that solving (3.21) is equivalent to seeking a fixed point of T, that is $\phi \in \mathrm{H}_0^1(\Omega)$ such that

$$
T(\phi) \,=\, \phi \,. \tag{3.25}
$$

3.3 Well-posedness of the uncoupled problems

3.3.1 Some preliminary results

We begin with the Babuška-Brezzi theorem in Banach spaces.

Theorem 3.1. *Let H_1, H_2, Q_1 and Q_2 be real reflexive Banach spaces, and let $a : H_2 \times H_1 \to \mathrm{R}$ and $b_i : H_i \times Q_i \to \mathrm{R}$, $i \in \{1, 2\}$, be bounded bilinear forms with boundedness constants given by $\|a\|$ and $\|b_i\|$, $i \in \{1, 2\}$, respectively. In addition, for each $i \in \{1, 2\}$, let K_i be the kernel of the operator induced by b_i, that is*

$$
K_i \,:=\, \Big\{ \tau \in H_i : \quad b_i(\tau, v) = 0 \quad \forall\, v \in Q_i \Big\} \,.
$$

Assume that

i) *there exists $\alpha > 0$ such that*

$$
\sup_{\substack{\tau \in K_1 \\ \tau \neq 0}} \frac{a(\zeta, \tau)}{\|\tau\|_{H_1}} \,\geq\, \alpha \, \|\zeta\|_{H_2} \qquad \forall\, \zeta \in K_2 \,,
$$

ii) *there holds*

$$\sup_{\zeta\in K_2} a(\zeta,\tau) \,>\, 0 \qquad \forall\,\tau\in K_1\,,\ \tau\neq 0\,,$$

iii) *for each* $i\in\{1,2\}$ *there exists* $\beta_i>0$ *such that*

$$\sup_{\substack{\tau\in H_i\\ \tau\neq 0}} \frac{b_i(\tau,v)}{\|\tau\|_{H_i}} \,\geq\, \beta_i\,\|v\|_{Q_i} \qquad \forall\,v\in Q_i\,.$$

Then, for each $(F,G)\in H_1'\times Q_2'$ *there exists a unique* $(\sigma,u)\in H_2\times Q_1$ *such that*

$$\begin{aligned} a(\sigma,\tau)+b_1(\tau,u) &= F(\tau) \qquad \forall\,\tau\in H_1\,,\\ b_2(\sigma,v) &= G(v) \qquad \forall\,v\in Q_2\,, \end{aligned} \tag{3.26}$$

and the following a priori estimates hold:

$$\begin{aligned} \|\sigma\|_{H_2} &\leq \frac{1}{\alpha}\,\|F\|_{H_1'} + \frac{1}{\beta_2}\left(1+\frac{\|a\|}{\alpha}\right)\|G\|_{Q_2'}\,,\\ \|u\|_{Q_1} &\leq \frac{1}{\beta_1}\left(1+\frac{\|a\|}{\alpha}\right)\|F\|_{H_1'} + \frac{\|a\|}{\beta_1\beta_2}\left(1+\frac{\|a\|}{\alpha}\right)\|G\|_{Q_2'}\,. \end{aligned} \tag{3.27}$$

Moreover, i)*,* ii)*, and* iii) *are also necessary conditions for the well-posedness of* (3.26)*.*

Proof. See [4, Theorem 2.1, Corollary 2.1, Section 2.1] for details. □

The results provided by the following two lemmas, which are originally stated and proved in [18, Lemmas 3.1 and 3.3], will serve to establish the well-posedness of (3.15) for a given ϕ (equivalently the well-definedness of the operator S).

The first lemma introduces a suitable linear operator mapping $\mathbb{L}^t(\Omega)$ into itself for a range of t.

Lemma 3.2. *Let* Ω *be a bounded Lipschitz-continuous domain of* R^n*,* $n\in\{2,3\}$*, and let* $t,t'\in(1,+\infty)$ *conjugate to each other with* t *satisfying the range specified by* [18, Theorem 3.1]*. Then, there exists a linear and bounded operator* $D_t:\mathbb{L}^t(\Omega)\to\mathbb{L}^t(\Omega)$ *such that*

$$\mathbf{div}\big(D_t(\boldsymbol{\tau})\big) \,=\, \mathbf{0} \quad \text{in} \quad \Omega\,, \tag{3.28}$$

and

$$\int_\Omega \mathrm{tr}\big(D_t(\boldsymbol{\tau})\big) \,=\, \int_\Omega \mathrm{tr}(\boldsymbol{\tau})\,, \tag{3.29}$$

for all $\boldsymbol{\tau}\in\mathbb{L}^t(\Omega)$*. In addition, for each* $\boldsymbol{\zeta}\in\mathbb{L}^{t'}(\Omega)$ *such that* $\mathbf{div}(\boldsymbol{\zeta})=\mathbf{0}$ *in* Ω*, there holds*

$$\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \big(D_t(\boldsymbol{\tau})\big)^{\mathtt{d}} \,=\, \int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{L}^t(\Omega)\,. \tag{3.30}$$

For later use, we remark in advance here that a particular case in which both t and t' satisfy the range specified by [18, Theorem 3.1] is when they lie in $[\frac{2n}{n+1},\frac{2n}{n-1}]$. More precisely, it is easy to see that t belongs to this closed interval if and only if t' does as well.

The second lemma announced previously generalizes from $t=2$ to any $t\in(1,+\infty)$ the inequality stated in [6, Chapter IV, Proposition 3.1] (see also [14, Lemma 2.3]), which is employed for the solvability analysis of the Hilbertian dual-mixed formulation of linear elasticity.

Lemma 3.3. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2, 3\}$, which is star-shaped with respect to a ball, and let $t \in (1, +\infty)$. Then, there exist positive constants $\widetilde{C}_{\mathtt{t}}$ and $\widehat{C}_{\mathtt{t}}$ such that*

$$\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,t;\Omega} \,\le\, \widetilde{C}_t \left\{ \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,t;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \right\} \tag{3.31}$$

and

$$\|\boldsymbol{\tau}\|_{0,t;\Omega} \,\le\, \widehat{C}_t \left\{ \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,t;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \right\} \tag{3.32}$$

for all $\boldsymbol{\tau} \in \mathbb{H}_0^t(\mathbf{div}_t; \Omega)$.

3.3.2 Well-definedness of the operator S

In what follows we employ some of the preliminary results provided in Section 3.3.1, along with Theorem 3.1, to prove that the operator S (cf. (3.22)) is well-defined. We begin by checking that the bilinear forms and linear functionals involved are all bounded. Indeed, we first observe from (3.17) that a can be rewritten as

$$a(\boldsymbol{\zeta}, \boldsymbol{\tau}) \,=\, \frac{1}{\mu} \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau} - \frac{\lambda + \mu}{\mu(n\lambda + (n+1)\mu)} \int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\, \mathrm{tr}(\boldsymbol{\tau})\,,$$

from which, noting that $\frac{\lambda+\mu}{n\lambda+(n+1)\mu} < \frac{1}{n}$, and employing, thanks to the triangle and Hölder inequalities, that for each $t \in (1, +\infty)$ there holds

$$\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,t;\Omega} \,\le\, n^{1/t'} \|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{L}^t(\Omega)\,, \tag{3.33}$$

we find, using again Hölder's inequality, that

$$\begin{aligned} |a(\boldsymbol{\zeta}, \boldsymbol{\tau})| \,&\le\, \frac{1}{\mu}\|\boldsymbol{\zeta}\|_{0,r;\Omega}\, \|\boldsymbol{\tau}\|_{0,s;\Omega} + \frac{1}{n\mu}\|\mathrm{tr}(\boldsymbol{\zeta})\|_{0,r;\Omega}\, \|\mathrm{tr}(\boldsymbol{\tau})\|_{0,s;\Omega} \\ &\le\, \frac{2}{\mu}\|\boldsymbol{\zeta}\|_{0,r;\Omega}\, \|\boldsymbol{\tau}\|_{0,s;\Omega} \,\le\, \frac{2}{\mu}\|\boldsymbol{\zeta}\|_{X_2}\, \|\boldsymbol{\tau}\|_{X_1} \qquad \forall\, (\boldsymbol{\zeta}, \boldsymbol{\tau}) \in X_2 \times X_1\,. \end{aligned} \tag{3.34}$$

In turn, invoking once more the aforementioned inequality, it follows from (3.18) that

$$|b_1(\boldsymbol{\tau}, \boldsymbol{v})| \,\le\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,s;\Omega}\, \|\boldsymbol{v}\|_{0,r;\Omega} \,\le\, \|\boldsymbol{\tau}\|_{\mathrm{div}_s,s;\Omega}\, \|\boldsymbol{v}\|_{0,r;\Omega} \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in X_1 \times M_2\,, \tag{3.35}$$

and similarly

$$|b_2(\boldsymbol{\tau}, \boldsymbol{v})| \,\le\, \|\boldsymbol{\tau}\|_{\mathrm{div}_r,r;\Omega}\, \|\boldsymbol{v}\|_{0,s;\Omega} \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in X_2 \times M_1\,. \tag{3.36}$$

In addition, bearing in mind the upper bound for $\boldsymbol{f}$ (cf. (2.11)) and the estimate (3.9), we deduce from (3.19) and (3.20), respectively, that

$$|G(\boldsymbol{\tau})| \,\le\, C_r\, \|\boldsymbol{u}_D\|_{1/s,r;\Gamma}\, \|\boldsymbol{\tau}\|_{X_1} \qquad \forall\, \boldsymbol{\tau} \in X_1\,, \tag{3.37}$$

and, for each $\phi \in \mathrm{H}_0^1(\Omega)$,

$$|F_\phi(\boldsymbol{v})| \,\le\, |\Omega|^{1/r}\, f_2\, \|\boldsymbol{v}\|_{0,s;\Omega} \qquad \forall\, \boldsymbol{v} \in M_2\,. \tag{3.38}$$

In this way, and as a straightforward consequence of (3.34) - (3.38), we conclude that a, b_1, b_2, G and F_ϕ are all bounded with respective constants satisfying

$$\|a\| \le \frac{2}{\mu}\,, \quad \|b_1\|\,,\ \|b_2\| \le 1\,, \quad \|G\| \le C_r\, \|\boldsymbol{u}_D\|_{1/s,r;\Gamma}\,, \quad \text{and} \quad \|F_\phi\| \le |\Omega|^{1/r}\, f_2\,. \tag{3.39}$$

Next, we let $\mathcal{K}_i$, $i \in \{1, 2\}$, be the kernel of the bilinear form b_i, $i \in \{1, 2\}$ (cf. (3.18)), that is

$$\mathcal{K}_i \,:=\, \left\{ \boldsymbol{\tau} \in X_i : \quad b_i(\boldsymbol{\tau}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in M_i \right\},$$

which, according to the definitions of X_1, X_2 and b_i (cf. (3.18)), yields

$$\mathcal{K}_1 \,=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}_0^s(\mathbf{div}_s;\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) = 0 \Big\} \tag{3.40}$$

and

$$\mathcal{K}_2 \,=\, \Big\{ \boldsymbol{\zeta} \in \mathbb{H}_0^r(\mathbf{div}_r;\Omega): \quad \mathbf{div}(\boldsymbol{\zeta}) = 0 \Big\}\,. \tag{3.41}$$

The continuous inf-sup conditions required for the bilinear forms a (cf. (3.17)) and b_i (cf. (3.18)), $i \in \{1,2\}$, are established next. While these results were already stated and proved in [18, Lemmas 4.1 and 4.3] by following similar approaches to those employed in [20, Lemmas 2.6 and 2.7], we provide them again here for sake of completeness of our presentation.

Lemma 3.4. *Assume that r and s satisfy the particular range specified by* [18, Theorem 3.1], *that is $r,\, s \in [\frac{2n}{n+1}, \frac{2n}{n-1}]$. Then, there exist positive constants M and α such that for each $\lambda > M$ there hold*

$$\sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \alpha\,\|\boldsymbol{\zeta}\|_{X_2} \qquad \forall\,\boldsymbol{\zeta}\in\mathcal{K}_2\,, \tag{3.42}$$

and

$$\sup_{\boldsymbol{\zeta}\in\mathcal{K}_2} a(\boldsymbol{\zeta},\boldsymbol{\tau}) \,>\, 0 \qquad \forall\,\boldsymbol{\tau}\in\mathcal{K}_1\,,\ \boldsymbol{\tau}\neq\mathbf{0}\,. \tag{3.43}$$

Proof. We begin by noticing, thanks to Hölder's inequality and (3.33), that for each pair $(\boldsymbol{\zeta},\boldsymbol{\tau}) \in X_2\times X_1 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega)\times\mathbb{H}_0^s(\mathbf{div}_s;\Omega)$ there holds

$$\left|\int_\Omega \operatorname{tr}(\boldsymbol{\zeta})\operatorname{tr}(\boldsymbol{\tau})\right| \,\leq\, n^{1/r}\,\|\operatorname{tr}(\boldsymbol{\zeta})\|_{0,r;\Omega}\,\|\boldsymbol{\tau}\|_{0,s;\Omega}\,. \tag{3.44}$$

Now, we consider $\boldsymbol{\zeta}\in\mathcal{K}_2$, that is $\boldsymbol{\zeta}\in X_2 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega)$ and $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{0}$, such that $\boldsymbol{\zeta}\neq\mathbf{0}$. Then, according to the definition of a (cf. (3.17)) and the estimates (3.44) and (3.31) (cf. Lemma 3.2), we obtain

$$\sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \frac{1}{\mu}\sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}\|_{X_1}} \;-\; \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)}\,\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,. \tag{3.45}$$

Next, in order to derive a lower bound for the supremum on the right hand side of (3.45), we let

$$\boldsymbol{\zeta}_s \,:=\, \begin{cases} |\boldsymbol{\zeta}^{\mathrm{d}}|^{r-2}\,\boldsymbol{\zeta}^{\mathrm{d}} & \text{if} \quad \boldsymbol{\zeta}^{\mathrm{d}}\neq\mathbf{0},\\ \mathbf{0} & \text{if} \quad \boldsymbol{\zeta}^{\mathrm{d}}=\mathbf{0}, \end{cases} \tag{3.46}$$

and observe that $\boldsymbol{\zeta}_s\in\mathbb{L}^s(\Omega)$ and

$$\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\zeta}_s \,=\, \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}^r \,=\, \|\boldsymbol{\zeta}_s\|_{0,s;\Omega}^s \,=\, \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_s\|_{0,s;\Omega}\,. \tag{3.47}$$

In addition, it is clear that $\operatorname{tr}(\boldsymbol{\zeta}_s) = 0$, and thus, thanks to Lemma 3.2, it follows that $D_s(\boldsymbol{\zeta}_s)$ belongs to $\mathcal{K}_1$. Moreover, using (3.30) and (3.47), we find that

$$\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\big(D_s(\boldsymbol{\zeta}_s)\big)^{\mathrm{d}} \,=\, \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\zeta}_s^{\mathrm{d}} \,=\, \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\zeta}_s \,=\, \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_s\|_{0,s;\Omega}\,,$$

and hence, noting that $\|D_s(\boldsymbol{\zeta}_s)\|_{X_1} = \|D_s(\boldsymbol{\zeta}_s)\|_{0,s;\Omega}$, and invoking the boundedness of D_s (cf. Lemma 3.2), we deduce that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\big(D_s(\boldsymbol{\zeta}_s)\big)^{\mathrm{d}}}{\|D_s(\boldsymbol{\zeta}_s)\|_{X_1}} \,=\, \frac{\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_s\|_{0,s;\Omega}}{\|D_s(\boldsymbol{\zeta}_s)\|_{0,s;\Omega}} \,\geq\, \frac{1}{\|D_s\|}\,\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,. \tag{3.48}$$

Consequently, replacing (3.48) back into (3.45), we get

$$\sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \left\{\frac{1}{\mu\|D_s\|} \,-\, \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)}\right\}\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,, \tag{3.49}$$

from which, choosing λ sufficiently large such that

$$\frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)} \,<\, \frac{1}{2\mu\|D_s\|}\,,$$

which reduces to

$$\lambda \,>\, M_s \,:=\, \frac{\mu}{n^{1+1/s}}\,\max\Big\{2\|D_s\|\widetilde{C}_r - n^{1/s}(n+1), 0\Big\}\,,$$

and applying (3.32) to $\boldsymbol{\zeta}$, we arrive at (3.42) with $\alpha := \frac{1}{2\,\mu\,\|D_s\|\,\widehat{C}_r}$. On the other hand, given now $\boldsymbol{\tau}\in\mathcal{K}_1$, $\boldsymbol{\tau}\neq\mathbf{0}$, we exchange the roles of $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$ in the above analysis, so that we obtain

$$\sup_{\boldsymbol{\zeta}\in\mathcal{K}_2} a(\boldsymbol{\zeta},\boldsymbol{\tau}) \,\geq\, \sup_{\substack{\boldsymbol{\zeta}\in\mathcal{K}_2\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\zeta}\|_{X_2}} \,\geq\, \frac{1}{2\,\mu\,\|D_r\|\,\widehat{C}_s}\,\|\boldsymbol{\tau}\|_{X_1} \,>\, 0 \tag{3.50}$$

for

$$\lambda \,>\, M_r \,:=\, \frac{\mu}{n^{1+1/r}}\,\max\Big\{2\|D_r\|\widetilde{C}_s - n^{1/r}(n+1), 0\Big\}\,,$$

which shows (3.43). In this way, the proof is completed by choosing $M := \max\big\{M_s, M_r\big\}$. □

From now on we assume that the Lamé parameter λ is such that

$$\lambda \,>\, M\,,$$

with M defined at the end of the foregoing proof.

Lemma 3.5. *Assume that r and s satisfy the particular range specified by* [18, Theorem 3.1], *that is, $r,\, s\in[\frac{2n}{n+1},\frac{2n}{n-1}]$. Then, there exist positive constants β_1, β_2 such that for each $i\in\{1,2\}$ there hold*

$$\sup_{\substack{\boldsymbol{\zeta}\in X_i\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{b_i(\boldsymbol{\zeta},\boldsymbol{v})}{\|\boldsymbol{\zeta}\|_{X_i}} \,\geq\, \beta_i\,\|\boldsymbol{v}\|_{M_i} \qquad \forall\,\boldsymbol{v}\in M_i\,. \tag{3.51}$$

Proof. Since b_1 and b_2 have the same algebraic structure (cf. (3.18)), and the pairs (X_1, M_1) and (X_2, M_2) are obtained from each other by exchanging r and s, it suffices to show (3.51) for either $i=1$ or $i=2$. We proceed here with $i=2$, for which, given $\boldsymbol{v}\in M_2 := \mathbf{L}^s(\Omega)$, we first set

$$\boldsymbol{v}_r \,:=\, \begin{cases} |\boldsymbol{v}|^{s-2}\,\boldsymbol{v} & \text{if} \quad \boldsymbol{v}\neq\mathbf{0}\,,\\ \quad\mathbf{0} & \text{if} \quad \boldsymbol{v}=\mathbf{0}\,. \end{cases} \tag{3.52}$$

It follows that $\boldsymbol{v}_r\in\mathbf{L}^r(\Omega)$, and similarly to (3.47), there holds

$$\int_\Omega \boldsymbol{v}\cdot\boldsymbol{v}_r \,=\, \|\boldsymbol{v}\|_{0,s;\Omega}^s \,=\, \|\boldsymbol{v}_r\|_{0,r;\Omega}^r \,=\, \|\boldsymbol{v}\|_{0,s;\Omega}\,\|\boldsymbol{v}_r\|_{0,r;\Omega}\,. \tag{3.53}$$

Next, we let $\boldsymbol{z}\in\mathbf{W}_0^{1,r}(\Omega)$ be the unique solution, guaranteed by [18, Theorem 3.2], of the vector Poisson equation [18, eq. (3.19)] with $\mathbf{f}=\mathbf{0}$ and $\boldsymbol{g}=\boldsymbol{v}_r$, that is

$$\Delta\boldsymbol{z} \,=\, \boldsymbol{v}_r \quad \text{in} \quad \Omega\,, \qquad \boldsymbol{z}=\mathbf{0} \quad \text{on} \quad \Gamma\,,$$

whose weak formulation reduces to: Find $\boldsymbol{z} \in \mathbf{W}_0^{1,r}(\Omega)$ such that

$$\int_\Omega \nabla \boldsymbol{z} \cdot \nabla \boldsymbol{w} = - \int_\Omega \boldsymbol{v}_r \cdot \boldsymbol{w} \qquad \forall\, \boldsymbol{w} \in \mathbf{W}_0^{1,s}(\Omega)\,.$$

Note that the corresponding continuous dependence result establishes the existence of a positive constant $\bar{c}_r$ such that

$$\|\boldsymbol{z}\|_{1,r;\Omega} \,\le\, \bar{c}_r\, \|\boldsymbol{v}_r\|_{0,r;\Omega}\,. \tag{3.54}$$

Furthermore, we observe that $\mathbf{div}\big(\nabla \boldsymbol{z}\big) = \boldsymbol{v}_r$ in Ω, which proves that $\nabla \boldsymbol{z} \in \mathbb{H}^r(\mathbf{div}_r;\Omega)$, and hence we let $\widehat{\boldsymbol{\zeta}}$ be the $\mathbb{H}_0^r(\mathbf{div}_r;\Omega)$-component of $\nabla \boldsymbol{z}$. Thus, employing (3.54) and noting that $\mathbf{div}(\widehat{\boldsymbol{\zeta}}) = \boldsymbol{v}_r$, we obtain

$$\|\widehat{\boldsymbol{\zeta}}\|_{X_2} \,=\, \|\widehat{\boldsymbol{\zeta}}\|_{0,r;\Omega} \,+\, \|\mathbf{div}(\widehat{\boldsymbol{\zeta}})\|_{0,r;\Omega} \,\le\, |\boldsymbol{z}|_{1,r;\Omega} \,+\, \|\boldsymbol{v}_r\|_{0,r;\Omega} \,\le\, \big(1 + \bar{c}_r\big)\, \|\boldsymbol{v}_r\|_{0,r;\Omega}\,.$$

Finally, bearing in mind the definition of b_2 (cf. (3.18), $i = 2$), and making use of (3.53) and the foregoing inequality, we conclude that

$$\sup_{\substack{\boldsymbol{\zeta}\in X_2\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{b_2(\boldsymbol{\zeta},\boldsymbol{v})}{\|\boldsymbol{\zeta}\|_{X_2}} \,\ge\, \frac{b_2(\widehat{\boldsymbol{\zeta}},\boldsymbol{v})}{\|\widehat{\boldsymbol{\zeta}}\|_{X_2}} \,=\, \frac{\displaystyle\int_\Omega \boldsymbol{v}\cdot\boldsymbol{v}_r}{\|\widehat{\boldsymbol{\zeta}}\|_{X_2}} \,\ge\, \frac{1}{1+\bar{c}_r}\, \|\boldsymbol{v}\|_{0,s;\Omega}\,, \tag{3.55}$$

which proves (3.51) for $i = 2$ with $\beta_2 := \big(1 + \bar{c}_r\big)^{-1}$. $\square$

For the rest of the paper we assume meanwhile that r and s lie in the range stipulated in Lemmas 3.4 and 3.5, that is

$$r,\, s \,\in\, \Big[\frac{2n}{n+1}, \frac{2n}{n-1}\Big]\,. \tag{3.56}$$

The following result establishes that the operator S (cf. (3.9)) is well defined.

Lemma 3.6. *For each $\varphi \in \mathrm{H}_0^1(\Omega)$ there exists a unique $\mathrm{S}(\varphi) = \big(\mathrm{S}_1(\varphi), \mathrm{S}_2(\varphi)\big) := (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}}) \in X_2 \times M_1$ solution to (3.22). Moreover, there hold*

$$\begin{aligned}
\|\mathrm{S}_1(\varphi)\|_{X_2} \,&=\, \|\widetilde{\boldsymbol{\sigma}}\|_{X_2} \,\le\, \frac{C_r}{\alpha}\, \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\beta_2}\left(1 + \frac{2}{\alpha\mu}\right) f_2\,, \quad \textit{and}\\
\|\mathrm{S}_2(\varphi)\|_{M_1} \,&=\, \|\widetilde{\boldsymbol{u}}\|_{M_1} \,\le\, \frac{C_r}{\beta_1}\left(1 + \frac{2}{\alpha\mu}\right) \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\beta_1\beta_2}\left(1 + \frac{2}{\alpha\mu}\right) f_2\,.
\end{aligned} \tag{3.57}$$

Proof. Thanks to the fact that X_1, X_2, M_1 and M_2 are all reflexive Banach spaces, along with the boundedness of all the forms and functionals involved, and the inf-sup conditions provided by Lemmas 3.4 and 3.5, the proof reduces to a direct application of Theorem 3.1. In particular, the a priori estimates (3.57) follow from (3.27) and (3.39). $\square$

3.3.3 Well-definedness of operator $\widetilde{\mathrm{S}}$

In this section we use the classical Lax-Milgram lemma to prove that $\widetilde{\mathrm{S}}$ (cf. (3.23)) is well defined. In fact, we first notice from (3.2) and (2.10) that, given $\boldsymbol{\zeta} \in X_2$, there holds

$$A_{\boldsymbol{\zeta}}(\phi,\varphi) \,\le\, \vartheta_2\, \|\phi\|_{1,\Omega}\, \|\varphi\|_{1,\Omega} \qquad \forall\, \phi,\varphi \in \mathrm{H}_0^1(\Omega)\,, \tag{3.58}$$

which says that $A_{\boldsymbol{\zeta}}$ is bounded independently of $\boldsymbol{\zeta}$ with

$$\|A_{\boldsymbol{\zeta}}\| \,\le\, \vartheta_2\,. \tag{3.59}$$

In turn, using now that ϑ is uniformly positive definite (cf. (2.9)), and denoting by c_p the constant of the Poincaré inequality in $\mathrm{H}_0^1(\Omega)$, which says that $\|\phi\|_{1,\Omega} \le c_p |\phi|_{1,\Omega} \quad \forall \phi \in \mathrm{H}_0^1(\Omega)$, we deduce that

$$A_{\boldsymbol{\zeta}}(\phi, \phi) = \int_\Omega \vartheta(\boldsymbol{\zeta}) \nabla\phi \cdot \nabla\phi \ge \widetilde{\alpha} \|\phi\|_{1,\Omega}^2 \quad \forall \phi \in \mathrm{H}_0^1(\Omega), \tag{3.60}$$

where

$$\widetilde{\alpha} := \frac{\vartheta_0}{c_p^2}, \tag{3.61}$$

thus establishing the $\mathrm{H}_0^1(\Omega)$-ellipticity of $A_{\boldsymbol{\zeta}}$ independently of $\boldsymbol{\zeta}$ as well. Furthermore, given $\boldsymbol{w} \in M_1$, and bearing in mind (3.3), we employ the upper bound of g (cf. (2.12)) and the Cauchy-Schwarz inequality to arrive at

$$|G_{\boldsymbol{w}}(\psi)| \le |\Omega|^{1/2} g_2 \|\psi\|_{0,\Omega} \qquad \forall \psi \in \mathrm{H}_0^1(\Omega), \tag{3.62}$$

which yields $G_{\boldsymbol{w}} \in \mathrm{H}_0^1(\Omega)'$ with $\|G_{\boldsymbol{w}}\| \le |\Omega|^{1/2} g_2$.

Consequently, we are in a position to state that the operator $\widetilde{\mathrm{S}}$ is well-defined.

Lemma 3.7. *For each $(\boldsymbol{\zeta}, \boldsymbol{w}) \in X_2 \times M_1$ there exists a unique $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) := \widetilde{\phi} \in \mathrm{H}_0^1(\Omega)$ solution to (3.23). Moreover, there holds*

$$\|\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w})\|_{1,\Omega} = \|\widetilde{\phi}\|_{1,\Omega} \le \widetilde{\mathtt{r}} := \frac{1}{\widetilde{\alpha}} |\Omega|^{1/2} g_2. \tag{3.63}$$

Proof. Thanks to the previous analysis, it is a straightforward application of Lax-Milgram's lemma (cf. [14, Theorem 1.1]). $\square$

3.4 Solvability of the fixed-point equation

In this section we address the solvability analysis of the fixed-point equation (3.25). For this purpose, the hypotheses of the Banach fixed-point theorem are verified in what follows. We begin by defining the ball

$$W := \left\{ \phi \in \mathrm{H}_0^1(\Omega) : \quad \|\phi\|_{1,\Omega} \le \widetilde{\mathtt{r}} \right\}, \tag{3.64}$$

where $\widetilde{\mathtt{r}} > 0$ is the constant specified in (3.63). The following result states that T maps W into itself.

Lemma 3.8. *There holds $T(W) \subseteq W$.*

Proof. It follows directly from the definition of T (cf. (3.24)) and the a priori estimate for the operator $\widetilde{\mathrm{S}}$ provided by (3.63). $\square$

The next goal is to establish the continuity of T, for which we previously prove the corresponding properties of S and $\widetilde{\mathrm{S}}$. We begin with the one of S.

Lemma 3.9. *There exists a positive constant C_{S}, depending only on μ, α, β_1, β_2, and the norm of the continuous injection $i_r : \mathrm{H}^1(\Omega) \to \mathrm{L}^r(\Omega)$, such that*

$$\|\mathrm{S}(\phi) - \mathrm{S}(\varphi)\|_{X_2 \times M_1} \le C_{\mathrm{S}} L_f \|\phi - \varphi\|_{1,\Omega} \qquad \forall \phi, \varphi \in \mathrm{H}_0^1(\Omega). \tag{3.65}$$

Proof. Given $\varphi, \psi \in \mathrm{H}_0^1(\Omega)$, we let $\mathrm{S}(\varphi) := (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}}) \in X_2 \times M_1$ and $\mathrm{S}(\psi) := (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{u}}) \in X_2 \times M_1$, which satisfy (3.22) with φ itself and with $\varphi = \psi$, respectively. Then, subtracting the corresponding equations of these systems, we obtain

$$\begin{aligned} a(\widetilde{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + b_1(\boldsymbol{\tau}, \widetilde{\boldsymbol{u}} - \bar{\boldsymbol{u}}) &= 0 && \forall \boldsymbol{\tau} \in X_1, \\ b_2(\widetilde{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}, \boldsymbol{v}) &= (F_\varphi - F_\psi)(\boldsymbol{v}) && \forall \boldsymbol{v} \in M_2, \end{aligned} \tag{3.66}$$

which says, thanks to the analysis and results from Section 3.3.2, particularly the inf-sup conditions satisfied by a, b_1 and b_2, along with Theorem 3.1, that $(\widetilde{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}} - \bar{\boldsymbol{u}}) \in X_2 \times M_1$ is the unique solution of (3.22) with G given by the null functional and F_φ replaced by $F_\varphi - F_\psi$. Next, having in mind the definitions of F_φ and F_ψ (cf. (3.20)), employing the Lipschitz-continuity of $\boldsymbol{f}$ (cf. (2.11)), applying Hölder's inequality, and invoking the continuous injection $i_r : \mathrm{H}^1(\Omega) \to \mathrm{L}^r(\Omega)$, which is valid in particular for $r \in [\frac{2n}{n+1}, \frac{2n}{n-1}]$, we readily find that

$$|(F_\varphi - F_\psi)(\boldsymbol{v})| \,\le\, L_f\, \|\varphi - \psi\|_{0,r;\Omega}\, \|\boldsymbol{v}\|_{0,s;\Omega} \,\le\, L_f\, \|i_r\|\, \|\varphi - \psi\|_{1,\Omega}\, \|\boldsymbol{v}\|_{0,s;\Omega} \quad \forall\, v \in M_2\,, \tag{3.67}$$

which implies $\|F_\varphi - F_\psi\|_{M_2'} \le L_f\, \|i_r\|\, \|\varphi - \psi\|_{1,\Omega}$. In this way, this latter inequality and the abstract estimate (3.27) applied to problem (3.66), yield (3.65) and end the proof. □

On the other hand, in order to establish a continuity property for $\widetilde{\mathrm{S}}$, we follow the approach of diverse previous works (see, e.g. [1], [9], [16], [17], and [20]), and introduce a regularity assumption on the solutions of the problem defining this operator. More precisely, from now on we suppose that there exists $\varepsilon \ge \frac{n}{r}$ and a constant $C_\varepsilon > 0$, such that

(**RA**) for each $(\boldsymbol{\zeta}, \boldsymbol{w}) \in X_2 \times M_1$ there holds $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) = \widetilde{\phi} \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^{1+\varepsilon}(\Omega)$ and

$$\|\widetilde{\phi}\|_{1+\varepsilon,\Omega} \,\le\, C_\varepsilon\, g_2\,. \tag{3.68}$$

The reason of the aforementioned lower bound of ε is clarified within the proof of the next lemma, which provides the Lipschitz-continuity of the operator $\widetilde{\mathrm{S}}$. In connection to this, and to be employed in the aforementioned proof as well, we recall now, thanks to the embedding between fractional Sobolev spaces, that for each $\varepsilon < \frac{n}{2}$ there holds $\mathrm{H}^\varepsilon(\Omega) \subset \mathrm{L}^{\varepsilon^*}(\Omega)$, with continuous injection

$$i_\varepsilon : \mathrm{H}^\varepsilon(\Omega) \longrightarrow \mathrm{L}^{\varepsilon^*}(\Omega)\,, \quad \text{where} \quad \varepsilon^* = \frac{2n}{n - 2\varepsilon}\,. \tag{3.69}$$

In this regard, we notice that the indicated lower and upper bounds for the additional regularity ε, which turn out to require that $\varepsilon \in [\frac{n}{r}, \frac{n}{2})$, are compatible if and only if $r > 2$, which is coherent with the fact that initially (cf. (3.5)) $r = 2p$, with $p \in (1, +\infty)$. Then, intersecting this constraint with the one stated previously in (3.56), we deduce that the feasible range for r becomes

$$r \,\in\, \Big(2, \frac{2n}{n-1}\Big] \,=\, \begin{cases} (2,4] & \text{if } n = 2\,, \\ (2,3] & \text{if } n = 3\,, \end{cases} \tag{3.70}$$

which we assume from now on. As a consequence, the range for the conjugate s of r is

$$s \,\in\, \Big[\frac{2n}{n+1}, 2\Big) \,=\, \begin{cases} \big[\frac{4}{3}, 2\big) & \text{if } n = 2\,, \\ \big[\frac{3}{2}, 2\big) & \text{if } n = 3\,. \end{cases} \tag{3.71}$$

Lemma 3.10. *There exists a positive constant $C_{\widetilde{\mathrm{S}}}$, depending only on $\widetilde{\alpha}$, the norm of the continuous injection $i_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^s(\Omega)$, $|\Omega|$, r, ε, $\|i_\varepsilon\|$ (cf. (3.69)), and C_ε (cf. (3.68)), such that*

$$\begin{aligned} &\|\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) - \widetilde{\mathrm{S}}(\boldsymbol{\tau}, \boldsymbol{v})\|_{1,\Omega} \\ &\qquad \le\, C_{\widetilde{\mathrm{S}}} \left\{ L_g + L_\vartheta\, g_2 \right\} \|(\boldsymbol{\zeta}, \boldsymbol{w}) - (\boldsymbol{\tau}, \boldsymbol{v})\|_{X_2 \times M_1} \quad \forall\, (\boldsymbol{\zeta}, \boldsymbol{w}),\, (\boldsymbol{\tau}, \boldsymbol{v}) \in X_2 \times M_1\,. \end{aligned} \tag{3.72}$$

Proof. Given $(\boldsymbol{\zeta}, \boldsymbol{w}), (\boldsymbol{\tau}, \boldsymbol{v}) \in X_2 \times M_1$, we let $\widetilde{\phi} := \widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w})$ and $\widetilde{\varphi} := \widetilde{\mathrm{S}}(\boldsymbol{\tau}, \boldsymbol{v})$, which means, according to the definition of $\widetilde{\mathrm{S}}$ (cf. (3.23)), that $\widetilde{\phi}$ and $\widetilde{\varphi}$ are the unique elements in $\mathrm{H}_0^1(\Omega)$ such that

$$A_{\boldsymbol{\zeta}}(\widetilde{\phi}, \psi) \,=\, G_{\boldsymbol{w}}(\psi) \qquad \forall\, \psi \in \mathrm{H}_0^1(\Omega)\,, \tag{3.73}$$

and

$$A_{\boldsymbol{\tau}}(\widetilde{\varphi},\psi) \,=\, G_{\boldsymbol{v}}(\psi) \qquad \forall\,\psi\in \mathrm{H}^1_0(\Omega)\,. \tag{3.74}$$

Thus, applying the $\mathrm{H}^1_0(\Omega)$-ellipticity of $A_{\boldsymbol{\zeta}}$, adding and subtracting $A_{\boldsymbol{\tau}}(\widetilde{\varphi},\widetilde{\phi}-\widetilde{\varphi})$, and then employing (3.73) and (3.74), we first obtain

$$\widetilde{\alpha}\,\|\widetilde{\phi}-\widetilde{\varphi}\|^2_{1,\Omega} \,\le\, A_{\boldsymbol{\zeta}}(\widetilde{\phi}-\widetilde{\varphi},\widetilde{\phi}-\widetilde{\varphi}) \,=\, (A_{\boldsymbol{\tau}}-A_{\boldsymbol{\zeta}})(\widetilde{\varphi},\widetilde{\phi}-\widetilde{\varphi}) \,+\, (G_{\boldsymbol{w}}-G_{\boldsymbol{v}})(\widetilde{\phi}-\widetilde{\varphi})\,. \tag{3.75}$$

Next, using the Lipschitz-continuity of g (cf. (2.12)), applying Hölder's inequality, and invoking the continuous injection $i_s : \mathrm{H}^1(\Omega)\to \mathrm{L}^s(\Omega)$, which is also valid for the present range of s, we find that

$$\begin{aligned} |(G_{\boldsymbol{w}}-G_{\boldsymbol{v}})(\widetilde{\phi}-\widetilde{\varphi})| \,&\le\, \int_\Omega |g(\boldsymbol{w})-g(\boldsymbol{v})|\,|\widetilde{\phi}-\widetilde{\varphi}| \,\le\, L_g \int_\Omega |\boldsymbol{w}-\boldsymbol{v}|\,|\widetilde{\phi}-\widetilde{\varphi}| \\ &\le\, L_g\,\|\boldsymbol{w}-\boldsymbol{v}\|_{0,r;\Omega}\,\|\widetilde{\phi}-\widetilde{\varphi}\|_{0,s;\Omega} \,\le\, L_g\,\|i_s\|\,\|\boldsymbol{w}-\boldsymbol{v}\|_{0,r;\Omega}\,\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega}\,. \end{aligned} \tag{3.76}$$

In turn, employing now the Lipschitz-continuity of ϑ (cf. (2.10)), and making use again of Hölder's inequality, we get

$$\begin{aligned} |(A_{\boldsymbol{\tau}}-A_{\boldsymbol{\zeta}})(\widetilde{\varphi},\widetilde{\phi}-\widetilde{\varphi})| &= \Big|\int_\Omega \big(\vartheta(\boldsymbol{\tau})-\vartheta(\boldsymbol{\zeta})\big)\,\nabla\widetilde{\varphi}\cdot\nabla(\widetilde{\phi}-\widetilde{\varphi})\Big| \\ &\le\, L_\vartheta\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,2q;\Omega}\,\|\nabla\widetilde{\varphi}\|_{0,2p;\Omega}\,\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega} \end{aligned} \tag{3.77}$$

where $p,\,q\in(1,+\infty)$ are conjugate to each other. Now, choosing p such that $2p\,=\,\varepsilon^*$ (cf. (3.69)), we get $2q\,=\,\frac{n}{\varepsilon}$, which, according to the range stipulated for ε, yields $2q\,\le\,r$, so that the norm of the embedding of the respective Lebesgue spaces is given by $C_{r,\varepsilon} := |\Omega|^{\frac{r\varepsilon-n}{rn}}$. In this way, using additionally the continuity of i_ε (cf. (3.69)) along with the regularity assumption (3.68), the estimate (3.77) becomes

$$\begin{aligned} |(A_{\boldsymbol{\tau}}-A_{\boldsymbol{\zeta}})(\widetilde{\varphi},\widetilde{\phi}-\widetilde{\varphi})| &\le\, L_\vartheta\,C_{r,\varepsilon}\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,r;\Omega}\,\|i_\varepsilon\|\,\|\nabla\widetilde{\varphi}\|_{\varepsilon,\Omega}\,\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega} \\ &\le\, L_\vartheta\,C_{r,\varepsilon}\,\|i_\varepsilon\|\,C_\varepsilon\,g_2\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,r;\Omega}\,\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega}\,. \end{aligned} \tag{3.78}$$

Finally, replacing the resulting estimates from (3.76) and (3.78) back into (3.75), simplifying $\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega}$ on both sides, and dividing by $\widetilde{\alpha}$, we arrive at (3.72) and finish the proof. □

We are now in a position to establish the Lipschitz-continuity of the fixed point operator T. More precisely, we have the following result.

Lemma 3.11. *There exists a positive constant C_T, depending only on C_{S} and $C_{\widetilde{\mathrm{S}}}$, such that*

$$\|T(\phi)-T(\varphi)\|_{1,\Omega} \,\le\, C_T\,L_f\,\big\{L_g+L_\vartheta\,g_2\big\}\,\|\phi-\varphi\|_{1,\Omega} \qquad \forall\,\phi,\,\varphi\in\mathrm{H}^1_0(\Omega)\,. \tag{3.79}$$

Proof. Given $\phi,\,\varphi\,\in\,\mathrm{H}^1_0(\Omega)$, and bearing in mind the definition of T (cf. (3.24)), straightforward applications of Lemmas 3.10 and 3.9 yield

$$\begin{aligned} \|T(\phi)-T(\varphi)\|_{1,\Omega} \,&\le\, C_{\widetilde{\mathrm{S}}}\,\big\{L_g+L_\vartheta\,g_2\big\}\,\|\mathrm{S}(\phi)-\mathrm{S}(\varphi)\|_{X_2\times M_1} \\ &\le\, C_{\widetilde{\mathrm{S}}}\,C_{\mathrm{S}}\,L_f\,\big\{L_g+L_\vartheta\,g_2\big\}\,\|\phi-\varphi\|_{1,\Omega}\,, \end{aligned}$$

which yields (3.79) with $C_T := C_{\mathrm{S}}\,C_{\widetilde{\mathrm{S}}}$. □

Consequently, the main result of this section is stated as follows.

Theorem 3.12. *Assume the regularity assumption* (**RA**) (*cf.* (3.68)) *and that the data* L_f, L_g, L_ϑ *and* g_2 *are sufficiently small so that*

$$C_T\, L_f\,\big\{L_g + L_\vartheta\, g_2\big\} \,<\, 1\,. \tag{3.80}$$

Then, the coupled problem (3.21) *has a unique solution* $(\boldsymbol{\sigma},\boldsymbol{u},\phi)\,\in\,X_2\times M_1\times \mathrm{H}_0^1(\Omega)$, *with* $\phi\in W$ (*cf.* (3.64)). *Moreover, there hold*

$$\begin{aligned}
\|\boldsymbol{\sigma}\|_{X_2} \,&\le\, \frac{C_r}{\alpha}\,\|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\beta_2}\left(1+\frac{2}{\alpha\mu}\right) f_2\,, \quad \textit{and}\\
\|\boldsymbol{u}\|_{M_1} \,&\le\, \frac{C_r}{\beta_1}\left(1+\frac{2}{\alpha\mu}\right)\|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\beta_1\beta_2}\left(1+\frac{2}{\alpha\mu}\right) f_2\,.
\end{aligned} \tag{3.81}$$

Proof. Thanks to Lemmas 3.8 and 3.11, and the assumption (3.80), the existence of a unique $\phi\in W$ solution to (3.25), and hence, equivalently, the existence of a unique $(\boldsymbol{\sigma},\boldsymbol{u},\phi)\,\in\,X_2\times M_1\times\mathrm{H}_0^1(\Omega)$ solution to (3.21), is merely an application of the Banach fixed point Theorem. In addition, the fact that $(\boldsymbol{\sigma},\boldsymbol{u})=\mathrm{S}(\phi)$ along with the a priori estimates provided by (3.57), yield (3.81) and conclude the proof. □

4 The Galerkin scheme

In this section we introduce the Galerkin scheme of the mixed-primal formulation (3.21), and analyze its solvability by employing a discrete version of the fixed point strategy developed in Section 3.2. For this purpose, we begin by considering arbitrary finite element subspaces $X_{2,h}\subseteq X_2$, $M_{1,h}\subseteq M_1$, $X_{1,h}\subseteq X_1$, $M_{2,h}\subseteq X_2$, and $\mathrm{H}_h\subseteq \mathrm{H}_0^1(\Omega)$, whose specific choices satisfying all the required stability conditions will be introduced later on in Section 5. In this way, the Galerkin scheme associated with (3.21) reads: Find $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\in X_{2,h}\times M_{1,h}$ and $\phi_h\in\mathrm{H}_h$ such that

$$\begin{aligned}
a(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+b_1(\boldsymbol{\tau}_h,\boldsymbol{u}_h) \;&=\; G(\boldsymbol{\tau}_h) && \forall\,\boldsymbol{\tau}_h\in\,X_{1,h}\,,\\
b_2(\boldsymbol{\sigma}_h,\boldsymbol{v}_h) \;&=\; F_{\phi_h}(\boldsymbol{v}_h) && \forall\,\boldsymbol{v}_h\in\,M_{2,h}\,,\\
A_{\boldsymbol{\sigma}_h}(\phi_h,\psi_h) \;&=\; G_{\boldsymbol{u}_h}(\psi_h) && \forall\,\psi_h\in\mathrm{H}_h\,.
\end{aligned} \tag{4.1}$$

4.1 The discrete fixed point strategy

Here we adopt the discrete analogue of the fixed point strategy introduced in Section 3.2 to analyse the solvability of (4.1). According to it, we now let $\mathrm{S}_h:\mathrm{H}_h\to X_{h,2}\times M_{h,1}$ be the operator defined for each $\varphi_h\in\mathrm{H}_h$ as $\mathrm{S}_h(\varphi_h)\,:=\,(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\boldsymbol{u}}_h)$, where $(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\boldsymbol{u}}_h)\in X_{2,h}\times M_{1,h}$ is the unique solution (to be confirmed below) of the first two equations of (4.1) with φ_h instead of ϕ_h, that is

$$\begin{aligned}
a(\widetilde{\boldsymbol{\sigma}}_h,\boldsymbol{\tau}_h)+b_1(\boldsymbol{\tau}_h,\widetilde{\boldsymbol{u}}_h) \;&=\; G(\boldsymbol{\tau}_h) && \forall\,\boldsymbol{\tau}_h\in\,X_{1,h}\,,\\
b_2(\widetilde{\boldsymbol{\sigma}}_h,\boldsymbol{v}_h) \;&=\; F_{\varphi_h}(\boldsymbol{v}_h) && \forall\,\boldsymbol{v}_h\in\,M_{2,h}\,.
\end{aligned} \tag{4.2}$$

In addition, we also let $\widetilde{\mathrm{S}}_h:X_{2,h}\times M_{1,h}\to\mathrm{H}_h$ be the operator defined for each $(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\in X_{2,h}\times M_{1,h}$ as $\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h,\boldsymbol{w}_h):=\widetilde{\phi}_h$, where $\widetilde{\phi}_h\in\mathrm{H}_h$ is the unique solution of the last equation of (4.1) with $(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)$ instead of $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)$, that is

$$A_{\boldsymbol{\zeta}_h}(\widetilde{\phi}_h,\psi_h)\,=\,G_{\boldsymbol{w}_h}(\psi_h)\quad\forall\,\psi_h\in\mathrm{H}_h\,. \tag{4.3}$$

Then, we define the operator $T_h:\mathrm{H}_h\to\mathrm{H}_h$ as

$$T_h(\varphi_h)\,:=\,\widetilde{\mathrm{S}}_h\big(\mathrm{S}_h(\varphi_h)\big)\quad\forall\,\varphi_h\in\,\mathrm{H}_h\,, \tag{4.4}$$

and realise that solving (4.1) is equivalent to seeking a fixed point of T_h, that is $\phi_h \in \mathrm{H}_h$ such that

$$T_h(\phi_h) \,=\, \phi_h \,. \tag{4.5}$$

4.2 Well-posedness of the operators S_h and $\widetilde{\mathrm{S}}_h$

We now apply the discrete versions of Theorem 3.1 and Lax-Milgram's lemma to show that the discrete operators S_h and $\widetilde{\mathrm{S}}_h$ are well defined, equivalently that problems (4.2) and (4.3) are well-posed. For this purpose, we now let $\mathcal{K}_{1,h}$ and $\mathcal{K}_{2,h}$ be the discrete kernels of the operators induced by the bilinear forms b_1 and b_2, respectively, that is

$$\mathcal{K}_{1,h} \,:=\, \Big\{ \boldsymbol{\tau}_h \in X_{1,h} : \quad b_1(\boldsymbol{\tau}_h, \boldsymbol{v}_h) = 0 \quad \forall\, \boldsymbol{v}_h \in M_{1,h} \Big\}\,, \tag{4.6}$$

$$\mathcal{K}_{2,h} \,:=\, \Big\{ \boldsymbol{\zeta}_h \in X_{2,h} : \quad b_2(\boldsymbol{\zeta}_h, \boldsymbol{v}_h) = 0 \quad \forall\, \boldsymbol{v}_h \in M_{2,h} \Big\}\,. \tag{4.7}$$

Next, we introduce some hypotheses involving the arbitrary spaces $X_{2,h}$, $M_{1,h}$, $X_{1,h}$, and $M_{2,h}$, as well as $\mathcal{K}_{1,h}$ and $\mathcal{K}_{2,h}$. More precisely, from now on we assume the following:

(**H.1**) there exists a constant $\alpha_{\mathrm{d}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in X_{1,h} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\geq\, \alpha_{\mathrm{d}}\, \|\boldsymbol{\sigma}_h\|_{X_2} \qquad \forall\, \boldsymbol{\sigma}_h \in K_{2,h}\,, \quad \text{and}$$

$$\sup_{\boldsymbol{\zeta}_h \in K_{2,h}} a(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h) \,>\, 0 \qquad \forall\, \boldsymbol{\tau}_h \in K_{1,h},\, \boldsymbol{\tau}_h \neq \mathbf{0}\,.$$

(**H.2**) there exist constants $\beta_{1,\mathrm{d}}$, $\beta_{2,\mathrm{d}} > 0$, independent of h, such that for each $i \in \{1,2\}$ there holds

$$\sup_{\substack{\boldsymbol{\tau}_h \in X_{i,h} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{b_i(\boldsymbol{\tau}_h, \boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\|_{X_i}} \,\geq\, \beta_{i,\mathrm{d}}\, \|\boldsymbol{v}_h\|_{M_i} \qquad \forall\, \boldsymbol{v}_h \in M_{i,h}\,.$$

Specific finite element subspaces satisfying (**H.1**) and (**H.2**) will be defined later on in Section 5.2. Thus, as a straightforward consequence of these assumptions, we obtain the following result.

Lemma 4.1. *For each $\varphi_h \in \mathrm{H}_h$ there exists a unique $\mathrm{S}_h(\varphi_h) = \big(\mathrm{S}_{1,h}(\varphi_h), \mathrm{S}_{2,h}(\varphi_h)\big) := (\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{u}}_h) \in X_{2,h} \times M_{1,h}$ solution to (4.2). Moreover, there hold*

$$\begin{aligned} \|\mathrm{S}_{1,h}(\varphi_h)\|_{X_2} = \|\widetilde{\boldsymbol{\sigma}}_h\|_{X_2} \,&\leq\, \frac{C_r}{\alpha_{\mathrm{d}}}\, \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\beta_{2,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\, \mu}\right) f_2\,, \quad \textit{and} \\ \|\mathrm{S}_{2,h}(\varphi_h)\|_{M_1} = \|\widetilde{\boldsymbol{u}}_h\|_{M_1} \,&\leq\, \frac{C_r}{\beta_{1,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\, \mu}\right) \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu \beta_{1,\mathrm{d}}\, \beta_{2,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\, \mu}\right) f_2\,. \end{aligned} \tag{4.8}$$

Proof. Invoking (**H.1**) and (**H.2**), the proof reduces to a direct application of the discrete version of Theorem 3.1 (see, e.g. [4, Corollary 2.2]). In particular, the a priori estimates given by (4.8) follow from the discrete analogue of (3.57). $\square$

Having proved that S_h is well-defined, we now establish the same property for $\widetilde{\mathrm{S}}_h$ with an arbitrary finite element subspace H_h of $\mathrm{H}^1(\Omega)$.

Lemma 4.2. *For each* $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in X_{2,h} \times M_{1,h}$ *there exists a unique* $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) := \widetilde{\phi}_h \in \mathrm{H}_h$ *solution to* (4.3)*. Moreover, with the same constant* $\widetilde{\mathtt{r}}$ *introduced in Lemma* 3.7*, there holds*

$$\|\widetilde{\mathrm{S}}(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\|_{1,\Omega} = \|\widetilde{\phi}_h\|_{1,\Omega} \,\leq\, \widetilde{\mathtt{r}}\,. \tag{4.9}$$

Proof. It suffices to note that the bilinear form $A_{\boldsymbol{\zeta}_h}$ is H_h-elliptic with the same constant $\widetilde{\alpha}$ given by (3.61), and that $G_{\boldsymbol{w}_h}$ restricted to H_h belongs to H_h' with $\|G_{\boldsymbol{w}_h}\| \,\leq\, |\Omega|^{1/2}\, g_2$ (cf. (3.62)). In this way, the proof is a direct application of Lax-Milgram's lemma. □

4.3 Discrete solvability analysis

Having proved that the discrete operators S_h, $\widetilde{\mathrm{S}}_h$, and hence T_h, are all well defined, we now address the solvability of the corresponding fixed point equation (4.5). To this end, and similarly to (3.64), we first introduce the discrete ball

$$W_h \,:=\, \Big\{ \phi_h \in \mathrm{H}_h : \quad \|\phi_h\|_{1,\Omega} \,\leq\, \widetilde{\mathtt{r}} \Big\}\,, \tag{4.10}$$

and establish the discrete analogue of Lemma 3.8.

Lemma 4.3. *There holds* $T_h(W_h) \subseteq W_h$*.*

Proof. Similarly to the proof of Lemma 3.8, it follows from the definition of T_h (cf. (4.4)) and the a priori estimate for the operator $\widetilde{\mathrm{S}}_h$ provided by (4.9). □

Next, we aim to state the continuity of the operators S_h, $\widetilde{\mathrm{S}}_h$, and T_h. We begin with S_h by proceeding analogously to the proof of Lemma 3.9. Indeed, considering the Galerkin scheme associated with (3.66), the inf-sup conditions provided by (**H.1**) and (**H.2**), the continuous injection $i_r : \mathrm{H}^1(\Omega) \to \mathrm{L}^r(\Omega)$, and the discrete version of the abstract estimate (3.27) (cf. [4, Corollary 2.2]), we readily deduce that there exists a positive constant $C_{\mathrm{S,d}}$, depending only on μ, α_{d}, $\beta_{1,\mathrm{d}}$, $\beta_{2,\mathrm{d}}$, and $\|i_r\|$, and hence independent of h, such that

$$\|\mathrm{S}_h(\phi_h) - \mathrm{S}_h(\varphi_h)\|_{X_2 \times M_1} \,\leq\, C_{\mathrm{S,d}}\, L_f\, \|\phi_h - \varphi_h\|_{1,\Omega} \qquad \forall\, \phi,\, \varphi \in \mathrm{H}_h\,. \tag{4.11}$$

In turn, for the continuity of $\widetilde{\mathrm{S}}_h$ we slightly modify the reasoning of the proof of Lemma 3.10. In fact, instead of the regularity assumption (**RA**), which is certainly not applicable in the present discrete context, we just employ an $\mathrm{L}^{2q} - \mathrm{L}^{2p} - \mathrm{L}^2$ argument to derive the discrete version of (3.72), where $p,\, q \in (1, +\infty)$ conjugate to each other, are chosen such that $2q = r$. Note that this is a feasible choice since, as stipulated in (3.70), there holds $r > 2$, which yields $r^* := 2p = \frac{2r}{r-2}$. In this way, given $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)$, $(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in X_{2,h} \times M_{1,h}$, and denoting $\widetilde{\phi}_h = \widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathrm{H}_h$ and $\widetilde{\varphi}_h = \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathrm{H}_h$, the discrete analogue of (3.77) becomes

$$|(A_{\boldsymbol{\tau}_h} - A_{\boldsymbol{\zeta}_h})(\widetilde{\varphi}_h, \widetilde{\phi}_h - \widetilde{\varphi}_h)| \,\leq\, L_\vartheta\, \|\boldsymbol{\tau}_h - \boldsymbol{\zeta}_h\|_{0,r;\Omega}\, \|\nabla \widetilde{\varphi}_h\|_{0,r^*;\Omega}\, \|\widetilde{\phi}_h - \widetilde{\varphi}_h\|_{1,\Omega}\,, \tag{4.12}$$

which, along with the discrete versions of (3.75) and (3.76), imply the existence of a positive constant $C_{\widetilde{\mathrm{S}},\mathrm{d}}$, depending only on $\widetilde{\alpha}$ and the norm of the continuous injection $i_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^s(\Omega)$, and hence independent of h, such that

$$\begin{aligned} &\|\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) - \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_{1,\Omega} \\ &\quad \leq\, C_{\widetilde{\mathrm{S}},\mathrm{d}} \left\{ L_g + L_\vartheta\, \|\nabla \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_{0,r^*;\Omega} \right\} \|(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) - (\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_{X_2 \times M_1} \end{aligned} \tag{4.13}$$

for all $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)$, $(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in X_{2,h} \times M_{1,h}$.

In this way, recalling the definition of T_h (cf. (4.4)), and employing the estimates (4.11) and (4.13), we conclude that

$$\|T_h(\phi_h) - T_h(\varphi_h)\|_{1,\Omega} \,\le\, C_{T,\mathrm{d}}\, L_f \left\{L_g + L_\vartheta\, \|T_h(\varphi_h)\|_{0,r^*;\Omega}\right\} \|\phi_h - \varphi_h\|_{1,\Omega} \qquad \forall\, \phi_h,\, \varphi_h \in \mathrm{H}_h\,, \quad (4.14)$$

with the positive constant $C_{T,\mathrm{d}} := C_{\mathrm{S,d}}\, C_{\widetilde{\mathrm{S}},\mathrm{d}}$. Regarding the estimate (4.14), we emphasize here that, while it proves the continuity of T_h, the lack of control of the term $\|T_h(\varphi_h)\|_{0,r^*;\Omega}$ does not allow us to conclude Lipschitz-continuity and hence nor contractivity of this operator. Consequently, we are able to establish next only the existence of a fixed point of T_h.

Theorem 4.4. *The Galerkin scheme (4.1) has at least one solution $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \phi_h) \in X_{2,h} \times M_{1,h} \times \mathrm{H}_h$, with $\phi_h \in W_h$ (cf. (4.10)). Moreover, there hold*

$$\begin{aligned} \|\boldsymbol{\sigma}_h\|_{X_2} \,&\le\, \frac{C_r}{\alpha_{\mathrm{d}}}\, \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\beta_{2,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\,\mu}\right) f_2\,, \quad \textit{and} \\ \|\boldsymbol{u}_h\|_{M_1} \,&\le\, \frac{C_r}{\beta_{1,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\,\mu}\right) \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\beta_{1,\mathrm{d}}\,\beta_{2,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\,\mu}\right) f_2\,. \end{aligned} \tag{4.15}$$

Proof. Thanks to Lemma 4.3 and the continuity of T_h (cf. (4.14)), and bearing in mind the equivalence between (4.1) and (4.5), a straightforward application of Brouwer's theorem (cf. [8, Theorem 9.9-2]) implies the first conclusion of this theorem. In turn, the fact that $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) = \mathrm{S}_h(\phi_h)$ and the a priori estimates from (4.8) yield (4.15), thus completing the proof. ☐

4.4 A priori error analysis

We now aim to derive an a priori error estimate for the Galerkin scheme (4.1) with arbitrary finite element subspaces satisfying the hypotheses introduced in Section 4.2. In other words, we are interested in establishing a Céa estimate for the global error

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} \,+\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{M_1} \,+\, \|\phi - \phi_h\|_{1,\Omega}\,,$$

where $(\boldsymbol{\sigma}, \boldsymbol{u}, \phi) \in X_2 \times M_1 \times \mathrm{H}^1_0(\Omega)$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \phi_h) \in X_{2,h} \times M_{1,h} \times \mathrm{H}_h$ are the unique solutions of (3.21) and (4.1) , respectively, with $\phi \in W$ (cf. (3.64)) and $\phi_h \in W_h$ (cf. (4.10)). For this purpose, and in order to employ suitable Strang estimates, we rewrite (3.21) and (4.1) as the following pairs of corresponding continuous and discrete formulations

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b_1(\boldsymbol{\tau}, \boldsymbol{u}) \;&=\; G(\boldsymbol{\tau}) \qquad && \forall\, \boldsymbol{\tau} \in X_1\,, \\ b_2(\boldsymbol{\sigma}, \boldsymbol{v}) \;&=\; F_\phi(\boldsymbol{v}) \qquad && \forall\, \boldsymbol{v} \in M_2\,, \\ a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_1(\boldsymbol{\tau}_h, \boldsymbol{u}_h) \;&=\; G(\boldsymbol{\tau}_h) \qquad && \forall\, \boldsymbol{\tau}_h \in X_{1,h}, \\ b_2(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) \;&=\; F_{\phi_h}(\boldsymbol{v}_h) \qquad && \forall\, \boldsymbol{v}_h \in M_{2,h}, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} A_{\boldsymbol{\sigma}}(\phi, \psi) \;&=\; G_{\boldsymbol{u}}(\psi) \qquad && \forall\, \psi \in \mathrm{H}^1_0(\Omega)\,, \\ A_{\boldsymbol{\sigma}_h}(\phi_h, \psi_h) \;&=\; G_{\boldsymbol{u}_h}(\psi_h) \qquad && \forall\, \psi_h \in \mathrm{H}_h\,. \end{aligned} \tag{4.17}$$

In what follows, given a subspace Z_h of a generic Banach space $(Z, \|\cdot\|_Z)$, we set for each $z \in Z$

$$\mathrm{dist}(z, Z_h) \,:=\, \inf_{z_h \in Z_h} \|z - z_h\|_Z\,.$$

Then, applying the Strang a priori error estimate from [4, Proposition 2.1, Corollary 2.3, and Theorem 2.3] to the context given by (4.16), we deduce that there exists a positive constant $\widehat{C}_{ST}$, depending only on $\alpha_{\rm d}$, $\beta_{1,\rm d}$, $\beta_{1,\rm d}$, $\|a\|$, $\|b_1\|$, and $\|b_2\|$, where $\|a\| \le \frac{2}{\mu}$ and $\|b_1\|$, $\|b_2\| \le 1$ (cf. (3.39)), such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{M_1} \,\le\, \widehat{C}_{ST}\left\{\operatorname{dist}(\boldsymbol{\sigma}, X_{2,h}) + \operatorname{dist}(\boldsymbol{u}, M_{1,h}) + \|F_\phi - F_{\phi_h}\|_{M'_{2,h}}\right\}. \tag{4.18}$$

Then, proceeding as for the derivation of (3.67) (cf. proof of Lemma 3.9), we readily find that

$$\|F_\phi - F_{\phi_h}\|_{M'_{2,h}} \,\le\, L_f\,\|i_r\|\,\|\phi - \phi_h\|_{1,\Omega}\,, \tag{4.19}$$

which, replaced back into (4.18), gives

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{M_1} \\ &\qquad \le\, \widehat{C}_{ST}\left\{\operatorname{dist}(\boldsymbol{\sigma}, X_{2,h}) + \operatorname{dist}(\boldsymbol{u}, M_{1,h}) + L_f\,\|i_r\|\,\|\phi - \phi_h\|_{1,\Omega}\right\}. \end{aligned} \tag{4.20}$$

On the other hand, applying now the classical first Strang Lemma for elliptic variational problems (cf. [11, Lemma 2.27]) to the context given by (4.17), and then adding and subtracting ϕ to the first components of the expressions involving $A_{\boldsymbol{\sigma}}$ and $A_{\boldsymbol{\sigma}_h}$ in the corresponding consistent term, and finally employing the boundedness of these bilinear forms (cf. (3.58) - (3.59)), we arrive at

$$\|\phi - \phi_h\|_{1,\Omega} \,\le\, \widetilde{C}_{ST}\left\{\operatorname{dist}(\phi, \mathrm{H}_h) \,+\, \|G_{\boldsymbol{u}} - G_{\boldsymbol{u}_h}\|_{\mathrm{H}'_h} \,+\, \|A_{\boldsymbol{\sigma}}(\phi,\cdot) - A_{\boldsymbol{\sigma}_h}(\phi,\cdot)\|_{\mathrm{H}'_h}\right\}, \tag{4.21}$$

where $\widetilde{C}_{ST}$ is a positive constant depending only on $\widetilde{\alpha}$ (cf. (3.60) - (3.61)) and the upper bound ϑ_2 of $\|A_{\boldsymbol{\sigma}_h}\|$ (cf. (3.58) - (3.59)). Next, proceeding exactly as for the derivations of (3.76) and (3.78), we find that for each $\varphi_h \in \mathrm{H}_h$ there hold

$$|(G_{\boldsymbol{u}} - G_{\boldsymbol{u}_h})(\varphi_h)| \,\le\, L_g\,\|i_s\|\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,r;\Omega}\,\|\varphi_h\|_{1,\Omega}$$

and

$$|A_{\boldsymbol{\sigma}}(\phi,\varphi_h) - A_{\boldsymbol{\sigma}_h}(\phi,\varphi_h)| \,\le\, L_\vartheta\,C_{r,\varepsilon}\,\|i_\varepsilon\|\,C_\varepsilon\,g_2\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,r;\Omega}\,\|\varphi_h\|_{1,\Omega}\,,$$

respectively, from which it follows that

$$\|G_{\boldsymbol{u}} - G_{\boldsymbol{u}_h}\|_{\mathrm{H}'_h} \,\le\, L_g\,\|i_s\|\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{M_1} \tag{4.22}$$

and

$$\|A_{\boldsymbol{\sigma}}(\phi,\cdot) - A_{\boldsymbol{\sigma}_h}(\phi,\cdot)\|_{\mathrm{H}'_h} \,\le\, L_\vartheta\,C_{r,\varepsilon}\,\|i_\varepsilon\|\,C_\varepsilon\,g_2\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2}\,. \tag{4.23}$$

In this way, replacing (4.22) and (4.23) back into (4.21), we conclude that

$$\|\phi - \phi_h\|_{1,\Omega} \,\le\, \widetilde{C}_{ST}\left\{\operatorname{dist}(\phi, \mathrm{H}_h) \,+\, L_g\,\|i_s\|\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{M_1} \,+\, L_\vartheta\,C_{r,\varepsilon}\,\|i_\varepsilon\|\,C_\varepsilon\,g_2\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2}\right\}. \tag{4.24}$$

In turn, using the foregoing bound in (4.20), and performing some algebraic arrangements, we get

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{M_1} \,&\le\, \bar{C}_0\left\{\operatorname{dist}(\boldsymbol{\sigma}, X_{2,h}) + \operatorname{dist}(\boldsymbol{u}, M_{1,h}) + \operatorname{dist}(\phi, \mathrm{H}_h)\right\} \\ &\quad + \bar{C}_1\,L_f\,L_g\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{M_1} \,+\, \bar{C}_2\,L_f\,L_\vartheta\,g_2\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2}\,, \end{aligned} \tag{4.25}$$

where $\bar{C}_0 \,:=\, \widehat{C}_{ST}\,\max\left\{1, L_f\,\|i_r\|\,\widetilde{C}_{ST}\right\}$, and $\bar{C}_1$ and $\bar{C}_2$ are positive constants depending only on $\widehat{C}_{ST}$, $\widetilde{C}_{ST}$, $\|i_r\|$, $\|i_s\|$, $\|i_\varepsilon\|$, $C_{r,\varepsilon}$, and C_ε.

According to the previous analysis, we are now in a position to establish the announced Céa estimate.

Theorem 4.5. *Assume that the data satisfy*

$$\bar{C}_1\, L_f\, L_g \,\leq\, \frac{1}{2} \quad \text{and} \quad \bar{C}_2\, L_f\, L_\vartheta\, g_2 \,\leq\, \frac{1}{2}\,. \tag{4.26}$$

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned} & \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{M_1} + \|\phi - \phi_h\|_{1,\Omega} \\ & \quad \leq\; C\Big\{\mathrm{dist}(\boldsymbol{\sigma}, X_{2,h}) + \mathrm{dist}(\boldsymbol{u}, M_{1,h}) + \mathrm{dist}(\phi, \mathrm{H}_h)\Big\}\,. \end{aligned} \tag{4.27}$$

Proof. It suffices to employ the assumptions from (4.26) in (4.25), and then combine the resulting estimate with (4.24). □

5 Specific finite element subspaces

We now restrict our analysis to the 2D case and define specific finite element subspaces $X_{2,h} \subseteq X_2$, $M_{2,h} \subseteq M_2$, $X_{1,h} \subseteq X_1$, $M_{1,h} \subseteq M_1$, and $\mathrm{H}_h \subseteq \mathrm{H}^1_0(\Omega)$, satisfying the abstract hypotheses (**H.1**) and (**H.2**) that were introduced in Section 4.2 in order to guarantee the well-posedness of the Galerkin scheme (4.1).

5.1 Preliminaries

We begin by letting $\big\{\mathcal{T}_h\big\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$, which are made of triangles K of diameters h_K, and define the meshsize $h := \max\big\{h_K : \quad K \in \mathcal{T}_h\big\}$, which also serves as the index of $\mathcal{T}_h$. Then, given an integer $k \geq 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials defined on K of degree $\leq k$, and denote its vector version by $\mathbf{P}_k(K)$. In addition, we let $\mathbf{RT}_k(K) = \mathbf{P}_k(K) \oplus \mathrm{P}_k(K)\boldsymbol{x}$ be the local Raviart-Thomas space defined on K of order k, where $\boldsymbol{x}$ stands for a generic vector in R^2, and denote by $\mathbb{RT}_k(K)$ its corresponding tensor counterpart, that is, letting $\boldsymbol{\tau}_i$ be the i-th row of a tensor $\boldsymbol{\tau}$, we set

$$\mathbb{RT}_k(K) \,:=\, \Big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; K) : \quad \boldsymbol{\tau}_i \in \mathbf{RT}_k(K) \quad \forall\, i \in \big\{1, 2\big\}\Big\}\,.$$

In turn, we let $\mathbf{P}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathbf{P}_k(K)$ and $\mathbb{RT}_k(K)$, respectively, that is

$$\mathbf{P}_k(\mathcal{T}_h) \,:=\, \Big\{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega) : \quad \boldsymbol{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h\Big\}\,,$$

and

$$\mathbb{RT}_k(\mathcal{T}_h) \,:=\, \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}; \Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall K \in \mathcal{T}_h\Big\}\,.$$

We stress here that for each $t \in [1, +\infty]$ there hold $\mathbf{P}_k(\mathcal{T}_h) \subseteq \mathbf{L}^t(\Omega)$ and $\mathbb{RT}_k \subseteq \mathbb{H}^t(\mathbf{div}_t; \Omega)$ (cf. (3.6)), which is implicitly utilized below in Section 5.2 to define the announced specific finite element subspaces. Some useful properties concerning $\mathbf{P}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ are needed first. For this purpose, we now introduce for each $t \in (1, +\infty)$ the space

$$\mathbb{H}_t \,:=\, \Big\{\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t; \Omega) : \quad \boldsymbol{\tau}|_K \in \mathbb{W}^{1,t}(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,$$

and let $\Pi^k_h : \mathbb{H}_t \to \mathbb{RT}_k(\mathcal{T}_h)$ be the global Raviart-Thomas interpolation operator (cf. [5, Section 2.5]). Then, we recall from [5, Proposition 2.5.2 and eq. (2.5.27)] that the commuting diagram property states that

$$\mathbf{div}(\Pi^k_h(\boldsymbol{\tau})) \,=\, \boldsymbol{\mathcal{P}}^k_h(\mathbf{div}(\boldsymbol{\tau})) \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_t\,, \tag{5.1}$$

where $\boldsymbol{\mathcal{P}}_h^k : \mathbf{L}^1(\Omega) \to \mathbf{P}_k(\mathcal{T}_h)$ is the usual orthogonal projector with respect to the $\mathbf{L}^2(\Omega)$-inner product, that is given $\boldsymbol{w} \in \mathbf{L}^1(\Omega)$, $\boldsymbol{\mathcal{P}}_h^k(\boldsymbol{w})$ is the unique element in $\mathbf{P}_k(\mathcal{T}_h)$ satisfying

$$\int_\Omega \boldsymbol{\mathcal{P}}_h^k(\boldsymbol{w}) \cdot \boldsymbol{v}_h \,=\, \int_\Omega \boldsymbol{w} \cdot \boldsymbol{v}_h \qquad \forall\, \boldsymbol{v}_h \in \mathbf{P}_k(\mathcal{T}_h)\,. \tag{5.2}$$

Regarding the approximation properties of $\boldsymbol{\mathcal{P}}_h^k$ and Π_h^k in the present context of the Banach spaces $\mathbf{L}^t(\Omega)$ and $\mathbb{H}_0^t(\mathbf{div}_t;\Omega)$, we remark that they follow in the usual way by employing now the $\mathrm{W}^{m,t}$ version of the Deny-Lions Lemma (cf. [11, Lemma B.67] with integer $m \geq 0$ and $t \in (1,+\infty)$, the associated scaling estimates (cf. [11, Lemma 1.101]), and the regularity of $\{\mathcal{T}_h\}_{h>0}$. Indeed, one deduces the existence of positive constants C_1, C_2, independent of h, such that for integers l and m verifying $0 \leq l \leq k+1$ and $0 \leq m \leq l$, there hold

$$|\boldsymbol{w} - \boldsymbol{\mathcal{P}}_h^k(\boldsymbol{w})|_{m,t;\Omega} \,\leq\, C_1\, h^{l-m}\, |\boldsymbol{w}|_{l,t;\Omega} \qquad \forall\, \boldsymbol{w} \in \mathbf{W}^{l,t}(\Omega)\,, \tag{5.3}$$

and

$$|\mathbf{div}(\boldsymbol{\tau}) - \mathbf{div}(\Pi_h^k(\boldsymbol{\tau}))|_{m,t;\Omega} \,\leq\, C_1\, h^{l-m}\, |\mathbf{div}(\boldsymbol{\tau})|_{l,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{l,t}(\Omega) \text{ with } \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,t}(\Omega)\,, \tag{5.4}$$

whereas for integers l and m verifying $1 \leq l \leq k+1$ and $0 \leq m \leq l$, there holds

$$|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{m,t;\Omega} \,\leq\, C_2\, h^{l-m}\, |\boldsymbol{\tau}|_{l,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \,\in\, \mathbb{W}^{l,t}(\Omega)\,. \tag{5.5}$$

Note that actually (5.4) follows from (5.1) and a direct application of (5.3) to $\boldsymbol{w} = \mathbf{div}(\boldsymbol{\tau})$. Also, we highlight that (5.3) is first derived for $1 \leq l \leq k+1$, and then using only the scaling estimates one proves the stability of $\boldsymbol{\mathcal{P}}_h^k$, that is the existence of a positive constant c, independent of h, such that

$$\|\boldsymbol{\mathcal{P}}_h^k(\boldsymbol{w})\|_{0,t;\Omega} \,\leq\, c\, \|\boldsymbol{w}\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{w} \in \mathbf{L}^t(\Omega)\,. \tag{5.6}$$

In turn, employing the triangle inequality and (5.5) with $l = 1$ and $m = 0$, we conclude the boundedness of $\Pi_h^k : \mathbb{W}^{1,t}(\Omega) \to \mathbb{L}^t(\Omega)$, which means that there exists a positive constant C, independent of h, such that

$$\|\Pi_h^k(\boldsymbol{\tau})\|_{0,t;\Omega} \,\leq\, C\, \|\boldsymbol{\tau}\|_{1,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{1,t}(\Omega)\,. \tag{5.7}$$

Finally, taking in particular $m = 0$ in (5.5) and (5.4), we readily find that there exists a positive constant C_3, independent of h, such that for $1 \leq l \leq k+1$ there holds

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{t,\mathrm{div}_t;\Omega} \,\leq\, C_3\, h^l \left\{ |\boldsymbol{\tau}|_{l,t;\Omega} + |\mathbf{div}(\boldsymbol{\tau})|_{l,t;\Omega} \right\} \tag{5.8}$$

for all $\boldsymbol{\tau} \in \mathbb{W}^{l,t}(\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,t}(\Omega)$.

5.2 The finite element subspaces

Appropriate finite element subspaces approximating the unknowns of the pseudostress-based mixed variational formulation for the elasticity problem are defined as follows

$$\begin{aligned}
X_{2,h} \,&:=\, X_2 \,\cap\, \mathbb{RT}_k(\mathcal{T}_h) \,:=\, \left\{ \boldsymbol{\zeta}_h \in \mathbb{H}_0^r(\mathbf{div}_r;\Omega) : \quad \boldsymbol{\zeta}_h|_K \in \mathbb{RT}_k(K) \quad \forall K \in \mathcal{T}_h \right\}, \\
M_{2,h} \,&:=\, M_2 \,\cap\, \mathbf{P}_k(\mathcal{T}_h) \,:=\, \left\{ \boldsymbol{v}_h \in \mathbf{L}^s(\Omega) : \quad \boldsymbol{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h \right\}, \\
X_{1,h} \,&:=\, X_1 \,\cap\, \mathbb{RT}_k(\mathcal{T}_h) \,:=\, \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_0^s(\mathbf{div}_s;\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall K \in \mathcal{T}_h \right\}, \\
M_{1,h} \,&:=\, M_1 \,\cap\, \mathbf{P}_k(\mathcal{T}_h) \,:=\, \left\{ \boldsymbol{v}_h \in \mathbf{L}^r(\Omega) : \quad \boldsymbol{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h \right\}.
\end{aligned} \tag{5.9}$$

In turn, the unknown of the diffusion problem is approximated by Lagrange finite elements of degree $\le\ k+1$, that is

$$\mathrm{H}_h \ :=\ \Big\{\psi_h \in \mathcal{C}(\Omega)\cap \mathrm{H}^1_0(\Omega): \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall\, K\in\mathcal{T}_h\Big\}\,. \tag{5.10}$$

Regarding the definitions in (5.9) we stress that, while the pairs $\big(X_{2,h}, M_{2,h}\big)$ and $\big(X_{1,h}, M_{1,h}\big)$ are topologically different, they do coincide algebraically, and hence the stiffness matrices associated to the bilinear forms b_1 and b_2 are exactly the same. Moreover, since $\mathbf{div}(X_{i,h})\ \subseteq\ M_{i,h}$, $i\in\big\{1,2\big\}$, it follows that the corresponding discrete kernels of the bilinear forms b_1 and b_2 coincide as well, and that they are given by the space

$$\mathcal{K}^k_{h,0} \ :=\ \Big\{\boldsymbol{\tau}_h\in\mathcal{K}^k_h: \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) = 0\Big\}\,, \tag{5.11}$$

where

$$\mathcal{K}^k_h \ :=\ \Big\{\boldsymbol{\tau}_h\in\mathbb{RT}_k(\mathcal{T}_h): \quad \mathbf{div}(\boldsymbol{\tau}_h)\ =\ 0\Big\}\,. \tag{5.12}$$

Moreover, similarly as derived for the vector version in [10, Lemma 2.1] (see also [20, Lemma 4.1] for a slight variant of it), one can show that

$$\mathcal{K}^k_h \ =\ \mathbf{curl}\big(\mathbf{P}_{k+1,0}(\mathcal{T}_h)\big)\,, \tag{5.13}$$

where

$$\mathbf{P}_{k+1,0}(\mathcal{T}_h)\ :=\ \Big\{\boldsymbol{\phi}_h\in\mathbf{H}^1(\Omega): \quad \boldsymbol{\phi}_h|_K\in\mathbf{P}_{k+1}(K)\quad \forall\,K\in\mathcal{T}_h\,,\quad \int_\Omega\boldsymbol{\phi}_h=\mathbf{0}\Big\}\,,$$

and $\mathbf{curl}$ is the usual curl operator acting component-wise.

Now, we let $\Theta^k_h:\mathbb{L}^1(\Omega)\to\mathcal{K}^k_h$ be the $\mathrm{L}^2(\Omega)$-orthogonal projector, that is, given $\boldsymbol{\zeta}\in\mathbb{L}^1(\Omega)$, $\Theta^k_h(\boldsymbol{\zeta})$ is the unique element in $\mathcal{K}^k_h$ satisfying

$$\int_\Omega \Theta^k_h(\boldsymbol{\zeta}):\boldsymbol{\tau}_h \ =\ \int_\Omega\boldsymbol{\zeta}:\boldsymbol{\tau}_h \qquad \forall\,\boldsymbol{\tau}_h\in\mathcal{K}^k_h\,. \tag{5.14}$$

Then, proceeding analogously to the vector version in [10, Theorem 3.1] (see also [20, Lemma 4.2] for a slight variant of it), and employing now (5.13), it can be proved in the present tensor version that for each $t\in(1,+\infty)$ and for each integer $k\ge 0$, there exist positive constants C^k_t and $\bar{C}^k_t$, independent of h, such that, defining

$$c^k_t \ :=\ \begin{cases} C^k_t & \text{if } \Omega \text{ is convex}\,,\\[2mm] \bar{C}^k_t\,\big\{-\log(h)\big\}^{|1-2/t|} & \text{if } \Omega \text{ is non-convex and } k=0\,,\\[2mm] \bar{C}^k_t & \text{if } \Omega \text{ is non-convex and } k\ge 1\,,\end{cases} \tag{5.15}$$

there holds

$$\|\Theta^k_h(\boldsymbol{\tau})\|_{0,t;\Omega}\ \le\ c^k_t\,\|\boldsymbol{\tau}\|_{0,t;\Omega}\qquad \forall\,\boldsymbol{\tau}\in\widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega)\,, \tag{5.16}$$

where

$$\widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega)\ :=\ \Big\{\boldsymbol{\tau}\in\mathbb{H}^t(\mathbf{div}_t;\Omega): \quad \mathbf{div}(\boldsymbol{\tau})=0 \quad \text{in}\quad \Omega\Big\}\,. \tag{5.17}$$

Whether the boundedness property (5.16) is satisfied or not in 3D is, up to our knowledge, still an open problem, and this fact is precisely the reason why we have restricted the analysis in the present Section 5 to the 2D case.

5.3 The discrete inf-sup conditions for S_h

In this section we show that the specific finite element subspaces introduced in Section 5.2 (cf. (5.9)) verify the hypotheses (**H.1**) and (**H.2**). To this end, we first introduce the deviatoric of $\mathcal{K}_h^k$, that is

$$\mathcal{K}_h^{k,\mathtt{d}} := \left\{ \boldsymbol{\tau}_h^{\mathtt{d}} : \quad \boldsymbol{\tau}_h \in \mathcal{K}_h^k \right\}, \tag{5.18}$$

and let $\Theta_h^{k,\mathtt{d}} : \mathbb{L}^1(\Omega) \to \mathcal{K}_h^{k,\mathtt{d}}$ be the projector defined for each $\boldsymbol{\tau} \in \mathbb{L}^1(\Omega)$ as

$$\int_\Omega \Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau}) : \boldsymbol{\zeta}_h \,=\, \int_\Omega \boldsymbol{\tau} : \boldsymbol{\zeta}_h \qquad \forall\, \boldsymbol{\zeta}_h \in \mathcal{K}_h^{k,\mathtt{d}}. \tag{5.19}$$

Then, we have the following identity relating $\Theta_h^{k,\mathtt{d}}$ and Θ_h^k.

Lemma 5.1. *There holds*

$$\Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big) \,=\, \big(\Theta_h^k(\boldsymbol{\tau})\big)^{\mathtt{d}} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{L}^1(\Omega). \tag{5.20}$$

Proof. Given $\boldsymbol{\tau} \in \mathbb{L}^1(\Omega)$, it follows from (5.18) and (5.19) that for each $\boldsymbol{\tau}_h \in \mathcal{K}_h^k$ there holds

$$\int_\Omega \Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big) : \boldsymbol{\tau}_h^{\mathtt{d}} \,=\, \int_\Omega \Theta_h^k(\boldsymbol{\tau}) : \boldsymbol{\tau}_h^{\mathtt{d}} \,=\, \int_\Omega \big(\Theta_h^k(\boldsymbol{\tau})\big)^{\mathtt{d}} : \boldsymbol{\tau}_h^{\mathtt{d}}.$$

Hence, since both $\Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big)$ and $\big(\Theta_h^k(\boldsymbol{\tau})\big)^{\mathtt{d}}$ belong to $\mathcal{K}_h^{k,\mathtt{d}}$, the identity (5.20) is concluded. $\square$

We suppose from now on that the operators Θ_h^k satisfy the following asymptotic property: for each $t \in (1, +\infty)$ and for each integer $k \geq 0$ there exists $h_t^k > 0$ such that

$$|||\, \mathrm{I} - \Theta_h^k \,|||_t \,:=\, \sup_{\substack{\boldsymbol{\tau} \in \widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\|\boldsymbol{\tau} - \Theta_h^k(\boldsymbol{\tau})\|_{0,t;\Omega}}{\|\boldsymbol{\tau}\|_{0,t;\Omega}} \,<\, 1 \qquad \forall\, h \,\leq h_t^k. \tag{5.21}$$

Numerical evidences supporting this assumption are provided later on in Section 6.

As a consequence of Lemma 5.1 and (5.21), we are able to provide next the $\mathbb{L}^t(\Omega)$-stability of $\Theta_h^{k,\mathtt{d}}$ when restricted to $\widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega)$.

Lemma 5.2. *For each $t \in (1, +\infty)$ and for each integer $k \geq 0$, there exists a positive constant $c_t^{k,\mathtt{d}}$ such that*

$$\|\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau})\|_{0,t;\Omega} \,\leq\, c_t^{k,\mathtt{d}}\, \|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega), \quad \forall\, h \leq h_t^k. \tag{5.22}$$

Proof. Given $\boldsymbol{\tau} \in \widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega)$, we first observe, thanks to the idempotence property of $\mathrm{I} - \Theta_h^k$, that

$$\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau}) - \Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big) = \Theta_h^{k,\mathtt{d}}\big((\mathrm{I} - \Theta_h^k)(\boldsymbol{\tau})\big) \,=\, \Theta_h^{k,\mathtt{d}}\big((\mathrm{I} - \Theta_h^k)^m(\boldsymbol{\tau})\big) \qquad \forall\, m \in \mathbb{N},$$

from which it follows that

$$\|\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau}) - \Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big)\|_{0,t;\Omega} \,\leq\, \|\Theta_h^{k,\mathtt{d}}\|_t \,|||\, \mathrm{I} - \Theta_h^k \,|||_t^m \,\|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall\, m \in \mathbb{N}, \tag{5.23}$$

where

$$\|\Theta_h^{k,\mathtt{d}}\|_t \,:=\, \sup_{\substack{\boldsymbol{\tau} \in \mathbb{L}^1(\Omega) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\|\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau})\|_{0,t;\Omega}}{\|\boldsymbol{\tau}\|_{0,t;\Omega}}.$$

In this way, invoking (5.21), taking $\lim\limits_{m\to+\infty}$ in (5.23), and employing (5.20) (cf. Lemma 5.1), we conclude that

$$\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau}) \,=\, \big(\Theta_h^k(\boldsymbol{\tau})\big)^{\mathtt{d}} \qquad \forall\, h \le h_t^k\,. \tag{5.24}$$

On the other hand, simple algebraic computations and (3.33) give

$$\|\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}\|_{0,t;\Omega} \,=\, n^{1/t}\,\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,t;\Omega} \,\le\, n\,\|\boldsymbol{\tau}\|_{0,t;\Omega}\,,$$

which readily implies

$$\|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,t;\Omega} \,\le\, 2\,\|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)\,. \tag{5.25}$$

Hence, employing (5.25) and (5.16), we deduce from (5.24) that

$$\|\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau})\|_{0,t;\Omega} \,\le\, 2\,c_t^k\,\|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall\, h \le h_t^k\,, \tag{5.26}$$

which constitutes the required inequality (5.22) with $c_t^{k,\mathtt{d}} = 2\,c_t^k$. $\square$

Having proved Lemma 5.2, we proceed in what follows to establish the discrete analogues of Lemmas 3.4 and 3.5, for which we suitably adapt their respective proofs to the present context. We begin with the discrete inf-sup conditions for a.

Lemma 5.3. *Assume that r and s satisfy the final ranges specified by (3.70) and (3.71), that is $r \in \big(2, \frac{2n}{n-1}\big]$ and $s \in \big[\frac{2n}{n+1}, 2\big)$. Then, there exist positive constants M_{d} and α_{d} such that for each $\lambda > M_{\mathrm{d}}$ and for each $h \le h_0 := \min\big\{h_r^k, h_s^k\big\}$, there hold*

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\ge\, \alpha_{\mathrm{d}}\,\|\boldsymbol{\zeta}_h\|_{X_2} \qquad \forall\,\boldsymbol{\zeta}_h \in \mathcal{K}_{h,0}^k\,, \tag{5.27}$$

and

$$\sup_{\boldsymbol{\zeta}_h\in\mathcal{K}_{h,0}^k} a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h) \,>\, 0 \qquad \forall\,\boldsymbol{\tau}_h \in \mathcal{K}_{h,0}^k,\ \boldsymbol{\tau}_h \neq \mathbf{0}\,. \tag{5.28}$$

Proof. Similarly to the proof of Lemmas 3.4, we first observe that, given $\boldsymbol{\zeta}_h \in \mathcal{K}_{h,0}^k$, there holds the discrete analogue of (3.45), namely

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\ge\, \frac{1}{\mu}\sup_{\substack{\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}} : \boldsymbol{\tau}_h^{\mathtt{d}}}{\|\boldsymbol{\tau}_h\|_{X_1}} \,-\, \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda + (n+1)\mu\big)}\,\|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega}\,, \tag{5.29}$$

whence the rest of the proof reduces to get a suitable lower bound for the supremum on the right hand side of (5.29). To this end, we proceed as in (3.46) and set

$$\boldsymbol{\zeta}_{h,s} \,:=\, \begin{cases} |\boldsymbol{\zeta}_h^{\mathtt{d}}|^{r-2}\,\boldsymbol{\zeta}_h^{\mathtt{d}} & \text{if} \quad \boldsymbol{\zeta}_h^{\mathtt{d}} \neq \mathbf{0},\\ \mathbf{0} & \text{if} \quad \boldsymbol{\zeta}_h^{\mathtt{d}} = \mathbf{0}\,, \end{cases} \tag{5.30}$$

which belongs to $\mathbb{L}^s(\Omega)$ and satisfies (cf. (3.47))

$$\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}} : \boldsymbol{\zeta}_{h,s} \,=\, \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega}^r \,=\, \|\boldsymbol{\zeta}_{h,s}\|_{0,s;\Omega}^s \,=\, \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_{h,s}\|_{0,s;\Omega}\,. \tag{5.31}$$

Then, we recall the definition of the operator D_s (cf. Lemma 3.2) and let $\widetilde{\boldsymbol{\tau}}_h \in \mathcal{K}_h^k$ (cf. (5.12)) such that $\widetilde{\boldsymbol{\tau}}_h^{\mathtt{d}} = \Theta_h^{k,\mathtt{d}}(D_s(\boldsymbol{\zeta}_{h,s})) \in \mathcal{K}_h^{k,\mathtt{d}}$ (cf. (5.18)). In this way, defining the constant

$$c_h \,:=\, \frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(\widetilde{\boldsymbol{\tau}}_h) \in \mathrm{R}\,,$$

it follows that $\widetilde{\boldsymbol{\tau}}_h - c_h \mathbb{I} \in \mathcal{K}_{h,0}^k$, and hence

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathcal{K}_{h,0}^k \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}} : \boldsymbol{\tau}_h^{\mathtt{d}}}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\geq\, \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}} : (\widetilde{\boldsymbol{\tau}}_h - c_h\mathbb{I})^{\mathtt{d}}}{\|\widetilde{\boldsymbol{\tau}}_h - c_h\mathbb{I}\|_{0,s;\Omega}} \,=\, \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}} : \widetilde{\boldsymbol{\tau}}_h^{\mathtt{d}}}{\|\widetilde{\boldsymbol{\tau}}_h - c_h\mathbb{I}\|_{0,s;\Omega}}\,. \tag{5.32}$$

Now, employing the characterization of $\Theta_h^{k,\mathtt{d}}$ (cf. (5.19)), the identity (3.30) satisfied by D_s, and (5.31), we find that

$$\begin{aligned} \int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}} : \widetilde{\boldsymbol{\tau}}_h^{\mathtt{d}} \,&=\, \int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}} : \Theta_h^{k,\mathtt{d}}(D_s(\boldsymbol{\zeta}_{h,s})) \,=\, \int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}} : D_s(\boldsymbol{\zeta}_{h,s}) \\ &=\, \int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}} : \boldsymbol{\zeta}_{h,s} \,=\, \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega}\, \|\boldsymbol{\zeta}_{h,s}\|_{0,s;\Omega}\,. \end{aligned} \tag{5.33}$$

In turn, applying (3.32) (cf. Lemma 3.3) to $\widetilde{\boldsymbol{\tau}}_h - c_h\mathbb{I}$, and making use of the boundedness of $\Theta_h^{k,\mathtt{d}}$ (cf. (5.22)) and D_s (cf. Lemma 3.2), we get

$$\begin{aligned} \|\widetilde{\boldsymbol{\tau}}_h - c_h\mathbb{I}\|_{0,s;\Omega} \,&\leq\, \widehat{C}_s\, \|\widetilde{\boldsymbol{\tau}}_h^{\mathtt{d}}\|_{0,s;\Omega} \,=\, \widehat{C}_s\, \|\Theta_h^{k,\mathtt{d}}(D_s(\boldsymbol{\zeta}_{h,s}))\|_{0,s;\Omega} \\ &\leq\, \widehat{C}_s\, c_s^{k,\mathtt{d}}\, \|D_s\|\, \|\boldsymbol{\zeta}_{h,s}\|_{0,s;\Omega} \qquad \forall\, h \leq h_s^k\,. \end{aligned} \tag{5.34}$$

Therefore, replacing (5.33) and (5.34) back into (5.32), and then the resulting estimate in (5.29), we arrive at

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathcal{K}_{h,0}^k \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{a(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\geq\, \left\{ \frac{1}{\mu\, \widehat{C}_s\, c_s^{k,\mathtt{d}} \|D_s\|} \,-\, \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda + (n+1)\mu\big)} \right\} \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega} \qquad \forall\, h \leq h_s^k\,, \tag{5.35}$$

from which, choosing λ sufficiently large such that

$$\frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda + (n+1)\mu\big)} \,<\, \frac{1}{2\,\mu\, \widehat{C}_s\, c_s^{k,\mathtt{d}}\, \|D_s\|}\,,$$

that is

$$\lambda \,>\, M_{s,\mathrm{d}} := \frac{\mu}{n^{1+1/s}} \max\left\{ 2\,\mu\, \widehat{C}_s\, \widetilde{C}_r\, c_s^{k,\mathtt{d}}\, \|D_s\| - n^{1/s}(n+1), 0 \right\}\,,$$

and applying (3.32) to $\boldsymbol{\zeta}_h$, we conclude (5.27), with $\alpha_{\mathrm{d}} \,:=\, \frac{1}{2\,\mu\, \widehat{C}_s\, \widehat{C}_r\, c_s^{k,\mathtt{d}}\, \|D_s\|}$, for each $h \leq h_s^k$. Similarly, given $\boldsymbol{\tau}_h \in \mathcal{K}_{h,0}^k$, $\boldsymbol{\tau}_h \neq 0$, we proceed analogously as above, but exchanging the roles of $\boldsymbol{\tau}_h$ and $\boldsymbol{\zeta}_h$, and obtain

$$\sup_{\boldsymbol{\zeta}_h \in \mathcal{K}_{h,0}^k} a(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h) \,\geq\, \sup_{\substack{\boldsymbol{\zeta}_h \in \mathcal{K}_{h,0}^k \\ \boldsymbol{\zeta}_h \neq \mathbf{0}}} \frac{a(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\zeta}_h\|_{X_2}} \,\geq\, \frac{1}{2\,\mu\, \widehat{C}_r\, \widehat{C}_s\, c_r^{k,\mathtt{d}}\, \|D_r\|} \|\boldsymbol{\tau}_h\|_{X_1} \,>\, 0 \qquad \forall\, h \leq h_r^k\,, \tag{5.36}$$

for

$$\lambda \,>\, M_{r,\mathrm{d}} \,:=\, \frac{\mu}{n^{1+1/r}} \max\left\{ 2\,\mu\, \widehat{C}_r\, \widetilde{C}_s\, c_r^{k,\mathtt{d}}\, \|D_r\| - n^{1/r}(n+1), 0 \right\}\,,$$

which proves (5.28) for each $h \leq h_r^k$. Finally, defining $M_{\mathtt{d}} \,:=\, \max\{M_{s,\mathtt{d}}, M_{r,\mathtt{d}}\}$, the proof is completed. $\square$

The discrete inf-sup conditions for the bilinear forms b_i, $i \in \{1,2\}$, are provided next.

Lemma 5.4. *Assume that r and s satisfy the final ranges specified by (3.70) and (3.71), that is $r \in \big(2, \frac{2n}{n-1}\big]$ and $s \in \big[\frac{2n}{n+1}, 2\big)$. Then, there exist positive constants $\beta_{1,\mathrm{d}}$, $\beta_{2,\mathrm{d}}$, independent of h, such that for each $i \in \{1,2\}$ there holds*

$$\sup_{\substack{\tau_h \in X_{i,h} \\ \tau \neq \mathbf{0}}} \frac{b_i(\boldsymbol{\tau}_h, \boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\|_{X_i}} \,\geq\, \beta_{i,\mathrm{d}}\, \|\boldsymbol{v}_h\|_{M_i} \qquad \forall\, \boldsymbol{v}_h \in M_{i,h}\,. \tag{5.37}$$

Proof. We adapt the proof of Lemma 3.5 to show (5.37) only for $i = 2$ since the case $i = 1$ is analogous. Indeed, given $\boldsymbol{v}_h \in M_{2,h} \subseteq M_2 = \mathbf{L}^s(\Omega)$, we follow (3.52) and define first

$$\boldsymbol{v}_{h,r} \,:=\, \begin{cases} |\boldsymbol{v}_h|^{s-2}\, \boldsymbol{v}_h & \text{if} \quad \boldsymbol{v}_h \neq \mathbf{0}\,, \\ \quad \mathbf{0} & \text{if} \quad \boldsymbol{v}_h = \mathbf{0}\,, \end{cases} \tag{5.38}$$

which belongs to $\mathbf{L}^r(\Omega)$ and, as in (3.53), satisfies

$$\int_\Omega \boldsymbol{v}_h \cdot \boldsymbol{v}_{h,r} \,=\, \|\boldsymbol{v}_h\|_{0,s;\Omega}^s \,=\, \|\boldsymbol{v}_{h,r}\|_{0,r;\Omega}^r \,=\, \|\boldsymbol{v}_h\|_{0,s;\Omega}\, \|\boldsymbol{v}_{h,r}\|_{0,r;\Omega}\,. \tag{5.39}$$

Next, proceeding similarly to the proof of [20, Lemma 5.4], we let $\mathcal{O}$ be a bounded convex polygonal domain containing $\bar{\Omega}$, and introduce

$$\mathbf{g} \,=\, \begin{cases} \boldsymbol{v}_{h,r} & \text{in} \qquad \Omega\,, \\ \ \mathbf{0} & \text{on} \qquad \mathcal{O}\backslash\bar{\Omega}\,, \end{cases} \tag{5.40}$$

which is clearly seen to belong to $\mathbf{L}^r(\mathcal{O})$ with $\|\mathbf{g}\|_{0,r;\mathcal{O}} \,=\, \|\boldsymbol{v}_{h,r}\|_{0,r;\Omega}$. Then, applying the elliptic regularity result provided in [13, Corollary 1], we deduce that there exists a unique $\boldsymbol{z} \in \mathbf{W}^{2,r}(\mathcal{O}) \cap \mathbf{W}_0^{1,r}(\mathcal{O})$ solution of

$$\Delta \boldsymbol{z} \,=\, \mathbf{g} \quad \text{in} \quad \mathcal{O}\,, \qquad \boldsymbol{z} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{O}\,, \tag{5.41}$$

and that there exists a positive constant $C_{\mathtt{reg}}$, depending only on $\mathcal{O}$, such that

$$\|\boldsymbol{z}\|_{2,r;\mathcal{O}} \,\leq\, C_{\mathtt{reg}}\, \|\boldsymbol{v}_{h,r}\|_{0,r;\Omega}\,. \tag{5.42}$$

In this way, defining now $\boldsymbol{\zeta} := \nabla \boldsymbol{z}|_\Omega \in \mathbf{W}^{1,r}(\Omega)$, it follows from (5.40), (5.41), and (5.42) that

$$\mathbf{div}(\boldsymbol{\zeta}) \,=\, \boldsymbol{v}_{h,r} \quad \text{in} \quad \Omega \quad \text{and} \quad \|\boldsymbol{\zeta}\|_{1,r;\Omega} \,\leq\, C_{\mathtt{reg}}\, \|\boldsymbol{v}_{h,r}\|_{0,r;\Omega}\,. \tag{5.43}$$

Thus, letting $\boldsymbol{\zeta}_h$ be the $\mathbb{H}_0^r(\mathbf{div}_r;\Omega)$-component (cf. (3.13)) of $\Pi_h^k(\boldsymbol{\zeta})$, and employing the commuting diagram property (5.1) and the identity from (5.43), we observe that

$$\mathbf{div}(\boldsymbol{\zeta}_h) \,=\, \mathbf{div}(\Pi_h^k(\boldsymbol{\zeta})) \,=\, \boldsymbol{\mathcal{P}}_h^k(\mathbf{div}(\boldsymbol{\zeta})) \,=\, \boldsymbol{\mathcal{P}}_h^k(\boldsymbol{v}_{h,r}) \quad \text{in} \quad \Omega\,, \tag{5.44}$$

so that, applying the stability estimate of $\boldsymbol{\mathcal{P}}_h^k$ (cf. (5.6)), it follows that

$$\|\mathbf{div}(\boldsymbol{\zeta}_h)\|_{0,r;\Omega} \,\leq\, c\, \|\boldsymbol{v}_{h,r}\|_{0,r;\Omega}\,. \tag{5.45}$$

On the other hand, according to (3.13) and the notations introduced there, and using the triangle and Holder inequality, and (3.33), it is easy to show that for each $t \in (1, +\infty)$ there holds

$$\|\boldsymbol{\tau}_0\|_{0,t;\Omega} \,\leq\, 2\, \|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\,. \tag{5.46}$$

Hence, employing now (5.46), the stability estimate of Π_h^k (cf. (5.7)), and the inequality from (5.43), we find that

$$\|\boldsymbol{\zeta}_h\|_{0,r;\Omega} \,\leq\, 2\, \|\Pi_h^k(\boldsymbol{\zeta})\|_{0,r;\Omega} \,\leq\, 2\, C\, \|\boldsymbol{\zeta}\|_{1,r;\Omega} \,\leq\, 2\, C\, C_{\mathtt{reg}}\, \|\boldsymbol{v}_{h,r}\|_{0,r;\Omega}\,, \tag{5.47}$$

which, jointly with (5.45), yield the existence of a positive constant $\widehat{C}$, independent of h, such that (cf. (3.7))

$$\|\boldsymbol{\zeta}_h\|_{X_2} \,=\, \|\boldsymbol{\zeta}_h\|_{0,r;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\zeta}_h)\|_{0,r;\Omega} \,\le\, \widehat{C}\,\|\boldsymbol{v}_{h,r}\|_{0,r;\Omega}\,. \tag{5.48}$$

Finally, bearing in mind (5.44), (5.2), (5.39), and (5.48), we obtain

$$\begin{aligned}\sup_{\substack{\boldsymbol{\tau}_h\in X_{2,h}\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{b_2(\boldsymbol{\tau}_h,\boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\|_{X_2}} \,&\ge\, \frac{\displaystyle\int_\Omega \boldsymbol{v}_h\cdot\mathbf{div}(\boldsymbol{\zeta}_h)}{\|\boldsymbol{\zeta}_h\|_{X_2}} \,=\, \frac{\displaystyle\int_\Omega \boldsymbol{v}_h\cdot\boldsymbol{\mathcal{P}}_h^k(\boldsymbol{v}_{h,r})}{\|\boldsymbol{\zeta}_h\|_{X_2}}\\
&=\, \frac{\displaystyle\int_\Omega \boldsymbol{v}_h\cdot\boldsymbol{v}_{h,r}}{\|\boldsymbol{\zeta}_h\|_{X_2}} \,=\, \frac{\|\boldsymbol{v}_h\|_{0,s;\Omega}\,\|\boldsymbol{v}_{h,r}\|_{0,r;\Omega}}{\|\boldsymbol{\zeta}_h\|_{X_2}} \,\ge\, \frac{1}{\widehat{C}}\,\|\boldsymbol{v}_h\|_{M_2}\,,\end{aligned}$$

which yields (5.37) for $i=2$ with $\beta_{2,\mathrm{d}} := \frac{1}{\widehat{C}}$. □

5.4 The rates of convergence

The rates of convergence of the Galerkin scheme (4.1) with the specific finite element subspaces introduced in Section 5.2 are provided next. To this end, we first collect the approximation properties of $X_{2,h}$ and $M_{1,h}$ (cf. (5.9)), which follow from (5.8) (for $t=r$) and (5.3) (for $m=0$ and $t=r$), respectively, along with interpolation estimates of Sobolev spaces. More precisely, they are given as follows:

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists $C>0$, independent of h, such that for each $l\in[1,k+1]$, and for each $\boldsymbol{\tau}\in\mathbb{W}^{l,r}(\Omega)$ with $\mathbf{div}(\boldsymbol{\tau})\in\mathbf{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau},X_{2,h}) \,:=\, \inf_{\boldsymbol{\tau}_h\in X_{2,h}} \|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_{r,\mathrm{div}_r;\Omega} \,\le\, C\,h^l\left\{\|\boldsymbol{\tau}\|_{l,r;\Omega}+\|\mathbf{div}(\boldsymbol{\tau})\|_{l,r;\Omega}\right\}.$$

$(\mathbf{AP}_h^{\boldsymbol{u}})$ there exists $C>0$, independent of h, such that for each $l\in[0,k+1]$, and for each $\boldsymbol{v}\in\mathbf{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{v},M_{1,h}) \,:=\, \inf_{\boldsymbol{v}_h\in M_{1,h}} \|\boldsymbol{v}-\boldsymbol{v}_h\|_{0,r;\Omega} \,\le\, C\,h^l\,\|\boldsymbol{v}\|_{l,r;\Omega}\,.$$

In turn, the approximation property of H_h, which makes use of interpolation estimates of Sobolev spaces as well, is stated as indicated below (cf. [11, Corollary 1.109]):

$(\mathbf{AP}_h^{\phi})$ there exists $C>0$, independent of h, such that for each $l\in(0,k+1]$, and for each $\varphi\in\mathrm{H}^{l+1}(\Omega)$, there holds

$$\mathrm{dist}(\varphi,\mathrm{H}_h) \,:=\, \inf_{\varphi_h\in\mathrm{H}_h} \|\varphi-\varphi_h\|_{1,\Omega} \,\le\, C\,h^l\,\|\varphi\|_{l+1,\Omega}\,.$$

Consequently, we can state the following main theorem.

Theorem 5.5. *Let $(\boldsymbol{\sigma},\boldsymbol{u},\phi)\in X_2\times M_1\times\mathrm{H}_0^1(\Omega)$ be the unique solution of (3.21) with $\phi\in W$ (cf. (3.64)), and let $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\phi_h)\in X_{2,h}\times M_{1,h}\times\mathrm{H}_h$ be a solution of (4.1) with $\phi_h\in W_h$ (cf. (4.10)), whose existences are guaranteed by Theorems 3.12 and 4.4, respectively. Assume that (4.26) (cf. Theorem 4.5) holds, and that there exists $l\in[1,k+1]$ such that $\boldsymbol{\sigma}\in\mathbb{W}^{l,r}(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma})\in\mathbf{W}^{l,r}(\Omega)$, $\boldsymbol{u}\in\mathbf{W}^{l,r}(\Omega)$ and $\phi\in\mathrm{H}^{l+1}(\Omega)$. Then there exists a constant $C>0$, independent of h, such that*

$$\begin{aligned}&\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{X_2}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{M_1}+\|\phi-\phi_h\|_{1,\Omega}\\
&\quad\le\, C\,h^l\left\{\|\boldsymbol{\sigma}\|_{l,r;\Omega}\,+\,\|\mathbf{div}(\boldsymbol{\sigma})\|_{l,r;\Omega}\,+\,\|\boldsymbol{u}\|_{l,r;\Omega}\,+\,\|\phi\|_{l+1,\Omega}\right\}.\end{aligned}$$

Proof. It follows directly from the Céa estimate (4.27) and the above approximation properties. □

6 Numerical results

In this section we report numerical experiments illustrating the performance of the Galerkin scheme (4.1) with the specific finite element spaces defined in (5.9), and confirming the theoretical rates of convergence provided by Theorem 5.5. We begin by recalling that part of the analysis developed in Section 5, namely the one referring to the discrete inf-sup conditions for the bilinear form a, depends on the hypothesis (5.21), which establishes an asymptotic behavior of the operators Θ_h^k. However, since proving this assumption has remained elusive, in what follows we present numerical evidence supporting its eventual validity. To this end, we now consider the convex and non-convex domains given by $\Omega_S := (0,1)^2$ and $\Omega_L := (-1,1)^2 \setminus [0,1]^2$, respectively, and let $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and $\boldsymbol{\tau}_3$ be the tensor fields defined for each $\boldsymbol{x} := (x_1, x_2)^{\mathrm{t}} \in \Omega_S \cup \Omega_L$ as:

$$\boldsymbol{\tau}_1 \;:=\; \mathbf{curl}\begin{pmatrix} \exp(-x_1^2 - x_2^2) \\ \exp(-x_1 x_2) \end{pmatrix} \;=\; \begin{pmatrix} -2x_2 \exp(-x_1^2 - x_2^2) & 2x_1 \exp(-x_1^2 - x_2^2) \\ -x_1 \exp(-x_1 x_2) & x_2 \exp(-x_1 x_2) \end{pmatrix},$$

$$\boldsymbol{\tau}_2 \;:=\; \mathbf{curl}\begin{pmatrix} \pi^{-1} \sin(\pi x_1)\cos(\pi x_2) \\ \pi^{-1} \cos(\pi x_1)\sin(\pi x_2) \end{pmatrix} \;=\; \begin{pmatrix} -\sin(\pi x_1)\sin(\pi x_2) & -\cos(\pi x_1)\cos(\pi x_2) \\ \cos(\pi x_1)\cos(\pi x_2) & \sin(\pi x_1)\sin(\pi x_2) \end{pmatrix}$$

and

$$\boldsymbol{\tau}_3 \;:=\; \mathbf{curl}\left(\frac{1}{3}\Big\{(x_1-2)^2 + (x_2-2)^2\Big\}^{3/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \;=\; \sqrt{(x_1-2)^2 + (x_2-2)^2}\begin{pmatrix} x_2 - 2 & 2 - x_1 \\ x_2 - 2 & 2 - x_1 \end{pmatrix},$$

which are clearly all divergence-free. Then, for $p = \frac{4}{3}$, $k \in \{0,1\}$, and five regular triangulations $\mathcal{T}_h$ of Ω_S and Ω_L, respectively, we compute the expressions

$$\mathsf{c}_{h,S}^k(\boldsymbol{\tau}) \;:=\; \frac{\|\boldsymbol{\tau} - \Theta_h^k(\boldsymbol{\tau})\|_{0,p;\Omega_S}}{\|\boldsymbol{\tau}\|_{0,p;\Omega_S}} \quad \text{and} \quad \mathsf{c}_{h,L}^k(\boldsymbol{\tau}) \;:=\; \frac{\|\boldsymbol{\tau} - \Theta_h^k(\boldsymbol{\tau})\|_{0,p;\Omega_L}}{\|\boldsymbol{\tau}\|_{0,p;\Omega_L}} \quad \forall\, \boldsymbol{\tau} \in \big\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3\big\},$$

which are displayed below in Table 6.1. We observe there that these values remain not only below 1, as requested by (5.21), but they actually approach 0 as the meshsize h tends to 0.

	Ω_S			Ω_L		
$\boldsymbol{\tau}$	h	$\mathsf{c}_{h,S}^0(\boldsymbol{\tau})$	$\mathsf{c}_{h,S}^1(\boldsymbol{\tau})$	h	$\mathsf{c}_{h,L}^0(\boldsymbol{\tau})$	$\mathsf{c}_{h,L}^1(\boldsymbol{\tau})$
$\boldsymbol{\tau}_1$	0.1414	6.91e-02	1.58e-03	0.1414	8.43e-02	2.21e-03
	0.0707	3.49e-02	4.00e-04	0.0471	2.85e-02	2.50e-04
	0.0471	2.33e-02	1.78e-04	0.0283	1.71e-02	9.05e-05
	0.0202	1.00e-02	3.29e-05	0.0202	1.22e-02	4.63e-05
	0.0109	5.40e-03	9.56e-06	0.0177	1.07e-02	3.54e-05
$\boldsymbol{\tau}_2$	0.1414	1.52e-01	8.71e-03	0.1414	1.53e-01	8.79e-03
	0.0707	7.66e-02	2.21e-03	0.0471	5.11e-02	9.88e-04
	0.0471	5.11e-02	9.85e-04	0.0283	3.07e-02	3.56e-04
	0.0202	2.19e-02	1.82e-04	0.0202	2.19e-02	1.82e-04
	0.0109	1.18e-02	5.28e-05	0.0177	1.92e-02	1.39e-04
$\boldsymbol{\tau}_3$	0.1414	2.07e-02	1.63e-04	0.1414	1.44e-02	7.89e-05
	0.0707	1.04e-02	4.08e-05	0.0471	4.81e-03	8.80e-06
	0.0471	6.90e-03	1.82e-05	0.0283	2.89e-03	3.17e-06
	0.0202	2.96e-03	3.34e-06	0.0202	2.06e-03	1.62e-06
	0.0109	1.59e-03	9.68e-07	0.0177	1.81e-03	1.24e-06

Table 6.1: Numerical evidence eventually supporting (5.21).

Next, we consider the finite element subspaces defined in (5.9) with $k \in \{0,1,2\}$, to illustrate the performance of the mixed-primal finite element scheme (4.1) and confirm the rates of convergence provided by Theorem 5.5, through three numerical examples. We begin by noticing that the total number of degrees of freedom (or unknowns) of (4.1) is given for $n = 2$ by

$$\begin{aligned} N := &\big\{\text{number of nodes of } \mathcal{T}_h\big\} \,+\, \big(2(k+1)+k\big) \times \big\{\text{number of edges of } \mathcal{T}_h\big\} \\ &+ \big(2k(k+1) + (k+1)(k+2) + \tfrac{1}{2}k(k-1)\big) \times \big\{\text{number of elements of } \mathcal{T}_h\big\} \,+\, 1\,, \end{aligned}$$

whereas for $n = 3$ it becomes

$$\begin{aligned} N := &\big\{\text{number of nodes of } \mathcal{T}_h\big\} \,+\, k \times \big\{\text{number of edges of } \mathcal{T}_h\big\} \\ &+ \big(2k^2+4k+3\big) \times \big\{\text{number of faces of } \mathcal{T}_h\big\} \\ &+ \big(\tfrac{13k^3+42k^2+53k+18}{6}\big) \times \big\{\text{number of elements of } \mathcal{T}_h\big\} \,+\, 1\,. \end{aligned}$$

Now, regarding the resolution itself of (4.1), we remark that the null integral mean condition for the traces of tensors in the space $X_{2,h}$ (cf. (5.9)) is imposed via a real Lagrange multiplier, and that the nonlinear algebraic systems obtained are solved following the discrete fixed-point strategy suggested by (4.5), whose computational implementation is given by a C^{++} code. We take as initial guess the trivial solution, and remark in advance that for each one of the examples to be reported below, three iterations are required to achieve a tolerance of 10^{-6}.

Furthermore, given r as specified in (3.70), we introduce the individual errors:

$$\begin{aligned} \mathtt{e}(\boldsymbol{\sigma}) \;&:=\; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{r,\mathbf{div}_r;\Omega}\,, \qquad \mathtt{e}(\boldsymbol{u}) \;:=\; \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,r;\Omega}\,, \\ \mathtt{e}(\phi) \;&:=\; \|\phi - \phi_h\|_{1,\Omega} \qquad \text{and} \qquad \mathtt{e}(\boldsymbol{\rho}) \;:=\; \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,r;\Omega}\,, \end{aligned}$$

where, according to (2.7) and (3.14), $\boldsymbol{\rho}_h$ is computed as:

$$\boldsymbol{\rho}_h \;:=\; \boldsymbol{\sigma}_h \,+\, \boldsymbol{\sigma}_h^{\mathrm{t}} \,-\, \left(\frac{\lambda+2\mu}{n\lambda+(n+1)\mu}\,\operatorname{tr}(\boldsymbol{\sigma}_h) \,-\, \frac{n\lambda+2\mu}{n|\Omega|}\int_\Gamma \boldsymbol{u}_D\cdot\boldsymbol{\nu}\right)\mathbb{I}\,. \tag{6.1}$$

In this way, the respective experimental rates of convergence are defined as:

$$\mathtt{r}(*) \;:=\; \frac{\log(\mathtt{e}(*)\,/\,\mathtt{e}'(*))}{\log(h\,/\,h')} \qquad \forall\, * \in \big\{\boldsymbol{\sigma}, \boldsymbol{u}, \phi, \boldsymbol{\rho}\big\}\,,$$

where $\mathtt{e}(*)$ and $\mathtt{e}'(*)$ denote errors computed on two consecutive meshes of sizes h and h', respectively.

In what follows we proceed to report on the numerical experiments obtained. The first example uses a smooth manufactured solution to illustrate that the optimal rates of convergence of our method are indeed attained in this case. The second one considers a singular solution to confirm that precisely the lack of smoothness directly affects the order of convergence. Finally, and while, as shown in Section 5, the discrete analysis using the specific finite element subspaces introduced in Section 5.2 has been guaranteed only in 2D, the third example illustrates the applicability of the method to a three-dimensional problem as well. In each case we let E and ν be the Young modulus and Poisson ratio, respectively, of the isotropic linear elastic solid occupying the region Ω, so that the corresponding Lamé parameters are given by:

$$\mu \;:=\; \frac{E}{2(1+\nu)} \qquad \text{and} \qquad \lambda \;:=\; \frac{E\nu}{(1+\nu)(1-2\nu)}\,. \tag{6.2}$$

Example 1. We consider the very same example from [16, Example 1, Section 5], which means that we let $\Omega = (0,1)^2$, and adequately manufacture the data so that the exact solution of (2.1) is given by the smooth functions

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; \begin{pmatrix} \frac{1}{20}\sin(\pi x_1)\cos(\pi x_2) + \frac{1}{2\lambda}x_1^2 \\[1ex] \frac{1}{20}\cos(\pi x_1)\sin(\pi x_2) + \frac{1}{2\lambda}x_2^2 \end{pmatrix} \quad \text{and} \quad \phi(\boldsymbol{x}) \;=\; x_1 x_2 (x_1 - 1)(x_2 - 1)\,,$$

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$
0	0.0333	19982	5.26e+01	$--$	9.33e-04	$--$	7.82e-03	$--$	2.07e+01	$--$
	0.0270	30342	4.26e+01	1.00	7.54e-04	1.02	6.54e-03	0.85	1.68e+01	1.00
	0.0217	46830	3.43e+01	1.00	6.05e-04	1.01	5.34e-03	0.93	1.35e+01	1.00
	0.0185	64478	2.92e+01	1.00	5.14e-04	1.01	4.60e-03	0.93	1.15e+01	1.00
	0.0164	82230	2.59e+01	1.00	4.55e-04	1.01	4.06e-03	1.01	1.02e+01	1.00
	0.0139	114482	2.19e+01	1.00	3.85e-04	1.00	3.44e-03	1.01	8.63e+00	1.00
	0.0122	148422	1.92e+01	1.00	3.38e-04	1.00	3.02e-03	1.01	7.58e+00	1.00
1	0.0333	65162	6.67e-01	$--$	1.21e-05	$--$	6.77e-05	$--$	2.39e-01	$--$
	0.0270	99014	4.38e-01	2.00	7.93e-06	2.01	4.22e-05	2.25	1.58e-01	1.99
	0.0217	152906	2.84e-01	2.00	5.13e-06	2.00	2.82e-05	1.85	1.02e-01	1.99
	0.0185	210602	2.06e-01	2.00	3.72e-06	2.00	2.02e-05	2.07	7.42e-02	1.99
	0.0164	268646	1.61e-01	2.00	2.91e-06	2.00	1.54e-05	2.23	5.82e-02	1.99
2	0.0333	135542	5.74e-03	$--$	1.06e-07	$--$	9.87e-07	$--$	1.97e-03	$--$
	0.0270	206018	3.06e-03	3.00	5.63e-08	3.00	4.75e-07	3.49	1.05e-03	3.00
	0.0217	318230	1.59e-03	3.00	2.93e-08	3.00	2.48e-07	2.99	5.48e-04	3.00
	0.0185	438374	9.85e-04	3.00	1.81e-08	3.00	1.54e-07	2.94	3.39e-04	3.00
	0.0164	559250	6.83e-04	3.00	1.26e-08	3.00	1.08e-07	2.96	2.35e-04	3.00

Table 6.2: History of convergence for Example 1 with $r = 3$.

for all $\boldsymbol{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$, whereas the body load, the diffusive source, and the tensorial diffusivity, are given, respectively, by

$$\boldsymbol{f}(\phi) \;=\; \begin{pmatrix} \frac{1}{10}\cos^2(\phi) \\ -\frac{1}{10}\sin(\phi) \end{pmatrix}, \quad g(\boldsymbol{u}) \;=\; \frac{1}{10}\left(1 + \frac{1}{1+|\boldsymbol{u}|}\right) \quad \text{and} \quad \vartheta(\boldsymbol{\sigma}) \;=\; \mathbb{I} + \frac{1}{10}\boldsymbol{\sigma}^2 .$$

It is important to remark here that the second and fifth equations of (2.8) actually include additional explicit source terms that are added to $\boldsymbol{f}(\phi)$ and $g(\boldsymbol{u})$, respectively. However, yielding only slight modifications of the functionals F_ϕ and $G_{\boldsymbol{u}}$ in (3.21), this fact does not compromise the continuous and discrete analyses. In addition, we take Young's modulus $E = 10^3$ and Poisson's ratio $\nu = 0.4$, which, according to (6.2), implies that $\mu = 357.1429$ and $\lambda = 1428.5714$. Thus, in Tables 6.2 and 6.3 we summarize the convergence history of the Galerkin scheme (4.1) with $r = 3$ and $r = 4$, respectively. In particular, we stress that the optimal order of convergence $O(h^{k+1})$ predicted by Theorem 5.5 is attained by all the unknowns. Some components and magnitudes of the discrete solutions are displayed in Figure 6.1. Furthermore, in order to compare our discrete scheme (4.1) with those proposed in [16], beyond the fact that they all confirm their theoretical rates of convergence, we first point out that the unknowns $\boldsymbol{\sigma}$ from the present paper and [16] do not coincide, and hence their numerical approximations $\boldsymbol{\sigma}_h$ and associated errors are not comparable. Actually, the stress $\boldsymbol{\sigma}$ from [16] corresponds to our $\boldsymbol{\rho}$, whose discrete approximation $\boldsymbol{\rho}_h$ (cf. (6.1)) lies only in $\mathbb{L}^r(\Omega)$. Consequently, we extract from Tables 6.2 and 6.3, and [16, Table 1], the necessary information to display in Figures 6.2 and 6.3 the error history for the unknowns $\boldsymbol{u}$ and ϕ only. The methods from [16] are referred to as "PEERS-Lagrange scheme with $k = 0$", "Augmented scheme with $k = 0$", and "Augmented scheme with $k = 1$", whereas those regarding (4.1) are named "Pseudostress-based scheme with $k = 0$" and "Pseudostress-based scheme with $k = 1$", additionally indicating for the latter the value of r (and hence of s) with which the corresponding norms are defined. Nevertheless, we observe from Figures 6.2 and 6.3 that, at least for this example, there is almost no difference between the curves obtained with $r = 3$ and $r = 4$ for both values of k. Finally, according to the aforementioned figures, and based on the comparison between schemes that use the same polynomial degree k, we infer that in general (4.1) requires less degrees of freedom than the methods from [16] to achieve a given accuracy. This fact is particularly notorious for the unknown $\boldsymbol{u}$ with $k \in \{0, 1\}$, and specially with $k = 1$, whereas for ϕ it is observed only with $k = 0$ since with $k = 1$ the respective curves are very close to each other and therefore no substantial difference is noticed.

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$
	0.0333	19982	5.48e+01	$--$	9.73e-04	$--$	7.82e-03	$--$	2.15e+01	$--$
	0.0270	30342	4.44e+01	1.00	7.86e-04	1.02	6.24e-03	1.08	1.75e+01	1.00
	0.0217	46830	3.57e+01	1.00	6.30e-04	1.01	4.94e-03	1.07	1.41e+01	1.00
0	0.0185	64478	3.04e+01	1.00	5.36e-04	1.01	4.20e-03	1.02	1.20e+01	1.00
	0.0164	82230	2.70e+01	1.00	4.74e-04	1.01	3.71e-03	1.00	1.06e+01	1.00
	0.0139	114482	2.28e+01	1.00	4.02e-04	1.00	3.12e-03	1.05	8.99e+00	1.00
	0.0122	148422	2.00e+01	1.00	3.53e-04	1.00	2.74e-03	1.01	7.89e+00	1.00
	0.0333	65162	7.08e-01	$--$	1.28e-05	$--$	6.77e-05	$--$	2.55e-01	$--$
	0.0270	99014	4.66e-01	2.00	8.43e-06	2.01	4.22e-05	2.25	1.68e-01	1.99
1	0.0217	152906	3.01e-01	2.00	5.45e-06	2.00	2.82e-05	1.85	1.09e-01	1.99
	0.0185	210602	2.19e-01	2.00	3.95e-06	2.00	2.02e-05	2.07	7.92e-02	1.99
	0.0164	268646	1.71e-01	2.00	3.10e-06	2.00	1.54e-05	2.23	6.21e-02	1.99
	0.0333	135542	6.21e-03	$--$	1.14e-07	$--$	9.87e-07	$--$	2.17e-03	$--$
	0.0270	206018	3.31e-03	3.00	6.09e-08	3.00	5.45e-07	2.83	1.16e-03	3.00
2	0.0217	318230	1.72e-03	3.00	3.17e-08	3.00	2.88e-07	2.94	6.02e-04	3.00
	0.0185	438374	1.07e-03	3.00	1.96e-08	3.00	1.77e-07	3.01	3.72e-04	3.00
	0.0164	559250	7.39e-04	3.00	1.36e-08	3.00	1.22e-07	3.09	2.58e-04	3.00

Table 6.3: History of convergence for Example 1 with $r = 4$.

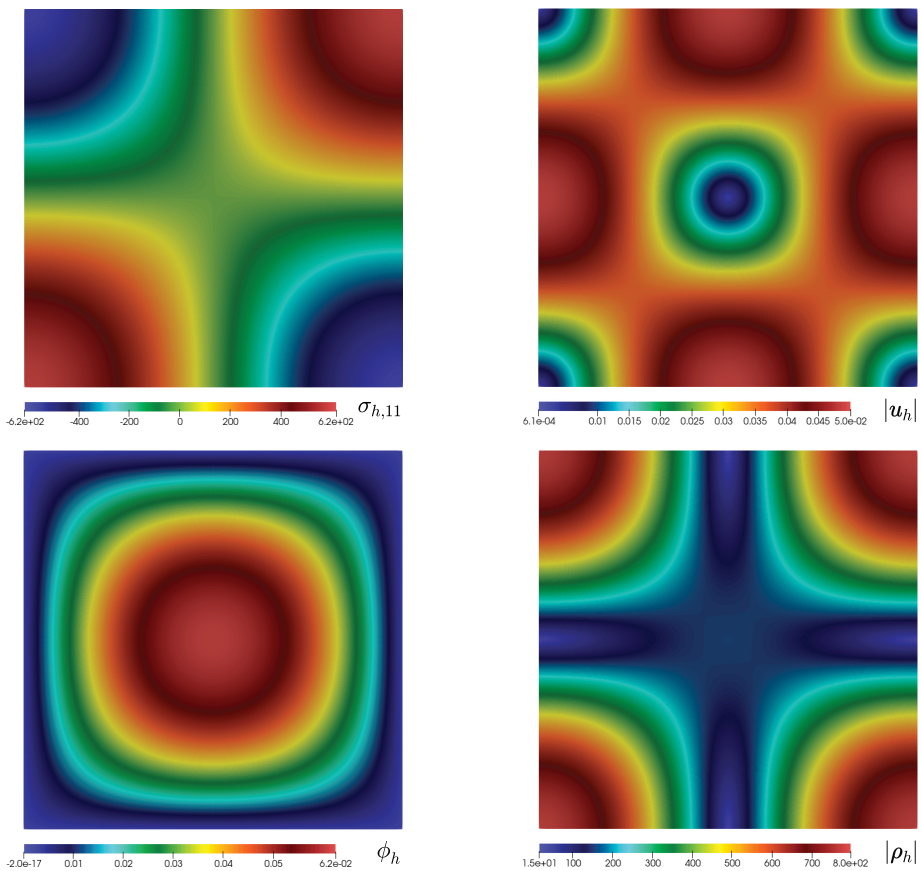

Figure 6.1: Some components and magnitudes of the solution of Example 1 with $k = 2$ and $N = 559250$

Example 2. We let Ω be the L-shaped (and hence non-convex) domain given by $(-1,1)^2 \setminus [0,1]^2$, and, again, suitably perturb the definition of the functionals F_ϕ and $G_{\boldsymbol{u}}$, so that, letting $\theta := \arctan\left(\frac{x_2}{x_1}\right)$, the exact solution of (2.1) reduces to:

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; \begin{pmatrix} |\boldsymbol{x}|^{2/3}\sin(\theta) \\ -|\boldsymbol{x}|^{2/3}\cos(\theta) \end{pmatrix} \qquad \text{and} \qquad \phi(\boldsymbol{x}) \;=\; e^{x_2}\left(x_1 - \frac{1}{2}\right)^3,$$

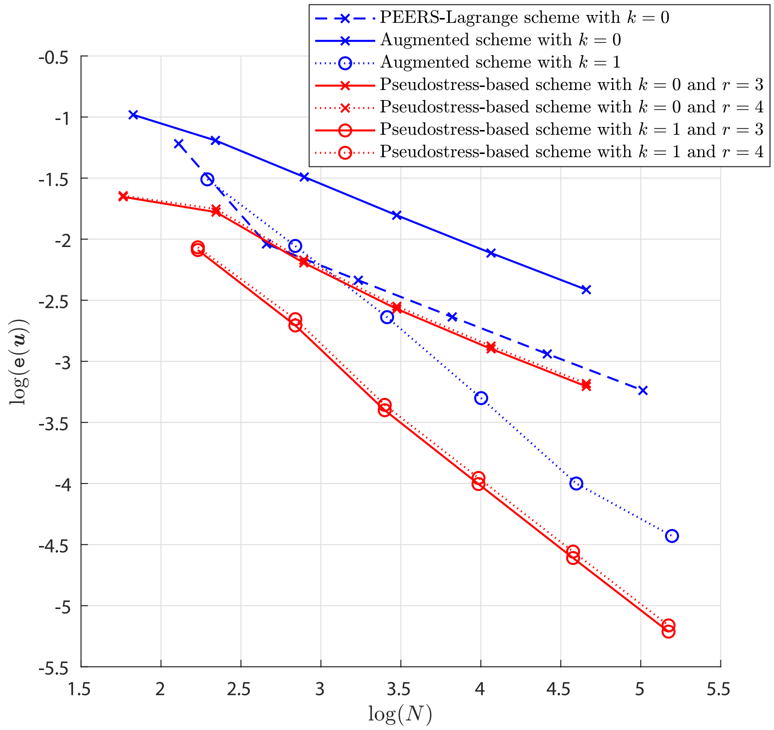

Figure 6.2: Example 1, $\log(\mathsf{e}(\boldsymbol{u}))$ vs. $\log(N)$ for the present scheme (4.1) and those from [16].

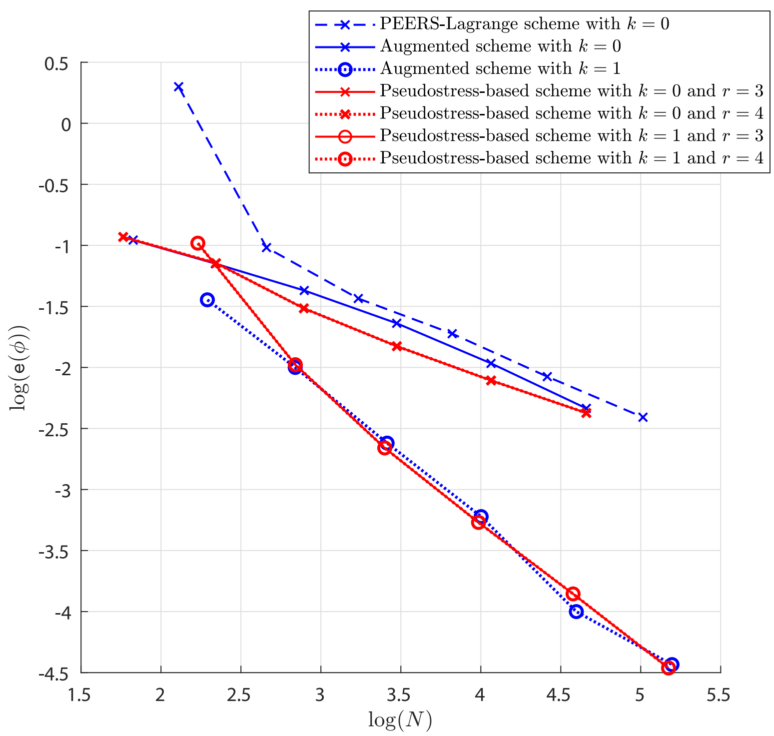

Figure 6.3: Example 1, $\log(\mathsf{e}(\phi))$ vs. $\log(N)$ for the present scheme (4.1) and those from [16].

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$
	0.0566	20927	4.51e+02	$--$	1.98e-02	$--$	2.72e-01	$--$	7.75e+00	$--$
	0.0471	30062	5.09e+02	-0.66	1.65e-02	1.00	2.26e-01	1.00	7.30e+00	0.33
	0.0372	48110	5.94e+02	-0.66	1.30e-02	1.00	1.79e-01	1.00	6.75e+00	0.33
0	0.0321	64418	6.54e+02	-0.66	1.12e-02	1.00	1.54e-01	1.00	6.43e+00	0.33
	0.0283	83102	7.12e+02	-0.66	9.90e-03	1.00	1.36e-01	1.00	6.16e+00	0.33
	0.0240	115583	7.94e+02	-0.66	8.39e-03	1.00	1.15e-01	1.00	5.83e+00	0.33
	0.0208	153410	8.73e+02	-0.66	7.28e-03	1.00	9.99e-02	1.00	5.56e+00	0.33
	0.0566	68102	4.49e+02	$--$	7.47e-04	$--$	2.77e-03	$--$	4.97e+00	$--$
	0.0471	97922	5.06e+02	-0.66	5.87e-04	1.32	1.93e-03	2.00	4.67e+00	0.33
1	0.0372	156866	5.92e+02	-0.66	4.30e-04	1.32	1.20e-03	2.00	4.32e+00	0.33
	0.0321	210146	6.52e+02	-0.66	3.55e-04	1.32	8.95e-04	2.00	4.11e+00	0.33
	0.0283	271202	7.10e+02	-0.66	3.00e-04	1.32	6.93e-04	2.00	3.94e+00	0.33
	0.0566	141527	4.60e+02	$--$	2.50e-04	$--$	1.68e-05	$--$	3.93e+00	$--$
	0.0471	203582	5.19e+02	-0.66	1.97e-04	1.32	1.08e-05	2.41	3.69e+00	0.33
2	0.0372	326270	6.07e+02	-0.66	1.44e-04	1.32	6.58e-06	2.10	3.41e+00	0.33
	0.0321	437186	6.69e+02	-0.66	1.19e-04	1.32	5.04e-06	1.82	3.25e+00	0.33
	0.0283	564302	7.28e+02	-0.66	1.00e-04	1.32	4.08e-06	1.65	3.12e+00	0.33

Table 6.4: History of convergence for Example 2 with $r = 3$.

for all $\boldsymbol{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$, whereas the body load, the diffusive source, and the tensorial diffusivity, are given, respectively, by

$$\boldsymbol{f}(\phi) \;=\; \begin{pmatrix} \frac{1}{40}\phi \\ \\ \frac{1}{40}\phi(1-\phi) \end{pmatrix}, \quad g(\boldsymbol{u}) \;=\; -|\boldsymbol{u}| \quad \text{and} \quad \vartheta(\boldsymbol{\sigma}) \;=\; \left(1 + \frac{1}{10}\big(1 + |\boldsymbol{\sigma}|^2\big)^{-1/2}\right)\mathbb{I}\,.$$

In addition, we take $E = 100$ and $\nu = 0.33$, whence the resulting Lamé parameters are given in this case (cf. (6.2)) by $\mu = 37.5940$ and $\lambda = 72.9766$. Due to the singularity of the vector field $\boldsymbol{u}$ at the origin, in this example we do not expect to attain the theoretical orders of convergence guaranteed by Theorem 5.5. In fact, in Tables 6.4 and 6.5 we display the corresponding convergence history with $r = 3$ and $r = 4$, respectively, from which we realize that sub-optimal, and even negative experimental rates of convergence are obtained. In turn, it is interesting to observe in this case that, differently from Example 1, these rates change not only with k but also with r, which must be certainly connected to the $\mathbf{W}^{l,r}(\Omega)$-regularity of the solution, most likely with a non-integer l depending on r. For instance, this was obtained for the regularity result of the Poisson problem in a non-convex domain, with homogeneous Neumann boundary conditions, and source term in $\mathrm{L}^r(\Omega)$ (see [20, Lemma B.1] for details). Anyhow, the usual way of recovering optimal rates of convergence in theses cases is by applying an adaptive strategy based on a posteriori error estimates. This is precisely the subject of an undergoing work to be communicated in a forthcoming contribution.

Example 3. Finally, and while not supported by the theory, we consider the three dimensional domain $\Omega = (0,1)^3$, and choose the data so that the exact solution is given by

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; e^{x_1+x_2+x_3}\begin{pmatrix} \sin(\pi x_1) \\ \sin(\pi x_2) \\ \sin(\pi x_3) \end{pmatrix} \quad \text{and} \quad \phi(\boldsymbol{x}) \;=\; -64x_1x_2x_3(x_1-1)(x_2-1)(x_3-1)\,,$$

for all $\boldsymbol{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \Omega$. In addition, the body load, the diffusive source, and the tensorial diffusivity are given, respectively, by

$$\boldsymbol{f}(\phi) \;=\; \begin{pmatrix} \phi \\ 1-\phi \\ \phi \end{pmatrix}, \quad g(\boldsymbol{u}) \;=\; x_1 + x_2 + x_3 \quad \text{and} \quad \vartheta(\boldsymbol{\sigma}) \;=\; \mathbb{I} + \frac{1}{10}\boldsymbol{\sigma}^2\,.$$

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$
	0.0566	20927	7.91e+02	−−	1.91e-02	−−	2.72e-01	−−	1.21e+01	−−
	0.0471	30062	9.19e+02	-0.82	1.60e-02	0.99	2.26e-01	1.00	1.17e+01	0.17
	0.0372	48110	1.12e+03	-0.83	1.26e-02	0.99	1.79e-01	1.00	1.13e+01	0.17
0	0.0321	64418	1.26e+03	-0.83	1.09e-02	0.99	1.54e-01	1.00	1.10e+01	0.17
	0.0283	83102	1.40e+03	-0.83	9.61e-03	1.00	1.36e-01	1.00	1.08e+01	0.17
	0.0240	115583	1.61e+03	-0.83	8.15e-03	1.00	1.15e-01	1.00	1.05e+01	0.17
	0.0208	153410	1.81e+03	-0.83	7.08e-03	1.00	9.99e-02	1.00	1.02e+01	0.17
	0.0566	68102	8.29e+02	−−	1.20e-03	−−	2.77e-03	−−	9.05e+00	−−
	0.0471	97922	9.64e+02	-0.83	9.74e-04	1.16	1.93e-03	2.00	8.78e+00	0.17
1	0.0372	156866	1.17e+03	-0.83	7.40e-04	1.16	1.20e-03	2.00	8.44e+00	0.17
	0.0321	210146	1.32e+03	-0.83	6.24e-04	1.16	8.95e-04	2.00	8.24e+00	0.17
	0.0283	271202	1.47e+03	-0.83	5.37e-04	1.16	6.93e-04	2.00	8.06e+00	0.17
	0.0566	141527	8.91e+02	−−	4.38e-04	−−	1.68e-05	−−	7.87e+00	−−
	0.0471	203582	1.04e+03	-0.83	3.54e-04	1.16	1.08e-05	2.41	7.63e+00	0.17
2	0.0372	326270	1.26e+03	-0.83	2.69e-04	1.16	6.58e-06	2.10	7.34e+00	0.17
	0.0321	437186	1.42e+03	-0.83	2.27e-04	1.17	5.04e-06	1.82	7.16e+00	0.17
	0.0283	564302	1.58e+03	-0.83	1.95e-04	1.17	4.08e-06	1.65	7.01e+00	0.17

Table 6.5: History of convergence for Example 2 with $r = 4$.

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$
	0.3542	9487	3.17e+03	−−	1.17e+00	−−	8.33e-01	−−	1.52e+03	−−
	0.3130	13844	2.81e+03	0.97	1.04e+00	0.91	7.37e-01	0.99	1.35e+03	0.93
	0.2804	18203	2.52e+03	0.99	9.26e-01	1.08	6.60e-01	0.99	1.21e+03	1.04
0	0.2657	22188	2.39e+03	1.00	8.76e-01	1.05	6.24e-01	1.06	1.14e+03	1.00
	0.2519	26479	2.26e+03	1.00	8.30e-01	1.01	5.90e-01	1.04	1.08e+03	1.09
	0.1832	82222	1.55e+03	1.18	5.86e-01	1.09	4.35e-01	0.96	7.78e+02	1.03
	0.1475	152258	1.25e+03	1.02	4.71e-01	1.00	3.55e-01	0.94	6.35e+02	0.94
	0.3542	40854	2.37e+02	−−	1.00e-01	−−	2.23e-02	−−	1.05e+02	−−
	0.3130	59920	1.85e+02	1.99	7.88e-02	1.94	1.75e-02	1.97	8.15e+01	2.07
1	0.2804	78912	1.49e+02	2.00	6.32e-02	2.00	1.40e-02	2.01	6.58e+01	1.95
	0.2657	96220	1.33e+02	2.00	5.68e-02	2.00	1.26e-02	2.00	5.90e+01	2.00
	0.2519	114932	1.20e+02	2.00	5.10e-02	2.02	1.13e-02	2.01	5.31e+01	2.01
	0.3542	106570	1.32e+01	−−	5.08e-03	−−	3.92e-03	−−	5.96e+00	−−
	0.3130	156755	9.09e+00	3.00	3.53e-03	2.95	2.71e-03	2.99	4.11e+00	3.01
2	0.2804	206621	6.53e+00	3.01	2.55e-03	2.97	1.95e-03	3.01	2.95e+00	3.01
	0.2657	251985	5.55e+00	3.00	2.17e-03	2.99	1.66e-03	3.00	2.51e+00	3.00
	0.2519	301137	4.74e+00	3.00	1.85e-03	3.00	1.41e-03	2.99	2.14e+00	3.00

Table 6.6: History of convergence for Example 3 with $r = 3$.

As for Example 2, we take again $E = 100$ and $\nu = 0.33$, which yields $\mu = 37.5940$ and $\lambda = 72.9766$. In addition, we employ the software TetGen (cf. [24]) to generate triangulations of Ω made of tetrahedrons. In this way, in Table 6.6 we present the convergence history of (4.1) with $k \in \{0, 1, 2\}$ and $r = 3$, from which we observe that the same orders from 2D (cf. Example 1) are attained in all these cases. This fact suggests, in coherence with the remark at the end of Section 5.2, that only some technical issues might be stopping us of extending the theoretical analysis to the 3D case. Finally, some components of the approximate solution are depicted in Figure 6.4.

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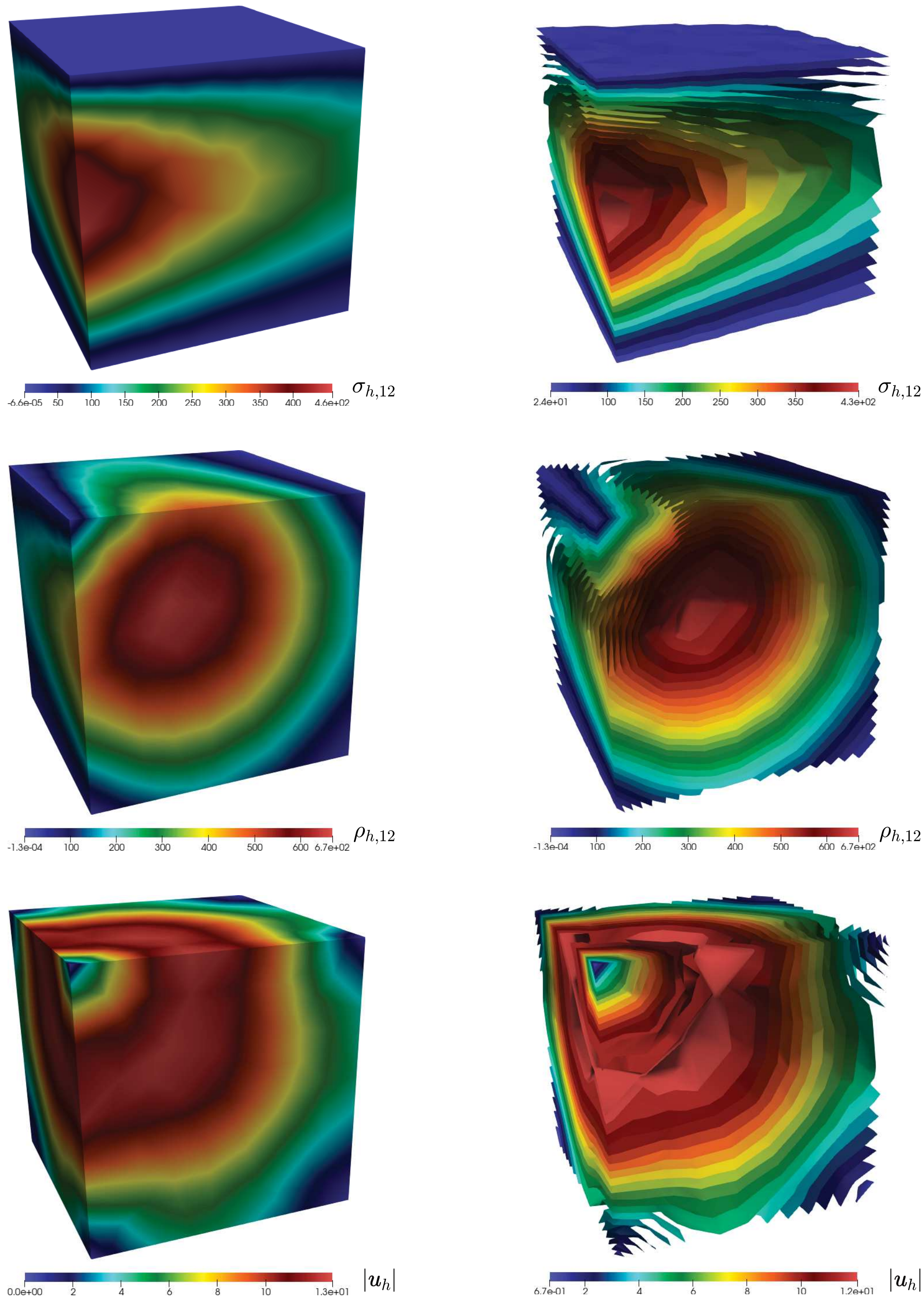

Figure 6.4: Some components and norms of the solution of Example 3 obtained with $k = 2$ and $N = 301137$ degrees of freedom. Surface (left) and contours (right).

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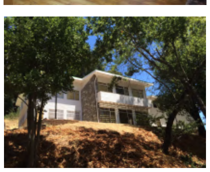

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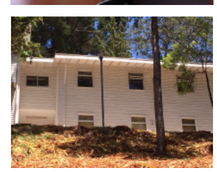
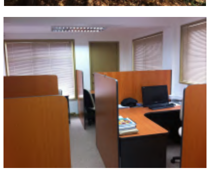

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