UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^2MA)

A fully-DG method for the stationary Boussinesq system

Eligio Colmenares, Ricardo Oyarzúa, Francisco Piña

PREPRINT 2022-03

SERIE DE PRE-PUBLICACIONES

A fully-DG method for the stationary Boussinesq system *

Eligio Colmenares[†] Ricardo Oyarzúa[‡] Francisco Piña[§]

Dedicated to the memory of Francisco–Javier Sayas

Abstract

In this work we present and analyze a finite element scheme yielding discontinuous Galerkin approximations to the solutions of the stationary Boussinesq system for the simulation of non-isothermal flow phenomena. The model consists of a Navier-Stokes type system, describing the velocity and the pressure of the fluid, coupled to an advection-diffusion equation for the temperature. The proposed numerical scheme is based on the standard interior penalty technique and an upwind approach for the nonlinear convective terms and employs the divergence-conforming Brezzi-Douglas-Marini (BDM) elements of order k for the velocity, discontinuous elements of order $k-1$ for the pressure and discontinuous elements of order k for the temperature. Existence and uniqueness results are shown and stated rigorously for both the continuous problem and the discrete scheme, and optimal a priori error estimates are also derived. Numerical examples back up the theoretical expected convergence rates as well as the performance of the proposed technique.

Key words: Boussinesq equations, finite element methods, discontinuous Galerkin method, divergence-conforming elements, fixed-point theory, a priori error analysis.

Mathematics subject classifications (2020): 35Q30,35Q35, 35Q79, 65N12, 65N15, 65N22, 65N30, 76D05, 76R10 80M10.

1 Introduction

Non-isothermal flows refer to a basic physical process of fluid flows with varying temperature. This phenomenon commonly appears, and its research is of crucial relevance, in several situations in engineering and applied sciences, in particular in desalination processes based on sweeping gas membrane distillation (see [38]).

The Boussinesq hypothesis allows to study such flows under the assumption that variations in fluid density have no effect on the flow field, except that they give rise to buoyancy forces [7, 40].

The mathematical model based on the Boussinesq approximation for non-isothermal flows is then a system that consists of the Navier-Stokes equations (for describing the velocity and the pressure of the

*This research was partially supported by ANIDChile through Fondecyt 11190241 and by Centro de Modelamiento Matemtico (CMM), ACE210010 and FB210005, BASAL funds for center of excellence from ANID-Chile.

[†]Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, Chillán, Chile, email: `ecolmenares@ubiobio.cl`.

[‡]GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, o, Casilla 5-C, Concepción, Chile, and CI2MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: `royarzua@ubiobio.cl`.

[§]GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, email: `fpina@ubiobio.cl`.

fluid) and an advection-diffusion equation (for modeling the temperature field) coupled by means of a buoyancy term and convective heat transfer. Because of its relevance and complexity, a wide variety of numerical techniques to approximate this and related models have been proposed so far (see, e.g., [2, 3, 4, 6, 12, 13, 15, 16, 17, 18, 21, 22, 23, 35, 36, 37, 41] and references therein).

To the best of our knowledge, [6] is the first method developed for this problem based on a finite element primal formulation. There the topological degree theory is applied to state existence results of solutions and, at discrete level, it is showed that the use of equal-order finite element subspaces for the velocity and the temperature leads to optimal–order convergence. The same results are later extended in [18] using a dual-mixed finite element technique, employing the Leray-Shauder theory and, at discrete level, inf-sup compatible finite element spaces constructed over triangulations with a macroelement structure [29, 30]. Other related primal techniques including projection-based strategies, adaptivity and divergence-free velocity approximations are proposed in [2, 12, 23, 35, 36, 37, 41] whereas the works [3, 4, 15, 16, 17, 21, 22] deal with the design of mixed finite element methods. These references consider the model with constant and/or temperature-dependent parameters and different boundary conditions.

In particular, [35, 36, 37] are proposed divergence-conforming schemes, based on a discontinuous Galerkin method for the Navier-Stokes equations, that yield exactly divergence-free velocity approximations; an essential constraint of the governing equations since it particularly guarantees that the solutions to the flow equations are locally conservative as well as energy stable (see [10, 11, 32], for more details in this regard). Regarding the discretization of the corresponding heat equation, three different approaches have been employed. In [36] it is considered a conforming scheme for the standard primal formulation, where the derivation of the corresponding a priori estimates relies on the introduction of a suitable lifting of the Dirichlet datum which may lead to an unstable discretization of (2.5) (see [36, Section 4.2]). In addition, a mixed-primal approach (introducing a Lagrange multiplier) and a mixed (introducing an additional vector variable) formulation have been proposed in [35, 37] to overcome this drawback, at the price of assuming that the non-homogeneous Dirichlet boundary condition for the temperature is sufficiently small.

According to the above discussion, in this work we propose a fully-DG finite element method for the Boussinesq problem with the following two main features:

1. As in [36, 37], it provides an exactly divergence-free approximation of the velocity.
2. The a priori estimates are derived without assuming any sufficiently small data assumption and without employing any suitable lifting of the Dirichlet datum, thus avoiding the possibility of obtaining an unstable discretization.

More precisely, we introduce a fully discontinuous Galerkin scheme based on the standard interior penalty technique and an upwind approach (that is, for both the fluid and the temperature equations), such as in [36]. The finite element subspaces are given by the divergence-conforming Brezzi-Douglas-Marini (BDM) elements of order k for the velocity, and discontinuous elements of order $k-1$ and order k for the pressure and the temperature, respectively.

The rest of the work is structured as follows. In Section 2 we introduce the model problem and the analysis of its weak formulation. In particular, for the sake of a better understanding of our approach, in Section 2.4 we carry out the solvability analysis of the continuous problem. In particular we follow the analysis in [18] to show that the a priori estimates explicitly depend on the inverse of the viscosity and the thermal conductivity and on a negative power of a positive parameter which is chosen close to zero (see Theorem 2.2).

Throughout Section 3 we introduce and analyze the corresponding discrete problem. Next, in

Section 4 we derive the a priori error analysis, and finally in Section 5 provide some numerical examples to illustrate the performance of the numerical scheme and to confirm the theoretical convergence rates.

2 The model problem

2.1 Preliminary notations and definitions

The model to be considered will be set in an open and bounded spatial domain $\Omega \subseteq \mathbb{R}^d$, with $d = 2$ or $d = 3$, with polyhedral boundary Γ with outward unit normal $\boldsymbol{n}$. Also, we suppose that $\Gamma_{\rm D}\,, \Gamma_{\rm N} \subseteq \Gamma$ satisfy $\Gamma_{\rm D} \cap \Gamma_{\rm N} = \emptyset$, $|\Gamma_{\rm D}| > 0$ and $\Gamma = \overline{\Gamma}_{\rm D} \cup \overline{\Gamma}_{\rm N}$. Standard notations for Lebesgue and Sobolev spaces will be employed. In particular, $W^{s,p}(\Omega)\,(s \geq 0)$ stands for the space of all the $L^p(\Omega)\,(p \geq 1)$ functions whose distributional derivatives up to order s are in $L^p(\Omega)$, and their respective norm and seminorm are denoted by $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$. When $p = 2$, we denote by $H^s(\Omega) := W^{s,2}(\Omega)$, $\|\cdot\|_{s,\Omega} := \|\cdot\|_{s,2,\Omega}$ and $|\cdot|_{s,\Omega} := |\cdot|_{s,p,\Omega}$, respectively. The case $s = 1/2$ on the domain $\Gamma_{\rm D}$ corresponds to the space of traces, denoted by $H^{1/2}(\Gamma_{\rm D})$, on $\Gamma_{\rm D}$ with norm defined as

$$\|\phi\|_{1/2,\Gamma_{\rm D}} = \inf\left\{\|\psi\|_{1,\Omega} : \quad \psi \in H^1(\Omega),\ \psi|_{\Gamma_{\rm D}} = \phi\right\}.$$

The space of functions with trace zero on subdomain $\Gamma_\star \subseteq \Gamma$, with $|\Gamma_\star| > 0$, will be denoted by $H^1_{\Gamma_\star}(\Omega)$ (or $H^1_0(\Omega)$ when $\Gamma_\star = \Gamma$), for which, thanks to the generalized Poincaré inequality, there exists $C_{\rm FP} > 0$ (depending only on Ω and $\Gamma_\star$), such that

$$\|\psi\|_{1,\Omega} \,\leq\, C_{\rm FP}\,|\psi|_{1,\Omega} \qquad \forall\,\psi \in H^1_{\Gamma_\star}(\Omega)\,. \tag{2.1}$$

Also, we will use and denote by $L^2_0(\Omega)$ to the space of L^2-functions with zero mean value over Ω. Likewise, we will make reference to the vector-valued Hilbert spaces

$$\begin{aligned}
\boldsymbol{H}(\mathrm{div},\Omega) &:= \left\{\boldsymbol{w} \in [L^2(\Omega)]^d \,: \qquad \mathrm{div}\,\boldsymbol{w} \in L^2(\Omega)\,\right\},\\
\boldsymbol{H}_0(\mathrm{div},\Omega) &:= \left\{\boldsymbol{w} \in \boldsymbol{H}(\mathrm{div},\Omega) \,: \qquad \boldsymbol{w}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma\,\right\},\\
\boldsymbol{H}_0(\mathrm{div}^0,\Omega) &:= \left\{\boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div},\Omega) \,: \qquad \mathrm{div}\,\boldsymbol{w} = 0 \quad \text{in} \quad \Omega\,\right\}.
\end{aligned}$$

We further recall that the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ holds for $1 \leq q < \infty$ when $d = 2$ and $1 \leq q \leq 6$ when $d = 3$. In particular, there exists a constant, let us say $C_{Sob}(q,d) > 0$, depending only on the domain, such that

$$\|\psi\|_{0,q,\Omega} \,\leq\, C_{Sob}(q,d)\|\psi\|_{1,\Omega} \quad \text{for} \quad \begin{cases} q \geq 1 & \text{if } d = 2\,,\\ q \in [1,6] & \text{if } d = 3. \end{cases} \tag{2.2}$$

The norm $\|\cdot\|$, with no subscripts, will be use for denoting the natural norm of an element or an operator in any product function space. A generic, positive constant is denoted by C which, unless labeled, is independent of any mesh parameters and data parameters.

2.2 The stationary Boussinesq problem: strong and weak forms

The equations for describing steady thermally driven flows in an enclosure Ω, using the Boussinesq approximation, reads: Find the velocity $\boldsymbol{u} = (u_i)_{1\leq i\leq d} : \Omega \longrightarrow \mathrm{R}^d$, the pressure $p : \Omega \longrightarrow \mathrm{R}$ and the temperature $\theta : \Omega \longrightarrow \mathrm{R}$ satisfying the following system of partial differential equations

$$\left.\begin{aligned}
-\,\nu\,\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla p - \theta\,\boldsymbol{g} \,=\, 0\,, \qquad \mathrm{div}\,\boldsymbol{u} \,=\, 0,\\
-\,\kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta \,=\, 0 \qquad\qquad\qquad
\end{aligned}\right\} \quad \text{in} \quad \Omega\,, \tag{2.3}$$

where $\nu > 0$ stands for the kinematic viscosity of the fluid, $\boldsymbol{g} \in [L^2(\Omega)]^d$ is the acceleration due to gravity and $\kappa > 0$ is the thermal conductivity of the fluid.

Equations in the first row of (2.3) express the momentum and mass conservation, respectively, and the latter one particularly enforces the divergence-free constraint on the velocity. The term $\theta\,\boldsymbol{g}$ appearing there is known as a buoyancy force and drives the fluid flow. In turn, the diffusion–convection equation appearing in the second row of (2.3) expresses the energy conservation and describes the heat distribution in the fluid.

To complete the system (2.3) we need to specify appropriate boundary conditions. To do so, we assume that the fluid has zero velocity relative to the boundary Γ (a no-slip condition/homogeneous Dirichlet condition for the fluid velocity), that its temperature is given and prescribed on the boundary Γ_{D} (a non-homogeneous Dirichlet condition for the temperature) and that there is no heat flow across Γ_{N} (an isolated surface/homogeneous Neumann condition for the temperature). Hence, denoting the prescribed Dirichlet temperature by $\theta_{\mathrm{D}} \in H^{1/2}(\Gamma_{\mathrm{D}})$, we then arrive at the following physical boundary conditions

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \quad \theta = \theta_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}} \quad \text{and} \quad \kappa \frac{\partial \theta}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}\,. \tag{2.4}$$

To attain the standard weak formulation of (2.3)-(2.4), we multiply the equations in (2.3) by appropriate test functions $\boldsymbol{v} \in [H_0^1(\Omega)]^d$, $q \in L_0^2(\Omega)$ and $\psi \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, respectively, integrate in the domain, then use integration-by-parts formulae in the diffusion terms and after incorporating the boundary conditions (2.4), we obtain the following variational formulation: Find $(\boldsymbol{u}, p, \theta) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega) \times H^1(\Omega)$ with $\theta|_{\Gamma_{\mathrm{D}}} = \theta_{\mathrm{D}}$ such that

$$\begin{aligned} \mathscr{A}^S(\boldsymbol{u}, \boldsymbol{v}) + \mathscr{C}^S(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) - \mathscr{B}^S(\boldsymbol{v}, p) &= \mathscr{D}^S(\theta, \boldsymbol{v}), \\ \mathscr{B}^S(\boldsymbol{u}, q) &= 0\,, \\ \mathscr{A}^T(\theta, \psi) + \mathscr{C}^T(\boldsymbol{u}; \theta, \psi) &= 0, \end{aligned} \tag{2.5}$$

for all $(\boldsymbol{v}, q, \psi) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega) \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, where the bilinear forms $\mathscr{A}^S : [H_0^1(\Omega)]^d \times [H_0^1(\Omega)]^d \to \mathbb{R}$ and $\mathscr{A}^T : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ are defined by

$$\mathscr{A}^S(\boldsymbol{u}, \boldsymbol{v}) = \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \quad \text{and} \quad \mathscr{A}^T(\theta, \psi) = \kappa \int_\Omega \nabla \theta \cdot \nabla \psi\,, \tag{2.6}$$

$\mathscr{B}^S : [H_0^1(\Omega)]^d \times L_0^2(\Omega) \to \mathbb{R}$ is the bilinear form associated to the divergence operator, namely

$$\mathscr{B}^S(\boldsymbol{v}, q) = \int_\Omega q \,\mathrm{div}\, \boldsymbol{v} \tag{2.7}$$

whereas $\mathscr{C}^S : [H_0^1(\Omega)]^d \times [H_0^1(\Omega)]^d \times [H_0^1(\Omega)]^d \to \mathbb{R}$ and $\mathscr{C}^T : [H_0^1(\Omega)]^d \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ are the trilinear forms linked to the convective terms, given by

$$\mathscr{C}^S(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\nabla \boldsymbol{u}) \boldsymbol{w} \cdot \boldsymbol{v}\,, \quad \text{and} \quad \mathscr{C}^T(\boldsymbol{w}; \theta, \psi) = \int_\Omega (\boldsymbol{w} \cdot \nabla \theta)\, \psi\,. \tag{2.8}$$

Finally, the form $\mathscr{D} : H^1(\Omega) \times [H^1(\Omega)]^d \longrightarrow \mathbb{R}$ is associated to the buoyancy term and it is defined by

$$\mathscr{D}^S(\theta, \boldsymbol{v}) = \int_\Omega \theta\, \boldsymbol{g} \cdot \boldsymbol{v}. \tag{2.9}$$

2.3 Stability properties

We state now some properties of the forms defining the variational problem (2.5). Firstly, by their own definitions and using the Cauchy-Schwarz inequality, it is easy to check that the forms $\mathscr{A}^S$ and $\mathscr{A}^T$ defined in (2.6) are bounded on their respective spaces, satisfying

$$\left|\mathscr{A}^S(\boldsymbol{u},\boldsymbol{v})\right| \,\le\, \nu\,\|\boldsymbol{u}\|_{1,\Omega}\,\|\boldsymbol{v}\|_{1,\Omega} \qquad \forall\,\boldsymbol{u},\boldsymbol{v}\in[H^1(\Omega)]^d \tag{2.10}$$

$$\left|\mathscr{A}^T(\theta,\psi)\right| \,\le\, \kappa\,\|\theta\|_{1,\Omega}\,\|\psi\|_{1,\Omega} \qquad \forall\,\theta,\psi\in H^1(\Omega)\,. \tag{2.11}$$

In addition, thanks to the Poincaré inequality with constant $C_{\rm FP}>0$ (see (2.1)) it follows that

$$\mathscr{A}^S(\boldsymbol{v},\boldsymbol{v}) \,=\, \nu\,|\boldsymbol{v}|^2_{1,\Omega} \,\ge\, \nu\,C_{\rm FP}^{-1}\|\boldsymbol{u}\|^2_{1,\Omega} \qquad \forall\,\boldsymbol{v}\in[H^1_0(\Omega)]^d \tag{2.12}$$

$$\mathscr{A}^T(\psi,\psi) \,\ge\, \kappa\,|\psi|^2_{1,\Omega} \,\ge\, \kappa\,C_{\rm FP}^{-1}\|\psi\|^2_{1,\Omega} \qquad \forall\,\psi\in H^1_{\Gamma_{\rm D}}(\Omega)\,, \tag{2.13}$$

and so the bilinear forms $\mathscr{A}^S$ and $\mathscr{A}^T$ are elliptic on $[H^1_0(\Omega)]^d$ and $H^1_{\Gamma_{\rm D}}(\Omega)$, respectively. In addition, by the Cauchy-Schwarz inequality and the fact that $\|{\rm div}\,\boldsymbol{v}\|_{0,\Omega}\,\le\,d^{1/2}\,\|\nabla\boldsymbol{v}\|_{0,\Omega}$, we have that the bilinear form $\mathscr{B}^S$ defined in (2.7) is clearly continuous, that is,

$$\left|\mathscr{B}^S(q,\boldsymbol{v})\right| \,\le\, d^{1/2}\,\|q\|_{0,\Omega}\,\|\boldsymbol{v}\|_{1,\Omega} \qquad \forall\,q\in L^2_0(\Omega),\,\forall\,\boldsymbol{v}\in[H^1(\Omega)]^d\,,$$

and satisfies the inf-sup condition

$$\sup_{\substack{\boldsymbol{v}\in[H^1_0(\Omega)]^d\\ \boldsymbol{v}\neq\boldsymbol{0}}} \frac{\mathscr{B}^S(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{1,\Omega}} \,\ge\, \beta\,\|q\|_{0,\Omega} \qquad \forall\,q\in L^2_0(\Omega)\,, \tag{2.14}$$

for some positive constant β, only depending on Ω, which comes from the the well-known fact that the divergence operator is an isomorphism from $\boldsymbol{X}^\perp$ onto $L^2_0(\Omega)$ (and thus surjective), where $\boldsymbol{X}$ stands for the kernel of $\mathscr{B}^S$ (see [26, Corollary I.2.4]), that is,

$$\boldsymbol{X} \,=\, \Big\{\boldsymbol{v}\in[H^1_0(\Omega)]^d\,:\quad \mathscr{B}^S(\boldsymbol{v},q)\,=\,0\,,\ \forall q\in L^2_0(\Omega)\Big\} \,=\, \Big\{\boldsymbol{v}\in[H^1_0(\Omega)]^d\,:\quad {\rm div}\,\boldsymbol{v}\,=\,0\Big\}. \tag{2.15}$$

In turn, regarding the trilinear forms $\mathscr{C}^S$ and $\mathscr{C}^T$ (cf. (2.8)), we use the Hölder's inequality to find that

$$\left|\mathscr{C}^S(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})\right| \le \|\boldsymbol{w}\|_{0,q,\Omega}\|\nabla\boldsymbol{u}\|_{0,\Omega}\|\boldsymbol{v}\|_{0,q',\Omega}, \tag{2.16}$$

$$\left|\mathscr{C}^T(\boldsymbol{w};\theta,\psi)\right| \,\le\, \|\boldsymbol{w}\|_{0,q,\Omega}\,\|\nabla\theta\|_{0,\Omega}\,\|\psi\|_{0,q',\Omega}\,, \tag{2.17}$$

for all $\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}\in[H^1(\Omega)]^d$ and $\theta,\psi\in H^1(\Omega)$, where q and q' satisfy $\frac{1}{q}+\frac{1}{q'}=\frac{1}{2}$. In particular, taking q and q' consistently with the dimension d in agreement with (2.2), from these estimates we obtain

$$\left|\mathscr{C}^S(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})\right| \le C_S\,\|\boldsymbol{w}\|_{1,\Omega}\,\|\boldsymbol{u}\|_{1,\Omega}\,\|\boldsymbol{v}\|_{1,\Omega}, \tag{2.18}$$

$$\left|\mathscr{C}^T(\boldsymbol{w};\theta,\psi)\right| \le C_T\,\|\boldsymbol{w}\|_{1,\Omega}\,\|\theta\|_{1,\Omega}\,\|\psi\|_{1,\Omega}\,. \tag{2.19}$$

for all $\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}\in[H^1(\Omega)]^d$ and $\theta,\psi\in H^1(\Omega)$. Moreover, by using an integration-by-parts formula, it follows that both $\mathscr{C}^S$ and $\mathscr{C}^T$ are skew-symmetric with respect to the last two components whenever the first argument is divergence-free and has normal component zero on the boundary, that is,

$$\mathscr{C}^S(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) \,=\, -\mathscr{C}^S(\boldsymbol{w};\boldsymbol{v},\boldsymbol{u}) \quad\text{and}\quad \mathscr{C}^T(\boldsymbol{w};\theta,\psi) \,=\, -\mathscr{C}^T(\boldsymbol{w};\psi,\theta)\,, \tag{2.20}$$

for all $\boldsymbol{w} \in \boldsymbol{X}$, $\boldsymbol{u}, \boldsymbol{v} \in [H^1(\Omega)]^d$ and $\theta, \psi \in H^1(\Omega)$. In particular, taking $\boldsymbol{v} = \boldsymbol{u}$ and $\psi = \theta$ in (2.20)

$$\mathscr{C}^S(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v}) = 0 \quad \text{and} \quad \mathscr{C}^T(\boldsymbol{w}; \psi, \psi) = 0\,, \tag{2.21}$$

for all $\boldsymbol{w} \in \boldsymbol{X}$, $\boldsymbol{v} \in [H^1(\Omega)]^d$ and $\psi \in H^1(\Omega)$.

As for the form $\mathscr{D}^S$ defined in (2.9), by using again the Hölder's inequality, we easily find that

$$\left|\mathscr{D}^S(\theta, \boldsymbol{v})\right| \leq \|\theta\|_{0,q,\Omega} \|\boldsymbol{g}\|_{0,\Omega} \|\boldsymbol{v}\|_{0,q',\Omega}\,, \tag{2.22}$$

for all $\boldsymbol{v} \in [H^1(\Omega)]^d$ and $\theta \in H^1(\Omega)$, with q and q' satisfying $\frac{1}{q} + \frac{1}{q'} = \frac{1}{2}$. Then, applying the Sobolev inequality (2.2) with q and q' taken consistently with the dimension d, there holds

$$\left|\mathscr{D}^S(\theta, \boldsymbol{v})\right| \leq \|\theta\|_{0,q,\Omega} \|\boldsymbol{g}\|_{0,\Omega} \|\boldsymbol{v}\|_{0,q',\Omega} \leq C_{\mathscr{D}} \, \|\boldsymbol{g}\|_{0,\Omega} \, \|\theta\|_{1,\Omega} \, \|\boldsymbol{v}\|_{1,\Omega}\,, \tag{2.23}$$

for all $\boldsymbol{v} \in [H^1(\Omega)]^d$ and $\theta \in H^1(\Omega)$.

2.4 Well-posedness of the weak formulation

In this section we revisit the analysis of existence and uniqueness of solution of problem (2.5), originally studied in [33] and [34] for sufficiently smooth boundaries (see also [6]).

We begin by noticing that, as in the standard velocity-pressure formulation for the Navier-Stokes equations, the divergence-free constraint given by the second equation of (2.5) implies that an eventual solution $\boldsymbol{u}$ must belong to the kernel of the bilinear form $\mathscr{B}^S$ (defined in (2.15)) and so the fluid problem can be equivalently reduced to $\boldsymbol{X}$, without pressure, thanks to the inf-sup condition (2.14). More precisely, problem 2.5 is equivalent to the reduced one: Find $(\boldsymbol{u}, \theta) \in \boldsymbol{X} \times H^1(\Omega)$, such that $\theta|_{\Gamma_{\rm D}} = \theta_{\rm D}$ and:

$$\begin{aligned} \mathscr{A}^S(\boldsymbol{u}, \boldsymbol{v}) + \mathscr{C}^S(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) &= \mathscr{D}^S(\theta, \boldsymbol{v}), \\ \mathscr{A}^T(\theta, \psi) + \mathscr{C}^T(\boldsymbol{u}; \theta, \psi) &= 0, \end{aligned} \tag{2.24}$$

for all $(\boldsymbol{v}, \psi) \in \boldsymbol{X} \times H^1_{\Gamma_{\rm D}}(\Omega)$.

According to the above, in what follows we focus on establishing the well-posedness of (2.24). We begin by deriving the corresponding a priori estimates which later on will help us to prove existence of solution by means of the Leray-Schauder Principle/Shafer's Theorem (see Section 2.4.2).

2.4.1 A priori estimates

In what follows we derive a priori bounds for weak solutions to the reduced problem (2.24). To that end, in particular for handling the non-homogeneous Dirichlet condition on the temperature, we introduce the following result.

Lemma 2.1 *Let Ω be a bounded domain in R^d, $d = 2$ or $d = 3$, with Lipschitz continuous boundary. Then for any $\delta > 0$, there exists an extension operator $E_\delta : H^{1/2}(\Gamma_{\rm D}) \to H^1(\Omega)$, such that*

$$\|E_\delta \psi\|_{0,3,\Omega} \leq C_{E,1} \delta \|\psi\|_{1/2,\Gamma_{\rm D}} \quad \text{and} \quad \|E_\delta \psi\|_{1,\Omega} \leq C_{E,2} \left(\delta^{-4} + 1\right) \|\psi\|_{1/2,\Gamma_{\rm D}}\,, \tag{2.25}$$

for all $\psi \in H^{1/2}(\Gamma_{\rm D})$, with $C_{E,1}, C_{E,2} > 0$ independent of δ.

Proof. The result follows from a slight modification of the proof of [18, Lemma 3.2]. □

Using this extension operator, for the forthcoming analysis we fix a suitable $\delta > 0$ (to be specified below in Theorem 2.2), and write the temperature θ as

$$\theta = \theta_0 + \theta_\delta, \quad \text{with} \quad \theta_\delta = E_\delta(\theta_{\mathrm{D}}) \quad \text{and} \quad \theta_0 \in H^1_{\Gamma_{\mathrm{D}}}(\Omega). \tag{2.26}$$

The main result of this section is established now.

Theorem 2.2 *Let $(\boldsymbol{u}, \theta)$ be a solution to (2.24). Then, for a fixed $\delta > 0$, satisfying*

$$\widehat{C}\, \nu^{-1}\, \kappa^{-1}\, \delta\, \|\boldsymbol{g}\|_{0,\Omega}\, \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \,\leq\, \frac{1}{2}\,, \tag{2.27}$$

with $\widehat{C} > 0$ independent of the physical parameters (see (2.34)), the following a priori estimates hold

$$\|\boldsymbol{u}\|_{1,\Omega} \,\leq\, C_1(\delta, \nu, \boldsymbol{g}, \theta_{\mathrm{D}}) \quad \textit{and} \quad \|\theta\|_{1,\Omega} \,\leq C_2(\delta, \theta_{\mathrm{D}})\,, \tag{2.28}$$

with

$$C_1(\delta, \nu, \boldsymbol{g}, \theta_{\mathrm{D}}) := C_{AP,1}\, (\delta^{-4} + 1)\, \nu^{-1}\, \|\boldsymbol{g}\|_{0,\Omega}\, \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \quad \textit{and} \quad C_2(\delta, \theta_{\mathrm{D}}) := C_{AP,2}\, (\delta^{-4} + 1)\, \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}, \tag{2.29}$$

where $C_{AP,1}, C_{AP,2}$ are positive constants independent of the physical parameters.

Proof. Let us suppose that $(\boldsymbol{u}, \theta)$ is a solution to (2.24) and set $\theta_\delta = E_\delta(\theta_{\mathrm{D}})$ and $\theta_0 = \theta - \theta_\delta$. Hence, taking $\boldsymbol{v} = \boldsymbol{u}$ and $\psi = \theta_0$ in (2.24), and using the properties (2.20) and (2.21) of the trilinear forms $\mathscr{C}^S$ and $\mathscr{C}^T$, we find

$$\begin{aligned} \mathscr{A}^S(\boldsymbol{u}, \boldsymbol{u}) \quad &= \quad \mathscr{D}^S(\theta_0 + \theta_\delta, \boldsymbol{u}), \\ \mathscr{A}^T(\theta_0, \theta_0) \quad &= \quad -\mathscr{A}^T(\theta_\delta, \theta_0) \,-\, \mathscr{C}^T(\boldsymbol{u}; \theta_\delta, \theta_0) = -\mathscr{A}^T(\theta_\delta, \theta_0) \,+\, \mathscr{C}^T(\boldsymbol{u}; \theta_0, \theta_\delta). \end{aligned} \tag{2.30}$$

Using the ellipticity of $\mathscr{A}^S$ and the continuity of the bilinear form $\mathscr{D}^S$ (see (2.12) and (2.23)) in the first equation of (2.30), we have that

$$\nu\, C_{\mathrm{FP}}^{-1}\, \|\boldsymbol{u}\|^2_{1,\Omega} \,\leq C_{\mathscr{D}}\, \|\boldsymbol{g}\|_{0,\Omega}\, \|\theta_0 + \theta_\delta\|_{1,\Omega}\, \|\boldsymbol{u}\|_{1,\Omega}\,,$$

and then, after simplifying and using the triangle inequality, we obtain

$$\|\boldsymbol{u}\|_{1,\Omega} \,\leq C_{\mathrm{FP}} C_{\mathscr{D}}\, \nu^{-1}\, \|\boldsymbol{g}\|_{0,\Omega} \Big(\|\theta_0\|_{1,\Omega} \,+\, \|\theta_\delta\|_{1,\Omega} \Big)\,. \tag{2.31}$$

On the other hand, using the continuity and the ellipticity of $\mathscr{A}^T$ (see (2.11) and (2.13)), the second identity in (2.21), the estimate (2.19) (with $q = 6$ and $q' = 3$) of the trilinear form $\mathscr{C}^T$, and subsequently the Sobolev embeddings $[H^1(\Omega)]^d \hookrightarrow [L^6(\Omega)]^d$ and $H^1(\Omega) \hookrightarrow L^3(\Omega)$ (see (2.2)) in the second equation of (2.30), and finally the inequalities in (2.25), it follows that

$$\begin{aligned} \kappa\, C_{\mathrm{FP}}^{-1}\, \|\theta_0\|^2_{1,\Omega} \,&\leq\, \kappa \|\theta_0\|_{1,\Omega}\, \|\theta_\delta\|_{1,\Omega} \,+\, \|\boldsymbol{u}\|_{0,6,\Omega} \|\nabla\theta_0\|_{0,\Omega} \|\theta_\delta\|_{0,3,\Omega} \\ &\leq\, \kappa \|\theta_0\|_{1,\Omega}\, \|\theta_\delta\|_{1,\Omega} \,+\, C_{Sob}(3,d) C_{E,1} \delta\, \|\boldsymbol{u}\|_{1,\Omega} \|\theta_0\|_{1,\Omega} \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,, \end{aligned}$$

and, after simplifying, we obtain

$$\|\theta_0\|_{1,\Omega} \,\leq\, C_{\mathrm{FP}} \|\theta_\delta\|_{1,\Omega} \,+\, \kappa^{-1} C_{\mathrm{FP}} C_{Sob}(3,d) C_{E,1} \delta \|\boldsymbol{u}\|_{1,\Omega} \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,. \tag{2.32}$$

Using (2.32) in (2.31) we get

$$\|\boldsymbol{u}\|_{1,\Omega} \,\leq\, C_{\mathscr{D}}\,\nu^{-1}\,\|\boldsymbol{g}\|_{0,\Omega}\,C_{\mathrm{FP}}(C_{\mathrm{FP}}+1)\|\theta_\delta\|_{1,\Omega} + \kappa^{-1}\nu^{-1}C_{\mathrm{FP}}^2 C_{Sob}(3,d)C_{E,1}\delta\|\boldsymbol{u}\|_{1,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,. \quad (2.33)$$

Thus, defining the positive constant $\widehat{C}$ in (2.27) as

$$\widehat{C} = C_{\mathscr{D}} C_{\mathrm{FP}}^2 C_{Sob}(3,d) C_{E,1}, \tag{2.34}$$

which is clearly independent of the physical parameters, from (2.27), (2.33), and the second estimate in (2.25), we readily obtain

$$\|\boldsymbol{u}\|_{1,\Omega} \,\leq\, 2C_{\mathscr{D}}\,C_{\mathrm{FP}}(C_{\mathrm{FP}}+1)\,\nu^{-1}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|\theta_\delta\|_{1,\Omega}\,. \tag{2.35}$$

In turn, by replacing (2.35) in (2.32), we get

$$\|\theta_0\|_{1,\Omega} \,\leq\, \frac{1}{2}(3C_{\mathrm{FP}}+1)\,\|\theta_\delta\|_{1,\Omega}\,, \tag{2.36}$$

which together with (2.26) and the triangle inequality, implies

$$\|\theta\|_{1,\Omega} \,\leq\, \frac{3}{2}(C_{\mathrm{FP}}+1)\,\|\theta_\delta\|_{1,\Omega}. \tag{2.37}$$

According to the above, the result follows from (2.35), (2.37) and the second estimate in (2.25).

□

Remark 2.1 *It is important to observe here that, as long as δ approaches to* 0*, the right-hand sides in estimates* (2.28) *exploit, and this issue is clearly inherited when the temperature θ is approximated through a conforming scheme, which may lead to an unstable discretization of* (2.5) *(see [36, Section 4.2]). As we will see next in Section 3.3, this is no longer an issue when using a non-conforming approach to approximate the temperature θ, since the Dirichlet datum θ_{D} appears on the right hand side of the system, thus no lifting operators must be used in the analysis.*

2.4.2 Existence of solution

In this section we provide the existence analysis to problem (2.24) by using the following special case of the Leray–Schauder Principle, known as Schaefer's Theorem. The statement is as follows:

Theorem 2.3 *Let B be a Banach space and $L : B \longrightarrow B$ a continuous and compact operator. Assume further that the set*

$$\Big\{ x \in B : \quad x = \lambda\, L(x) \quad \textit{for some } \lambda \in [0,1] \Big\},$$

is bounded. Then L has a fixed point.

To put the problem (2.24) in the context of Theorem 2.3, let us firstly consider a linearized version of (2.24) defined as: Given $(\boldsymbol{w},\phi) \in \boldsymbol{X} \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ and $\theta_\delta = E_\delta\theta_D$, with $\delta > 0$ defined according to (2.27), find $(\boldsymbol{u},\theta_0) \in \boldsymbol{X} \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ satisfying

$$\begin{aligned} \mathscr{A}^S(\boldsymbol{u},\boldsymbol{v}) + \mathscr{C}^S(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) &= \mathscr{D}^S(\phi+\theta_\delta,\boldsymbol{v}),\\ \mathscr{A}^T(\theta_0,\psi) + \mathscr{C}^T(\boldsymbol{w};\theta_0,\psi) &= -\mathscr{A}^T(\theta_\delta,\psi) - \mathscr{C}^T(\boldsymbol{w};\theta_\delta,\psi)\,, \end{aligned} \tag{2.38}$$

or equivalently,

$$\mathscr{A}_{\boldsymbol{w}}((\boldsymbol{u},\theta_0),(\boldsymbol{v},\psi)) \,=\, \mathscr{F}_{(\boldsymbol{w},\phi)}(\boldsymbol{v},\psi) \quad \forall\,(\boldsymbol{v},\psi)\in\,\boldsymbol{X}\times H^1_0(\Omega). \tag{2.39}$$

where,

$$\mathscr{A}_{\boldsymbol{w}}((\boldsymbol{u},\theta_0),(\boldsymbol{v},\psi)) \,=\, \mathscr{A}^S(\boldsymbol{u},\boldsymbol{v}) \,+\, \mathscr{C}^S(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) \,+\, \mathscr{A}^T(\theta_0,\psi) \,+\, \mathscr{C}^T(\boldsymbol{w};\theta_0,\psi) \tag{2.40}$$

and

$$\mathscr{F}_{(\boldsymbol{w},\phi)}(\boldsymbol{v},\psi) \,=\, \mathscr{D}^S(\phi+\theta_\delta,\boldsymbol{v}) \,-\, \mathscr{A}^T(\theta_\delta,\psi) \,-\, \mathscr{C}^T(\boldsymbol{w};\theta_\delta,\psi) \tag{2.41}$$

for all $(\boldsymbol{v},\psi)\in\,\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$. Secondly, let us consider the following fixed-point operator:

$$\begin{array}{rcl} \mathscr{L}\,:\,\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega) & \longrightarrow & \boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega) \\ (\boldsymbol{w},\phi) & \longmapsto & (\boldsymbol{u},\theta_0) := (\mathscr{L}_1(\boldsymbol{w},\phi),\mathscr{L}_2(\boldsymbol{w},\phi))\,, \end{array} \tag{2.42}$$

where, given $(\boldsymbol{w},\phi)\in\,\boldsymbol{X}\,\times\,H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, $(\boldsymbol{u},\theta_0) := (\mathscr{L}_1(\boldsymbol{w},\phi),\mathscr{L}_2(\boldsymbol{w},\phi))$ is the unique solution (to be verified next in Lemma 2.4) of problem 2.38. Then, it is quite clear that to prove existence and uniqueness of solution of problem (2.24) is equivalent to prove that there exists a unique $(\boldsymbol{w},\phi)\in\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, such that

$$(\boldsymbol{w},\phi) \,=\, \mathscr{L}(\boldsymbol{w},\phi). \tag{2.43}$$

Then, in what follows we firstly apply Theorem 2.3 to prove that there exists at least one $(\boldsymbol{w},\phi)\in\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ satisfying (2.43) and later on we provide suitable assumptions on the data to prove uniqueness of solution. Before doing that, in what follows we establish that problem 2.38 is well-posed, which equivalently means that $\mathscr{L}$ is well-defined.

Lemma 2.4 *The operator $\mathscr{L}$ (cf.* (2.42)*) is well-defined. Moreover, there holds*

$$\|\mathscr{L}(\boldsymbol{w},\phi)\| \,\le\, \frac{1}{C_{\mathscr{A}_{\boldsymbol{w}}}}\Big\{\,C_3(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,\|(\boldsymbol{w},\phi)\| \,+\, C_4(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,\Big\}\,, \tag{2.44}$$

for any $(\boldsymbol{w},\phi)\in\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, where $C_{\mathscr{A}_{\boldsymbol{w}}}$, $C_3(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})$ and $C_4(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})$ are the constants specified in (2.46), (2.51) *and* (2.52) *below, respectively.*

Proof. Let $(\boldsymbol{w},\phi)\in\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$. By using the estimates (2.10), (2.11), (2.18) and (2.19), and the Cauchy-Schwarz inequality, we find that the bilinear form $\mathscr{A}_{\boldsymbol{w}}$ is bounded as follows

$$\begin{aligned} &\big|\mathscr{A}_{\boldsymbol{w}}((\boldsymbol{u},\theta_0),(\boldsymbol{v},\psi))\big| \,\le\, \big|\mathscr{A}^S(\boldsymbol{u},\boldsymbol{v})\big| \,+\, \big|\mathscr{C}^S(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})\big| \,+\, \big|\mathscr{A}^T(\theta_0,\psi)\big| \,+\, \big|\mathscr{C}^T(\boldsymbol{w};\theta_0,\psi)\big| \\ &\le\, \nu\,\|\boldsymbol{u}\|_{1,\Omega}\,\|\boldsymbol{v}\|_{1,\Omega} \,+\, C_S\|\boldsymbol{w}\|_{1,\Omega}\,\|\boldsymbol{u}\|_{1,\Omega}\,\|\boldsymbol{v}\|_{1,\Omega} \,+\kappa\|\theta_0\|_{1,\Omega}\,\|\psi\|_{1,\Omega} \,+\, C_T\|\boldsymbol{w}\|_{1,\Omega}\,\|\theta_0\|_{1,\Omega}\,\|\psi\|_{1,\Omega} \\ &\le\, \big\{\nu+\kappa+(C_S+C_T)\|\boldsymbol{w}\|_{1,\Omega}\big\}\,\big(\|\boldsymbol{u}\|^2_{1,\Omega}+\|\theta_0\|^2_{1,\Omega}\big)^{1/2}\,\big(\|\boldsymbol{v}\|^2_{1,\Omega}+\|\psi\|^2_{1,\Omega}\big)^{1/2}\,, \end{aligned}$$

which implies

$$\big|\mathscr{A}_{\boldsymbol{w}}((\boldsymbol{u},\theta_0),(\boldsymbol{v},\psi))\big| \,\le\, \|\mathscr{A}_{\boldsymbol{w}}\|\,\|(\boldsymbol{u},\theta_0)\|\,\|(\boldsymbol{v},\psi)\|\,,$$

with

$$\big\|\mathscr{A}_{\boldsymbol{w}}\big\| \le \big\{\nu+\kappa+(C_S+C_T)\|\boldsymbol{w}\|_{1,\Omega}\big\}.$$

Additionally, thanks to the ellipticity of $\mathscr{A}^S$ and $\mathscr{A}^T$ (see (2.12) and (2.13)) and the skew-symmetric property of $\mathscr{C}^S$ and $\mathscr{C}^T$ (see (2.21)), there holds

$$\mathscr{A}_{\boldsymbol{w}}((\boldsymbol{v},\psi),(\boldsymbol{v},\psi)) \,\ge\, \nu\,C_{\mathrm{FP}}^{-1}\,\|\boldsymbol{v}\|^2_{1,\Omega} \,+\, \kappa\,C_{\mathrm{FP}}^{-1}\,\|\psi\|^2_{1,\Omega} \,\ge C_{\mathscr{A}_{\boldsymbol{w}}}\,\|(\boldsymbol{v},\psi)\|^2 \tag{2.45}$$

and so, $\mathscr{A}_{\boldsymbol{w}}$ is coercive on $\boldsymbol{X} \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ with constant

$$C_{\mathscr{A}_{\boldsymbol{w}}} = C_{\mathrm{FP}}^{-1} \min\{\nu, \kappa\}. \tag{2.46}$$

Finally, regarding the linear functional $\mathscr{F}_{(\boldsymbol{w},\phi)}$ (see (2.41)), we observe by convenience that it can be rewritten as

$$\mathscr{F}_{(\boldsymbol{w},\phi)}(\boldsymbol{v},\psi) \,=\, \mathscr{F}_{1,(\boldsymbol{w},\phi)}(\boldsymbol{v},\psi) \,+\, \mathscr{F}_2(\boldsymbol{v},\psi)\,, \tag{2.47}$$

where

$$\mathscr{F}_{1,(\boldsymbol{w},\phi)}(\boldsymbol{v},\psi) = \mathscr{D}^S(\phi,\boldsymbol{v}) \,-\, \mathscr{C}^T(\boldsymbol{w};\theta_\delta,\psi)\,, \tag{2.48}$$

and

$$\mathscr{F}_2(\boldsymbol{v},\psi) = \mathscr{D}^S(\theta_\delta,\boldsymbol{v}) \,-\, \mathscr{A}^T(\theta_\delta,\psi)\,. \tag{2.49}$$

Hence, using the estimates (2.19) and (2.23) as well as the second bound for θ_δ given in (2.25), we find that

$$\begin{aligned}
\big|\mathscr{F}_{1,(\boldsymbol{w},\phi)}(\boldsymbol{v},\psi)\big| \,&\leq\, \big|\mathscr{D}^S(\phi,\boldsymbol{v})\big| \,+\, \big|\mathscr{C}^T(\boldsymbol{w};\theta_\delta,\psi)\big| \\
&\leq\, C\left(\|\boldsymbol{g}\|_{0,\Omega}\|\phi\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} + \|\boldsymbol{w}\|_{1,\Omega}\|\theta_\delta\|_{1,\Omega}\|\psi\|_{1,\Omega}\right) \\
&\leq C_3(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,\|(\boldsymbol{w},\phi)\|\,\|(\boldsymbol{v},\psi)\|\,,
\end{aligned} \tag{2.50}$$

where

$$C_3(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g}) = C\max\left\{(\delta^{-4}+1)\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}, \|\boldsymbol{g}\|_{0,\Omega}\right\}. \tag{2.51}$$

In turn, combining now the estimates (2.11) and (2.23), and using once again the bound for $\|\theta_\delta\|_{1,\Omega}$ (as in the previous estimate) we obtain from (2.49)

$$\begin{aligned}
\big|\mathscr{F}_2(\boldsymbol{v},\psi)\big| \,&\leq\, \big|\mathscr{D}^S(\theta_\delta,\boldsymbol{v})\big| \,+\, \big|\mathscr{A}^T(\theta_\delta,\psi)\big| \,\leq\, C\left(\|\boldsymbol{g}\|_{0,\Omega}\|\boldsymbol{v}\|_{1,\Omega} \,+\, \kappa\|\psi\|_{1,\Omega}\right)\|\theta_\delta\|_{1,\Omega} \\
&\leq C_4(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,\|(\boldsymbol{v},\psi)\|\,,
\end{aligned}$$

with

$$C_4(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g}) = C\,(\delta^{-4}+1)\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\left\{\|\boldsymbol{g}\|^2_{0,\Omega} \,+\, \kappa^2\right\}^{1/2}. \tag{2.52}$$

Then, by combining the last two estimates we obtain

$$\big|\mathscr{F}_{(\boldsymbol{w},\phi)}(\boldsymbol{v},\psi)\big| \,\leq\, C_3(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,\|(\boldsymbol{w},\phi)\|\,\|(\boldsymbol{v},\psi)\| \,+\, C_4(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,\|(\boldsymbol{v},\psi)\|\,,$$

and therefore $\mathscr{F}_{(\boldsymbol{w},\phi)} \in \left(\boldsymbol{X} \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)\right)'$ with

$$\big\|\mathscr{F}_{(\boldsymbol{w},\phi)}\big\| \,\leq\, C_3(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,\|(\boldsymbol{w},\phi)\| \,+\, C_4(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,.$$

In this way, owing to the Lax-Milgram Lemma [20, Lemma 1.4], for any $(\boldsymbol{w},\phi) \in \boldsymbol{X} \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, there exists a unique $(\boldsymbol{u},\theta_0) \in \boldsymbol{X} \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ solution to (2.38) or, equivalently, such that $\mathscr{L}(\boldsymbol{w},\phi) = (\boldsymbol{u},\theta_0)$, which implies that the operator $\mathscr{L}$ is well-defined, and the estimate (2.44) is an immediate consequence.

□

The next result states key properties of the operator $\mathscr{L}$.

Lemma 2.5 *The operator $\mathscr{L} : \boldsymbol{X} \times H^1_{\Gamma_{\mathrm{D}}}(\Omega) \to \boldsymbol{X} \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ defined by (2.42) is compact. Moreover, the operator is Lipschitz continuous, that is, for all $(\boldsymbol{w},\phi)\,,(\widetilde{\boldsymbol{w}},\widetilde{\phi}) \in \boldsymbol{X} \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, there holds*

$$\|\mathscr{L}(\boldsymbol{w},\phi) - \mathscr{L}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \,\leq C_{\mathrm{LIP}}\,\|(\boldsymbol{w},\phi) \,-\, (\widetilde{\boldsymbol{w}},\widetilde{\phi})\|, \tag{2.53}$$

with $C_{\mathrm{LIP}} = C^{-1}_{\mathscr{A}_{\boldsymbol{w}}}\,C_3(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})$ where $C_{\mathscr{A}_{\boldsymbol{w}}} = C_{\mathrm{FP}}^{-1}\min\{\nu,\kappa\}$ is the coercivity constant of the bilinear form $\mathscr{A}_{\boldsymbol{w}}$ (defined in (2.46)) and $C_3(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})$ is given by (2.51).

Proof. Let us start by showing that the operator $\mathscr{L}$ defined by (2.42) is compact. To do so, we let $\{(\boldsymbol{w}_n, \phi_n)\}_{n\geq 1} \subseteq \boldsymbol{X} \times H^1_{\Gamma_\mathrm{D}}(\Omega)$ be a sequence weakly convergent to $(\boldsymbol{w}, \phi) \in \boldsymbol{X} \times H^1_{\Gamma_\mathrm{D}}(\Omega)$, and let $(\boldsymbol{u}, \theta_0) = \mathscr{L}(\boldsymbol{w}, \phi)$ and $\{(\boldsymbol{u}_n, \theta_{0,n})\}_{n\geq 1} \subseteq \boldsymbol{X} \times H^1_{\Gamma_\mathrm{D}}(\Omega)$, be the sequence given by

$$(\boldsymbol{u}_n, \theta_{0,n}) \,=\, \mathscr{L}(\boldsymbol{w}_n, \phi_n) \quad \text{for each } n \geq 1\,. \tag{2.54}$$

From the definition of $\mathscr{L}$ (cf. (2.42)), and applying the coercivity of $\mathscr{A}_{\boldsymbol{w}}$ (cf. (2.45)), its linearity in the first component, the definition of $\mathscr{F}_1$ (cf. (2.48)), and the estimates (2.17) and (2.22), with $q = q' = 4$, it follows that

$$\begin{aligned}
&\|\mathscr{L}(\boldsymbol{w}_n, \phi_n) - \mathscr{L}(\boldsymbol{w}, \phi)\|^2 \,=\, \|(\boldsymbol{u}_n, \theta_{0,n}) - (\boldsymbol{u}, \theta_0)\|^2 \\
&\quad \leq C^{-1}_{\mathscr{A}_{\boldsymbol{w}}} \mathscr{A}_{\boldsymbol{w}}((\boldsymbol{u}_n, \theta_{0,n}) - (\boldsymbol{u}, \theta_0), (\boldsymbol{u}_n, \theta_{0,n}) - (\boldsymbol{u}, \theta_0)) \\
&\quad \leq C^{-1}_{\mathscr{A}_{\boldsymbol{w}}} \Big\{ \mathscr{A}_{\boldsymbol{w}}((\boldsymbol{u}_n, \theta_{0,n}), (\boldsymbol{u}_n - \boldsymbol{u}, \theta_{0,n} - \theta_0)) - \mathscr{A}_{\boldsymbol{w}}((\boldsymbol{u}, \theta_0), (\boldsymbol{u}_n - \boldsymbol{u}, \theta_{0,n} - \theta_0)) \Big\} \\
&\quad = C^{-1}_{\mathscr{A}_{\boldsymbol{w}}} \mathscr{F}_{1,(\boldsymbol{w}_n - \boldsymbol{w}, \phi_n - \phi)}(\boldsymbol{u}_n - \boldsymbol{u}, \theta_{0,n} - \theta_0) \\
&\quad \leq C^{-1}_{\mathscr{A}_{\boldsymbol{w}}} \Big\{ \|\boldsymbol{g}\|_{0,\Omega} \|\phi_n - \phi\|_{0,4,\Omega} \|\boldsymbol{u}_n - \boldsymbol{u}\|_{0,4,\Omega} \,-\, \|\boldsymbol{w}_n - \boldsymbol{w}\|_{0,4,\Omega} \|\nabla\theta_\delta\|_{0,\Omega} \|\theta_{0,n} - \theta_0\|_{0,4,\Omega} \Big\}\,.
\end{aligned} \tag{2.55}$$

Now, since $(\boldsymbol{w}_n, \phi_n) \rightharpoonup (\boldsymbol{w}, \phi)$ in $[H^1(\Omega)]^d \times H^1(\Omega)$, it follows that $\|\boldsymbol{w}_n - \boldsymbol{w}\|_{0,4,\Omega} \xrightarrow{n\to\infty} 0$ and also $\|\phi_n - \phi\|_{0,4,\Omega} \xrightarrow{n\to\infty} 0$ according to the Rellich-Kondrachov compactness Theorem (see e.g., [1, Theorem 6.3] and [39, Theorem 1.3.5]). Also, note that $(\boldsymbol{u}_n, \theta_{0,n})$ and $(\boldsymbol{u}, \theta_0)$, coming from (2.54), satisfy (2.44) and therefore, they are bounded in their respective norms since $\{(\boldsymbol{w}_n, \phi_n)\}_{n\geq 1}$ is bounded as it is weakly convergent, as a consequence of the Banach-Steinhaus Theorem [8, Theorem 2.2]. In view of these facts and the estimate (2.55), we can conclude that $\|\mathscr{L}(\boldsymbol{w}_n, \phi_n) - \mathscr{L}(\boldsymbol{w}, \phi)\| \xrightarrow{n\to\infty} 0$, that is, the sequence $\{\mathscr{L}(\boldsymbol{w}_n, \phi_n)\}_{n\geq 1}$ is strongly, i.e. norm, convergent to $\mathscr{L}(\boldsymbol{w}, \phi)$, and so $\mathscr{L}$ is compact.

Next, in order to show the Lipschitz continuity property we consider $(\boldsymbol{w}, \phi), (\widetilde{\boldsymbol{w}}, \widetilde{\phi}) \in \boldsymbol{X} \times H^1_{\Gamma_\mathrm{D}}(\Omega)$ and set

$$(\boldsymbol{u}, \theta_0) \,=\, \mathscr{L}(\boldsymbol{w}, \phi) \quad \text{and} \quad (\widetilde{\boldsymbol{u}}, \widetilde{\theta}_0) \,=\, \mathscr{L}(\widetilde{\boldsymbol{w}}, \widetilde{\phi})\,. \tag{2.56}$$

By proceeding similarly to (2.55), and then using (2.50) we find

$$\begin{aligned}
\|\mathscr{L}(\boldsymbol{w}, \phi) - \mathscr{L}(\widetilde{\boldsymbol{w}}, \widetilde{\phi})\|^2 \,&\leq\, C^{-1}_{\mathscr{A}_{\boldsymbol{w}}} \mathscr{F}_{1,(\boldsymbol{w} - \widetilde{\boldsymbol{w}}, \phi - \widetilde{\phi})}(\boldsymbol{u} - \widetilde{\boldsymbol{u}}, \theta_0 - \widetilde{\theta}_0) \\
&\leq C^{-1}_{\mathscr{A}_{\boldsymbol{w}}} C_3(\delta, \kappa, \theta_\mathrm{D}, \boldsymbol{g}) \|(\boldsymbol{w}, \phi) - (\widetilde{\boldsymbol{w}}, \widetilde{\phi})\| \|(\boldsymbol{u}, \theta_0) - (\widetilde{\boldsymbol{u}}, \widetilde{\theta}_0)\|\,,
\end{aligned}$$

and after using (2.56) and simplifying we get (2.53). □

In the following result we show that the set

$$\mathcal{K} := \Big\{ (\boldsymbol{u}, \theta_0) \in \boldsymbol{X} \times H^1_{\Gamma_\mathrm{D}}(\Omega) : \qquad (\boldsymbol{u}, \theta_0) \,=\, \lambda\, \mathscr{L}(\boldsymbol{u}, \theta_0) \quad \text{for} \quad \lambda \in [0, 1] \Big\}\,,$$

is bounded.

Lemma 2.6 *$\mathcal{K}$ is a bounded set.*

Proof. We first observe that if $(\boldsymbol{u}, \theta_0) \in \mathcal{K}$, for some $\lambda \in (0, 1]$, then according to the definition of $\mathscr{L}$, it follows that

$$\mathscr{A}_{\boldsymbol{u}}((\lambda^{-1}\boldsymbol{u}, \lambda^{-1}\theta_0), (\boldsymbol{v}, \psi)) \,=\, \mathscr{F}_{(\boldsymbol{u},\theta_0)}(\boldsymbol{v}, \psi) \quad \forall\, (\boldsymbol{v}, \psi) \in \boldsymbol{X} \times H^1_{\Gamma_\mathrm{D}}(\Omega)\,,$$

or equivalently

$$\mathscr{A}_{\boldsymbol{u}}((\boldsymbol{u},\theta_0),(\boldsymbol{v},\psi)) \,=\, \lambda\,\mathscr{F}_{(\boldsymbol{u},\theta_0)}(\boldsymbol{v},\psi) \quad \forall\,(\boldsymbol{v},\psi)\in\,\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega)\,.$$

In particular, by using the definition of $\mathscr{A}_{\boldsymbol{w}}$ and $\mathscr{F}_{(\boldsymbol{w},\phi)}$, according to (2.40) and (2.41), with $(\boldsymbol{u},\theta_0)$ in place of $(\boldsymbol{w},\phi)$, taking there $(\boldsymbol{v},\psi)=(\boldsymbol{u},\theta_0)$, using the skew-symmetry property of $\mathscr{C}^S$ and $\mathscr{C}^T$ given by (2.21), and decoupling, we get

$$\begin{aligned}\mathscr{A}^S(\boldsymbol{u},\boldsymbol{u}) \;&=\; \lambda\,\mathscr{D}^S(\theta_0+\theta_\delta,\boldsymbol{u}),\\ \mathscr{A}^T(\theta_0,\theta_0) \;&=\; -\lambda\,\mathscr{A}^T(\theta_\delta,\theta_0)\,-\,\lambda\,\mathscr{C}^T(\boldsymbol{u};\theta_\delta,\theta_0)\end{aligned}$$

which coincides with (2.30) in Theorem 2.2 up to the multiplicative constant λ on the right hand side of these expressions, and therefore we can immediately deduce that

$$\|\boldsymbol{u}\|_{1,\Omega}\,\leq\,\lambda C_{\mathrm{FP}}C_{\mathscr{D}}\,\nu^{-1}\,\|\boldsymbol{g}\|_{0,\Omega}\Big(\|\theta_0\|_{1,\Omega}+\|\theta_\delta\|_{1,\Omega}\Big)\,\leq\,C_{\mathrm{FP}}C_{\mathscr{D}}\nu^{-1}\,\|\boldsymbol{g}\|_{0,\Omega}\Big(\|\theta_0\|_{1,\Omega}+\|\theta_\delta\|_{1,\Omega}\Big)\,,\tag{2.57}$$

and

$$\begin{aligned}\|\theta_0\|_{1,\Omega}\,&\leq\,\lambda\,\Big(C_{\mathrm{FP}}\|\theta_\delta\|_{1,\Omega}\,+\,\kappa^{-1}C_{\mathrm{FP}}C_{Sob}(3,d)C_{E,1}\delta\|\boldsymbol{u}\|_{1,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\Big)\\ &\leq\,C_{\mathrm{FP}}\|\theta_\delta\|_{1,\Omega}\,+\,\kappa^{-1}C_{\mathrm{FP}}C_{Sob}(3,d)C_{E,1}\delta\|\boldsymbol{u}\|_{1,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,,\end{aligned}\tag{2.58}$$

where we have used that $\lambda\leq 1$. Next, we realize that the estimates (2.57) and (2.58) are the same of (2.31) and (2.32), respectively. Thus, we can proceed exactly as in the proof of Theorem 2.2 to obtain that $\boldsymbol{u}$ and θ_0 satisfy estimates (2.35) and (2.36), respectively. Finally, if $(\boldsymbol{u},\theta_0)\in\mathcal{K}$, for $\lambda=0$, then $(\boldsymbol{u},\theta_0)=(0,0)$, and evidently estimates (2.35) and (2.36) hold, which concludes the proof. □

Now, we are in position to establishing existence of solutions to problem (2.24).

Theorem 2.7 *There exists a solution* $(\boldsymbol{u},\theta)$ *to* (2.24).

Proof. First, we observe that Lemma 2.5 implies that operator $\mathscr{L}:\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega)\longrightarrow\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ is continuous and compact. Furthermore, Lemma 2.6 states that $\mathcal{K}$ is bounded. Therefore, Shaefer's Theorem (see Theorem 2.3) guarantees the existence of a fixed point $(\boldsymbol{u},\theta_0)\in\boldsymbol{X}\times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ of $\mathscr{L}$; that is,

$$(\boldsymbol{u},\theta_0)=\mathscr{L}(\boldsymbol{u},\theta_0)\,,$$

and so, satisfying

$$\begin{aligned}\mathscr{A}^S(\boldsymbol{u},\boldsymbol{v})\,+\,\mathscr{C}^S(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) \;&=\; \mathscr{D}^S(\theta_0+\theta_\delta,\boldsymbol{v}),\\ \mathscr{A}^T(\theta_0,\psi)\,+\,\mathscr{C}^T(\boldsymbol{u};\theta_0,\psi) \;&=\; -\mathscr{A}^T(\theta_\delta,\psi)\,-\,\mathscr{C}^T(\boldsymbol{u};\theta_\delta,\psi)\,,\end{aligned}$$

which implies that $(\boldsymbol{u},\theta_0+\theta_\delta)$ is a solution to (2.24). □

We close the section by mentioning that the existence of the pressure p solution to problem (2.5) follows from the inf–sup condition (2.14) and the properties (2.10) (2.18) and (2.23) of the forms $\mathscr{A}^S$, $\mathscr{C}^S$ and $\mathscr{D}^S$, respectively (see also e.g., [24, Theorem 1.4]).

2.4.3 Uniqueness of solution

Now we establish the uniqueness result for (2.24) under a sufficiently small data assumption.

Theorem 2.8 *Assume that the Lipschitz continuity constant (see Lemma 2.5) satisfies*

$$C_{\mathrm{LIP}} = C_{\mathrm{FP}}^{-1}\min\{\nu,\kappa\}C_3(\delta,\kappa,\theta_{\mathrm{D}},\boldsymbol{g}) < 1. \tag{2.59}$$

Then, there exists a unique solution $(\boldsymbol{u},p,\theta) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega) \times H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ *to* (2.5).

Proof. Let $(\boldsymbol{u},p,\theta)$ and $(\widetilde{\boldsymbol{u}},\widetilde{p},\widetilde{\theta})$ be two solutions of problem (2.5). Then, since $(\boldsymbol{u},\theta)$ and $(\widetilde{\boldsymbol{u}},\widetilde{\theta})$ are solutions to (2.24), we let $\theta_0 = \theta - \theta_\delta$ and $\widetilde{\theta}_0 = \widetilde{\theta} - \theta_\delta$ and conclude that $(\boldsymbol{u},\theta_0)$ and $(\widetilde{\boldsymbol{u}},\widetilde{\theta}_0)$ are fixed points of the operator $\mathscr{L}$. Thus, from the Lipschitz continuity of $\mathscr{L}$ (cf. Lemma 2.5), we deduce that

$$\|(\boldsymbol{u},\theta_0) - (\widetilde{\boldsymbol{u}},\widetilde{\theta}_0)\| = \|\mathscr{L}(\boldsymbol{u},\theta_0) - \mathscr{L}(\widetilde{\boldsymbol{u}},\widetilde{\theta}_0)\| \le C_{\mathrm{LIP}}\|(\boldsymbol{u},\theta_0) - (\widetilde{\boldsymbol{u}},\widetilde{\theta}_0)\|,$$

which together with (2.59), implies that $\boldsymbol{u} - \widetilde{\boldsymbol{u}} = \mathbf{0}$ and $\theta - \widetilde{\theta} = \theta_0 - \widetilde{\theta}_0 = 0$. In turn, using the fact that $(\boldsymbol{u},p,\theta)$ and $(\widetilde{\boldsymbol{u}},\widetilde{p},\widetilde{\theta})$ satisfy the first equation of (2.5), it readily follows that

$$\mathscr{B}^S(\boldsymbol{v},p-\widetilde{p}) = \mathscr{D}^S(\theta-\widetilde{\theta},\boldsymbol{v}) - \mathscr{A}^S(\boldsymbol{u}-\widetilde{\boldsymbol{u}},\boldsymbol{v}) - (\mathscr{C}^S(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) - \mathscr{C}^S(\widetilde{\boldsymbol{u}};\widetilde{\boldsymbol{u}},\boldsymbol{v})) = 0,$$

for all $\boldsymbol{v} \in [H_0^1(\Omega)]^d$, which combined with the inf-sup condition (2.14) imply that $p - \widetilde{p} = 0$, which concludes the proof. □

3 The fully-dG finite element discretization

In this section, we present and analyze the discrete scheme based on discontinuous Galerkin approximations to find the solution to problem (2.5). In this way, after introducing some preliminary notations and definitions in Section 3.1, we then set and analyze the discrete scheme throughout Section 3.2.

3.1 Preliminaries

Let $\mathcal{T}_h$ be a shape-regular partition of Ω, made up of simplices K, where K is a triangle in 2D or a tetrahedron in 3D, with unit outward normal vector $\boldsymbol{n}_K$ and element diameter h_K. As usual, the mesh size is defined as $h := \max\limits_{K\in\mathcal{T}_h} h_K$. For simplicity, we further assume that if $\partial K \cap \partial\Omega \neq \emptyset$ then either $|\partial K \cap \Gamma_{\mathrm{D}}| = 0$ or $|\partial K \cap \Gamma_{\mathrm{N}}| = 0$ and that the intersection of two elements is either empty, a vertex, an edge, or a face. The set of edges/faces of the mesh $\mathcal{T}_h$ will be denoted by $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$, where $\mathcal{E}_h^i$ and $\mathcal{E}_h^b$ stand for the sets of all interior and boundary edges/faces, respectively, and $\mathcal{E}^b_{h,\Gamma_{\mathrm{D}}} = \mathcal{E}_h^b \cap \Gamma_{\mathrm{D}}$. For any edge/face $e \in \mathcal{E}_h$, we denote by h_e its respective $(d-1)$-diameter, that is, its length in 2D or its maximum diameter in 3D.

Jumps and average operators to be used in the sections that follow are introduced next. Firstly, let $e \in \mathcal{E}_h^i$ be a common edge/face of two neighbor elements $K^+, K^- \in \mathcal{T}_h$ satisfying $e = \partial K^+ \cap \partial K^-$, and let $\boldsymbol{n}_e^\pm$ be the unit outward normal vector to e on $K^\pm$. If ψ and $\boldsymbol{v}$ are sufficiently regular scalar or vector piecewise functions on $\mathcal{T}_h$, respectively, denote by $\psi^\pm$ and $\boldsymbol{v}^\pm$ their traces taking from the interior of $K^\pm$. We then define the jump $[\![\cdot]\!]$ acting on ψ and $\boldsymbol{v}$ as

$$[\![\psi]\!] = \begin{cases} \psi^+\boldsymbol{n}_e^+ + \psi^-\boldsymbol{n}_e^-, & e \in \mathcal{E}_h^i \\ \psi\,\boldsymbol{n}, & e \in \mathcal{E}_h^b \end{cases} \quad \text{and} \quad [\![\boldsymbol{v}]\!] = \begin{cases} \boldsymbol{v}^+ \otimes \boldsymbol{n}_e^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}_e^-, & e \in \mathcal{E}_h^i \\ \boldsymbol{v} \otimes \boldsymbol{n} & e \in \mathcal{E}_h^b, \end{cases}$$

where $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$. In turn, For any smooth enough piecewise (scalar-, vector- or tensor-valued) function η we define its average across $e \in \mathcal{E}_h^i$ as $\{\{\eta\}\} = \frac{1}{2}(\eta^+ + \eta^-)$ and $\{\{\eta\}\} = \eta$ if $e \in \mathcal{E}_h^b$.

For $r \geq 0$, we set the standard broken Sobolev space

$$H^r(\mathcal{T}_h) \,=\, \left\{ \phi \in L^2(\Omega) : \quad \phi\big|_K \,\in\, H^r(K) \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{3.1}$$

and the mesh-dependent broken norms

$$\begin{aligned} \|\psi_h\|^2_{1,\mathcal{T}_h} \,&=\, \sum_{K\in\mathcal{T}_h} \|\nabla_h \psi\|^2_{0,K} \,+\, \sum_{e\in\mathcal{E}_h} \frac{a_0}{h_e} \|[\![\psi]\!]\|^2_{0,e} \quad \forall\, \psi \in H^1(\mathcal{T}_h)\,, \\ \|\psi_h\|^2_{2,\mathcal{T}_h} \,&=\, \|\psi_h\|^2_{1,\mathcal{T}_h} \,+\, \sum_{K\in\mathcal{T}_h} h_K^2 |\psi|^2_{2,K} \quad \forall\, \psi \in H^2(\mathcal{T}_h)\,, \end{aligned} \tag{3.2}$$

where $\nabla_h(\,\cdot\,)$ is the broken gradient operator and a_0 is a fixed parameter. An inverse inequality allows to guarantee the existence of a positive constant C, independent of the meshsize, such that (see [36, Section 3.3.1]):

$$\|\psi_h\|_{2,\mathcal{T}_h} \,\leq\, C\, \|\psi_h\|_{1,\mathcal{T}_h} \quad \forall\, \psi_h \,\in\, \Psi_h\,, \tag{3.3}$$

where Ψ_h is any piece-wise polynomial space.

Also, we recall the broken version of the Sobolev embedding (2.2) (see e.g., [25, 42]): there exists a constant $\widetilde{C}_{Sob} > 0$ such that

$$\|\psi\|_{0,q,\Omega} \,\leq\, \widetilde{C}_{Sob} \|\psi\|_{1,\mathcal{T}_h} \quad \forall\, \psi \in H^1(\mathcal{T}_h)\,, \quad \text{where} \quad \begin{cases} q \geq 1 & \text{if } d = 2\,, \\ q \in [1,6] & \text{if } d = 3, \end{cases} \tag{3.4}$$

which is useful to proof the discrete estimates and stability properties of the forms defining the discrete problem to be defined next in Section 3.2. The respective vector versions of (3.1)-(3.4) are extended in a natural way.

3.2 Discontinuous Galerkin finite element scheme

For an approximation of order $k \geq 1$ and a mesh $\mathcal{T}_h$ on Ω as in Section (3.1), let $P_k(K)$ be the local space spanned by polynomials of degree $\leq k$ on K. We then consider the following finite dimensional spaces

$$\boldsymbol{V}_h \,:=\, \left\{ \boldsymbol{v}_h \in \boldsymbol{H}_0(\mathrm{div};\Omega) \,:\, \boldsymbol{v}_h\big|_K \in [P_k(K)]^d, \quad \forall\, K \in \mathcal{T}_h \right\}, \quad Q_h \,:=\, S_h^{k-1} \cap L_0^2(\Omega)\,, \quad W_h \,:=\, S_h^k\,, \tag{3.5}$$

where

$$S_h^l \,:=\, \left\{ r_h \in L^2(\Omega) \,:\, r_h\big|_K \in P_l(K), \quad \forall\, K \in \mathcal{T}_h \right\}, \quad \text{for} \quad l \geq 0. \tag{3.6}$$

Note that $\boldsymbol{V}_h$ is the space of divergence-conforming BDM elements [9]. In this way, based on the discrete spaces (3.5), we propose the following fully discontinuous Galerkin finite element method for problem (2.5): Find $(\boldsymbol{u}_h, p_h, \theta_h) \in \boldsymbol{V}_h \times Q_h \times W_h$, tal que

$$\begin{aligned} \mathscr{A}_h^S(\boldsymbol{u}_h, \boldsymbol{v}_h) \,+\, \mathscr{C}_h^S(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \,+\, \mathscr{B}^S(\boldsymbol{v}_h, p_h) \,&=\, \mathscr{D}^S(\theta_h, \boldsymbol{v}_h), \\ \mathscr{B}^S(\boldsymbol{u}_h, q_h) \,&=\, 0\,, \\ \mathscr{A}_h^T(\theta_h, \psi_h) \,+\, \mathscr{C}_h^T(\boldsymbol{u}_h; \theta_h, \psi_h) \,&=\, \mathscr{D}^T_{\theta_{\mathrm{D}}}(\psi_h), \end{aligned} \tag{3.7}$$

for all $(\boldsymbol{v}_h, q_h, \psi_h) \in \boldsymbol{V}_h \times Q_h \times W_h$. Here, the discrete bilinear forms $\mathscr{A}_h^S$ and $\mathscr{A}_h^T$ we consider are based on the interior penalty method [5] and given, respectively, by

$$\begin{aligned}\mathscr{A}_h^S(\boldsymbol{u}_h, \boldsymbol{v}_h) = &\int_\Omega \nu \nabla_h \boldsymbol{u}_h : \nabla_h \boldsymbol{v}_h - \sum_{e\in\mathcal{E}_h} \int_e \{\{\nu \nabla_h \boldsymbol{u}_h\}\} : [\![\boldsymbol{v}_h]\!] \\ &- \sum_{e\in\mathcal{E}_h} \int_e \{\{\nu \nabla_h \boldsymbol{v}_h\}\} : [\![\boldsymbol{u}_h]\!] + \sum_{e\in\mathcal{E}_h} \frac{\nu a_0}{h_e} \int_e [\![\boldsymbol{u}_h]\!] : [\![\boldsymbol{v}_h]\!] ,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\mathscr{A}_h^T(\theta_h, \psi_h) = &\int_\Omega \kappa \nabla_h \theta_h \cdot \nabla_h \psi_h - \sum_{e\in\mathcal{E}_h^i \cup \mathcal{E}_{h,\Gamma_D}^b} \int_e \{\{\kappa \nabla_h \theta_h\}\} \cdot [\![\psi_h]\!] \\ &- \sum_{e\in\mathcal{E}_h^i \cup \mathcal{E}_{h,\Gamma_D}^b} \int_e \{\{\kappa \nabla_h \psi_h\}\} \cdot [\![\theta_h]\!] + \sum_{e\in\mathcal{E}_h^i \cup \mathcal{E}_{h,\Gamma_D}^b} \frac{\kappa a_0}{h_e} \int_e [\![\theta_h]\!] \cdot [\![\psi_h]\!] ,\end{aligned}$$

where a_0 is the interior penalty parameter taken large enough so that the bilinear forms $\mathscr{A}_h^S$ and $\mathscr{A}_h^T$ are both coercive (see [5] for further details). In turn, the discrete forms linked to the nonlinear convective terms, based on an upwind approach [31], are defined by

$$\mathscr{C}_h^S(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega (\boldsymbol{w}_h \cdot \nabla_h) \boldsymbol{u}_h \cdot \boldsymbol{v}_h + \frac{1}{2} \sum_{K\in\mathcal{T}_h} \int_{\partial K \setminus \Gamma} (\boldsymbol{w}_h \cdot \boldsymbol{n}_K - |\boldsymbol{w}_h \cdot \boldsymbol{n}_K|)(\boldsymbol{u}_h^e - \boldsymbol{u}_h) \cdot \boldsymbol{v}_h,$$

and

$$\mathscr{C}_h^T(\boldsymbol{w}_h; \theta_h, \psi_h) = \int_\Omega (\boldsymbol{w}_h \cdot \nabla_h \theta_h)\, \psi_h + \frac{1}{2} \sum_{K\in\mathcal{T}_h} \int_{\partial K \setminus \Gamma} (\boldsymbol{w}_h \cdot \boldsymbol{n}_K - |\boldsymbol{w}_h \cdot \boldsymbol{n}_K|) \left(\theta_h^e - \theta_h\right) \psi_h,$$

where $\boldsymbol{u}_h^e$ and θ_h^e stand for the traces of $\boldsymbol{u}_h$ and θ_h, respectively, taken from within the exterior of K. Finally, the functional $\mathscr{D}_{\theta_D}^T$ is defined by

$$\mathscr{D}_{\theta_D}^T(\psi_h) = \sum_{e\in\mathcal{E}_{h,\Gamma_D}^b} \int_e \Big(\frac{a_0}{h_e} \psi_h - \kappa \nabla_h \psi_h \cdot \boldsymbol{n} \Big) \theta_D , \tag{3.9}$$

and the bilinear form $\mathscr{B}^S$ and the linear functional $\mathscr{D}^S$ are defined by (2.7) and (2.9), respectively.

3.2.1 Discrete estimates and stability properties

Here we state some properties of the forms defining (3.7) and that are required for the discrete analysis. Their proofs can be found in the previous related works [10, 25, 36, 37] and therefore we omit them.

$$\Big|\mathscr{A}_h^S(\boldsymbol{u}_h, \boldsymbol{v}_h)\Big| \le \nu\, \widetilde{C}_{\mathscr{A}^S} \|\boldsymbol{u}_h\|_{1,\mathcal{T}_h} \|\boldsymbol{v}_h\|_{1,\mathcal{T}_h} \qquad \forall\, \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h , \tag{3.10}$$

$$\Big|\mathscr{A}_h^S(\boldsymbol{u}, \boldsymbol{v}_h)\Big| \le \nu\, \widehat{C}_{\mathscr{A}^S} \|\boldsymbol{u}\|_{2,\mathcal{T}_h} \|\boldsymbol{v}_h\|_{1,\mathcal{T}_h} \qquad \forall\, \boldsymbol{u} \in [H^2(\mathcal{T}_h)]^n \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h , \tag{3.11}$$

$$\Big|\mathscr{A}_h^T(\theta_h, \psi_h)\Big| \le \kappa\, \widetilde{C}_{\mathscr{A}^T} \|\theta_h\|_{1,\mathcal{T}_h} \|\psi_h\|_{1,\mathcal{T}_h} \qquad \forall\, \theta_h, \psi_h \in W_h , \tag{3.12}$$

$$\Big|\mathscr{A}_h^T(\theta, \psi_h)\Big| \le \kappa\, \widehat{C}_{\mathscr{A}^T} \|\theta\|_{2,\mathcal{T}_h} \|\psi_h\|_{1,\mathcal{T}_h} \qquad \forall\, \theta \in H^2(\mathcal{T}_h) \quad \forall\, \psi_h \in W_h , \tag{3.13}$$

$$\Big|\mathscr{B}^S(\boldsymbol{v}_h, q)\Big| \le \widetilde{C}_{\mathscr{B}^S} \|\boldsymbol{v}_h\|_{1,\mathcal{T}_h} \|q\|_{0,\Omega} \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h \quad \forall\, q \in L_0^2(\Omega) , \tag{3.14}$$

$$\left|\mathscr{C}_h^S(\boldsymbol{w}_h;\boldsymbol{u}_h,\boldsymbol{v}_h)\right| \,\le\, \widetilde{C}_{\mathscr{C}^S}\,\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h}\,\|\boldsymbol{u}_h\|_{1,\mathcal{T}_h}\,\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h} \qquad \forall\,\boldsymbol{w}_h,\boldsymbol{u}_h,\boldsymbol{v}_h\in\boldsymbol{V}_h\,, \tag{3.15}$$

$$\left|\mathscr{C}_h^T(\boldsymbol{w}_h;\theta_h,\psi_h)\right| \,\le\, \widetilde{C}_{\mathscr{C}^T}\,\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h}\,\|\theta_h\|_{1,\mathcal{T}_h}\,\|\psi_h\|_{1,\mathcal{T}_h} \qquad \forall\,\boldsymbol{w}_h\in\boldsymbol{V}_h \quad \forall\,\theta_h,\psi_h\in W_h\,. \tag{3.16}$$

Also, it is well-known that for a sufficiently large parameter a_0 the bilinear forms $\mathscr{A}_h^S$ and $\mathscr{A}_h^T$ are coercive (see [5, 20] for further details). More precisely, there exist positive and $h-$independent constants $\widetilde{\alpha}_S$ and $\widetilde{\alpha}_T$ such that

$$\mathscr{A}_h^S(\boldsymbol{v}_h,\boldsymbol{v}_h) \,\ge\, \nu\,\widetilde{\alpha}_S\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}^2 \quad \forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h\,, \quad \text{and} \quad \mathscr{A}_h^T(\psi_h,\psi_h)\ge\kappa\,\widetilde{\alpha}_T\|\psi_h\|_{1,\mathcal{T}_h}^2 \quad \forall\,\psi_h\in W_h. \tag{3.17}$$

Regarding the bilinear form $\mathscr{B}^s$, we recall from [27] the discrete inf-sup condition: there exists an h-independent constant $\widetilde{\beta}>0$ such that

$$\sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h\\ \boldsymbol{v}_h\neq\boldsymbol{0}}}\frac{\mathscr{B}^S(\boldsymbol{v}_h,q_h)}{\|v_h\|_{1,\mathcal{T}_h}} \,\ge\, \widetilde{\beta}\,\|q_h\|_{0,\Omega} \qquad \forall\,q_h\in Q_h\,. \tag{3.18}$$

As in the continuous case, we define the discrete kernel $\boldsymbol{X}_h$ of the bilinear form $\mathscr{B}^S$ as

$$\boldsymbol{X}_h := \Big\{\boldsymbol{v}_h\in\boldsymbol{V}_h:\quad \mathscr{B}^S(\boldsymbol{v}_h,q_h)\,=\,0\,,\ \forall q_h\in Q_h\Big\} = \Big\{\boldsymbol{v}_h\in\boldsymbol{V}_h:\quad \operatorname{div}\boldsymbol{v}_h\,=\,0 \quad \text{in} \quad \Omega\Big\}\,,$$

where the last equality follows from the fact that $\boldsymbol{V}_h\subset\boldsymbol{H}_0(\operatorname{div};\Omega)$ and $\operatorname{div}\boldsymbol{V}_h\subset Q_h$ (see [11]). This particularity implies that $\boldsymbol{X}_h\subset\boldsymbol{H}_0(\operatorname{div}^0;\Omega)$. Incidentally, according to [11, 36] we deduce that

$$\mathscr{C}_h^S(\boldsymbol{w}_h;\boldsymbol{v}_h,\boldsymbol{v}_h) \,=\, \frac{1}{2}\sum_{e\in\mathcal{E}_h^i}\int_e|\boldsymbol{w}_h\cdot\boldsymbol{n}_e|\,|[\![\boldsymbol{v}_h]\!]| \,\ge\, 0 \qquad \forall\,\boldsymbol{w}_h\in\boldsymbol{H}_0(\operatorname{div}^0;\Omega)\,,\forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h\,, \tag{3.19}$$

$$\mathscr{C}_h^T(\boldsymbol{w}_h;\psi_h,\psi_h) \,=\, \frac{1}{2}\sum_{e\in\mathcal{E}_h^i}\int_e|\boldsymbol{w}_h\cdot\boldsymbol{n}_e|\,|[\![\psi_h]\!]| \,\ge\, 0 \qquad \forall\,\boldsymbol{w}_h\in\boldsymbol{H}_0(\operatorname{div}^0;\Omega)\,,\forall\,\psi_h\in W_h\,. \tag{3.20}$$

Note further that the convective forms $\mathscr{C}_h^S$ and $\mathscr{C}_h^T$ are not linear in their first argument. However, they satisfy the following Lipschitz continuity property: For all $\boldsymbol{w}_h,\widetilde{\boldsymbol{w}}_h,\boldsymbol{u}_h\in[H^2(\mathcal{T}_h)]^d$, $\theta_h\in H^2(\mathcal{T}_h)$, $\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $\psi_h\in W_h$, there exist positive constants $\widetilde{C}_{S,\mathrm{LIP}}$ and $\widetilde{C}_{T,\mathrm{LIP}}$, independent of the meshsize, such that

$$\begin{aligned}\left|\mathscr{C}_h^S(\boldsymbol{w}_h;\boldsymbol{u}_h,\boldsymbol{v}_h)-\mathscr{C}_h^S(\widetilde{\boldsymbol{w}}_h;\boldsymbol{u}_h,\boldsymbol{v}_h)\right| &\le \widetilde{C}_{S,\mathrm{LIP}}\|\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h\|_{1,\mathcal{T}_h}\|\boldsymbol{u}_h\|_{1,\mathcal{T}_h}\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}\,,\\ \left|\mathscr{C}_h^T(\boldsymbol{w}_h;\theta_h,\psi_h)-\mathscr{C}_h^T(\widetilde{\boldsymbol{w}}_h;\theta_h,\psi_h)\right| &\le \widetilde{C}_{T,\mathrm{LIP}}\|\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h\|_{1,\mathcal{T}_h}\|\theta_h\|_{1,\mathcal{T}_h}\|\psi_h\|_{1,\mathcal{T}_h}\,.\end{aligned} \tag{3.21}$$

As for the forms $\mathscr{D}^S$ and $\mathscr{D}_{\theta_{\mathrm{D}}}^T$ defined in (2.9) and (3.9), we easily find that

$$\left|\mathscr{D}^S(\psi_h,\boldsymbol{v}_h)\right| \,\le\, \widetilde{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|\psi_h\|_{1,\mathcal{T}_h}\,\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}\,, \qquad \forall\psi_h\in W_h\,,\forall\,v_h\in\boldsymbol{V}_h\,, \tag{3.22}$$

$$\left|\mathscr{D}^S(\psi,\boldsymbol{v}_h)\right| \,\le\, \widehat{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|\psi\|_{2,\mathcal{T}_h}\,\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}\,, \qquad \forall\psi\in H^2(\mathcal{T}_h)\,,\forall\,v_h\in\boldsymbol{V}_h\,, \tag{3.23}$$

$$\left|\mathscr{D}_{\theta_{\mathrm{D}}}^T(\psi_h)\right| \,\le\, \widetilde{C}_{\mathscr{D}^T}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,\|\psi_h\|_{1,\mathcal{T}_h}\,, \qquad \forall\psi_h\in W_h\,. \tag{3.24}$$

3.3 Well-posedenss of the discrete problem

In this section we adapt the analysis developed in Section 2.4 to prove the well-posedness of problem (3.7). We begin by observing that, analogously to the continuous case, and owing to the discrete inf-sup condition (3.18), problem (3.7) is equivalent to the reduced problem: Find $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{X}_h \times W_h$ such that:

$$
\begin{aligned}
\mathscr{A}_h^S(\boldsymbol{u}_h, \boldsymbol{v}_h) \,+\, \mathscr{C}_h^S(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \;&=\; \mathscr{D}^S(\theta_h, \boldsymbol{v}_h), \\
\mathscr{A}_h^T(\theta_h, \psi_h) \,+\, \mathscr{C}_h^T(\boldsymbol{u}_h; \theta_h, \psi_h) \;&=\; \mathscr{D}_{\theta_{\mathrm{D}}}^T(\psi_h)\,,
\end{aligned}
\tag{3.25}
$$

for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{X}_h \times W_h$. Consequently, in what follows we focus on analyzing problem (3.25).

We start by deriving the corresponding a priori estimates.

Theorem 3.1 *Let $(\boldsymbol{u}_h, \theta_h)$ be a solution to (3.25). Then, the following a priori estimates hold*

$$
\|\boldsymbol{u}_h\|_{1,\mathcal{T}_h} \,\leq\, \widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}) \qquad \text{and} \qquad \|\theta_h\|_{1,\mathcal{T}_h} \,\leq \widetilde{C}_2(\kappa, \theta_{\mathrm{D}})\,,
\tag{3.26}
$$

where

$$
\widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}) = c_1\,\nu^{-1}\,\kappa^{-1}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \quad \text{and} \quad \widetilde{C}_2(\kappa, \theta_{\mathrm{D}}) = c_2\,\kappa^{-1}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,,
\tag{3.27}
$$

with $c_1, c_2 > 0$, independent of h and the physical parameters.

Proof. Assuming that $(\boldsymbol{u}_h, \theta_h)$ is a solution to (3.25) and taking there $(\boldsymbol{v}_h, \psi_h) = (\boldsymbol{u}_h, \theta_h)$ yields

$$
\begin{aligned}
\mathscr{A}_h^S(\boldsymbol{u}_h, \boldsymbol{u}_h) \,+\, \mathscr{C}_h^S(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{u}_h) \;&=\; \mathscr{D}^S(\theta_h, \boldsymbol{u}_h), \\
\mathscr{A}_h^T(\theta_h, \psi_h) \,+\, \mathscr{C}_h^T(\boldsymbol{u}_h; \theta_h, \theta_h) \;&=\; \mathscr{D}_{\theta_{\mathrm{D}}}^T(\theta_h)\,.
\end{aligned}
$$

Invoking now the coercivity of the bilinear forms $\mathscr{A}_h^S$ and $\mathscr{A}_h^T$ (cf. (3.17)), the non-negativity property of the forms $\mathscr{C}_h^S$ and $\mathscr{C}_h^T$ (cf. (3.19)-(3.20)), and the continuity of $\mathscr{D}^S$ and $\mathscr{D}_{\theta_{\mathrm{D}}}^T$ (cf. (3.22)-(3.24)) we immediately obtain

$$
\begin{aligned}
\nu\,\widetilde{\alpha}_S \|\boldsymbol{u}_h\|_{1,\mathcal{T}_h}^2 \;&\leq\; \widetilde{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|\theta_h\|_{1,\mathcal{T}_h}\|\boldsymbol{u}_h\|_{1,\mathcal{T}_h}, \\
\kappa\,\widetilde{\alpha}_T \|\theta_h\|_{1,\mathcal{T}_h}^2 \;&\leq\; \widetilde{C}_{\mathscr{D}^T}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,\|\theta_h\|_{1,\mathcal{T}_h}\,.
\end{aligned}
$$

Then, estimates (3.26) follow after simplifying and using the bound obtained for $\|\theta_h\|_{1,\mathcal{T}_h}$ in the respective one for $\boldsymbol{u}_h$. $\square$

Remark 3.1 *Notice that, differently from (2.28), here the estimates for $\boldsymbol{u}_h$ and θ_h do not depend on any other parameter but the inverse of the viscosity and the thermal conductivity.*

In order to state existence of discrete solutions, similarly to the continuous case, we firstly define the discrete version of operator $\mathscr{L}$ defined in (2.42), namely,

$$
\begin{aligned}
\mathscr{L}_h: \qquad \boldsymbol{X}_h \times W_h \;&\longrightarrow\; \boldsymbol{X}_h \times W_h \\
(\boldsymbol{w}_h, \phi_h) \;&\longmapsto\; (\boldsymbol{u}_h, \theta_h) := (\mathscr{L}_{h,1}(\boldsymbol{w}_h, \phi_h), \mathscr{L}_{h,2}(\boldsymbol{w}_h, \phi_h))\,,
\end{aligned}
\tag{3.28}
$$

where, given $(\boldsymbol{w}_h, \phi_h) \in \boldsymbol{X}_h \times W_h$, $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{X}_h \times W_h$ is the unique solution (to be verified next in Lemma 3.2) of problem: Find $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{X}_h \times W_h$, such that

$$
\begin{aligned}
\mathscr{A}_h^S(\boldsymbol{u}_h, \boldsymbol{v}_h) \,+\, \mathscr{C}_h^S(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \;&=\; \mathscr{D}^S(\phi_h, \boldsymbol{v}_h)\,, \\
\mathscr{A}_h^T(\theta_h, \psi_h) \,+\, \mathscr{C}_h^T(\boldsymbol{w}_h; \theta_h, \psi_h) \;&=\; \mathscr{D}_{\theta_{\mathrm{D}}}^T(\psi_h)\,,
\end{aligned}
\tag{3.29}
$$

for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{X}_h \times W_h$.

According to the above, it is clear that $(\boldsymbol{u}_h, \theta_h)$ is a solution to (3.25), if and only if $(\boldsymbol{u}_h, \theta_h)$ is a fixed-point of $\mathscr{L}_h$, namely,

$$\mathscr{L}_h(\boldsymbol{u}_h, \theta_h) = (\boldsymbol{u}_h, \theta_h).$$

Then in what follows we prove the existence of a fixed-point of $\mathscr{L}_h$ by means of the well-known Brouwer's fixed-point theorem given in the following form (see e.g. [8]):

Let B be a nonempty compact convex subset of a finite-dimensional normed space, and let L be a continuous mapping of B into itself. Then L has a fixed point in B. Before doing that, in the following result we establish that operator $\mathscr{L}_h$ is well defined.

Lemma 3.2 *Operator $\mathscr{L}_h$ (cf. (3.28)) is well-defined, or equivalently, for all $(\boldsymbol{w}_h, \phi_h) \in \boldsymbol{X}_h \times W_h$, there exists a unique $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{X}_h \times W_h$ solution to (3.29). Moreover, the pairs $(\boldsymbol{w}_h, \phi_h)$ and $(\boldsymbol{u}_h, \theta_h) = \mathscr{L}_h(\boldsymbol{w}_h, \phi_h)$, satisfy*

$$\|\boldsymbol{u}_h\|_{1,\mathcal{T}_h} \leq \widetilde{\alpha}_S^{-1} \widetilde{C}_{\mathscr{D}^S}\, \nu^{-1}\, \|\boldsymbol{g}\|_{0,\Omega}\, \|\phi_h\|_{1,\mathcal{T}_h} \quad \text{and} \quad \|\theta_h\|_{1,\mathcal{T}_h} \leq \widetilde{C}_2(\kappa, \theta_{\mathrm{D}})\, \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}, \tag{3.30}$$

with $\widetilde{C}_2(\theta_{\mathrm{D}})$ defined in (3.27).

Proof. Given $(\boldsymbol{w}_h, \phi_h) \in \boldsymbol{X}_h \times W_h$, using the ellipticity of $\mathscr{A}_h^S$ and $\mathscr{A}_h^T$ (cf. (3.17)) and estimates (3.19) and (3.20), we deduce that

$$\nu\, \widetilde{\alpha}_S \|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}^2 \leq \mathscr{A}_h^S(\boldsymbol{v}_h, \boldsymbol{v}_h) + \mathscr{C}_h^S(\boldsymbol{w}_h; \boldsymbol{v}_h, \boldsymbol{v}_h) \quad \text{and} \quad \kappa\, \widetilde{\alpha}_T \|\psi_h\|_{1,\mathcal{T}_h}^2 \leq \mathscr{A}_h^T(\psi_h, \psi_h) + \mathscr{C}_h^T(\boldsymbol{w}_h; \psi_h, \psi_h), \tag{3.31}$$

for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{X}_h \times W_h$, that is, $\mathscr{A}_h^S(\cdot,\cdot) + \mathscr{C}_h^S(\boldsymbol{w}_h;\cdot,\cdot)$ and $\mathscr{A}_h^T(\cdot,\cdot) + \mathscr{C}_h^T(\boldsymbol{w}_h;\cdot,\cdot)$ are elliptic and bounded bilinear forms on $\boldsymbol{X}_h \times W_h$, where the boundedness follows immediately from estimates (3.10), (3.12), (3.15) and (3.16). Then, since $\mathscr{D}^S(\phi_h, \cdot)$ and $\mathscr{D}_{\theta_{\mathrm{D}}}^T(\cdot)$ are bounded and linear functionals on $\boldsymbol{X}_h$ and W_h, respectively, an straightforward application of the Lax-Milgram Lemma [20, Lemma 1.4] to the uncoupled problem (3.29) yields the well-definedness of $\mathscr{L}_h$.

Finally, we observe that (3.30) follows straightforwardly from (3.29), (3.31), (3.22) and (3.24), which concludes the proof. □

Now, to apply the Brouwer's fixed point theorem in our context, we define the compact and convex set $\boldsymbol{B}_h \subset \boldsymbol{X}_h \times W_h$, defined by

$$\boldsymbol{B}_h := \Big\{ (\boldsymbol{w}_h, \phi_h) \in \boldsymbol{X}_{\boldsymbol{h}} \times W_h : \|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} \leq \widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}) \quad \text{and} \quad \|\phi_h\|_{1,\mathcal{T}_h} \leq \widetilde{C}_2(\kappa, \theta_{\mathrm{D}}) \Big\} \tag{3.32}$$

where $\widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g})$ and $\widetilde{C}_2(\kappa, \theta_{\mathrm{D}})$ are the constants given in (3.27).

It is not difficult to see from (3.31), that $\mathscr{L}(\boldsymbol{B}_h) \subseteq \boldsymbol{B}_h$. In fact, given $(\boldsymbol{w}_h, \phi_h) \in \boldsymbol{B}_h$, we let $(\boldsymbol{v}_h, \psi_h) = \mathscr{L}(\boldsymbol{w}_h, \phi_h)$ and observe from (3.29) and (3.31), that there hold

$$\nu\, \widetilde{\alpha}_S \|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}^2 \leq \mathscr{D}^S(\phi_h, \boldsymbol{v}_h) \quad \text{and} \quad \kappa\, \widetilde{\alpha}_T \|\psi_h\|_{1,\mathcal{T}_h}^2 \leq \mathscr{D}_{\theta_{\mathrm{D}}}^T(\psi_h),$$

Then, from the continuity of $\mathscr{D}^S$ and $\mathscr{D}_{\theta_{\mathrm{D}}}^T$ (cf. (3.22) and (3.24)), the latter yields

$$\nu\, \widetilde{\alpha}_S \|\boldsymbol{v}_h\|_{1,\mathcal{T}_h} \leq \widetilde{C}_{\mathscr{D}^S}\, \|\boldsymbol{g}\|_{0,\Omega}\, \|\phi_h\|_{1,\mathcal{T}_h}, \quad \text{and} \quad \kappa\, \widetilde{\alpha}_T \|\psi_h\|_{1,\mathcal{T}_h} \leq \widetilde{C}_{\mathscr{D}^T}\, \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}},$$

which together with the fact that $\|\phi_h\|_{1,\mathcal{T}_h} \leq \widetilde{C}_2(\kappa, \theta_{\mathrm{D}})$, implies that $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{B}_h$.

Now we state the Lipschitz continuity of the operator $\mathscr{L}_h$ on $\boldsymbol{B}_h$.

Lemma 3.3 *The operator $\mathscr{L}_h$ defined in* (3.28) *is Lipschitz continuous on $\boldsymbol{B}_h$, that is*

$$\|\mathscr{L}_h(\boldsymbol{w}_h,\phi_h)-\mathscr{L}_h(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\|_{1,\mathcal{T}_h} \le \widetilde{C}_{\mathrm{LIP}}\,\|(\boldsymbol{w}_h,\phi_h)\,-\,(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\|_{1,\mathcal{T}_h}, \tag{3.33}$$

for all $(\boldsymbol{w}_h,\phi_h)\,,(\widetilde{\boldsymbol{w}_h},\widetilde{\phi}_h)\in\boldsymbol{B}_h$, where $\widetilde{C}_{\mathrm{LIP}}>0$, independent of the meshsize, is defined in (3.34).

Proof. Let $(\boldsymbol{w}_h,\phi_h),(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h),(\boldsymbol{u}_h,\theta_h),(\widetilde{\boldsymbol{u}}_h,\widetilde{\theta}_h)\in\boldsymbol{B}_h$ be such that

$$(\boldsymbol{u}_h,\theta_h)\,=\,\mathscr{L}_h(\boldsymbol{w}_h,\phi_h)\quad\text{and}\quad(\widetilde{\boldsymbol{u}}_h,\widetilde{\theta}_h)\,=\,\mathscr{L}_h(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\,.$$

According to the definition of $\mathscr{L}_h$ (cf. (3.28)), it follows that

$$\begin{aligned}\mathscr{A}_h^S(\boldsymbol{u}_h,\boldsymbol{v}_h)\,+\,\mathscr{C}_h^S(\boldsymbol{w}_h;\boldsymbol{u}_h,\boldsymbol{v}_h)&=\mathscr{D}^S(\phi_h,\boldsymbol{v}_h)\,,\\ \mathscr{A}_h^T(\theta_h,\psi_h)\,+\,\mathscr{C}_h^T(\boldsymbol{w}_h;\theta_h,\psi_h)&=\mathscr{D}_{\theta_{\mathrm{D}}}^T(\psi_h)\,,\end{aligned}$$

and

$$\begin{aligned}\mathscr{A}_h^S(\widetilde{\boldsymbol{u}}_h,\boldsymbol{v}_h)\,+\,\mathscr{C}_h^S(\widetilde{\boldsymbol{w}}_h;\widetilde{\boldsymbol{u}}_h,\boldsymbol{v}_h)&=\mathscr{D}^S(\phi_h,\boldsymbol{v}_h)\,,\\ \mathscr{A}_h^T(\widetilde{\theta}_h,\psi_h)\,+\,\mathscr{C}_h^T(\widetilde{\boldsymbol{w}}_h;\widetilde{\theta}_h,\psi_h)&=\mathscr{D}_{\theta_{\mathrm{D}}}^T(\psi_h)\,,\end{aligned}$$

for all $(\boldsymbol{v}_h,\psi_h)\in\boldsymbol{X}_h\times W_h$. Then, by subtracting these systems of equations and manipulating the resulting expression, we find that

$$\mathscr{A}_h^S(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}_h,\boldsymbol{v}_h)\,+\,\mathscr{C}_h^S(\boldsymbol{w}_h;\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}_h,\boldsymbol{v}_h)=-\mathscr{C}_h^S(\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h;\widetilde{\boldsymbol{u}}_h,\boldsymbol{v}_h)+\mathscr{D}^S(\phi_h-\widetilde{\phi}_h,\boldsymbol{v}_h)\,,$$

and

$$\mathscr{A}_h^T(\theta_h-\widetilde{\theta}_h,\psi_h)\,+\,\mathscr{C}_h^T(\boldsymbol{w}_h;\theta_h-\widetilde{\theta}_h,\psi_h)=-\mathscr{C}_h^T(\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h;\widetilde{\theta}_h,\psi_h)\,,$$

for all $(\boldsymbol{v}_h,\psi_h)\in\boldsymbol{X}_h\times W_h$. Then, using (3.31), (3.15), (3.22) and (3.16), from the previous identities with $(\boldsymbol{v}_h,\psi_h)=(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}_h,\theta_h-\widetilde{\theta}_h)$, we obtain

$$\nu\,\widetilde{\alpha}_S\|\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}_h\|_{1,\mathcal{T}_h}\le\widetilde{C}_{\mathscr{C}^S}\,\|\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h\|_{1,\mathcal{T}_h}\,\|\widetilde{\boldsymbol{u}}_h\|_{1,\mathcal{T}_h}\,+\widetilde{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|\phi_h-\widetilde{\phi}_h\|_{1,\mathcal{T}_h}$$

and

$$\kappa\,\widetilde{\alpha}_T\|\theta_h-\widetilde{\theta}_h\|_{1,\mathcal{T}_h}\le\widetilde{C}_{\mathscr{C}^T}\,\|\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h\|_{1,\mathcal{T}_h}\,\|\widetilde{\theta}_h\|_{1,\mathcal{T}_h}.$$

But, since $(\widetilde{\boldsymbol{u}}_h,\widetilde{\theta}_h)\in\boldsymbol{B}_h$, which means that $\|\widetilde{\boldsymbol{u}}_h\|_{1,\mathcal{T}_h}\le\widetilde{C}_1(\nu,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})$ and $\|\widetilde{\theta}_h\|_{1,\mathcal{T}_h}\le\widetilde{C}_2(\kappa,\theta_{\mathrm{D}})$, from the previous inequalities, we obtain

$$\nu\,\widetilde{\alpha}_S\|\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}_h\|_{1,\mathcal{T}_h}\le\widetilde{C}_{\mathscr{C}^S}\,\widetilde{C}_1(\nu,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,\|\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h\|_{1,\mathcal{T}_h}+\widetilde{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|\phi_h-\widetilde{\phi}_h\|_{1,\mathcal{T}_h}$$

and

$$\kappa\,\widetilde{\alpha}_T\|\theta_h-\widetilde{\theta}_h\|_{1,\mathcal{T}_h}\le\widetilde{C}_{\mathscr{C}^T}\,\widetilde{C}_2(\kappa,\theta_{\mathrm{D}})\,\|\boldsymbol{w}_h-\widetilde{\boldsymbol{w}}_h\|_{1,\mathcal{T}_h},$$

which imply (3.33), with $\widetilde{C}_{\mathrm{LIP}}>0$, given by

$$\widetilde{C}_{\mathrm{LIP}}:=C\max\left\{\nu^{-1}\widetilde{C}_1(\nu,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\,+\,\kappa^{-1}\widetilde{C}_2(\kappa,\theta_{\mathrm{D}}),\nu^{-1}\|\boldsymbol{g}\|_{0,\Omega}\right\}, \tag{3.34}$$

with $C>0$, independent of h and the physical parameters.

□

Now we are in position of establishing the existence result.

Theorem 3.4 *There exists at least one* $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{B}_h$*, solution to* (3.25)*. Furthermore, there exists* $p_h \in Q_h$ *so that* $(\boldsymbol{u}_h, p_h, \theta_h) \in \mathbf{V}_h \times Q_h \times W_h$ *is a solution to* (3.7)*, with* p_h *satisfying the estimate*

$$\|p_h\|_{0,\Omega} \leq \widetilde{C}_3(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}), \tag{3.35}$$

with

$$\widetilde{C}_3(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}) := \widetilde{\beta}^{-1}\widetilde{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}\widetilde{C}_2(\kappa, \theta_{\mathrm{D}}) + \widetilde{\beta}^{-1}\nu\,\widetilde{C}_{\mathscr{A}^S}\widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}) \,+\, \widetilde{\beta}^{-1}\widetilde{C}_{\mathscr{C}^S}\widetilde{C}_1^2(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}),$$

with $\widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g})$ *and* $\widetilde{C}_2(\kappa, \theta_{\mathrm{D}})$ *the parameter-dependent constants defined in* (3.27)*.*

Proof. Recalling that operator $\mathscr{L}_h$ is Lipschitz continuous on $\boldsymbol{B}_h$ and satisfies $\mathscr{L}(\boldsymbol{B}_h) \subseteq \boldsymbol{B}_h$, the existence of $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{B}_h$ solution to (3.25) is a direct consequence of the Brouwer's fixed-point theorem. In turn, similarly to the continuous case, the existence of a discrete pressure is a direct consequence of the inf-sup compatibility condition (3.18) (see [36, Lemma 3.6], for further details).

Finally, to derive estimate (3.35) we employ the inf-sup condition (3.18), the first equation of (3.7), estimates (3.10), (3.15), (3.22), and the fact that $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{B}_h$ (cf. (3.32)), so that the estimates in (3.26) hold, to obtain

$$\begin{aligned}
\widetilde{\beta}\,\|p_h\|_{0,\Omega} &\leq \sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h\\ \boldsymbol{v}_h \neq \boldsymbol{0}}} \frac{\mathscr{B}^S(\boldsymbol{v}_h, p_h)}{\|v_h\|_{1,\mathcal{T}_h}} = \sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h\\ \boldsymbol{v}_h \neq \boldsymbol{0}}} \frac{\mathscr{D}^S(\theta_h, \boldsymbol{v}_h) \,-\, \mathscr{A}_h^S(\boldsymbol{u}_h, \boldsymbol{v}_h) \,-\, \mathscr{C}_h^S(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)}{\|v_h\|_{1,\mathcal{T}_h}},\\
&\leq \widetilde{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}\,\|\theta_h\|_{1,\mathcal{T}_h} + \nu\,\widetilde{C}_{\mathscr{A}^S}\,\|\boldsymbol{u}_h\|_{1,\mathcal{T}_h} \,+\, \widetilde{C}_{\mathscr{C}^S}\|\boldsymbol{u}_h\|_{1,\mathcal{T}_h}^2,\\
&\leq \widetilde{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}\widetilde{C}_2(\kappa, \theta_{\mathrm{D}}) + \nu\,\widetilde{C}_{\mathscr{A}^S}\widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}) \,+\, \widetilde{C}_{\mathscr{C}^S}\widetilde{C}_1^2(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}),
\end{aligned}$$

which concludes the proof. □

Now, as in the continuous case, we are further able to derive the uniqueness result as a straightforward consequence of the Lipschitz continuity property of operator $\mathscr{L}_h$. In fact, if $(\boldsymbol{u}_h, \theta_h)$ and $(\widetilde{\boldsymbol{u}}_h, \widetilde{\theta}_h)$ are both two different solutions to problem (3.25), then they satisfy

$$(\boldsymbol{u}_h, \theta_h) = \mathscr{L}_h(\boldsymbol{u}_h, \theta_h) \quad \text{and} \quad (\widetilde{\boldsymbol{u}}_h, \widetilde{\theta}_h) = \mathscr{L}_h(\widetilde{\boldsymbol{u}}_h, \widetilde{\theta}_h)\,,$$

Then, from Lemma 3.3 we obtain

$$\|(\boldsymbol{u}_h, \theta_h) - (\widetilde{\boldsymbol{u}}_h, \widetilde{\theta})_h\|_{1,\mathcal{T}_h} = \|\mathscr{L}_h(\boldsymbol{u}_h, \theta_h) - \mathscr{L}_h(\widetilde{\boldsymbol{u}}_h, \widetilde{\theta}_h)\|_{1,\mathcal{T}_h} \leq \widetilde{C}_{\mathrm{LIP}}\|(\boldsymbol{u}_h, \theta_h) - (\widetilde{\boldsymbol{u}}_h, \widetilde{\theta}_h)\|_{1,\mathcal{T}_h}\,,$$

with $\widetilde{C}_{\mathrm{LIP}}$ being the constant defined in (3.34) that explicitly depends on data. Therefore, by assuming that $\widetilde{C}_{\mathrm{LIP}} < 1$ we readily obtain the desired uniqueness result. More precisely, we have the following Theorem whose proof is omitted since it is analogous to the proof of Theorem 2.8.

Theorem 3.5 *Assume that the data is small enough so that the constant* $\widetilde{C}_{\mathrm{LIP}}$ *(see* (3.34)*) satisfies* $\widetilde{C}_{\mathrm{LIP}} < 1$*. Then there exists a unique solution* $(\boldsymbol{u}_h, p_h, \theta_h) \in \mathbf{V}_h \times Q_h \times W_h$ *to* (3.7)*.*

4 A priori error analysis

In this section we proceed to derive a priori error estimates for the numerical scheme introduced and analyzed in Section 3. To do that, we assume that the hypotheses of Theorems 2.8 and 3.5 hold, and let $(\boldsymbol{u}, p, \theta)$ and $(\boldsymbol{u}_h, p_h, \theta_h)$ be the unique solutions to problems (2.5) and (3.7), respectively and assume that the exact solution satisfies the following additional regularity:

$$\boldsymbol{u} \in [H^{k+1}(\Omega)]^d \cap [H^1_0(\Omega)]^d \cap \boldsymbol{X}, \quad p \in H^k(\Omega) \cap L^2_0(\Omega) \quad \text{and} \quad \theta \in H^{k+1}(\Omega), \quad \text{for } k \geq 1.$$

To derive the theoretical rate of convergence of our scheme, we let Π_h^{BDM} be the BDM interpolation operator from $[H^{k+1}(\Omega)]^d$ into $\mathbf{V}_h$, and for $l \geq 0$, we denote by $\mathcal{P}^l$ and $\mathcal{P}^l_0$ the L^2-projections into S^l_h (cf. (3.6)) and $S^l_h \cap L^2_0(\Omega)$, respectively. This operators satisfy the following approximation properties (see [9, 20]):

$$\begin{aligned} \|\boldsymbol{u} - \Pi_h^{\mathrm{BDM}} u\|_{2,\mathcal{T}_h} &\leq C\, h^k \|\boldsymbol{u}\|_{k+1,\Omega}\,, \\ \|p - \mathcal{P}_0^{k-1}(p)\|_{0,\Omega} &\leq C\, h^k \|p\|_{k,\Omega}\,, \\ \|\theta - \mathcal{P}^k(\theta)\|_{2,\mathcal{T}_h} &\leq C\, h^k \|\theta\|_{k+1,\Omega}\,. \end{aligned} \tag{4.1}$$

Then, denoting by

$$\begin{aligned} &\boldsymbol{e}_{\boldsymbol{u}} := \boldsymbol{u} - \boldsymbol{u}_h\,, && \boldsymbol{\xi}_{\boldsymbol{u}} := \boldsymbol{u} - \Pi_h^{\mathrm{BDM}}(\boldsymbol{u}), && \boldsymbol{\chi}_{\boldsymbol{u}} := \Pi_h^{\mathrm{BDM}}(\boldsymbol{u}) - \boldsymbol{u}_h, \\ &\boldsymbol{e}_p := p - p_h, && \boldsymbol{\xi}_p := p - \mathcal{P}_0^{k-1}(p), && \boldsymbol{\chi}_p := \mathcal{P}_0^{k-1}(p) - p_h, \\ &\boldsymbol{e}_\theta := \theta - \theta_h, && \boldsymbol{\xi}_\theta := \theta - \mathcal{P}^k(\theta), && \boldsymbol{\chi}_\theta := \mathcal{P}^k(\theta) - \theta_h, \end{aligned} \tag{4.2}$$

and observing that

$$\boldsymbol{e}_{\boldsymbol{u}} := \boldsymbol{\xi}_{\boldsymbol{u}} + \boldsymbol{\chi}_{\boldsymbol{u}}\,, \qquad \boldsymbol{e}_p = \boldsymbol{\xi}_p + \boldsymbol{\chi}_p\,, \qquad \boldsymbol{e}_\theta = \boldsymbol{\xi}_\theta + \boldsymbol{\chi}_\theta\,, \tag{4.3}$$

from the triangle inequality, the approximation properties (4.1) and the inverse inequality (3.3), we have that

$$\begin{aligned} \|\boldsymbol{e}_{\boldsymbol{u}}\|_{2,\mathcal{T}_h} &\leq \|\boldsymbol{\xi}_{\boldsymbol{u}}\|_{2,\mathcal{T}_h} + \|\boldsymbol{\chi}_{\boldsymbol{u}}\|_{2,\mathcal{T}_h} \leq Ch^k\|\boldsymbol{u}\|_{k+1,\Omega} + C\|\boldsymbol{\chi}_{\boldsymbol{u}}\|_{1,\mathcal{T}_h}\,, \\ \|\boldsymbol{e}_p\|_{0,\Omega} &\leq \|\boldsymbol{\xi}_p\|_{0,\Omega} + \|\boldsymbol{\chi}_p\|_{0,\Omega} \leq Ch^k\|p\|_{k,\Omega} + \|\boldsymbol{\chi}_p\|_{0,\Omega}\,, \\ \|\boldsymbol{e}_\theta\|_{2,\mathcal{T}_h} &\leq \|\boldsymbol{\xi}_\theta\|_{2,\mathcal{T}_h} + \|\boldsymbol{\chi}_\theta\|_{2,\mathcal{T}_h} \leq Ch^k\|\theta\|_{k+1,\Omega} + C\|\boldsymbol{\chi}_\theta\|_{1,\mathcal{T}_h}\,, \end{aligned} \tag{4.4}$$

which means that the estimation of the individual errors $\boldsymbol{e}_{\boldsymbol{u}}$, $\boldsymbol{e}_p$ and $\boldsymbol{e}_\theta$ is reduced to estimate the discrete individual errors $\boldsymbol{\chi}_{\boldsymbol{u}}$, $\boldsymbol{\chi}_p$ and $\boldsymbol{\chi}_\theta$.

With these ingredients at hand we are now in position to state and prove the main result of this section.

Theorem 4.1 *Assume that the hypotheses of Theorems 2.8 and 3.5 hold and let $(\boldsymbol{u}, p, \theta)$ and $(\boldsymbol{u}_h, p_h, \theta_h)$ be the unique solutions to problems* (2.5) *and* (3.7)*, respectively, and suppose that $\boldsymbol{u} \in [H^{k+1}(\Omega)]^d \cap [H^1_0(\Omega)]^d \cap \boldsymbol{X}$, $p \in H^k(\Omega) \cap L^2_0(\Omega)$ and $\theta \in H^{k+1}(\Omega)$ for $k \geq 1$. Assume further that*

$$\widetilde{C}_4(\delta, \kappa, \nu, \boldsymbol{g}, \theta_{\mathrm{D}}) := \widetilde{C}_{S,\mathrm{LIP}} C_1(\delta, \nu, \boldsymbol{g}, \theta_{\mathrm{D}}) + \frac{\widetilde{C}_{T,\mathrm{LIP}} C_2(\delta, \theta_{\mathrm{D}}) \widetilde{C}_{\mathscr{D}^S}\, \|\boldsymbol{g}\|_{0,\Omega}}{\kappa\, \widetilde{\alpha}_T} \leq \frac{1}{2}\nu \widetilde{\alpha}_S, \tag{4.5}$$

with $C_1(\delta, \nu, \boldsymbol{g}, \theta_{\mathrm{D}})$ and $C_2(\delta, \theta_{\mathrm{D}})$ being the constants defined in (2.29)*. Then, there exists a constant $C > 0$, independent of h, such that*

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{2,\mathcal{T}_h} + \|\theta - \theta_h\|_{2,\mathcal{T}_h} &\leq C\, h^k \left(\|\boldsymbol{u}\|_{k+1,\Omega} + \|\theta\|_{k+1,\Omega} \right), \\ \|p - p_h\|_{0,\Omega} &\leq C\, h^k \left(\|p\|_{k,\Omega} + \|\boldsymbol{u}\|_{k+1,\Omega} + \|\theta\|_{k+1,\Omega} \right). \end{aligned} \tag{4.6}$$

Proof. We start by noticing that, since $\boldsymbol{u} \in [H^{k+1}(\Omega)]^d \cap [H_0^1(\Omega)]^d \cap \boldsymbol{X}$ and $\theta \in H^{k+1}(\Omega)$, then the following Galerking orthogonality properties hold:

$$\mathscr{A}_h^S(\boldsymbol{e_u}, \boldsymbol{v}_h) + \mathscr{C}_h^S(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) - \mathscr{C}_h^S(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - \mathscr{B}^S(\boldsymbol{v}_h, \boldsymbol{e}_p) - \mathscr{D}^S(\boldsymbol{e}_\theta, \boldsymbol{v}_h) = 0 \tag{4.7}$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\mathscr{B}^S(\boldsymbol{e_u}, q_h) = 0,$$

for all $q_h \in Q_h$, and

$$\mathscr{A}_h^T(\boldsymbol{e}_\theta, \psi_h) + \mathscr{C}_h^T(\boldsymbol{u}; \theta, \psi_h) - \mathscr{C}_h^T(\boldsymbol{u}_h; \theta_h, \psi_h) = 0, \tag{4.8}$$

for all $\psi_h \in W_h$. In particular, from (4.7), noticing that $\mathbf{X}_h \subseteq \mathbf{X}$, and applying some suitable algebraic manipulations, we obtain

$$\mathscr{A}_h^S(\boldsymbol{e_u}, \boldsymbol{v}_h) + \mathscr{C}_h^S(\boldsymbol{e_u}; \boldsymbol{u}, \boldsymbol{v}_h) + \mathscr{C}_h^S(\boldsymbol{u}_h; \boldsymbol{e_u}, \boldsymbol{v}_h) - \mathscr{D}^S(\boldsymbol{e}_\theta, \boldsymbol{v}_h) = 0,$$

for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$, which together with (4.3), implies

$$\begin{aligned}\mathscr{A}_h^S(\boldsymbol{\chi_u}, \boldsymbol{v}_h) + \mathscr{C}_h^S(\boldsymbol{u}_h; \boldsymbol{\chi_u}, \boldsymbol{v}_h) = \;& \mathscr{D}^S(\boldsymbol{e}_\theta, \boldsymbol{v}_h) - \mathscr{A}_h^S(\boldsymbol{\xi_u}, \boldsymbol{v}_h) - \mathscr{C}_h^S(\boldsymbol{u}_h; \boldsymbol{\xi_u}, \boldsymbol{v}_h)\\ & - \mathscr{C}_h^S(\boldsymbol{\xi_u}; \boldsymbol{u}, \boldsymbol{v}_h) - \mathscr{C}_h^S(\boldsymbol{\chi_u}; \boldsymbol{u}, \boldsymbol{v}_h),\end{aligned}$$

for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$. Then, taking $\boldsymbol{v}_h = \boldsymbol{\chi_u} \in \boldsymbol{X}_h$ in the latter identity and employing (3.11), the first estimate in (3.21), (3.22) and the first estimate in (3.31), we obtain

$$\begin{aligned}\nu\, \widetilde{\alpha}_S \|\boldsymbol{\chi_u}\|_{1,\mathcal{T}_h} \leq \;& \widetilde{C}_{\mathscr{D}^S} \|\boldsymbol{g}\|_{0,\Omega} \|\boldsymbol{\xi}_\theta\|_{1,\mathcal{T}_h} + \widetilde{C}_{\mathscr{D}^S} \|\boldsymbol{g}\|_{0,\Omega} \|\boldsymbol{\chi}_\theta\|_{1,\mathcal{T}_h} + \nu\, \widehat{C}_{\mathscr{A}^S} \|\boldsymbol{\xi_u}\|_{2,\mathcal{T}_h}\\ & + \widetilde{C}_{S,\mathrm{LIP}} \|\boldsymbol{u}_h\|_{1,\mathcal{T}_h} \|\boldsymbol{\xi_u}\|_{1,\mathcal{T}_h} + \widetilde{C}_{S,\mathrm{LIP}} \|\boldsymbol{\xi_u}\|_{1,\mathcal{T}_h} \|\boldsymbol{u}\|_{1,\mathcal{T}_h} + \widetilde{C}_{S,\mathrm{LIP}} \|\boldsymbol{\chi_u}\|_{1,\mathcal{T}_h} \|\boldsymbol{u}\|_{1,\mathcal{T}_h},\end{aligned}$$

and recalling that $\boldsymbol{u}$ and $\boldsymbol{u}_h$ satisfy the a priori estimates (cf. (2.28) and (3.26), respectively)

$$\|\boldsymbol{u}\|_{1,\Omega} \leq C_1(\delta, \nu, \boldsymbol{g}, \theta_{\mathrm{D}}) \quad \text{and} \quad \|\boldsymbol{u}_h\|_{1,\mathcal{T}_h} \leq \widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}),$$

we readily obtain

$$(\nu\, \widetilde{\alpha}_S - \widetilde{C}_{S,\mathrm{LIP}} C_1(\delta, \nu, \boldsymbol{g}, \theta_{\mathrm{D}})) \|\boldsymbol{\chi_u}\|_{1,\mathcal{T}_h} \leq \widetilde{C}_{\mathscr{D}^S} \|\boldsymbol{g}\|_{0,\Omega} \|\boldsymbol{\chi}_\theta\|_{1,\mathcal{T}_h} + K_1(\boldsymbol{\xi}_\theta, \boldsymbol{\xi_u}) \tag{4.9}$$

with

$$\begin{aligned}K_1(\boldsymbol{\xi}_\theta, \boldsymbol{\xi_u}) := \;& \widetilde{C}_{\mathscr{D}^S} \|\boldsymbol{g}\|_{0,\Omega} \|\boldsymbol{\xi}_\theta\|_{1,\mathcal{T}_h} + \nu\, \widehat{C}_{\mathscr{A}^S} \|\boldsymbol{\xi_u}\|_{2,\mathcal{T}_h}\\ & + \widetilde{C}_{S,\mathrm{LIP}} \widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}) \|\boldsymbol{\xi_u}\|_{1,\mathcal{T}_h} + \widetilde{C}_{S,\mathrm{LIP}} C_1(\delta, \nu, \boldsymbol{g}, \theta_{\mathrm{D}}) \|\boldsymbol{\xi_u}\|_{1,\mathcal{T}_h}.\end{aligned}$$

Similarly, from (4.8) with $\psi_h = \boldsymbol{\chi}_\theta$, combined with (4.2), the second estimate in (3.31), (3.13), (3.21) and the bounds $\|\boldsymbol{u}_h\|_{1,\mathcal{T}_h} \leq \widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g})$ and $\|\theta\|_{1,\Omega} \leq C_2(\delta, \theta_{\mathrm{D}})$ (cf. (2.28) and (3.26)), we obtain

$$\kappa\, \widetilde{\alpha}_T \|\boldsymbol{\chi}_\theta\|_{1,\mathcal{T}_h} \leq \widetilde{C}_{T,\mathrm{LIP}} C_2(\delta, \theta_{\mathrm{D}}) \|\boldsymbol{\chi_u}\|_{1,\mathcal{T}_h} + K_2(\boldsymbol{\xi}_\theta, \boldsymbol{\xi_u}) \tag{4.10}$$

with

$$K_2(\boldsymbol{\xi}_\theta, \boldsymbol{\xi_u}) := \kappa\, \widehat{C}_{\mathscr{A}^T} \|\boldsymbol{\xi}_\theta\|_{2,\mathcal{T}_h} + \widetilde{C}_{T,\mathrm{LIP}} \widetilde{C}_1(\nu, \kappa, \theta_{\mathrm{D}}, \boldsymbol{g}) \|\boldsymbol{\xi}_\theta\|_{1,\mathcal{T}_h} + \widetilde{C}_{T,\mathrm{LIP}} C_2(\delta, \theta_{\mathrm{D}}) \|\boldsymbol{\xi_u}\|_{1,\mathcal{T}_h}.$$

Then, combining (4.9) with (4.10), we obtain

$$\left(\nu\, \widetilde{\alpha}_S - \widetilde{C}_4(\delta, \kappa, \nu, \boldsymbol{g}, \theta_{\mathrm{D}})\right) \|\boldsymbol{\chi_u}\|_{1,\mathcal{T}_h} \leq K_1(\boldsymbol{\xi}_\theta, \boldsymbol{\xi_u}) + \frac{\widetilde{C}_{\mathscr{D}^S} \|\boldsymbol{g}\|_{0,\Omega}}{\kappa\, \widetilde{\alpha}_T} K_2(\boldsymbol{\xi}_\theta, \boldsymbol{\xi_u})$$

which together with (4.5), implies

$$\frac{\nu\,\widetilde{\alpha}_S}{2}\|\boldsymbol{\chi}_{\boldsymbol{u}}\|_{1,\mathcal{T}_h} \;\le\; K_1(\boldsymbol{\xi}_\theta,\boldsymbol{\xi}_{\boldsymbol{u}}) \;+\; \frac{\widetilde{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}}{\kappa\,\widetilde{\alpha}_T}K_2(\boldsymbol{\xi}_\theta,\boldsymbol{\xi}_{\boldsymbol{u}}). \tag{4.11}$$

Finally, employing (4.11) in (4.10) we deduce that

$$\kappa\,\widetilde{\alpha}_T\|\boldsymbol{\chi}_\theta\|_{1,\mathcal{T}_h} \;\le\; \frac{2\widetilde{C}_{T,\mathrm{LIP}}C_2(\delta,\theta_{\mathrm{D}})}{\nu\,\widetilde{\alpha}_S}K_1(\boldsymbol{\xi}_\theta,\boldsymbol{\xi}_{\boldsymbol{u}}) \;+\; \left(\frac{2\widetilde{C}_{T,\mathrm{LIP}}C_2(\delta,\theta_{\mathrm{D}})\widetilde{C}_{\mathscr{D}^S}\,\|\boldsymbol{g}\|_{0,\Omega}}{\nu\,\widetilde{\alpha}_S\kappa\,\widetilde{\alpha}_T}+1\right)K_2(\boldsymbol{\xi}_\theta,\boldsymbol{\xi}_{\boldsymbol{u}}). \tag{4.12}$$

In this way, from (4.4), (4.11), (4.12) we easily obtain the first estimate in (4.6).

Now, for the second estimate in (4.4), we first note from the discrete inf-sup condition (3.18) with $q_h = \boldsymbol{e}_p^h$, using then that $\boldsymbol{e}_p^h = \boldsymbol{e}_p^{\Pi} - \boldsymbol{e}_p$ and subsequently employing the bound (3.14) for $\mathscr{B}^S$, we find

$$\begin{aligned}\widetilde{\beta}\,\|\boldsymbol{\chi}_p\|_{0,\Omega} \le \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h\\ \boldsymbol{v}_h\neq\boldsymbol{0}}}\frac{\mathscr{B}^S(\boldsymbol{v}_h,\boldsymbol{\chi}_p)}{\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}} &\le \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h\\ \boldsymbol{v}_h\neq\boldsymbol{0}}}\frac{\mathscr{B}^S(\boldsymbol{v}_h,\boldsymbol{\xi}_p)}{\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}} + \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h\\ \boldsymbol{v}_h\neq\boldsymbol{0}}}\frac{\mathscr{B}^S(\boldsymbol{v}_h,-\boldsymbol{e}_p)}{\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}}\\ &\le \widetilde{C}_{\mathscr{B}^S}\,\|\boldsymbol{\xi}_p\|_{0,\Omega} + \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h\\ \boldsymbol{v}_h\neq\boldsymbol{0}}}\frac{\mathscr{B}^S(\boldsymbol{v}_h,-\boldsymbol{e}_p)}{\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}}\,.\end{aligned} \tag{4.13}$$

Now, to handle the second term at the right-hand side of the latter expression, we use the Galerkin orthogonality relation (4.7) and after adding and subtracting $\mathscr{C}_h^S(\boldsymbol{u}_h;\boldsymbol{u},\boldsymbol{v}_h)$ it follows that

$$\begin{aligned}\mathscr{B}^S(\boldsymbol{v}_h,-\boldsymbol{e}_p) &= -\mathscr{A}_h^S(\boldsymbol{e}_{\boldsymbol{u}},\boldsymbol{v}_h) - \mathscr{C}_h^S(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h) + \mathscr{C}_h^S(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h) + \mathscr{D}^S(\boldsymbol{e}_\theta,\boldsymbol{v}_h)\\ &= -\mathscr{A}_h^S(\boldsymbol{e}_{\boldsymbol{u}},\boldsymbol{v}_h) - \mathscr{C}_h^S(\boldsymbol{e}_{\boldsymbol{u}};\boldsymbol{u},\boldsymbol{v}_h) - \mathscr{C}_h^S(\boldsymbol{u}_h;\boldsymbol{e}_{\boldsymbol{u}},\boldsymbol{v}_h) + \mathscr{D}^S(\boldsymbol{e}_\theta,\boldsymbol{v}_h)\,.\end{aligned}$$

In this way, a straightforward application of the estimates (3.11), (3.21) and (3.23), and the fact that $\|\cdot\|_{1,\mathcal{T}_h} \le \|\cdot\|_{2,\mathcal{T}_h}$ in $H^2(\mathcal{T}_h)$, yield

$$\begin{aligned}|\mathscr{B}^S(\boldsymbol{v}_h,-\boldsymbol{e}_p)| \;\le\; &(\widehat{C}_{\mathscr{A}^S} + \widetilde{C}_{S,\mathrm{LIP}}\,\|\boldsymbol{u}\|_{1,\mathcal{T}_h} + \widetilde{C}_{S,\mathrm{LIP}}\,\|\boldsymbol{u}_h\|_{1,\mathcal{T}_h})\,\|\boldsymbol{e}_{\boldsymbol{u}}\|_{2,\mathcal{T}_h}\,\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}\\ &+\widehat{C}_{\mathscr{D}^S}\|\boldsymbol{g}\|_{0,\Omega}\,\|\boldsymbol{e}_\theta\|_{2,\mathcal{T}_h}\,\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}.\end{aligned} \tag{4.14}$$

Therefore, after replacing (4.14) into (4.13), simplifying by $\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}$, employing the a priori estimates (2.28) and (3.26) for $\boldsymbol{u}$ and $\boldsymbol{u}_h$, respectively, and grouping terms we get

$$\|\boldsymbol{\chi}_p\|_{0,\Omega} \;\le\; \widetilde{\beta}^{-1}\widetilde{C}_{\mathscr{B}^S}\,\|\boldsymbol{\xi}_p\|_{0,\Omega} + \widetilde{\beta}^{-1}\widetilde{C}_5(\delta,\nu,\kappa,\boldsymbol{g},\theta_{\mathrm{D}})\Big[\,\|\boldsymbol{e}_{\boldsymbol{u}}\|_{2,\mathcal{T}_h} + \|\boldsymbol{e}_\theta\|_{2,\mathcal{T}_h}\Big], \tag{4.15}$$

where

$$\widetilde{C}_5(\delta,\nu,\kappa,\theta_{\mathrm{D}},\boldsymbol{g}) \;=\; \max\Big\{\widehat{C}_{\mathscr{A}^S} + \widetilde{C}_{S,\mathrm{LIP}}\Big[C_1(\delta,\nu,\boldsymbol{g},\theta_{\mathrm{D}}) + \widetilde{C}_1(\nu,\kappa,\theta_{\mathrm{D}},\boldsymbol{g})\Big], \widehat{C}_{\mathscr{D}^S}\|\boldsymbol{g}\|_{0,\Omega}\Big\}\,.$$

Then, the result follows by inserting (4.15) in (4.4) and using the estimates for $\|\boldsymbol{e}_{\boldsymbol{u}}\|_{2,\mathcal{T}_h} + \|\boldsymbol{e}_\theta\|_{2,\mathcal{T}_h}$ obtained in the first part of the proof. □

5 Numerical results

This section presents a couple of examples in order to illustrate the performance of the fully discontinuous Galerkin method (3.7) constructed and analyzed in Section 3 for approximating the solutions to the stationary Boussinesq system (2.3), and to confirm the theoretical convergence rates (4.6) predicted by the theory according to the Theorem 4.1.

The first example below is a problem with manufactured smooth solution considering only non-homogeneous Dirichlet boundary conditions for the temperature whereas the second one deals with a setting that involves physical boundary conditions such as in (2.4); more general case that was analyzed in the present work.

The computational implementation was carried out using a **FreeFem++** code (see [28]) and the linear solver **UMFPACK** (see [19]). The experimental errors and convergence rates for the velocity vector field, for the pressure and the temperature are the result of iterations based on a Picard method as fixed-point strategy over a family of triangulations $\mathcal{T}_h$ of the respective domain. This process ends when the relative error of the entire coefficients vector given by two consecutive iterations is small enough, that is,

$$\frac{||\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}||_{l^2}}{||\mathbf{coeff}^{m+1}||_{l^2}} \leq tol,$$

where tol is a specific tolerance and $||\cdot||_{\ell^2}$ stands for the Euclidean ℓ^2-norm in R^N with N denoting the total number of degrees of freedom defined by the finite element family $(\boldsymbol{V}_h, Q_h, W_h)$ specified in Section 3.2 with $k=1$, that is, the velocity $\boldsymbol{u}$, the pressure p and the temperature θ are approximated by means of the discrete subspaces $\mathbf{BDM}_1$, $\mathbb{P}_0(\mathcal{T}_h)$ and $\mathbb{P}_1^{\mathrm{disc}}(\mathcal{T}_h)$, respectively, where the former corresponds to the Brezzi-Douglas-Marini finite element space of first order, the second to piecewise constant functions, and the last one to linear discontinuous piecewise polynomials. In all the cases, we have chosen the penalization parameter as $a_0 = 5$.

The individual experimental errors and the convergence rates associated to each variable are given by

$$e(\boldsymbol{u}) := \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\mathcal{T}_h}, \quad e(p) := \|p-p_h\|_{0,\Omega}, \quad e(\theta) := \|\theta-\theta_h\|_{1,\mathcal{T}_h},$$

$$r(\boldsymbol{u}) := \frac{\log(e(\boldsymbol{u})/e(\boldsymbol{u})')}{\log(h/h')}, \quad r(p) := \frac{\log(e(p)/e(p)')}{\log(h/h')}, \quad r(\theta) := \frac{\log(e(\theta)/e(\theta)')}{\log(h/h')},$$

where h and h' denote the size of two consecutive meshes with their respective errors e and e'.

5.1 Example 1: a vortex in the unit box

In the first example we work on the domain $\Omega = (0,1)^2$ and the physical parameters $\nu = \kappa = 1$ and $\boldsymbol{g} = (0,-1)^t$. The manufactured solutions are given by

$$\boldsymbol{u}(x,y) = (\boldsymbol{u}_1(x,y), \boldsymbol{u}_2(x,y)), \qquad \theta(x,y) = \boldsymbol{u}_1(x,y) + \boldsymbol{u}_2(x,y),$$

$$p(x,y) = rs\,\sin\left(\frac{2\pi(e^{rx}-1)}{e^r-1}\right)\sin\left(\frac{2\pi(e^{sy}-1)}{e^s-1}\right)\frac{e^{rx+sy}}{(e^r-1)(e^s-1)},$$

where,

$$\boldsymbol{u}_1(x,y) = \frac{se^{sy}}{2\pi(e^s-1)}\left(1-\cos\left(\frac{2\pi(e^{rx}-1)}{e^r-1}\right)\right)\sin\left(\frac{2\pi(e^{sy}-1)}{e^s-1}\right),$$

$$\boldsymbol{u}_2(x,y) = \frac{-re^{rx}}{2\pi(e^r-1)}\left(1-\cos\left(\frac{2\pi(e^{sy}-1)}{e^s-1}\right)\right)\sin\left(\frac{2\pi(e^{rx}-1)}{e^r-1}\right),$$

with $r,s>0$. Here, the velocity vector field $\boldsymbol{u}$ is similar to a counter clockwise vortex in a unit-box whose coordinates depend on the choice of the parameters r and s (cf. [14, 41]). In our example, we take $r = 3.5$ and $s = 9.1$ which corresponds to a vortex located near the upper right corner of the domain. Table 5.1 presents the errors and the convergence rates obtained with a fixed tolerance $tol < 1E-06$. We confirm there that the individual errors of all the variables decrease with an optimal $\mathcal{O}(h)$ convergence as predicted by Theorem 4.1 with $k=1$ and that the discrete velocities are divergence free (see 9th. column). In Figure 5.1 we display the streamlines of velocity $\boldsymbol{u}_h$, the pressure p_h and the temperature θ_h obtained with a mesh with $N = 89920$ degrees of freedom.

Finite element approximation $\mathbf{BDM}_1 - \mathrm{P}_0(\mathcal{T}_h) - \mathrm{P}_1^{\mathrm{disc}}(\mathcal{T}_h)$									
N	h	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(p)$	$r(p)$	$e(\theta)$	$r(\theta)$	$\lVert\mathrm{div}\,\boldsymbol{u}_h\rVert_{\infty,\Omega}$	Iter
2064	0.1179	16.5374	-	3.6655	-	17.8977	-	3.0407e-11	11
3648	0.0884	14.0742	0.5606	2.8195	0.9121	14.4985	0.7322	2.5919e-11	5
8160	0.0589	9.8316	0.8848	2.1387	0.6815	9.8172	0.9616	2.6303e-11	6
32448	0.0295	4.3685	1.1937	1.2322	0.8203	4.3631	1.1783	2.4174e-11	32
57600	0.0221	3.1019	1.1902	0.9599	0.8681	3.1290	1.1557	2.2162e-11	17
89920	0.0177	2.3862	1.1711	0.7849	0.9094	2.4276	1.1330	2.2453e-11	8

Table 5.1: Example 1: Convergence history for the Boussinesq system using the fully discontinuos Galerkin Family $\mathbf{BDM}_1 - \mathrm{P}_0(\mathcal{T}_h) - \mathrm{P}_1^{\mathrm{disc}}(\mathcal{T}_h)$ $(k=1)$.

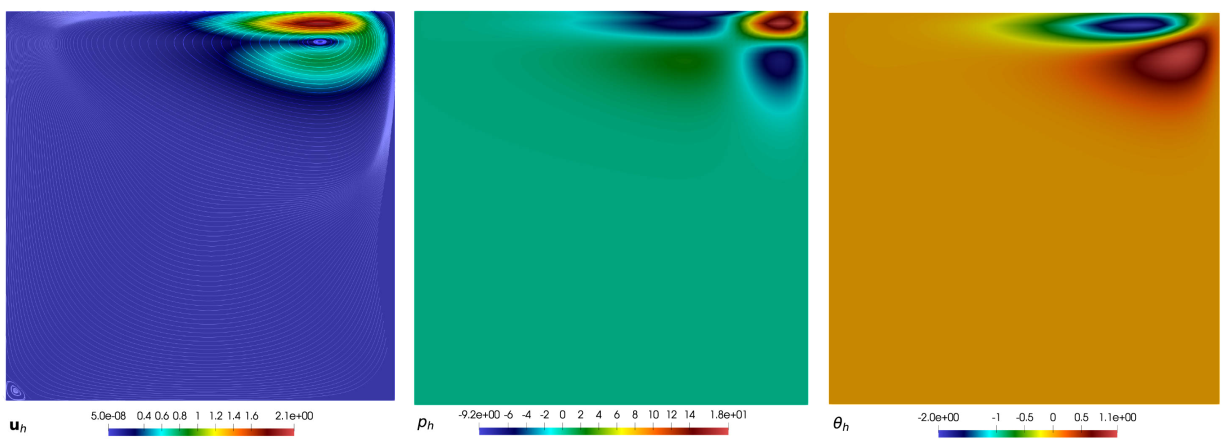


Figure 5.1: Example 1: Streamlines of the velocity $\boldsymbol{u}_h$, pressure p_h and temperature θ_h of the Boussinesq system obtained with the discontinuous finite element family $\mathbf{BDM}_1 - \mathrm{P}_0(\mathcal{T}_h) - \mathrm{P}_1^{\mathrm{disc}}(\mathcal{T}_h)$ $(k=1)$ and $N = 89920$ degrees of freedom.

5.2 Example 2: square cavity stationary flow.

In this example we consider a stationary flow problem in the unit box $(0,1)^2$ with the physical boundary conditions

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma\,, \quad \theta = \begin{cases} 0 & \text{on} \quad \Gamma_{\mathrm{D}}^{(1)} \\ 4y(1-y) & \text{on} \quad \Gamma_{\mathrm{D}}^{(2)} \end{cases} \quad \text{and} \quad \frac{\partial\theta}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}\,,$$

where the Dirichlet boundary $\Gamma_{\mathrm{D}} = \Gamma_{\mathrm{D}}^{(1)} \cup \Gamma_{\mathrm{D}}^{(2)}$ with

$$\Gamma_{\mathrm{D}}^{(1)} = \{(x,y) \in \mathrm{R}^2 : \quad x = 0 \quad \text{and} \quad 0 \leq y \leq 1\,, \quad \text{or} \quad y = 0 \quad \text{and} \quad 0 \leq x \leq 1\,\}\,,$$

$$\Gamma_{\mathrm{D}}^{(2)} = \{(x,y) \in \mathrm{R}^2 : \quad x = 1 \quad \text{and} \quad 0 \leq y \leq 1\,\}\,,$$

and the Neumann boundary is defined as

$$\Gamma_{\mathrm{N}} = \{(x,y) \in \mathrm{R}^2 : \quad y = 1 \quad \text{and} \quad 0 \leq x \leq 1\,\}\,.$$

In this example, we set $\kappa = 1$, $\nu = 0.5$ and $\boldsymbol{g} = (0,-1)^t$. Since non analytical solution is known in this case, we will compute the errors and the convergence rates by considering the discrete solution obtained with a finer mesh ($N = 273554$) as the exact solution. In Table 5.2, we report the results we obtained for a sequence of uniform triangulations considering the discontinuous finite element family $\mathbf{BDM}_1 - \mathrm{P}_0(\mathcal{T}_h) - \mathrm{P}_1^{\mathrm{disc}}(\mathcal{T}_h)$. Once again, it is observed that the rate of convergence $O(h)$ is attained by all the unknowns in agreement with Theorem 4.1 and that the discrete velocities are divergence free. We further display in Figure 5.2 the approximate velocity, the pressure and the temperature. Our results are concordance with [41]. All the figures presented there were obtained with $N = 67792$ degrees of freedom.

Finite element approximation $\mathbf{BDM}_1 - \mathrm{P}_0(\mathcal{T}_h) - \mathrm{P}_1^{\mathrm{disc}}(\mathcal{T}_h)$									
N	h	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(p)$	$r(p)$	$e(\theta)$	$r(\theta)$	$\\|\mathrm{div}\,\boldsymbol{u}_h\\|_{\infty,\Omega}$	Iter
268	0.7454	6.3201	-	5.6810	-	1.0258	-	1.2190e-12	15
1110	0.3802	3.6201	0.8277	4.3212	0.4067	0.5852	0.8337	1.9250e-15	12
4278	0.1901	1.9850	0.8669	2.5850	0.7412	0.3192	0.8749	1.6210e-14	12
17040	0.0951	1.0875	0.8686	1.4652	0.8195	0.1724	0.8888	1.8870e-14	13
67792	0.0530	0.5820	1.0690	0.8789	0.8739	0.1002	0.9279	1.6988e-13	12

Table 5.2: Example 2: Convergence history for the square cavity stationary flow using the fully discontinuos Galerkin Family $\mathbf{BDM}_1 - \mathrm{P}_0(\mathcal{T}_h) - \mathrm{P}_1^{\mathrm{disc}}(\mathcal{T}_h)$ ($k = 1$).

References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces*. Second edition. Pure and Applied Mathematics (Amsterdan), 140. Elsevier/Academic Press, Amsterdam, 2003.

[2] K. Allali, *A priori and a posteriori error estimates for Boussinesq equations*. Int. J. Numer. Anal. Model. 2 (2005), no. 2, pp. 179–196.

[3] J.A. Almonacid, G.N. Gatica and R.Oyarzúa, *A mixed-primal finite element method for the Boussinesq problem with temperature-dependent viscosity*. Calcolo 55 (2018), no. 3, pp. 1–42.

[4] J.A. Almonacid, G.N. Gatica and R.Oyarzúa , *A posteriori error analysis of a mixed-primal finite element method for the Boussinesq problem with temperature-dependent viscosity*. J. Sci. Comp. 78 (2019), no. 2, pp. 887–917.

[5] D. N. Arnold, *An interior penalty finite element method with discontinuous elements*. SIAM J. Numer. Anal. 19 (1982), no. 4, pp. 742–760.

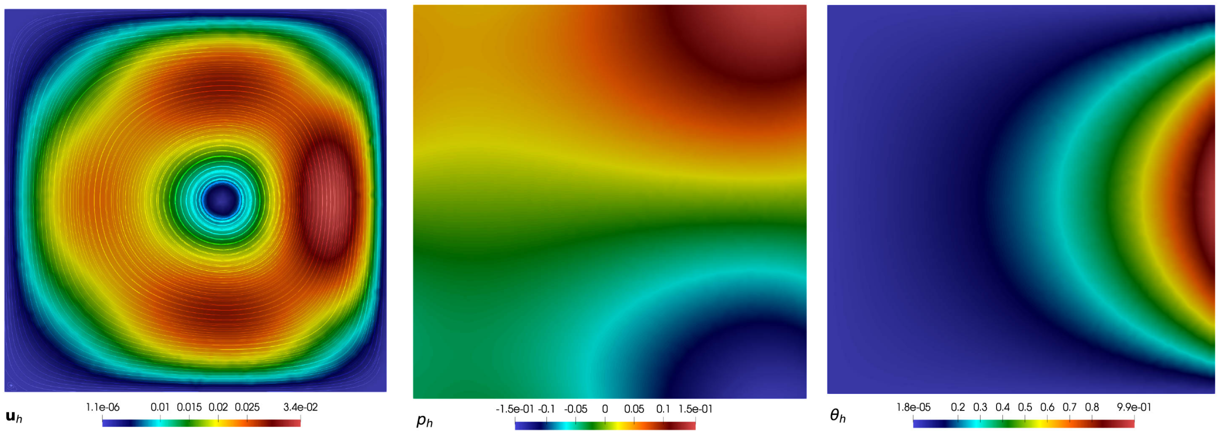


Figure 5.2: Example 2: Streamlines of the velocity $\boldsymbol{u}_h$, pressure p_h and temperature θ_h of square cavity stationary flow obtained with the discontinuous finite element family $\mathbf{BDM}_1 - \mathrm{P}_0(\mathcal{T}_h) - \mathrm{P}_1^{\mathrm{disc}}(\mathcal{T}_h)$ $(k = 1)$ and $N = 89920$ degrees of freedom.

[6] C. Bernardi, B. Métivet and B. Pernaud-Thomas, *Couplage des équations de Navier-Stokes et de la chaleur: le modèle et son approximation par éléments finis.* ESAIM: Math Model Numer Anal 29 (1995), pp. 871–921.

[7] J. Boussinesq, *Théorie de lécoulement tourbillonnant et tumultueux des liquides dans les lits rectilignes a grande section,*, Nabu Press, 2010.

[8] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. MR 2759829

[9] F. Brezis and M. Fortin, *Mixed and Hybrid Finite Element Methods*, vol. 15, Springer Series of Computational Mathematics, Springer, New York. 1991.

[10] B. Cockburn, G. Kanschat and D. Schötzau, *A locally conservative LDG method for the incompressible Navier-Stokes equations.* Math. Comp. 74 (2005), pp. 1067–1095.

[11] B. Cockburn, G. Kanschat and D. Schötzau, *A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations.* J. Sci. Comp. 31 (2007), pp. 61–73.

[12] A. Çibik and K. Songl, *A projection–based stabilized finite element method for steady-state natural convection problem.* J Math Anal Appl 381 (2011), pp 469–484.

[13] E. Colmenares, G. N. Gatica and R. Oyarzúa, *A posteriori error analysis of an augmented mixed–primal formulation for the stationary Boussinesq Problem.* Calcolo 54 (2017), no. 3, pp. 1055–1095.

[14] E. Colmenares, G. N. Gatica and R. Oyarzúa, *A posteriori error analysis of an augmented fully–mixed formulation for the stationary Boussinesq Problem.* Computers & Mathematics with Applications 77 (2019), no. 3, pp. 693–714.

[15] E. Colmenares, G. N. Gatica and R. Oyarzúa, *An augmented fully-mixed finite element method for the stationary Boussinesq problem.* Calcolo 54 (2017), no. 1 , pp. 167–205.

[16] E. Colmenares, G. N. Gatica and R. Oyarzúa, *Analysis of an augmented mixed–primal formulation for the stationary Boussinesq problem.* Numer. Methods Partial Differ. Equ. 32 (2016), no. 2, pp. 445–478.

[17] E. Colmenares, G. N. Gatica and R. Oyarzúa, *Fixed point strategies for mixed variational formulations of the stationary Boussinesq problem.* C.R. Math. Acad. Sci. Paris 354 (2016), no. 1, pp. 57–62.

[18] E. Colmenares and M. Neilan, *Dual–mixed formulations for the stationary Boussinesq problem.* Computers & Mathematics with Applications 72 (2016), pp. 1828-1850.

[19] T.A. Davis, *Algorithm 832: UMFPACK V4.3–an unsymmetric-pattern multifrontal method.* ACM Trans. Math. Software 30 (2004), no. 2, pp. 196–199.

[20] A. Ern and D. A. Di Pietro, *Mathematical Apects of Discontinuous Galerkin Methods*, vol. 69 of Mathématiques et Applications, Springer-Verlag, 2012.

[21] M. Farhloul, S. Nicaise and L. Paquet *A mixed formulation of Boussinesq equations: analysis of nonsingular solutions.* Math. Comp. 69 (2000), no. 231, pp. 965–986.

[22] M. Farhloul, S. Nicaise and L. Paquet *A refined mixed finite element method for the Boussinesq equations in polygonal domains.* IMA Journal of Numerical Analysis 21 (2001), no. 2, pp. 525–551.

[23] K. J. Galvin, A. Linke, L. G. Rebholz, N. E. Wilson, *Stabilizing poor mass conservation in incompressible flow problems with large irrotecional forcing and application to thermal convection.* Comput. Methods Appl. Mech. Engrg. 237-240 (2012), pp. 166–176.

[24] V. Girault and P-A. Raviart, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms.* Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, 1986.

[25] V. Girault, B. Rivière and M. Wheeler, *A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems.* Math. Comput. 74 (2005), pp. 53–83

[26] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations*, vol. 5 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1986. Theory and algorithms.

[27] P. Hansbo, and M. G. Larson, *Discontinuous Galerkin methods for incompressible and nearly incompressibility elasticity by Nitsche's method.* Comput. Methods Appl. Mech. Engrg. 191 (2002), pp. 1895–1908.

[28] F. Hecht, *New development in FreeFem++.* J. Numer. Math. 20 (2012), no. 3-4, pp. 251–265.

[29] J. Howell and N. Walkington, *Dual-mixed finite element methods for the Navier-Stokes equations.* ESAIM: Mathematical Modelling and Numerical Analysis 47 (2013), pp. 789–805.

[30] J. Howell and N. Walkington, *Dual-mixed finite element methods for the Navier-Stokes equations.* arXiv:1603.09231 [math.NA], (2016)

[31] Lesaint and P.A. Raviart, *On a finite element method for solving the neutron transport equation*, Mathematical Aspects of Finite Elements in Partial Differential Equations, de Boor C. (ed.). Academic Press: New York, 1974; 89–145.

[32] A. Linke, *Collision in a cross-shaped domain: A steady 2d Navier-Stokes example demonstrating the importance of mass conservation in CFD* . Comput. Methods Appl. Mech. Engrg. 198 (2009), pp. 3278–3286.

[33] H. Morimoto, *On the existence of weak solutions of equation of natural convection*, J Fac. Sci. Univ. Tokyo. Sect. IA, Math. 36, pp. 87–102, (1989).

[34] H. Morimoto, *On the Existence and Uniqueness of the Stationary Solution to the Equations of Natural Convection*, Tokyo J. Math. 14, no. 1, 217–226, (1991).

[35] R. Oyarzúa and M. Serón, *A divergence-conforming DG-mixed finite element method for the stationary Boussinesq problem.* J. Sci. Comp. 85, no. 14, (2020).

[36] R. Oyarzúa, T. Qin and D. Schötzau, *An exactly divergence-free finite element method for a generalized Boussinesq problem.* IMA J. Numer. Anal. 34 (2014), no. 3, pp. 1104–1135.

[37] R. Oyarzúa and P. Zuñiga, *Analysis of a conforming finite element method for the Boussinesq problem with temperature-dependent parameters,*. J. Comput. Appl. Math. 323 (2017), pp. 71-94.

[38] V. Perfilov, A. Ali and V. Fila, *A general predictive model for direct contact membrane distillation.* Desalination, 445 (2018), pp. 181–196.

[39] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, vol. 23 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1994. Theory and algorithms.

[40] D. J. Tritton, *Physical Fluid Dynamics,* NeVan Nostrand Reinhold Co., New York, 1977.

[41] Y. Zhang, Y. Hou and H. Zuo. *A posteriori error estimation and adaptive computation of conduction convection problems.* Appl. Math. Model. 35 (2011), pp. 2336–2347.

[42] C. Waluga, *Analysis of hybrid discontinuous Galerkin methods for incompressible flow problems.* Ph.D. Thesis, RWTH Aachen, Germany, (2012)

Centro de Investigación en Ingeniería Matemática (CI²MA)

PRE-PUBLICACIONES 2021 - 2022

2021-22 Ricardo Oyarzúa, Manuel Solano, Paulo Zuñiga: *Analysis of an unfitted mixed finite element method for a class of quasi-Newtonian Stokes flow*

2021-23 Felisia A. Chiarello, Harold D. Contreras, Luis M. Villada: *Nonlocal reaction traffic flow model with on-off ramps*

2021-24 Sergio Caucao, Ricardo Oyarzúa, Segundo Villa-Fuentes, Ivan Yotov: *A three-field Banach spaces-based mixed formulation for the unsteady Brinkman-Forchheimer equations*

2021-25 Reinaldo Campos-Vargas, Esther Carrera, Bruno G. Defilippi, Claudia Fuentealba, Ignacia Hernández, Maarten Hertog, Claudio Meneses, Gerardo Nuñez, Diego Paredes, Romina Pedreschi, Virgilio Uarrota: *Transcriptome and hormone analyses reveals differences in physiological age of "Hass" avocado fruit*

2021-26 Tomás Barrios, Edwin Behrens, Rommel Bustinza: *Numerical analysis of a stabilized mixed method applied to incompressible elasticity problems with Dirichlet and with mixed boundary conditions*

2021-27 Liliana Camargo, Manuel Solano: *A high order unfitted HDG method for the Helmholtz equation with first order absorbing boundary condition*

2021-28 Ana Alonso-Rodriguez, Jessika Camaño: *A graph-based algorithm for the approximation of the spectrum of the curl operator*

2021-29 Veronica Anaya, Ruben Caraballo, Ricardo Ruiz-Baier, Hector Torres: *Augmented finite element formulation for the Navier-Stokes equations with vorticity and variable viscosity*

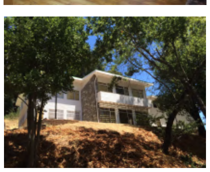
2021-30 Claudio I. Correa, Gabriel N. Gatica: *On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces*

2022-01 Nestor Sánchez, Tonatiuh Sanchez-Vizuet, Manuel Solano: *Afternote to Coupling at a distance: convergence analysis and a priori error estimates*

2022-02 Gabriel N. Gatica, Cristian Inzunza, Filander A. Sequeira: *A pseudostress-based mixed-primal finite element method for stress-assisted diffusion problems in Banach spaces*

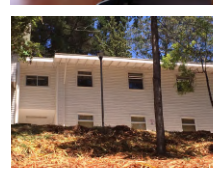
2022-03 Eligio Colmenares, Ricardo Oyarzúa, Francisco Piña: *A fully-DG method for the stationary Boussinesq system*

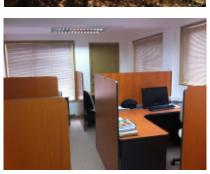
Para obtener copias de las Pre-Publicaciones, escribir o llamar a: Director, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, Tel.: 41-2661324, o bien, visitar la página web del centro: `http://www.ci2ma.udec.cl`



Centro de Investigación en Ingeniería Matemática (CI²MA)

Universidad de Concepción





Casilla 160-C, Concepción, Chile

Tel.: 56-41-2661324/2661554/2661316

http://www.ci2ma.udec.cl