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Abstract

In this paper, we deal the following decision problem: given a conjunctive Boolean network defined by its interaction digraph, does it have a limit cycle of a given length k? We prove that this problem is NP-complete in general if k is a parameter of the problem and in P if the interaction digraph is strongly connected. The case where k is a constant, but the interaction digraph is not strongly connected remains open.

Furthermore, we study the variation of the decision problem: given a conjunctive Boolean network, does there exist a block-sequential (resp. sequential) update schedule such that there exists a limit cycle of length k? We prove that this problem is NP-complete for any constant $k \geq 2$.

Keywords: Boolean network, limit cycle, update schedule, update digraph, NP-Hardness.

1. Introduction

A Boolean network with n components is a discrete and finite dynamical system whose dynamics can be described by a map from $\{0,1\}^n$ to $\{0,1\}^n$. Boolean networks have many applications. In particular, they are classical models for the dynamics of gene networks Kauffman (1969); Thomas (1973); Thomas and d'Ari (1990); Thomas and Kaufman (2001); De Jong (2002), neural networks McCulloch and Pitts (1943); Hopfield (1982); Goles (1985); Goles and Martinez (2013) and social interactions Poljak and Sura (1983); Poindron (2021). They are also essential tools in information theory, for the binary network coding problem Riis (2007); Gadouleau and Riis (2011); Gadouleau et al. (2016).

A conjunctive network is a particular type or Boolean network where the local function of each component (the function that behaves the evolution of the state of the component) realizes a conjunction on the state of a subset of components of the network. The study of conjunctive networks in the context of gene networks has captured special interest in the last time due to increasing evidence that the synergistic regulation of a gene by several transcription factors, corresponding to a conjunctive function, is a common mechanism in regulatory networks Nguyen and D'haeseleer (2006); Gummow et al. (2006). Besides, they have been used in the combinatorial influence of deregulated gene sets on disease phenotype classification Park et al. (2010) and in the construction of synthetic gene networks Shis and Bennett (2013). Also conjunctive networks have been studied with an analytic point of view Aracena et al. (2017); Goles and Hernández (2000); Gadouleau (2021); Goles and Noual (2012); Jarrah et al. (2010a); Aledo et al. (2012); Gao et al. (2018).

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The update schedule in a Boolean network, that is the order in which each node is updated, is of great importance in its dynamical behavior. In general, Boolean networks are usually studied with synchronous (parallel) or sequential schemes. A generalization of these schemes, known as block-sequential update schedules, was introduced by F. Robert Robert (1986, 1995), and they are currently used in the modeling of regulatory networks Goles et al. (2013); Ruz et al. (2014).

Many analytic studies have been done about the limit cycles of a Boolean network with different block-sequential update schedules Aracena et al. (2013b, 2009); Bridoux et al. (2021); Demongeot et al. (2008); Goles and Noual (2012); Goles and Montealegre (2014); Gómez (2015); Macauley and Mortveit (2009); Mortveit (2012). Most of them show that the limit cycles are very sensitive to changes in the updating scheme of a network. In particular, some of these articles exhibit examples of Boolean networks where the existence of limit cycles depends on the used update schedule Aracena et al. (2013b, 2009, 2013a); Bridoux et al. (2021); Demongeot et al. (2008); Mortveit (2012). Some papers study the complexity of problems related to Boolean networks and their limit-cycles Bridoux et al. (2021); Gómez (2015); Aracena et al. (2013b).

In this paper, we study the algorithmic complexity of problems of existence of limit cycles in conjunctive networks, highlighting the differences between the parallel schedule and other block-sequential update schedules.

This paper is organized as follows. In sections 4, we study the problem of knowing if a given conjunctive network updated in parallel has a limit cycle of length k. Theorem 8 states that the problem is in P if the interaction digraph of the conjunctive network is strongly connected. Next, we give two side results. Corollary 11 states that the length of the limit cycles of a conjunctive network of size n has no prime factor strictly greater than n. Corollary 12 states that the bigger length of a limit cycle that a conjunctive network of size n can have is $\mathrm{L}(n)$ where L is the Landau's function, that is the largest lcm of numbers $t_1, \dots, t_m$ whose sum is n. Theorem 15 states that the problem is NP-complete if k is a parameter of the problem given in binary. Finally, Proposition 16 states that the problem is in P when k is a constant and a power of a prime. The question of the complexity class of the problem remains open when k is a constant but not a power of a prime and the interaction digraph is not strongly connected.

In section 5, we study the problem of knowing if, given a conjunctive network, does there exist a block-sequential (*resp.* sequential) update schedule s such that f^s (f updated in the order given by s) has a limit cycle of length k? Theorem 22 states that if f has a block-sequential update schedule s such that f^s has a limit cycle of length $k > 2$ then f^s has a sequential (and therefore block-sequential) update schedule such that f^s has a limit cycle of length $k-1$. Theorem 23 states that the problem is NP-complete of all $k \geq 2$ for the two versions of the problem: block-sequential and sequential.

2. Definitions and notation

A *Boolean network* (BN) N of size n can be represented as a (global) function $f : \{0,1\}^n \to \{0,1\}^n$ and an update schedule s. The global function $f : \{0,1\}^n \to \{0,1\}^n$ can be decomposed into n *local functions* $f_1, \dots, f_n : \{0,1\}^n \to \{0,1\}$, each local function describing the behavior of a component of the BN and for all *configurations* $x \in \{0,1\}^n$, we have $f(x) = (f_1(x), \dots f_n(x))$. The update schedule defines the order in which the components of the BN are updated between two time steps. This paper is focused on the *block-sequential update schedules* which were introduced in Robert (1986): the coordinates $[n] := \{1, \dots, n\}$ are partitioned into p blocks $s = (B_1, \dots, B_p)$ and the dynamics of $N = (f, s)$ is defined by:

$$f^s = f^{B_p} \circ \dots \circ f^{B_1}$$

with $f^{B_i} : \{0,1\}^n \to \{0,1\}^n$ such that:

$$f_j^{B_i}(x) = \begin{cases} f_j(x) & \text{if } j \in B_i \\ x_j & \text{if } j \notin B_i \end{cases}$$

The dynamics of a BN $N = (f, s)$ is given by the function f^s and two Boolean networks $N_1 = (f, s)$ and $N_2 = (f', s')$ have the same dynamical behavior if $f^s = (f')^{s'}$. There are two extreme cases of block-sequential update schedules:

- Parallel update schedule (denoted s^p): there is only one block $[n]$. In other words, all components are updated all together and the dynamics of $N = (f, s^p)$ is simply f.
- Sequential update schedule: all components are updated one at the time. The dynamics of $N = (f, s)$ is then given by $f^s = f^{s_n} \circ \cdots \circ f^{s_1}$ (with $f^i := f^{\{i\}}$ for all $i \in [n]$).

Note that the BN $N = (f, s)$ with a block-sequential update schedule has the same dynamics f^s that the BN $N' = (f^s, s^p)$. In other words, all dynamics can be realized with a parallel update schedule. Also, the block-sequential update schedule was called Serial-Parallel in Robert (1986), and in the particular case of sequential updates, f^s was called Gauss-Seidel operator. In the following, for any $i \in [n]$, $s(i) \in [p]$ is the number of the block in which i is updated.

Since $\{0,1\}^n$ is a finite set, for any BN $f : \{0,1\}^n \to \{0,1\}^n$ and update schedule s, we have two limit behaviors for the iteration of a network:

- *Fixed Point.* A fixed point is a configuration $x \in \{0,1\}^n$ such that $f^s(x) = x$.
- *Limit Cycle.* A limit cycle of length $\ell > 1$ is a tuple $(x^0, x^1, x^2, \ldots, x^{\ell-1})$ such that
  - for all $i, j \in [0, \ell[$, if $i \neq j$ then $x^i \neq x^j$, and
  - for all $i \in [0, \ell[$, $x^{(i+1) \bmod \ell} = f^s(x^i)$.

Fixed points and configurations in limit cycles are called *attractors* of the network. In the following, we use these notations: given a BN of dynamics f and $k \geq 2$,

- $\Phi_k(f)$: set of configurations in a limit cycle of length k of f.

We also note: $\Phi(f) = \bigcup_{k \geq 2} \Phi_k(f)$.

For any attractor $x \in \Phi_k(f)$ and for any $u \in [n]$, the periodic trace of u is the word $\rho_u(x) = (x_u, f_u(x), f_u^2(x), \ldots, f_u^{\ell-1}(x))$ with $\ell \in [1, k]$ the minimum number such that for all $t \in \mathbb{N}$, $f_u^{t+\ell}(x) = f_u^t(x)$.

2.1. Interaction digraphs

A digraph $D = (V, A)$ is composed of a set of vertices V and a set of arcs $A \subseteq V \times V$. An arc $(v, v) \in A$ is called a *loop*. Given a vertex $v \in V$, the set of its incoming neighbors is denoted as $\mathcal{N}_D^{\text{in}}(v) = \{u \in V : (u, v) \in A\}$. Analogously, the set of outgoing vertices from v is denoted as $\mathcal{N}_D^{\text{out}}(v) = \{u \in V : (v, u) \in A\}$. A strongly connected component is *not trivial* if it has at least one arc.

Given a non-trivial strongly connected component H of a digraph D, the *index of cyclicity* of H, denoted $c(H)$, is defined as the greatest common divisor of the lengths of the cycles of H. If a digraph D has non-trivial strongly connected components $H_1, \ldots H_m$, then its index of cyclicity is defined as $c(D) = \text{lcm}\{c(H_i) : 1 \leq i \leq m\}$ (with lcm the *least common multiple*), or one if it does not have any cycles. The index of cyclicity was referred to as the *loop number* in Jarrah et al. (2010b) where it was proved that it can be computed on polynomial time in Colón-Reyes et al. (2005).

Lemma 1 (Colón-Reyes et al. (2005)). *The index of cyclicity $c(H)$ of a non-trivial strongly connected component H of a digraph D can be computed in polynomial time.*

The number $c(D)$ is also referred to as the *index of cyclicity* of D in Schutter and Moor (2000) and as the *index of imprimitivity* of the adjacency matrix of D in Brualdi et al. (1991) and Berman and Plemmons (1994). We say that D is *primitive* if $c(D) = 1$. The *girth* of a digraph D (*resp.* a component H), denoted $\mathrm{g}(D)$ (*resp.* $\mathrm{g}(H)$), is defined as the length of the shortest cycle in D (*resp.* in $\mathrm{g}(H)$). Naturally, we have $c(H) \leq \mathrm{g}(H)$.

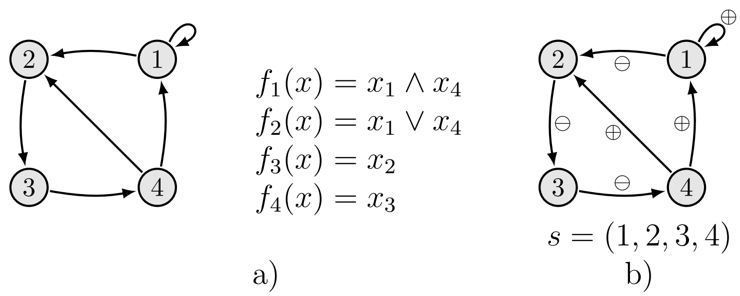


Figure 1: a) Digraph associated to a global function f. b) Update Digraph associated to a Boolean network $N = (f, s)$.

The *interaction digraph* of a global function $f : \{0,1\}^n \to \{0,1\}^n$, is the directed digraph $D^f = (V, A)$, where $V = [n]$ and $(u, v) \in A$ if and only if $f_v(x)$ depends on x_u. More formally, $(u, v) \in A$ iff there exists $x \in \{0,1\}^n$ such that $f_v(x) \neq f_v(\bar{x}^u)$ (where $\bar{x}^u \in \{0,1\}^n$ is a configuration different of x in the coordinate u and identical everywhere else). Note that if f_v is constant, then $\mathcal{N}^{\mathrm{in}}_{D^f}(j) = \emptyset$. See an example of a interaction digraph in Figure 1.

A *k-labeling* function lab of a digraph $D = (V, A)$ is a labeling function of D such that all cycles of D have a multiple of k positive arcs. The labeled digraph D_{lab} is then called a *k-labeled* digraph.

Remark 1. *For any integer k and any digraph $D = (V, A)$, there is always a k-labeling function: the function* $\mathrm{lab} : A \mapsto \ominus$ *that associates a negative sign to all arcs of D. However, if it is not an acyclic digraph, this function does not correspond to an update digraph.*

Consider a digraph $D = (V, A)$ of size n and a block-sequential update schedule $s = (B_1, \ldots, B_p)$. We denote $D_s = (D, \mathrm{lab}_s)$ the labeled digraph, called *update digraph*, where the function $\mathrm{lab}_s : A \to \{\ominus, \oplus\}$ is defined as: $\forall (u, v) \in A,\ u \in B_i \wedge v \in B_j$:

$$\mathrm{lab}_s(u, v) = \begin{cases} \oplus & \text{if } i \geq j, \\ \ominus & \text{if } i < j. \end{cases} \tag{1}$$

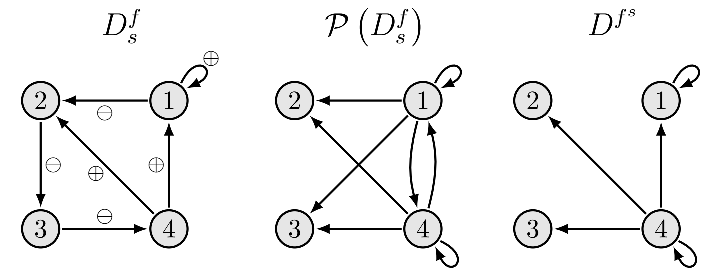
If s is a sequential update schedule, D_s is then called a *sequential update digraph.*

The update digraph associated to a Boolean network $N = (f, s)$ is defined by $D^f_s = (D^f, \mathrm{lab}_s)$ (see an example of update digraph D_s in Figure 1).

Note that the label on a loop will always be $\oplus$. It was proven in Aracena et al. (2009) that if two Boolean networks with the same update function and different updates schedules give the same update digraph, then they also have the same dynamical behavior (*i.e.* $D^f_s = D^f_{s'} \implies f^s = f^{s'}$).

Given an update digraph D_s, with $D = (V, A)$, we define the operator $\mathcal{P}$ as $\mathcal{P}(D_s) = (V, A')$, where $(u, v) \in A'$ if and only if there exists a path $(u_1, u_2, \ldots, u_m)$ with $u = u_1$ and $v = u_m$ in D such that $\mathrm{lab}(u_1 = u, u_2) = \oplus$ and $\mathrm{lab}(u_2, u_3) = \mathrm{lab}(u_3, u_4) = \cdots = \ominus$ (see also Goles and Noual (2012)). See an example in Figure 2. $\mathcal{P}(D_s)$ is referred to as the *parallel digraph of D_s*.

A local function $f_v : \{0,1\}^n \to \{0,1\}$ is said *conjunctive* if there exists a set $I \subseteq [n]$ such that $f_v(x) = 0 \iff \exists u \in I$ such that $x_u = 0$ (if $I = \emptyset$ then $f_v(x) = 1$ for any $x \in \{0,1\}^n$). A conjunctive



$$
\begin{aligned}
f_1(x) &= x_1 \wedge x_4 \\
f_2(x) &= x_1 \vee x_4 \\
f_3(x) &= x_2 \\
f_4(x) &= x_3 \\
s &= (1,2,3,4)
\end{aligned}
\qquad
\begin{aligned}
f_1^s(x) &= x_1 \wedge x_4 \\
f_2^s(x) &= x_4 \\
f_3^s(x) &= x_4 \\
f_4^s(x) &= x_4
\end{aligned}
$$

Figure 2: Example of D_s^f, $\mathcal{P}(D_s^f)$ and D^{f^s}

global function $f\colon \{0,1\}^n \to \{0,1\}^n$ is a global function where each local function is conjunctive. A conjunctive global function f can be completely described by its interaction digraph D^f. That is, given a digraph $D = (V, A)$, we have for every $v \in V$:

$$f_v(x) = \bigwedge_{u \in \mathcal{N}_D^{\text{in}}(v)} x_u.$$

Note that if $\mathcal{N}_D^{\text{in}}(v) = \emptyset$, then $f_v(x) = 1$.

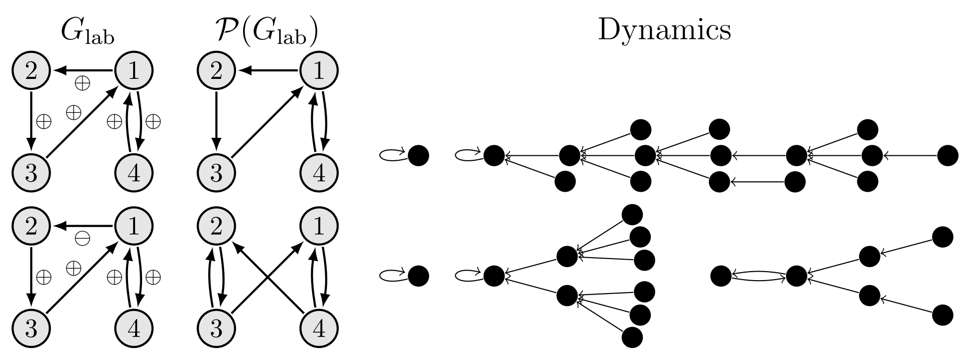


Figure 3: Example of (from left to right): two updates digraphs, their associated parallel digraphs, and their dynamical behavior.

2.2. 3-SAT

A 3-CNF formula ψ is composed of n variables $\lambda = \{\lambda_1, \dots, \lambda_n\}$ and m clauses $\mu = \{\mu_1, \dots, \mu_m\}$. Each clause $\mu_j \in \mu$ is a set of three literals $\{\mu_{j,1}, \mu_{j,2}, \mu_{j,3}\}$. A literal $\mu_{j,p}$ is a couple composed of a variable $\lambda_i \in \lambda$ and a polarity $\rho \in \{\top, \bot\}$. A clause $\mu_j \in \mu$ is satisfied by a valuation $g : \lambda \to \{\bot, \top\}$ if there is a literal $\mu_{j,p} = (\lambda_i, \rho) \in \mu_j$ such that $g(\lambda_i) = \rho$. By abuse of notation, we then write $g(\mu_{j,p}) = \top$ (the valuation satisfies the literal) and $g(\mu_j) = \top$ (the valuation satisfies the clause). The decision problem 3-SAT is to know if, given a 3-CNF formula ψ, there exists a valuation $g : \lambda \to \{\bot, \top\}$ which satisfies ψ (*i.e.* each clause of ψ). By Cook's theorem, the problem 3-SAT is NP-hard (see Theorem 8.2 of Papadimitriou (1994)).

3. Decision problems

In this section, we focus on conjunctive networks and define the following three decision problems for any constant k.

Parallel Limit Cycle Existence (k-PLCE) Problem:
Input: A global function $f : \{0,1\}^n \to \{0,1\}^n$.
Question: Does $\Phi_k(f) \neq \emptyset$?

Block-sequential Limit Cycle Existence (k-BLCE) Problem:
Input: A global function $f : \{0,1\}^n \to \{0,1\}^n$.
Question: Does there exist a **block-sequential** update schedule s such that $\Phi_k(f) \neq \emptyset$?

Sequential Limit Cycle Existence (k-SLCE) Problem:
Input: A global function $f : \{0,1\}^n \to \{0,1\}^n$.
Question: Does there exist a **sequential** update schedule s such that $\Phi_k(f^s) \neq \emptyset$?

We will also study a variant of these three decision problems where k is not part of the problem, but a parameter of the problem encoded in binary. These three new problems are then simply named PLCE, BLCE and SLCE.

In general, for these problems, f has to be encoded efficiently because there are $(2^n)^{(2^n)}$ distinct global functions from $\{0,1\}^n$ to $\{0,1\}^n$. A possible solution is to only consider global functions with an interaction digraph with bounded incoming degree. In the following, we consider these problems restricted to **conjunctive** global function f. This type of function can be represented efficiently by its interaction digraph D^f and D^f itself can be represented by an adjacency matrix encoded in n^2 bits.

Remark 2. *The configuration $(1)^n$ is always a fixed point of a conjunctive function f and therefore the cases $k = 1$ is trivial and not interesting.*

A digraph D is *cyclically k-partite* if its vertex set can be partitioned into k parts $V_0, \ldots, V_{k-1}$ such that the graph obtained by collapsing each part into a single vertex is a cycle of length k, that is, every arc of D goes from V_i to $V_{(i+1) \bmod k}$ for some $0 \leq i \leq k-1$. The same definition can be used to a non-trivial strongly connected component of a digraph D.

There is a direct relation between the index of cyclicity and this notion of "cyclically k-partite".

Lemma 2 (Brualdi et al. (1991))**.** *If D is a strongly connected digraph with index of cyclicity $c(D) = k$, then D is cyclically k-partite.*

Furthermore, there is a relation between the index of cyclicity and the length of the limit cycles.

Lemma 3 (Goles and Hernández (2000); Jarrah et al. (2010b))**.** *Let D with a unique strongly connected component with index of cyclicity ℓ, and let f be the conjunctive network on D. Then, $\Phi_k(f) \neq \emptyset$ if and only if k divide ℓ.*

Consequently, if the index of cyclicity is 1 then f has only fixed point as attractors.

Conjunctive networks updated under another kind of update schedules were studied in Goles and Noual (2012), where, in our wording, was shown the following result:

Proposition 4. *Let f be a conjunctive global function with symmetric interaction digraph D^f. Then, if D^f is strongly connected, for all $s \neq s^p$ such that $D^f_s \neq D^f_{s^p}$, $\Phi_{\geq 2}(f^s) = \emptyset$. Furthermore, all limit cycles of f are of length 2 if D^f is bipartite and $\Phi_{\geq 2}(f) = \emptyset$ otherwise.*

Observe that, if f is a conjunctive global function with D^f a complete digraph, then for every update schedule s, $\Phi_{\geq 2}(f^s) = \emptyset$.

It is clear that the condition in the above proposition can be tested in polynomial time, and therefore the problems with symmetric interaction digraphs are polynomial. We are now considering the problem in the general case.

In Lemma 3, it is claimed that the dynamics of the limit cycles of conjunctive networks with strongly connected interaction digraph updated in parallel is polynomial characterized. Hence, our approach will consist in studying:

- conjunctive global function f with not strongly connected digraph D^f, and
- conjunctive global function f updated with other block-sequential update schedules s by studying D^{f^s}.

Constructing the interaction digraph from the global function f^s is not an easy task in general, since several compositions of conjunctive functions must be made. However, it is enough to work with the most easily to construct $\mathcal{P}(D^f_s)$, since next lemma shows that they are both the same digraph.

Lemma 5 (Goles and Noual (2012)). *If f is a conjunctive global function, then f^s is also a conjunctive global function and $D^{f^s} = \mathcal{P}(D^f_s)$.*

Proof. Straightforward from the definition of operator $\mathcal{P}$ and the fact that the composition of conjunctive functions is also a conjunctive function (and therefore f^s is a conjunctive function). $\square$

4. Complexity of the PLCE and k-PLCE problems

In this section we study the problem of knowing if a given conjunctive network f with a parallel update schedule has a limit cycle of length k.

We define the circular right shift σ as the function that take a word $a = (a_1, \dots, a_m)$ on the alphabet $\{0, 1\}$ and return a word $b \in \{0, 1\}^m$ where $b_i = a_{i-1}$ for all $2 \leq i \leq m$ and $b_1 = a_m$. Furthermore, for all $t \in \mathbb{N}$, σ^t is the composition of t times the function σ.

Proposition 6 below gives some proprieties on the behavior of the cycle c of a configuration x in a limit cycles.

Proposition 6. *Let f be a conjunctive global function, and let $x \in \Phi_k(f)$. Then, for any cycle $c = (u_1, \dots, u_m)$ of D we have:*

1. *For all $t \geq 0$, $|\{u_i \in c : f^t(x)_{u_i} = 0\}| = b$ with b a constant.*
2. *For all $t \geq 0$, $f^t_c(x) = \sigma^t(x_c)$.*
3. *$f^m_c(x) = x_c$.*

Proof. Remark that for any configuration $y \in \{0, 1\}^n$, the function f can only increase the number of 0 present in the cycle c. Indeed, for any $u_i \in c$ such that $x_{u_i} = 0$,

$$f_{u_{i+1}}(x) = \bigwedge_{v \in \mathcal{N}^{\text{in}}(u_{i+1})} x_v = x_{u_i} \wedge \cdots = 0 \wedge \cdots = 0$$

(where if $i = m$, $i + 1 = 1$). If $f_c(x)$ has more 0 than x_c then $f^k_c(x) = f^{k-1}_c(f(x))$ has also more 0 than x_c and $f^k(x) \neq x$. It would be a contradiction because $x \in \Phi_k(f)$. So the number of 0 present in the cycle c in a configuration $x \in \Phi_k(f)$ is invariant by f and this proves item 1.

Now, since by item 1, the number of 0 in x_c is the same as in $f_c(x)$, the vertices u_i such that $f_{u_i}(x) = 0$ are exactly those that $x_{u_{i-1}} = 0$ (again, with $i - 1 = m$ if $i = 1$). This means that if $x_{u_{i-1}} = 1$ then $f_{u_i}(x) = 1$ and therefore $f_{u_i}(x) = x_{u_{i-1}}$ and $f_{(u_1,u_2,\dots,u_m)}(x) = x_{(u_m,u_1,u_2,\dots,u_{m-1})} = \sigma(x_c)$.

Lastly, since the function f shifts circularly the cycle c ($f_c(x) = \sigma(x_c)$) to the right, by applying m times the function f, we shift m times to the right and get back to $f^m_c(x) = \sigma^m(x_c) = x_c$. $\square$

Proposition 7 below gives some proprieties on the behavior of a non-trivial strongly connected component H of a configuration x in a limit cycles.

Proposition 7. *Let f be a conjunctive global function, let H be a non-trivial strongly connected component of D^f and let $x \in \Phi_k(f)$. Let t_H be the smallest integer such that $(f^{t_H}(x))_H = x_H$. Then we have:*

1. *For all $v \in H$, $u \in \mathcal{N}_H^{in}(v)$, $\rho_v(x) = \sigma(\rho_u(x))$.*
2. *For all $v \in H$, either:*
   - *For all $u \in \mathcal{N}^{in}(v)$, $x_u = 1$ (and $f_v(x) = 1$), or*
   - *for $u \in \mathcal{N}_H^{in}(v)$, $x_u = 0$ (and $f_v(x) = 0$).*
3. *$f_H(x)$ can be computed from only x_H.*
4. *The periodic trace $\rho_H(x)$ is of length t_H.*
5. *Let $u \in H$. For all $v \in H$, the periodic trace of v is of length t_H and $\rho_v(x) = \sigma^i(\rho_u(x))$ where i is the length of any path between u and v modulo t_H.*
6. *D_H is cyclically t_H-partite with a partition $H_0, \dots, H_{t_H-1}$.*
7. *$\forall i \in [0, t_H - 1]$, either $x_{H_i} = 0^{|H_i|}$ or $x_{H_i} = 1^{|H_i|}$.*
8. *If $t \neq 1$ then there exists $i, j \in [0, t_H - 1]$ with $x_{H_i} = 0^{|H_i|}$ and $x_{H_j} = 1^{|H_j|}$.*
9. *t_H divides k.*
10. *t_H divides $c(D_H)$ and therefore $1 \leq t_H \leq g(H) \leq |H| \leq n$.*

Proof. Let $v \in H$ and $u \in \mathcal{N}_H^{\text{in}}(v)$. Since H is a not trivial strongly connected component, there exists a path from v to u. Thus, u and v are in a cycle $c = (c_0, c_{m-1}, c_{m-2}, \dots, c_2, c_1)$ of H where $v = c_{m-1}$ and where $u = c_0$ is the predecessor of v in c. By item 2 of Proposition 6, for all $t \geq 0$, $f_u^t(x) = x_{c_{t \bmod m}}$ and $f_v^t(x) = x_{c_{t-1 \bmod m}}$. Therefore, we have $\rho_u(x) = x_{(c_0, c_1, \dots, c_{t_c-1})}$ with t_c dividing m and $\rho_v(x) = x_{(c_1, c_2, \dots, c_{t_c-1}, c_0)} = \sigma(\rho_u(x))$. This proves item 1.

Let $v \in H$ and $u, u' \in \mathcal{N}_H^{\text{in}}(v)$. By item 1, we have $\rho_v(x) = \sigma(\rho_u(x))$ and $\rho_v(x) = \sigma(\rho_{u'}(x))$. Therefore, $\rho_u(x) = \rho_{u'}(x) = \sigma^{-1}(\rho_v(x))$ and in particular $x_u = x_{u'}$.

Now, let $v \in H$ and $u \in \mathcal{N}_H^{\text{in}}(v)$ with $x_u = 1$. By item 1, $\rho_v(x) = \sigma(\rho_u(x))$ and in particular $f_v(x) = x_u = 1$. Furthermore, $f_v(x) = \bigwedge_{u' \in \mathcal{N}^{\text{in}}(v)} x_{u'}$. So for all $u' \in \mathcal{N}^{\text{in}}(v)$, $x_{u'} = 1$ and this prove item 2.

For all $v \in H$, let $u \in \mathcal{N}_H^{\text{in}}(v)$, by item 2, we have $f_v(x) = x_u$. Therefore, we can compute all $f_H(x)$ simply from x_H and this proves item 3.

Since by item 3 $f_H(y)$ depends only of y_H for all $y \in \Phi_k(f)$ and since $f_H^{t_H}(x) = x_H$, we can show by induction that for all $t \geq 0$,

$$f_H^t(x) = f_H^{t \bmod t_H}(x).$$

Therefore, the periodic trace $\rho_H(x)$ is of length at most t_H. Furthermore, $t_H \leq |\rho_H(x)|$ by minimality of t_H. This proves item 4.

Let $u \in H$. Let $t' = |\rho_u(x)|$. By induction, let us prove that all vertices v such that there is a path of length d from u to v, we have $\rho_v(x) = \sigma^i(\rho_u(x))$ with $i = d \bmod t'$. It is true for $d = 0$ because $\rho_u(x) = \sigma^0(\rho_u(x))$. Suppose this is true for d. Let $v \in H$ with a path $(u, \dots, v', v)$ of length $d + 1$ from u. Then, there exists a path $(u, \dots, v')$ of size d between u and $v' \in \mathcal{N}_H^{\text{in}}(v)$. Therefore,

$\rho_{v'}(x) = \sigma^{d \bmod t'}(\rho_u(x))$. Now, by item 1, $\rho_v(x) = \sigma(\rho_{v'}(x)) = \sigma(\sigma^{d \bmod t'}(\rho_u(x)))$. Therefore, $|\rho_v(x)| = t'$ and $\rho_v(x) = \sigma^{(d+1) \bmod t'}(\rho_u(x))$. Now, one can see that $t' = t_H$. Indeed, since the trace of all $v \in H$ are of same length t', we have $f_H^{t'}(x) = x_H$ and therefore $t_H \leq t'$. Furthermore, by item 4, $t' = |\rho_u(x)| \leq |\rho_H(x)| \leq t_H$. This proves item 5.

To prove item 6, it is sufficient to fix $u \in H$ and to partition H into t_H parts $H_0, \ldots, H_{t_H-1}$ where $H_i = \{v \in H : \rho_v(x) = \sigma^i(\rho_u(x))\}$ (the parts are disjoint and their union is H by item 5). If $v \in V_i$ and $v' \in \mathcal{N}_H^{\text{out}}(v)$, then by item 1, $\rho_{v'}(x) = \sigma(\rho_v(x)) = \sigma^{i+1}(\rho_u(x))$ and therefore $v' \in H_{i+1 \bmod t_H}$.

To prove item 7, it is sufficient to see that for all $v \in H_i$, we have $x_v = (\rho_u(x))_i$.

To prove item 8, by item 7, for all parts H_i we have either $x_{H_i} = 0^{|H_i|}$ or $x_{H_i} = 1^{|H_i|}$. Now, if for all parts H_i, we have $x_{H_i} = 0^{|H_i|}$ then $x_H = 0^{|H|}$, $f_H(x) = 0^{|H|} = x_H$ and $t = 1$. The same reasoning can be used if for all parts H_i we have $x_{H_i} = 1^{|H_i|}$ and this proves item 8.

To prove item 9, it is sufficient to see that k is the length of the periodic trace $\rho_V(x)$ which is a least common multiple of the length of the periodic traces of all $v \in V$. Since, t_H is the length of the periodic trace $\rho_u(x)$ then t_H divides k.

To prove item 10, it is sufficient to prove that t_H divides the length of any cycle c of H (since $c(H)$ is the lcm of all these lengths). Let c be a cycle of H. Let $v \in c$. The cycle c is a path of length $|c|$ from v to v. Therefore, by item 5, $\rho_v(x) = \sigma^i(\rho_v(x))$ where $i = |c| \bmod t_H$. If $|c|$ does not divide t_H then $\rho_v(x) = \sigma^i(\rho_v(x))$ with $0 < i < t_H$ with contradicts the minimality of $\rho_v(x)$. Therefore, $|c|$ divides t_H and this proves item 10.

□

The following result is straightforward from Jarrah et al. (2010b).

Theorem 8. *The PLCE problem (and therefore the k-PLCE problem) is in P when we restrict to conjunctive networks f such that D^f is strongly connected.*

Proof. Consider the instance of the PLCE problem with f a conjunctive network such that D^f is strongly connected and an integer k. Then by Lemma 3 it is sufficient to compute the index of cyclicity ℓ of D^f and then to check that k divides ℓ.

In order to prove the NP-completeness of PLCE problem in the general case we previously prove some results in Lemmas 9 10 13 and Proposition 14 below.

Lemma 9 below states that it is polynomial to check if a given configuration is part of a limit cycle of a given length k.

Lemma 9. *For all conjunctive global functions f, for all integers k and for all configurations $x \in \{0,1\}^n$, it is polynomial to check if $f^k(x) = x$.*

Proof. Remember that by Lemma 5, the composition of conjunctive global functions is a conjunctive global function and that a conjunctive global function can be represented efficiently (for example, with its interaction digraph). First, even if k is exponential in n, it is polynomial to compute f^k. Indeed, one can write k in a sum of a logarithmic number of powers of 2: $k = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_m}$ and we have $f^k = f^k = f^{k_1} \circ f^{k_2} \circ \cdots \circ f^{k_m}$. Furthermore, any function f^{k_i} can be computed in a logarithmic number of compositions. For example, $f^2 = f \circ f$, $f^4 = f^2 \circ f^2$, *etc.* As a result, f^k can be computed in polynomial time and it is then sufficient to check that $f^k(x) = x$ which is also polynomial (and even linear). □

Lemma 10 below show that if a configuration is in a limit cycle of length k, then k is exactly the lcm of the periods of the limit not-trivial strongly connected components of D^f (the period of a not-trivial strongly connected component H being the smallest integer t such that $(f^t(x))_H = x_H$ by Proposition 7, item 4).

Lemma 10. *Let $H_1, \ldots, H_m$ be the not trivial strongly connected components of D^f. Let $x \in \Phi(f)$ and for all $i \in [m]$, let $t_i > 0$ be the smallest integer such that $x_{H_i} = (f^{t_i}(x))_{H_i}$. Then $x \in \Phi_k(f)$ with $k = \operatorname{lcm}(t_1, \ldots, t_m)$.*

Proof. In Proposition 7, item 4, it is said that the periodic trace $\rho_{H_i}(x)$ is of length t_i. To prove that the periodic trace $\rho(x)$ is of length k, it is then sufficient to prove that the period $t_v = |\rho_v(x)|$ of each vertex v out of a not trivial strongly connected component is a divisor of k. Consider an order of the vertices of $[n]$ not in a not trivial strongly connected component such that if there is a path from u to v then $u < v$. Consider the smallest v such that $t_v = |\rho_v(x)|$ does not divides k. So, q such that $f^{q+k}(x) \neq f$ □

A consequence of Lemma 10 is that the length of a limit cycle of a conjunctive network cannot be divided by an integer strictly greater than n.

Corollary 11. *For all conjunctive global functions f, for all prime $p > n$ and for all q multiple of p, $\Phi_q(f) = \emptyset$.*

Proof. Let $x \in \Phi_k(f)$ for any integer k. Let $H_1, \ldots, H_m$ be the not trivial strongly connected components of D^f and let t_i be the smallest integer such that $x_{H_i} = (f^{t_i}(x))_{H_i}$ for all $1 \leq i \leq m$. By Lemma 10, $k = \operatorname{lcm}(t_1, \ldots, t_m)$. Furthermore, by item 10 of Proposition 7, for all $1 \leq i \leq m$, $t_i \leq |H_i| \leq n$ so k is not a multiple of any prime $p \geq n$. □

Since for any update schedule s, f^s is a conjunctive network if f is a conjunctive network, then the result of Corollary 11 remains true for any update schedule.

Another consequence of Lemma 10 is that the maximum length of a limit cycle of any conjunctive network of length n corresponds to the Landau's function $\mathrm{L}(n)$, that is the largest lcm of numbers $t_1, \ldots, t_m$ whose sum is n and is corresponds to the OEIS sequence A000793.

Corollary 12. *Let $n \in \mathbb{N}^*$, and let k be the greatest integer such that there exists a conjunctive network $f : \{0,1\}^n \to \{0,1\}^n$ such that $\Phi_k(f) \neq \emptyset$. Then $k = L(n)$.*

Proof. First, let us prove that $k \leq \mathrm{L}(n)$. Let $x \in \Phi_k(f)$ for any integer k. Let $H_1, \ldots, H_m$ be the not trivial strongly connected components of D^f and let t_i be the smallest integer such that $x_{H_i} = (f^{t_i}(x))_{H_i}$ for all $1 \leq i \leq m$. By Lemma 10, $k = \operatorname{lcm}(t_1, \ldots, t_m)$. Let $t = \sum\limits_{1 \leq i \leq m} t_i$. By item 10 of Proposition 7, for all $1 \leq i \leq m$, $t_i \leq |H_i|$ so $t \leq n$. Since, the Landau's function is an increasing function then $k \leq \mathrm{L}(t) \leq \mathrm{L}(n)$.

Now, let us prove that $k \geq \mathrm{L}(n)$. We take $t_1, \ldots, t_m$ a partition of n such that $\operatorname{lcm}(t_1, \ldots, t_m) = \mathrm{L}(n)$. Now consider f a conjunctive network such that D^f is composed of m disjoint cycles $c_1, \ldots, c_m$ such that $|c_i| = t_i$ for all $1 \leq i \leq m$. We take $x \in \{0,1\}^n$ such that x has exactly one 1 in each cycle c_i. So the period trace ρ_{H_i} is of length t_i. As a result, $x \in \phi_k(x)$ with $k = \operatorname{lcm}(t_1, \ldots, t_m) = \mathrm{L}(n)$.

□

More generally, the lengths of all the limit cycle of all conjunctive networks of length n are exactly all the lcm of numbers $t_1, \ldots, t_m$ whose sum is n.

Lemma 13. *For all conjunctive global functions f, for all configuration $x \in \{0,1\}^n$, for all integer k, it is polynomial to check if $x \in \Phi_k(f)$.*

Proof. By Lemma 9, it is polynomial to check that $f^k(x) = x$. It is not sufficient to conclude that $x \in \Phi_k(f)$ because it could exist an integer $1 \le q < k$ such that $f^q(x) = x$, but it proves that $x \in \Phi(f)$. Next, one can decompose D^f into no trivial strongly connected components $H_1, \ldots, H_m$ and compute $t_1, \ldots t_m$, the smallest strictly positive integers such that $(f^{t_i}(x))_{H_i} = x_{H_i}$. Any integer t_i can be computed in polynomial time because by Proposition 7, item 9, t_i is smaller than n. By Lemma 10, $x \in \Phi_{\mathrm{lcm}(t_1,\ldots,t_m)}(f)$ and it is then sufficient to check that $k = \mathrm{lcm}(t_1, \ldots, t_m)$. □

Proposition 14 below show that if there is a path between two non-trivial strongly connected component H_1 and H_2 then in all limit cycle, either one of the component is stable or the two component's period are not co-prime.

Proposition 14. *Let f be a conjunctive global function with at least two strongly connected components H_1 and H_2 with a path between H_1 and H_2. Let $x \in \Phi(f)$. Then either $t_{H_1} = 1$ or $t_{H_2} = 1$ or $\gcd(t_{H_1}, t_{H_2}) \neq 1$ with t_{H_1} and t_{H_2} the smallest integers such that $f_{H_1}^{t_{H_1}}(x) = x_{H_1}$ and $f_{H_2}^{t_{H_2}}(x) = x_{H_2}$.*

Proof. Suppose that $t_{H_1} \neq 1$ and $\gcd(t_{H_1}, t_{H_2}) = 1$. Let $u \in H_1$ and $v \in H_2$. Since $t_{H_1} \neq 1$, by item 8 of proposition 7, there exists $u \in H_1$ such that $x_u = 0$ and by item 5, for all $k \ge 0$, $f_u^{kt_{H_1}}(x) = 0$. Now, consider the path $u_0 = u, u_1, \ldots, u_\ell = v$ between u and v with $v \in H_2$.

By induction, one can show that for all $k \ge 0$, $f_{u_i}^{kt_1+i}(x) = 0$. Indeed, $f_{u_0}^{kt_1+0}(x) = f_u^{kt_1}(x) = 0$ and for all $i > 0$, $f_{u_i}^{kt_1+i}(x) = f_{u_{i-1}}^{kt_1+(i-1)}(x) \wedge \cdots = 0 \wedge \cdots = 0$. As a result, for all $k \ge 0$, $f_v^{kt_1+\ell}(x) = 0$.

Now, for the sake of contradiction, suppose that $t_{H_2} \neq 1$. As a result, there exists q such that for all $k' \ge 0$, $f_v^{k't_{H_2}+q}(x) = 1$. However, since t_1 and t_2 are co-primes then there exists k and k' such that $kt_{H_1} + \ell = k't_{H_2} + q$ and therefore, for this couple $(k,k)'$, $f_v^{kt_{H_1}+\ell}(x) = 0$ and $f_v^{kt_{H_1}+\ell}(x) = 1$. This is a contradiction. Therefore, $t_{H_2} = 1$. □

Theorem 15 below shows that the general problem of knowing if a conjunctive network has a limit cycle of length k, with k an entry of the problem given in binary is NP-complete.

Theorem 15. *The PLCE problem is NP-complete in general.*

Proof. In this proof, the prime numbers are denoted by $p_1, p_2, \ldots$. Consider a conjunctive global function f and an integer k. To prove that $\Phi_k(f) \neq \emptyset$, is is sufficient to exhibit a configuration $x \in \Phi_k(f)$. By Lemma 13, it is then polynomial to check that x really belongs to $\Phi_k(f)$. Therefore, the PLCE problem is in NP.

Now, let us prove that the PLCE problem in NP-hard. For this purpose we will reduce from 3-SAT. Consider a 3-CNF formula ψ composed of n variables $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ and m clauses $\mu = \{\mu_1, \ldots, \mu_m\}$. Consider the following conjunctive global functions f described by its digraph H. First, each variable λ_i is represented by two isolated cycles of length p_i. We will refer to the first one as $\Lambda_{i,\top}$ and the second one as $\Lambda_{i,\perp}$. For example, λ_2 is represented by two cycles of length $p_2 = 3$. Second, each clause μ_j is represented by three cycles of length p_{n+j} referred as $M_{j,1}$, $M_{j,2}$ and $M_{j,3}$ (each one corresponding to a literal of the clause). Now, for each literal $\mu_{j,\ell} = (\lambda_i, \rho)$ with $j \in [m], \ell \in [3], i \in [n]$ and $\rho \in \{\top, \perp\}$, there are two cases. If $\rho = \top$, then there is an arc from the component $\Lambda_{i,\top}$ to the component $M_{j,\ell}$. Otherwise, then there is an arc from the component $\Lambda_{i,\perp}$ to the component $M_{j,\ell}$. Finally, let $k = p_1 p_2 \ldots p_{n+m}$.

This complete the description of this reduction. Note that $p_{n+m} \le (n+m)(\ln(n+m) + \ln\ln(n+m)) \le (n+m)^2$ for $(n+m)$ big enough Robin (1983). Furthermore, there are two cycles of size of length $\le p_{n+m}$ for each variable and at most 3 cycles of length $\le p_{n+m}$. So D^f is of polynomial size $\le 3(n+m)^3$.

We claim that $\Phi_k(f) \neq \emptyset$ if and only if ψ has a solution.

First, suppose that there exists a valid valuation $v : \lambda \to \{\top, \perp\}$ of ψ. Then we consider the following configuration x. First, for all $i \in [n]$, there are two cases.

- if $v(\lambda_i) = \top$, then $x_{\Lambda_{i,\perp}} = 1(0)^{p_i-1}$ and $x_{\Lambda_{i,\top}} = (0)^{p_i}$, and
- if $v(\lambda_i) = \perp$, then $x_{\Lambda_{i,\perp}} = (0)^{p_i}$ and $x_{\Lambda_{i,\top}} = 1(0)^{p_i-1}$.

Similarly, for any literal $\mu_{j,\ell} = (\lambda_i, \rho)$, if $v(\mu_{j,\ell}) = \top$ (*i.e.* if $v(\lambda_i) = \rho$), then $x_{M_{j,\ell}} = 1(0)^{p_{n+j}-1}$ and $x_{M_{j,\ell}} = (1)^{p_{n-j}}$ otherwise. We will now prove that if v is a valid valuation then $x \in \Phi_k(f)$.

For all, H a strongly connected component of D (*i.e.* $H = \Lambda_{i,\rho}$ or $H = M_{j,\ell}$), let t_H be the smallest strictly positive integer such that $f^{t_H}(x) = x_H$ (we prove that such an integer exists all the time next). Let T be the set of all these integers. Let $i \in [n]$ and $\rho = v(\lambda_i)$. Note that the cycles $\Lambda_{i,\top}$ and $\Lambda_{i,\perp}$ are isolated: the only in-neighbors of the vertices are their predecessors in the cycles. As a consequence we have

$$\begin{aligned} x_{\Lambda_{i,p}} &= 1(0)^{p_i-1} \\ (f(x))_{\Lambda_{i,p}} &= 01(0)^{p_i-2} \neq x_{\Lambda_{i,p}} \\ &\dots \\ (f^{p_i-1}(x))_{\Lambda_{i,p}} &= (0)^{p_i-1}1 \neq x_{\Lambda_{i,p}} \\ (f^{p_i}(x))_{\Lambda_{i,p}} &= 1(0)^{p_i-1} = x_{\Lambda_{i,p}}. \end{aligned}$$

and

$$(f(x))_{\Lambda_{i,\neg p}} = (0)^{p_i} \qquad\qquad = x_{\Lambda_{i,\neg p}}.$$

We can see that if $v(\rho) = \top$ then $t_{\Lambda_{i,\top}} = 1$ and $t_{\Lambda_{i,\perp}} = p_i$. Otherwise, $t_{\Lambda_{i,\top}} = p_i$ and $t_{\Lambda_{i,\perp}} = 1$. In both cases, we have $\mathrm{lcm}(t_{\Lambda_{i,\top}}, t_{\Lambda_{i,\perp}}) = p_i$.

The same way, for all $\mu_{j,\ell} = (\mu_i, \rho) \in \mu$, their only in-neighbors are their predecessor in the cycle $M_{j,\ell}$ and, for one of them, a vertex of $\Lambda_{i,\rho}$. They are two cases: If $v(\mu_{j,\ell}) = \top$ (*i.e.* $v(\lambda_i) = \rho$) then

$$x_{\Lambda_{i,\rho}} = (f(x))_{\Lambda_{i,\rho}} = (f^2(x))_{\Lambda_{i,\rho}} = \dots = (0)^{p_i}.$$

Furthermore,

$$\begin{aligned} x_{M_{j,\ell}} &= 1(0)^{p_{j+n}-1} \\ (f(x))_{M_{j,\ell}} &= 01(0)^{p_{j+n}-2} \neq x_{M_{j,\ell}} \\ &\dots \\ (f^{p_{j-n}}(x))_{M_{j,\ell}} &= 1(0)^{p_{j+n}-1} = x_{M_{j,\ell}} \end{aligned}$$

Therefore, we have $t_{M_{j,\ell}} = p_{j+n}$. Otherwise, if $v(\mu_{j,\ell}) = \perp$, then $x_{M_{j,\ell}} = (1)^{p_{j+n}}$ and $(f(x))_{M_{j,\ell}} = (1)^{p_{j+n}}$ and we have $t_{M_{j,\ell}} = 1$.

Note that since v is a valid valuation of ψ, we have $v(\mu_{j,1}) = \top$ or $v(\mu_{j,2}) = \top$ or $v(\mu_{j,3}) = \top$. This signifies that $\mathrm{lcm}(t_{M_{j,1}}, t_{M_{j,2}}, t_{M_{j,3}}) = p_{j+n}$. As a result,

$$\mathrm{lcm}(T) = \prod_{i=1}^{n} \mathrm{lcm}(t_{\Lambda_{i,\top}}, t_{\Lambda_{i,\perp}}). \prod_{j=1}^{m} \mathrm{lcm}(t_{M_{j,1}}, t_{M_{j,2}}, t_{M_{j,3}}) = \prod_{i=1}^{n} p_i. \prod_{j=1}^{m} p_{j+n} = \prod_{i=1}^{n+m} p_i = k.$$

Hence, $f^k(x) = x$, $x \in \Phi(f)$. Now, by Lemma 10, we have $x \in \Phi_q(f)$ with $q = \mathrm{lcm}(T) = k$. As a result, $x \in \Phi_k(f)$.

In the other hands, suppose that there exists $x \in \Phi_k(f)$. Let us prove that there exists a valid valuation $v : \lambda \to \{\top, \perp\}$. First, Lemma 10, there exists a set of integers $T = \{t_H :$ H a strongly connected component of $D^f\}$ such that for all $t_H \in T$, t_H is the smallest strictly positive integer such that $(f^{t_H}(x))_H = x_H$. By item 7 of proposition 7, for all $i \in [n]$ and $\rho \in \{\top, \perp\}$, $t_{\Lambda_{i,\rho}}$

must divides $c(\Lambda_{i,\rho}) = p_i$. Since p_i is prime then $t_{\Lambda_{i,\rho}} = 1$ or $t_{\Lambda_{i,\rho}} = p_i$. The same way, for all $j \in [m]$ and $\ell \in [3]$, $t_{m_{j,\ell}} = 1$ or $t_{m_{j,\ell}} = p_{j+n}$.

Second, for all $i \in [n]$ and for all $\rho \in \{\top, \bot\}$, we have $t_{\Lambda_{i,\top}} = p_i$ or $t_{\Lambda_{i,\bot}} = p_i$ because otherwise p_i is not a factor of $k = \prod_{i=1}^{n+m} p_i$ which is a contradiction. The same way, for all $j \in [m]$, we have $t_{M_{j,1}} = p_{j+n}$ or $t_{M_{j,2}} = p_{j+n}$ or $t_{M_{j,3}} = p_{j+n}$.

We define $v : \lambda \to \{\top, \bot\}$ such that for all $i \in [n]$, $v(\lambda_i) = \top$ if $t_{\Lambda_{i,\bot}} = p_i$ and $v(\lambda_i) = \bot$ otherwise (note that in this second case we have $t_{\Lambda_{i,\top}} = p_i$). Let us prove that v is a valid valuation of ψ.

Let $\mu_{j,\ell} = (\lambda_i, \rho) \in$ be a literal such that $v(\mu_{j,\ell}) = \bot$ (*i.e.* $v(\lambda_i) \neq \rho$). Then, we have a path from $\Lambda_{i,\rho}$ to $M_{j,\ell}$ (by definition of D^f). By proposition 14, $t_{\Lambda_{i,\rho}} = 1$ or $t_{M_{j,\ell}} = 1$ or $\mathrm{lcm}(t_{\Lambda_{i,\rho}}, t_{M_{j,\ell}}) \neq 1$. We know that $t_{M_{j,\ell}} = 1$ or $t_{M_{j,\ell}} = p_{j+n}$ and $t_{\Lambda_{i,\rho}} = p_i$ (by definition of v). The primes p_i and p_{j+n} are different and therefore $\mathrm{lcm}(t_{\Lambda_{i,\rho}}, t_{M_{j,\ell}}) \neq 1$. As a result, $t_{\Lambda_{i,\rho}}$ must be equal to 1. This signifies that for all literal $\mu_{j,\ell}$, $v(\mu_{j,\ell}) = \bot$ implies $t_{\Lambda_{i,\rho}} = 1$.

Therefore, if v is not a valid valuation, then there exists $j \in [m]$ such that $v(\mu_{j,1}) = v(\mu_{j,2}) = v(\mu_{j,3}) = \bot$ and therefore $t_{M_{j,1}} = t_{M_{j,2}} = t_{M_{j,3}} = 1$. This is a contradiction because we already proved that $t_{M_{j,1}} = p_{j+n}$ or $t_{M_{j,2}} = p_{j+n}$ or $t_{M_{j,3}} = p_{j+n}$. As a result, v is a valid valuation of ψ.

This concludes the proof that PLCE is NP-hard and therefore NP-complete.

□

A question still open is the complexity of the k-PLCE problems when the interaction digraph is not strongly connected.

We know by Lemma 10 and item 6 of Proposition 7, that a necessary condition for k-PLCE(f) to have a solution is that k equals $\mathrm{lcm}(t_1, \dots, t_m)$ for $t_1, \dots, t_m$ dividing respectively $c(H_1), \dots, c(H_m)$ with $H_1, \dots, H_m$ the strongly connected component of D^f.

For some values of k, this condition is sufficient and that makes the problem in P as shown in Proposition 16 below.

Proposition 16. *If k is a power of a prime, then k-PLCE(f) has a solution if and only if D^f has a strongly connected component H such that k divides $c(H)$.*

Proof. The left to right direction is a direct consequence of the Lemma 10 and item 6 of Proposition 7. For the right to left direction, using the strongly connected component H, we can create a configuration $x \in \Phi_k(f)$ the following way. First, we know that H is k-partite for a partition $H^1, \dots, H^k$. We take $x_{H^1} = 1^{|H^1|}$ and $x_{H \setminus H^1} = 0^{|H \setminus H^1|}$ and we have $t_H = k$. For all the other strongly connected components H' we take $x_{H'} = 0^{|H'|}$ and we have $t_{H'} = 1$. For the vertices v out of any strongly connected component, if there is a path from a strongly connected component other than H to v then $x_v = 0$. Else $x_v = 0$ iff and only if there is path between H^1 and v and all path between H^1 and v have lengths multiple of k. One can check that $x \in \Phi_k(f)$. □

However, there are more complicated cases. For example, by Proposition 14, if D^f is composed of 2 cycles of length 2 and 3 connected by an arc then f has no limit cycle of size 6 because 2 and 3 are co-primes. However, if D^f is composed of 2 cycles of length 6 and 10 connected by an arc, then f has a limit cycle of length 30. The exact characterization of when k-PLCE(f) has a solution is unknown and, therefore it is an open problem to know for which values of k, k-PLCE is in P.

5. Complexity of the BLCE, SLCE, *k*-BLCE and *k*-SLCE problems

In this section, we study the problem of the existence of limit cycle of a given length for a block-sequential or sequential update schedule.

Lemma 17 (Goles and Noual (2012)). *Let f be a conjunctive network such that D^f is strongly connected and let s be a block-sequential update schedule. Then, $\mathcal{P}(D^f_s)$ has a unique strongly connected component.*

Proposition 18. *Let D_{lab} be an update digraph. Then, every circuit in D_{lab} produces a circuit in $\mathcal{P}(D_{\text{lab}})$ with length the number of the $\oplus$-labeled arcs of it. Conversely, every circuit in $\mathcal{P}(D_{\text{lab}})$ comes from a circuit in D_{lab} with a number of $\oplus$-labeled arcs equal to the length of the cycle.*

Proof. Let us consider $D = (V, A)$ and $\mathcal{P}(D_{\text{lab}}) = (V, A')$.

Let $c = (c_1, \ldots, c_m)$ a circuit in D_{lab} with ℓ $\oplus$-labeled arcs. Let us prove that there is a circuit of length ℓ in D. Let $c_{i_1}, \ldots, c_{i_\ell}$ be the ℓ vertices of c such that (c_{i_j-1}, c_{i_j}) is a $\oplus$-labeled arc for all $1 \leq j \leq \ell$ with $i_1 \leq i_2 \leq \cdots \leq i_\ell$. For all $1 \leq j \leq m$, such that $i_j + 1 < i_{j+1}$, we have $\text{lab}(c_{i_j}, c_{i_j+1}) = \cdots = (c_{i_{j+1}-2}, c_{i_{j+1}-1}) = \ominus$ and $(c_{i_{j+1}-1}, c_{i_{j+1}}) = \oplus$. Hence, by definition of $\mathcal{P}(D_{\text{lab}})$, $(c_{i_j}, c_{i_{j+1}}) \in \mathcal{P}(D_{\text{lab}})$ and $(c_{i_1}, \ldots, c_{i_\ell})$ is a circuit in $\mathcal{P}(D_{\text{lab}})$ which is also a cycle if c is a cycle.

In the other hand, take $(c_1, \ldots, c_\ell)$ a circuit in $\mathcal{P}(D_{\text{lab}})$. Then, by definition of $\mathcal{P}(D_{\text{lab}})$, for all $1 \leq j \leq \ell$, $\text{lab}(c_j, c_{j+1}) = \oplus$ or there is a path $(c_j = v_1, v_2, \ldots, c_{j+1} = v_r)$ in D_{lab} such that $\text{lab}(v_1, v_2) = \cdots = (v_{r-2}, v_{r-1}) = \ominus$ and $(v_{r-1}, v_r) = \oplus$. As a result, there is a circuit in D_{lab} with exactly ℓ $\oplus$-labeled arcs. □

Proposition 19. *Let f be a global conjunctive function and s be a block-sequential update schedule network with strongly connected interaction digraph. Then, $\Phi_k(f^s) \neq \emptyset$ if and only if D^f_s is k-labeled.*

Proof. First, by Proposition 17, $D^{f^s} = \mathcal{P}(D^f_s)$ has a unique strongly connected component.

Suppose that D^f_s is k-labeled. Then every circuit in D^f_s has a multiple of k $\oplus$-labeled arcs. Therefore, every circuit in $\mathcal{P}(D^f_s)$ is of length multiple of k. Indeed, if there is a circuit of length no multiple of k in $\mathcal{P}(D^f_s)$ then there is a circuit in D^f_s with a number of $\oplus$-labeled arcs no multiple of k which is a contradiction by Proposition 18. Since any circuit in $\mathcal{P}(D^f_s)$ has a length multiple of k then k divides the index of cyclicity of the strongly connected component. As a result, by Lemma 3, $\Phi_k(f^s) \neq \emptyset$.

The other direction is symmetrical. Suppose that $\Phi_k(f^s) \neq \emptyset$. Then, by Proposition 3 the index of cyclicity of the strongly connected component of D^{f^s} is a multiple of k. Therefore, every circuit in D^{f^s} is of length multiple of k and by Proposition 18, every circuit in D^f_s has a number of $\oplus$-labeled multiple of k. Then D^f_s is k-labeled. □

Proposition 20. *Suppose that f is a conjunctive global function such that D^f is strongly connected. Then, k-BLCE(f) has a solution if and only if there exists a k-labeling* lab *of D^f such that D^f_{lab} is an update digraph. Furthermore, k-SLCE(f) has a solution if and only if there exists a k-labeling of D^f* lab *such that D^f_{lab} is a sequential update digraph.*

Proof. k-BLCE(f) (*resp.* k-SLCE(f)) has a solution $\Leftrightarrow$ there exists a block sequential update schedule s such that $\Phi_k(f^s) \neq \emptyset \Leftrightarrow D^f_s$ is an update k-labeled digraph (sequential update k-labeled digraph if s is sequential)

□

To prove Theorem 22 below which states that if k-BLCE(f) has a solution then so do k-SLCE(f), we first need to introduce the concept of *reversed* digraph which permit a characterization of the (sequential) update digraph.

Given a labeled digraph $D_{\text{lab}} = (D = (V, A), \text{lab})$, we define $D^R_{\text{lab}} = (D^R = (V, A^R), \text{lab}^R)$, the *reverse digraph*, as follows:

- $(u, v) \in A^R \iff ((u, v) \in A \wedge \text{lab}(u, v) = \oplus) \vee ((v, u) \in A \wedge \text{lab}(v, u) = \ominus)$.

- $\text{lab}^R(u,v) = \ominus$ if $\text{lab}(v,u) = \ominus$ and $\text{lab}^R(u,v) = \oplus$ otherwise.

Basically, each $\ominus$-arc (u,v) in D_{lab} gives a $\ominus$-arc (v,u) in D^R_{lab} and each $\oplus$-arc (u,v) in D_{lab} gives a $\oplus$-arc (u,v) in D^R_{lab} if D^R not already have a $\ominus$-arc (u,v). See an example of a interaction digraph in Figure 4).

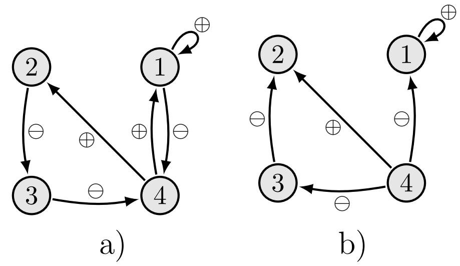


Figure 4: In a) a partial labeled digraph and in b) its reverse digraph.

A cycle in D^R_{lab} is called *forbidden* if it contains at least one $\ominus$-arc.

It was shown in Aracena et al. (2011) that there is a characterization of the (sequential) update digraph by the reverse digraph.

Lemma 21 (Aracena et al. (2011)). *D_{lab} is an update digraph if and only if D^R_{lab} does not contain any forbidden cycle. Furthermore, D_{lab} is a sequential update digraph if and only if D^R_{lab} is acyclic. Besides, these properties can be tested in polynomial time.*

Theorem 22. *For any conjunctive global function f such that D^f is strongly connected and for any $k \geq 3$, k-BLCE(f) $\implies$ $(k-1)$-SLCE(f) $\implies$ $(k-1)$-BLCE(f).*

Proof. Let $k \geq 3$, and consider a conjunctive global function f and $D^f = ([n], A)$. Suppose that k-BLCE(f) (resp. k-SLCE(f)) has a solution. Then there exists a labeling lab and a partition $V_0, \dots, V_{k-1}$ of $[n]$ such that for every arc $(i,j) \in A$, if $i \in V_p$, we have $j \in V_{(p+1) \bmod k}$ if (i,j) is a positive arc and $j \in V_p$ otherwise (and without cycle in V_ℓ for all $0 \leq \ell \leq k-1$). Therefore, a $(k-1)$-labeling lab$'$ can be obtained from lab by switching all arc (i,j) with $i \in V_{k-2}$ and $j \in V_{k-1}$ ($V_{k-2} \cup V_{k-1}$ is a part in the new partition) see Figure 5. Furthermore, the arcs from V_{k-2} to V_{k-1} is a feedback arc set of D^f, and all arcs from V_{k-2} to V_{k-1} are $\ominus$ arcs in $D^f_{\text{lab}'}$. Hence, $D^f_{\text{lab}'}$ has no cycle with only $\oplus$ arcs, and therefore it corresponds to a sequential update schedule by Lemma 21. Therefore, $(k-1)$-SLCE(f) has a solution. $\square$

Figure 6 shows an example of D^f such that 2-BLCE(f) has a solution, but not 2-SLCE(f) and not 2-PLCE(f).

Theorem 23. *The BLCE and SLCE problems are NP-complete and for all $k \geq 2$, the problems k-BLCE and k-SLCE are NP-complete.*

Theorem 23 is a direct consequence of Lemmas 24 and 28 below.

Lemma 24. *The BLCE, k-BLCE, SLCE and k-SLCE problems are in NP.*

Proof. A possible certificate is to give a block-sequential (*resp.* sequential) update schedule s and a configuration $x \in \Phi_k(f^s)$. Reminder that f^s is a conjunctive global function. By Lemma 13 it is polynomial to check that $x \in \Phi_k(f^s)$. $\square$

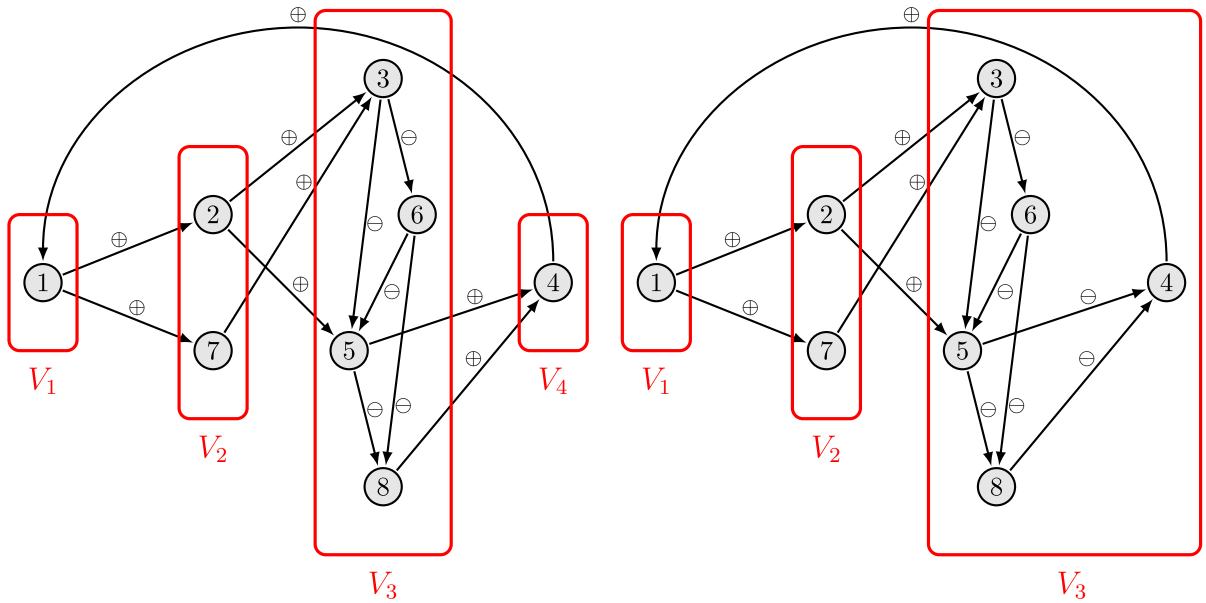


Figure 5: If D_s is k-labeled update schedule then exists $D_{s'}$ $(k-1)$-labeled sequential update schedule

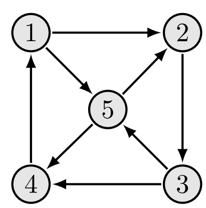


Figure 6: Example of D^f such that 2-BLCE(f) has a solution, but not 2-SLCE(f) and not 2-PLCE(f).

Lemma 25 (Macauley and Mortveit (2009))**.** *Let f be a global function and $s = (B^1, \ldots, B^{p-1}, B^p)$ be a block-sequential update schedule, then for any $k \geq 1$, $|\Phi_k(f^s)| = |\Phi_k(f^{(B^p, B^1, \ldots, B^{p-1})})|$.*

Corollary 26. *If for one instance f, the BLCE (resp. SLCE) problem has a solution with a block-sequential (resp. sequential) update schedule s, then, for each block X of s, there exist another solution s' which update X first.*

Proof. Direct from Lemma 25. $\square$

To prove that these problems are NP-hard, we do a reduction from 3-SAT. We will first consider $k = 2$ and then generalize.

Lemma 27. *The 2-BLCE and 2-SLCE problems are NP-hard.*

Proof. Consider a 3-CNF formula ψ composed of n variables $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ and m clauses $\mu = \{\mu_1, \ldots, \mu_m\}$. Construct the following digraph $D = (V, A)$ whose answer to the k-BLCE and k-SLCE problems are positives if and only if the formula ψ can be satisfied.

First, each variable $\lambda_i \in \lambda$ is represented in D by four vertices $x_i, t_i, \bar{x}_i, f_i$ and five arcs $(x_i, t_i), (t_i, \bar{x}_i), (\bar{x}_i, f_i), (f_i, x_i)$ and (t_i, f_i) as shown in Figure 7. Let $X = \{x_i : i \in [n]\} \cup \{\bar{x}_i : i \in [n]\}$.

Second, each clause $\mu_j \in \mu$ is represented by a cycle of seven vertices $c_{j,1}, \ell_{j,1}, c_{j,2}, \ell_{j,2}, c_{j,3}, \ell_{j,3}$ and $c_{j,4}$ (in this order) as represented in Figure 8. Each node $\ell_{j,p}$ corresponds to a literal $\mu_{j,p} = (\lambda_i, \rho)$. If $\rho = \top$,

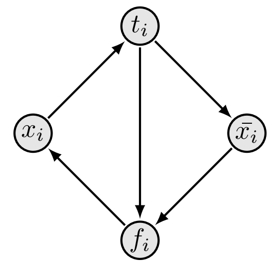


Figure 7: Representation of a variable of a 3-CNF ψ in the digraph D^{ψ}.

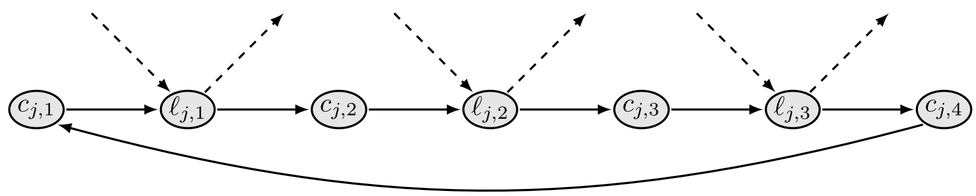


Figure 8: Representation of a clause of a 3-CNF ψ in the digraph D^{ψ}.

then we add the two arcs $(x_i, \ell_{j,p})$ and $(\ell_{j,p}, \bar{x_i})$. Otherwise, we add two arcs $(\bar{x_i}, \ell_{j,p})$ and $(\ell_{j,p}, x_i)$. Let $C = \{c_{j,p} : j \in [m] \text{ and } p \in [4]\}$ and $L = \{\ell_{j,p} : j \in [m] \text{ and } p \in [3]\}$.

Finally, we add two vertices a_1, a_k, an arc (a_1, a_k) and, for each vertex $v \in X \cup C$, we add two arcs (v, a_1) and (a_k, v). An example of digraph D^{ψ} is represented Figure 9.

Now, let us prove that if the 2-SLCE problem has a solution for an instance, then the corresponding 3-SAT too. Note that if the 2-SLCE problem has a solution then the 2-BLCE problem too because a sequential update schedule is a special block-sequential update schedule.

Now, let us prove that if ψ can be satisfied, there is a solution to the corresponding instance 2-SLCE instance. Let g be a valuation of λ that satisfies ψ. Consider the following partition of V. In V_1 we put:

- a_1
- For all variables λ_i, t_i if $g(\lambda_i) = \top$ or f_i if $g(\lambda_i) = \bot$.
- For all literal $\mu_{j,p}$, $\ell_{j,p}$ $g(\mu_{j,p}) = \top$ ($i.e$ $g(\lambda_i) = \rho$ with $\mu_{j,p} = (\lambda_i, \rho)$).

In V_0 we put the others. An example of partition is given in Figure 10.

For all u, let $V(u) = i$ such that $u \in V_i$. Then, let lab be the labeling such that for all $(u, v) \in A$, $\mathrm{lab}\,(u, v) = \ominus$ if $V(u) = V(v)$ and $\mathrm{lab}\,(u, v) = \oplus$ otherwise. To prove that this is a solution for 2-SLCE, it is sufficient to prove that the reversed digraph $D' = D^R_{\mathrm{lab}}$, is acyclic by Lemma 21.

First, one can show that there is no cycle in D' passing through a_k or b. Indeed, the only incoming arc of a_k in D is (b, a) which is positive, and the outgoing arcs of a_k are all negatives. Therefore, a_k has no outgoing arcs in D' and no cycle go through a_k. Similarly, the only outgoing arcs of a_1 in D' go to a_k. This signifies that no cycle can go through b.

Second, one can show that no cycle in D' passes through vertices $x_i, \bar{x_i}$ with $i \in [n]$. From now on, we will ignore the arcs (a_k, x_i), $(\bar{x_i}, a_1)$, $(a_k, \bar{x_i})$ and (x_i, a_1) because we already said that no cycle go through a_1 or a_k. There are two cases to consider. First, consider that $g(\lambda_i) = \top$. Then, all the out-neighbors of x_i in D are in V_1. Indeed, t_i is in V_1 and the vertices $\ell_{j,p}$ such that $\mu_{j,p} = (\lambda_i, \top)$ are also in V_1. Furthermore, all in-neighbors of x_i are in V_0. Indeed, f_i is in V_0 and the vertices $\ell_{j,p}$ such that $\mu_{j,p} = (\lambda_i, \bot)$ are also in V_0. Therefore, all in-arcs of x_i in D are negative arcs and all out-arc are positive. This signifies that x_i has no in-neighbors in D' and therefore no cycle go through x_i. The same way, all

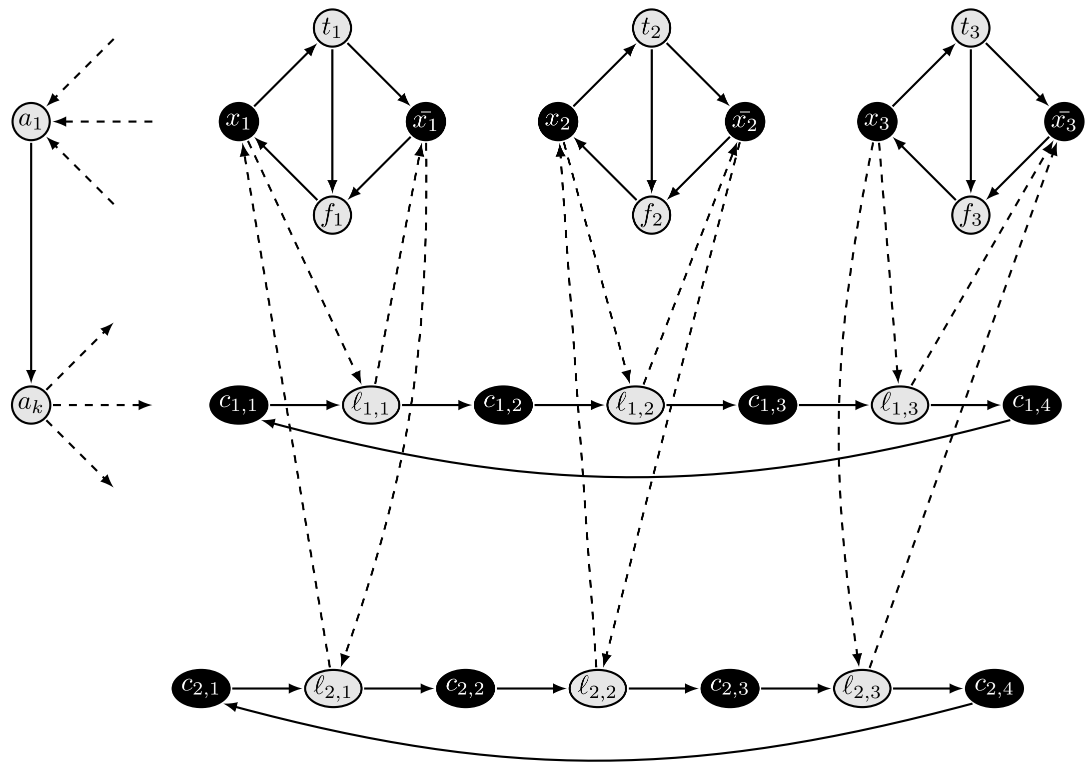


Figure 9: Representation of the digraph D^{ψ} with ψ the 3-CNF formula $\lambda_1 \vee \lambda_2 \vee \lambda_3 \wedge \neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3$. To not put too much arcs, the out-arcs of a_k and the in-arcs of a_1 are not represented. The vertices v such that there exists an arc (v, a_1) and another arc (a_k, v) are fill in black.

in-arcs of $\bar{x}_i$ in D are positive and all out-arc are negative arcs. Therefore, $\bar{x}_i$ has no out-neighbors in D' and therefore no cycle go through $\bar{x}_i$. The case where $g(\lambda_i) = \perp$ is totally symmetrical. Because no cycle go through x_i and $\bar{x}_i$ in D', one can see that no cycle can go through t_i or f_i neither.

Finally, the only remaining vertices are those in C and L. The graph D_{ψ} restricted to these vertices is only composed of m disjoint cycles corresponding to the m clauses. Consider the cycle that correspond to a clause μ_j. One can see that the arc $(c_{j,4}, c_{j,1})$ is negative (because $c_{j,4}$ and $c_{j,1}$ are in V_0). Hence, for a cycle in D' exists in the component corresponding to the μ_j, all the arcs have to be negative and therefore, all the vertices have to be in V_0. However, since g is a valid solution of ψ, one of the vertices $\ell_{j,1}$, $\ell_{j,2}$ or $\ell_{j,3}$ is in V_1. As a result, D' has no cycles and 2-SLCE (and therefore 2-BLCE) has a solution.

Now, let us consider a partition of 2-BLCE on a instance D_{ψ} and prove that there is also a solution of 2-SLCE and that it gives a solution for the corresponding 3-CNF.

Conversely, suppose that there exists a block-sequential update schedule s' such that $\Phi_k(f^{s'}) \neq \emptyset$.

First, let us prove that there exists a block-sequential update schedule s which update first a block $B_1 = \{a_k\}$ and such that $\Phi_k(f^s) \neq \emptyset$. First, by Lemma 25, it is possible to update the block B_1 which contains a_k in first. Second, by Aracena et al. (2011), if is impossible to update a_k alone, then it is because there is a cycle in D^f_s containing a_k with only positive arcs. All cycles in D^f going through a_k are passing through a_1 because $\mathcal{N}^{\text{in}}_{D^f}(a_k) = \{a_1\}$. Furthermore, the positive cycles are necessarily going through a vertex $v \in X \cup C$ because $\mathcal{N}^{\text{out}}_{D^f}(a_k) = X \cup C$. This means that a_1, a_k and v are updated in the same block B_1 and $s(a_k) = s(a_1) = s(v)$. Furthermore, because $\mathcal{N}^{\text{in}}(a_1) = X \cup C$, there exists an arc (v, a_1) and because $s(a_1) = s(v)$, the arc (v, a_1) is positive like the arcs (a_1, a_k) and (a_k, v). As a result, the cycle (a_1, a_k, v) has 3 positive arcs, which is not a multiple of $k = 2$ and D is not a k-labeling which is a contradiction. As a result, we can find a block-sequential update schedule $s = B_1, \ldots, B_m$ such that $B_1 = \{a_k\}$ such that $\Phi_2(f^s) \neq \emptyset$ by Aracena et al. (2011).

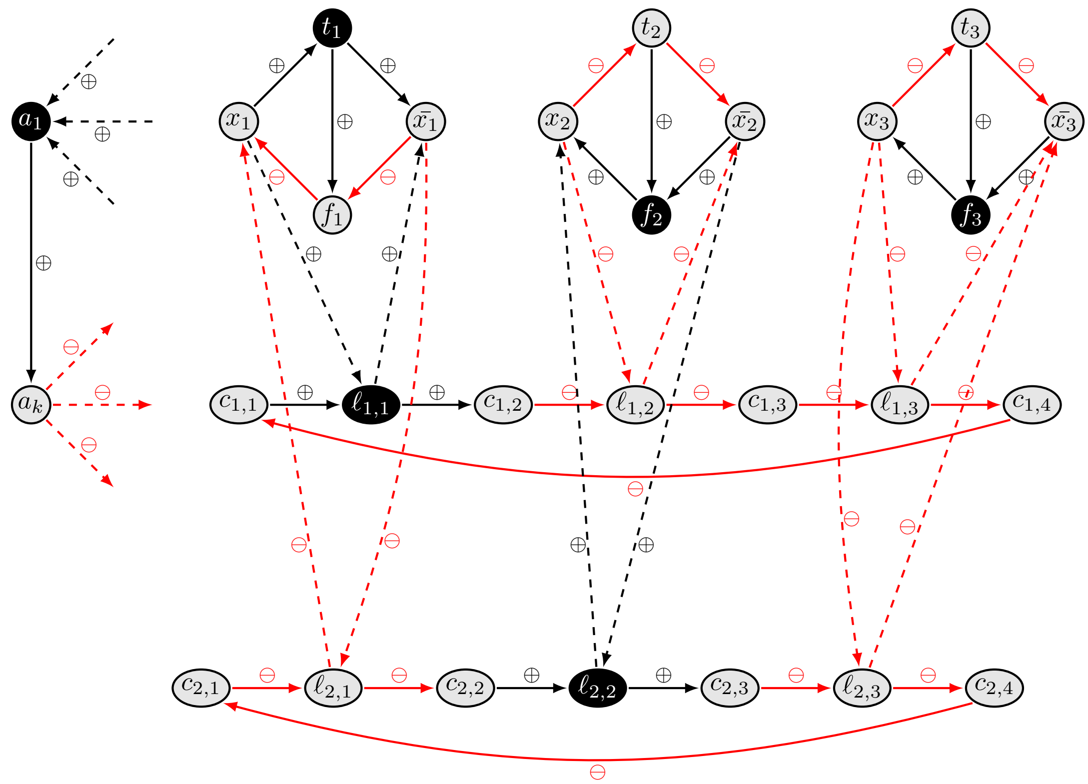


Figure 10: Representation of a partition of D^ψ with ψ the 3-CNF formula $\lambda_1 \vee \lambda_2 \vee \lambda_3 \wedge \neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3$ which make it a 2-labeled sequential update digraph. In this example we take $g(\lambda_1) = \top, g(\lambda_2) = \bot$ and $g(\lambda_3) = \bot$ and we represent V_1 in black and V_0 in light gray).

Therefore, $D^s = (D, \text{lab})$ is a 2-labeling and therefore there is a partition $\{V_0, V_1\}$ of V such that for all $(u, v) \in A$, $\text{lab}(u, v)$ is positive iff $V(u) \neq V(v)$.

For all $u \in V$, let $V(u) = i$ such that $u \in B_i$. Without loss of generality, suppose $V(a_k) = 0$. Since, $B_1 = \{a_k\}$, $s(a_k) = 1$ then for all $v \in X \cup C$, we have $s(a_k) < s(v)$ and $(a_k, v) \in A$. Therefore, $\text{lab}(a_k, v) = \ominus$ and then $V(v) = V(a_k) = 0$.

Now, let us prove that for all $i \in [n]$, $V(t_i) \neq V(f_i)$. For the sake of contradiction, suppose that $V(t_i) = V(f_i)$. There are two cases.

- If $V(t_i) = V(f_i) = 0$, then $\text{lab}(x_i, t_i) = \text{lab}(t_i, f_i) = \text{lab}(f_i, x_i) = \ominus$ and (x_i, f_i, t_i) is a cycle of $(D^f_s)^R$ with only negative arcs. It is a contradiction, because D^f_s would not be an update digraph.

- If $V(t_i) = V(f_i) = 1$ then $\text{lab}(t_i, \overline{x_i}) = \text{lab}(\overline{x_i}, f_i) = \oplus$, $\text{lab}(t_i, f_i) = \ominus$ and $(\overline{x_i}, f_i, t_i)$ is a cycle of $(D^f_s)^R$ with a negative (f_i, ti) arc. It is a contradiction, because D^f_s would not be an update digraph.

As a result, we have $V(t_i) \neq V(f_i)$ for all $i \in [n]$. Note that there are two cases:

- $V(t_i) = 0$, $V(f_i) = 1$, $\text{lab}(x_i, t_i) = \text{lab}(t_i, \overline{x_i}) = \ominus$ and $\text{lab}(\overline{x_i}, f_i) = \text{lab}(f_i, x_i) = \oplus$.

- $V(t_i) = 1$, $V(f_i) = 0$, $\text{lab}(x_i, t_i) = \text{lab}(t_i, \overline{x_i}) = \oplus$ and $\text{lab}(\overline{x_i}, f_i) = \text{lab}(f_i, x_i) = \ominus$.

In these two cases, since $V(t_i) \neq V(f_i)$, $\text{lab}(f_i, t_i) = \oplus$.

Consider the valuation $g : \lambda_i \mapsto \begin{cases} \top & \text{if } V(t_i) = 1 \\ \bot & \text{if } V(t_i) = 0 \end{cases}$. In the following, we prove that g is a valuation that satisfies ψ.

Let us prove that, for all $j \in m$, $p \in [3]$, we have $g(\mu_{j,p}) = \top$ if and only if $V(\ell_{j,p}) = 1$. To prove that, it is sufficient to show that $V(\ell_{j,p}) = V(t_i)$ if $\mu_{j,p} = (\lambda_i, \top)$ and $V(\ell_{j,p}) = V(f_i)$ if $\mu_{j,p} = (\lambda_i, \bot)$.

First, suppose that $\mu_{j,p} = (\lambda_i, \top)$. Recall that wa have two arcs $(x_i, \ell_{j,p})$ and $(\ell_{j,p}, \overline{x_i})$. Now, for the sake of contradiction, suppose that $V(\ell_{j,p}) \neq V(t_i)$. There are two cases.

- If $V(t_i) = 0$ and $V(\ell_{j,p}) = 1$, then $\text{lab}(x_i, t_i) = \text{lab}(t_i, \overline{x_i}) = \ominus$ and $\text{lab}(x_i, \ell_{j,p}) = \text{lab}(\ell_{j,p}, \overline{x_i}) = \oplus$. As a result, $(x_i, \ell_{j,p}, \overline{x_i}, t_i, x_i)$ is a cycle in $(D_s^f)^R$ with two negative arcs ($(\overline{x_i}, t_i)$ and (t_i, x_i)).
- If $V(t_i) = 1$ and $V(\ell_{j,p}) = 0$, then $\text{lab}(x_i, t_i) = \text{lab}(t_i, \overline{x_i}) = \oplus$ and $\text{lab}(x_i, \ell_{j,p}) = \text{lab}(\ell_{j,p}, \overline{x_i}) = \ominus$. As a result, $(x_i, t_i, \overline{x_i}, \ell_{j,p}, x_i)$ is a cycle in $(D_s^f)^R$ with two negative arcs ($(\overline{x_i}, \ell_{j,p})$ and (t_i, x_i)).

Both cases are impossible because it would mean that D_s^f is not an update digraph. Therefore, if $\mu_{j,p} = (\lambda_i, \top)$ then $g(\mu_{j,p}) = \top$ if and only if $V(\ell_{j,p}) = 1$. The case where $\mu_{j,p} = (\lambda_i, \bot)$ is totally symmetrical.

Finally, let us prove that all clauses are satisfied by g (and then that the formula ψ is satisfied by g). In other word, let us prove that for all $j \in [m]$ there exists $p \in [3]$ such that $g(\mu_{j,p}) = \top$. By contradiction, suppose that for some $j \in [m]$, for all $p \in [3]$, we have $g(\mu_{j,p}) = \bot$. This means that

$$V(c_{j,1}) = V(\ell_{j,1}) = V(c_{j,2}) = V(\ell_{j,2}) = V(c_{j,3}) = V(\ell_{j,3}) = V(c_{j,4}) = 0$$

and then that $(c_{j,4}, \ell_{j,3}, c_{j,3}, \ell_{j,2}, c_{j,2}, \ell_{j,1}, c_{j,1})$ is a cycle of $(D_s^f)^R$ with only negative arcs which is a contradiction.

Then g is a valid valuation of ψ. Furthermore, with the same reasoning as earlier, we can show that D_s^f is an sequential update digraph. Indeed:

- No positive cycles of D_s^f go through a_k and then through b.
- For any $i \in [n]$, no positive cycles of D_s^f go through x_i or $\overline{x_i}$ and then through t_i or f_i.
- For any $j \in [m]$, no positive cycle can go through $c_{j,1}$ because $\text{lab}(c_{j,4}, c_{j,1}) = \ominus$ and then through $L \cup C$.

It means that if there is a block-sequential solution, then there is a sequential one and this solution corresponds to a valuation of ψ. Hence, we finished the reduction from 3-SAT to 2-PLCE and 2-BLCE and they are NP-hard.

□

Lemma 28. *For any $k \geq 2$, k-BLCE and k-SLCE problems are NP-hard.*

Proof. Consider the digraph $D^{\psi,k} = (V', A')$ which is constructed from the digraph $D^\psi = (V, A)$ of Lemma 27. More precisely, the following arcs of D^ψ will be replaced by isolated paths of length $k-1$ in $D^{\psi,k}$:

- (a_1, a_k)
- $(t_i, \overline{x_i})$ for any $i \in [n]$
- (t_i, f_i) for any $i \in [n]$
- (f_i, x_i) for any $i \in [n]$
- $(c_{j,p}, \ell_{j,p})$ for any $j \in [m]$

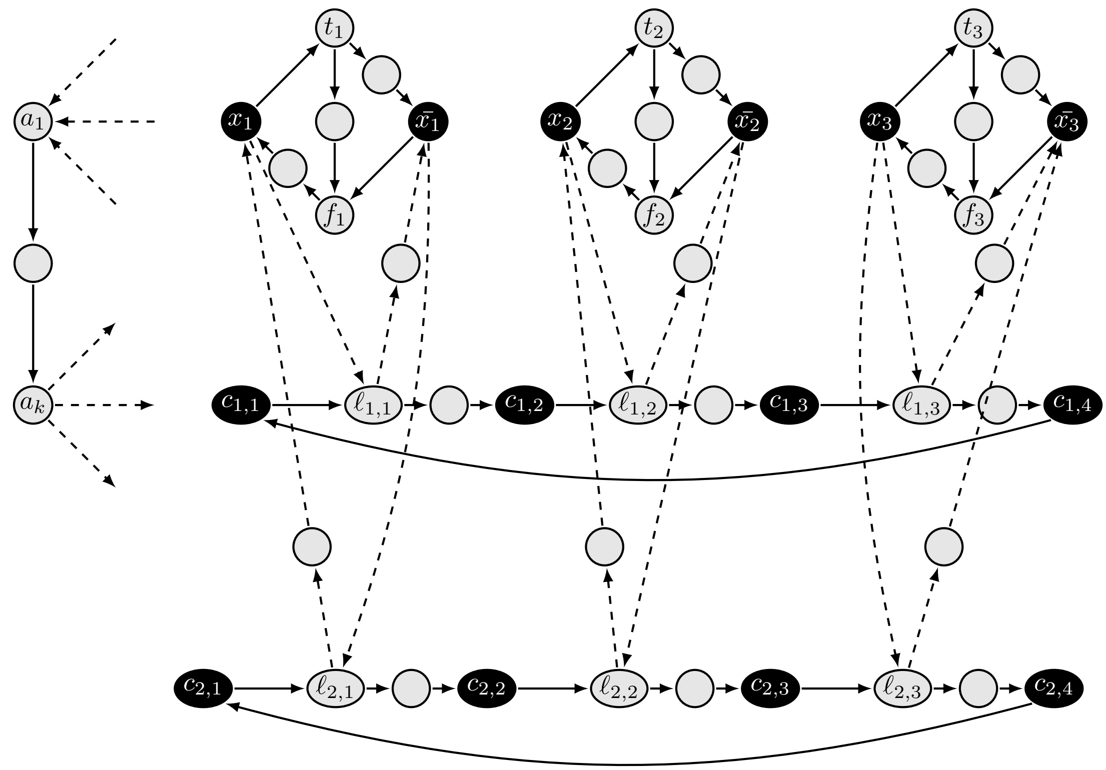


Figure 11: Representation of the digraph $D^{\psi,k}$ with ψ the 3-CNF formula $\lambda_1 \vee \lambda_2 \vee \lambda_3 \wedge \neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3$ and $k = 3$. It differs from D^{ψ} because some arcs are replace by isolated paths of length $k - 1 = 2$.

- $(c_{j,p}, x_i)$ or $(c_{j,p}, \overline{x_i})$ for $j \in [m]$ and $i \in [n]$ such that the arc exists.

An example of digraph $D^{\psi,k}$ for $k = 3$ is given Figure 11.

The proof is the same that Lemma 27, but instead of the digraph D^{ψ}, we use the digraph $D' = D^{\psi,k}$.

If ψ has a solution then there is a labeling lab of D^{ψ} such that D^{ψ}_{lab} is a 2-labeling sequential update digraph. From this construction, we can construct a labeling lab$'$ such that $D'_{\text{lab}'}$ is a k-labeling sequential update digraph.

Consider the partition function $h : V \to [0, 1]$ such that for all $\ell \in \{0, 1\}$, and for all $i \in V_\ell$, we have $h(i) = \ell$. We define the new partition function $h' : V' \to [0, k[$ as follows. First, for all $v \in V$, $h'(v) = h(v) \in \{0, 1\}$. Second, for all arc $(v^1, v^k) \in A$ replaced by a path $(v^1, v^2, \ldots, v^k)$ in D' there are three cases:

- If $h(v^1) = h(v^k)$ (and therefore the arc (v^1, v^k) is negative) then $h'(v^1) = h'(v^2) = \cdots = h'(v^k)$ (and therefore all the path is negative).

- If $h(v^1) = 1$ and $h(v^k) = 0$ (and therefore, in D_ψ, the arc (v^1, v^k) is positive), then for all $i \in [2, k]$, we have $h'(v^i) = h'(v^{i-1}) + 1 \bmod k$. In other words, $h'(v^1) = 1, h'(v^2) = 2, \ldots, h'(v^{k-1}) = k - 1$ and $h'(v^k) = 0$ and all the path is positive in D'.

- If $h(v^1) = 0$ and $h(v^1) = 1$ (and therefore the arc (v^1, v^k) is positive), then $h'(v^1) = 0$ and $h'(v^2) = h'(v^3) = \cdots = h'(v^k) = 1$ and the path is composed of one positive arc and $k - 2$ negative arcs.

One can check that for all arc $(v, v') \in A'$ either $h'(v') = h'(v)$ (and therefore $\text{lab}'(v, v') = \ominus$) or $h'(v') = (h'(v) + 1) \bmod k$ (and therefore $\text{lab}'(v, v') = \oplus$). As a result, $D'_{\text{lab}'}$ is k-labeled. Furthermore, one can see that $D'_{\text{lab}'}$ is a sequential update digraph. Indeed, as seen in Lemma 27, D_{lab} is a sequential update digraph and therefore D^R_{lab} is acyclic. Moreover, there are few differences between D^R_{lab} and $D'^R_{\text{lab}'}$: some arcs (v^1, v^k) of D_{lab} are replaced by a path $(v^1, v^2, \ldots, v^k)$ in $D'_{\text{lab}'}$. However, as we have seen, if $\text{lab}(v^1, v^k) = \oplus$ then the entire path $(v^1, v^2, \ldots, v^k)$ is positive in $D'_{\text{lab}'}$ and will therefore not add any cycle in $D'^R_{\text{lab}'}$. The same way, if $\text{lab}(v^1, v^k) = \oplus$ then either the entire path $(v^1, v^2, \ldots, v^k)$ is negative (and then the negative arc (v^k, v^1) of D^R_{lab} is replaced by a negative path in $D'^R_{\text{lab}'}$), or (v^1, v^2) is positive and all the path $(v^2, \ldots, v^k)$ is negative (and then the negative arc (v^k, v^1) of D^R_{lab} is replace by the two paths (v^1, v^2) and $(v^k, \ldots, v^2)$ in $D'^R_{\text{lab}'}$). In both cases, there are no possible cycles added. Therefore, $D'^R_{\text{lab}'}$ is acyclic and $D'_{\text{lab}'}$ is a sequential update digraph.

Conversely, consider a block-sequential update schedule s' such that of $D'_{s'}$ is an update digraph. One can prove, similarly that in Lemma 27 that there exists another block-sequential s which updates $\{a_k\}$ first. Indeed, from Lemma 25, we know that we can update the block containing a_k first. Furthermore, we can prove that we can update a_k only in the first block because if a_k is in a positive cycle then it is in a positive cycle of length $k+1$ which would contradict the fact that D'_s is a k labeling.

As a result, for all $v \in X \cup C$, we have $h'(v) = h'(a_k) = 0$. On the other hands, for all $v' \in V$ there exists $v \in X \cup C$ such that $(v, v') \in A'$. Therefore, for all $v \in V$, we have $h'(v) \in \{0, 1\}$.

Now, we prove that for all $i \in [n]$, $h'(t_i) \neq h'(f_i)$. For the sake of contradiction, let us suppose that $h'(t_i) = h'(f_i)$. Note that the path between t_i and f_i is only composed of negative arcs because there are only $k-1$ arcs between t_i and f_i and $h'(t_i) = h'(f_i)$. There are two cases. First, if $h'(t_i) = h'(f_i) = 0$ then the path $(f_i, \ldots, x_i)$ and the arc (x_i, t_i) are full negative because $h'(x_i) = h'(t_i) = h'(f_i) = 0$. This means that there is a cycle $(t_i, x_i, \ldots, f_i, \ldots, t_i)$ in $D'^R_{\text{lab}'}$ with only negative arcs. Second, if $h'(t_i) = h'(f_i) = 1$ then the path $(t_i, \ldots, \bar{x}_i)$ and the path $(\bar{x}_i, f_i)$ are positive $h'(\bar{x}_i)$ and $h'(t_i) = h'(f_i) = 1$. This means that there is a cycle $(t_i, \ldots, \bar{x}_i, f_i, \ldots, t_i)$ in $D'^R_{\text{lab}'}$ with negative arcs in the path $(f_i, \ldots, t_i)$. As a result, we have $h'(t_i) \neq h'(f_i)$.

Consider the valuation $g : \lambda_i \mapsto \begin{cases} \top & \text{if } h'(t_i) = 1 \\ \bot & \text{if } h'(t_i) = 0 \end{cases}$. In the following, we prove that g is a valid valuation of ψ.

Let us prove that, for all $j \in m$, $p \in [3]$, we have $g(\mu_{j,p}) = \top$ if and only if $h(\ell_{j,p}) = 1$.

To prove that, it is sufficient to show that $h'(\ell_{j,p}) = h'(t_i)$ if $\mu_{j,p} = (\lambda_i, \top)$ and $h'(\ell_{j,p}) = h'(f_i)$ if $\mu_{j,p} = (\lambda_i, \bot)$.

First, suppose that $\mu_{j,p} = (\lambda_i, \top)$. Recall that wa have an arc $(x_i, \ell_{j,p})$ and a path $(\ell_{j,p}, \ldots, \overline{x_i})$. Now, for the sake of contradiction, suppose that $h(\ell_{j,p}) \neq h(t_i)$. There are two cases.

- If $h'(t_i) = 0$ and $h'(\ell_{j,p}) = 1$, then the path $(x_i, t_i, \ldots, \overline{x_i})$ is fully negative when the path $(x_i, \ell_{j,p}, \ldots, \overline{x_i})$ is fully positive. As a result, $(x_i, \ell_{j,p}, \ldots, \overline{x_i}, \ldots, t_i, x_i)$ is a cycle in $(D')^R_{\text{lab}'}$ with negative arcs.

- If $h'(t_i) = 1$ and $h'(\ell_{j,p}) = 0$, then the path $(x_i, t_i, \ldots, \overline{x_i})$ is fully positive when the path $(x_i, \ell_{j,p}, \ldots, \overline{x_i})$ is fully negative. As a result, $(x_i, t_i, \ldots, \overline{x_i}, \ldots, \ell_{j,p}, x_i)$ is a cycle in $(D')^R_{\text{lab}'}$ with negative arcs.

Both cases are impossible because it would mean that $D'_{\text{lab}'}$ is not an update digraph and then if $\mu_{j,p} = (\lambda_i, \top)$ then $g(\mu_{j,p}) = \top$ if and only if $h'(\ell_{j,p}) = 1$. The $\mu_{j,p} = (\lambda_i, \bot)$ case is totally symmetrical.

Finally, let us prove that all clauses are satisfied by g (and then that the formula ψ is satisfied by g). In other word, let us prove that for all $j \in [m]$ there exists $p \in [3]$ such that $g(\mu_{j,p}) = \top$. By contradiction, suppose that for some $j \in [m]$, for all $p \in [3]$, we have $g(\mu_{j,p}) = \bot$. This signifies that

$$h(c_{j,1}) = h(\ell_{j,1}) = h(c_{j,2}) = h(\ell_{j,2}) = h(c_{j,3}) = h(\ell_{j,3}) = h(c_{j,4}) = 0$$

and therefore that all paths $(\ell_{j,p}, \ldots, c_{j,p+1})$ are fully negative and then that $(c_{j,4}, \ldots, \ell_{j,3}, c_{j,3}, \ldots, \ell_{j,2}, c_{j,2}, \ldots, \ell_{j,1}, c_{j,1})$ is a cycle of $(D')^R$ with only negative arcs which is a contradiction.

Then g is a valid valuation of ψ. Furthermore, with the same reasoning than earlier, we can show that D' is an sequential update digraph. Indeed:

- No positive cycles of D' go through a_k and then through b.
- For any $i \in [n]$, no positive cycles of D' go through x_i or $\overline{x_i}$ and then through t_i or f_i.
- For any $j \in [m]$, no positive cycle can go through $c_{j,1}$ because $\text{lab}(c_{j,4}, c_{j,1}) = \ominus$ and then through $L \cup C$.

It means that if there is a block-sequential solution, then there is a sequential one and this solution corresponds to a valuation of ψ. Hence, we finished the reduction from 3-SAT to k-PLCE and k-BLCE and they are NP-hard. □

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7. Conclusion

In this paper, we study the complexity of the problem of determining if a conjunctive network has a limit cycle of a given length k.

In a first part, we study this problem with a parallel update schedule. We show that the problem is in P when the interaction digraph of the conjunctive network is strongly connected. Furthermore, we also prove that the problem is NP-complete when k is a parameter of the problem and the interaction digraph is not strongly connected. However, the case where the interaction digraph is not strongly connected, but k is fixed remains open. As a side result, we proved that the lengths of the limit cycle of a conjunctive network of length n cannot divide a prime greater than n and that the maximum length of a limit cycle of a conjunctive network of length n corresponds to the Landau's function $g(n)$.

In a second part, we study this problem with block-sequential and sequential update schedules. The problem is then: is there a block-sequential (*resp.* a sequential) update schedule s such that f^s has a limit cycle of length k. We prove that this problem, even with k fixed and with sequential or block-sequential update schedule is NP-complete for all $k \geq 2$.

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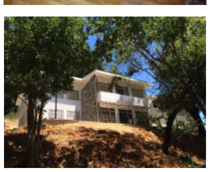

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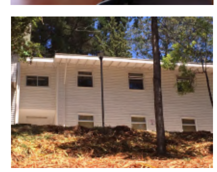

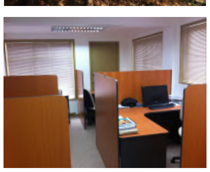

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