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A review on some discrete variational techniques for the approximation of essential boundary conditions

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Abstract

We review different techniques to enforce essential boundary conditions, such as the (nonhomogeneous) Dirichlet boundary condition, within a discrete variational framework, and especially techniques that allow to account for them in a weak sense. Those are of special interest for discretizations such as geometrically unfitted finite elements or high order methods, for instance. Some of them remain primal, and add extra terms in the discrete weak form without adding a new unknown: this is the case of the boundary penalty and Nitsche techniques. Others are mixed, and involve a Lagrange multiplier with or without stabilization terms. For a simple setting, we detail the different associated formulations, and recall what is known about their stability and convergence properties.

Keywords: essential boundary conditions, Dirichlet boundary condition, finite elements, penalty, Nitsche, mixed finite elements, stabilized finite elements.

AMS Subject Classification: 65N12, 65N15, 65N30.

1 Introduction

Among the first papers that contributed to the mathematical theory of the finite element method, some already proposed various techniques to handle essential boundary conditions, such as the (nonhomogeneous) Dirichlet boundary conditions. Early contributions have been made by I. Babuška in 1972 and 1973 with two seminal papers about Lagrange multipliers [14] and penalty [15] formulations. In the same period, in 1971, J.A. Nitsche published an alternative boundary penalty technique [118], consistent and with no additional unknown.

These pioneering works were followed by studies that allowed to refine the first analysis, to extend these techniques to more complex problems, and to invent alternatives, as well. The first motivation for some improvements in the approximation of essential boundary conditions concerns discrete variational techniques for which the degrees of

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freedom on the boundary are not nodal values, and in which it is not direct to approximate a Dirichlet condition. This is notably the case for unfitted finite elements or geometrically nonconforming finite elements, where the mesh boundary and the domain boundary do not match, as it occurs in fictitious domain methods, the extended finite element method or the cut finite element method: see, *e.g.*, [47, 91, 98, 116, 119]. It is also the case for IsoGeometric Analysis (IGA) [67] where degrees of freedom are control poins of B-spline functions, and are not nodes on the boundary [25]. A similar situation occurs in domain decomposition techniques with non-matching meshes, where the mortar method [30, 33, 34, 133] or Nitsche's method [28, 87, 104] have revealed their usefulness. Another motivation has been the design of more and more sophisticated numerical methods to solve efficiently problems involving more complex essential boundary or interface conditions, such as fluid-structure interaction [13, 84, 89], contact and friction [58, 134] or even fluid-structure-contact now [10, 48, 72, 85, 114].

A survey on the topic has been published by R. Stenberg in 1995 [129], that presented some mixed, stabilized and Nitsche's formulations. Notably this survey presented in a simplified form the stabilization technique of H. Barbosa and T.J.R. Hughes [17, 18], and it revealed a fundamental relationship between Barbosa & Hughes stabilization and Nitsche's original symmetric formulation (see also [104] for domain decomposition). The aim of the present survey is to complete and update this previous one with new techniques that emerged since then, as for instance variants of Nitsche's method or new stabilized formulations. Moreover we include a short description of two common techniques: the discrete lifting and boundary penalty formulations.

This paper is outlined as follows. In Section 2 we introduce Poisson's problem with a nonhomogeneous Dirichlet boundary condition as a model problem representing an elliptic partial differential equation with an essential boundary condition. In Section 3 we describe primal methods for which the only unknown is the solution to Poisson's problem, and no extra unknown is introduced. In Section 4 we present methods coming from a duality argument and introducing a Lagrange multiplier that represents the flux on the boundary. We conclude in Section 5.

Let us introduce some useful notations. For D an open set in $\mathbb{R}^d$, we denote by $H^s(D)$, $s \in \mathbb{R}$, the Sobolev space of real-valued function on D (see, *e.g.*, [2]). The standard scalar product (resp. norm) of $H^s(D)$ is denoted by $(\cdot,\cdot)_{s,D}$ (resp. $\|\cdot\|_{s,D}$). When $s = 0$ we drop the index s and note $(\cdot,\cdot)_D$ (resp. $\|\cdot\|_D$) the scalar product (resp. norm) in $L^2(D)$. The notations $C > 0$, $c > 0$ and $\beta > 0$ stand for arbitrary constants (the value of which may vary from one place to another) independent of the mesh size.

2 Setting

Let Ω be an open and bounded polytope of $\mathbb{R}^d$ ($d \geq 1$), of Lipschitz boundary $\Gamma := \partial\Omega$. The notation $\mathbf{n}$ stands for the unit outer normal to Γ. For a smooth enough function $v : \Omega \to \mathbb{R}$, $\partial_{\mathbf{n}} v := \nabla v \cdot \mathbf{n}$ stands for its normal derivative on Γ.

2.1 Poisson's problem

As a model problem, we focus on Poisson's problem with (nonhomogeneous) Dirichlet boundary condition:

$$\text{Find } u : \Omega \to \mathbb{R} \text{ solution to}$$
$$\begin{cases} -\Delta u = f & \text{in } \Omega, \quad (i) \\ \quad u = g & \text{on } \Gamma, \quad (ii) \end{cases} \tag{1}$$

with given source term $f \in L^2(\Omega)$ and boundary datum $g \in H^{\frac{1}{2}}(\Gamma)$. The equivalent weak form is:

$$\begin{aligned}&\text{Find } u \in H^1(\Omega) \text{ with } u|_\Gamma = g \text{ and}\\ &a(u,v) = (f,v)_\Omega \quad \text{for all } v \in H^1_0(\Omega),\end{aligned} \tag{2}$$

where $u|_\Gamma$ is the trace of u on the boundary Γ, $H^1_0(\Omega)$ is the subspace of functions in $H^1(\Omega)$ with vanishing trace on Γ, and with the notation

$$a(v,w) := (\nabla v, \nabla w)_\Omega$$

for $v, w \in H^1(\Omega)$. Problem (2) admits one unique solution u in $H^1(\Omega)$ (see [124, Section 4.4] or [83, Proposition 31.12]) that is also the minimum on $H^1(\Omega)$ of the quadratic convex functional

$$\mathcal{J} : H^1(\Omega) \ni v \mapsto \frac{1}{2}a(v,v) - (f,v)_\Omega \in \mathbb{R}$$

under the equality constraint $v_\Gamma = g$. Moreover, with $\mathcal{L}g \in H^1(\Omega)$ a lifting of g [115], Problem (2) can be reformulated equivalently:

$$\begin{aligned}&\text{Find } u \in H^1(\Omega),\ u = \mathcal{L}g + u_0, \text{ with } u_0 \in H^1_0(\Omega) \text{ solution to}\\ &a(u_0,v) = (f,v)_\Omega - a(\mathcal{L}g, v) \quad \text{for all } v \in H^1_0(\Omega).\end{aligned} \tag{3}$$

2.2 Finite element setting

We denote by $\mathbb{P}_k$ the vector space of d-variate real polynomials of maximal degree $k \geq 1$. Let $(\mathcal{T}^h)_{h>0}$ be a family of simplicial meshes of the domain Ω. We define the mesh size h as follows: $h := \max_{T\in\mathcal{T}^h} h_T$, where h_T is the diameter of the simplex T. The mesh is supposed to be regular in Ciarlet's sense: there exists $\sigma > 0$ such that

$$\frac{h_T}{\rho_T} \leq \sigma, \quad \forall T \in \mathcal{T}^h, \tag{4}$$

where ρ_T is the radius of the largest ball contained in the simplex T. We call facets either the edges of the triangles (when $d = 2$) or the faces of the tetrahedra (when $d = 3$). The notation E is used for a generic facet in $\mathcal{T}^h$ and the notation h_E stands for the diameter of the facet E. Remember that the regularity of the mesh implies there exists $c, C > 0$ (independent of the mesh size) such that

$$ch_T \leq h_E \leq Ch_T$$

for any facet E that belongs to a simplex $T \in \mathcal{T}_h$. We suppose finally that the mesh resolves the boundary Γ: for each simplex, each of its facets that intersects Γ is contained completely in one and only one of the facets of Γ.

The Lagrange finite element space of degree $k \geq 1$ is (see, *e.g.*, [40, 63, 82]):

$$V^{h,k} := \left\{v^h \in \mathscr{C}^0(\overline{\Omega}) \mid v^h|_T \in \mathbb{P}_k(T), \forall T \in \mathcal{T}^h\right\}. \tag{5}$$

We will omit the subscript k when there is no ambiguity and note simply V^h instead of $V^{h,k}$. We define also

$$V^h_0 := V^h \cap H^1_0(\Omega) = \{v^h \in V^h \mid v^h|_\Gamma = 0\}$$

the vector space of functions with vanishing discrete trace.

For the analysis of some methods, it will be convenient to make use of the following discrete norms. For this purpose let us take ζ^α ($\alpha \in \mathbb{R}$) as a piecewise constant function on the boundary Γ, that represents locally the power α of the boundary mesh size, and which restriction is defined as

$$\zeta^{\alpha}|_E = h_E^{\alpha},$$

for any boundary facet E. For $v \in L^2(\Gamma)$ we define

$$\|v\|_{-1/2,h,\Gamma} := \|\zeta^{\frac{1}{2}} v\|_{0,\Gamma_C}, \qquad \|v\|_{1/2,h,\Gamma} := \|\zeta^{-\frac{1}{2}} v\|_{0,\Gamma_C}.$$

For $v \in H^1(\Omega)$ we set

$$\|v\|_h := \left(\|\nabla v\|_{0,\Omega}^2 + \|v\|_{1/2,h,\Gamma}^2\right)^{\frac{1}{2}}$$

which is an equivalent norm of the $H^1(\Omega)$-norm.

For the stability and convergence analysis of some methods, the following discrete trace inequality (often called discrete trace inverse inequality) will be needed

Lemma 2.1. *There exists a constant $c_I > 0$ independent of the mesh size h such that, for any $v^h \in V^h$:*

$$\left\|\partial_{\mathbf{n}} v^h\right\|_{-1/2,h,\Gamma}^2 \leq c_I \|\nabla v^h\|_{0,\Omega}^2. \tag{6}$$

Moreover, for finite elements of degree k on a simplicial mesh in dimension d there holds:

$$c_I \leq \frac{c_\rho}{d}(k+1)(k+d) \tag{7}$$

where c_ρ is a constant that depends only on ρ and is independent of the polynomial degree.

Proof. See [82, Lemma 12.8] for (6) (or also [130, Lemma 2.1] when the mesh is quasi-uniform). The estimate (7) is established in [132] (see also [82, Lemma 12.2]). □

3 Primal methods

This section focuses on primal methods. Primal means that we do not use duality techniques so as to reformulate the problem. Duality techniques introduce some extra unknowns (here associated to fluxes on the boundary). So primal methods are characterized by the fact that there is just one unknown, u, that is directly approximated. In this framework, there are various possibilities to enforce the nonhomogeneous Dirichlet boundary condition (1)–(ii). We describe first the most standard one, that is presented in most of the basic textbooks and classnotes, and that is based upon a discrete lifting operator. Then we focus on the penalty method, which is also widespread, and even simpler in some sense than the discrete lifting technique, but with the fundamental issue that consistency is lost. We end this presentation with Nitsche's technique, in which supplementary terms are added, that allow to recover consistency.

3.1 The discrete lifting

We summarize here the standard method for imposing nonhomogeneous Dirichlet boundary condition, that relies on a discrete lifting or direct nodal imposition [24, 80, 83, 127].

3.1.1 Formulation

To simplify, we suppose that, in Equation (1)–(ii), the Dirichlet data g is continuous: $g \in \mathscr{C}^0(\Gamma) \cap H^{\frac{1}{2}}(\Gamma)$. As a result, we can take its Lagrange interpolant: $g^h := \mathcal{I}_\Gamma^h(g)$, where

$\mathcal{I}_\Gamma^h$ denotes the Lagrange interpolant on the trace space of V^h. A discrete counterpart of Problem (2) is then:

$$\begin{aligned} &\text{Find } u^h \in V^h \text{ that satisfies } u^h|_\Gamma = g^h \text{ and} \\ &a(u^h, v^h) = L(v^h) \quad \text{for all } v^h \in V_0^h. \end{aligned} \tag{8}$$

As for the continuous case, it is equivalent to find the minimum on V^h of the quadratic convex functional $\mathcal{J}(\cdot)$ under the equality constraint $v^h|_\Gamma = g^h$.

Let $\mathcal{L}^h g^h \in V^h$ be a discrete lifting of g^h. It can be obtained for instance by setting $\mathcal{L}^h g^h(\mathbf{a}_i) = g(\mathbf{a}_i)$ if $\mathbf{a}_i$ is a boundary node ($\mathbf{a}_i \in \Gamma$), and $\mathcal{L}^h g^h(\mathbf{a}_i) = 0$ if $\mathbf{a}_i$ is an interior node ($\mathbf{a}_i \in \Omega$). An equivalent formulation of Problem (8) is:

$$\begin{aligned} &\text{Find } u^h \in V^h,\ u^h = \mathcal{L}^h g^h + u_0^h, \text{ with } u_0^h \in V_0^h(\Omega) \text{ solution to} \\ &a(u_0^h, v^h) = (f, v^h)_\Omega - a(\mathcal{L}^h g^h, v^h) \quad \text{for all } v^h \in V_0^h(\Omega). \end{aligned} \tag{9}$$

Problem (9) is a discrete counterpart of Problem (3).

3.1.2 Well-posedness and convergence

Problem (8) is well-posed and an optimal H^1-error estimate can be derived [83, Lemma 33.1,Theorem 33.2]. More precisely, let $u \in H^s(\Omega)$ be the solution to Problem (2), with $d/2 < s \le k+1$ and let u^h be the solution to Problem (8), there holds:

$$\|u - u^h\|_{1,\Omega} \le C h^{s-1} \|u\|_{s,\Omega}.$$

Moreover, an obvious bound can be obtained for the trace on the boundary. If we suppose that g is in $H^\tau(\Gamma)$, with $(d-1)/2 < \tau \le k$ and using the condition $u^h|_\Gamma = g^h$, we get

$$\|u - u^h\|_{0,\Gamma} = \|g - g^h\|_{0,\Gamma} = \|g - \mathcal{I}_\Gamma^h(g)\|_{0,\Gamma} \le C h^\tau \|g\|_{\tau,\Gamma}$$

from the interpolation error of g.

Optimal error estimates in the L^2-norm can be derived using the Aubin-Nitsche duality trick, see for instance [24, 127] for the $\mathbb{P}_1$ Lagrange finite element, and [83] for $\mathbb{P}_k$, $k \ge 1$. For *a posteriori* error estimates, the interested reader can refer for instance to [3, 24, 123].

For the practical solution of the linear system associated to (8), see [83, Section 33.1.3]. For an estimate of the related condition number, followed by numerical experiments, in the case of a quasi-uniform mesh and $\mathbb{P}_1$ finite elements, see [127].

Remark finally that the assumption of continuity on the boundary data g can be alleviated, if one uses for instance a L^2-projection operator (or local L^2-projections) instead of Lagrange interpolation. In this case, optimal H^1- and L^2-error estimates still hold [24].

This first method, though simple and efficient, is not easy to generalize to more complex boundary conditions. Furthermore, it is not suited to other discretization approaches, in which degrees of freedom are not directly linked to the value of the discrete solution u^h at some nodes located on the boundary Γ.

3.2 Boundary penalty

There are various possibilities to regularize the Dirichlet condition (1)-(*ii*) and the most simple and widespread one is one of the earliest (see, *e.g.*, [11, 15]).

3.2.1 Continuous formulation

In its most widespread form, the penalty method, or boundary penalty method can be formulated at the continuous level and reads (see, *e.g.*, [11, 15]):

$$\begin{aligned}&\text{Find } u_\varepsilon \in H^1(\Omega) \text{ solution to}\\ &a(u_\varepsilon, v) + \frac{1}{\varepsilon}(u_\varepsilon, v)_\Gamma = (f, v)_\Omega + \frac{1}{\varepsilon}(g, v)_\Gamma \quad \text{for all } v \in H^1(\Omega),\end{aligned} \tag{10}$$

where the penalty parameter is denoted by $\varepsilon > 0$. When going back to strong form, we verify that u_ε satisfies Poisson's equation complemented with a Robin-Fourier boundary condition

$$\partial_{\mathbf{n}} u_\varepsilon = -\frac{1}{\varepsilon}(u_\varepsilon - g) \tag{11}$$

that approximates the Dirichlet boundary condition (1)–(*ii*) for ε small enough. The above weak form (10) is also the first order optimality condition associated to the minimization of the functional

$$\mathcal{J}_\varepsilon : V \ni v \mapsto \mathcal{J}(v) + \frac{1}{2\varepsilon}\|v - g\|_{0,\Gamma}^2 \in \mathbb{R}. \tag{12}$$

So, for a symmetric weak form, we recover the usual technique from convex optimization, which consists in reformulating the minimization problem under an equality constraint into an unconstrained problem where the constraint is penalized.

Using a Poincaré-Friedrichs inequality, we check that the modified bilinear form in (10) is elliptic on $H^1(\Omega)$, so Problem (10) is well-posed by application of the Lax-Milgram lemma. Moreover, if we suppose that $u \in H^2(\Omega)$ we can derive the bound

$$\|u - u_\varepsilon\|_{1,\Omega} \leq C\varepsilon \|u\|_{2,\Omega}$$

from which we conclude that u_ε tends to u when ε vanishes [23]. Remark also that Problem (10) remains well-defined and well-posed even if we weaken the regularity assumptions on the boundary datum g, that needs only to belong to $L^2(\Omega)$.

3.2.2 Discrete formulation

The discrete boundary penalty formulation is a direct rewriting of (10) and reads:

$$\begin{cases}\text{Find } u_\varepsilon^h \in V^h \text{ such that:}\\ a(u_\varepsilon^h, v^h) + \frac{1}{\varepsilon}(u_\varepsilon^h, v^h)_\Gamma = (f, v^h)_\Omega + \frac{1}{\varepsilon}(g, v^h)_\Gamma \text{ for all } v^h \in V^h.\end{cases} \tag{13}$$

In such a formulation, it is usual to assume that the mesh is quasi-uniform and that the the penalty parameter is of the form

$$\varepsilon = \varepsilon_0 h^\lambda \tag{14}$$

where the coefficient $\varepsilon_0 > 0$ and the power $\lambda \geq 0$ are independent of the mesh and chosen appropriately by the user (of course, for nonuniform meshes, ε can be set as a function of the local edge size h_E for each boundary edge E, as follows: $\varepsilon = \varepsilon_0 \zeta^\lambda$).

Remark 3.1. *Since conditions* (11) *and* (1)–(*ii*) *are not equivalent* ($u_\varepsilon \neq u$)*, the discrete penalty method* (13) *is not consistent.*

3.2.3 Well-posedness and convergence

The same argument as for the continuous formulation ensures well-posedness of Problem (13). The numerical analysis of the boundary penalty method has been carried out thoroughly in a paper of J.W. Barrett and C.M. Elliott [23] and error estimates for the H^1- and L^2-norms on the domain, and for the L^2-norm on the boundary have been obtained as functions of the power λ in (14), the degree k of the finite element space and the regularity of the solution u, in different situations. Notably, when the domain Ω is a convex polyhedron, they manage to obtain the following bounds:

- If $\lambda = k$, there holds:

$$\|u - u_\varepsilon^h\|_{0,\Omega} + h^{\frac{1}{2}}\|u - u_\varepsilon^h\|_{1,\Omega} + \|u - u_\varepsilon^h\|_{0,\Gamma} \leq Ch^k\|u\|_{1+k,\Omega}, \tag{15}$$

  which ensure convergence but is suboptimal.

- If $\lambda = k + \frac{1}{2}$, there holds:

$$h^{-\frac{1}{2}}\|u - u_\varepsilon^h\|_{0,\Omega} + \|u - u_\varepsilon^h\|_{1,\Omega} + h^{-\frac{1}{2}}\|u - u_\varepsilon^h\|_{0,\Gamma} \leq Ch^k\|u\|_{1+k,\Omega}, \tag{16}$$

  which is optimal for the H^1-norm but remains suboptimal for the L^2-norm on the domain.

- If $\lambda = k + 1$, there holds:

$$h^{-1}\|u - u_\varepsilon^h\|_{0,\Omega} + \|u - u_\varepsilon^h\|_{1,\Omega} + h^{-1}\|u - u_\varepsilon^h\|_{0,\Gamma} \leq Ch^k\|u\|_{2+k,\Omega}, \tag{17}$$

  which are the desired optimal convergence rates.

At first glance, the bound (17) seems to be satisfying but it requires more regularity on the solution u than what is usually expected. Moreover, setting the power λ to $k + 1$ instead of k has an impact on the conditioning on the linear system and to the convergence of iterative solvers, especially for fine meshes. Nevertheless, it is the best available results in such a configuration to the best of our knowledge, and it improves the first result obtained by I. Babuška in [14], for which no value of λ provided optimal estimates for polynomial degree $k \geq 2$. Note however that I. Babuška obtained an almost optimal convergence rate in the H^1 norm for the lowest polynomial degree $k = 1$ and $\lambda = 1$, which is the usual choice in practice. Nevertheless, the error estimate in L^2 norm of the domain was suboptimal. Under some restrictive assumptions on the regularity of the solution, Z.C. Shi [126] provided also an optimal bound for the lower polynomial degree $k = 1$. The case of a curved boundary with variational crimes due to the geometric approximation of the boundary and numerical integration is also studied thoroughly in the paper of J.W. Barrett and C.M. Elliott. In such a situation, some improved results have been published recently by I. Dione [71].

Remark 3.2. *Since it imposes weakly the Dirichlet boundary condition, the penalty method has been considered for fictitious domains techniques, see,* e.g., *[7, 113].*

In practice, a critical issue about the boundary penalty method is the best choice of the penalty parameter ε_0. If its value is too small, the Dirichlet condition is approximated accurately but the conditioning of the global stiffness matrix deteriorates, and if the value is too large, the Dirichlet condition is approximated poorely. As a result, a compromise between these two situations must be adopted. Another issue is an accurate prediction of the boundary flux $\partial_{\mathbf{n}} u_\varepsilon^h$, which can be inaccurate and depends on ε_0. This is fundamentally due to (11). This point has been object of recent work by V. Garg and S. Prudhomme [88].

Remark 3.3. *A penalty technique can also be performed directly at the algebraic level, after assembly of the stiffness matrix corresponding to pure (homogeneous) Neumann boundary conditions. This technique is sometimes called* exact penalty*: see [99] for more details or also [83, Remark 33.5].*

3.3 Nitsche

We present now a consistent boundary penalty technique to incorporate the nonhomogeneous Dirichlet boundary condition (1)–(*ii*). This technique has been originally proposed by J.A Nitsche in 1971 [118]. It has gained first its popularity in the Discontinuous Galerkin community, see, *e.g.*, [9, 70], and later on in the finite element community as a method to treat various boundary and interface conditions [95]. As we will see, there is in fact a whole family of methods that can be derived systematically and share some common features of well-posedness and optimal accuracy (but with different conditions on the numerical parameter). The most widespread member of this family is the original one of J.A. Nitsche, that derives of a functional and preserves the symmetry of the original weak form (2). Nonsymmetric variants have been derived since then. For the symmetric variant, there are various references that present the method and detail its numerical analysis, particularly the survey of R. Stenberg [129] already mentioned in the introduction, a chapter of P. Hansbo [95] and the book of V. Thomée [130, Chapter 2]. A detailed presentation of two variants, the symmetric and the incomplete ones, can be found in the second volume of A. Ern & J.L. Guermond [83, Chapter 37].

3.3.1 Formulation

Let us first derive the method formally, and for this purpose take $\gamma > 0$ a positive function on the boundary Γ and $\theta \in \mathbb{R}$ a fixed parameter. As a starting point to obtain a Nitsche's method for (1), we can reformulate the Dirichlet boundary condition (1)–(*ii*) as:

$$\partial_{\mathbf{n}} u = -\gamma((u-g) - \gamma^{-1}\partial_{\mathbf{n}} u). \tag{18}$$

Note that, in opposition to (11), the reformulation (18) is formally equivalent to (1)–(*ii*), provided that $\gamma > 0$.

Let u be the solution to (1), and v a test function, that we suppose both regular enough so that the following calculations make sense. From (1)–(*i*) and using Green formula we get first:

$$a(u,v) - (\partial_{\mathbf{n}} u, v)_\Gamma = (f,v)_\Omega.$$

Then we rewrite $v = (v - \theta\gamma^{-1}\partial_{\mathbf{n}} v) + \theta\gamma^{-1}\partial_{\mathbf{n}} v$ and obtain:

$$a(u,v) - \theta(\gamma^{-1}\partial_{\mathbf{n}} u, \partial_{\mathbf{n}} v)_\Gamma - (\partial_{\mathbf{n}} u, v - \theta\gamma^{-1}\partial_{\mathbf{n}} v)_\Gamma = (f,v)_\Omega.$$

Now the reformulation (18) yields

$$\begin{aligned} & a(u,v) - \theta(\gamma^{-1}\partial_{\mathbf{n}} u, \partial_{\mathbf{n}} v)_\Gamma + (\gamma(u - \gamma^{-1}\partial_{\mathbf{n}} u), v - \theta\gamma^{-1}\partial_{\mathbf{n}} v)_\Gamma \\ = \; & (f,v)_\Omega + (\gamma g, v - \theta\gamma^{-1}\partial_{\mathbf{n}} v)_\Gamma. \end{aligned}$$

We develop the second boundary term and simplify. We get

$$\begin{aligned} & a(u,v) - (\partial_{\mathbf{n}} u, v)_\Gamma - \theta(u, \partial_{\mathbf{n}} v)_\Gamma + (u, \gamma v)_\Gamma \\ = \; & (f,v)_\Omega + (g, \gamma v - \theta\partial_{\mathbf{n}} v)_\Gamma. \end{aligned} \tag{19}$$

The above weak form may have no meaning at the continuous level but has a well-defined discrete counterpart. For this purpose let us take γ as a piecewise constant function on the boundary Γ:

$$\gamma = \gamma_0 \zeta^{-1},$$

where γ_0 is a positive constant. To lighten the writing of the weak form (19) we introduce the bilinear form

$$A_N(u^h, v^h) := a(u^h, v^h) - (\partial_{\mathbf{n}} u^h, v^h)_\Gamma - \theta(u^h, \partial_{\mathbf{n}} v^h)_\Gamma + (\gamma u^h, v^h)_\Gamma$$

and the linear form

$$l_N(v^h) := (f, v^h)_\Omega + (g, \gamma v^h - \theta \partial_{\mathbf{n}} v^h)_\Gamma.$$

Nitsche's method for Poisson's problem (1) reads:

$$\begin{cases} \text{Find } u^h \in V^h \text{ solution to} \\ A_N(u^h, v^h) = l_N(v^h), \quad \forall\, v^h \in V^h. \end{cases} \tag{20}$$

Three notable variants of the method, for different values of the parameter θ can be obtained, as for the discontinuous Galerkin Interior Penalty (dGIP) method [70]:

1. For $\theta = 1$, the formulation (20) is symmetric and identical to Nitsche's original formulation [118]. It can be obtained as the first order optimality condition associated to the functional

$$\mathcal{J}_N : V^h \ni v^h \mapsto \mathcal{J}(v^h) - (v^h - g, \partial_{\mathbf{n}} v^h)_\Gamma + \frac{1}{2}\|\gamma^{\frac{1}{2}}(v^h - g)\|_\Gamma^2 \in \mathbb{R}.$$

   It can be obtained alternatively from an augmented lagrangian formalism, see [52, Section 5.2.2]

   Remark that the expression of the Nitsche's functional $\mathcal{J}_N(\cdot)$ is indeed similar to that of an augmented lagrangian, and that it includes an extra term in comparison to the functional $\mathcal{J}_\varepsilon(\cdot)$ of the boundary penalty. This term allows to recover consistency.

2. For $\theta = 0$, we get the simplest, incomplete, formulation, that has the less terms, and that is presented for instance in [83, Section 37.1].

3. For $\theta = -1$, we recover the skew-symmetric formulation of J. Freund and R. Stenberg [86], where discrete ellipticity is ensured whatever the value of $\gamma_0 > 0$ is.

Remark 3.4. *The steps above can be generalized to derive Nitsche's method for other essential boundary conditions and other differential operators, see [101] for more details.*

For $u \in H^s(\Omega)$ with $s > \frac{3}{2}$, it is direct to check that Nitsche's method (20) is consistent, and, from this point, it differs drastically from the boundary penalty method (13). Fundamentally this difference comes from the fact that the reformulation (18) is not a regularization of the Dirichlet boundary condition.

3.3.2 Well-posedness and convergence

Thanks to the discrete trace inequality (6) we deduce that Problem (20) is well-posed provided that

$$1 \geq \frac{(1+\theta)^2 c_I}{\gamma_0}, \tag{21}$$

where c_I is the constant of the discrete trace inequality (6). The complete statement for well-posedness and its proof are provided in the Appendix A. See also [56, 83, 95, 118, 129, 130] for specific values of θ (and sometimes of k), and the generalization to an arbitrary θ makes no specific difficulty. For the symmetric variant $\theta = 1$ this result implies notably the strong convexity of Nitsche's functional $J_N(\cdot)$ when γ_0 is large enough. As a result for the skew-symmetric variant $\theta = -1$, well-posedness is ensured irrespectively of the value of $\gamma_0 > 0$, and, otherwise, γ_0 needs to be large enough.

Remark 3.5. *For more complex problems, a similar condition as* (21) *holds, but with a constant that depends for instance of the physical parameters. It can be interesting to establish precise quantitative bounds for this constant to fix the value of γ_0 as low as possible (for $\theta \neq 1$). It can be done through solving eigenvalue problems (see,* e.g., *[95, 101]) or obtained by other methods (such as dimensional studies). In case of small strain elasticity, see for instance [57, 96].*

Nitsche's method converges optimally in the H^1-norm, or equivalently in the discrete norm defined above, provided that γ_0 be large enough ($\gamma_0 > 0$ when $\theta \neq -1$). So for $u \in H^s(\Omega)$ with $\frac{3}{2} < s \leq 1 + k$, we have

$$\|u - u^h\|_h + \left\|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h\right\|_{-1/2,h,\Gamma} \leq Ch^s \|u\|_{s,\Omega}. \tag{22}$$

The complete statement and its proof are given in Appendix B. See also [56, 83, 95, 118, 129, 130] for alternative proofs for some specific values of θ (and sometimes of k). Remark that the above estimate provides also an optimal convergence rate for the boundary flux.

Error estimates in the L^2-norm can also be obtained, see, *e.g.*, [83, 130]. Recently optimal error estimates in the H^1-norm have been obtained even for solutions with low regularity $1 \leq s \leq 3/2$, see, *e.g.*, [81, 94]. For residual *a posteriori* error estimates, see for instance [28, 94]: the results are for interface problems but it can be straightforwardly adapted for a Dirichlet boundary condition.

Of course, the value of γ_0 influences the condition number of the global stiffness matrix associated to $A_N(\cdot,\cdot)$, and for this reason it does not have to be taken too large. Anyway, since the method is consistent, the impact of the numerical parameter γ_0 has on the approximation of the Dirichlet condition is not as important as for the boundary penalty method (13). Moreover, since the function γ scales as $\mathcal{O}(h^{-1})$ whatever the polynomial order k is, this does not deteriorate too much the conditioning, that remains in $\mathcal{O}(h^{-2})$ [95]. This is not the case for the boundary penalty (see Section 3.2). This property makes Nitsche's method particularly relevant for high order discretizations, and for this reason it has been considered for isogeometric analysis (IGA) [8, 55, 101] or also for Hybrid High Order (HHO) [54, 57] variational approximations.

Remark finally that, as it was already the case for the boundary penalty method, Nitsche's method enforces weakly the Dirichlet condition and is also a candidate for fictitious domain methods. Various works have been dedicated to this topic: see for instance [47, 49, 110], to mention just a few.

Remark 3.6. *Nitsche's method can be adapted as well for Robin boundary conditions. Of course Robin boundary conditions are not difficult to take into account in a weak formulation, but the Nitsche formulation of [105] is robust irrespectively of the value of the Robin coefficient, that can be arbitrarily large, and this is its main interest. Recently the first analysis of [105] has been improved and extended for solutions with low regularity [112].*

3.4 New variants of Nitsche

We present here some other Nitsche-type methods that emerged in the last decade.

3.4.1 A penalty free Nitsche method

Note that the skew-symmetric version $\theta = -1$ still makes sense if we set $\gamma_0 = 0$ and in this case we get from (20):

$$\begin{cases} \text{Find } u^h \in V^h \text{ solution to} \\ a(u^h, v^h) - (\partial_{\mathbf{n}} u^h, v^h)_\Gamma + (u^h, \partial_{\mathbf{n}} v^h)_\Gamma = (f, v^h)_\Omega + (g, \partial_{\mathbf{n}} v^h)_\Gamma, \quad \forall\, v^h \in V^h. \end{cases}$$

This version has no extra numerical parameter and is called penalty-free Nitsche. In this case the method remains stable, since a discrete inf-sup condition holds, and converges optimally in the H^1 norm, see [45]. In reference [45] an estimate in the L^2 norm is also proven. It is suboptimal of order $\mathcal{O}(h^{\frac{1}{2}})$ since this nonsymmetric variant is not consistent for the adjoint problem. Various extensions of this variant have been made later on, for fictitious domains with cut elements [38], for elasticity [37], for Signorini [51], for Brinkman [35] and for isogeometric analysis [101, 125].

3.4.2 Other Nitsche-type methods

In [39, 76] a variant of Nitsche has been suggested, that adds an extra term involving the Laplace-Beltrami differential operator on the boundary. In [107] another variant of Nitsche has been suggested, where another variable is added for the approximation of the gradient, and later on eliminated by static condensation. This approach is close in some sense to mixed methods.

In [26] a variant of Nitsche that does not involve any boundary term has been suggested. The idea, is, roughly speaking, to use a representation of the boundary flux with help of a variational residual. More precisely, the first step is to rewrite V^h as the following direct sum

$$V^h = V_i^h \oplus V_b^h,$$

where V_i^h is the subspace of functions with vanishing trace on Γ and V_b^h is the complementary subspace. Take $g_b^h \in V_b^h$ an approximation of the boundary data g (it can be the discrete lifting, but also other kind of reconstruction). We want to find $u^h = u_i^h + u_b^h$ that minimizes the following modified Nitsche's energy functional

$$\mathcal{J}_B : V^h \ni u^h \mapsto \mathcal{J}(u^h) + (f, u_b^h - g_b^h)_\Omega - (\nabla u^h, \nabla(u_b^h - g_b^h))_\Omega + \|\nabla(u_b^h - g_b^h)\|_\Omega^2 \in \mathbb{R}.$$

Note that it mimics exactly the original functional $\mathcal{J}_N$ but involves only integral terms on the domain Ω. Moreover there is no discrete parameter. The discrete weak problem comes from the first order optimality condition of the above functional and reads:

$$\begin{aligned}
&\text{Find } (u_i^h, u_b^h) \in V_i^h \times V_b^h \text{ solution to} \\
&a(u_i^h, v_i^h) = (f, v_i^h)_\Omega - a(g_b^h, v_i^h) \quad \text{ for all } v_i^h \in V_i^h, \\
&a(u_b^h, v_b^h) = a(g_b^h, v_b^h) \quad \text{ for all } v_b^h \in V_b^h.
\end{aligned}$$

Some nonsymmetric variants can also be obtained. The main interest of this method is for singularly perturbed problems such as reaction-diffusion or convection-diffusion, where it becomes nonconforming [26]. Numerical results in [26] illustrate indeed that it performs as Nitsche's method and better than the discrete lifting in the limit cases.

4 Methods with Lagrange multipliers

In mixed methods, the boundary condition in (1) is treated weakly thanks to a Lagrange multiplier λ that represents either the normal derivative $\partial_\mathbf{n} u$ or its opposite $-\partial_\mathbf{n} u$ [14, 41, 120, 121] (see also [36, 80, 82, 83] for general references about the discretization of mixed and hybrid methods). The continuous weak problem that serves for further numerical approximations is

$$\begin{aligned}
&\text{Find } (u, \lambda) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \text{ solution to} \\
&B((u,\lambda),(v,\mu)) = L(v,\mu) \qquad \text{for all } (v,\mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma),
\end{aligned} \tag{23}$$

with the notations

$$B((u,\lambda),(v,\mu)) := a(u,v) + \langle \lambda, v\rangle_\Gamma + \langle \mu, u\rangle_\Gamma, \quad L(v,\mu) := (f,v)_\Omega + \langle \mu, g\rangle_\Gamma,$$

where $\langle \cdot,\cdot\rangle_\Gamma$ is the duality product $H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$. Note that we used a sign convention that allows to preserve symmetry, and that corresponds to $\lambda = -\partial_\mathbf{n} u$. Problem (23) is well-posed and a continuous inf-sup condition holds

$$\inf_{\mu \in H^{-\frac{1}{2}}(\Gamma)} \sup_{v \in H^1(\Omega)} \frac{\langle \mu, v\rangle_\Gamma}{\|\mu\|_{-\frac{1}{2},\Gamma} \|v\|_{1,\Omega}} \geq \beta, \tag{24}$$

with $\beta > 0$ [124, Section 10.3]. Problem (23) is equivalent to the original problem (2). Moreover, its solution is the unique saddle-point of the following Lagrangian:

$$\mathcal{L}(v,\mu) := \mathcal{J}(v) + \langle \mu, v - g\rangle_\Gamma, \tag{25}$$

i.e. it verifies

$$\mathcal{L}(u,\mu) \leq \mathcal{L}(u,\lambda) \leq \mathcal{L}(v,\lambda)$$

for all $(v,\mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$. See for instance [124, Section 10.3] for a detailed presentation of the above setting.

4.1 Compatible pairs of spaces

The most simple technique to approximate Problem (23) consists in chosing a finite element space for the Lagrange multiplier on the boundary, that we denote by

$$M^H \ \subset \ H^{-\frac{1}{2}}(\Gamma).$$

The notation $H > 0$ stands for the mesh size for the boundary mesh. The discrete counterpart of (23) reads then:

$$\begin{aligned}&\text{Find } (u^h,\lambda^H) \in V^h \times M^H \text{ solution to}\\ &B((u^h,\lambda^H),(v^h,\mu^H)) = L(v^h,\mu^H) \qquad \text{for all } (v^h,\mu^H) \in V^h \times M^H.\end{aligned} \tag{26}$$

From this point, we see the first advantages of this approach: 1) as it was already the case for the boundary penalty and for Nitsche, the weak treatment of the Dirichlet boundary condition allows extensions for fictitious domains, see for instance [91, 92, 93] or more recently [98] for a framework based on the extended finite element method; 2) there is more flexibility to approximate the boundary flux, since it is related to the choice of M^H. Particularly, the boundary mesh, that we denote by $\mathcal{E}_H$, does not need to be the trace mesh of the global mesh $\mathcal{T}_h$, and can be chosen independently, as well as the polynomial order for the approximation of λ^H, that can be different from $k-1$. Moreover, since $M^H \subset H^{-\frac{1}{2}}(\Gamma)$, the dual space for the Lagrange multiplier can be made either of continuous or discontinuous functions while preserving conformity. This flexibility is however limited, since a discrete inf-sup compatibility condition between V^h and M^H needs to be satisfied to recover stability and optimal accuracy, and to avoid spurious modes in the solution.

4.1.1 Some simple particular cases

Let us discuss the above point in detail, first in the particular case where the boundary mesh is the trace mesh of $\mathcal{T}_h$, *i.e.* when $h = H$ and

$$\mathcal{E}_H = \mathcal{E}_h = \mathcal{T}_h|_\Gamma.$$

A first natural and simple choice consists in taking $V^h = V^{h,1}$ (finite elements with polynomial order $k = 1$) and to approximate the dual space with piecewise constant discontinuous functions on the trace mesh:

$$M_0^{h,d} := \left\{\mu^h \in L^2(\Omega) \ \mid \ \mu^h|_E \in \mathbb{P}_0(E), \forall\, E \in \mathcal{E}^h\right\}. \tag{27}$$

With this choice ($M^H = M_0^{h,d}$) we get a well-posed discrete problem (26). Indeed, since we can easily verify

$$\{\mu^h \in M_0^h \ \mid \ \langle \mu^h, v^h\rangle_\Gamma = 0, \ \forall v^h \in V^h\} = \{0\} \tag{28}$$

and since $a(\cdot,\cdot)$ is elliptic on $\{v^h \in V^h \mid \langle \mu^h, v^h\rangle_\Gamma = 0,\ \forall \mu^h \in M_0^h\} = V_0^h$, we can apply for instance Proposition 2.42 from [80] to ensure well-posedness. In fact, condition (28) is equivalent to:

$$\inf_{\mu^h \in M_0^h} \sup_{v^h \in V^h} \frac{\langle \mu^h, v^h\rangle_\Gamma}{\|\mu^h\|_{-\frac{1}{2},\Gamma}\|v^h\|_{1,\Omega}} \geq \beta_h, \tag{29}$$

with $\beta_h > 0$ a constant that possibly depends of the mesh. The above condition is sometimes called *rank condition* [120] or an inf-sup condition nonuniform in h (or even a weak inf-sup condition). Though it is enough for well-posedness, it leads to suboptimal error estimates unless we can show that $\beta_h > 0$ can be set independently of h, *i.e.*, if we can exhibit an inf-sup condition uniform in h (or strong inf-sup condition) that is a discrete counterpart of (24). Unfortunately, it can be shown that β_h necessarily tends to 0 when h vanishes [134, Lemma 3.1], so the choice $M^H = M_0^h$ is not numerically correct.

Remark 4.1. *Alternatively, other spaces with discontinuous functions can be considered:*

$$M_l^{h,d} := \left\{\mu^h \in L^2(\Omega) \mid \mu^h|_E \in \mathbb{P}_l(E), \forall\, E \in \mathcal{E}^h\right\},$$

with $l \geq 0$, but the pair $V^h \times M_l^{h,d}$ does not satisfy necessarily a discrete inf-sup condition, even weak, especially for the lowest values of k ($k = 1, 2$).

Another natural choice is to set the discrete dual space as the trace space of V^h, and to use continuous piecewise polynomial functions on the trace mesh. So for $l \geq 1$ the polynomial order of the dual variables, we set

$$M_l^h := \left\{\mu^h \in \mathscr{C}^0(\overline{\Omega}) \mid \mu^h|_T \in \mathbb{P}_l(E), \forall\, E \in \mathcal{E}^h\right\}. \tag{30}$$

Let us focus first on the simplest situation when $k = l = 1$. In this case, it is still straightforward to verify the rank condition

$$\{\mu^h \in M_1^h \mid \langle \mu^h, v^h\rangle_\Gamma = 0,\ \forall v^h \in V^h\} = \{0\} \tag{31}$$

and thus a non-uniform inf-sup condition and well-posedness of the discrete problem. Moreover, under an extra assumption on the trace mesh, that needs to satisfy the Crouzeix-Thomée criterion [68], it is possible to establish a uniform inf-sup condition [32, Lemma 3.1]:

$$\inf_{\mu^h \in M_1^h} \sup_{v^h \in V^h} \frac{\langle \mu^h, v^h\rangle_\Gamma}{\|\mu^h\|_{-\frac{1}{2},\Gamma}\|v^h\|_{1,\Omega}} \geq \beta, \tag{32}$$

where this time, $\beta > 0$ is independent of h. Notably, when the trace mesh is quasi-uniform, it satisfies the Crouzeix-Thomée criterion, but this is one of the most restrictive situations, and a large class of non quasi-uniform meshes satisfies this criterion.

4.1.2 The general situation

For the more general case, I. Babuška proved that, under the assumption of quasi-uniform meshes on the boundary and under the geometric condition

$$H \geq Ch,$$

where $C > 0$ is independent of H and h, there holds a strong discrete inf-sup condition [14]

$$\inf_{\mu^H \in M^H} \sup_{v^h \in V^h} \frac{\langle \mu^H, v^h\rangle_\Gamma}{\|\mu^H\|_{-\frac{1}{2},\Gamma}\|v^h\|_{1,\Omega}} \geq \beta, \tag{33}$$

with $\beta > 0$ independent of h and H, for the pair $V^h \times M_l^H$, $k, l \geq 1$, where M_l^H is the space of continuous piecewise polynomial functions of degree l on the boundary mesh $\mathcal{E}_H$.

Since the conditions provided by I. Babuška are only sufficient and may be too restrictive in practice, notably because the constant C is not easy to determine, much effort has been devoted later on to characterize compatible pairs of spaces, particularly it has been the object of a serie of papers by J. Pitkäranta [120, 121, 122]. Among the contributions of J. Pitkäranta, there is also an error analysis using the mesh-dependent norm introduced previously in the finite element setting, and the following result can be proven [121] (see also [129, Theorem 1B]): assume that the mesh sizes h and H are such that $ch \leq H \leq Ch$, and assume that

$$\inf_{\mu^H \in M^H} \sup_{v^h \in V^h} \frac{\langle \mu^H, v^h \rangle_\Gamma}{\|\mu^H\|_{-1/2,h,\Gamma} \, \|v^h\|_h} \geq \beta,$$

with $\beta > 0$ independent of h. Then there holds, for $u \in H^{1+k}(\Omega)$ and $\lambda \in \Pi_{j=1}^J H^{1+l}(\Gamma_j)$:

$$\|u - u^h\|_h + \|\lambda - \lambda_h\|_{-1/2,h,\Gamma} \leq C(h^k \|u\|_{1+k,\Omega} + h^{l+\frac{3}{2}} \|\lambda\|_{1+l,\Gamma}).$$

Above we denoted by Γ_j, $j = 1, \dots, J$ the boundary facets of Γ, and $\|\cdot\|_{1+l,\Gamma}$ should be understood as a broken Sobolev norm (this technical point disappears when the boundary is smooth, but then the estimate does not take into account the approximation error for the curved boundary). A similar result holds when using the natural norms [14]. Note the similarity of the above error estimate with those of Nitsche's method, and that there is again an optimal error bound on the flux (Lagrange multiplier).

Remark 4.2. *In the case of the fictitious domain formulation [91], and for the pair $\mathbb{P}_1/\mathbb{P}_0$, a similar sufficient condition $H \geq Ch$ (with $C \geq 3$) needs to be satisfied to ensure a strong inf-sup condition, as proven in [90].*

There are also other possibilities to design compatible pairs of spaces, for instance introducing bubble functions [32] or using biorthogonal basis for the dual space [134]. This last approach can be successfully extended for domain decomposition with nonmatching meshes [133] or contact and friction problems [134].

4.2 Barbosa & Hughes stabilization

Stabilized mixed methods start from the mixed formulation (23), that they complement with extra terms at the discrete level. These terms allow to preserve well-posedness and optimal accuracy, even for pairs of finite element spaces that do not satisfy the discrete inf-sup compatibility condition (33). For the original idea of stabilized mixed methods, see for instance the seminal paper of L. Franca and T.J.R. Hughes [103], where these ideas were applied first to the Stokes equations. In this category, the stabilized method of H. Barbosa and T.J.R. Hughes [17, 18] introduces a residual-least squares stabilization (see also [131] for Navier-Stokes with slip boundary conditions and [16] for domain decomposition). The Barbosa & Hughes method can be formulated as:

$$\begin{aligned} &\text{Find } (u^h, \lambda^H) \in V^h \times M^H \text{ solution to} \\ &B_h((u^h, \lambda^H), (v^h, \mu^H)) = L(v^h, \mu^H) \qquad \text{for all } (v^h, \mu^H) \in V^h \times M^H, \end{aligned} \tag{34}$$

where

$$\begin{aligned} B_h((u^h, \lambda^H), (v^h, \mu^H)) \quad := \quad & B((u^h, \lambda^H), (v^h, \mu^H)) \\ & -(\delta(\lambda^h + \partial_{\mathbf{n}} u^h), \mu^h + \partial_{\mathbf{n}} v^h)_\Gamma. \end{aligned}$$

In the above definition, δ is a piecewise constant function on the boundary Γ:

$$\delta = \delta_0 \zeta^1,$$

where $\delta_0 \geq 0$ is the stabilization parameter, that needs to be small enough. The original formulation [17, 18] contains some extra terms, but that are not necessary for stability and optimal convergence. The formulation (34) above comes from R. Stenberg [129].

A stability and convergence analysis of the above formulation is done in [129], when the boundary mesh is the trace mesh and when the space of Lagrange multipliers is the space M_l^h introduced previously. Under the condition

$$0 < \delta_0 < c_I,$$

where c_I is the discrete trace constant already involved in Nitsche's method, well-posedness with a uniform stability bound can be obtained [129, Lemma 6] and the same optimal convergence rates as the mixed method with a compatible pair follow. Again assume that the mesh sizes h and H are such that $ch \leq H \leq Ch$, then, for $u \in H^{1+k}(\Omega)$ and $\lambda \in \Pi_{j=1}^J H^{1+l}(\Gamma_j)$:

$$\|u - u^h\|_h + \|\lambda - \lambda_h\|_{-1/2,h,\Gamma} \leq C(h^k \|u\|_{1+k,\Omega} + h^{l+\frac{3}{2}} \|\lambda\|_{1+l,\Gamma}).$$

Moreover if we eliminate the discrete multiplier through static condensation, we get the symmetric variant of Nitsche's method [129], in which the stabilization parameter δ_0 plays the same role as the inverse of the Nitsche parameter γ_0. See also Appendix C for a detailed proof in a simple situation.

This formulation allows any choice for the pair of discrete spaces: see for instance some numerical examples in [100] in the case of Signorini contact.

Remark 4.3. *In [22] is suggested a simplification of Barbosa & Hughes stabilization, where some terms have been removed, in the context of a fictitious domain method for a perfored domain. This modification makes the method nonconsistent, but still well-posedness and error estimates can be proven.*

Remark 4.4. *The Barbosa & Hughes stabilization technique has been extended for variational inequalities in [19]. This allowed further extensions for Signorini contact [100], friction [109] and contact with extended finite elements for a crack [4]. The relationship between Barbosa and Hughes stabilization and Nitsche's method allowed later on to design Nitsche's methods for contact and friction [58, 59, 60].*

Remark 4.5. *Recently augmented lagrangian formulations have been considered, not as a solution technique, as usual, but as a discretization technique* per se *[50, 52]. In [52] for instance, the link between augmented lagrangian formulations and residual-least squares stabilization is studied, and in case of Dirichlet boundary condition, this allows to recover the symmetric variant of Nitsche's formulation (see also Section 3.3.1).*

4.3 Minimal stabilization

Many other stabilization techniques have been designed to relax the inf-sup compatibility condition. In the 2000s, minimal stabilization procedures have been proposed and studied, see, *e.g.*, [27, 42, 43]. The adaptation of such techniques to treat essential boundary conditions has been carried out later on, in the 2010s, and has been thoroughly studied by E. Burman [46]. For Problem (23), minimal stabilization methods can be formulated, in a very general form, as:

$$\begin{aligned}
&\text{Find } (u^h, \lambda^H) \in V^h \times M^H \text{ solution to} \\
&\widetilde{B}_h((u^h, \lambda^H), (v^h, \mu^H)) = L(v^h, \mu^H) \qquad \text{for all } (v^h, \mu^H) \in V^h \times M^H,
\end{aligned} \tag{35}$$

where

$$\widetilde{B}_h((u^h, \lambda^H), (v^h, \mu^H)) \quad := \quad B((u^h, \lambda^H), (v^h, \mu^H)) - \delta_0 s(\lambda_H, \mu_H).$$

Above the numerical parameter $\delta_0 > 0$ is still the stabilization parameter. Conversely to Barbosa & Hughes stabilization (34) presented previously, the stabilization term $s(\cdot, \cdot)$ involves solely the Lagrange multiplier λ^H, and does not requires the primal variable u^h or its normal derivative.

Broadly speaking, $s(\cdot, \cdot)$ is designed to penalize the gap between the Lagrange multiplier and an underlying stable space. More fundamentally this class of methods relies on the following observation [46, Lemma 2.1]:

$$\sup_{v^h \in V^h} \frac{\langle \mu^H, v^h \rangle_\Gamma}{\|v^h\|_{1,\Omega}} \geq \beta \|\mu^H\|_{-\frac{1}{2},\Gamma} - c \left\| \mu^H - \widetilde{\pi} \mu^H \right\|_{-1/2,h,\Gamma}, \tag{36}$$

where $\widetilde{\pi}$ is the (global) L^2-projection onto a discrete space, $\widetilde{M}^H$, coarser than the space M^H and for which the discrete (strong) inf-sup compatibility condition (33) is verified.

There are various possibilities to write the stabilization term. For instance it can be

$$s(\lambda^H, \mu^H) = (\zeta^1 (I - \widetilde{\pi}_H)\lambda^H, (I - \widetilde{\pi}_H)\mu^H)_\Gamma,$$

where $\widetilde{\pi}_H$ is a local projection operator, easier to compute in practice. When such choice is made, the corresponding method is usually named local projection stabilization (LPS). More conveniently, under appropriate conditions, we can use a term that involves only the jumps of the multiplier and its derivative at the interface between the facets [46, Section III]. Notably, we have seen above that the pair $V^{h,1} \times M_0^h$ is not uniformly inf-sup stable. In two dimensions, if we denote by $\mathcal{X}_h$ the set of boundary nodes, at the exception of corner nodes, let us consider the term:

$$s(\lambda^H, \mu^H) = h^2 \sum_{x \in \mathcal{X}_h} [\![\lambda^H(x)]\!] [\![\mu^H(x)]\!],$$

where we used the notation $[\![\cdot(x)]\!]$ to denote the jump of a quantity between the two neighbouring edges of each node x. With such an appropriate stabilization term, still for the pair $V^{h,1} \times M_0^h$, formulation (35) is well-posed and the optimal accuracy of the solution can be recovered [46].

Some fictitious domains methods have been designed accordingly to this paradigm. From the analysis of V. Girault and R. Glowinski [90] where a geometric condition has been exhibited between the global mesh and the mesh for the Lagrange multiplier (see Remark 4.2 above), a fictitious domain method with the Lagrange multiplier on the trace of the global mesh combined with a local projection stabilization term has been designed in [20] (see also [21] for an extension to time-dependent parabolic problems). A fictitious domain method based on extended finite elements and a minimal stabilization procedure has been made in [5], and later on extended for a crack geometry with Tresca friction condition [6].

Remark 4.6. *In [46, Section C] it is shown that the framework of minimal stabilization allows to interpret the unsymmetric version of Barbosa & Hughes method as a minimal stabilization with respect to the penalty-free Nitsche method, seen as an inf-sup stable mixed method.*

5 Conclusion

There are no longer problems that are solved and others that are not, there are only problems that are more or less solved. Henri Poincaré[1]

[1] Il n'y a plus des problèmes résolus et d'autres qui ne le sont pas, il y a seulement des problèmes plus ou moins résolus. (Henri Poincaré, Conférence prononcée au Congrès international des Mathématiciens, Rome, 1908 ; t. I, p. 173 des Actes du Congrès.)

There is now an embarrassingly choice of mathematically sound technologies to incorporate essential boundary conditions within a variational discretization such as the finite element method (or its extensions or its alternatives). An important motivation for the design of most of these methods has been to incorporate essential boundary conditions in a weak sense, in order to mimic in some way what happens with Neumann or Robin boundary conditions. This weak treatment can not be done directly, and this explains why different pathes have been explored to do this. This weak incorporation has however some major advantages in terms of flexibility to account the essential boundary conditions at the discrete level.

At first glance, the boundary penalty method seems to be the most attractive since it may be the easiest to understand and to implement. For this reason, it is still widespread, particularly in some industrial codes. Nevertheless, when applied to some specific problems, its performance in terms of accuracy may not always be satisfactory: see for instance some numerical evidence in [134] for static contact and friction problems or in [61, 73] for contact in elastodynamics. Nitsche method may represent a better alternative, and remains simple to implement. It still involves a numerical parameter, but which does not play the same role as a regularization parameter and is closest to a stabilization parameter. Particularly, it does not need to be very large to ensure the stability and optimal accuracy, but it needs only to be fixed above a threshold. Alternatively, mixed methods with compatible pairs can be relevant to represent accurately the boundary flux, and do not need an extra numerical parameter. However ensuring in practice the strong discrete inf-sup condition needed for stability and accuracy is not always an easy task, particularly for complex geometries and nonlinear problems. Stabilized mixed method allow to circumvent this issue. However, once again an extra numerical parameter appears. Some families of least squares stabilized mixed methods are closely related to Nitsche's method and perform the same way.

A major field of applications is concerned, broadly speaking, with, on one side, high order discretization techniques such as isogeometric analysis [67] or polytopal methods [29, 64, 69, 65, 108, 111], and, on the other side, geometrically unfitted finite element methods, such as fictitious domains, extended or cut finite element methods. In this last case the interested reader may refer to, *e.g.*, [12, 117] for some surveys with numerical comparisons. In order to make this presentation as simple as possible, we restricted the setting to a mesh conforming to the exact boundary, and did not mentioned the difficulties and specificities of dissociating the physical and the numerical boundaries. These difficulties are mostly related to numerical integration on cut elements, and also to the preservation of a reasonable condition number (see, *e.g.*, [44]). Note that, in the last years, have emerged new methods, called *phi-FEM*, that allow to incorporate very easily essential boundary conditions with help of the level set function that defines the boundary [79]. One major advantage of this strategy is that it requires no numerical integration on cut elements. However it requires special care to take into account Neumann or Robin boundary conditions [78].

Furthermore, various techniques we reviewed have been extended for more challenging problems, particularly interface problems with unfitted meshes, and problems with nonlinear boundary conditions associated with variational inequalities. For the emblematic Signorini contact problem where boundary conditions are inequations and a complementarity condition, penalized and mixed methods have been suggested very early [97, 106]. In the last two decades, there have been some improvements for mixed techniques [1, 31, 32, 66, 75, 102, 134], but also the first adaptation of stabilized formulations [100] and Nitsche formulations to contact conditions [58, 59, 60]. For variational inequalities, the numerical analysis may be much more challenging, and for instance, optimal error estimates in the H^1-norm have been obtained only a few years ago for the Signorini problem [59, 74], and the obtention of optimal error estimates in the L^2-norm is still an open issue [53, 62, 66, 128].

Most of the techniques presented here have in common some kind of genericity and an ability to be extended to many settings. Particularly they do not depend that much of the differential operator under consideration. When the weak form is symmetric and can be obtained as the first order optimality condition associated to the minimization of a functional, there is a systematic way to recover some of the methods thanks to a minimization argument, after having added extra terms to the original functional. Otherwise we shown there is another path for nonsymmetric problems. Most of the discretizations preserve symmetry when needed, but for Nitsche's method, we presented unsymmetric variants that can be of interest for nonsymmetric problems and allow more freedom to choose the numerical parameter.

A Well-posedness of Nitsche's method

We provide here the precise statement of well-posedness for Nitsche's method, and a detailed proof.

Lemma A.1. *Suppose that*

$$1 \geq \frac{(1+\theta)^2 c_I}{\gamma_0}, \tag{37}$$

where c_I is the constant of the discrete trace inequality (6)*, then the bilinear form $A_N(\cdot,\cdot)$ is elliptic on V^h:*

$$A_N(v^h, v^h) \geq c\|v^h\|_h^2, \tag{38}$$

for any $v^h \in V^h$ and where $c > 0$ does not depend on h. As a result, Problem (20) *is well-posed.*

Proof. Take v^h in V^h and write

$$\begin{aligned}
A_N(v^h, v^h) &= a(v^h, v^h) - (1+\theta)(\partial_{\mathbf{n}} v^h, v^h)_\Gamma + (\gamma v^h, v^h)_\Gamma \\
&= \|\nabla v^h\|_{0,\Omega}^2 - (1+\theta)(\gamma^{-\frac{1}{2}}\partial_{\mathbf{n}} v^h, \gamma^{\frac{1}{2}} v^h)_\Gamma + \gamma_0 \left\|v^h\right\|_{1/2,h,\Gamma}^2 \\
&\geq \|\nabla v^h\|_{0,\Omega}^2 - |1+\theta| \|\gamma^{-\frac{1}{2}}\partial_{\mathbf{n}} v^h\|_{0,\Gamma} \|\gamma^{\frac{1}{2}} v^h\|_{0,\Gamma} + \gamma_0 \left\|v^h\right\|_{1/2,h,\Gamma}^2 \\
&\geq \|\nabla v^h\|_{0,\Omega}^2 - \frac{(1+\theta)^2}{2\gamma_0} \left\|\partial_{\mathbf{n}} v^h\right\|_{-1/2,h,\Gamma}^2 + \frac{1}{2}\gamma_0 \left\|v^h\right\|_{1/2,h,\Gamma}^2 .
\end{aligned}$$

We applied Cauchy-Schwarz inequality on the third line and Young inequality on the fourth. It remains to use the discrete trace inequality (6) to bound the remining boundary term:

$$A_N(v^h, v^h) \geq \left(1 - \frac{c_I(1+\theta)^2}{2\gamma_0}\right) \|\nabla v^h\|_{0,\Omega}^2 + \frac{1}{2}\gamma_0 \left\|v^h\right\|_{1/2,h,\Gamma}^2 .$$

So provided that

$$\left(1 - \frac{c_I(1+\theta)^2}{2\gamma_0}\right) \geq \frac{1}{2}$$

we get (38). Well-posedness of Problem (20) follows from the Lax-Milgram lemma. □

B Error estimate for Nitsche's method

We detail here the convergence result for Nitsche's method.

Theorem B.1. *Let $u \in H^s(\Omega)$ with $3/2 < s \leq 1 + k$, be the solution to Problem* (2) *and let $u^h \in V^h$ be the solution to Problem* (20)*. Suppose that the Nitsche's parameter γ_0 is large, or that $\gamma_0 > 0$ when $\theta \neq -1$. Then there holds:*

$$\|u - u^h\|_h + \left\|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h\right\|_{-1/2,h,\Gamma} \leq C h^s \|u\|_{s,\Omega}. \tag{39}$$

where the constant $C > 0$ does not depend on h, but depends on θ and γ_0.

Proof. The proof is in fact an adaptation from [60, Theorem 3.6 and Theorem 3.8] in case of Signorini contact with elasticity. Let $u \in H^1(\Omega)$ be the solution to Problem (2) and $u^h \in V^h$ be the solution to Nitsche formulation (20). For any $v^h \in V^h$ there holds:

$$\begin{aligned}
\|\nabla u - \nabla u^h\|_{0,\Omega}^2 &= a(u-u^h, u(-v^h+v^h)-u^h) \\
&= a(u-u^h, u-v^h) + a(u-u^h, v^h-u^h) \\
&\leq \|\nabla u - \nabla u^h\|_{0,\Omega}\|\nabla u - \nabla v^h\|_{0,\Omega} + a(u-u^h, v^h-u^h),
\end{aligned}$$

where we used Cauchy-Schwarz inequality at the last line. Then with Young inequality we get

$$\frac{1}{2}\|\nabla u - \nabla u^h\|_{0,\Omega}^2 \leq \frac{1}{2}\|\nabla u - \nabla v^h\|_{0,\Omega}^2 + a(u-u^h, v^h-u^h). \tag{40}$$

We now focus on the discretization error term $a(u-u^h, v^h-u^h)$ and we use both formulations (1) and (20), the Green formula as well as the properties

$$v^h - u^h \subset V^h \subset H^1(\Omega)$$

to get

$$\begin{aligned}
a(u-u^h, v^h-u^h) &= a(u, v^h-u^h) - a(u^h, v^h-u^h) \\
&= (f, v^h-u^h)_\Omega + (\partial_{\mathbf{n}} u, v^h-u^h)_\Gamma \\
&\quad -(f, v^h-u^h)_\Omega - (g, \gamma(v^h-u^h) - \theta(\partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h))_\Gamma. \\
&\quad -(\partial_{\mathbf{n}} u^h, v^h-u^h)_\Gamma - \theta(u^h, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h)_\Gamma + (\gamma u^h, v^h-u^h)_\Gamma.
\end{aligned}$$

Note that, because of the regularity assumptions on u, there holds $\partial_{\mathbf{n}} u \in L^2(\Gamma)$ and the corresponding boundary term does not need to be a duality product but an integral. We can simplify the above expression as follows

$$\begin{aligned}
& a(u-u^h, v^h-u^h) \\
= \;& \underbrace{(\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, v^h-u^h)_\Gamma - \theta(u^h-g, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h)_\Gamma}_{\mathscr{T}_1} + \underbrace{(\gamma(u^h-g), v^h-u^h)_\Gamma}_{\mathscr{T}_2}.
\end{aligned} \tag{41}$$

The last term is rewritten, using the condition $u = g$ from (1) and the splitting $v^h - u^h = v^h - u + u - u^h$:

$$\mathscr{T}_2 = (\gamma(u^h-u), v^h-u)_\Gamma - (\gamma(u^h-u), u^h-u)_\Gamma.$$

Let $\omega_1 > 0$ be an arbitrary weight, the value of which will be fixed later on, and let us use Cauchy-Schwarz and Young inequalities:

$$\mathscr{T}_2 \leq \frac{\omega_1}{2}\gamma_0 \left\|v^h - u\right\|_{1/2,h,\Gamma}^2 + \gamma_0\left(\frac{1}{2\omega_1} - 1\right)\left\|u^h - u\right\|_{1/2,h,\Gamma}^2. \tag{42}$$

There remains to treat the other term $\mathcal{T}_1$ and, using once again the Dirichlet condition $u = g$ from (1) we transform it as below:

$$\begin{aligned}
\mathscr{T}_1 &= (\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, v^h-u^h)_\Gamma - \theta(u^h-g, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h)_\Gamma \\
&= (\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, v^h-u)_\Gamma + (\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, u-u^h)_\Gamma \\
&\quad -\theta(u^h-g, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u)_\Gamma - \theta(u^h-g, \partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h)_\Gamma \\
&= (\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, v^h-u)_\Gamma + (1+\theta)(\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h, u-u^h)_\Gamma \\
&\quad +\theta(u-u^h, \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u)_\Gamma.
\end{aligned}$$

With $\omega_2, \omega_3, \omega_4 > 0$ other weights and still with Cauchy-Schwarz and Young inequalities, we bound:

$$\begin{aligned}\mathscr{T}_1 \leq\ & \frac{\omega_2}{2}\left\|v^h - u\right\|^2_{1/2,h,\Gamma} + \frac{1}{2\omega_2}\left\|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h\right\|^2_{-1/2,h,\Gamma} \\ & + \frac{|\theta|\omega_3}{2}\left\|\partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u\right\|^2_{-1/2,h,\Gamma} + \frac{|\theta|}{2\omega_3}\left\|u - u^h\right\|^2_{1/2,h,\Gamma} \\ & + \frac{|1+\theta|\omega_4}{2}\left\|\partial_{\mathbf{n}} u^h - \partial_{\mathbf{n}} u\right\|^2_{-1/2,h,\Gamma} + \frac{|1+\theta|}{2\omega_4}\left\|u - u^h\right\|^2_{1/2,h,\Gamma}. \end{aligned} \tag{43}$$

Now we rewrite

$$\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h = \partial_{\mathbf{n}} u - \partial_{\mathbf{n}} v^h + \partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h,$$

and then use the discrete trace inequality (6) as well as a triangular inequality as follows

$$\left\|\partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h\right\|_{-1/2,h,\Gamma} \leq c_I^{\frac{1}{2}}\|\nabla v^h - \nabla u^h\|_{0,\Omega} \leq C(\|\nabla v^h - \nabla u\|_{0,\Omega} + \|\nabla u - \nabla u^h\|_{0,\Omega}).$$

We combine the above bounds with (40)–(43) and get:

$$\begin{aligned} & \left(\frac{1}{2} - \frac{C}{2\omega_2} - \frac{C|1+\theta|\omega_4}{2}\right)\|\nabla u - \nabla u^h\|^2_{0,\Omega} \\ & + \left[\gamma_0\left(1 - \frac{1}{2\omega_1}\right) - \frac{|\theta|}{2\omega_3} - \frac{|1+\theta|}{2\omega_4}\right]\left\|u^h - u\right\|^2_{1/2,h,\Gamma} \\ \leq\ & \left(\frac{1}{2} + \frac{C}{2\omega_2} + \frac{C|1+\theta|\omega_4}{2}\right)\|\nabla u - \nabla v^h\|^2_{0,\Omega} \\ & + \left(\frac{\omega_1}{2}\gamma_0 + \frac{\omega_2}{2}\right)\left\|v^h - u\right\|^2_{1/2,h,\Gamma} \\ & + \left(\frac{C}{2\omega_2} + \frac{|\theta|\omega_3}{2} + \frac{C|1+\theta|\omega_4}{2}\right)\left\|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} v^h\right\|^2_{-1/2,h,\Gamma}. \end{aligned}$$

Now take for instance $\omega_1 = 1$, $\omega_2 = 4C$ and ω_4 small enough. For θ arbitrary, take moreover $\omega_3 = 1$ for instance and γ_0 large enough. For $\theta = -1$ and any value of γ_0, take ω_3 large enough. With the above choices, we get the abstract estimate:

$$\begin{aligned} & \|\nabla u - \nabla u^h\|^2_{0,\Omega} + \left\|u^h - u\right\|^2_{1/2,h,\Gamma} \\ \leq\ & C\left(\|\nabla u - \nabla v^h\|^2_{0,\Omega} + \left\|v^h - u\right\|^2_{1/2,h,\Gamma} + \left\|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} v^h\right\|^2_{-1/2,h,\Gamma}\right). \end{aligned} \tag{44}$$

We then use standard interpolation estimates for the Lagrange operator [77] (for the estimate associated to the interpolation of the normal derivative, see for instance [60, 87]). The estimate on the normal derivative comes from

$$\begin{aligned} & \left\|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} u^h\right\|_{-1/2,h,\Gamma} \\ \leq\ & \left\|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} v^h\right\|_{-1/2,h,\Gamma} + \left\|\partial_{\mathbf{n}} v^h - \partial_{\mathbf{n}} u^h\right\|_{-1/2,h,\Gamma} \\ \leq\ & \left\|\partial_{\mathbf{n}} u - \partial_{\mathbf{n}} v^h\right\|_{-1/2,h,\Gamma} + C(\|\nabla v^h - \nabla u\|_{0,\Omega} + \|\nabla u - \nabla u^h\|_{0,\Omega}). \end{aligned}$$

This ends the proof. □

C Barbosa & Hughes and Nitsche

Following [104, 129] we detail the equivalence between the Barbosa & Hughes stabilized method (34) and Nitsche's formulation (20), but in a simple situation where we suppose $g = 0$ and take the pair $V^h = V^{h,1}$ $(k = 1)$ and $M^h = M_1^{h,d}$ $(l = 1)$ for the solution and the multiplier. In (34) take $v^h = 0 \in V^h$ and, for an arbitrary boundary facet $E \in \mathcal{E}_h$, and an arbitrary polynomial $\mu_E \in \mathbb{P}_1(E)$, take $\mu^h|_E = \mu_E$ and $\mu^h = 0$ elsewhere. We get then:

$$(\mu_E, u^h)_E - (\delta(\lambda^h + \partial_{\mathbf{n}} u^h), \mu_E)_E = 0.$$

We reformulate

$$(\mu_E, u^h - \delta(\lambda^h + \partial_{\mathbf{n}} u^h))_E = 0.$$

Remark that, since $u^h - \delta(\lambda^h + \partial_{\mathbf{n}} u^h) \in \mathbb{P}_1(E)$, this implies the equality

$$u^h - \delta(\lambda^h + \partial_{\mathbf{n}} u^h) = 0$$

or, identically:

$$u^h = \delta(\lambda^h + \partial_{\mathbf{n}} u^h), \quad \lambda^h = \frac{1}{\delta} u^h - \partial_{\mathbf{n}} u^h. \tag{45}$$

Now from (34) which is

$$a(u^h, v^h) + (\lambda^h, v^h)_\Gamma + (\mu^h, u^h)_\Gamma - (\delta(\lambda^h + \partial_{\mathbf{n}} u^h), \mu^h + \partial_{\mathbf{n}} v^h)_\Gamma = (f, v^h)_\Omega$$

we use (45) and we recover

$$a(u^h, v^h) + (\lambda^h, v)_\Gamma + (\mu^h, u^h)_\Gamma - (u^h, \mu^h + \partial_{\mathbf{n}} v^h)_\Gamma = (f, v^h)_\Omega.$$

We develop and the two terms in $(\mu^h, u^h)_\Gamma$ cancel:

$$a(u^h, v^h) + (\lambda^h, v^h)_\Gamma - (u^h, \partial_{\mathbf{n}} v^h)_\Gamma = (f, v^h)_\Omega.$$

We replace λ with the expression above in (45) and get

$$a(u^h, v^h) - (\partial_{\mathbf{n}} u^h, v^h)_\Gamma - (u^h, \partial_{\mathbf{n}} v^h)_\Gamma + (\delta^{-1} u^h, v^h)_\Gamma = (f, v^h)_\Omega.$$

This is exactly the symmetric Nitsche's formulation (20), with $\theta = 1$ and $\gamma_0 = \delta_0^{-1}$. Let us do this the other way round. Let us start from the Nitsche's formulation (20) (with $\theta = 1$):

$$a(u^h, v^h) - (\partial_{\mathbf{n}} u^h, v^h)_\Gamma - (u^h, \partial_{\mathbf{n}} v^h)_\Gamma + (\gamma u^h, v^h)_\Gamma = (f, v^h)_\Omega.$$

We define λ^h as in (45) (still with $\gamma_0 = \delta_0^{-1}$), we check that $\lambda^h \in M^h$ and rewrite (45) as

$$(\delta(\lambda^h + \partial_{\mathbf{n}} u^h), \mu^h + \partial_{\mathbf{n}} v^h)_\Gamma = (u^h, \mu^h + \partial_{\mathbf{n}} v^h)_\Gamma$$

for all μ^h (note that $\partial_{\mathbf{n}} v^h \in M^h$). Nitsche formulation is rewritten

$$a(u^h, v^h) + (\underbrace{\gamma u^h - \partial_{\mathbf{n}} u^h}_{\lambda^h}, v^h)_\Gamma - \underbrace{(u^h, \partial_{\mathbf{n}} v^h)_\Gamma}_{(\delta(\lambda^h + \partial_{\mathbf{n}} u^h), \mu^h + \partial_{\mathbf{n}} v^h)_\Gamma - (u^h, \mu^h)_\Gamma} = (f, v^h)_\Omega.$$

And we recover

$$a(u^h, v^h) + (\lambda^h, v^h)_\Gamma + (u^h, \mu^h)_\Gamma - (\delta(\lambda^h + \partial_{\mathbf{n}} u^h), \mu^h + \partial_{\mathbf{n}} v^h)_\Gamma = (f, v^h)_\Omega.$$

This is exactly Barbosa & Hughes formulation (34). When $\theta = -1$, we proceed the same way and we recover another consistent, but skewsymmetric stabilized formulation equivalent to skewsymmetric Nitsche:

$$a(u^h, v^h) + (\lambda^h, v^h)_\Gamma - (u^h, \mu^h)_\Gamma + (\delta(\lambda^h + \partial_{\mathbf{n}} u^h), \mu^h + \partial_{\mathbf{n}} v^h)_\Gamma = (f, v^h)_\Omega.$$

For different spaces of functions, of if $g \neq 0$ the equivalence between the two formulation may not be exact anymore and some projection operators need to be introduced [104, 129].

Remark C.1. *Alternatively, for any value of θ, we can do as follows: we rewrite* (45) *as*

$$(\delta(\lambda^h + \partial_{\mathbf{n}} u^h), \mu^h + \theta\partial_{\mathbf{n}} v^h)_\Gamma = (u^h, \mu^h + \theta\partial_{\mathbf{n}} v^h)_\Gamma$$

for all μ^h. Nitsche's formulation is rewritten

$$a(u^h, v^h) + (\underbrace{\gamma u^h - \partial_{\mathbf{n}} u^h}_{\lambda^h}, v^h)_\Gamma - \underbrace{(u^h, \theta\partial_{\mathbf{n}} v^h)_\Gamma}_{(\delta(\lambda^h+\partial_{\mathbf{n}} u^h),\mu^h+\theta\partial_{\mathbf{n}} v^h)_\Gamma-(u^h,\mu^h)_\Gamma} = (f, v^h)_\Omega.$$

We find

$$a(u^h, v^h) + (\lambda^h, v^h)_\Gamma + (u^h, \mu^h)_\Gamma - (\delta(\lambda^h + \partial_{\mathbf{n}} u^h), \mu^h + \theta\partial_{\mathbf{n}} v^h)_\Gamma = (f, v^h)_\Omega.$$

For $\theta = 1$ it is still the symmetric Barbosa & Hughes formulation, and for other values of θ, it is a consistent, but unsymmetric stabilized method.

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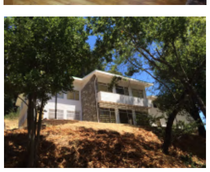

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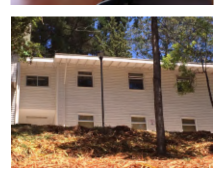
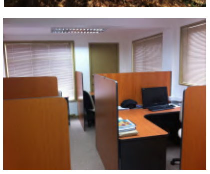

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