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**Study of avalanche models using well balanced finite volume schemes**

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# Study of Avalanche Models Using Well Balanced Finite Volume Schemes

# Estudio de modelos de avalancha usando esquemas de volúmenes finitos bien balanceados

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*Avalanches are natural disasters that have a significant human and economic impact on a global scale. Chile is a mountainous country and is significantly affected by these events. In this research, we are interested in using a numerical technique based on a well-balanced Finite Volume method to examine snow avalanche behaviour. As an avalanche model, we investigate the Saint-Venant system with Voellmy-Salm friction. We analyze the finite volume approach using a hydrostatic reconstruction scheme. The Rigopiano avalanche in Italy was used to test this strategy. The numerical model is explained and the results for the real avalanche case are presented graphically. Finally, some conclusions and suggestions for further research are presented.*

*Keywords: Avalanches; natural disasters; Saint Venant's equations; well balanced finite volumes; hydrostatic reconstruction*

*Las avalanchas son desastres naturales que tienen un impacto humano y económico significativo a escala global. Chile es un país montañoso y se ve significativamente afectado por estos eventos. En esta investigación, estamos interesados en utilizar una técnica numérica basada en un método de Volúmenes Finitos bien balanceado para examinar el comportamiento de las avalanchas de nieve. Como modelo de avalancha, investigamos el sistema de Saint-Venant con la fricción de Voellmy-Salm. Analizamos el enfoque de volumen finito utilizando un esquema de reconstrucción hidrostática. La avalancha de Rigopiano en Italia se utilizó para probar esta estrategia. Se explica el modelo numérico y se presentan gráficamente los resultados para el caso real de avalancha. Finalmente, se presentan algunas conclusiones y sugerencias para futuras investigaciones.*

*Palabras Claves: Avalanchas; desastres naturales; ecuaciones de Saint Venant; volúmenes finitos bien balanceados; reconstrucción hidrostática*

## Introduction

The effects of climate change on natural disasters are attracting considerable attention. These changes are susceptible to trigger snow avalanches. Snow avalanches are defined as the rapid descent of snow masses down steep slopes as result of gravity, often dragging soil, rocks or vegetation (Pudasaini and Hutter, 2007).

An estimated 250 people per year fatalities are due to snow avalanches worldwide (Schweizer et al., 2015). In certain regions, the economic cost to avoid the effects of snow avalanches can be very high. For example, it is estimated that the average annual cost in Canada exceeds $5 billion (Schweizer et al., 2015).

In Chile, there are large areas with high altitudes where snow avalanches can occur. According to published statistics on fatalities in central Chile between 1906 and 2001, of the 378 total victims, 241 (63.8%) were related to mining activities, while 52 (13.8%) were related to tourism (Ramírez and Mery, 2007).

Currently, only the mining sector in the Center-North zone uses meteorological records, data analysis, and avalanche simulation for avalanche risk management (Ramírez and Mery, 2007).. In Chile, there is no governmental avalanche warning service, and only private groups such as ski resorts and mining enterprises take preventative steps.

Using a numerical model to simulate snow height and flow velocity is one method for analyzing avalanche dynamics. Numerous physical models exist to describe avalanches. For this purpose, the Saint-Venant system of differential equations is widely used (Pudasaini and Hutter, 2007). The model includes friction effect as a source term. The friction model or rheology used varies depending on the fluid characteristics. Consideration will be given to the Voellmy-Salm rheology proposed by Salm (1993) and Voelmy (1955). However, we can mention that other physical model can be used. For example, the Savage-Hutter equations of various types are used to model avalanches (Savage and Hutter, 1991). The numerical model that we use is the finite volume method, which uses a non-conservative scheme. The main approach is described by Bouchut (2004), along with the hydrostatic reconstruction scheme. This approach will be used to conduct our simulations. The scheme is well-balanced, consistent, and stable (Bouchut, 2004).

We might list a few publications that complement and work with this strategy. The hydrostatic reconstruction is utilized by Audusse (2004) for a well-balanced approach for the Saint-Venant problem with topography, including proofs and numerical examples, as well as an extension to second order. The enhanced second-order approach shown by Kurganov and Petrova (2007) preserves steady states and fluid height positivity. In reference to pyroclastic avalanches, a numerical technique using the Voelmy-Salm rheology is discussed by Michieli Vitturi et al. (2018) along with other numerical examples.

The Rigopiano avalanche in Italy was investigated using the numerical technique in this study. There is abundant literature about this disaster, and numerous studies have been conducted in the zone to establish the event's characteristics. The meteorological conditions and fluid dynamics are detailed by Frigo et al. (2020). The velocity and departure distance estimates are provided by Issler (2020). We will provide graphs of the numerical simulations performed with the scheme, using estimates for the physical parameters and assumptions about the initial conditions that we consider reasonable according to the available data. Finally, we will conclude, describe various numerical approach enhancements that may be investigated, and provide some suggestions for future studies.

# Methodology

We consider the Saint-Venant system with Voellmy-Salm rheology as avalanche model. The model is given by

$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\dfrac{h^2}{2}\right) + \partial_y(huv) + gh\partial_x z = \\ \dfrac{u}{\sqrt{u^2+v^2}}\left(\mu hg + \dfrac{g}{\xi}(u^2+v^2)\right), \\ \partial_t(hv) + \partial_x(huv) + \partial_y\left(hv^2 + g\dfrac{h^2}{2}\right) + gh\partial_y z = \\ \dfrac{v}{\sqrt{u^2+v^2}}\left(\mu hg + \dfrac{g}{\xi}(u^2+v^2)\right), \end{cases} \tag{1}$$

where $h = h(x, y, t)$ is the fluid height, $u = u(x, y, t)$ and $v = v(x, y, t)$ are the components of the velocity, $z = z(x, y)$ is the topography height, $g$ is the gravitational constant, $\mu$ is Coulomb's coefficient of friction (Popov, 2010) and $\xi$ is turbulent friction coefficient (Ferziger and Peric, 2002). We also consider the initial conditions

$$h_0 = h(x, y, 0), \quad u_0 = u(x, y, 0), \quad v_0 = (x, y, 0) \tag{2}$$

We define $Z = gz$ and we set $U = (h, hu, hv)$ as the system solution.

We have made the following assumptions for the avalanche model:

1. The avalanche can be treated as an homogeneous fluid (the density is constant in space and time).
2. The velocity in the vertical direction is negligible.
3. The pressure distribution is hydrostatic in the vertical direction.
4. The curvature of the bed is negligible.
5. Normal and shear stresses on the free surface are negligible.
6. We can consider that the bed has a gentle slope concerning the horizontal plane of reference. This means we can approximate the normal to the bed with the vertical direction.

The last assumption is not realistic in everyday avalanche events with steep terrain. However, first, we will develop the scheme for this model. Later, we will show a more accurate physical model that considers this problem. Then, we will discuss a way to adapt the numerical scheme for the new model.
In this work we use a non-conservative finite volume scheme. We study this method in parts, first developing the one-dimensional model, then using this scheme for the two-dimensional problem, and finally, we include friction in the model.

## One-dimensional frictionless model

The one-dimensional model without friction is given by

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x \left( hu^2 + g\dfrac{h^2}{2} \right) + gh\partial_x z = 0, \end{cases} \tag{3}$$

We define $U = (h, hu)$. We can write the equations as a quasi-lineal system in the variable $\widetilde{U} = (U, Z)$.

$$\begin{cases} \partial_t U + \partial_x F(U,Z) + B(U,Z)\partial_x Z = 0, \\ \partial_t Z = 0, \end{cases} \tag{4}$$

with

$$F = \begin{pmatrix} hu \\ hu^2 + g\dfrac{h^2}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ h \end{pmatrix} \tag{5}$$

We consider a critical point for this system a point $(U, Z)$ such that $F_U(U, Z)$ is not invertible. We can write the equations in the form

$$\partial_t(U,Z) + A(U,Z)\partial_x(U,Z) = 0 \tag{6}$$

with

$$A(U,Z) = \begin{pmatrix} F_U & F_Z + B \\ 0 & 0 \end{pmatrix} \tag{7}$$

The eigenvalues of $A(U, Z)$ are

$$\lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = 0 \quad \lambda_3 = u + \sqrt{gh} \tag{8}$$

Then, we have that at every noncritical point, the system is hyperbolic ($A(U, Z)$ is diagonalizable).
The stationary states are the solutions $U(x)$ independent of time. This states are relevant because generally represent the solution when time tend to infinity. The stationary states are the functions $h(x), u(x)$ that satisfy

$$\begin{cases} hu = \text{cte.}, \\ \dfrac{u^2}{2} + gh + Z = \text{cte.}, \end{cases} \tag{9}$$

The stationary states at rest are given by

$$\begin{cases} u = 0, \\ h + z = \text{cte.} \end{cases} \tag{10}$$

In the finite volume method, a mesh of points $x_{i+1/2}$, $i \in \mathbb{Z}$, in space is created. Finite volumes are defined by $C_i = ]x_{i-1/2}, x_{i+1/2}[$. We consider a time step $\Delta t$ and define $t_{n+1} = t_n + \Delta t$, $n \in \mathbb{N}$. We want to approximate the solution $U(x, t)$ by discrete values $U_i^n$, $i \in \mathbb{Z}$, $n \in \mathbb{N}$. This is,

$$U_i^n \approx \frac{1}{\Delta x_i} \int_{C_i} U(t_n, x)dx \tag{11}$$

We consider a first order non-conservative finite volume scheme given by

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{\Delta x_i}(F_{i+1/2-} - F_{i-1/2}) = 0, \tag{12}$$

with

$$F_{i+1/2-} = F_l(\tilde{U}_i^n, \tilde{U}_{i+1}^n), \quad F_{i+1/2+} = F_r(\tilde{U}_i^n, \tilde{U}_{i+1}^n) \tag{13}$$

where $\tilde{U}_i^n = (U_i^n, Z_i)$, and $F_l$ and $F_r$ are the left and right numerical fluxes, respectively.
We impose a CFL condition for the time step to prevent that the numerical values explode. The condition have the following form

$$\Delta t a \leq \Delta x \tag{14}$$

where $a = \text{máx}|\lambda|$, is the maximum modulus of the eigenvalues of the matrix system, evaluated for all cells at time step $n$ (Audusse, 2004).
A well balanced scheme for this problem is the hydrostatic reconstruction scheme (Bouchut, 2004). Due to the presence of topography, to calculate the fluxes between the mesh elements it is necessary to reconstruct the left and right solution states at each interface. We denote this states $U_l = (h_l, h_l u_l)$ and $U_r = (h_r, h_r u_r)$, respectively. We also denote the reconstructed states as $U_l^* = (h_l^*, h_l^* u_l)$ and $U_r^* = (h_r^*, h_r^* u_r)$, respectively.
In the hydrostatic reconstruction scheme, the steady state relations are replaced by

$$\begin{cases} u = \text{cte.}, \\ gh + Z = \text{cte.}, \end{cases} \tag{15}$$

With these relations, the reconstructed states are obtained by

$$\begin{aligned} gh_l^* &= (gh_l - (\Delta Z)_+)_+, \\ gh_r^* &= (gh_r - (-\Delta Z)_+)_+, \end{aligned} \tag{16}$$

where $\Delta Z = Z_r - Z_l$. The fluxes will be given by

$$\begin{aligned} F_l(U_l, U_r, Z_l, Z_r) &= \mathcal{F}(U_l^*, U_r^*) \\ &+ \begin{pmatrix} 0 \\ \frac{gh_l^2}{2} - \frac{g(h_l^*)^2}{2} \end{pmatrix}, \\ F_r(U_l, U_r, Z_l, Z_r) &= \mathcal{F}(U_l^*, U_r^*) \\ &+ \begin{pmatrix} 0 \\ \frac{gh_r^2}{2} - \frac{g(h_r^*)^2}{2} \end{pmatrix}, \end{aligned} \tag{17}$$

where $\mathcal{F}$ is a consistent numerical flux for the Saint-Venant problem without topography. In our case, we will use the Lax-Friedrichs flux, given by

$$\mathcal{F}(U_l, U_r) = \frac{1}{2}(F(U_l) + F(U_r)) - \frac{\Delta x_i}{2\Delta t}(U_r - U_l), \tag{18}$$

It can be proven that with this flux, the scheme is conservative in $h$, preserves the non-negativity of $h$ in the interface, and is well balanced, consistent and stable. Further details and technical explanations of concepts and the proofs of these propositions are presented by Bouchut (2004).

## Two-dimensional frictionless model

Now we can proceed with the avalanche model in two dimensions without friction force. The two-dimensional Saint-Venant problem with frictionless topography is given by

$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\frac{h^2}{2}\right) \\ +\partial_y(huv) + gh\partial_x z = 0, \\ \partial_t(hv) + \partial_x(huv) + \\ \partial_y\left(hv^2 + g\frac{h^2}{2}\right) + gh\partial_y z = 0, \end{cases} \tag{19}$$

The solutions of the system can develop discontinuities, which means we have to consider weak solutions. This solutions are well defined under the assumption that the topography $z \in W^{1,\infty}(\mathbb{R})$ (Dafermos, 2000). To find a unique physical solution we use an entropy condition as an additional admissibility criteria. The details of the theory about uniqueness of solution are explained by Fjordholm et al. (2011).

For the finite volume method, we consider a mesh of elements $C_i$ in two dimensions. Let $\Gamma_{ij}$ be the edge between the volumes $C_i$ and $C_j$, and $n_{ij}$ the unitary normal vector with orientation from $C_i$ to $C_j$. Let $U_i^n$ be the values of the solution in some interior point of the element $C_i$ at time $t_n$. The finite volume method is given by

$$U_i^{n+1} - U_i + \frac{\Delta t}{|C_i|}\sum_{j\in K_i}|\Gamma_{ij}|F_{ij} = 0, \tag{20}$$

where $|C_i|$ is the area of the control volume $C_i$, $|\Gamma_{ij}|$ is the length of the edge $\Gamma_{ij}$, $K_i$ is the set of indices of the cells that share edges with $C_i$, and $F_{ij}$ is the flux between $C_i$ and $C_j$ with

$$F_{ij} = F(U_i, U_j, Z_i, Z_j, n_{ij}). \tag{21}$$

Let $n = (n_1, n_2)$ be the unit vector with its rotation matrix given by

$$R_n = \begin{pmatrix} n_1 & -n_2 \\ n_2 & n_1 \end{pmatrix} \tag{22}$$

Let $x' = R_n x$ and $(u', v') = R_n^{-1}(u, v)$. Then $U' = (h, hu', hv')$ is a solution to the two-dimensional problem. We can compute the numerical fluxes through the following one-dimensional problem

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\frac{h^2}{2}\right) + gh\partial_x z = 0, \\ \partial_t(hv) + \partial_x(huv) = 0, \end{cases} \tag{23}$$

Let $F_l(U_l', U_r', \Delta Z) = (F_l^0, F_l^1, F_l^2)$ be the flux obtained from the one-dimensional problem with $U'$. Them the left flux of the original two-dimensional problem is given by

$$F_l(U_l, U_r, \Delta Z, n) = \begin{pmatrix} F_l^0(U_l', U_r', \Delta Z) \\ R_n\begin{pmatrix} F_l^1(U_l', U_r', \Delta Z) \\ F_l^2(U_l', U_r', \Delta Z) \end{pmatrix} \end{pmatrix} \tag{24}$$

Let $(h, hu, hv)^* = (h, -hu, -hv)$. By symmetry we have

$$F_r(U_l, U_r, \Delta Z) = -F_l(U_r^*, U_l^*, -\Delta Z)^*, \tag{25}$$

The right flux of the original two-dimensional problem is given by

$$-F_r(U_r, U_l, -\Delta Z, n) = \begin{pmatrix} F_r^0(U_l', U_r', \Delta Z) \\ R_n\begin{pmatrix} F_r^1(U_l', U_r', \Delta Z) \\ F_r^2(U_l', U_r', \Delta Z) \end{pmatrix} \end{pmatrix} \tag{26}$$

Now we explain how we solve the one-dimensional problem in (**??**). We can obtain the numerical flux for the problem with the first and second equation with the method for one-dimensional problems shown in the previous section.

The third equation is a passive transport equation

$$\partial_t(hv) + \partial_x(huv) = 0 \tag{27}$$

We obtain the flux for this part using

$$F_l^2 = \begin{cases} F_l^0 v_l & \text{if} \quad F_l^0 \geq 0, \\ F_l^0 v_r & \text{if} \quad F_l^0 \leq 0, \end{cases} \tag{28}$$

And analogously with $F_r^2$.

## Two-dimensional model with friction

Now we can consider the two-dimensional problem with friction given by (**??**). Friction in the Voelmy-Salm rheological model includes two parts:

1. Coulomb friction:

$$F_c = \frac{(u,v)}{\sqrt{u^2+v^2}}(\mu h g) \tag{29}$$

2. Turbulent friction:

$$F_t = \frac{(u,v)}{\sqrt{u^2+v^2}}\left(\frac{g}{\xi}(u^2+v^2)\right) \tag{30}$$

Each type of friction is treated differently in the numerical scheme.
We can include the Coulomb friction in the numerical method by modifying the source term in our scheme. To do this, the numerical fluxes are computed with

$$F_{ij} = F(U_i, U_j, \Delta Z_{ij} - f_1^{ij}(x_j - x_i)_1 - f_2^{ij}(x_j - x_i)_2, n_{ij}) \tag{31}$$

where $x_i$ and $x_j$ are arbitrary points in the interior of the elements $C_i$ and $C_j$, respectively. We define

$$f^{ij} = -\varphi_{g\mu}\left((gh_i - gh_j - \Delta Z_{ij})\frac{x_j - x_i}{|x_j - x_i|^2}, \frac{(u^{ij}, v^{ij})}{\Delta t}\right) \tag{32}$$

In the above expression, we use

$$u^{ij} = \frac{h_i u_i + h_j u_j}{h_i + h_j}, \qquad v^{ij} = \frac{h_i v_i + h_j v_j}{h_i + h_j} \tag{33}$$

$$\varphi_{g\mu}(X,Y) = \operatorname*{proj}_{g\mu}\left(\operatorname*{proj}_{g\mu}(X) + \frac{2}{1+\max(1, -X\cdot Y/g\mu|Y|)}Y\right) \tag{34}$$

with

$$\operatorname*{proj}_{g\mu}(X) = \begin{cases} X & \text{if} \quad |X| \leq g\mu, \\ g\mu\dfrac{X}{|X|} & \text{if} \quad |X| > g\mu, \end{cases} \tag{35}$$

For the turbulent friction, we use a splitting method to ensure the scheme stability (Bouchut et al., 2020). Once the solution of the finite volume method is obtained, which we will denote as $h^*, u^*, v^*$, we proceed to include the turbulent friction. The final solution is given by

$$\begin{aligned} h &= h^* \\ u &= \frac{u^* h^* \xi}{g\sqrt{w^*}\Delta t + h^*\xi} \\ v &= \frac{v^* h^* \xi}{g\sqrt{w^*}\Delta t + h^*\xi} \end{aligned} \tag{36}$$

where $w = u^2 + v^2$.

## Model in global coordinates

As we explained before, the model studied considers a smooth slope. In real situations, this assumption can be unsatisfactory. A way to solve this problems is to use a model in global coordinates that considers the effect on vertical velocity given by steep terrain. The scheme derived from this flow analysis is explained in detail by Zugliani and Rosatti (2021). In Figure 1 we show a control volume for the one-dimensional model with a slope angle $\theta$.

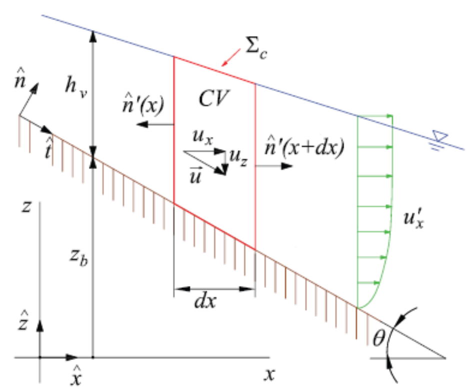


Figure 1: Control volume ($x$ is horizontal axis and $z$ is vertical axis). $h_v$ is the height of the fluid, $z_b$ is the height of the terrain topography and $u_x$ is the velocity component in $x$.

The system of equations in global coordinates is given by

$$\begin{cases} \partial_t h_v + \partial_x(h_v u_x) = 0, \\ \partial_t(h_v u_x) + \partial_x\left(h_v u_x^2 + g\cos^2\theta \dfrac{h_v^2}{2}\right) \\ +g\cos^2\theta h_v \partial_x z_b = -\dfrac{\tau_0}{\rho}\dfrac{s_x^\tau}{\cos\theta}, \end{cases} \tag{37}$$

where

$$\frac{\tau_0}{\rho} = \mu g h_v \cos^2\theta + g\frac{u_x^2}{\xi\cos^2\theta} \tag{38}$$

$$s_x^\tau = \frac{u_x}{|u_x|}\cos\theta \tag{39}$$

This model can be extended to two dimensions, giving the following system

$$
\begin{cases}
\partial_t h_v + \partial_x(h_v u_x) + \partial_y(h_v u_y) = 0, \\
\\
\partial_t(h_v u_x) + \partial_x\left(h_v u_x^2 + gcos^2\theta \dfrac{h_v^2}{2}\right) + \partial_y(h_v u_x u_y) \\
+ g\cos^2\theta h_v \partial_x z_b = \dfrac{u_x}{|u|}\left(\mu g h_v \cos^2\theta + g\left(\dfrac{u_x^2+u_y^2}{\xi\cos^2(\theta)}\right)\right), \\
\\
\partial_t(h_v u_y) + \partial_x(h_v u_x u_y) + \partial_y\left(h_v u_y^2 + g\cos^2\theta \dfrac{h_v^2}{2}\right) \\
+ g\cos^2\theta h_v \partial_x z_b = \dfrac{u_y}{|u|}\left(\mu g h_v \cos^2\theta + g\left(\dfrac{u_x^2+u_y^2}{\xi\cos^2(\theta)}\right)\right)
\end{cases}
\tag{40}
$$

We can rewrite this system in the form of the avalanche model (**??**) as follows

$$
\begin{cases}
\partial_t h_v + \partial_x(h_v u_x) + \partial_y(h_v u_y) = 0, \\
\\
\partial_t(h_v u_x) + \partial_x\left(h_v u_x^2 + \widetilde{g}\dfrac{h_v^2}{2}\right) + \partial_y(h_v u_x u_y) \\
+ \widetilde{g} h_v \partial_x z_b = \dfrac{u_x}{|u|}\left(\mu \widetilde{g} h_v + \dfrac{g}{\widetilde{\xi}}(u_x^2+u_y^2)\right), \\
\partial_t(h_v u_y) + \partial_x(h_v u_x u_y) + \partial_y\left(h_v u_y^2 + \widetilde{g}\dfrac{h_v^2}{2}\right) \\
+ \widetilde{g} h_v \partial_x z_b = \dfrac{u_y}{|u|}\left(\mu \widetilde{g} h_v + \dfrac{g}{\widetilde{\xi}}(u_x^2+u_y^2)\right).
\end{cases}
\tag{41}
$$

where $\widetilde{g} = g\cos^2\theta$ and $\widetilde{\xi} = \xi\cos^2\theta$. In this way we can use the hydrostatic reconstruction method for this system with the new parameters $\widetilde{g}$ and $\widetilde{\xi}$. We recall that this method allows incorporating the Coulomb friction. In the case of turbulent friction, it is added by means of splitting, and to do so the original parameter $g$ is used, as can be seen from the system of equations.

# Results

The numerical scheme was applied for the case of the snow avalanche of January 18, 2017 at Rigopiano, Gran Sasso National Park (Frigo et al., 2020). This natural disaster destroyed the Rigopiano hotel, resulting in the death of 29 people. The avalanche was a mixture of snow and wood, displacing rocks and trees in its path. The damage generated in the event shows that the avalanche was of great intensity. Figure 2 shows an aerial view of the path of the avalanche before and after the event.

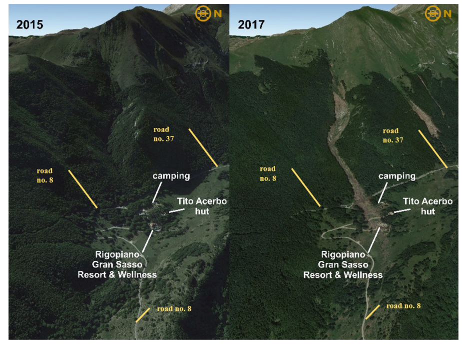


Figure 2: Aerial view of the path of the avalanche over Rigopiano Hotel, before (2015) and after (2017) the catastrophic event. The image is taken from Frigo et al. (2020).

Figure 3 shows a map of the Abrazzo region, where the avalanche happened. The position of the Rigopiano Hotel is indicated. The location of snow measurement points in the area is also shown. We also show a map of Italy with the position of the Abrazzo region.

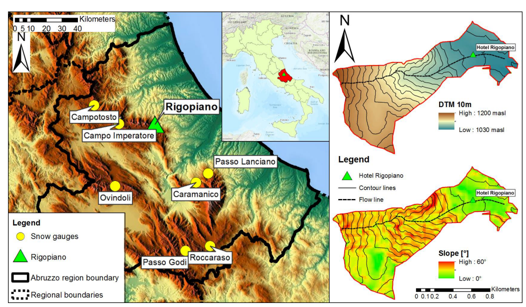


Figure 3: Geographical area of the Abrazzo region with the Rigopiano Hotel (green triangle) position. Snow gauges in the area are also shown (yellow circles). Italy map on top with Abrazzo region marked in red. The right diagrams show the relief's height and slope, indicating the avalanche's main flow. The image is taken from Bocchiola et al. (2020).

A simulation of the avalanche at Rigopiano using the system of equations in global coordinates given in (**??**) is now presented. The numerical method consists of finite volumes with a hydrostatic reconstruction scheme, with the parameters for the new physical model being modified. Figure 4 displays the topography $z$ of the Rigopiano area under study.

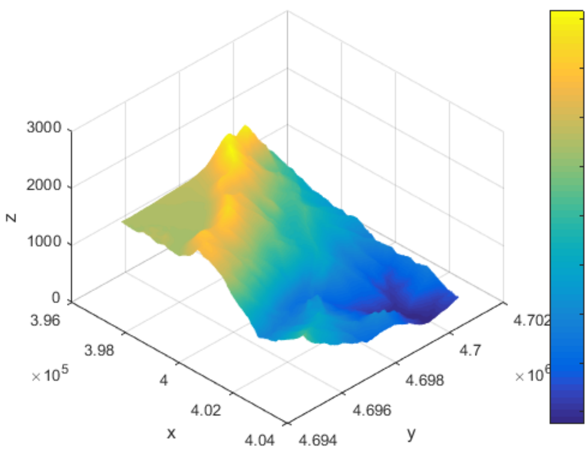


Figure 4: Rigopiano topography with height $z$. This surface is considered as the relief without snow of the area. Right scale is given in m.

We worked with a sub-domain of this domain for the numerical simulation, based on the area shown in the Figure 3. This sub-domain's dimensions are approximately 989 m x 989 m, with approximates coordinates in Figure 4 given by $[4.0001 \cdot 10^5, 4.0100 \cdot 10^5] \times [4.6970 \cdot 10^6, 4.6980 \cdot 10^6]$. The topography in this sub-domain is shown in Figure 5.

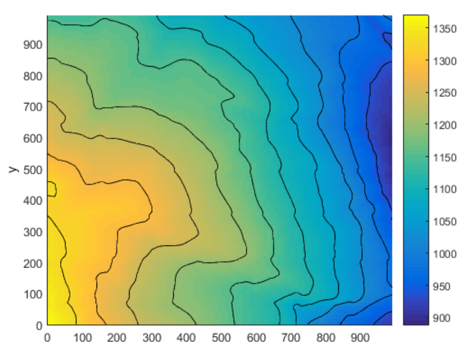


Figure 5: Topography in sub-domain with height $z$. The isolevels are shown. Right scale is given in m.

The initial condition for the snow height is considered to be a constant slope with a maximum height of 2 m that goes to 0 m at the isolevel $z = 1600$ m. As data we used $g = 9.8$ m/s$^2$, $\mu = 0.15$, $k = g/\xi = 0.002$ and an angle of slope $\theta = 30°$. Figures 6-10 show the results at different times.

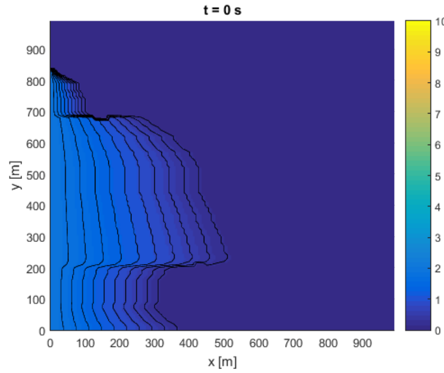


Figure 6: Simulation at $t = 0$ s. The snow depth $h$ is shown with the isolevels. The scale is in m.

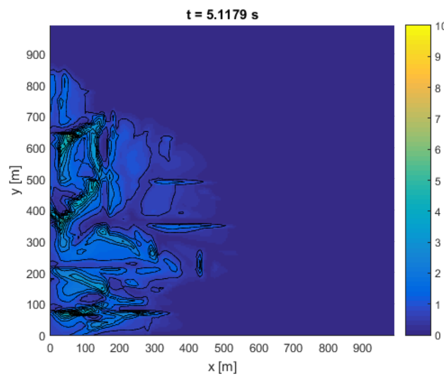


Figure 7: Simulation at $t = 5$ s. The snow depth $h$ is shown with the isolevels. The scale is in m.

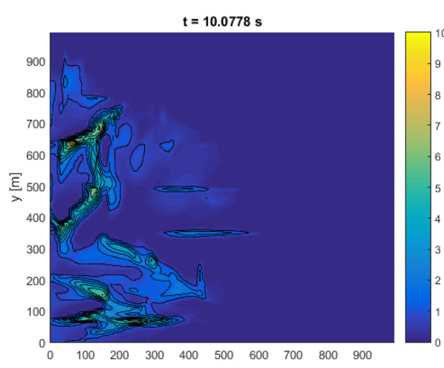


Figure 8: Simulation at $t = 10$ s. The snow depth $h$ is shown with the isolevels. The scale is in m.

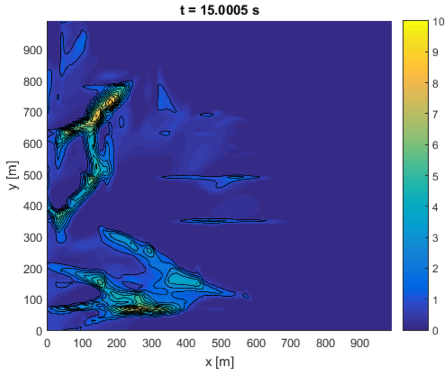


Figure 9: Simulation at $t = 15$ s. The snow depth $h$ is shown with the isolevels. The scale is in m.

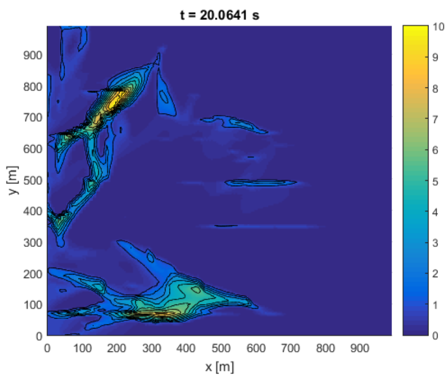


Figure 10: Simulation at $t = 20$ s. The snow depth $h$ is shown with the isolevels. The scale is in m.

# Conclusions

In this work, we model an avalanche using the Saint-Venant system of differential equations with topography and friction given by the Voellmy-Salm rheology. We use the finite volume method with hydrostatic reconstruction to analyze a real case. This scheme was evaluated for different examples taken from the literature, considering the one-dimensional and two-dimensional cases, as well as the frictional and frictionless cases. These testing allowed us to calibrate and validate the numerical model.

The study's primary focus and so called "real" event was the Rigopiano avalanche. We found that the steep terrain is an essential factor in the results. It is necessary to consider this conditions' effect on the flow and incorporate it's impact in the system of equations. Due to the lack of specific information, the exact information is unavailable, we assumed the initial snow depth at the start of the avalanche. Therefore, we considered setting a maximum snow height of 2 m as a reasonable assumption. We have credible estimations for the friction characteristics and the slope angle to consider for the system, which we put at 30 degrees, based on the relevant literature and research. We first selected a smaller domain for the simulations, but discovered later that the actual avalanche began near the edge of the original dominion. With these considerations, our initial numerical results have been improved. We can still consider other improvements. We may search other more sophisticated physical models that more realistically describe the avalanche's flow characteristics. In addition, it is feasible to work with more precise numerical models, particularly for the inclusion of friction and the influence of slope angle on velocity.

For the numerical scheme and the hydrostatic reconstruction, a Lax-Friedrichs flux was used, which is simple to implement but has the disadvantage of having too much numerical dissipation. This can be improved by replacing the flow with an upwind HLLC flow and by means of second order extensions employing the Riemann solver of Osher-Solomon-Toro. Some numerical experiments are being done in this regard, which could be used in future works.

Future development opportunities exist in Chile for applications connected to avalanches, particularly due to the growing influence of rising temperatures. In certain locations of Chile, winter snowfall accumulations are substantial, and avalanches pose already a serious threat. To tackle these catastrophes, it is essential to use a range of investigation methods. Numerical models such as those provide in this research work may be used to augment field observations and laboratory studies. There is further work to be done in this field of research in Chile. This is the first effort of this kind in Chile, to the best of our knowledge, and we hope that these ideas will be refined and applied to real-world situations in the near future.

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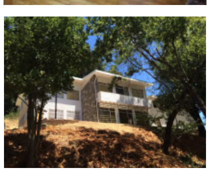

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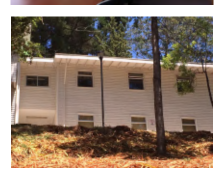
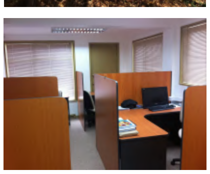

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