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A five-field mixed formulation for stationary magnetohydrodynamic flows in porous media*

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Abstract

We introduce and analyze a new mixed variational formulation for a stationary magnetohydrodynamic flows in porous media problem, whose governing equations are given by the steady Brinkman–Forchheimer equations coupled with the Maxwell equations. Besides the velocity, magnetic field and a Lagrange multiplier asssociated to the divergence-free condition of the magnetic field, a convenient translation of the velocity gradient and the pseudostress tensor are introduced as further unknowns. As a consequence, we obtain a five-field Banach spaces-based mixed variational formulation, where the aforementioned variables are the main unknowns of the system. The resulting mixed scheme is then written equivalently as a fixed-point equation, so that the well-known Banach theorem, combined with classical results on nonlinear monotone operators and a sufficiently small data assumption, are applied to prove the unique solvability of the continuous and discrete systems. In particular, the analysis of the discrete scheme requires a quasi-uniformity assumption on mesh. The finite element discretization involves Raviart–Thomas elements of order $k \geq 0$ for the pseudostress tensor, discontinuous piecewise polynomial elements of degree k for the velocity and the translation of the velocity gradient, Nédélec elements of degree k for the magnetic field and Lagrange elements of degree $k+1$ for the associated Lagrange multiplier. Stability, convergence, and optimal *a priori* error estimates for the associated Galerkin scheme are obtained. Numerical tests illustrate the theoretical results.

Key words: Brinkman–Forchheimer equations, Maxwell equations, mixed finite element methods, fixed point theory, *a priori* error analysis

Mathematics subject classifications (2020): 65N30, 65N12, 65N15, 76M10

1 Introduction

Magnetohydrodynamics (MHD) is the study of the flow of electrically conducting fluids in the presence of magnetic fields. The interest in the study of MHD has increased with respect to scientific and engineering problems in recent years. In fact, the MHD applications cover a very wide range of physical objects, from liquid metals to cosmic plasmas. Concerning to the mathematical model of MHD, it is

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based on the equations governing fluid motion in the presence of magnetic fields and the equations governing electromagnetics fields in moving fluids. Briefly speaking, it is a coupled system where the Navier–Stokes equations are coupled with the Maxwell equations trought the Lorentz force and Ohm's law. However, several physical situations require, sometimes, a modification or a simplification of these equations in order to capture in a more realistic way the physical phenomena of interest.

As it is explained in [35], nowadays, it is common to use Darcy's law in the modeling of the fluid motion through a porous medium. Darcy's empirical law represents a simple linear relationship between the flow rate and the pressure drop in a porous medium. Nevertheless, this fundamental equation may be inaccurate for modeling fluid flow through porous media with high Reynolds numbers or through media with high porosity (see, e.g., [30, 32, 18] and references therein). To overcome this limitation, it is possible to consider the Brinkman–Forchheimer equations (see for instance [10, 9]), where terms are added to Darcys law in order to take into account high velocity flow and high porosity. The latter and the increasing interest in the modelling of MHD in a porous media has motivated the introduction of the coupled problem between the Brinkman–Forchheimer and Maxwell equations.

Concerning literature, we can find some papers devoted to the mathematical analysis of the coupled Brinkman-Forchheimer and Maxwell equations (see, for instance, [1] and [35]). We begin mentioning [1], where, for the stationary coupled problem, the authors prove existence of weak solutions and uniqueness under small data assumptions. In addition, a convergence result, as the Brinkman coefficient tends to 0, of the weak solutions to a solution of the system formed by the Darcy–Forchheimer equations and the magnetic induction equation is also established in [1]. Later on, in [35] the authors show that the transient problem is also well-posed. However, neither [1] nor in [35] numerical analysis is developed. Up to the author's knowledge, there are no literature focused on the numerical analysis of this coupled problem. On the other hand, several papers have been devoted to the design and the analysis of numerical schemes for the simulation of the classical MHD. In fact, we can start mentioning [24] where the authors study well-posedness and convergence analysis of a conforming FEM for MHD. They use inf-sup stable velocity-pressure elements for the hydrodynamic variables and standard H^1-conforming finite elements for the magnetic field. In [21] and [26] we also observe that the authors look for the magnetic field in $H^1(\Omega)^3$. However, in a non-convex polyhedral domain, the magnetic induction may have regularity below $H^1(\Omega)^3$ and a nodal finite element approximation can converge to a magnetic field that misses certain singular solution components induced by reentrant vertices or edges (see [13]). To circumvent this inconvenient, in [34] was proposed to impose weakly the divergence-free condition of the magnetic field and by doing that, the magnetic field can be approximated by curl-conforming Nédélec elements. Thus, the convex domain assumption is not longer required. There exist other alternatives overcoming this difficulty. Meanwhile, we can mention [25] (see also [14]) where the authors introduced a mixed finite element method based on weighted regularization for the incompressible MHD system.

There exist many different discretizations for the classical incompressible MHD problem. A fully-DG method is proposed in [29] for a linearized variant of the classical incompressible MHD system, whereby all the variables are approximated through discontinuous finite element spaces. However, this approach requires a large number of degrees of freedom. In [23] the authors design a new finite element discretization, in an attempt to overcome the above mentioned difficulties. The velocity field is discretized using divergence-conforming Brezzi–Douglas–Marini (BDM) elements and the magnetic field is approximated by curl-conforming Nédélec elements. Recently in [36] the authors have studied a mixed finite element scheme for stationary inductionless magnetohydrodynamic equations on a general Lipschitz domain. They approximate the velocity and the current density by $H(\mathrm{div})$-conforming finite elements. The H^1-continuity of the velocity is enforced by discontinuous approach. However, in this approach the magnetic field is considered as a datum.

The goal of the present paper is to contribute to the development of a new numerical method for the MHD model in porous media. We carry out in this article mathematical and numerical analysis of the coupled system. The main advantages of our proposed scheme are the optimal order of convergence reached even in non-convex domains and the possibility of computing further variables of interest, in which no numerical differentiation is applied, and hence no further sources of error arise. Our scheme is based on a new mixed finite element method for the steady Brinkman–Forchheimer and double-diffusion equations recently introduced in [8]. In fact, the main novelty introduced in that paper is that no augmentation procedure needs to be incorporated into the formulation. This has been possible thanks to the introduction of mixed methods of finite elements based on suitable Banach spaces. A similar idea was applied in [12] for the steady Boussinesq problem. On the other hand, recently in [9] the authors have extended the result presented in [8] to the transient Brinkman–Forchheimer equations introducing the velocity, the velocity gradient, and the pseudostress tensor as the main unknowns of the system. In the same spirit of the previous works, we can mention [6]. In there the authors have proposed a new mixed finite element method for the classical MHD system. This article introduces non-standard Banach spaces for approximating the hydrodynamic unknowns, and Hilbert spaces for the electromagnetics variables. Our work is an adaptation of the analysis realized in [8] for the Brinkman–Forchheimer problem and the analysis presented for the Maxwell equations in [34] (see also[6]) to our stationary MHD problem in porous media.

The work is organized as follows. The remainder of this section describes standard notation and function spaces to be employed throughout the paper. In Section 2 we introduce the model problem, reformulate it as an equivalent set of equations and derive our mixed variational formulation. Next, in Section 3 we establish the well-posedness of this continuous scheme by means of classical results on nonlinear monotone operators and the Banach fixed point theorem. The associated Galerkin scheme is introduced and analyzed in Section 4. Its well-posedness is attained by mimicking the theory developed for the continuous problem under a quasi-uniformity assumption on the mesh. In Section 5 we establish the corresponding Céa's estimate and the consequent rates of convergence. Finally, in Section 6 we present some numerical examples illustrating the good performance of our mixed finite element method and confirming the theoretical rates of convergence.

Preliminary notations

Let $\Omega \subset \mathrm{R}^3$, denote a bounded domain with polyhedral boundary Γ, and denote by $\mathbf{n}$ the outward unit normal vector on Γ. Standard notations will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$, with $p \in [1,\infty]$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$ with $s \geq 0$, endowed with the norms $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. Note that $\mathrm{W}^{0,p} = \mathrm{L}^p(\Omega)$ and if $p = 2$, we write $\mathrm{H}^s(\Omega)$ in place of $\mathrm{W}^{s,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{s,\Omega}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. With $\langle\cdot,\cdot\rangle_\Gamma$ we denote the corresponding product of duality between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M. In turn, for any vector field $\mathbf{v} = (v_i)_{i=1,3}$, we set the gradient, divergence, and curl operators, respectively, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,3}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}, \quad \text{and}$$

$$\mathrm{curl}(\mathbf{v}) := \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)^{\mathrm{t}}.$$

The cross product of two vectors $\mathbf{u} = (u_i)_{i=1,3}$ and $\mathbf{v} = (v_i)_{i=1,3}$ in R^3 is given by

$$\mathbf{u} \times \mathbf{v} = (u_2\, v_3 - u_3\, v_2, u_3\, v_1 - u_1\, v_3, u_1\, v_2 - u_2\, v_1)^{\mathrm{t}}.$$

In addition, it can be proved that for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in R^3, the following identity is true

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \,=\, -(\mathbf{w} \times \mathbf{v}) \cdot \mathbf{u}\,. \tag{1.1}$$

Furthermore, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,3}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,3}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,3}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{3} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{3} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathrm{R}^{3\times 3}$. Additionally, we recall the Hilbert spaces

$$\mathbf{H}(\mathrm{div};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^2(\Omega) \Big\}$$

and

$$\mathbf{H}(\mathrm{curl};\Omega) := \Big\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \quad \mathrm{curl}(\mathbf{v}) \in \mathbf{L}^2(\Omega) \Big\}$$

endowed with the norms

$$\|\boldsymbol{\tau}\|_{\mathrm{div};\Omega}^2 := \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \quad \text{and} \quad \|\mathbf{v}\|_{\mathrm{curl};\Omega}^2 := \|\mathbf{v}\|_{0,\Omega}^2 + \|\mathrm{curl}(\mathbf{v})\|_{0,\Omega}^2\,,$$

respectively. Both spaces are standard in mixed and electromagnetism problems, respectively. We denote by $\mathbf{H}(\mathrm{div}^0;\Omega)$ the subspace of $\mathbf{H}(\mathrm{div};\Omega)$ with divergence zero. In addition, in the sequel we will make use of the well-known Hölder inequality given by

$$\int_\Omega |fg| \le \|f\|_{0,p;\Omega}\|g\|_{0,q;\Omega} \quad \forall\, f \in \mathrm{L}^p(\Omega),\, \forall\, g \in \mathrm{L}^q(\Omega), \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1\,.$$

Finally, we recall the continuous injection i_p of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^p(\Omega)$ for $p \in [1,6]$ in R^3 (cf. [33, Theorem 1.3.4]). More precisely, we have the following inequality

$$\|w\|_{0,p;\Omega} \,\le\, \|i_p\|\,\|w\|_{1,\Omega} \quad \forall\, w \in \mathrm{H}^1(\Omega)\,, \tag{1.2}$$

with $\|i_p\| > 0$ depending only on $|\Omega|$ and p. We will denote by $\mathbf{i}_p$ the vectorial version of i_p.

2 The continuous formulation

In this section we introduce the model problem and derive its corresponding weak formulation.

2.1 The model problem

We are interested in analyzing the behaviour of stationary magnetohydrodynamic flows in a fluid-saturated porous medium. To that end, we consider a slight modification of the model analyzed in [1] (see also [35]) and, for simplicity, we assume that the bounded Lipschitz polyhedral domain $\Omega \subset \mathrm{R}^3$ is simply-connected and with a connected boundary Γ. More precisely, we focus on solving the coupling of the Brinkman–Forchheimer and Maxwell equations, which reduces to finding a velocity field $\mathbf{u}$, a pressure field p, and a magnetic field $\mathbf{b}$, such that

$$-\nu\,\Delta\mathbf{u} + \alpha\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} + \nabla p - \frac{1}{\mu}\,\mathrm{curl}(\mathbf{b}) \times \mathbf{b} = \mathbf{f}_f \quad \text{in} \quad \Omega\,, \tag{2.1a}$$

$$\mathrm{div}(\mathbf{u}) = g_f \quad \text{in} \quad \Omega\,, \tag{2.1b}$$

$$\frac{1}{\varrho\,\mu}\,\mathrm{curl}(\mathrm{curl}(\mathbf{b})) + \nabla\lambda - \mathrm{curl}(\mathbf{u}\times\mathbf{b}) = \mathbf{f}_m \quad \text{in} \quad \Omega\,, \tag{2.1c}$$

$$\mathrm{div}(\mathbf{b}) = 0 \quad \text{in} \quad \Omega\,, \tag{2.1d}$$

where, the unknown λ is the corresponding Lagrange multiplier associated with (2.1d) (see [34] and [6] for similar approaches), whereas $g_f \in \mathrm{L}^2(\Omega)$ denotes a nonzero mass source, and $\mathbf{f}_f \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{f}_m \in \mathbf{L}^2(\Omega)$ are external forces, which in particular are taken as $\mathbf{0}$ and $\frac{1}{\varrho}\mathrm{curl}(\mathbf{J}_0)$ in [1, eq. (13)], respectively, with $\mathbf{J}_0$ denoting the source electric current density and $\varrho > 0$, the electric conductivity. In turn, the constant $\nu > 0$ is the Brinkman coefficient, $\alpha > 0$ is the Darcy coefficient, $\mathtt{F} > 0$ is the Forchheimer coefficient, $\mathrm{p} \in [3,4]$ is a given number, and $\mu > 0$ is the magnetic permeability. In addition, we consider the following boundary conditions:

$$\mathbf{u} = \mathbf{u}_\mathrm{D}, \quad \mathbf{n}\times\mathbf{b} = \mathbf{0}, \quad \text{and} \quad \lambda = 0 \quad \text{on} \quad \Gamma\,, \tag{2.2}$$

where $\mathbf{u}_\mathrm{D} \in \mathbf{H}^{1/2}(\Gamma)$ is the prescribed velocity on Γ satisfying the compatibility condition

$$\int_\Gamma \mathbf{u}_\mathrm{D}\cdot\mathbf{n} \;=\; \int_\Omega g_f\,. \tag{2.3}$$

In addition, due to (2.1a), and in order to guarantee uniqueness of the pressure, this unknown will be sought in the space

$$\mathrm{L}^2_0(\Omega) := \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\}.$$

Next, in order to derive a new mixed formulation for (2.1)–(2.3), we proceed as in [12] and [8]. More precisely, we now introduce as further unknowns a translation of the velocity gradient $\mathbf{t}$ and the pseudostress tensor $\boldsymbol{\sigma}$, which are defined, respectively, by

$$\mathbf{t} := \nabla\mathbf{u} - \frac{1}{3}\,g_f\,\mathbb{I} \quad \text{and} \quad \boldsymbol{\sigma} := \nu\,\nabla\mathbf{u} - p\,\mathbb{I} \quad \text{in} \quad \Omega\,. \tag{2.4}$$

In this way, applying the matrix trace to the tensors $\mathbf{t}$ and $\boldsymbol{\sigma}$, and utilizing the condition (2.1b), one arrives at $\mathrm{tr}(\mathbf{t}) = 0$ in Ω and

$$p = -\frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}) + \frac{\nu}{3}\,g_f \quad \text{in} \quad \Omega\,. \tag{2.5}$$

Hence, replacing back (2.5) in the second equation of (2.4) and after simple computations, we find that the model problem (2.1)–(2.2) can be rewritten, equivalently, as follows: Find $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma})$ and $(\mathbf{b},\lambda)$, in suitable spaces to be indicated below, such that

$$\nabla\mathbf{u} - \frac{1}{3}\,g_f\,\mathbb{I} = \mathbf{t} \quad \text{in} \quad \Omega\,, \tag{2.6a}$$

$$\nu\,\mathbf{t} = \boldsymbol{\sigma}^\mathrm{d} \quad \text{in} \quad \Omega\,, \tag{2.6b}$$

$$\alpha\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) - \frac{1}{\mu}\,\mathrm{curl}(\mathbf{b})\times\mathbf{b} = \mathbf{f}_f \quad \text{in} \quad \Omega\,, \tag{2.6c}$$

$$\int_\Omega \big(\mathrm{tr}(\boldsymbol{\sigma}) - \nu\,g_f\big) = 0\,, \tag{2.6d}$$

$$\frac{1}{\varrho\,\mu^2}\,\mathrm{curl}(\mathrm{curl}(\mathbf{b})) + \frac{1}{\mu}\nabla\lambda - \frac{1}{\mu}\mathrm{curl}(\mathbf{u}\times\mathbf{b}) = \frac{1}{\mu}\mathbf{f}_m \quad \text{in} \quad \Omega\,, \tag{2.6e}$$

$$\mathrm{div}(\mathbf{b}) = 0 \quad \text{in} \quad \Omega\,, \tag{2.6f}$$

$$\mathbf{u} = \mathbf{u}_{\mathrm{D}}, \quad \mathbf{n}\times\mathbf{b} = \mathbf{0}, \quad \text{and} \quad \lambda = 0 \quad \text{on} \quad \Gamma\,. \tag{2.6g}$$

At this point we stress that, as suggested by (2.5), p is eliminated from the present formulation and computed afterwards in terms of $\boldsymbol{\sigma}$ and g_f by using that identity. This fact justifies (2.6d), which aims to ensure that the resulting p does belong to $\mathrm{L}^2_0(\Omega)$. Notice also that further variables of interest, such as the velocity gradient $\mathbf{G} = \nabla\mathbf{u}$, the vorticity $\boldsymbol{\omega} = \frac{1}{2}\big(\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{t}}\big)$, and the stress $\widetilde{\boldsymbol{\sigma}} := \nu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{t}}) - p\,\mathbb{I}$ can be computed, respectively, as follows

$$\mathbf{G} \,=\, \mathbf{t} + \frac{1}{3}\,g_f\,\mathbb{I}, \quad \boldsymbol{\omega} \,=\, \frac{1}{2}\big(\mathbf{t} - \mathbf{t}^{\mathrm{t}}\big), \quad \text{and} \quad \widetilde{\boldsymbol{\sigma}} \,=\, \boldsymbol{\sigma} + \nu\,\mathbf{t}^{\mathrm{t}} + \frac{\nu}{3}\,g_f\,\mathbb{I}\,. \tag{2.7}$$

2.2 The variational formulation

In this section we derive our five-field mixed variational formulation for the system (2.6). To that end, we first proceed as in [12] and [8] to derive the mixed formulation associated to the Brinkman–Forchheimer equations. In fact, multiplying (2.6a), (2.6b) and (2.6c) by suitable test functions $\boldsymbol{\tau}, \mathbf{s}$, and $\mathbf{v}$, respectively, integrating by parts and using the Dirichlet boundary condition $\mathbf{u} = \mathbf{u}_{\mathrm{D}}$ on Γ, we get

$$\int_\Omega \mathbf{t} : \boldsymbol{\tau} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = -\frac{1}{3}\int_\Omega g_f\,\mathrm{tr}(\boldsymbol{\tau}) + \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle_\Gamma\,, \tag{2.8a}$$

$$\nu\int_\Omega \mathbf{t} : \mathbf{s} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s} = 0\,, \tag{2.8b}$$

$$\alpha\int_\Omega \mathbf{u}\cdot\mathbf{v} + \mathtt{F}\int_\Omega |\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}\cdot\mathbf{v} - \int_\Omega \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{v} - \frac{1}{\mu}\int_\Omega \big(\mathrm{curl}(\mathbf{b})\times\mathbf{b}\big)\cdot\mathbf{v} = \int_\Omega \mathbf{f}_f\cdot\mathbf{v}\,, \tag{2.8c}$$

for all $(\boldsymbol{\tau}, \mathbf{s}, \mathbf{v}) \in \mathbb{X}\times\mathbb{Q}\times\mathbf{M}$, where $\mathbb{X}, \mathbb{Q}$ and $\mathbf{M}$ are spaces to be defined below.

On the other hand, for the Maxwell equations (2.6e)–(2.6f), we proceed as in [34] (see also [6] for a similar approach), that is, we introduce the space

$$\mathbf{H}_0(\mathrm{curl};\Omega) := \Big\{\mathbf{d}\in\mathbf{H}(\mathrm{curl};\Omega) : \quad \mathbf{n}\times\mathbf{d} = 0 \quad \text{on} \quad \Gamma\Big\}\,,$$

and multiply (2.6e) by $\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega)$, and integrate by parts, to get

$$\frac{1}{\varrho\,\mu^2}\int_\Omega \mathrm{curl}(\mathbf{b})\cdot\mathrm{curl}(\mathbf{d}) + \frac{1}{\mu}\int_\Omega \nabla\lambda\cdot\mathbf{d} - \frac{1}{\mu}\int_\Omega (\mathbf{u}\times\mathbf{b})\cdot\mathrm{curl}(\mathbf{d}) \;=\; \frac{1}{\mu}\int_\Omega \mathbf{f}_m\cdot\mathbf{d}\,.$$

Then, applying the identity (1.1) to $\mathbf{u}$, $\mathbf{b}$, and $\mathrm{curl}(\mathbf{d})$ in the third term of the foregoing equation, and testing (2.6f) by $\xi\in\mathrm{H}^1_0(\Omega)$, integrating by parts, and multiplying the resulting equation by $1/\mu$, we obtain

$$\frac{1}{\varrho\,\mu^2}\int_\Omega \mathrm{curl}(\mathbf{b})\cdot\mathrm{curl}(\mathbf{d}) + \frac{1}{\mu}\int_\Omega \nabla\lambda\cdot\mathbf{d} + \frac{1}{\mu}\int_\Omega (\mathrm{curl}(\mathbf{d})\times\mathbf{b})\cdot\mathbf{u} \;=\; \frac{1}{\mu}\int_\Omega \mathbf{f}_m\cdot\mathbf{d}\,, \tag{2.9a}$$

$$\frac{1}{\mu}\int_\Omega \mathbf{b}\cdot\nabla\xi \;=\; 0\,, \tag{2.9b}$$

for all $(\mathbf{d},\xi)\in\mathbf{H}_0(\mathrm{curl};\Omega)\times\mathrm{H}^1_0(\Omega)$. In this way, at first we are interested in finding $\boldsymbol{\sigma}\in\mathbb{X}$, $\mathbf{t}\in\mathbb{Q}$, $\mathbf{u}\in\mathbf{M}$, $\mathbf{b}\in\mathbf{H}_0(\mathrm{curl};\Omega)$ and $\lambda\in\mathrm{H}^1_0(\Omega)$ satisfying (2.8)–(2.9) and the condition (2.6d).

Now, we turn to specify the spaces $\mathbb{X}$, $\mathbb{Q}$, and $\mathbf{M}$. We begin by noting that the first term in (2.8b) is well defined for $\mathbf{t}, \mathbf{s} \in \mathbb{L}^2(\Omega)$, but due to the condition $\operatorname{tr}(\mathbf{t}) = 0$ in Ω, it makes sense to look for $\mathbf{t}$, and consequently the test function $\mathbf{s}$, in $\mathbb{Q} = \mathbb{L}^2_{\operatorname{tr}}(\Omega)$, with

$$\mathbb{L}^2_{\operatorname{tr}}(\Omega) := \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \operatorname{tr}(\mathbf{s}) = 0 \quad \text{in} \quad \Omega \Big\}.$$

This implies that (2.8b) can be rewritten equivalently as

$$\nu \int_\Omega \mathbf{t} : \mathbf{s} - \int_\Omega \boldsymbol{\sigma} : \mathbf{s} = 0 \quad \forall\, \mathbf{s} \in \mathbb{L}^2_{\operatorname{tr}}(\Omega)\,. \tag{2.10}$$

In turn, we let

$$\mathbf{C} := \left\{ \mathbf{d} \in \mathbf{H}_0(\operatorname{curl};\Omega) : \quad \int_\Omega \mathbf{d} \cdot \nabla \xi = 0 \quad \forall\, \xi \in \mathrm{H}^1_0(\Omega) \right\} = \mathbf{H}_0(\operatorname{curl};\Omega) \cap \mathbf{H}(\operatorname{div}^0;\Omega)\,, \tag{2.11}$$

and observe that, since $\mathbf{b}$ satisfies (2.9b) with constant $\mu > 0$, then $\mathbf{b} \in \mathbf{C}$ (see [22, Section I.2.2]). Then, since $\mathbf{C}$ is continuously embedded into $\mathbf{H}^s(\Omega)$ for some $s > 1/2$ (cf. [2, Proposition 3.7]), which in turn is continuously embedded into $\mathbf{L}^{3+\delta}(\Omega)$, for some $\delta > 0$ (see [33, Theorem 1.3.4]), we obtain

$$\|\mathbf{b}\|_{0,3+\delta;\Omega} \,\leq\, c_1\, \|\mathbf{b}\|_{\operatorname{curl};\Omega} \quad \forall\, \mathbf{b} \in \mathbf{C}\,.$$

Therefore, using the well-known embedding inequality

$$\|\mathbf{v}\|_{0,q;\Omega} \,\leq\, c_2\, \|\mathbf{v}\|_{0,6;\Omega} \quad \forall\, q \in [1,6)\,, \tag{2.12}$$

and defining $\delta^* := \frac{4\,\delta}{1+\delta} > 0$, it follows that

$$\left| \int_\Omega (\operatorname{curl}(\mathbf{d}) \times \mathbf{b}) \cdot \mathbf{v} \right| \leq \|\operatorname{curl}(\mathbf{d})\|_{0,\Omega}\, \|\mathbf{b}\|_{0,3+\delta;\Omega}\, \|\mathbf{v}\|_{0,6-\delta^*;\Omega} \leq C_s\, \|\mathbf{d}\|_{\operatorname{curl};\Omega}\, \|\mathbf{b}\|_{\operatorname{curl};\Omega}\, \|\mathbf{v}\|_{0,6;\Omega}\,, \tag{2.13}$$

for all $\mathbf{d} \in \mathbf{H}(\operatorname{curl};\Omega), \mathbf{b} \in \mathbf{C}$ and $\mathbf{v} \in \mathbf{L}^6(\Omega)$, with C_s the resulting constant from the aforementioned embedding inequalities. According to the above, the fourth and third terms in (2.8c) and (2.9a), respectively, are well defined if we set $\mathbf{M} := \mathbf{L}^6(\Omega)$, which, thanks to (2.12), is consistent with the first and second terms of (2.8c), and consequently, the second and third terms in (2.8a) and (2.8c), respectively, are well defined if $\mathbf{div}(\boldsymbol{\sigma})$ and $\mathbf{div}(\boldsymbol{\tau})$ belong to $\mathbf{L}^{6/5}(\Omega)$. In addition, using the fact that the first and second terms in (2.8a) and (2.8b) (or (2.10)), respectively, are well defined if $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$, we introduce the Banach space

$$\mathbb{H}(\mathbf{div}_{6/5};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^{6/5}(\Omega) \Big\}\,,$$

equipped with the norm $\|\boldsymbol{\tau}\|^2_{\mathbf{div}_{6/5};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,6/5;\Omega}$, and deduce that (2.8) is well defined if we choose the spaces $\mathbb{Q} := \mathbb{L}^2_{\operatorname{tr}}(\Omega)$, $\mathbf{M} := \mathbf{L}^6(\Omega)$, and $\mathbb{X} := \mathbb{H}(\mathbf{div}_{6/5};\Omega)$, with their respective norms: $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{0,6;\Omega}$, and $\|\cdot\|_{\mathbf{div}_{6/5};\Omega}$.

Now, for convenience of the subsequent analysis and similarly as in [6] (see also [5, 12]) we consider the decomposition

$$\mathbb{H}(\mathbf{div}_{6/5};\Omega) \,=\, \mathbb{H}_0(\mathbf{div}_{6/5};\Omega) \oplus \mathrm{R}\,\mathbb{I}\,,$$

where

$$\mathbb{H}_0(\mathbf{div}_{6/5};\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{6/5};\Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0 \right\}\,,$$

that is, $\mathrm{R}\,\mathbb{I}$ is a topological supplement for $\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$. More precisely, each $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{6/5};\Omega)$ can be decomposed uniquely as

$$\boldsymbol{\tau}=\boldsymbol{\tau}_0+d\,\mathbb{I}\,,\quad \text{with}\quad \boldsymbol{\tau}_0\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\quad\text{and}\quad d:=\frac{1}{3\,|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau})\in\mathrm{R}\,.$$

In particular, using from (2.6d) that $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma})=\nu\int_\Omega g_f$, we obtain

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}_0+c_0\,\mathbb{I}\quad\text{with}\quad \boldsymbol{\sigma}_0\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\quad\text{and}\quad c_0=\frac{\nu}{3\,|\Omega|}\int_\Omega g_f\,. \tag{2.14}$$

In this way, knowing explicitly c_0 in terms of g_f, it remains to find the $\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$-component $\boldsymbol{\sigma}_0$ of $\boldsymbol{\sigma}$ to fully determine it. In this regard, using the fact that $\mathbf{div}(\boldsymbol{\sigma})=\mathbf{div}(\boldsymbol{\sigma}_0)$ and $\boldsymbol{\sigma}:\mathbf{s}=\boldsymbol{\sigma}_0:\mathbf{s}$, for all $\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, we deduce that (2.8b)–(2.8c) remain unchanged if $\boldsymbol{\sigma}$ is replaced there by $\boldsymbol{\sigma}_0$. Moreover it is easy to see, thanks to the compatibility condition (2.3) satisfied by the Dirichlet datum $\mathbf{u}_{\mathrm{D}}$, that both sides of (2.8a) vanish for $\boldsymbol{\tau}=\mathbb{I}$, and hence, testing this equation against $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{6/5};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$.

According to the above, and redenoting from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$, we arrive to the variational problem: Find $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma})\in\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{b},\lambda)\in\mathbf{H}_0(\mathrm{curl};\Omega)\times\mathrm{H}^1_0(\Omega)$, such that

$$[a_f(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})]+[c_f(\mathbf{b})(\mathbf{b}),\mathbf{v}]+[b_f(\mathbf{v},\mathbf{s}),\boldsymbol{\sigma}]=[F_1,(\mathbf{v},\mathbf{s})]\quad\forall\,(\mathbf{v},\mathbf{s})\in\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{2.15a}$$

$$[b_f(\mathbf{u},\mathbf{t}),\boldsymbol{\tau}]=[F_2,\boldsymbol{\tau}]\quad\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,, \tag{2.15b}$$

$$[a_m(\mathbf{b}),\mathbf{d}]+[c_m(\mathbf{b})(\mathbf{u}),\mathbf{d}]+[b_m(\mathbf{d}),\lambda]=[F_3,\mathbf{d}]\quad\forall\,\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega)\,, \tag{2.15c}$$

$$[b_m(\mathbf{b}),\xi]=0\quad\forall\,\xi\in\mathrm{H}^1_0(\Omega)\,, \tag{2.15d}$$

where the operators a_f, b_f, a_m, b_m, $c_f(\widehat{\mathbf{b}})$, $c_m(\widehat{\mathbf{b}})$, for a given $\widehat{\mathbf{b}}\in\mathbf{C}$ (cf. (2.11)), are defined, respectively, as

$$[a_f(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})]\,:=\,\alpha\int_\Omega \mathbf{u}\cdot\mathbf{v}+\mathrm{F}\int_\Omega |\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}\cdot\mathbf{v}+\nu\int_\Omega \mathbf{t}:\mathbf{s}\,, \tag{2.16}$$

$$[b_f(\mathbf{v},\mathbf{s}),\boldsymbol{\tau}]\,:=\,-\int_\Omega \boldsymbol{\tau}:\mathbf{s}-\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \tag{2.17}$$

$$[a_m(\mathbf{b}),\mathbf{d}]\,:=\,\frac{1}{\varrho\,\mu^2}\int_\Omega \mathrm{curl}(\mathbf{b})\cdot\mathrm{curl}(\mathbf{d})\,,\quad [b_m(\mathbf{d}),\xi]\,:=\,\frac{1}{\mu}\int_\Omega \mathbf{d}\cdot\nabla\xi\,, \tag{2.18}$$

and

$$[c_f(\widehat{\mathbf{b}})(\mathbf{b}),\mathbf{v}]\,:=\,-\frac{1}{\mu}\int_\Omega \big(\mathrm{curl}(\mathbf{b})\times\widehat{\mathbf{b}}\big)\cdot\mathbf{v}\,,\quad [c_m(\widehat{\mathbf{b}})(\mathbf{u}),\mathbf{d}]\,:=\,\frac{1}{\mu}\int_\Omega \big(\mathrm{curl}(\mathbf{d})\times\widehat{\mathbf{b}}\big)\cdot\mathbf{u}\,, \tag{2.19}$$

for all $(\mathbf{v},\mathbf{s},\boldsymbol{\tau})\in\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{d},\xi)\in\mathbf{H}_0(\mathrm{curl};\Omega)\times\mathrm{H}^1_0(\Omega)$. In turn, F_1,F_2, and F_3 are the bounded linear functionals defined by

$$[F_1,(\mathbf{v},\mathbf{s})]\,:=\,\int_\Omega \mathbf{f}_f\cdot\mathbf{v}\,,\quad [F_2,\boldsymbol{\tau}]\,:=\,\frac{1}{3}\int_\Omega g_f\,\mathrm{tr}(\boldsymbol{\tau})\,-\,\langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_{\mathrm{D}}\rangle_\Gamma\,, \tag{2.20}$$

and

$$[F_3,\mathbf{d}]\,:=\,\frac{1}{\mu}\int_\Omega \mathbf{f}_m\cdot\mathbf{d}\,. \tag{2.21}$$

In all the terms above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators.

Let us define the global unknown and space:

$$\vec{\mathbf{u}} := (\mathbf{u},\mathbf{t},\mathbf{b}) \in \mathbf{X} := \mathbf{L}^6(\Omega)\times \mathbb{L}^2_{\mathrm{tr}}(\Omega)\times \mathbf{C}\,, \tag{2.22}$$

where $\mathbf{X}$ is endowed with the norm

$$\|\vec{\mathbf{v}}\|^2_{\mathbf{X}} = \|(\mathbf{v},\mathbf{s},\mathbf{d})\|^2_{\mathbf{X}} = \|\mathbf{v}\|^2_{0,6;\Omega} + \|\mathbf{s}\|^2_{0,\Omega} + \|\mathbf{d}\|^2_{\mathrm{curl};\Omega} \quad \forall\,\vec{\mathbf{v}} := (\mathbf{v},\mathbf{s},\mathbf{d}) \in \mathbf{X}. \tag{2.23}$$

Now, recalling that the operator b_m (cf. (2.18)) satisfies the inf-sup condition (see [34, Section 2.4] or [28, Section 5.4]):

$$\sup_{\substack{\mathbf{d}\in \mathbf{H}_0(\mathrm{curl};\Omega)\\ \mathbf{d}\neq 0}} \frac{[b_m(\mathbf{d}),\xi]}{\|\mathbf{d}\|_{\mathrm{curl};\Omega}} \,\geq\, \beta_m\,\|\xi\|_{1,\Omega} \quad \forall\,\xi\in \mathrm{H}^1_0(\Omega)\,, \tag{2.24}$$

with $\beta_m > 0$, analogously to [34], it is not difficult to see that (2.15) can be rewritten equivalently (to be proved below in Lemma 2.1) as the following coupled problem: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{X}\times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] + [\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\sigma}] &= [\mathbf{F},\vec{\mathbf{v}}] \quad \forall\,\vec{\mathbf{v}}\in\mathbf{X}\,,\\ [\mathbf{B}(\vec{\mathbf{u}}),\boldsymbol{\tau}] &= [\mathbf{G},\boldsymbol{\tau}] \quad \forall\,\boldsymbol{\tau}\in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,, \end{aligned} \tag{2.25}$$

where, given $\widehat{\mathbf{b}}\in\mathbf{C}$, the operator $\mathbf{A}(\widehat{\mathbf{b}}) : \mathbf{X}\to\mathbf{X}'$ is defined by

$$[\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] \,:=\, [\mathbf{a}(\vec{\mathbf{u}}),\vec{\mathbf{v}}] + [\mathbf{c}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] \tag{2.26}$$

with

$$[\mathbf{a}(\vec{\mathbf{u}}),\vec{\mathbf{v}}] \,:=\, [a_f(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})] + [a_m(\mathbf{b}),\mathbf{d}]\,, \tag{2.27}$$

$$[\mathbf{c}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] \,:=\, [c_f(\widehat{\mathbf{b}})(\mathbf{b}),\mathbf{v}] + [c_m(\widehat{\mathbf{b}})(\mathbf{u}),\mathbf{d}]\,, \tag{2.28}$$

whereas the operator $\mathbf{B} : \mathbf{X}\to \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'$ is given by

$$[\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}] \,:=\, [b_f(\mathbf{v},\mathbf{s}),\boldsymbol{\tau}]\,. \tag{2.29}$$

In turn, the functionals $\mathbf{F}$ and $\mathbf{G}$ are set as

$$[\mathbf{F},\vec{\mathbf{v}}] \,:=\, [F_1,(\mathbf{v},\mathbf{s})] + [F_3,\mathbf{d}] \quad\text{and}\quad [\mathbf{G},\boldsymbol{\tau}] \,:=\, [F_2,\boldsymbol{\tau}]\,. \tag{2.30}$$

The following lemma establishes that problems (2.15) and (2.25) are in fact equivalents.

Lemma 2.1 *If* $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma}) \in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ *and* $(\mathbf{b},\lambda)\in\mathbf{H}_0(\mathrm{curl};\Omega)\times\mathrm{H}^1_0(\Omega)$ *is a solution of* (2.15)*, then* $\mathbf{b}\in\mathbf{C}$ *and* $(\vec{\mathbf{u}},\boldsymbol{\sigma}) = ((\mathbf{u},\mathbf{t},\mathbf{b}),\boldsymbol{\sigma})\in\mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ *is a solution of* (2.25)*. Conversely, if* $(\vec{\mathbf{u}},\boldsymbol{\sigma})\in\mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ *is a solution of* (2.25)*, then there exists* $\lambda\in\mathrm{H}^1_0(\Omega)$ *such that* $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma})$ *and* $(\mathbf{b},\lambda)$ *is a solution of* (2.15)*.*

Proof. The first assertion is evident. On the other hand, let $(\vec{\mathbf{u}},\boldsymbol{\sigma})\in\mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ be a solution of (2.25). Note that (2.25) directly implies (2.15a), (2.15b) and (2.15d). Thus it only remains to show the existence of $\lambda\in\mathrm{H}^1_0(\Omega)$ such that (2.15c) is satisfied. We prove this fact proceeding similarly as in [6, Corollary 3.8]. Indeed, let $F\in\mathbf{H}_0(\mathrm{curl};\Omega)$ be the unique element in $\mathbf{H}_0(\mathrm{curl};\Omega)$ (guaranteed by the Riesz representation theorem), such that

$$\langle F,\mathbf{d}\rangle_{\mathrm{curl}} = [F_3,\mathbf{d}] - [a_m(\mathbf{b}),\mathbf{d}] - [c_m(\mathbf{b})(\mathbf{u}),\mathbf{d}] \quad \forall\,\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega)\,,$$

with $\langle\cdot,\cdot\rangle_{\mathrm{curl}}$ being the inner product of $\mathbf{H}_0(\mathrm{curl};\Omega)$. Testing the first equation of (2.25), with $\vec{\mathbf{v}} = (\mathbf{0},\mathbf{0},\mathbf{d})$, we deduce that $\langle F,\mathbf{d}\rangle_{\mathrm{curl}} = 0$ for all $\mathbf{d}\in\mathbf{C}$, that is, $F\in\mathbf{C}^{\perp}$. Then, owing to the inf-sup condition (2.24), and according to [20, Lemma 2.1-(ii)], we deduce that there exists a unique $\lambda\in\mathrm{H}^1_0(\Omega)$, such that

$$[b_m(\mathbf{d}),\lambda] = \langle F,\mathbf{d}\rangle_{\mathrm{curl}} = [F_3,\mathbf{d}] - [a_m(\mathbf{b}),\mathbf{d}] - [c_m(\mathbf{b})(\mathbf{u}),\mathbf{d}] \quad \forall\,\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega)\,, \tag{2.31}$$

which implies that $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma})\in\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{b},\lambda)\in\mathbf{H}_0(\mathrm{curl};\Omega)\times\mathrm{H}^1_0(\Omega)$ is solution of (2.15), completing the proof. $\square$

As a consequence of the above, in what follows we focus on analyzing problem (2.25).

3 Analysis of the coupled problem

In this section we combine classical results on nonlinear monotone operators with the Banach fixed-point theorem, to prove the well-posedness of (2.25) (equivalently (2.15)) under suitable smallness assumptions on the data. To that end we first collect some previous results and notations that will serve for the forthcoming analysis.

3.1 Preliminaries

We begin by stating a slight adaptation of the abstract result established in [8, Theorem 3.1].

Theorem 3.1 *Let X_1, X_2, X_3 and Y be separable and reflexive Banach spaces, being X_1, X_2 and X_3 uniformly convex, and set $X := X_1\times X_2\times X_3$. Let $\mathcal{A}: X\to X'$ be a nonlinear operator, $\mathcal{B}\in\mathcal{L}(X,Y')$, and let V be the kernel of $\mathcal{B}$, that is,*

$$V \;:=\; \Big\{\vec{v} = (v_1,v_2,v_3)\in X: \quad \mathcal{B}(\vec{v}) \;=\; \mathbf{0}\Big\}\,.$$

Assume that

(i) *there exist constants $L>0$ and p_1, p_2, $p_3\geq 2$, such that*

$$\|\mathcal{A}(\vec{u}) - \mathcal{A}(\vec{v})\|_{X'} \;\leq\; L\sum_{j=1}^{3}\Big\{\|u_j - v_j\|_{X_j} + \big(\|u_j\|_{X_j} + \|v_j\|_{X_j}\big)^{p_j-2}\|u_j - v_j\|_{X_j}\Big\}$$

for all $\vec{u} = (u_1,u_2,u_3)$, $\vec{v} = (v_1,v_2,v_3)\in X$,

(ii) *the family of operators $\Big\{\mathcal{A}(\,\cdot + \vec{z}): V\to V': \quad \vec{z}\in X\Big\}$ is uniformly strongly monotone, that is there exists $\alpha>0$ such that*

$$[\mathcal{A}(\vec{u}+\vec{z}) - \mathcal{A}(\vec{v}+\vec{z}),\vec{u}-\vec{v}] \;\geq\; \alpha\,\|\vec{u}-\vec{v}\|^2_X\,,$$

for all $\vec{z}\in X$, and for all $\vec{u},\vec{v}\in V$, and

(iii) *there exists $\beta>0$ such that*

$$\sup_{\substack{\vec{v}\in X\\ \vec{v}\neq 0}} \frac{[\mathcal{B}(\vec{v}),\tau]}{\|\vec{v}\|_{\mathrm{X}}} \;\geq\; \beta\,\|\tau\|_Y \quad \forall\,\tau\in Y\,.$$

Then, for each $(\mathcal{F},\mathcal{G}) \in X' \times Y'$ *there exists a unique* $(\vec{u},\sigma) \in X \times Y$ *such that*

$$\begin{aligned} [\mathcal{A}(\vec{u}),\vec{v}] + [\mathcal{B}(\vec{v}),\sigma] &= [\mathcal{F},\vec{v}] \quad \forall\, \vec{v} \in X\,, \\ [\mathcal{B}(\vec{u}),\tau] &= [\mathcal{G},\tau] \quad \forall\, \tau \in Y\,. \end{aligned} \tag{3.1}$$

Moreover, there exist positive constants C_1 *and* C_2*, depending only on* L*,* α*, and* β*, such that*

$$\|\vec{u}\|_X \,\le\, C_1\, \mathcal{M}(\mathcal{F},\mathcal{G}) \tag{3.2}$$

and

$$\|\sigma\|_Y \,\le\, C_2 \left\{ \mathcal{M}(\mathcal{F},\mathcal{G}) + \sum_{j=1}^{3} \mathcal{M}(\mathcal{F},\mathcal{G})^{p_j-1} \right\}, \tag{3.3}$$

where

$$\mathcal{M}(\mathcal{F},\mathcal{G}) \,:=\, \|\mathcal{F}\|_{X'} + \|\mathcal{G}\|_{Y'} + \sum_{j=1}^{3} \|\mathcal{G}\|_{Y'}^{p_j-1} + \|\mathcal{A}(0)\|_{X'}\,. \tag{3.4}$$

Next, we establish the stability properties of some of the operators involved in (2.15) and (2.25). We begin by observing that the operators a_m, $\mathbf{B}$ and functionals F_3, $\mathbf{F}$, $\mathbf{G}$ are linear. In turn, from (2.18), (2.29), (2.21), (2.30), and employing Hölder and Cauchy–Schwarz inequalities, there holds

$$|[a_m(\mathbf{b}),\mathbf{d}]| \,\le\, \frac{1}{\varrho\,\mu^2}\, \|\mathbf{b}\|_{\mathrm{curl};\Omega}\, \|\mathbf{d}\|_{\mathrm{curl};\Omega} \quad \forall\, \mathbf{b},\mathbf{d} \in \mathbf{H}(\mathrm{curl};\Omega)\,, \tag{3.5}$$

$$|[\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}]| \,\le\, \|\vec{\mathbf{v}}\|_{\mathbf{X}}\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5};\Omega} \quad \forall\, \vec{\mathbf{v}} \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{H}(\mathrm{curl};\Omega),\, \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,, \tag{3.6}$$

$$|[F_3,\mathbf{d}]| \,\le\, \frac{1}{\mu} \|\mathbf{f}_m\|_{0,\Omega} \|\mathbf{d}\|_{\mathrm{curl};\Omega} \quad \forall\, \mathbf{d} \in \mathbf{H}(\mathrm{curl};\Omega)\,, \tag{3.7}$$

$$|[\mathbf{F},\vec{\mathbf{v}}]| \,\le\, C_{\mathbf{F}} \left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega}\right) \|\vec{\mathbf{v}}\|_{\mathbf{X}} \quad \forall\, \vec{\mathbf{v}} \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{H}(\mathrm{curl};\Omega)\,, \tag{3.8}$$

with $C_{\mathbf{F}} := \max\{1, 1/\mu\}$. Notice that (3.6) and (3.8) also hold for all $\vec{\mathbf{v}} \in \mathbf{X}$. We have written (3.6) and (3.8) in a more general form since both inequalities will be used later on to prove well-posedness of the Galerkin scheme proposed in Section 4 and to derive the *a priori* error analysis (cf. Lemma 5.1). In addition,

$$|[\mathbf{G},\boldsymbol{\tau}]| \,\le\, C_{\mathbf{G}} \left(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right) \|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5};\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,, \tag{3.9}$$

where $C_{\mathbf{G}} := \max\{1/\sqrt{3}, C_\Omega\}$ and C_Ω is a positive constant depending on $\|\mathbf{i}_6\|$ (for more details see [5, Lemma 3.5] and (1.2)).

Finally, using (2.13) and the definition of the operators $c_f(\widehat{\mathbf{b}})\,, c_m(\widehat{\mathbf{b}})\,, \mathbf{c}(\widehat{\mathbf{b}})$ (cf. (2.19), (2.28)), we observe that for any $\widehat{\mathbf{b}} \in \mathbf{C}$, there hold

$$\left|[c_f(\widehat{\mathbf{b}})(\mathbf{b}),\mathbf{v}]\right| \,\le\, \frac{C_s}{\mu}\, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\, \|\mathbf{b}\|_{\mathrm{curl};\Omega}\, \|\mathbf{v}\|_{0,6;\Omega} \quad \forall\, (\mathbf{b},\mathbf{v}) \in \mathbf{H}(\mathrm{curl};\Omega) \times \mathbf{L}^6(\Omega)\,, \tag{3.10}$$

$$\left|[c_m(\widehat{\mathbf{b}})(\mathbf{u}),\mathbf{d}]\right| \,\le\, \frac{C_s}{\mu}\, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\, \|\mathbf{u}\|_{0,6;\Omega}\, \|\mathbf{d}\|_{\mathrm{curl};\Omega} \quad \forall\, (\mathbf{u},\mathbf{d}) \in \mathbf{L}^6(\Omega) \times \mathbf{H}(\mathrm{curl};\Omega)\,, \tag{3.11}$$

$$\begin{aligned} \left|[\mathbf{c}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}]\right| &\le \frac{C_s}{\mu}\, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega} \left(\|\mathbf{u}\|^2_{0,6;\Omega} + \|\mathbf{b}\|^2_{\mathrm{curl};\Omega}\right)^{1/2} \left(\|\mathbf{v}\|^2_{0,6;\Omega} + \|\mathbf{d}\|^2_{\mathrm{curl};\Omega}\right)^{1/2} \\ &\le \frac{C_s}{\mu}\, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\, \|\vec{\mathbf{u}}\|_{\mathbf{X}}\, \|\vec{\mathbf{v}}\|_{\mathbf{X}} \quad \forall\, \vec{\mathbf{u}} = (\mathbf{u},\mathbf{t},\mathbf{b}), \vec{\mathbf{v}} = (\mathbf{v},\mathbf{s},\mathbf{d}) \in \mathbf{X}\,, \end{aligned} \tag{3.12}$$

and, in addition,

$$[\mathbf{c}(\widehat{\mathbf{b}})(\vec{\mathbf{v}}),\vec{\mathbf{v}}] \,=\, 0 \quad \forall\, \vec{\mathbf{v}} \in \mathbf{X}\,. \tag{3.13}$$

3.2 A fixed point strategy

We begin the solvability analysis of (2.25) (equivalently (2.15)) by defining the operator $\mathbf{T}:\mathbf{C}\to\mathbf{C}$ by

$$\mathbf{T}(\widehat{\mathbf{b}}) := \mathbf{b} \quad \forall\,\widehat{\mathbf{b}}\in\mathbf{C}\,, \tag{3.14}$$

where $\mathbf{b}$ is part of the element $(\vec{\mathbf{u}},\boldsymbol{\sigma}) = ((\mathbf{u},\mathbf{t},\mathbf{b}),\boldsymbol{\sigma})$ in $\mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ satisfying

$$\begin{aligned} [\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] + [\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\sigma}] &= [\mathbf{F},\vec{\mathbf{v}}] \quad \forall\,\vec{\mathbf{v}}\in\mathbf{X}\,,\\ [\mathbf{B}(\vec{\mathbf{u}}),\boldsymbol{\tau}] &= [\mathbf{G},\boldsymbol{\tau}] \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,. \end{aligned} \tag{3.15}$$

Notice that solving (2.25) is equivalent to finding $\mathbf{b}\in\mathbf{C}$ such that

$$\mathbf{T}(\mathbf{b}) = \mathbf{b}\,.$$

In this way, in what follows we focus on proving that $\mathbf{T}$ possesses a unique fixed-point. To that end, we first show that the coupled problem (3.15) is well-posed, which means, equivalently, that $\mathbf{T}$ (cf. 3.14) is indeed well-defined. We observe that, given $\widehat{\mathbf{b}}\in\mathbf{C}$, the problem (3.15) has the same structure as the one in Theorem 3.1 (cf. (3.1)). Therefore, in order to apply this abstract result, we notice that, thanks to the uniform convexity and separability of $\mathrm{L}^p(\Omega)$, for $p\in(1,+\infty)$, all the spaces involved in (3.15), that is, $\mathbf{L}^6(\Omega)$, $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbf{C}$ and $\mathbb{H}_0(\mathbf{div}_{6/5},\Omega)$, share the same properties.

We continue our analysis by proving that, given $\widehat{\mathbf{b}}\in\mathbf{C}$, the nonlinear operator $\mathbf{A}(\widehat{\mathbf{b}})$ (cf. (2.26)) satisfies hypothesis (i) of Theorem 3.1 with $p_1 = \mathrm{p}\in[3,4]$ and $p_2 = p_3 = 2$.

Lemma 3.2 *Let* $\mathrm{p}\in[3,4]$*. Given* $\widehat{\mathbf{b}}\in\mathbf{C}$*, there exists* $L_{\mathtt{MH}}>0$*, depending on* ν, $\mathtt{F}$, α, $|\Omega|$, C_s, ϱ*, and* μ*, such that*

$$\begin{aligned} \|\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}) - \mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{v}})\|_{\mathbf{X}'} \le L_{\mathtt{MH}}\Big\{&\big(1+\|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\big)\big(\|\mathbf{u}-\mathbf{v}\|_{0,6;\Omega} + \|\mathbf{b}-\mathbf{d}\|_{\mathrm{curl};\Omega}\big)\\ &+ \|\mathbf{t}-\mathbf{s}\|_{0,\Omega} + \big(\|\mathbf{u}\|_{0,6;\Omega}+\|\mathbf{v}\|_{0,6,\Omega}\big)^{\mathrm{p}-2}\|\mathbf{u}-\mathbf{v}\|_{0,6;\Omega}\Big\}\,, \end{aligned} \tag{3.16}$$

for all $\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t},\mathbf{b})$*,* $\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s},\mathbf{d})\in\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{H}(\mathrm{curl};\Omega)$.

Proof. Let $\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t},\mathbf{b})$, $\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s},\mathbf{d})$, and $\vec{\mathbf{w}} = (\mathbf{w},\mathbf{r},\mathbf{e})\in\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{H}(\mathrm{curl};\Omega)$. From the definition of the operator $\mathbf{A}(\widehat{\mathbf{b}})$ (cf. (2.26)), the Cauchy–Schwarz and Hölder inequalities, the continuity bound of $\mathbf{c}(\widehat{\mathbf{b}})$ (cf. (3.12)), and simple computations, we deduce that

$$\begin{aligned} [\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}) - \mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{v}}),\vec{\mathbf{w}}] &\le \mathtt{F}\,\||\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - |\mathbf{v}|^{\mathrm{p}-2}\mathbf{v}\|_{0,\mathrm{q};\Omega}\,\|\mathbf{w}\|_{0,\mathrm{p};\Omega}\\ &+ \alpha\,|\Omega|^{2/3}\,\|\mathbf{u}-\mathbf{v}\|_{0,6;\Omega}\|\mathbf{w}\|_{0,6;\Omega} + \nu\,\|\mathbf{t}-\mathbf{s}\|_{0,\Omega}\,\|\mathbf{r}\|_{0,\Omega} + \frac{1}{\varrho\mu^2}\,\|\mathbf{b}-\mathbf{d}\|_{\mathrm{curl};\Omega}\,\|\mathbf{e}\|_{\mathrm{curl};\Omega}\\ &+ \frac{C_s}{\mu}\,\|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\big(\|\mathbf{u}-\mathbf{v}\|_{0,6;\Omega} + \|\mathbf{b}-\mathbf{d}\|_{\mathrm{curl};\Omega}\big)\big(\|\mathbf{w}\|_{0,6;\Omega} + \|\mathbf{e}\|_{\mathrm{curl};\Omega}\big)\,, \end{aligned} \tag{3.17}$$

where $1/\mathrm{p} + 1/\mathrm{q} = 1$. In turn, using [3, Lemma 2.1, eq.(2.1a)] to bound the first term on the right hand side of (3.17), and the embedding (2.12) of $\mathbf{L}^6(\Omega)$ into $\mathbf{L}^{\mathrm{p}}(\Omega)$, with $\mathrm{p}\in[3,4]$, we deduce that there exists $c_{\mathrm{p}}>0$, depending only on $|\Omega|$ and p, such that

$$\begin{aligned} \||\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - |\mathbf{v}|^{\mathrm{p}-2}\mathbf{v}\|_{0,\mathrm{q};\Omega}\,\|\mathbf{w}\|_{0,\mathrm{p};\Omega} &\le c_{\mathrm{p}}\,\big(\|\mathbf{u}\|_{0,\mathrm{p};\Omega} + \|\mathbf{v}\|_{0,\mathrm{p};\Omega}\big)^{\mathrm{p}-2}\|\mathbf{u}-\mathbf{v}\|_{0,\mathrm{p};\Omega}\,\|\mathbf{w}\|_{0,\mathrm{p};\Omega}\\ &\le c_{\mathrm{p}}\,|\Omega|^{(6-\mathrm{p})/6}\big(\|\mathbf{u}\|_{0,6;\Omega} + \|\mathbf{v}\|_{0,6;\Omega}\big)^{\mathrm{p}-2}\,\|\mathbf{u}-\mathbf{v}\|_{0,6;\Omega}\,\|\mathbf{w}\|_{0,6;\Omega}\,. \end{aligned} \tag{3.18}$$

Thus, replacing back (3.18) into (3.17), we obtain (3.16) with

$$L_{\mathtt{MH}} := \max\left\{\alpha\,|\Omega|^{2/3}, \mathtt{F}\,c_{\rm p}\,|\Omega|^{(6-{\rm p})/6}, \nu, \frac{1}{\varrho\mu^2}, \frac{C_s}{\mu}\right\},$$

which completes the proof. $\square$

At this point we observe that since (3.16) holds for all $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in $\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\rm tr}(\Omega)\times\mathbf{H}(\mathrm{curl};\Omega)$, it is clear that it also holds for all $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in $\mathbf{X}$ (cf. (2.22)). We write (3.16) in the current general form since it will be used later on to derive the *a priori* error analysis (cf. Lemma 5.1).

Now, let us look at the kernel of the operator $\mathbf{B}$ (cf. (2.29)) that is

$$\mathbf{V} := \Big\{\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s},\mathbf{d}) \in \mathbf{X} : \quad [\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}] = 0 \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\Big\}$$

which, proceeding similarly to [12, eq. (3.34)] reduce to

$$\mathbf{V} := \mathbf{K}\times\mathbf{C}, \quad \text{where} \quad \mathbf{K} = \Big\{(\mathbf{v},\mathbf{s}) \in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\rm tr}(\Omega) : \quad \nabla\mathbf{v} = \mathbf{s} \quad \text{and} \quad \mathbf{v}\in\mathbf{H}^1_0(\Omega)\Big\}\,. \tag{3.19}$$

In addition, we recall from [31, Corollary 3.51] that

$$\|\mathrm{curl}(\mathbf{d})\|^2_{0,\Omega} \,\geq\, \alpha_m\,\|\mathbf{d}\|^2_{\mathrm{curl};\Omega} \quad \forall\,\mathbf{d}\in\mathbf{C}\,. \tag{3.20}$$

Thus, the following lemma shows that the operator $\mathbf{A}(\widehat{\mathbf{b}})$ satisfies hypothesis (ii) of Theorem 3.1 with $p_1 = {\rm p}\in[3,4]$ and $p_2 = p_3 = 2$.

Lemma 3.3 *Given $\widehat{\mathbf{b}}\in\mathbf{C}$, the family of operators $\big\{\mathbf{A}(\widehat{\mathbf{b}})(\cdot+\vec{\mathbf{z}}) : \mathbf{V}\to\mathbf{V}' : \vec{\mathbf{z}}\in\mathbf{X}\big\}$ is uniformly strongly monotone, that is, there exists $\alpha_{\mathtt{MH}}>0$, depending on ν, α, α_m, $\|\boldsymbol{i}_6\|$, ϱ, and μ such that*

$$[\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}+\vec{\mathbf{z}}) - \mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{v}}+\vec{\mathbf{z}}), \vec{\mathbf{u}}-\vec{\mathbf{v}}] \,\geq\, \alpha_{\mathtt{MH}}\,\|\vec{\mathbf{u}}-\vec{\mathbf{v}}\|^2_{\mathbf{X}} \tag{3.21}$$

for all $\vec{\mathbf{z}} = (\mathbf{z},\mathbf{r},\mathbf{e})\in\mathbf{X}$, and for all $\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t},\mathbf{b})$, $\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s},\mathbf{d})\in\mathbf{V}$.

Proof. Let $\vec{\mathbf{z}} = (\mathbf{z},\mathbf{r},\mathbf{e})\in\mathbf{X}$ and $\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t},\mathbf{b}), \vec{\mathbf{v}} = (\mathbf{v},\mathbf{s},\mathbf{d})\in\mathbf{V}$. Bearing in mind the definition of $\mathbf{A}(\widehat{\mathbf{b}})$, $\mathbf{a}$, and $\mathbf{c}(\widehat{\mathbf{b}})$ (cf. (2.26), (2.27), (2.28)), and the identity (3.13), we get

$$\begin{aligned}
&[\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}+\vec{\mathbf{z}}) - \mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{v}}+\vec{\mathbf{z}}), \vec{\mathbf{u}}-\vec{\mathbf{v}}] \,=\, [\mathbf{a}(\vec{\mathbf{u}}+\vec{\mathbf{z}}) - \mathbf{a}(\vec{\mathbf{v}}+\vec{\mathbf{z}}), \vec{\mathbf{u}}-\vec{\mathbf{v}}] + [\mathbf{c}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}-\vec{\mathbf{v}}), \vec{\mathbf{u}}-\vec{\mathbf{v}}]\\
&\quad=\, \alpha\,\|\mathbf{u}-\mathbf{v}\|^2_{0,\Omega} + \mathtt{F}\int_\Omega \big(|\mathbf{u}+\mathbf{z}|^{{\rm p}-2}(\mathbf{u}+\mathbf{z}) - |\mathbf{v}+\mathbf{z}|^{{\rm p}-2}(\mathbf{v}+\mathbf{z})\big)\cdot(\mathbf{u}-\mathbf{v})\\
&\qquad + \nu\,\|\mathbf{t}-\mathbf{s}\|^2_{0,\Omega} + \frac{1}{\varrho\,\mu^2}\,\|\mathrm{curl}(\mathbf{b}-\mathbf{d})\|^2_{0,\Omega}\,.
\end{aligned} \tag{3.22}$$

In turn, using [3, Lemma 2.1, eq.(2.1b)], there exists $C_{\rm p}>0$ depending only on $|\Omega|$ and p, such that

$$\int_\Omega \big(|\mathbf{u}+\mathbf{z}|^{{\rm p}-2}(\mathbf{u}+\mathbf{z}) - |\mathbf{v}+\mathbf{z}|^{{\rm p}-2}(\mathbf{v}+\mathbf{z})\big)\cdot(\mathbf{u}-\mathbf{v}) \,\geq\, C_{\rm p}\,\|\mathbf{u}-\mathbf{v}\|^{\rm p}_{0,{\rm p};\Omega} \,\geq\, 0\,,$$

which, together with (3.20) and (3.22), yields

$$[\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}+\vec{\mathbf{z}}) - \mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{v}}+\vec{\mathbf{z}}), \vec{\mathbf{u}}-\vec{\mathbf{v}}] \,\geq\, \alpha\,\|\mathbf{u}-\mathbf{v}\|^2_{0,\Omega} + \nu\,\|\mathbf{t}-\mathbf{s}\|^2_{0,\Omega} + \frac{\alpha_m}{\varrho\,\mu^2}\,\|\mathbf{b}-\mathbf{d}\|^2_{\mathrm{curl};\Omega}\,. \tag{3.23}$$

Next, employing the fact that $\mathbf{t}-\mathbf{s}=\nabla(\mathbf{u}-\mathbf{v})\in\Omega$ and $\mathbf{u}-\mathbf{v}\in\mathbf{H}_0^1(\Omega)$ in (cf. (3.19)), and using the continuous injection $\mathbf{i}_6$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^6(\Omega)$ (cf. (1.2)), we deduce that

$$[\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}+\vec{\mathbf{z}})-\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{v}}+\vec{\mathbf{z}}),\vec{\mathbf{u}}-\vec{\mathbf{v}}] \,\geq\, \min\left\{\alpha,\frac{\nu}{2}\right\}\|\mathbf{u}-\mathbf{v}\|_{1,\Omega}^2+\frac{\nu}{2}\|\mathbf{t}-\mathbf{s}\|_{0,\Omega}^2+\frac{\alpha_m}{\varrho\,\mu^2}\|\mathbf{b}-\mathbf{d}\|_{\mathrm{curl};\Omega}^2$$
$$\geq\,\min\left\{\alpha,\frac{\nu}{2}\right\}\|\mathbf{i}_6\|^{-2}\|\mathbf{u}-\mathbf{v}\|_{0,6;\Omega}^2+\frac{\nu}{2}\|\mathbf{t}-\mathbf{s}\|_{0,\Omega}^2+\frac{\alpha_m}{\varrho\,\mu^2}\|\mathbf{b}-\mathbf{d}\|_{\mathrm{curl};\Omega}^2\,,$$

which yields (3.21) with

$$\alpha_{\mathtt{MH}}\,:=\,\min\left\{\|\mathbf{i}_6\|^{-2}\min\left\{\alpha,\frac{\nu}{2}\right\},\frac{\nu}{2},\frac{\alpha_m}{\varrho\,\mu^2}\right\}. \tag{3.24}$$

□

As a corollary of Lemma 3.3, replacing $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}\in\mathbf{V}$ and $\vec{\mathbf{z}}\in\mathbf{X}$ in (3.21) by $\vec{\mathbf{u}}-\vec{\mathbf{v}}$, $\mathbf{0}\in\mathbf{V}$ and $\vec{\mathbf{v}}\in\mathbf{X}$, respectively, we arrive at

$$[\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{u}})-\mathbf{A}(\widehat{\mathbf{b}})(\vec{\mathbf{v}}),\vec{\mathbf{u}}-\vec{\mathbf{v}}]\,\geq\,\alpha_{\mathtt{MH}}\|\vec{\mathbf{u}}-\vec{\mathbf{v}}\|_{\mathbf{X}}^2\,, \tag{3.25}$$

for all $\vec{\mathbf{u}},\vec{\mathbf{v}}\in\mathbf{X}$ such that $\vec{\mathbf{u}}-\vec{\mathbf{v}}\in\mathbf{V}$.

We end the verification of the hypotheses of Theorem 3.1, with the corresponding inf-sup condition for the operator $\mathbf{B}$ (cf. (2.29), (2.17)).

Lemma 3.4 *There exists a positive constant $\beta_{\mathtt{MH}}$, such that*

$$\sup_{\substack{\vec{\mathbf{v}}\in\mathbf{X}\\ \vec{\mathbf{v}}\neq\mathbf{0}}}\frac{[\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}]}{\|\vec{\mathbf{v}}\|_{\mathbf{X}}}\,\geq\,\beta_{\mathtt{MH}}\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5};\Omega}\qquad\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,. \tag{3.26}$$

Proof. First, we note that from a slight adaptation of [12, Lemma 3.3] the following inf-sup condition for b_f holds

$$\sup_{\substack{(\mathbf{v},\mathbf{s})\in\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\\ (\mathbf{v},\mathbf{s})\neq\mathbf{0}}}\frac{[b_f(\mathbf{v},\mathbf{s}),\boldsymbol{\tau}]}{\|(\mathbf{v},\mathbf{s})\|}\,\geq\,\beta_{\mathtt{MH}}\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5};\Omega}\qquad\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,. \tag{3.27}$$

Thus, (3.26) follows straightforwardly from (3.27) and the definition of the operator $\mathbf{B}$ (cf. (2.29)). □

Now, we are in a position of establishing the solvability of the nonlinear problem (3.15).

Lemma 3.5 *For each $\widehat{\mathbf{b}}\in\mathbf{C}$, the problem (3.15) has a unique solution $(\vec{\mathbf{u}},\boldsymbol{\sigma})=((\mathbf{u},\mathbf{t},\mathbf{b}),\boldsymbol{\sigma})\in\mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$, and hence $\mathbf{T}(\widehat{\mathbf{b}}):=\mathbf{b}\in\mathbf{C}$ is well-defined. Moreover, there exists a positive constant $C_{\mathbf{T}}$, independent of $\widehat{\mathbf{b}}$, such that*

$$\|\mathbf{T}(\widehat{\mathbf{b}})\|_{\mathrm{curl};\Omega}\,\leq\,\|\vec{\mathbf{u}}\|_{\mathbf{X}}\,\leq\,C_{\mathbf{T}}\left\{\|\mathbf{f}_f\|_{0,6/5;\Omega}+\|\mathbf{f}_m\|_{0,\Omega}+\sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\Omega}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right\}. \tag{3.28}$$

Proof. Given $\widehat{\mathbf{b}}\in\mathbf{C}$, we first recall from (3.6), (3.8) and (3.9) that $\mathbf{B}$, $\mathbf{F}$ and $\mathbf{G}$ are all linear and bounded. Thus, thanks to Lemmas 3.2 and 3.3, the inf-sup condition of $\mathbf{B}$ given by (3.26), a straightforward application of Theorem 3.1, with $p_1=\mathrm{p}\in[3,4]$ and $p_2=p_3=2$ to problem (3.15) completes the proof. In particular, given $\widehat{\mathbf{b}}\in\mathbf{C}$, noting from (2.26) that $\mathbf{A}(\widehat{\mathbf{b}})(\mathbf{0})$ is the null functional, we get from (3.4) that

$$\mathcal{M}(\mathbf{F},\mathbf{G})=\|\mathbf{F}\|_{\mathbf{X}'}+3\|\mathbf{G}\|_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'}+\|\mathbf{G}\|_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'}^{\mathrm{p}-1}\,,$$

and hence the *a priori* estimate (3.2) yields

$$\|\vec{\mathbf{u}}\|_{\mathbf{X}} \,\leq\, C_1 \left\{ \|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{G}\|_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'} + \|\mathbf{G}\|^{\mathrm{p}-1}_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'} \right\} ,$$

with a positive constant C_1 depending only on $L_{\mathtt{MH}}$, $\alpha_{\mathtt{MH}}$ and $\beta_{\mathtt{MH}}$. The foregoing inequality together with the bounds of $\|\mathbf{F}\|_{\mathbf{X}'}$ and $\|\mathbf{G}\|_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'}$ (cf. (3.8), (3.9)) imply (3.28) with $C_{\mathbf{T}}$ depending on $\|\mathbf{i}_6\|$, $L_{\mathtt{MH}}$, $\alpha_{\mathtt{MH}}, \mu$ and $\beta_{\mathtt{MH}}$, thus completing the proof. □

For later use in the paper we note here that, applying (3.3), and using again the bounds (3.8) and (3.9) for $\|\mathbf{F}\|_{\mathbf{X}'}$ and $\|\mathbf{G}\|_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'}$, respectively, the *a priori* estimate for the second component of the solution to the problem defining $\mathbf{T}$ (*cf.* (3.15)) reduces to

$$\|\boldsymbol{\sigma}\|_{\mathbf{div}_{6/5};\Omega} \,\leq\, C_{\boldsymbol{\sigma}} \sum_{i\in\{\mathrm{p},2\}} \left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right)^{i-1} , \tag{3.29}$$

with $C_{\boldsymbol{\sigma}}$ depending on $L_{\mathtt{MH}}$, $\alpha_{\mathtt{MH}}$, μ and $\beta_{\mathtt{MH}}$.

3.3 Well-posedness of the continuous formulation

Having proved the well-posedness of the coupled problem (3.15) which ensures that the operator $\mathbf{T}$ is well defined, we now aim to establish the existence of a unique fixed-point of the operator $\mathbf{T}$. For this purpose, in what follows we will verify the hypothesis of the Banach fixed-point theorem. We begin by providing suitable conditions under which $\mathbf{T}$ maps a ball into itself.

Lemma 3.6 *Given $r > 0$, let $\mathbf{W}$ be the closed ball in $\mathbf{C}$ with center at the origin and radius r, and assume that the data satisfy*

$$C_{\mathbf{T}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right\} \,\leq\, r\,. \tag{3.30}$$

Then, there holds $\mathbf{T}(\mathbf{W}) \subseteq \mathbf{W}$.

Proof. It is a direct consequence of the *a priori* estimate (3.28) and the assumption (3.30). □

We now aim to prove that the operator $\mathbf{T}$ is Lipschitz continuous.

Lemma 3.7 *Let C_s, $\alpha_{\mathtt{MH}}$, and $C_{\mathbf{T}}$ be given by (2.13), (3.24), and (3.28), respectively. Then, there holds*

$$\begin{aligned} &\|\mathbf{T}(\widehat{\mathbf{b}}) - \mathbf{T}(\widehat{\mathbf{b}}_0)\|_{\mathrm{curl};\Omega} \\ &\quad\leq\, \frac{C_s\,C_{\mathbf{T}}}{\mu\,\alpha_{\mathtt{MH}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right\} \|\widehat{\mathbf{b}} - \widehat{\mathbf{b}}_0\|_{\mathrm{curl};\Omega}\,, \end{aligned} \tag{3.31}$$

for all $\widehat{\mathbf{b}}$, $\widehat{\mathbf{b}}_0 \in \mathbf{C}$.

Proof. Given $\widehat{\mathbf{b}}$, $\widehat{\mathbf{b}}_0 \in \mathbf{C}$, we let $(\vec{\mathbf{u}},\boldsymbol{\sigma}) := ((\mathbf{u},\mathbf{t},\mathbf{b}),\boldsymbol{\sigma})$ and $(\vec{\mathbf{u}}_0,\boldsymbol{\sigma}_0) := ((\mathbf{u}_0,\mathbf{t}_0,\mathbf{b}_0),\boldsymbol{\sigma}_0) \in \mathbf{X} \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ be the corresponding solutions of (3.15) so that $\mathbf{b} := \mathbf{T}(\widehat{\mathbf{b}})$ and $\mathbf{b}_0 := \mathbf{T}(\widehat{\mathbf{b}}_0)$. Then, subtracting the corresponding problems from (3.15), and using the definition of the operator $\mathbf{A}(\widehat{\mathbf{b}})$ (cf. (2.26)), we obtain

$$\begin{aligned} [\mathbf{A}(\widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}) - \mathbf{A}(\widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}_0), \vec{\mathbf{v}}] + [\mathbf{B}(\vec{\mathbf{v}}), \boldsymbol{\sigma} - \boldsymbol{\sigma}_0] &= -[\mathbf{c}(\widehat{\mathbf{b}} - \widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}), \vec{\mathbf{v}}]\,, \\ [\mathbf{B}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0), \boldsymbol{\tau}] &= 0\,, \end{aligned} \tag{3.32}$$

for all $\vec{\mathbf{v}} \in \mathbf{X}$ and $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$. We note from the second equation of (3.32) that $\vec{\mathbf{u}} - \vec{\mathbf{u}}_0 \in \mathbf{V}$ (cf (3.19)). Hence, taking $\vec{\mathbf{v}} := \vec{\mathbf{u}} - \vec{\mathbf{u}}_0 \in \mathbf{V}$ in the first equation of (3.32), applying (3.25) with $\vec{\mathbf{u}}$, $\vec{\mathbf{u}}_0$ $\in \mathbf{X}$, and using the continuity bound of $\mathbf{c}(\widehat{\mathbf{b}})$ (cf. (3.12)), we obtain

$$\begin{aligned} &\alpha_{\mathtt{MH}}\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{X}}^2 \,\le\, [\mathbf{A}(\widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}) - \mathbf{A}(\widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}_0), \vec{\mathbf{u}} - \vec{\mathbf{u}}_0] \,=\, -[\mathbf{c}(\widehat{\mathbf{b}} - \widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}), \vec{\mathbf{u}} - \vec{\mathbf{u}}_0] \\ &\le\, \frac{C_s}{\mu}\,\|\vec{\mathbf{u}}\|_{\mathbf{X}}\,\|\widehat{\mathbf{b}} - \widehat{\mathbf{b}}_0\|_{\mathrm{curl};\Omega}\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{X}}\,, \end{aligned}$$

which, together with (3.28) to bound $\|\vec{\mathbf{u}}\|_{\mathbf{X}}$, implies (3.31), completing the proof. □

We are now in position to establish the main result concerning the solvability of (2.25)

Theorem 3.8 *Given $r > 0$, let $\mathbf{W}$ be the closed ball in $\mathbf{C}$ with center at the origin and radius r, and assume that the data satisfy* (3.30) *and*

$$\frac{C_s\,C_{\mathbf{T}}}{\mu\,\alpha_{\mathtt{MH}}}\left\{\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right\} \,<\, 1\,. \tag{3.33}$$

Then the operator $\mathbf{T}$ has a unique fixed point $\mathbf{b} \in \mathbf{W}$. Equivalently, the coupled problem (2.25) *has a unique solution $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{X} \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$, with $\mathbf{b} \in \mathbf{W}$. Moreover, there exist positive constants $C_{\mathbf{T}}, C_{\boldsymbol{\sigma}}$, depending on C_s, ν, $\mathtt{F}$, α, α_m, $|\Omega|$, ϱ, μ, and $\beta_{\mathtt{MH}}$, such that the following a priori estimates hold*

$$\|\vec{\mathbf{u}}\|_{\mathbf{X}} \,\le\, C_{\mathbf{T}}\left\{\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right\}, \tag{3.34}$$

$$\|\boldsymbol{\sigma}\|_{\mathbf{div}_{6/5};\Omega} \,\le\, C_{\boldsymbol{\sigma}}\sum_{i\in\{\mathrm{p},2\}}\left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right)^{i-1}. \tag{3.35}$$

Proof. We begin by recalling from Lemma 3.6 that, under the assumption (3.30), $\mathbf{T}$ maps the ball $\mathbf{W}$ into itself, and hence, for each $\mathbf{b} \in \mathbf{W}$ we have that both $\|\mathbf{b}\|_{\mathrm{curl};\Omega}$ and $\|\mathbf{T}(\mathbf{b})\|_{\mathrm{curl};\Omega}$ are bounded by r. In turn, it is clear from (3.31) in Lemma 3.7 and Hypotheses (3.33) that $\mathbf{T}$ is a contraction. Therefore, the Banach fixed-point theorem provides the existence of a unique fixed point $\mathbf{b} \in \mathbf{W}$ of $\mathbf{T}$, equivalently, the existence of a unique solution $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{X} \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$, of the coupled problem (2.25), with $\mathbf{b} \in \mathbf{W}$. In addition, it is clear that the estimates (3.34) and (3.35) follow straightforwardly from (3.28) and (3.29), respectively, which finishes the proof. □

We end this section by establishing the well-posedness of (2.15).

Corollary 3.9 *Let $\mathbf{f}_f \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{f}_m \in \mathbf{L}^2(\Omega)$, $g_f \in \mathrm{L}^2(\Omega)$, and $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$, such that* (3.30) *and* (3.33) *hold. Then, there exist a unique $(\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{b}, \lambda) \in$*

$\mathbf{H}_0(\mathrm{curl};\Omega)\times \mathrm{H}_0^1(\Omega)$ *solution to* (2.15). *In addition,* $(\mathbf{u},\mathbf{t},\mathbf{b})$ *and* $\boldsymbol{\sigma}$ *satisfy* (3.34) *and* (3.35), *respectively, and for* λ*, there exits a constant* C_λ *depending on* C_s, ν, $\mathtt{F}$, α, α_m, $|\Omega|$, ϱ, μ, $\beta_{\mathtt{MH}}$, *and* β_m*, such that*

$$\|\lambda\|_{1,\Omega} \,\leq\, C_\lambda \sum_{i=1}^{2}\left(\|\mathbf{f}_f\|_{0,6/5;\Omega}+\|\mathbf{f}_m\|_{0,\Omega}+\sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\Omega}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right)^{i}. \tag{3.36}$$

Proof. We begin by recalling from Lemma 2.1 that the problems (2.15) and (2.25) are equivalents. Thus, the well-posedness of (2.15) and stability bounds for $(\mathbf{u},\mathbf{t},\mathbf{b})$ and $\boldsymbol{\sigma}$ follow from Theorem 3.8. On the other hand, using the identity (2.31), the inf-sup condition (2.24), and the continuity bounds of a_m, F_3 and c_m (cf. (3.5), (3.7), (3.11)), we deduce that

$$\begin{aligned}\beta_m\|\lambda\|_{1,\Omega} \,&\leq\, \sup_{\substack{\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega)\\ \mathbf{d}\neq 0}}\frac{[F_3,\mathbf{d}]-[a_m(\mathbf{b}),\mathbf{d}]-[c_m(\mathbf{b})(\mathbf{u}),\mathbf{d}]}{\|\mathbf{d}\|_{\mathrm{curl};\Omega}}\\ &\leq\, \frac{1}{\mu}\,\|\mathbf{f}_m\|_{0,\Omega}+\frac{1}{\varrho\,\mu^2}\,\|\mathbf{b}\|_{\mathrm{curl};\Omega}+\frac{C_s}{\mu}\,\|\mathbf{b}\|_{\mathrm{curl};\Omega}\,\|\mathbf{u}\|_{0,6;\Omega}\,.\end{aligned}$$

Finally, bounding $\|\mathbf{b}\|_{\mathrm{curl};\Omega}\,\|\mathbf{u}\|_{0,6;\Omega}$ by $\|\vec{\mathbf{u}}\|^2_{\mathbf{X}}$ in the foregoing inequality, and employing (3.34), we obtain (3.36), completing the proof. □

4 Galerkin scheme

In this section we introduce and analyze the corresponding Galerkin scheme for the five-field mixed formulation (2.15) (equivalently (2.25)). We mention in advance that, as we shall see in the forthcoming subsections, the well-posedness analysis follows straightforwardly by adapting the results derived for the continuous problem to the discrete case, so most of the details are omitted.

4.1 Discrete setting

We first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polyhedral region $\overline{\Omega}$ made up of tetrahedra T in R^3 of diameter h_T such that $\overline{\Omega}=\cup\big\{T:\,T\in\mathcal{T}_h\big\}$ and define $h:=\max\big\{h_T:\,T\in\mathcal{T}_h\big\}$. Given an integer $l\geq 0$ and a subset S of R^3, we denote by $P_l(S)$ the space of polynomials of total degree at most l defined on S, $\widetilde{P}_l(S)$ the space of homogeneous polynomials of degree exactly l on S and $\mathbf{M}_l(S)$ the space of polynomials $\mathbf{p}$ in $\widetilde{\mathbf{P}}_l(S)$ satisfying $\mathbf{p}(\mathbf{x})\cdot\mathbf{x}=0$ on S, where $\mathbf{x}:=(x_1,x_2,x_3)^t$ is a generic vector of R^3. Hence, for each integer $k\geq 0$ and for each $T\in\mathcal{T}_h$, we define the local Raviart–Thomas and Nédélec elements of order k (see for instance [4] and [31]), respectively, by

$$\mathbf{RT}_k(T)\,:=\,\mathbf{P}_k(T)\oplus\widetilde{P}_k(T)\mathbf{x}\quad\text{and}\quad\mathbf{ND}_k(T)\,:=\,\mathbf{P}_k(T)\oplus\mathbf{M}_{k+1}(T)\,.$$

In this way, introducing the finite element subspaces:

$$\begin{aligned}\mathbf{H}_h^{\mathbf{u}} &:= \big\{\mathbf{v}_h\in\mathbf{L}^6(\Omega):\quad \mathbf{v}_h|_T\in\mathbf{P}_k(T)\quad\forall\,T\in\mathcal{T}_h\big\}\,,\\ \mathbb{H}_h^{\mathbf{t}} &:= \big\{\mathbf{r}_h\in\mathbb{L}^2_{\mathrm{tr}}(\Omega):\quad \mathbf{r}_h|_T\in\mathbb{P}_k(T)\quad\forall\,T\in\mathcal{T}_h\big\}\,,\\ \mathbb{H}_h^{\boldsymbol{\sigma}} &:= \big\{\boldsymbol{\tau}_h\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega):\quad \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau}_h|_T\in\mathbf{RT}_k(T)\quad\forall\,\mathbf{c}\in\mathrm{R}^n,\quad\forall\,T\in\mathcal{T}_h\big\}\,,\\ \mathbf{H}_h^{\mathbf{b}} &:= \{\mathbf{d}_h\in\mathbf{H}_0(\mathrm{curl};\Omega):\quad \mathbf{d}_h|_T\in\mathbf{ND}_k(T)\quad\forall\,T\in\mathcal{T}_h\}\,,\\ \mathrm{H}_h^{\lambda} &:= \big\{\xi\in\mathrm{H}_0^1(\Omega):\quad \xi_h|_T\in P_{k+1}(T)\quad\forall\,T\in\mathcal{T}_h\big\}\,,\end{aligned}\tag{4.1}$$

the Galerkin scheme for (2.15) reads: Find $(\mathbf{u}_h, \mathbf{t}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\mathbf{b}_h, \lambda_h) \in \mathbf{H}_h^{\mathbf{b}} \times \mathrm{H}_h^{\lambda}$, such that

$$[a_f(\mathbf{u}_h, \mathbf{t}_h), (\mathbf{v}_h, \mathbf{s}_h)] + [c_f(\mathbf{b}_h)(\mathbf{b}_h), \mathbf{v}_h] + [b_f(\mathbf{v}_h, \mathbf{s}_h), \boldsymbol{\sigma}_h] = [F_1, (\mathbf{v}_h, \mathbf{s}_h)], \tag{4.2a}$$

$$[b_f(\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\tau}_h] = [F_2, \boldsymbol{\tau}_h]\,, \tag{4.2b}$$

$$[a_m(\mathbf{b}_h), \mathbf{d}_h] + [c_m(\mathbf{b}_h)(\mathbf{u}_h), \mathbf{d}_h] + [b_m(\mathbf{d}_h), \lambda_h] = [F_3, \mathbf{d}_h]\,, \tag{4.2c}$$

$$[b_m(\mathbf{b}_h), \xi_h] = 0\,, \tag{4.2d}$$

for all $(\mathbf{v}_h, \mathbf{s}_h, \boldsymbol{\tau}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and for all $(\mathbf{d}_h, \xi_h) \in \mathbf{H}_h^{\mathbf{b}} \times \mathrm{H}_h^{\lambda}$.

Now, analogously to the continuous case, from [28, Section 5.4] we recall that the bilinear form b_m satisfies the discrete inf-sup condition:

$$\sup_{\substack{\mathbf{d}_h \in \mathbf{H}_h^{\mathbf{b}} \\ \mathbf{d}_h \neq 0}} \frac{[b_m(\mathbf{d}_h), \xi_h]}{\|\mathbf{d}_h\|_{\mathrm{curl};\Omega}} \geq \beta_m \|\xi_h\|_{1,\Omega} \quad \forall\, \xi_h \in \mathrm{H}_h^{\lambda}\,, \tag{4.3}$$

with $\beta_m > 0$ being the same constant satisfying (2.24), which certainly is independent of h. Then, defining the discrete version of $\mathbf{C}$ (cf. (2.11)) as

$$\mathbf{C}_h := \left\{ \mathbf{d}_h \in \mathbf{H}_h^{\mathbf{b}} : \quad \int_{\Omega} \mathbf{d}_h \cdot \nabla \xi_h = 0 \quad \forall\, \xi_h \in \mathrm{H}_h^{\lambda} \right\}\,, \tag{4.4}$$

and denoting from now on

$$\vec{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h)\,,\ \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h, \mathbf{d}_h) \in \mathbf{X}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbf{C}_h\,,$$

the discrete version of (2.25) reads: Find $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ such that:

$$\begin{aligned} [\mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h] &= [\mathbf{F}, \vec{\mathbf{v}}_h] \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{X}_h\,, \\ [\mathbf{B}(\vec{\mathbf{u}}_h), \boldsymbol{\tau}_h] &= [\mathbf{G}, \boldsymbol{\tau}_h] \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,, \end{aligned} \tag{4.5}$$

where, as in the continuous case, given $\widehat{\mathbf{b}}_h \in \mathbf{C}_h$, the operator $\mathbf{A}(\widehat{\mathbf{b}}_h) : \mathbf{X}_h \to \mathbf{X}_h'$ is defined by

$$[\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] := [\mathbf{a}(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{c}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h]\,, \tag{4.6}$$

where $\mathbf{X}_h$ is endowed with the norm defined in (2.23).

At this point, we observe that owing to the discrete inf-sup condition (4.3), and using similar arguments to the ones employed in Lemma 2.1, the discrete problems (4.2) and (4.5) are equivalent. According to this, in what follows we focus on analyzing (4.5).

We now develop the discrete analogue of the fixed-point approach utilized in Section 3.2. To this end, we introduce the operator $\mathbf{T}_{\mathtt{d}} : \mathbf{C}_h \to \mathbf{C}_h$ defined by

$$\mathbf{T}_{\mathtt{d}}(\widehat{\mathbf{b}}_h) := \mathbf{b}_h \quad \forall\, \widehat{\mathbf{b}}_h \in \mathbf{C}_h\,, \tag{4.7}$$

where $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) = ((\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h), \boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ is the unique solution (to be confirmed below) of the problem

$$\begin{aligned} [\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h] &= [\mathbf{F}, \vec{\mathbf{v}}_h] \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{X}_h\,, \\ [\mathbf{B}(\vec{\mathbf{u}}_h), \boldsymbol{\tau}_h] &= [\mathbf{G}, \boldsymbol{\tau}_h] \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,. \end{aligned} \tag{4.8}$$

Therefore solving (4.5) is equivalent to seeking a fixed point of the operator $\mathbf{T}_{\mathtt{d}}$, that is: Find $\mathbf{b}_h \in \mathbf{C}_h$ such that

$$\mathbf{T}_{\mathtt{d}}(\mathbf{b}_h) = \mathbf{b}_h\,. \tag{4.9}$$

4.2 Solvability analysis

We begin by proving that (4.8) is well-posed, or equivalently that $\mathbf{T_d}$ (cf. (4.7)) is well defined. We remark in advance that the respective proof, being the discrete analogue of the one of Lemma 3.5, makes use again of the abstract result given by Theorem 3.1. We note also that the discrete kernel of b_m, namely $\mathbf{C}_h$ (cf. (4.4)), is not included in its continuous counterpart $\mathbf{C}$ (cf. (2.11)), and consequently, we can not employ the embedding $\mathbf{C} \subseteq \mathbf{H}^s(\Omega)$ for some $s > 1/2$. In order to overcome this drawback, as we shall see in the following lemma, from now on we need to assume that the mesh is quasi-uniform. Then, recalling the inverse inequality (see [11, Theorem 3.2.6]):

$$\|\xi\|_{0,q;\Omega} \,\leq\, C_I\, h^{3\,(1/q-1/p)} \|\xi\|_{0,p;\Omega}, \quad 1 \,\leq\, p \,\leq\, q \,\leq\, \infty\,, \tag{4.10}$$

for all piecewise polynomial functions ξ and $C_I > 0$ independent of h, we are able to establish general versions of (3.10), (3.11), and (3.12).

Lemma 4.1 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Given $\widehat{\mathbf{b}} \in \mathbf{C} + \mathbf{C}_h$, there exists a positive constant $C_{s,\mathbf{d}}$, independent of h and the physical parameters, such that*

$$\big|[c_f(\widehat{\mathbf{b}})(\mathbf{b}), \mathbf{v}]\big| \,\leq\, \frac{C_{s,\mathbf{d}}}{\mu}\, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega} \|\mathbf{b}\|_{\mathrm{curl};\Omega}\, \|\mathbf{v}\|_{0,6;\Omega} \quad \forall\, (\mathbf{b}, \mathbf{v}) \in \mathbf{H}_h^{\mathbf{b}} \times \mathbf{L}^6(\Omega)\,, \tag{4.11}$$

$$\big|[c_m(\widehat{\mathbf{b}})(\mathbf{u}), \mathbf{d}]\big| \,\leq\, \frac{C_{s,\mathbf{d}}}{\mu}\, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\, \|\mathbf{u}\|_{0,6;\Omega}\, \|\mathbf{d}\|_{\mathrm{curl};\Omega} \quad \forall\, (\mathbf{u}, \mathbf{d}) \in \mathbf{L}^6(\Omega) \times \mathbf{H}_h^{\mathbf{b}} \tag{4.12}$$

and

$$\begin{aligned} \big|[\mathbf{c}(\widehat{\mathbf{b}})(\vec{\mathbf{u}}), \vec{\mathbf{v}}]\big| \,&\leq\, \frac{C_{s,\mathbf{d}}}{\mu}\, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega} \left(\|\mathbf{u}\|_{0,6;\Omega}^2 + \|\mathbf{d}\|_{\mathrm{curl};\Omega}^2\right)^{1/2} \left(\|\mathbf{v}\|_{0,6;\Omega}^2 + \|\mathbf{d}\|_{\mathrm{curl};\Omega}^2\right)^{1/2} \\ &\leq\, \frac{C_{s,\mathbf{d}}}{\mu}\, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\, \|\vec{\mathbf{u}}\|_{\mathbf{X}}\, \|\vec{\mathbf{v}}\|_{\mathbf{X}} \end{aligned} \tag{4.13}$$

for all $\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t}, \mathbf{b}), \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}, \mathbf{d}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{H}_h^{\mathbf{b}}$.

Proof. In order to show (4.11), we proceed similarly to the proof of [34, Proposition 3.2]. First, notice that given $\widehat{\mathbf{b}} \in \mathbf{C}$, (4.11) follow straightforwardly from (3.10), since $\mathbf{b} \in \mathbf{H}_h^{\mathbf{b}} \subseteq \mathbf{H}(\mathrm{curl};\Omega)$. Now, let $\widehat{\mathbf{b}} \in \mathbf{C}_h$ and $(\mathbf{b}, \mathbf{v}) \in \mathbf{H}_h^{\mathbf{b}} \times \mathbf{L}^6(\Omega)$. In order to prove (4.11) we let $\mathbf{S} : \mathbf{C}_h \to \mathbf{C}$ be a linear operator such that (see [28, Section 4])

$$\mathrm{curl}(\mathbf{d}) = \mathrm{curl}(\mathbf{S}(\mathbf{d})) \quad \forall\, \mathbf{d} \in \mathbf{C}_h \tag{4.14}$$

satisfying

$$\|\mathbf{d} - \mathbf{S}(\mathbf{d})\|_{0,\Omega} \,\leq\, C_{\mathbf{S}}\, h^s\, \|\mathrm{curl}(\mathbf{d})\|_{0,\Omega} \quad \forall\, \mathbf{d} \in \mathbf{C}_h\,, \tag{4.15}$$

where $s > 1/2$ is the parameter such that $\mathbf{C} \subseteq \mathbf{H}^s(\Omega)$ (see [28, Lemma 4.5]). Next, adding and subtracting $\mathbf{S}(\widehat{\mathbf{b}})$ in the operator $c_f(\widehat{\mathbf{b}})$ and using triangle inequality, we obtain

$$\big|[c_f(\widehat{\mathbf{b}})(\mathbf{b}), \mathbf{v}]\big| \,\leq\, \big|[c_f(\widehat{\mathbf{b}} - \mathbf{S}(\widehat{\mathbf{b}}))(\mathbf{b}), \mathbf{v}]\big| + \big|[c_f(\mathbf{S}(\widehat{\mathbf{b}}))(\mathbf{b}), \mathbf{v}]\big|\,. \tag{4.16}$$

In order to bound the first term on the right-hand side of (4.16) we apply Hölder's inequality, the inverse inequality (4.10), with $q = 3$ and $p = 2$, and the estimate (4.15), to obtain

$$\begin{aligned} \big|[c_f(\widehat{\mathbf{b}} - \mathbf{S}(\widehat{\mathbf{b}}))(\mathbf{b}), \mathbf{v}]\big| \,&\leq\, \frac{1}{\mu}\, \|\widehat{\mathbf{b}} - \mathbf{S}(\widehat{\mathbf{b}})\|_{0,\Omega}\, \|\mathrm{curl}(\mathbf{b})\|_{0,3;\Omega}\, \|\mathbf{v}\|_{0,6;\Omega} \\ &\leq\, \frac{C_{\mathbf{S}}\, C_I}{\mu} h^{s-1/2} \|\mathrm{curl}(\widehat{\mathbf{b}})\|_{0,\Omega} \|\mathrm{curl}(\mathbf{b})\|_{0,\Omega} \|\mathbf{v}\|_{0,6;\Omega} \leq \frac{C_{\mathbf{S}}\, C_I}{\mu} h^{s-1/2} \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega} \|\mathbf{b}\|_{\mathrm{curl};\Omega} \|\mathbf{v}\|_{0,6;\Omega}\,. \end{aligned} \tag{4.17}$$

In turn, using the estimates (3.10) and (3.20), and the identity (4.14), we are able to deduce

$$
\begin{aligned}
\big|[c_f(\mathbf{S}(\widehat{\mathbf{b}}))(\mathbf{b},\mathbf{v}]\big| \,&\le\, \frac{C_s}{\mu}\|\mathbf{S}(\widehat{\mathbf{b}})\|_{\mathrm{curl};\Omega}\,\|\mathbf{b}\|_{\mathrm{curl};\Omega}\,\|\mathbf{v}\|_{0,6;\Omega} \\
&\le\, \frac{C_s}{\mu\,\alpha_m^{1/2}}\,\|\mathrm{curl}(\mathbf{S}(\widehat{\mathbf{b}}))\|_{0,\Omega}\,\|\mathbf{b}\|_{\mathrm{curl};\Omega}\,\|\mathbf{v}\|_{0,6;\Omega} \,\le\, \frac{C_s}{\mu\,\alpha_m^{1/2}}\,\|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\,\|\mathbf{b}\|_{\mathrm{curl};\Omega}\,\|\mathbf{v}\|_{0,6;\Omega}\,.
\end{aligned}
\tag{4.18}
$$

Thus, replacing back (4.17) and (4.18) into (4.16), and using the fact that $h^{s-1/2}<1$, since $s>1/2$, we obtain (4.11) with $C_{s,\mathtt{d}} = C_{\mathbf{S}}\,C_I + C_s/\alpha_m^{1/2}$ independent of h and the physical parameters. The proof of (4.12) follows analogously to (4.11), reason why is omitted, whereas (4.13) follows from the definition of the operator $\mathbf{c}(\widehat{\mathbf{b}})$ (cf. (2.28)) and estimates (4.11), (4.12). □

The following result establishes that the nonlinear operator $\mathbf{A}(\widehat{\mathbf{b}}_h)$ (cf. (4.6)) satisfies hypothesis (i) of Theorem 3.1 with $p_1 = \mathrm{p}\in[3,4]$ and $p_2 = p_3 = 2$.

Lemma 4.2 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Let $\mathrm{p}\in[3,4]$. Given $\widehat{\mathbf{b}}_h\in\mathbf{C}_h$, there exits $L_{\mathtt{MH,d}}>0$, depending on ν, $\mathtt{F}$, α, $|\Omega|$, $C_{s,\mathtt{d}}$, ϱ, and μ, such that*

$$
\begin{aligned}
\|\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h) - \mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h)\|_{\mathbf{X}_h'} \,&\le\, L_{\mathtt{MH,d}}\Big\{\big(1+\|\widehat{\mathbf{b}}_h\|_{\mathrm{curl};\Omega}\big)\big(\|\mathbf{u}_h-\mathbf{v}_h\|_{0,6;\Omega} + \|\mathbf{b}_h-\mathbf{d}_h\|_{\mathrm{curl};\Omega}\big) \\
&\quad + \|\mathbf{t}_h-\mathbf{s}_h\|_{0,\Omega} + \big(\|\mathbf{u}_h\|_{0,6;\Omega}+\|\mathbf{v}_h\|_{0,6,\Omega}\big)^{\mathrm{p}-2}\|\mathbf{u}_h-\mathbf{v}_h\|_{0,6;\Omega}\Big\}\,,
\end{aligned}
\tag{4.19}
$$

for all $\vec{\mathbf{u}}_h = (\mathbf{u}_h,\mathbf{t}_h,\mathbf{b}_h)$, $\vec{\mathbf{v}}_h = (\mathbf{v}_h,\mathbf{s}_h,\mathbf{d}_h)\in\mathbf{X}_h$.

Proof. First, given $\widehat{\mathbf{b}}_h\in\mathbf{C}_h$, we observe from the definition of the operator $\mathbf{A}(\widehat{\mathbf{b}}_h)$ (cf. (4.6)) that for $\vec{\mathbf{u}}_h = (\mathbf{u}_h,\mathbf{t}_h,\mathbf{b}_h)$, $\vec{\mathbf{v}}_h = (\mathbf{v}_h,\mathbf{s}_h,\mathbf{d}_h)\in\mathbf{X}_h$ there certainly holds

$$
\|\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h) - \mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h)\|_{\mathbf{X}_h'} \,\le\, \|\mathbf{a}(\vec{\mathbf{u}}_h)-\mathbf{a}(\vec{\mathbf{v}}_h)\|_{\mathbf{X}_h'} + \|\mathbf{c}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h)\|_{\mathbf{X}_h'}\,.
$$

Then, employing similar arguments to (3.16) and considering (4.13), we obtain (4.19), with

$$
L_{\mathtt{MH,d}} := \max\left\{\alpha\,|\Omega|^{2/3}, \mathtt{F}\,c_{\mathrm{p}}\,|\Omega|^{(6-\mathrm{p})/6}, \nu, \frac{1}{\varrho\mu^2}, \frac{C_{s,\mathtt{d}}}{\mu}\right\}.
$$

□

Next, in order to prove the hypotheses (ii) and (iii) of Theorem 3.1, we set the discrete kernel of the operator $\mathbf{B}$, which is given by $\mathbf{V}_h := \mathbf{K}_h\times\mathbf{C}_h$, with

$$
\mathbf{K}_h := \left\{(\mathbf{v}_h,\mathbf{s}_h)\in\mathbf{H}_h^{\mathbf{u}}\times\mathbb{H}_h^{\mathbf{t}} : \quad -\int_\Omega \boldsymbol{\tau}_h:\mathbf{s}_h - \int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\,\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\right\}. \tag{4.20}
$$

Then, from a slight adaptation of [8, Lemma 4.1], which in turn follows by using similar arguments to the ones developed in [12, Section 5], we now provide the discrete inf-sup condition for the operator b_f (cf. (2.17)) and an intermediate result that will be used to show later on the strong monotonicity of $\mathbf{A}(\widehat{\mathbf{b}}_h)$ on $\mathbf{V}_h$.

Lemma 4.3 *There exist positive constants $\beta_{\mathtt{MH,d}}$ and $C_{\mathtt{d}}$ such that*

$$
\sup_{\substack{(\mathbf{v}_h,\mathbf{s}_h)\in\mathbf{H}_h^{\mathbf{u}}\times\mathbb{H}_h^{\mathbf{t}}\\(\mathbf{v}_h,\mathbf{s}_h)\neq\mathbf{0}}} \frac{[b_f(\mathbf{v}_h,\mathbf{s}_h),\boldsymbol{\tau}_h]}{\|(\mathbf{v}_h,\mathbf{s}_h)\|} \,\ge\, \beta_{\mathtt{MH,d}}\,\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \qquad \forall\,\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}} \tag{4.21}
$$

and

$$
\|\mathbf{s}_h\|_{0,\Omega} \,\ge\, C_{\mathtt{d}}\|\mathbf{v}_h\|_{0,6;\Omega} \qquad \forall\,(\mathbf{v}_h,\mathbf{s}_h)\in\mathbf{K}_h\,. \tag{4.22}
$$

In addition, we recall from [28, Theorem 4.7] that

$$\|\mathrm{curl}(\mathbf{d}_h)\|_{0,\Omega}^2 \,\geq\, \alpha_{m,\mathtt{d}}\,\|\mathbf{d}_h\|_{\mathrm{curl};\Omega}^2 \quad \forall\, \mathbf{d}_h \in \mathbf{C}_h\,. \tag{4.23}$$

We now establish the discrete strong monotonicity property of $\mathbf{A}(\widehat{\mathbf{b}}_h)$ (cf. (4.6)).

Lemma 4.4 *Given $\widehat{\mathbf{b}}_h \in \mathbf{C}_h$, the family of operators $\big\{\mathbf{A}(\widehat{\mathbf{b}}_h)(\cdot + \vec{\mathbf{z}}_h) : \mathbf{V}_h \to \mathbf{V}_h' : \vec{\mathbf{z}}_h \in \mathbf{X}_h\big\}$ is uniformly strongly monotone, that is, there exists $\alpha_{\mathtt{MH,d}} > 0$, depending on ν, $\alpha_{m,\mathtt{d}}$, $C_{\mathtt{d}}$, ϱ, and μ such that*

$$[\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h + \vec{\mathbf{z}}_h) - \mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h + \vec{\mathbf{z}}_h), \vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h] \,\geq\, \alpha_{\mathtt{MH,d}}\,\|\vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h\|_{\mathbf{X}}^2\,, \tag{4.24}$$

for all $\vec{\mathbf{z}}_h = (\mathbf{z}_h, \mathbf{r}_h, \mathbf{e}_h) \in \mathbf{X}_h$, and for all $\vec{\mathbf{u}}_h = (\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h), \vec{\mathbf{v}}_h = (\mathbf{v}_h, \mathbf{s}_h, \mathbf{d}_h) \in \mathbf{V}_h$.

Proof. We follow an analogous reasoning to the proof of Lemma 3.3. In fact, let $\vec{\mathbf{z}}_h = (\mathbf{z}_h, \mathbf{r}_h, \mathbf{e}_h) \in \mathbf{X}_h$ and $\vec{\mathbf{u}}_h = (\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h), \vec{\mathbf{v}}_h = (\mathbf{v}_h, \mathbf{s}_h, \mathbf{d}_h) \in \mathbf{V}_h = \mathbf{K}_h \times \mathbf{C}_h$ (cf. (4.20)). Then, according to the definition of $\mathbf{A}(\widehat{\mathbf{b}}_h)$ (cf. (4.6)), and using the identity (3.13) (which is also true when $\widehat{\mathbf{b}} \in \mathbf{C}_h$ and $\vec{\mathbf{v}} \in \mathbf{X}_h$), [3, Lemma 2.1, eq.(2.1b)], and (4.23), we get, similarly to (3.23) that

$$\begin{aligned}&[\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h + \vec{\mathbf{z}}_h) - \mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h + \vec{\mathbf{z}}_h), \vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h] \\ &\quad\geq\, \alpha\,\|\mathbf{u}_h - \mathbf{v}_h\|_{0,\Omega}^2 + \nu\,\|\mathbf{t}_h - \mathbf{s}_h\|_{0,\Omega}^2 + \frac{\alpha_{m,\mathtt{d}}}{\varrho\,\mu^2}\,\|\mathbf{b}_h - \mathbf{d}_h\|_{\mathrm{curl};\Omega}^2\,.\end{aligned} \tag{4.25}$$

Next, bounding below the first term on the right hand side of (4.25) by 0, and using the fact that $\vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h := ((\mathbf{u}_h - \mathbf{v}_h, \mathbf{t}_h - \mathbf{s}_h), \mathbf{b}_h - \mathbf{d}_h) \in \mathbf{K}_h \times \mathbf{C}_h$ in combination with the estimate (4.22), we obtain

$$\begin{aligned}&[\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h + \vec{\mathbf{z}}_h) - \mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h + \vec{\mathbf{z}}_h), \vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h] \\ &\quad\geq\, \frac{\nu}{2} C_{\mathtt{d}}^2\,\|\mathbf{u}_h - \mathbf{v}_h\|_{0,6;\Omega}^2 + \frac{\nu}{2}\,\|\mathbf{t}_h - \mathbf{s}_h\|_{0,\Omega}^2 + \frac{\alpha_{m,\mathtt{d}}}{\varrho\,\mu^2}\,\|\mathbf{b}_h - \mathbf{d}_h\|_{\mathrm{curl};\Omega}^2\,,\end{aligned}$$

which yields (4.24) with

$$\alpha_{\mathtt{MH,d}} \,:=\, \min\left\{\frac{\nu}{2}, \frac{\nu\,C_{\mathtt{d}}^2}{2}, \frac{\alpha_{m,\mathtt{d}}}{\varrho\,\mu^2}\right\}\,. \tag{4.26}$$

□

Similar to the continuous case, replacing $\vec{\mathbf{u}}_h$, $\vec{\mathbf{v}}_h \in \mathbf{V}_h$ and $\vec{\mathbf{z}}_h \in \mathbf{X}_h$ by $\vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h$, $\mathbf{0} \in \mathbf{V}_h$ and $\vec{\mathbf{v}}_h \in \mathbf{X}_h$ in (4.24), we arrive at

$$[\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h) - \mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h), \vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h] \,\geq\, \alpha_{\mathtt{MH,d}}\,\|\vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h\|_{\mathbf{X}}^2\,, \tag{4.27}$$

for all $\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h \in \mathbf{X}_h$ such that $\vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h \in \mathbf{V}_h$.

We continue with the discrete inf-sup condition for the operator $\mathbf{B}$ (cf. (2.29), (2.17)).

Lemma 4.5 *There exists a positive constant $\beta_{\mathtt{MH,d}}$, such that*

$$\sup_{\substack{\vec{\mathbf{v}}_h \in \mathbf{X}_h \\ \vec{\mathbf{v}}_h \neq \mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\tau}_h]}{\|\vec{\mathbf{v}}_h\|_{\mathbf{X}}} \,\geq\, \beta_{\mathtt{MH,d}}\,\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,. \tag{4.28}$$

Proof. The statement follows directly from the definition of the operator $\mathbf{B}$ (cf. (2.29)) and (4.21). □

We are now in position of establishing the discrete analogue of Lemma 3.5.

Lemma 4.6 *Assume that* $\{\mathcal{T}_h\}_{h>0}$ *is a family of quasi-uniform triangulations. Then, for each* $\widehat{\mathbf{b}}_h \in \mathbf{C}_h$*, the problem* (4.8) *has a unique solution* $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) = ((\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h), \boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$*, and hence* $\mathbf{T}_{\mathtt{d}}(\widehat{\mathbf{b}}_h) := \mathbf{b}_h \in \mathbf{C}_h$ *is well-defined. Moreover, there exists a positive constant* $C_{\mathbf{T}_{\mathtt{d}}}$*, independent of* $\widehat{\mathbf{b}}_h$*, such that*

$$\|\mathbf{T}_{\mathtt{d}}(\widehat{\mathbf{b}}_h)\|_{\mathrm{curl};\Omega} \le \|\vec{\mathbf{u}}_h\|_{\mathbf{X}} \le C_{\mathbf{T}_{\mathtt{d}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right\}. \quad (4.29)$$

Proof. According to Lemmas 4.2 and 4.4 and the discrete inf-sup condition for $\mathbf{B}$ provided by (4.28) (cf. Lemma 4.5), the proof follows from a direct application of Theorem 3.1, with $p_1 = \mathrm{p} \in [3,4]$ and $p_2 = p_3 = 2$, to the discrete setting represented by (4.8). In particular, the *a priori* bound (4.29) is consequence of the abstract estimate (3.2) applied to (4.8), which makes use of the bounds for $\|\mathbf{F}\|_{\mathbf{X}_h'}$ and $\|\mathbf{G}\|_{\mathbb{H}_h^{\boldsymbol{\sigma}\prime}}$ (cf. (3.8), (3.9)). □

We remark here that, proceeding similarly to the derivation of (3.29), we obtain

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \le C_{\boldsymbol{\sigma}_{\mathtt{d}}} \sum_{i\in\{\mathrm{p},2\}} \left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right)^{i-1}, \quad (4.30)$$

with $C_{\boldsymbol{\sigma}_{\mathtt{d}}}$ depending only on $L_{\mathtt{MH,d}}$, $\alpha_{\mathtt{MH,d}}$ and $\beta_{\mathtt{MH,d}}$.

We now proceed to analyze the fixed-point equation (4.9). We begin with the discrete version of Lemma 3.6, whose proof, follows straightforwardly from Lemma 4.6.

Lemma 4.7 *Given* $r > 0$*, let* $\mathbf{W}_h$ *be the closed ball in* $\mathbf{C}_h$ *with center at the origin and radius* r*, and assume that the data satisfy*

$$C_{\mathbf{T}_{\mathtt{d}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right\} \le r. \quad (4.31)$$

Then, there holds $\mathbf{T}_{\mathtt{d}}(\mathbf{W}_h) \subseteq \mathbf{W}_h$.

Next, we address the discrete counterpart of Lemma 3.7, whose proof, being almost verbatim of the continuous ones, is omitted. We just remark that Lemma 4.8 below is derived using the strong monotonicity of $\mathbf{A}(\widehat{\mathbf{b}}_h)$ on $\mathbf{V}_h$ (cf. (4.24)) and the continuity bound of $\mathbf{c}(\widehat{\mathbf{b}}_h)$ (cf. (4.13)). Thus, we simply state the corresponding result as follow.

Lemma 4.8 *Assume that* $\{\mathcal{T}_h\}_{h>0}$ *is a family of quasi-uniform triangulations. Let* $C_{s,\mathtt{d}}$*,* $\alpha_{\mathtt{MH,d}}$*, and* $C_{\mathbf{T}_{\mathtt{d}}}$ *be given by* (4.13)*,* (4.26)*, and* (4.29)*, respectively. Then, there holds*

$$\begin{aligned} &\|\mathbf{T}_{\mathtt{d}}(\widehat{\mathbf{b}}_h) - \mathbf{T}_{\mathtt{d}}(\widehat{\mathbf{b}}_{0,h})\|_{\mathrm{curl};\Omega} \\ &\quad \le \frac{C_{s,\mathtt{d}}\, C_{\mathbf{T}_{\mathtt{d}}}}{\mu\, \alpha_{\mathtt{MH,d}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right\} \|\widehat{\mathbf{b}}_h - \widehat{\mathbf{b}}_{0,h}\|_{\mathrm{curl};\Omega}, \end{aligned} \quad (4.32)$$

for all $\widehat{\mathbf{b}}_h, \widehat{\mathbf{b}}_{0,h} \in \mathbf{C}_h$.

We are now in position of establishing the well-posedness of (4.5)

Theorem 4.9 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Given $r > 0$, let $\mathbf{W}_h$ be the closed ball in $\mathbf{C}_h$ with center at the origin and radius r, and assume that the data satisfy* (4.31) *and*

$$\frac{C_{s,\mathtt{d}}\, C_{\mathbf{T}_\mathtt{d}}}{\mu\, \alpha_{\mathtt{MH},\mathtt{d}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \big)^{j-1} \right\} < 1 . \tag{4.33}$$

Then the operator $\mathbf{T}_\mathtt{d}$ has a unique fixed point $\mathbf{b}_h \in \mathbf{W}_h$. Equivalently, the problem (4.5) *has a unique solution $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$, with $\mathbf{b}_h \in \mathbf{W}_h$. Moreover, there exist positive constants $C_{\mathbf{T}_\mathtt{d}}, C_{\boldsymbol{\sigma}_\mathtt{d}}$, depending on $C_{s,\mathtt{d}}$, ν, $\mathtt{F}$, α, $\alpha_{m,\mathtt{d}}$, $|\Omega|$, ϱ, μ, and $\beta_{\mathtt{MH},\mathtt{d}}$, such that the following a priori estimates hold*

$$\|\vec{\mathbf{u}}_h\|_{\mathbf{X}} \le C_{\mathbf{T}_\mathtt{d}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \big)^{j-1} \right\} , \tag{4.34}$$

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \le C_{\boldsymbol{\sigma}_\mathtt{d}} \sum_{i\in\{\mathrm{p},2\}} \left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \big)^{j-1} \right)^{i-1} . \tag{4.35}$$

Proof. It follows similarly to the proof of Theorem 3.8. Indeed, we first notice from Lemma 4.7 that $\mathbf{T}_\mathtt{d}$ maps the ball $\mathbf{W}_h$ into itself. Next, it is easy to see from (4.32) (cf. Lemma 4.8) and (4.33) that $\mathbf{T}_\mathtt{d}$ is a contraction, and hence the existence and uniqueness results follow from the Banach fixed-point theorem. In addition, it is clear that the estimates (4.34) and (4.35) follow straightforwardly from (4.29) and (4.30), which ends the proof. □

We end this section by establishing the well-posedness of (4.2), whose proof is omitted since it follows analogously to the proof of Corollary 3.9. We just remark that Corollary 4.10 below is derived using the discrete inf-sup condition of b_m (cf. (4.3)) and the continuity bound of $c_m(\widehat{\mathbf{b}}_h)$ (cf. (4.12)).

Corollary 4.10 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Let $\mathbf{f}_f \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{f}_m \in \mathbf{L}^2(\Omega), g_f \in \mathrm{L}^2(\Omega)$, and $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$, such that* (4.31) *and* (4.33) *hold. Then, there exist a unique $(\mathbf{u}_h, \mathbf{t}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\mathbf{b}_h, \lambda_h) \in \mathbf{H}_h^{\mathbf{b}} \times \mathrm{H}_h^{\lambda}$ solution to* (4.2)*. In addition, $(\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h)$ and $\boldsymbol{\sigma}_h$ satisfy* (4.34) *and* (4.35)*, respectively, and for λ_h, there exits a constant $C_{\lambda_\mathtt{d}}$ depending on $C_{s,\mathtt{d}}, \nu, \mathtt{F}, \alpha, |\Omega|, \varrho, \mu, \beta_{\mathtt{MH},\mathtt{d}}$, and β_m, such that*

$$\|\lambda_h\|_{1,\Omega} \le C_{\lambda_\mathtt{d}} \sum_{i=1}^{2} \left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \big)^{j-1} \right)^{i} .$$

5 A priori error analysis

In this section we derive Céa's estimate for the Galerkin scheme (4.2) with the finite element subspaces given by (4.1) (cf. Section 4.1), and then use the approximation properties of the latter to establish the corresponding rates of convergence. In fact, let $(\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{b}, \lambda) \in \mathbf{H}_0(\mathrm{curl};\Omega) \times \mathrm{H}^1_0(\Omega)$, with $\mathbf{b} \in \mathbf{W}$, be the unique solution of the problem (2.15) and let $(\mathbf{u}_h, \mathbf{t}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\mathbf{b}_h, \lambda_h) \in \mathbf{H}_h^{\mathbf{b}} \times \mathrm{H}_h^{\lambda}$, with $\mathbf{b}_h \in \mathbf{W}_h$, be the unique solution of the discrete problem (4.2). Then, we are interested in obtaining an *a priori* estimate for the global error

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,6;\Omega} + \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\mathbf{b} - \mathbf{b}_h\|_{\mathrm{curl};\Omega} + \|\lambda - \lambda_h\|_{1,\Omega} . \tag{5.1}$$

For this purpose, in what follows we introduce some definitions. Hereafter, given a subspace X_h of a generic Banach space $(X, \|\cdot\|_X)$, we set as usual

$$\operatorname{dist}(x, X_h) := \inf_{x_h \in X_h} \|x - x_h\|_X \qquad \forall\, x \in X\,.$$

We stress here that in order to derive an *a priori* bound for the global error (5.1), we first bound, separately, the terms $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$ and $\|\lambda - \lambda_h\|_{1,\Omega}$, being $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}, \mathbf{b}), \boldsymbol{\sigma}) \in \mathbf{X} \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ the unique solution of the problem (2.25), and $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) = ((\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h), \boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ the unique solution of the discrete problem (4.5). This is done below in Lemmas 5.1 and 5.2, respectively. We begin by bounding $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$. To that end, we first notice that differently to [8] we can not apply directly the Strang-type lemma derived in [8, Lemma 5.1] since $\mathbf{C}_h$ is not included in its continuous counterpart $\mathbf{C}$. Nevertheless, most of the arguments used to prove [8, Lemma 5.1] are employed below in Lemma 5.1 for the context given by (2.25) and (4.5), namely discrete strong monotocity of $\mathbf{A}(\mathbf{b}_h)$ (cf. (4.27)), continuity of the operator $\mathbf{c}(\mathbf{b}_h)$ (cf. (4.13)), and discrete inf-sup condition of $\mathbf{B}$ (cf. (4.28)).

Next, we define the set

$$\mathbf{V}_h^{\mathbf{G}} := \Big\{ \vec{\mathbf{w}}_h \in \mathbf{X}_h : \quad [\mathbf{B}(\vec{\mathbf{w}}_h), \boldsymbol{\tau}_h] = [\mathbf{G}, \boldsymbol{\tau}_h] \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \Big\}\,, \tag{5.2}$$

which is clearly nonempty, since (4.28) holds. Note from the second equation of (4.5) that $\vec{\mathbf{u}}_h \in \mathbf{V}_h^{\mathbf{G}}$ and then $\vec{\mathbf{u}}_h - \vec{\mathbf{w}}_h \in \mathbf{V}_h$ for all $\vec{\mathbf{w}}_h \in \mathbf{V}_h^{\mathbf{G}}$. In addition, we recall that the discrete inf-sup conditions (4.28) and (4.3), and a classical result on mixed methods (see, for instance, [20, eq. (2.89) in Theorem 2.6]) ensure the existence of $C_1, C_2 > 0$, independent of h, such that:

$$\operatorname{dist}(\vec{\mathbf{u}}, \mathbf{V}_h^{\mathbf{G}}) \,\le\, C_1 \operatorname{dist}(\vec{\mathbf{u}}, \mathbf{X}_h) \,\le\, C_1 \Big(\operatorname{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) + \operatorname{dist}(\mathbf{t}, \mathbb{H}_h^{\mathbf{t}}) + \operatorname{dist}(\mathbf{b}, \mathbf{C}_h) \Big) \tag{5.3}$$

and

$$\operatorname{dist}(\mathbf{b}, \mathbf{C}_h) \,\le\, C_2 \operatorname{dist}(\mathbf{b}, \mathbf{H}_h^{\mathbf{b}})\,. \tag{5.4}$$

Throughout the rest of the paper, given any $r > 0$, both $c(r)$ and $C(r)$, with or without sub-indexes, denote positive constants depending on r, and eventually on other constants or parameters.

The announced preliminary result regarding $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$ is established as follows.

Lemma 5.1 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Let $\mathbf{f}_f \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{f}_m \in \mathbf{L}^2(\Omega)$, $g_f \in \mathrm{L}^2(\Omega)$, and $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$, satisfying*

$$\frac{C_{s,\mathtt{d}}\, C_{\mathbf{T}}}{\mu\, \alpha_{\mathtt{MH},\mathtt{d}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \big)^{j-1} \right\} \,\le\, \frac{1}{2}\,. \tag{5.5}$$

Then, there exists a positive constant $C_1(r)$, independent of h, such that

$$\begin{aligned} &\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \\ &\quad \le C_1(r) \left\{ \sum_{j\in\{\mathrm{p},2\}} \big(\operatorname{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) + \operatorname{dist}(\mathbf{t}, \mathbb{H}_h^{\mathbf{t}}) + \operatorname{dist}(\mathbf{b}, \mathbf{H}_h^{\mathbf{b}}) \big)^{j-1} + \operatorname{dist}(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) \right\}. \end{aligned} \tag{5.6}$$

Proof. We begin by noting that the first equation in (2.25) is well-defined even dough for test functions in $\mathbf{X}_h$. Then, we subtract the first equations of (2.25) and (4.5), to obtain

$$[\mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{v}}_h] - [\mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma} - \boldsymbol{\sigma}_h] = 0 \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{X}_h\,. \tag{5.7}$$

Next, let $\vec{\mathbf{w}}_h = (\mathbf{w}_h, \mathbf{r}_h, \mathbf{e}_h)$ be an arbitrary element in $\mathbf{V}_h^{\mathbf{G}}$ (cf. (5.2)), adding and subtracting suitable terms in (5.7), we arrive at

$$\begin{aligned} &[\mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{w}}_h) - \mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] \\ &\quad = [\mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{w}}_h), \vec{\mathbf{v}}_h] - [\mathbf{A}(\mathbf{b})(\vec{\mathbf{w}}_h), \vec{\mathbf{v}}_h] + [\mathbf{A}(\mathbf{b})(\vec{\mathbf{w}}_h) - \mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{v}}_h] - [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\,, \end{aligned} \tag{5.8}$$

for all $\vec{\mathbf{v}}_h \in \mathbf{X}_h$. Testing (5.8) with $\vec{\mathbf{v}}_h = \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h \in \mathbf{V}_h$, using (4.27) (cf. Lemma 4.4) and the fact that $[\mathbf{B}(\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h), \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h] = 0$ for all $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$, we get

$$\begin{aligned} \alpha_{\mathtt{MH,d}}\, \|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}^2 \,&\le\, \big|[\mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{w}}_h), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h] - [\mathbf{A}(\mathbf{b})(\vec{\mathbf{w}}_h), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h]\big| \\ &+ \big|[\mathbf{A}(\mathbf{b})(\vec{\mathbf{w}}_h) - \mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h]\big| \,+\, \big|[\mathbf{B}(\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h), \boldsymbol{\sigma} - \boldsymbol{\tau}_h]\big|\,, \end{aligned} \tag{5.9}$$

where, using the definitions of $\mathbf{A}(\mathbf{b})$ (cf. (2.26)) and $\mathbf{A}(\mathbf{b}_h)$ (cf. (4.6)), and employing Lemma 4.1 and triangle inequality, we first deduce that

$$\begin{aligned} &\big|[\mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{w}}_h), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h] - [\mathbf{A}(\mathbf{b})(\vec{\mathbf{w}}_h), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h]\big| \,=\, \big|[\mathbf{c}(\mathbf{b}_h - \mathbf{b})(\vec{\mathbf{w}}_h), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h]\big| \\ &\le\, \frac{C_{s,\mathtt{d}}}{\mu}\, \|\mathbf{b} - \mathbf{b}_h\|_{\mathrm{curl};\Omega} \big(\|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{u}}\|_{\mathbf{X}}\big)\, \|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \\ &\le\, \frac{C_{s,\mathtt{d}}}{\mu} \Big\{ \big(\|\mathbf{b}\|_{\mathrm{curl};\Omega} + \|\mathbf{b}_h\|_{\mathrm{curl};\Omega}\big) \|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{u}}\|_{\mathbf{X}} \|\mathbf{b} - \mathbf{b}_h\|_{\mathrm{curl};\Omega} \Big\} \|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,. \end{aligned}$$

Then, using the fact that $\mathbf{b} \in \mathbf{W}$, $\mathbf{b}_h \in \mathbf{W}_h$, and bounding $\|\mathbf{b} - \mathbf{b}_h\|_{\mathrm{curl};\Omega}$ by $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}$, we arrive at

$$\begin{aligned} &\big|[\mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{w}}_h), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h] - [\mathbf{A}(\mathbf{b})(\vec{\mathbf{w}}_h), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h]\big| \\ &\quad \le\, \Big(c_1(r) \|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}} + \frac{C_{s,\mathtt{d}}}{\mu}\, \|\vec{\mathbf{u}}\|_{\mathbf{X}}\, \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \Big) \|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,, \end{aligned} \tag{5.10}$$

with $c_1(r)$ depending on $C_{s,\mathtt{d}}$, μ, and r. In turn, using Lemma 3.2, and simple computations, we get

$$\begin{aligned} &\big|[\mathbf{A}(\mathbf{b})(\vec{\mathbf{w}}_h) - \mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h]\big| \,\le\, \|\mathbf{A}(\mathbf{b})(\vec{\mathbf{w}}_h) - \mathbf{A}(\mathbf{b})(\vec{\mathbf{u}})\|_{\mathbf{X}'} \|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \\ &\le\, L_{\mathtt{MH}} \Big\{ \big(1 + \|\mathbf{b}\|_{\mathrm{curl};\Omega}\big) \big(\|\mathbf{u} - \mathbf{w}_h\|_{0,6;\Omega} \,+\, \|\mathbf{b} - \mathbf{e}_h\|_{\mathrm{curl};\Omega}\big) \\ &+\, \|\mathbf{t} - \mathbf{r}_h\|_{0,\Omega} \,+\, \big(2\, \|\mathbf{u}\|_{0,6;\Omega} + \|\mathbf{u} - \mathbf{w}_h\|_{0,6,\Omega}\big)^{\mathrm{p}-2} \|\mathbf{u} - \mathbf{w}_h\|_{0,6;\Omega} \Big\} \|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,, \end{aligned}$$

which combined with the property $(a+b)^p \,\le\, 2^{p-1}(a^p + b^p)$ for $a, b > 0$ and $p > 1$, the fact that $\mathbf{b} \in \mathbf{W}$, using (3.34) in conjunction with (3.30) to bound $\|\mathbf{u}\|_{0,6;\Omega}$ by r, and similar arguments to the ones employed in (5.10), we deduce

$$|[\mathbf{A}(\mathbf{b})(\vec{\mathbf{w}}_h) - \mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h]| \le c_2(r) \Big\{ \|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}}^{\mathrm{p}-1} \Big\} \|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,, \tag{5.11}$$

with $c_2(r)$ depending on $L_{\mathtt{MH}}$, p, and r. In addition, we observe from (3.6), that

$$\big|[\mathbf{B}(\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h), \boldsymbol{\sigma} - \boldsymbol{\tau}_h]\big| \,\le\, \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}\, \|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,. \tag{5.12}$$

Thus, replacing back (5.10), (5.11) and (5.12) into (5.9), and bounding $\|\vec{\mathbf{u}}\|_{\mathbf{X}}$ by (3.34), we obtain

$$\begin{aligned}\|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}} &\le c_3(r)\Big\{\|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}} \,+\, \|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}}^{\mathrm{p}-1} \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}\Big\} \\ &+ \frac{C_{s,\mathtt{d}}\,C_{\mathbf{T}}}{\mu\,\alpha_{\mathtt{MH},\mathtt{d}}}\left\{\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right\}\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,,\end{aligned}$$

with $c_3(r)$ depending on $\alpha_{\mathtt{MH},\mathtt{d}}$, $C_{s,\mathtt{d}}$, μ, $L_{\mathtt{MH}}$, p, and r. Hence, triangle inequality $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \le \|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{w}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}$, and the assumption (5.5), yields

$$\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \,\le\, c_4(r)\,\Big\{\|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}} \,+\, \|\vec{\mathbf{u}} - \vec{\mathbf{w}}_h\|_{\mathbf{X}}^{\mathrm{p}-1} \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}\Big\}\,, \tag{5.13}$$

with $c_4(r)$ only depending on $\alpha_{\mathtt{MH},\mathtt{d}}$, $C_{s,\mathtt{d}}$, μ, $L_{\mathtt{MH}}$, p, and r.

On the other hand, to estimate the term $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$, we consider an arbitrary element $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$ and use the discrete inf-sup condition (4.28), to get

$$\beta_{\mathtt{MH},\mathtt{d}}\,\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \,\le\, \sup_{\substack{\vec{\mathbf{v}}_h\in\mathbf{X}_h\\ \vec{\mathbf{v}}_h\neq\mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h - \boldsymbol{\sigma}] - [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma} - \boldsymbol{\tau}_h]}{\|\vec{\mathbf{v}}_h\|}\,, \tag{5.14}$$

where, using again (5.7) and adding and subtracting suitable terms, we obtain

$$[\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h - \boldsymbol{\sigma}] \,=\, [\mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] - [\mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}) - \mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h]\,.$$

In turn, similarly to (5.10) and (5.11), using (3.34) and (4.34) in conjunction with (3.30) and (4.31) to bound $\|\mathbf{u}\|_{0,6;\Omega}$, $\|\mathbf{u}_h\|_{0,6;\Omega}$, and $\|\vec{\mathbf{u}}_h\|_{\mathbf{X}}$ by r, and the fact that $\mathbf{b} \in \mathbf{W}$, we deduce, respectively, that

$$\big|[\mathbf{A}(\mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] - [\mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h]\big| \,\le\, \frac{C_{s,\mathtt{d}}}{\mu}\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,\|\vec{\mathbf{u}}_h\|_{\mathbf{X}}\,\|\vec{\mathbf{v}}_h\|_{\mathbf{X}} \,\le\, c_5(r)\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,\|\vec{\mathbf{v}}_h\|_{\mathbf{X}} \tag{5.15}$$

and

$$\begin{aligned}\big|[\mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}_h) - \mathbf{A}(\mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{v}}_h]\big| &\le L_{\mathtt{MH}}\sqrt{3}\Big\{1 + \|\mathbf{b}\|_{\mathrm{curl};\Omega} + \big(\|\mathbf{u}\|_{0,6;\Omega} + \|\mathbf{u}_h\|_{0,6;\Omega}\big)^{\mathrm{p}-2}\Big\}\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\|\vec{\mathbf{v}}_h\|_{\mathbf{X}} \\ &\le\, c_6(r)\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\|\vec{\mathbf{v}}_h\|_{\mathbf{X}}\,,\end{aligned} \tag{5.16}$$

with $c_5(r)$ and $c_6(r)$ only depending on $C_{s,\mathtt{d}}, \mu, L_{\mathtt{MH}}, \mathrm{p}$, and r. Thus, replacing back (5.15) and (5.16) into (5.14), using (3.6), triangle inequality, and some algebraic manipulations, we obtain

$$\begin{aligned}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} &\le\, \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \,+\, \|\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \\ &\le\, c_7(r)\,\Big\{\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\Big\}\,,\end{aligned} \tag{5.17}$$

with $c_7(r)$ only depending on $C_{s,\mathtt{d}}$, μ, $L_{\mathtt{MH}}$, p, $\beta_{\mathtt{MH},\mathtt{d}}$ and r. Finally, combining (5.13) and (5.17), using the fact that $\vec{\mathbf{w}}_h \in \mathbf{V}_h^{\mathbf{G}}$ and $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$ are arbitrary, taking infimum over the corresponding discrete subspaces $\mathbf{V}_h^{\mathbf{G}}$ and $\mathbb{H}_h^{\boldsymbol{\sigma}}$, and applying (5.3)–(5.4), we conclude (5.6) completing the proof. □

The aforementioned result regarding $\|\lambda - \lambda_h\|_{1,\Omega}$ is established as follows.

Lemma 5.2 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Assume further that the data satisfy* (5.5)*. Then, there exists a positive constant $C_2(r)$, independent of h, such that*

$$\begin{aligned}\|\lambda - \lambda_h\|_{1,\Omega} \,\leq\, C_2(r)\Bigg\{ \sum_{j\in\{\mathrm{p},2\}} \big(\mathrm{dist}\,(\mathbf{u},\mathbf{H}_h^{\mathbf{u}}) + \mathrm{dist}\,(\mathbf{t},\mathbb{H}_h^{\mathbf{t}}) + \mathrm{dist}\,(\mathbf{b},\mathbf{H}_h^{\mathbf{b}})\big)^{j-1} \\ + \mathrm{dist}\,(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}) + \mathrm{dist}\,(\lambda,\mathrm{H}_h^{\lambda})\Bigg\}. \end{aligned} \tag{5.18}$$

Proof. Let ξ_h be an arbitrary element in H_h^λ. From the discrete inf-sup condition (4.3) and simple computations, we have that

$$\beta_m\,\|\xi_h - \lambda_h\|_{1,\Omega} \,\leq\, \sup_{\substack{\mathbf{d}_h\in\mathbf{H}_h^{\mathbf{b}}\\ \mathbf{d}_h\neq 0}} \frac{[b_m(\mathbf{d}_h),\xi_h-\lambda] + [b_m(\mathbf{d}_h),\lambda-\lambda_h]}{\|\mathbf{d}_h\|}\,. \tag{5.19}$$

In turn, subtracting (4.2c) to (2.15c) and after adding and subtracting suitable terms there holds

$$[b_m(\mathbf{d}_h),\lambda-\lambda_h] \,=\, -\,[a_m(\mathbf{b}-\mathbf{b}_h),\mathbf{d}_h] \,+\, [c_m(\mathbf{b}_h-\mathbf{b})(\mathbf{u}_h),\mathbf{d}_h] \,+\, [c_m(\mathbf{b})(\mathbf{u}_h-\mathbf{u}),\mathbf{d}_h]\,, \tag{5.20}$$

for all $\mathbf{d}_h\in\mathbf{H}_h^{\mathbf{b}}$. Next, using (3.5), (3.11) and (4.12) to bound, respectively, the three terms on the right-hand side of (5.20), we deduce that

$$\big|[b_m(\mathbf{d}_h),\lambda-\lambda_h]\big| \,\leq\, \left\{\frac{1}{\varrho\,\mu^2}\,\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\|_{\mathbf{X}} + \Big(\frac{C_{s,\mathtt{d}}}{\mu}\|\vec{\mathbf{u}}_h\|_{\mathbf{X}} + \frac{C_s}{\mu}\,\|\vec{\mathbf{u}}\|_{\mathbf{X}}\Big)\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\|_{\mathbf{X}}\right\}\,\|\mathbf{d}_h\|_{\mathrm{curl};\Omega}\,. \tag{5.21}$$

Thus, replacing back (5.21) into (5.19), using triangle inequality, (3.34) and (4.34) in conjunction with (3.30) and (4.31) to bound both $\|\vec{\mathbf{u}}\|_{\mathbf{X}}$ and $\|\vec{\mathbf{u}}_h\|_{\mathbf{X}}$ by r, we get

$$\|\lambda-\lambda_h\|_{1,\Omega} \,\leq\, \|\lambda-\xi_h\|_{1,\Omega} \,+\, \|\xi_h-\lambda_h\|_{1,\Omega} \,\leq\, c(r)\,\Big\{\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\|_{\mathbf{X}} \,+\, \|\lambda-\xi_h\|_{1,\Omega}\Big\}\,, \tag{5.22}$$

with $c(r)$ depending on β_m, ϱ, μ, $C_{s,\mathtt{d}}$, C_s, p, and r. Finally, combining (5.22) and (5.6), and using the fact that $\xi_h\in\mathrm{H}_h^\lambda$ is arbitrary, we conclude (5.18) completing the proof. □

We are now in position of establishing the Céa estimate of (4.2). The aforementioned result follows straightforwardly from Lemmas 5.1 and 5.2.

Theorem 5.3 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Assume further that the data satisfy* (5.5)*. Then, there exists a positive constant $C(r)$, independent of h, but depending on r, $L_{\mathtt{MH,d}}$, $\alpha_{\mathtt{MH,d}}$, ϱ, μ, $C_{s,\mathtt{d}}$, C_s, $L_{\mathtt{MH}}$,* p*, β_m and $\beta_{\mathtt{MH,d}}$, such that*

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_h\|_{0,6;\Omega} + \|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega} + \|\mathbf{b}-\mathbf{b}_h\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\lambda-\lambda_h\|_{1,\Omega} \\ &\quad\leq C(r)\left\{\sum_{j\in\{\mathrm{p},2\}}\big(\mathrm{dist}\,(\mathbf{u},\mathbf{H}_h^{\mathbf{u}}) + \mathrm{dist}\,(\mathbf{t},\mathbb{H}_h^{\mathbf{t}}) + \mathrm{dist}\,(\mathbf{b},\mathbf{H}_h^{\mathbf{b}})\big)^{j-1} + \mathrm{dist}\,(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}) + \mathrm{dist}\,(\lambda,\mathrm{H}_h^{\lambda})\right\}.\end{aligned}$$

In order to establish the rate of convergence of the Galerkin scheme (4.2), we recall next the approximation properties of the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{b}}$ and H_h^λ (cf. (4.1)), whose derivations can be found in [4], [19], [20], [22], [31, Theorem 5.41] and [7, Section 3.1] (see also [12, Section 5]):

$(\mathbf{AP})_{\mathtt{BF}}$: there exist positive constants C_1, C_2, and C_3, independent of h, such that for each $\mathbf{v} \in \mathbf{W}^{k+1,6}(\Omega)$, $\mathbf{s} \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and $\boldsymbol{\tau} \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{k+1,6/5}(\Omega)$, there hold

$$\mathrm{dist}\,(\mathbf{v}, \mathbf{H}^{\mathbf{u}}_h) := \inf_{\mathbf{v}_h \in \mathbf{H}^{\mathbf{u}}_h} \|\mathbf{v} - \mathbf{v}_h\|_{0,6;\Omega} \,\le\, C_1\, h^{k+1}\, \|\mathbf{v}\|_{k+1,6;\Omega}\,,$$

$$\mathrm{dist}\,(\mathbf{s}, \mathbb{H}^{\mathbf{t}}_h) := \inf_{\mathbf{r}_h \in \mathbb{H}^{\mathbf{t}}_h} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \,\le\, C_2\, h^{k+1}\, \|\mathbf{s}\|_{k+1,\Omega}\,,$$

and

$$\mathrm{dist}\,(\boldsymbol{\tau}, \mathbb{H}^{\boldsymbol{\sigma}}_h) := \inf_{\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_h} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \,\le\, C_3\, h^{k+1} \Big\{ \|\boldsymbol{\tau}\|_{k+1,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{k+1,6/5;\Omega} \Big\}\,.$$

$(\mathbf{AP})_{\mathtt{M}}$: there exist positive constants C_4 and C_5, independent of h, such that for each $\mathbf{d} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0(\mathrm{curl};\Omega)$ with $\mathrm{curl}(\mathbf{d}) \in \mathbf{H}^{k+1}(\Omega)$, and $\xi \in \mathrm{H}^{k+2}(\Omega) \cap \mathrm{H}^1_0(\Omega)$, there hold

$$\mathrm{dist}\,(\mathbf{d}, \mathbf{H}^{\mathbf{b}}_h) := \inf_{\mathbf{d}_h \in \mathbf{H}^{\mathbf{b}}_h} \|\mathbf{d} - \mathbf{d}_h\|_{\mathrm{curl};\Omega} \,\le\, C_4\, h^{k+1} \Big\{ \|\mathbf{d}\|_{k+1,\Omega} + \|\mathrm{curl}(\mathbf{d})\|_{k+1,\Omega} \Big\}\,,$$

and

$$\mathrm{dist}\,(\xi, \mathrm{H}^{\lambda}_h) := \inf_{\xi_h \in \mathrm{H}^{\lambda}_h} \|\xi - \xi_h\|_{1,\Omega} \,\le\, C_5\, h^{k+1}\, \|\xi\|_{k+2,\Omega}\,.$$

Now we are in a position to provide the theoretical rate of convergence of the Galerkin scheme (4.2).

Theorem 5.4 *In addition to the hypotheses of Theorems 3.8, 4.9, and 5.3, given an integer $k \ge 0$, assume that $\mathbf{u} \in \mathbf{W}^{k+1,6}(\Omega)$, $\mathbf{t} \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ with $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{k+1,6/5}(\Omega)$, $\mathbf{b} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0(\mathrm{curl};\Omega)$ with $\mathrm{curl}(\mathbf{b}) \in \mathbf{H}^{k+1}(\Omega)$, and $\lambda \in \mathrm{H}^{k+2}(\Omega) \cap \mathrm{H}^1_0(\Omega)$. Then, there exists a positive constant $C_{\mathtt{rate}}$, independent of h, such that*

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|_{0,6;\Omega} + \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\mathbf{b} - \mathbf{b}_h\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\lambda - \lambda_h\|_{1,\Omega} \\
&\quad \le C_{\mathtt{rate}}\, h^{k+1} \Bigg\{ \sum_{j \in \{\mathrm{p},2\}} \Big(\|\mathbf{u}\|_{k+1,6;\Omega} + \|\mathbf{t}\|_{k+1,\Omega} + \|\mathbf{b}\|_{k+1,\Omega} + \|\mathrm{curl}(\mathbf{b})\|_{k+1,\Omega} \Big)^{j-1} \\
&\qquad + \|\boldsymbol{\sigma}\|_{k+1,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{k+1,6/5;\Omega} + \|\lambda\|_{k+2,\Omega} \Bigg\}\,.
\end{aligned}$$

Proof. The result follows from a direct application of Theorem 5.3 and the approximation properties provided by $(\mathbf{AP})_{\mathtt{BF}}$ and $(\mathbf{AP})_{\mathtt{M}}$. Further details are omitted. □

We end this section by introducing suitable approximations for other variables of interest, such as the pressure p, the velocity gradient $\mathbf{G} = \nabla\mathbf{u}$, the vorticity $\boldsymbol{\omega} = \frac{1}{2}\big(\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{t}}\big)$, and the stress $\widetilde{\boldsymbol{\sigma}} := \nu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{t}}) - p\,\mathbb{I}$, are all them written in terms of the solution of the discrete problem (4.2a)–(4.2b). In fact, using (2.5), (2.7), and (2.14), and after simple computations, we deduce that at the continuous level, there hold

$$\begin{gathered}
p = -\frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}) + \frac{\nu}{3}\, g_f - c_0\,, \quad \mathbf{G} = \mathbf{t} + \frac{1}{3}\, g_f\, \mathbb{I}\,, \quad \boldsymbol{\omega} = \frac{1}{2}\big(\mathbf{t} - \mathbf{t}^t\big)\,, \\
\text{and} \quad \widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \nu\, \mathbf{t}^t + \Big(\frac{\nu}{3}\, g_f + c_0\Big)\, \mathbb{I}\,, \quad \text{with} \quad c_0 = \frac{\nu}{3\,|\Omega|} \int_\Omega g_f\,,
\end{gathered} \tag{5.23}$$

provided the discrete solution $(\mathbf{u}_h, \mathbf{t}_h, \mathbf{u}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ of problem (4.2a)–(4.2b), we propose the following approximations for the aforementioned variables:

$$p_h = -\frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}_h) + \frac{\nu}{3}\,g_f - c_0\,, \quad \mathbf{G}_h = \mathbf{t}_h + \frac{1}{3}\,g_f\,\mathbb{I}\,, \quad \boldsymbol{\omega}_h = \frac{1}{2}\left(\mathbf{t}_h - \mathbf{t}_h^t\right), \\ \text{and} \quad \widetilde{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h + \nu\,\mathbf{t}_h^t + \left(\frac{\nu}{3}\,g_f + c_0\right)\mathbb{I}\,. \tag{5.24}$$

The following result, whose proof follows directly from Theorem 5.4, establishes the corresponding approximation result for this post-processing procedure.

Corollary 5.5 *Let* $(\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ *and* $(\mathbf{b}, \lambda) \in \mathbf{H}_0(\mathrm{curl};\Omega) \times \mathrm{H}^1_0(\Omega)$ *be the unique solution of the continuous problem* (2.15)*, and let* $p, \mathbf{G}, \boldsymbol{\omega}$ *and* $\widetilde{\boldsymbol{\sigma}}$ *given by* (5.23)*. In addition, let* $p_h, \mathbf{G}_h, \boldsymbol{\omega}_h$ *and* $\widetilde{\boldsymbol{\sigma}}_h$ *be the discrete counterparts introduced in* (5.24)*. Let an integer* $k \geq 0$ *and assume that the hypotheses of the Theorem 5.4 be hold. Then, there exists a positive constant* $C_{\mathtt{post}}$*, independent of* h*, such that*

$$\begin{aligned} &\|p - p_h\|_{0,\Omega} + \|\mathbf{G} - \mathbf{G}_h\|_{0,\Omega} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \\ &\quad \leq C_{\mathtt{post}}\,h^{k+1}\Bigg\{\sum_{j\in\{\mathrm{p},2\}}\Big(\|\mathbf{u}\|_{k+1,6;\Omega} + \|\mathbf{t}\|_{k+1,\Omega} + \|\mathbf{b}\|_{k+1,\Omega} + \|\mathrm{curl}(\mathbf{b})\|_{k+1,\Omega}\Big)^{j-1} \\ &\qquad + \|\boldsymbol{\sigma}\|_{k+1,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{k+1,6/5;\Omega} + \|\lambda\|_{k+2,\Omega}\Bigg\}. \end{aligned}$$

Proof. Recalling the formulae given in (5.23) and (5.24), and employing suitable algebraic manipulations it is not difficult to show that there exists $C > 0$, independent of h, such that the following estimate holds

$$\|p - p_h\|_{0,\Omega} + \|\mathbf{G} - \mathbf{G}_h\|_{0,\Omega} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \leq C\left\{\|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}\right\}.$$

Then, the result follows straightforwardly from Theorem 5.4. We omit further details. □

6 Numerical results

In this section we report two examples illustrating the performance of the mixed finite element method (4.2), on a set of quasi-uniform triangulations of the respective 3D domains, and considering the finite element subspaces defined by (4.1) (cf. Section 4.1). In what follows, we refer to the corresponding sets of finite element subspaces generated by $k = 0$ as simply $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$. Our implementation is based on a `FreeFemm++` code [27], in conjunction with the direct linear solver `UMFPACK` [17]. In order to solve the nonlinear problem (4.2), given $\mathbf{0} \neq \mathbf{w} \in \mathbf{L}^6(\Omega)$ we introduce the Gâteaux dirivative and functional associated, respectively, to a_f and F_1 (cf. (2.16), (2.20)), that is

$$[\mathcal{D}a_f(\mathbf{w})(\mathbf{u},\mathbf{t}), (\mathbf{v},\mathbf{s})] := \alpha\int_\Omega \mathbf{u}\cdot\mathbf{v} + \mathtt{F}\int_\Omega |\mathbf{w}|^{\mathrm{p}-2}\mathbf{u}\cdot\mathbf{v} + \mathtt{F}(\mathrm{p}-2)\int_\Omega |\mathbf{w}|^{\mathrm{p}-4}(\mathbf{w}\cdot\mathbf{u})(\mathbf{w}\cdot\mathbf{v}) + \nu\int_\Omega \mathbf{t}:\mathbf{s}$$

and

$$[F_1(\mathbf{w}), (\mathbf{v},\mathbf{s})] := [F_1, (\mathbf{v},\mathbf{s})] + \mathtt{F}\,(\mathrm{p}-2)\int_\Omega |\mathbf{w}|^{\mathrm{p}-2}\mathbf{w}\cdot\mathbf{v}\,,$$

for all $(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s}) \in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)$. In this way, we propose the Newton-type strategy: Given $\mathbf{0}\neq\mathbf{u}_h^0\in\mathbf{H}_h^{\mathbf{u}}$, for $i\geq 1$, solve

$$\begin{aligned}
[a_m(\mathbf{b}_h^i),\mathbf{d}_h] + [c_m(\mathbf{b}_h^i)(\mathbf{u}_h^{i-1}),\mathbf{d}_h] + [b_m(\mathbf{d}_h),\lambda_h^i] &= [F_3,\mathbf{d}_h]\,,\\
[b_m(\mathbf{b}_h^i),\xi_h] &= 0\,,
\end{aligned}\tag{6.1}$$

for all $\mathbf{d}_h\in\mathbf{H}_h^{\mathbf{b}}$ and $\xi_h\in\mathrm{H}_h^{\lambda}$, and

$$\begin{aligned}
[\mathcal{D}a_f(\mathbf{u}_h^{i-1})(\mathbf{u}_h^i,\mathbf{t}_h^i),(\mathbf{v}_h,\mathbf{s}_h)] + [b_f(\mathbf{v}_h,\mathbf{s}_h),\boldsymbol{\sigma}_h^i] &= [F_1(\mathbf{u}_h^{i-1}),(\mathbf{v}_h,\mathbf{s}_h)] - [c_f(\mathbf{b}_h^i)(\mathbf{b}_h^i),\mathbf{v}_h]\,,\\
[b_f(\mathbf{u}_h^i,\mathbf{t}_h^i),\boldsymbol{\tau}_h] &= [F_2,\boldsymbol{\tau}_h]\,,
\end{aligned}\tag{6.2}$$

for all $(\mathbf{v}_h,\mathbf{s}_h)\in\mathbf{H}_h^{\mathbf{u}}\times\mathbb{H}_h^{\mathbf{t}}$ and $\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}$. More precisely, we first solve the linear system (6.1) with the given $\mathbf{u}_h^0$, whose solution is denoted $(\mathbf{b}_h^1,\lambda_h^1)$. Next, we solve (6.2) with the given $(\mathbf{u}_h^0,\mathbf{b}_h^1)$, so that, starting from $\mathbf{u}_h^0 := (0, 1\mathrm{E}-6, 0)^{\mathrm{t}}$, we perform just one Newton iteration to obtain $(\mathbf{u}_h^1,\mathbf{t}_h^1,\boldsymbol{\sigma}_h^1)$ as an approximate solution of it. Then, the process continues with $\mathbf{u}_h^i$ for each $i\geq 1$. In this way, for a fixed tolerance $\mathsf{tol} = 1\mathrm{E}-6$, the above iterations are terminated, which yields the number of Newton iterations reported in the tables below, once the relative error between two consecutive iterates, say $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1}-\mathbf{coeff}^m\|}{\|\mathbf{coeff}^{m+1}\|} \leq \mathsf{tol}\,,$$

where $\|\cdot\|$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DOF}}$ with $\mathtt{DOF}$ denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{b}}$ and H_h^{λ}.

We now introduce some additional notations. The individual errors are denoted by:

$$\begin{gathered}
\mathrm{e}(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{0,6;\Omega}\,,\quad \mathrm{e}(\mathbf{t}) := \|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega}\,,\quad \mathrm{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}\,,\\
\mathrm{e}(\mathbf{b}) := \|\mathbf{b}-\mathbf{b}_h\|_{\mathrm{curl};\Omega}\,,\quad \mathrm{e}(\lambda) := \|\lambda-\lambda_h\|_{1,\Omega}\,,\quad \mathrm{e}(p) := \|p-p_h\|_{0,\Omega}\,,\\
\mathrm{e}(\mathbf{G}) := \|\mathbf{G}-\mathbf{G}_h\|_{0,\Omega}\,,\quad \mathrm{e}(\boldsymbol{\omega}) := \|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{0,\Omega}\,,\quad \mathrm{e}(\widetilde{\boldsymbol{\sigma}}) := \|\widetilde{\boldsymbol{\sigma}}-\widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,,
\end{gathered}$$

where the pressure p, the velocity gradient $\mathbf{G}$, the vorticity $\boldsymbol{\omega}$, and the shear stress tensor $\widetilde{\boldsymbol{\sigma}}$ are further variables of physical interest that are recovered by using the corresponding postprocessing formulae p_h, $\mathbf{G}_h$, $\boldsymbol{\omega}_h$, and $\widetilde{\boldsymbol{\sigma}}_h$ detailed in (5.23)–(5.24). Next, as usual, for each $\star\in\Big\{\mathbf{u},\mathbf{t},\boldsymbol{\sigma},\mathbf{b},\lambda,p,\mathbf{G},\boldsymbol{\omega},\widetilde{\boldsymbol{\sigma}}\Big\}$ we let $\mathsf{r}(\star)$ be the experimental rate of convergence given by

$$\mathsf{r}(\star) := \frac{\log(\mathrm{e}(\star)/\mathrm{e}'(\star))}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with errors e and e′ respectively.

The examples to be considered in this section are described next. In all them we take for sake of simplicity $\nu=1,\mu=1,\alpha=1,\mathtt{F}=10$, and $\varrho=1$, and set the vector $\mathbf{1} := (1,1,1)^{\mathrm{t}}\in\mathrm{R}^3$. In addition, the mean value of $\mathrm{tr}(\boldsymbol{\sigma}_h^i)$ over Ω, with $i\geq 1$, is fixed via a Lagrange multiplier strategy (adding one row and one column to the matrix system that solves (6.2) for $\mathbf{u}_h^i$, $\mathbf{t}_h^i$, and $\boldsymbol{\sigma}_h^i$).

Example 1: Accuracy assessment with a smooth solution in a convex domain.

In the first example we illustrate the performance of the Galerkin scheme (6.1)–(6.2) (cf. (4.2)) in a convex domain. We consider the domain $\Omega := (0,1)\times(0,0.5)\times(0,0.5)$, the inertial power $\mathrm{p}=3$, and choose the data $\mathbf{f}_f$, $\mathbf{f}_m$, g_f and $\mathbf{u}_{\mathrm{D}}$ such that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \sin(\pi x_1)\,\cos(\pi x_2)\,\cos(\pi x_3) \\ -2\cos(\pi x_1)\,\sin(\pi x_2)\,\cos(\pi x_3) \\ \cos(\pi x_1)\,\cos(\pi x_2)\,\sin(3\pi x_3) \end{pmatrix}, \quad p(\mathbf{x}) = x_2\,x_3\,(x_1 - 0.5)\,,$$

$$\mathbf{b}(\mathbf{x}) = \mathrm{curl}\big(x_1^2\,(x_2-0.5)^2 x_3^2 \cos(\pi x_3)^2 \mathbf{1}\big)\,, \quad \lambda(\mathbf{x}) = x_1\,x_2\,x_3\,(x_2-0.5)(x_3-0.5)(x_1-1)\,.$$

The model problem is then complemented with the appropriate Dirichlet boundary conditions. Table 6.1 shows the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations. Notice that we are able not only to approximate the original unknowns but also the pressure field, the velocity gradient tensor, the vorticity, and the shear stress tensor through the formula (5.24). Note also that $\mathrm{e}(\mathbf{t}) = \mathrm{e}(\mathbf{G})$ since $\mathbf{t}$ (resp. $\mathbf{t}_h$) is just a translation of $\mathbf{G}$ (resp. $\mathbf{G}_h$). The results confirm that the optimal rates of convergence $\mathcal{O}(h^{k+1})$ predicted by Theorem 5.4 and Corollary 5.5 are attained for $k = 0$. The Newton method exhibits a behavior independent of the meshsize, converging in four iterations in all cases. In Figure 6.1 we display some solutions obtained with the five-field mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ approximation with meshsize $h = 0.0505$ and $32,928$ tetrahedra elements (actually representing $613,593$ DOF).

Example 2: Accuracy assessment with a smooth solution in a non-convex domain.

In the second example we test the iterative method (6.1)–(6.2) (cf. (4.2)) in a non-convex domain. In fact, we consider the Fichera's corner domain $\Omega := (-1,1)^3 \setminus [0,1)^3$, where, due to the regularity of the Neumann problem (see [15] and [16] for details), there holds $\mathbf{H}_0(\mathrm{curl};\Omega) \cap \mathbf{H}(\mathrm{div}^0;\Omega) \subseteq \mathbf{H}^s(\Omega)$ for $s \in \big(1/2, 2/3\big)$. We consider the inertial power $\mathrm{p} = 4$ and choose the data $\mathbf{f}_f$, $\mathbf{f}_m$, g_f and $\mathbf{u}_{\mathrm{D}}$ so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} -\,x_1\,(x_2-x_3)\,(x_2+x_3) \\ 2x_2\,(x_1-x_3)\,(x_1+x_3) \\ -\,x_3\,(x_1-x_2)\,(x_1+x_2) \end{pmatrix}, \quad p(\mathbf{x}) := x_1\,x_2\,x_3 - c_p\,,$$

$$\mathbf{b}(\mathbf{x}) := \mathrm{curl}\big(\sin^2(\pi x_1)\,\sin^2(\pi x_2)\,\sin^2(\pi x_3)\mathbf{1}\big)\,, \quad \lambda(\mathbf{x}) := \sin(\pi x_1)\,\sin(\pi x_2)\,\sin(\pi x_3)\,,$$

where $c_p \in \mathrm{R}$ is chosen in such a way $p \in \mathrm{L}^2_0(\Omega)$. The convergence history for a set of quasi-uniform mesh refinements using $k = 0$ is shown in Table 6.2. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$, as it was proved by Theorem 5.4 and Corollary 5.5. In addition, some components of the numerical solution are displayed in Figure 6.2, which were built using the five-field mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ approximation with meshsize $h = 0.1414$ and $42,000$ tetrahedra elements (actually representing $782,121$ DOF). We observe that for this example the second diagonal components of the velocity gradient and its translation, namely $\mathbf{G}_{22,h}$ and $\mathbf{t}_{22,h}$, look quite similar since they only differ in the term $\frac{1}{3}\,g_f$ (cf. (5.24)), with $g_f = \mathrm{div}(\mathbf{u}) = x_1^2 - x_3^2$ small in Ω.

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DOF	h	iter	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{b}$)	r($\mathbf{b}$)
1977	0.3536	4	0.3140	–	1.0919	–	6.0263	–	0.0518	–
28791	0.1414	4	0.1405	0.877	0.4788	0.900	2.6277	0.906	0.0235	0.864
115905	0.0884	4	0.0889	0.974	0.3055	0.956	1.6499	0.990	0.0148	0.977
298959	0.0643	4	0.0649	0.990	0.2241	0.974	1.1985	1.004	0.0108	0.991
613593	0.0505	4	0.0510	0.995	0.1768	0.983	0.9402	1.006	0.0085	0.995

e(λ)	r(λ)	e(p)	r(p)	e($\mathbf{G}$)	r($\mathbf{G}$)	e($\boldsymbol{\omega}$)	r($\boldsymbol{\omega}$)	e($\tilde{\boldsymbol{\sigma}}$)	r($\tilde{\boldsymbol{\sigma}}$)
0.0014	–	0.4359	–	1.0919	–	0.5534	–	2.0283	–
0.0006	0.853	0.2133	0.780	0.4788	0.900	0.2463	0.883	0.9004	0.886
0.0004	0.964	0.1272	1.100	0.3055	0.956	0.1565	0.965	0.5692	0.976
0.0003	0.984	0.0879	1.158	0.2241	0.974	0.1145	0.981	0.4143	0.998
0.0002	0.991	0.0665	1.158	0.1768	0.983	0.0902	0.988	0.3252	1.004

Table 6.1: [Example 1] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the five-field mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ approximation for the coupling of the Brinkman–Forchheimer and Maxwell equations.

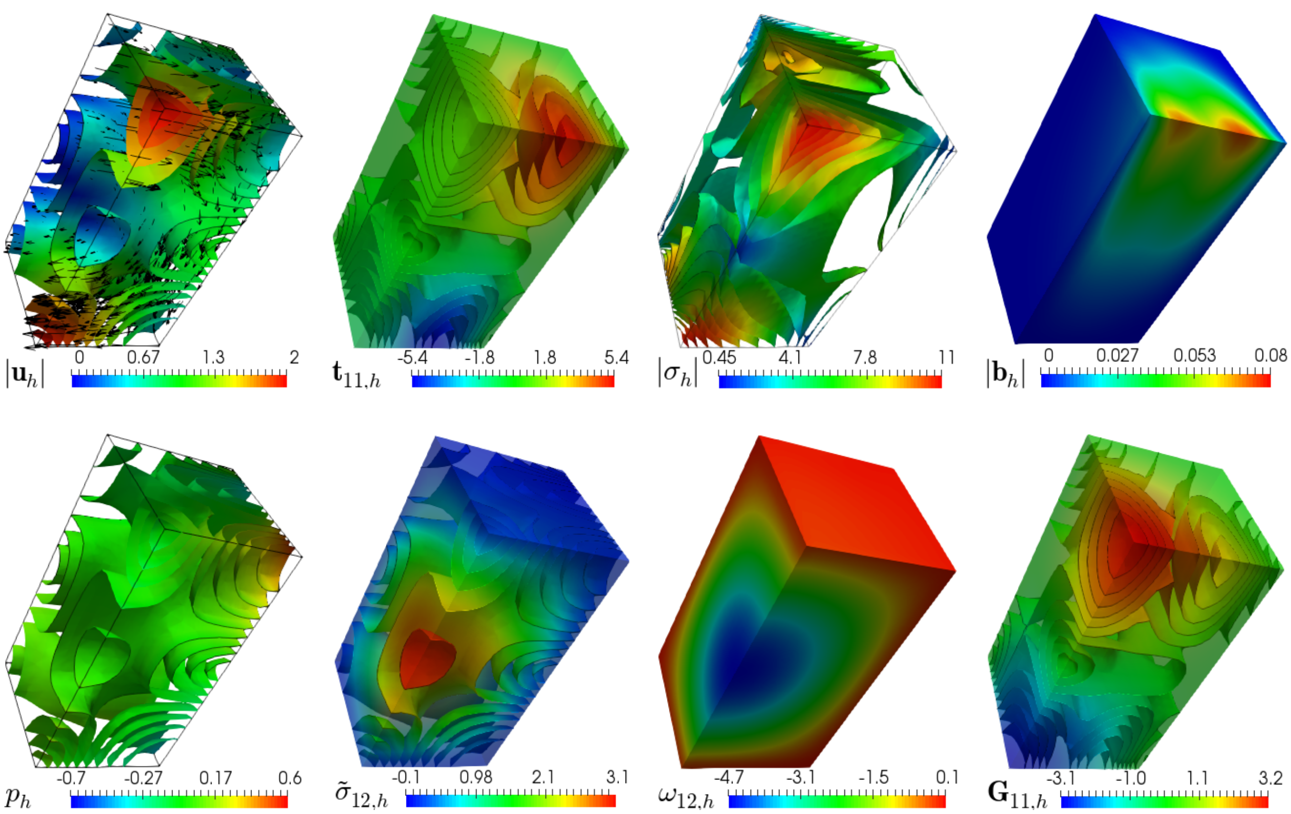

Figure 6.1: [Example 1] Computed magnitude of the velocity, translation of the velocity gradient component, computed magnitude of the pseudostress tensor and magnetic field (top plots); pressure field, shear stress tensor component, vorticity component, and velocity gradient component (bottom plots).

DOF	h	iter	$\mathrm{e}(\mathbf{u})$	$\mathsf{r}(\mathbf{u})$	$\mathrm{e}(\mathbf{t})$	$\mathsf{r}(\mathbf{t})$	$\mathrm{e}(\boldsymbol{\sigma})$	$\mathsf{r}(\boldsymbol{\sigma})$	$\mathrm{e}(\mathbf{b})$	$\mathsf{r}(\mathbf{b})$
6665	0.7071	5	0.6801	–	4.6074	–	84.4548	–	39.3474	–
51249	0.3536	6	0.3526	0.948	1.7824	1.370	47.1319	0.841	18.3178	1.103
170713	0.2357	6	0.2371	0.979	0.9743	1.490	30.2412	1.094	12.5493	0.933
402017	0.1768	6	0.1784	0.988	0.6758	1.271	22.7760	0.985	9.5022	0.967
782121	0.1414	6	0.1430	0.993	0.5186	1.187	18.2251	0.999	7.6352	0.980

$\mathrm{e}(\lambda)$	$\mathsf{r}(\lambda)$	$\mathrm{e}(p)$	$\mathsf{r}(p)$	$\mathrm{e}(\mathbf{G})$	$\mathsf{r}(\mathbf{G})$	$\mathrm{e}(\boldsymbol{\omega})$	$\mathsf{r}(\boldsymbol{\omega})$	$\mathrm{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathsf{r}(\widetilde{\boldsymbol{\sigma}})$
4.0629	–	4.0174	–	4.6074	–	2.6043	–	10.3055	–
2.4151	0.750	1.1982	1.745	1.7824	1.370	1.1494	1.180	3.4249	1.589
1.6693	0.911	0.6539	1.494	0.9743	1.490	0.6562	1.383	1.8323	1.543
1.2684	0.955	0.4065	1.652	0.6758	1.271	0.4623	1.218	1.2116	1.438
1.0210	0.973	0.2758	1.738	0.5186	1.187	0.3570	1.159	0.8913	1.376

Table 6.2: [Example 2] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the five-field mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ approximation for the coupling of the Brinkman–Forchheimer and Maxwell equations.

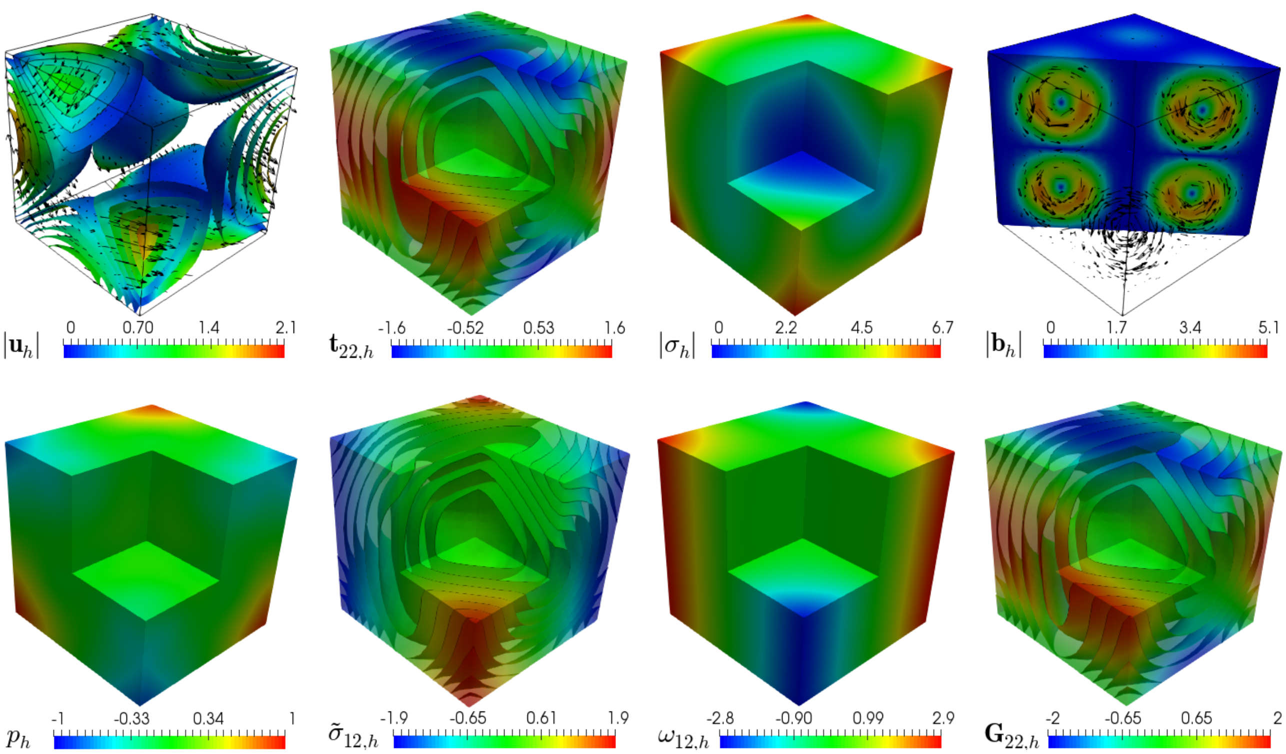

Figure 6.2: [Example 2] Computed magnitude of the velocity, translation of the velocity gradient component, computed magnitude of the pseudostress tensor, and magnetic field (top plots); pressure field, shear stress tensor component, vorticity component, and velocity gradient component (bottom plots).

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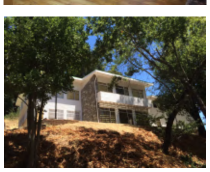

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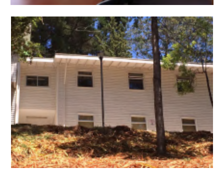

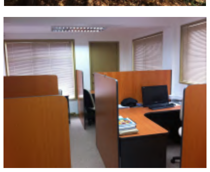

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