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## A conforming and mass conservative pseudostress-based mixed finite element method for Stokes


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# A conforming and mass conservative pseudostress-based mixed finite element method for Stokes *

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**Abstract**

In this paper, we propose a mass conservative pseudostress-based finite element method for solving the Stokes problem with both Dirichlet and mixed boundary conditions. We decompose the velocity by means of a Helmholtz decomposition and derive a three-field mixed variational formulation, where the pseudostress, the velocity, both in $\mathbf{H}(\mathrm{div})$, and an additional unknown representing the null function, are the main unknowns of the system. By employing suitable finite element spaces, the velocity is approximated using $\mathbf{H}(\mathrm{div})$-conforming finite elements, ensuring the desired mass conservation property. The proposed method offers several advantages, including simplicity of implementation and compatibility with existing software packages for partial differential equation solvers. Additionally, we extend the study to incorporate mixed boundary conditions for the Stokes problem and complement the analysis with the introduction of a reliable and efficient residual-based a posteriori error estimator. Numerical examples are provided to validate the theoretical results, demonstrating the effectiveness and accuracy of the proposed method.

**Key words**: Stokes problem; mass conservation; conforming scheme, mixed finite element method; Raviart–Thomas elements; BDM elements

**Mathematics Subject Classifications (1991)**: 65N15, 65N30, 76D05, 76M10

## 1 Introduction

The Navier-Stokes (NS) problem is a challenging problem in numerical analysis, and the scientific community has been focused on proposing more accurate and efficient methods to obtain good approximations of its solution for several decades. In its most basic form, NS is written as

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follows:

$$\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma := \partial\Omega, \quad \int_\Omega p = 0, \tag{1.1}$$

where the vector field $\mathbf{u} := (u_1, \ldots, u_d)^t$ and the scalar field $p$ are the velocity and pressure, respectively, of a fluid with a given viscosity $\nu > 0$ occupying a domain $\Omega \subseteq \mathbb{R}^d$ $(d = 2, 3)$. Here, $\mathbf{f}$ is a given source force, and $\mathbf{u}_D$ is a prescribed velocity on the boundary $\Gamma := \partial\Omega$ satisfying the compatibility condition:

$$\int_\Gamma \mathbf{u}_D \cdot \mathbf{n} = 0, \tag{1.2}$$

where $\mathbf{n}$ denotes the outward unit normal vector on $\Gamma$. The first equation of (1.1) represents the conservation of momentum of the system, whereas the second one represents the conservation of mass.

For several years, the most commonly employed methods by engineers to approximate the solution of (1.1) are based on conforming discretizations of the classical velocity-pressure formulation, mainly because they are relatively cheap and easy to implement. Actually, in most of the software designed to solve partial differential equations, such as *Freefem++* and *Fenics*, the classical families of finite elements for (1.1) are already available (see [21] for a detailed study of these classical families). However, since the aforementioned conservation laws are imposed weakly, these quantities are not exactly preserved, yielding instabilities of the numerical schemes (see, for instance, [23] and [28]).

To study the mass conservation property of (1.1), one can restrict the analysis to the simplified Stokes model:

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \qquad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma, \quad \int_\Omega p = 0. \tag{1.3}$$

In [26], the authors improved the lack of conservation of mass of conforming velocity-pressure formulations for (1.1) by introducing a family of divergence-free conforming finite elements for (1.3) on general triangular meshes in two dimensions. There, the divergence-free property is attained by enriching the polynomial space for the velocity with suitable rational functions, which makes the computational implementation more difficult and increases the computational cost. The approach in [26] was later extended to the three-dimensional case in [27].

Another possible approach for obtaining mass conservative numerical methods for (1.1) is to use nonconforming schemes, as demonstrated in [10], where the authors introduced a discontinuous Galerkin scheme for (1.1). There, the divergence constraint is exactly satisfied at the discrete level due to the use of divergence-conforming discrete spaces to approximate the velocity. Another example is the pressure-robust hybridized discontinuous Galerkin method proposed by [24].

On the other hand, several approaches based on reformulations of (1.1) or (1.3) have also been considered to improve the lack of conservation of laws. For instance, in [12] (see also [2]), the introduction of the vorticity as a further unknown enables the authors to obtain a variational formulation of (1.3) with the velocity in the Hilbert space $\mathbf{H}(\operatorname{div};\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div}\mathbf{v} \in \mathrm{L}^2(\Omega)\}$. This approach allows the velocity to be approximated by $\mathbf{H}(\operatorname{div};\Omega)$-conforming finite elements, thereby obtaining a conforming, mass conservative numerical scheme. However, the approach in [12] has not been extended to the Navier-Stokes problem (1.1), as the convective term does not allow the same spaces to be used for the variables introduced in [12].

More recently, in [22], the gradient of the velocity is introduced as an additional unknown, leading to a new formulation where the velocity can be approximated using $\mathbf{H}(\mathrm{div};\Omega)$-conforming elements and providing exact mass conservation. As in [12], the convective term prevents the extension of [22] to the Navier-Stokes problem (1.1).

An alternative method to approximate the solution of (1.3) is the so called pseudostress-based formulation. This method involves rewriting (1.3) using the pseudostress tensor $\boldsymbol{\sigma} := \nu\nabla\mathbf{u} - p\mathbb{I}$ (refer to [7], [18], [19] and [20] for further information). The momentum equation can be reformulated as $-\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f}$ in $\Omega$, and the divergence constraint implies that $p = -\frac{1}{d}\mathrm{tr}\,(\boldsymbol{\sigma})$. Then, the pressure can be eliminated from the system to obtain the first-order set of equations:

$$\boldsymbol{\sigma}^{\mathrm{d}} = \nu\nabla\mathbf{u} \quad \text{in} \quad \Omega \quad -\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega \quad \text{and} \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma, \quad \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) = 0. \tag{1.4}$$

Here, $\mathbb{I}$ denotes the identity matrix, $\mathrm{tr}\,(\boldsymbol{\sigma})$ is the trace of the tensor $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^{\mathrm{d}} := \boldsymbol{\sigma} - \frac{1}{d}\mathrm{tr}\,(\boldsymbol{\sigma})\mathbb{I}$ denotes the deviatoric part of $\boldsymbol{\sigma}$, and $\mathbf{div}\,\boldsymbol{\tau}$ is the divergence operator div acting along the rows of $\boldsymbol{\tau}$ for any tensor field $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,d}$.

The corresponding variational problem is given by: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}\,;\Omega) \times \mathbf{L}^2(\Omega)$, satisfying the variational equation

$$\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \nu \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} \quad = \nu\langle\boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma, \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega), \tag{1.5}$$

and the differential equation

$$\mathbf{div}\,\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \quad \text{in} \quad \Omega. \tag{1.6}$$

Here,

$$\begin{aligned} \mathbb{H}(\mathbf{div}\,;\Omega) &:= \left\{\boldsymbol{\tau} \in [\mathrm{L}^2(\Omega)]^{d\times d} : \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\right\}, \qquad \mathbf{L}^2(\Omega) := [\mathrm{L}^2(\Omega)]^d, \\ \mathbb{H}_0(\mathbf{div}\,;\Omega) &:= \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega) : \quad \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}) = 0\right\}, \end{aligned}$$

and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the corresponding product of duality between the trace space $\mathrm{H}^{1/2}(\Gamma)$ and its dual $\mathrm{H}^{-1/2}(\Gamma)$.

Based on (1.6), it is evident that conforming discretizations of (1.5)-(1.6) naturally conserve momentum. This is one of the main benefits of this approach. Additionally, other variables of interest, such as the gradient of velocity and vorticity, can be approximated through a simple postprocessing of $\boldsymbol{\sigma}$, without applying any numerical differentiation, thus, avoiding further sources of error. However, there is currently no literature on the conservation of mass of conforming discretizations of (1.5)-(1.6).

Motivated by the above, this paper presents a reformulation of (1.5)-(1.6) to obtain a mass conservative and conforming numerical scheme for the Stokes problem. Specifically, it utilizes a Helmholtz decomposition for $\mathbf{u}$ and derives a three-field mixed variational formulation for (1.4), where the main unknowns of the resulting system are $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$, $\mathbf{u} \in \mathbf{H}(\mathrm{div}^0;\Omega)$ and an additional unknown $\varphi \in \mathrm{H}_0^1(\Omega)$ representing the null function. Here,

$$\begin{aligned} \mathbf{H}(\mathrm{div}^0;\Omega) &:= \{\mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega) : \mathrm{div}\,\mathbf{v} = 0 \quad \text{in} \quad \Omega\}, \\ \mathrm{H}_0^1(\Omega) &:= \{\psi \in \mathrm{H}^1(\Omega) : \psi = 0 \quad \text{on} \quad \Gamma\}. \end{aligned}$$

Consequently, at the discrete level, the velocity can be approximated using $\mathbf{H}(\mathrm{div};\Omega)$-conforming finite elements, which ensures the desired mass conservation property. This constitutes one of the main advantages of this work. Additionally, the resulting Galerkin scheme can be easily implemented in two and three dimensions using softwares such as *Freefem++* and *Fenics*, among others. Furthermore, the study can be straightforwardly extended to the Stokes problem with mixed boundary conditions and can be complemented with a reliable and efficient residual-based a posteriori error estimator. Finally, we observe that, unlike most of the mass conservative numerical schemes available in the literature (see [23]), and similar to classical conforming schemes for Stokes, in this study, the velocity error is amplified by the inverse of the viscosity. This aspect constitutes a topic for further research.

The rest of the article is organized as follows: In Section 2, we introduce the three-field continuous problem and analyze its well-posedness. Then, in Section 3, we propose the mass conservative numerical scheme and study its well-posedness and convergence. Afterward, in Section 4, we derive a residual-based a posteriori error estimator and prove its reliability and efficiency. In Section 5, we extend the previous results to the Stokes problem with mixed boundary conditions.

We end this section by fixing some notations and introducing some preliminary results. We begin by recalling that for any vector field $\mathbf{v} = (v_i)_{i=1,d}$, the differential operators $\nabla\mathbf{v}$ and $\mathrm{div}\,\mathbf{v}$ introduced above, are given by

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,d} \quad \text{and} \quad \mathrm{div}\,\mathbf{v} := \sum_{j=1}^{d} \frac{\partial v_j}{\partial x_j}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,d}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,d}$, the transpose, the trace and the tensor inner product are defined, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,d}, \quad \mathrm{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{d} \tau_{ii} \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{d} \tau_{ij}\zeta_{ij}.$$

For simplicity, in what follows we denote

$$(v,w)_\Omega := \int_\Omega vw, \quad (\mathbf{v},\mathbf{w})_\Omega := \int_\Omega \mathbf{v}\cdot\mathbf{w}, \quad (\mathbf{v},\mathbf{w})_\Gamma := \int_\Gamma \mathbf{u}\cdot\mathbf{v} \quad \text{and} \quad (\boldsymbol{\tau},\boldsymbol{\zeta})_\Omega := \int_\Omega \boldsymbol{\tau} : \boldsymbol{\zeta}.$$

In the sequel, the norms of the well-known Lebesgue and Sobolev spaces $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^1(\Omega)$, will be denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{1,\Omega}$, respectively. We also write $|\cdot|_{1,\Omega}$ for the $\mathrm{H}^1(\Omega)$-seminorm. We additionally recall that the space $\mathbf{H}(\mathrm{div};\Omega)$ equipped with the usual norm $\|\mathbf{v}\|_{\mathrm{div},\Omega} := (\|\mathbf{v}\|_{0,\Omega}^2 + \|\mathrm{div}\,\mathbf{v}\|_{0,\Omega}^2)^{1/2}$ is a Hilbert space, as well as the space $\mathbb{H}(\mathbf{div}\,;\Omega)$ with the norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}\,,\Omega} := (\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\mathbf{div}\;\boldsymbol{\tau}\|_{0,\Omega}^2)^{1/2}. \tag{1.7}$$

Finally, by $\mathbf{S}$ and $\mathbb{S}$ we denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space $S$.

# 2 The continuous mass conservative mixed formulation

## 2.1 Derivation of the mass conservative mixed variational formulation

Here, we derive the mass conservative variational formulation and define the bilinear forms and functionals involved. To that end, let us introduce the auxiliary problem: Find $(\boldsymbol{\sigma}, \mathbf{u}, \varphi) \in$

$\mathbb{X}_0 \times \mathbf{V}_0 \times \Psi_0$ such that

$$(\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau})_\Omega + \nu(\mathbf{div}\ \boldsymbol{\tau}, \mathbf{u} + \nabla\varphi)_\Omega = \nu\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma \,-\, (\mathbf{f}, \mathbf{div}\,\boldsymbol{\tau})_\Omega \tag{2.1}$$

$\forall\, \boldsymbol{\tau} \in \mathbb{X}$, and

$$\mathbf{div}\ \boldsymbol{\sigma}\ + \mathbf{f} = \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.2}$$

where

$$\mathbb{X} := \mathbb{H}(\mathbf{div}\,;\Omega), \quad \mathbb{X}_0 := \mathbb{H}_0(\mathbf{div}\,;\Omega), \quad \mathbf{V}_0 := \mathbf{H}(\mathrm{div}^0;\Omega) \quad \text{and} \quad \Psi_0 := \mathrm{H}^1_0(\Omega). \tag{2.3}$$

Observe that if $(\boldsymbol{\sigma}, \mathbf{u}, \varphi) \in \mathbb{X}_0 \times \mathbf{V}_0 \times \Psi_0$ is a solution to (2.1)-(2.2), then it is clear that $(\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau})_\Omega$
$= -(\mathbf{f}, \mathbf{div}\,\boldsymbol{\tau})_\Omega$, for all $\boldsymbol{\tau} \in \mathbb{X}$, and then, taking $\boldsymbol{\tau} = \psi\mathbb{I}$, with $\psi \in \Psi_0$ in (2.1), and using the fact that $\boldsymbol{\tau}^{\mathrm{d}} = (\psi\mathbb{I})^{\mathrm{d}} = 0$, $\langle \psi\mathbf{n}, \mathbf{u}_D\rangle = 0$ and $\mathbf{div}\ (\psi\mathbb{I}) = \nabla\psi$, it follows that

$$(\nabla\varphi, \nabla\psi)_\Omega \,=\, 0\,, \quad \forall \psi \in \Psi_0,$$

which implies that $\varphi = 0$ in $\Omega$ and $(\boldsymbol{\sigma}, \mathbf{u})$ is a solution to (1.5)-(1.6).

Conversely, if $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X}_0 \times \mathbf{L}^2(\Omega)$ is a solution to (1.5)-(1.6), from the Helmholtz decomposition

$$\mathbf{L}^2(\Omega) = \mathbf{V}_0 \oplus \nabla\Psi_0, \tag{2.4}$$

it readily follows that the velocity $\mathbf{u}$ can decomposed as follows

$$\mathbf{u} = \mathbf{w} + \nabla\varphi \quad \text{in} \quad \Omega,$$

with $\mathbf{w} \in \mathbf{V}_0$ and $\varphi \in \Psi_0$. Then, noticing that (1.6) implies that $(\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau})_\Omega = -(\mathbf{f}, \mathbf{div}\,\boldsymbol{\tau})_\Omega$, for all $\boldsymbol{\tau} \in \mathbb{X}$, it is easy to see that $(\boldsymbol{\sigma}, \mathbf{w}, \varphi)$ satisfies equations (2.1)-(2.2). But, proceeding exactly as above one can deduce that $\varphi = 0$ in $\Omega$, which implies that $\mathbf{u} = \mathbf{w} \in \mathbf{V}_0$, and therefore $(\boldsymbol{\sigma}, \mathbf{u}, \varphi)$ is a solution to (2.1)-(2.2). In this way, we have proved the following lemma

**Lemma 2.1** *If* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X}_0 \times \mathbf{L}^2(\Omega)$ *is a solution to* (1.5)-(1.6)*, then* $\mathbf{u} \in \mathbf{V}_0$ *and* $(\boldsymbol{\sigma}, \mathbf{u}, 0)$ *is a solution to* (2.1)-(2.2)*. Conversely, if* $(\boldsymbol{\sigma}, \mathbf{u}, \varphi) \in \mathbb{X}_0 \times \mathbf{V}_0 \times \Psi_0$ *is a solution to* (2.1)-(2.2)*, then* $\varphi = 0$ *in* $\Omega$ *and* $(\boldsymbol{\sigma}, \mathbf{u})$ *is a solution to* (1.5)-(1.6)*.*

As a consequence of the previous lemma, in what follows we focus on studying and discretizing the system (2.1)-(2.2). To do that we first recall that the space $\mathbb{X}$ (cf. (2.3)) can be decomposed as follows:

$$\mathbb{X} = \mathbb{X}_0 \,\oplus\, P_0(\Omega)\mathbb{I},$$

where $P_0(\Omega)$ is the space of constant polynomials on $\Omega$. More precisely, each $\boldsymbol{\tau} \in \mathbb{X}$ can be decomposed uniquely as:

$$\boldsymbol{\tau} \,=\, \boldsymbol{\tau}_0 \,+\, c\,\mathbb{I}\,, \quad \text{with} \quad \boldsymbol{\tau}_0 \,\in\, \mathbb{X}_0 \quad \text{and} \quad c \,:=\, \frac{1}{d\,|\Omega|}(\mathrm{tr}\ \boldsymbol{\tau}, 1)_\Omega \,\in\, \mathbb{R}\,.$$

Then, owing to the compatibility condition (1.2), after simple computations we realize that the test space for (2.1) can be equivalently reduced to $\mathbb{X}_0$. In turn, as a consequence of the Helmholtz decomposition (2.4), equation (2.2) can be equivalently imposed weakly as follows:

$$(\mathbf{div}\ \boldsymbol{\sigma}, \mathbf{v} + \nabla\psi)_\Omega \,=\, -(\mathbf{f}, \mathbf{v} + \nabla\psi)_\Omega \qquad \forall\, (\mathbf{v}, \psi) \in \mathbf{V}_0 \times \Psi_0.$$

In this way, defining the bilinear forms $\mathbf{a} : \mathbb{X}_0 \times \mathbb{X}_0 \to \mathbb{R}$ and $\mathbf{b} : \mathbb{X}_0 \times \mathbf{M} \to \mathbb{R}$ and the functionals $F : \mathbb{X}_0 \to \mathbb{R}$ and $G : \mathbf{M} \to \mathbb{R}$, respectively as follows

$$\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := (\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{div}\, \boldsymbol{\sigma}, \mathbf{div}\, \boldsymbol{\tau})_\Omega, \quad \mathbf{b}(\boldsymbol{\tau}, (\mathbf{v}, \psi)) := \nu(\mathbf{div}\ \boldsymbol{\tau}, \mathbf{v} + \nabla\psi)_\Omega, \tag{2.5}$$

$$F(\boldsymbol{\tau}) \ := \ \nu\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma - (\mathbf{f}, \mathbf{div}\, \boldsymbol{\tau})_\Omega \quad \text{and} \quad G(\mathbf{v}, \psi) \ := \ -\nu(\mathbf{f}, \mathbf{v} + \nabla\psi)_\Omega,$$

where

$$\mathbf{M} = \mathbf{V}_0 \times \Psi_0,$$

we rewrite (2.1)-(2.2) equivalently as the variational problem: Find $(\boldsymbol{\sigma}, (\mathbf{u}, \varphi)) \in \mathbb{X}_0 \times \mathbf{M}$, such that:

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \ + \ \mathbf{b}(\boldsymbol{\tau}, (\mathbf{u}, \varphi)) \quad &= \ F(\boldsymbol{\tau}) \quad\ \ \forall\, \boldsymbol{\tau} \in \mathbb{X}_0, \\ \mathbf{b}(\boldsymbol{\sigma}, (\mathbf{v}, \psi)) \quad &= \ G(\mathbf{v}, \psi) \quad \forall\, (\mathbf{v}, \psi) \in \mathbf{M}. \end{aligned} \tag{2.6}$$

**Remark 2.2** *Later on, in Remark 3.4, we provide more details on the introduction of the term* $(\mathbf{div}\, \boldsymbol{\sigma}, \mathbf{div}\, \boldsymbol{\tau})_\Omega$ *in the definition of the bilinear form* $\mathbf{a}$*.*

## 2.2 Analysis of the continuous problem

In this section we prove the well–posedness of problem (2.6). To that end, we first recall from [14, Lemma 2.3] that the following inequality holds

$$C_d \|\boldsymbol{\tau}\|_{0,\Omega}^2 \leq \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}\ \boldsymbol{\tau}\|_{0,\Omega}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0, \tag{2.7}$$

which in particular implies that the seminorm

$$|\boldsymbol{\tau}|_{\mathbf{div},\Omega} := \left\{ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega}^2 \right\}^{1/2} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0,$$

is a norm in $\mathbb{X}_0$, equivalent to the norm (1.7). According to this, in what follows we equip the space $\mathbb{X}_0$ with the norm $|\cdot|_{\mathbf{div},\Omega}$. In turn, for the sake of the forthcoming analysis, and using the fact that the well-known Poincaré inequality (see eg. [13, B.61])

$$\|w\|_{0,\Omega} \leq C_P |w|_{1,\Omega} \quad \forall w \in \mathrm{H}_0^1(\Omega),$$

with $C_P > 0$ depending only on $|\Omega|$, implies that the seminorm $|\cdot|_{1,\Omega}$ is equivalent to the norm $\|\cdot\|_{1,\Omega}$ in $\mathrm{H}_0^1(\Omega)$, in what follows we endow the product space $\mathbf{M}$ with the viscosity-dependent norm

$$\|(\mathbf{v}, \psi)\|_{\mathbf{M},\nu} = \nu\{\|\mathbf{v}\|_{\mathrm{div};\Omega}^2 + |\psi|_{1,\Omega}^2\}^{1/2}.$$

As wee shall see next, this norm will allows us to derive the stability properties of the bilinear forms involved, with constants independent of $\nu$. In fact, from the Cauchy-Schwartz inequality we can easily deduce that

$$|\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau})| \leq |\boldsymbol{\sigma}|_{\mathbf{div},\Omega} |\boldsymbol{\tau}|_{\mathbf{div},\Omega} \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{X}_0,$$

$$|\mathbf{b}(\boldsymbol{\tau}, (\mathbf{v}, \psi))| \leq |\boldsymbol{\tau}|_{\mathbf{div},\Omega} \|(\mathbf{v}, \psi)\|_{\mathbf{M},\nu} \quad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0, \forall\, \mathbf{v} \in \mathbf{M},$$

where in the latter we utilized the fact that

$$\|\mathbf{v}+\nabla\psi\|_{0,\Omega}^2 = \|\mathbf{v}\|_{0,\Omega}^2 + |\psi|_{1,\Omega}^2 \qquad \forall\,(\mathbf{v},\psi)\in\mathbf{M}. \tag{2.8}$$

In turn, from the definition of the bilinear form $\mathbf{a}$, it is clear that

$$\mathbf{a}(\boldsymbol{\tau},\boldsymbol{\tau}) = |\boldsymbol{\tau}|_{\mathbf{div},\Omega}^2 \qquad \forall\,\boldsymbol{\tau}\in\mathbb{X}_0, \tag{2.9}$$

thus $\mathbf{a}$ is elliptic on $\mathbb{X}_0$.

Now we provide the corresponding inf-sup condition of the bilinear form $\mathbf{b}$.

**Lemma 2.3** *There exists $\beta>0$, independent of $\nu$, such that*

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{X}_0} \frac{\mathbf{b}(\boldsymbol{\tau},(\mathbf{v},\psi))}{|\boldsymbol{\tau}|_{\mathbf{div},\Omega}} \geq \beta\|(\mathbf{v},\psi)\|_{\mathbf{M},\nu} \qquad \forall\,(\mathbf{v},\psi)\,\in\,\mathbf{M}. \tag{2.10}$$

*Proof.* It follows analogously to the proof of the inf-sup condition in [18, Theorem 2.1]. In fact, given $(\mathbf{v},\psi)\in\mathbf{M}$, we let $\tilde{\boldsymbol{\tau}} := -\nabla\mathbf{z} + \frac{1}{d|\Omega|}(\operatorname{div}\mathbf{z},1)_\Omega\mathbb{I}$ in $\Omega$, with $\mathbf{z}\in\mathbf{H}_0^1(\Omega)$ being the unique weak solution of the auxiliary problem

$$-\Delta\mathbf{z} = \nu(\mathbf{v}+\nabla\psi) \quad \text{in} \quad \Omega, \qquad \mathbf{z}=\mathbf{0} \quad \text{on} \quad \Gamma,$$

satisfying

$$|\mathbf{z}|_{1,\Omega} \leq C\nu\|\mathbf{v}+\nabla\psi\|_{0,\Omega}. \tag{2.11}$$

Then, we observe that $\mathbf{div}\,\tilde{\boldsymbol{\tau}} = \nu(\mathbf{v}+\nabla\psi)$ in $\Omega$, which together with (2.11) implies $|\tilde{\boldsymbol{\tau}}|_{\mathbf{div},\Omega} \leq \tilde{C}\nu\|\mathbf{v}+\nabla\psi\|_{0,\Omega}$. From the latter, and the identity (2.8), we obtain

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{X}_0} \frac{\mathbf{b}(\boldsymbol{\tau},(\mathbf{v},\psi))}{|\boldsymbol{\tau}|_{\mathbf{div},\Omega}} \geq \frac{\mathbf{b}(\tilde{\boldsymbol{\tau}},(\mathbf{v},\psi))}{|\tilde{\boldsymbol{\tau}}|_{\mathbf{div},\Omega}} \geq \tilde{C}^{-1}\frac{\nu^2(\|\mathbf{v}\|_{\mathrm{div},\Omega}^2 + |\psi|_{1,\Omega}^2)}{\nu\|\mathbf{v}+\nabla\psi\|_{0,\Omega}} = \beta\|(\mathbf{v},\psi)\|_{\mathbf{M},\nu},$$

with $\beta = \tilde{C}^{-1}>0$. □

We now provide the well-posedness of problem (2.6).

**Theorem 2.4** *There exists a unique $(\boldsymbol{\sigma},(\mathbf{u},\varphi))\in\mathbb{X}_0\times\mathbf{M}$ solution to (2.6) with $\varphi=0$ in $\Omega$. Furthermore, the following a priori estimates hold:*

$$|\boldsymbol{\sigma}|_{\mathbf{div},\Omega} \leq \nu c_1\|\mathbf{u}_D\|_{1/2,\Gamma} + c_2\|\mathbf{f}\|_{0,\Omega} \quad \textit{and} \quad \nu\|\mathbf{u}\|_{0,\Omega} \leq c_3\nu\|\mathbf{u}_D\|_{1/2,\Gamma} + c_4\|\mathbf{f}\|_{0,\Omega}, \tag{2.12}$$

*with $c_1,c_2,c_3,c_4>0$, all of them independent of $\nu$.*

*Proof.* The well-posedness of (2.6) follows straightforwardly from (2.9), (2.10) and the Babuška–Brezzi theory, and from Lemma 2.1 we deduce that $\varphi=0$ in $\Omega$. In turn, to derive the estimates (2.12) we first observe that from [14, Theorem 1.7] and (2.7), there holds

$$|\langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle_\Gamma| \leq \nu C|\boldsymbol{\tau}|_{\mathbf{div},\Omega}\|\mathbf{u}_D\|_{1/2,\Gamma}, \qquad \forall\,\boldsymbol{\tau}\in\mathbb{X}_0,$$

with $C>0$ independent of $\nu$, which implies that

$$\|F\|_{\mathbb{X}_0'} \leq \nu C\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,\Omega}.$$

Then, noticing that

$$\|G\|_{\mathbf{M}'} \leq \|\mathbf{f}\|_{0,\Omega},$$

the estimates in (2.12) can be easily obtained from [13, Theorem 2.34, estimate (2.30)] and the fact that $\operatorname{div}\mathbf{u} = 0$ in $\Omega$. $\square$

We end this section by providing the converse of the derivation of (2.6).

**Theorem 2.5** *Let $(\boldsymbol{\sigma},(\mathbf{u},\varphi)) \in \mathbb{X}_0 \times \mathbf{M}$ be the unique solution of (2.6). Then, $-\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f}$ in $\Omega$, $\varphi = 0$ in $\Omega$, $\mathbf{u} = \mathbf{u}_D$ on $\Gamma$ and $\nu\nabla\mathbf{u} = \boldsymbol{\sigma}^{\mathrm{d}}$ in $\Omega$, which implies that $\mathbf{u} \in \mathbf{H}^1(\Omega)$.*

*Proof.* The identity $-\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f}$ in $\Omega$ follows from the second equation of (2.6) and the Helmholtz decomposition (2.4) whereas identity $\varphi = 0$ in $\Omega$ follows from Lemma 2.1. The rest of the identities follow from the first equation of (2.6), considering suitable test functions and integrating by parts backwardly. We omit further details.

$\square$

# 3 Galerkin Scheme

In this section we introduce the Galerkin scheme associated to problem (2.6), analyze its solvability and finally derive the corresponding a priori error estimate.

## 3.1 The discrete problem

Let $\mathcal{T}_h$ be a regular family of regular triangulations of the polyhedral region $\overline{\Omega}$ by triangles $T$ in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$ of diameter $h_T$, such that $\overline{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$ and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. Given an integer $l \geq 0$ and a subset $S$ of $\mathbb{R}^d$, we denote by $P_l(S)$ the space of polynomials of total degree at most $l$ defined on $S$. Hence, for each integer $k \geq 0$ and for each $T \in \mathcal{T}_h$, we define the local Raviart–Thomas space of order $k$ and the Brezzi–Douglas–Marini element of order $k+1$, respectively as (see, for instance [5]):

$$\mathbf{RT}_k(T) := [P_k(T)]^d \oplus \widetilde{P}_k(T)\mathbf{x}, \quad \text{and} \quad \mathbf{BDM}_{k+1}(T) = [P_{k+1}(T)]^d$$

where $\mathbf{x} := (x_1, \ldots, x_d)^t$ is a generic vector of $\mathbb{R}^d$ and $\widetilde{P}_k(T)$ is the space of polynomials of total degree equal to $k$ defined on $T$. In this way, we define the discrete spaces

$$\begin{aligned}
\mathbf{V}_h^k :=& \quad \{\mathbf{z}_h \in \mathbf{H}(\mathrm{div};\Omega) \, : \, \mathbf{z}_h|_T \in \mathbf{RT}_k(T), \quad \forall T \in \mathcal{T}_h\}, \\
\mathbf{X}_h^{k+1} :=& \quad \{\tau_h \in \mathbf{H}(\mathrm{div};\Omega) \, : \, \tau_h|_T \in \mathbf{BDM}_{k+1}(T), \quad \forall T \in \mathcal{T}_h\},
\end{aligned}$$

and let

$$\mathbb{X}_h^{k+1} := \left\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}\,;\Omega) : \mathbf{c}^{\mathrm{t}}\,\boldsymbol{\tau}_h \in \mathbf{X}_h^{k+1} \quad \forall\,\mathbf{c} \in \mathbb{R}^d\right\}, \quad \mathbb{X}_{h,0}^{k+1} := \mathbb{X}_h^{k+1} \cap \mathbb{H}_0(\mathbf{div}\,;\Omega),$$

$$\mathbf{V}_{h,0}^k := \mathbf{V}_h^k \cap \mathbf{H}(\mathrm{div}^0;\Omega),$$

$$\Psi_{h,0}^{k+1} := \left\{\varphi_h \in C(\overline{\Omega}) \, : \, \varphi_h|_T \in P_{k+1}(T), \quad \forall T \in \mathcal{T}_h\right\} \cap \mathrm{H}_0^1(\Omega),$$

$$\mathbf{M}_h^k := \mathbf{V}_{h,0}^k \times \Psi_{h,0}^{k+1},$$

in such a way, the Galerkin scheme associated to problem (2.6) reads: Find $(\boldsymbol{\sigma}_h, (\mathbf{u}_h, \varphi_h)) \in \mathbb{X}_{h,0}^{k+1} \times \mathbf{M}_h^k$, such that:

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, (\mathbf{u}_h, \varphi_h)) &= F(\boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{h,0}^{k+1}, \\ \mathbf{b}(\boldsymbol{\sigma}_h, (\mathbf{v}_h, \psi_h)) &= G(\mathbf{v}_h, \psi_h) \quad \forall\, (\mathbf{v}_h, \psi_h) \in \mathbf{M}_h^k. \end{aligned} \tag{3.1}$$

**Remark 3.1** *Observe that the discrete space* $\mathbf{V}_{h,0}^k$ *becomes*

$$\mathbf{V}_{h,0}^k = \{\mathbf{v}_h \in \mathbf{V}_h^k \,:\, \operatorname{div} \mathbf{v}_h = 0 \quad in \quad \Omega\},$$

*thus the numerical scheme* (3.1) *produces exactly divergence-free approximations for the velocity* $\mathbf{u}$.

In what follows we establish the well-posedness of (3.1) and derive the corresponding a priori error estimates.

## 3.2 Well-posedness

We begin by observing that the bilinear form $\mathbf{a}$ satisfies

$$\mathbf{a}(\boldsymbol{\tau}, \boldsymbol{\tau}) = |\boldsymbol{\tau}|^2_{\mathbf{div},\Omega} \qquad \forall \boldsymbol{\tau} \in \mathbb{X}_{h,0}^{k+1}.$$

Next, to prove the discrete version of (2.10) we recall from [5, Section 2.5] that there exist interpolator operators $\Pi_h^{RT} : \mathbf{H}^1(\Omega) \to \mathbf{V}_h^k$ and $\Pi_h^{BDM} : \mathbf{H}^1(\Omega) \to \mathbf{X}_h^{k+1}$ satisfying the approximation property

$$\|\Pi_h^\star(\tau) - \tau\|_{0,T} \leq c h_T^m |\tau|_{m,T}, \quad \forall \tau \in \mathbf{H}^m(T), \quad \forall\, T \in \mathcal{T}_h, \tag{3.2}$$

for all $1 \leq m \leq l_\star(k)$ and $\star \in \{RT, BDM\}$, with $l_{RT}(k) = k+1$ and $l_{BDM}(k) = k+2$, and the commutative property

$$\operatorname{div}(\Pi_h^\star(\tau)) = \mathcal{P}_k(\operatorname{div} \tau), \quad \forall \tau \in \mathbf{H}^1(\Omega), \quad \forall \star \in \{RT, BDM\}, \tag{3.3}$$

where $\mathcal{P}_h$ is the $L^2$-projection on

$$Q_h^k := \{q \in L^2(\Omega) \,:\, q|_T \in P_k(T), \quad \forall\, T \in \mathcal{T}_h\},$$

which satisfies

$$\int_\Omega (\mathcal{P}_h^k(v) - v) z_h = 0 \quad \forall\, z_h \in Q_h^k,$$

and the local approximation property

$$\|v - \mathcal{P}_h^k(v)\|_{0,T} \leq C h^m |v|_{m,T}, \quad \forall\, T \in \mathcal{T}_h, \tag{3.4}$$

for all $0 \leq m \leq k+1$ and for all $v \in \mathrm{H}^m(\Omega)$. Notice that from (3.3) and (3.4) we have that

$$\|\operatorname{div} \tau - \operatorname{div}(\Pi_h^\star(\tau))\|_{0,T} \leq C h^m |\operatorname{div} \tau|_{m,T}, \quad \forall\, T \in \mathcal{T}_h, \tag{3.5}$$

for all $0 \leq m \leq k+1$ and for all $\tau \in \mathbf{H}^1(\Omega)$ with $\operatorname{div} \tau \in \mathrm{H}^m(\Omega)$.

In what follows we will employ a tensor version of $\Pi_h^{BDM}$, denoted by $\boldsymbol{\Pi}_h^{BDM} : \mathbb{H}^1(\Omega) \to \mathbb{X}_h^k$, which is defined row-wise by $\Pi_h^{BDM}$, and the vector version of $\mathcal{P}_h^k$, denoted by $\mathbf{P}_h^k : \mathbf{L}^2(\Omega) \to \mathbf{Q}_h^k := [Q_h^k]^d$, defined component-wise by $\mathcal{P}_h^k$.

Now we adapt the proof of [18, Lemma 3.2] to deduce the discrete inf-sup condition of $\mathbf{b}$.

**Lemma 3.2** *There exists $\widehat{\beta} > 0$, independent of $h$ and $\nu$, such that*

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_h\in\mathbb{X}_{h,0}} \frac{\mathbf{b}(\boldsymbol{\tau}_h,(\mathbf{v}_h,\psi_h))}{|\boldsymbol{\tau}_h|_{\mathbf{div},\Omega}} \geq \widehat{\beta}\|(\mathbf{v}_h,\psi_h)\|_{\mathbf{M},\nu} \qquad \forall\,(\mathbf{v}_h,\psi_h) \,\in\, \mathbf{M}_h^k. \tag{3.6}$$

*Proof.* We let $B \subseteq \mathbb{R}^d$ be a bounded and open convex domain such that $\Omega \subset B$, and given $(\mathbf{v}_h,\psi_h) \in \mathbf{M}_h^k$, we let $\mathbf{z} \in \mathbf{H}_0^1(B)$ be the unique weak solution of the auxiliary problem

$$-\Delta\mathbf{z} = \mathbf{h}(\mathbf{v}_h,\psi_h) \quad \text{in} \quad B, \qquad \mathbf{z} = \mathbf{0} \quad \text{on} \quad \partial B,$$

with

$$\mathbf{h}(\mathbf{v}_h,\psi_h) := \begin{cases} \nu(\mathbf{v}_h + \nabla\psi_h), & \text{in } \Omega \\ \mathbf{0}, & \text{in } B\backslash\overline{\Omega}. \end{cases}$$

It is well known that $\mathbf{z} \in \mathbf{H}^2(B)$ (see [25]) and

$$\|\mathbf{z}\|_{2,\Omega} \leq C\nu\|\mathbf{h}(\mathbf{v}_h,\psi_h)\|_{0,B} = C\nu\|\mathbf{v}_h + \nabla\psi_h\|_{0,\Omega} = C\|(\mathbf{v}_h,\psi_h)\|_{\mathbf{M},\nu}. \tag{3.7}$$

Notice that, since $\mathbf{v}_h \in \mathbf{V}_{h,0}^k$, then according to [5, Corollary 2.3.1], $\mathbf{v}_h \in \mathbf{Q}_h^k$. Then, we let $\widehat{\boldsymbol{\tau}}_h := -\boldsymbol{\Pi}_h^{BDM}(\nabla\mathbf{z}|_\Omega) + \frac{1}{d|\Omega|}(\mathrm{tr}\,(\boldsymbol{\Pi}_h^{BDM}(\nabla\mathbf{z}|_\Omega)),1)_\Omega\mathbb{I}$ in $\Omega$ and apply (3.2), (3.3), (3.4) and (3.7), to deduce that

$$\mathbf{div}\,\widehat{\boldsymbol{\tau}}_h = \nu(\mathbf{v}_h + \nabla\psi_h) \in \mathbf{Q}_h^k \quad \text{and} \quad |\widehat{\boldsymbol{\tau}}_h|_{\mathbf{div},\Omega} \leq \hat{C}\|(\mathbf{v}_h,\psi_h)\|_{\mathbf{M},\nu}.$$

Therefore, we proceed as in the proof of Lemma 2.3 to obtain the desired estimate. □

These properties and the Babuška-Brezzi theory allow us to conclude the well-posedness of (3.1). This result is established next.

**Theorem 3.3** *There exists a unique $(\boldsymbol{\sigma}_h,(\mathbf{u}_h,\varphi_h)) \in \mathbb{X}_{h,0}^{k+1}\times\mathbf{M}_h^k$ solution to the Galerkin scheme* (3.1)*. In addition, there exist positive constants $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$ and $\tilde{c}_4$, independent of $h$ and $\nu$, such that*

$$|\boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} \leq \nu\tilde{c}_1\|\mathbf{u}_D\|_{1/2,\Gamma} + \tilde{c}_2\|\mathbf{f}\|_{0,\Omega} \quad \textit{and} \quad \|(\mathbf{u}_h,\varphi_h)\|_{\mathbf{M},\nu} \leq \tilde{c}_3\nu\|\mathbf{u}_D\|_{1/2,\Gamma} + \tilde{c}_4\|\mathbf{f}\|_{0,\Omega}, \tag{3.8}$$

*Proof.* The existence and uniqueness of solution is a straightforward application of the Babuška-Brezzi theory, whereas estimate (3.8) follows analogously to the proof of (2.12). We omit further details. □

**Remark 3.4** *Let $\mathbb{K}_h$ be the discrete kernel of the bilinear form $\mathbf{b}$, that is:*

$$\begin{aligned} \mathbb{K}_h \;&:= \left\{\boldsymbol{\tau} \in \mathbb{X}_{h,0}^{k+1} : \quad \mathbf{b}(\boldsymbol{\tau},(\mathbf{v},\psi)) = 0, \quad \forall\,(\mathbf{v},\psi) \in \mathbf{M}_h^k\right\} \\ &= \left\{\boldsymbol{\tau} \in \mathbb{X}_{h,0}^{k+1} : \quad (\mathbf{div}\;\boldsymbol{\tau},\mathbf{v} + \nabla\psi)_\Omega = 0, \quad \forall\,(\mathbf{v},\psi) \in \mathbf{M}_h^k\right\}. \end{aligned}$$

*It should be noted that if $\boldsymbol{\tau} \in \mathbb{K}_h$, it does not necessarily satisfy $\mathbf{div}\,\boldsymbol{\tau} = \mathbf{0}$ in $\Omega$. Therefore, without introducing the term $(\mathbf{div}\,\boldsymbol{\sigma},\mathbf{div}\,\boldsymbol{\tau})_\Omega$ in the definition of the bilinear form $\mathbf{a}$ (cf. (2.5)), the ellipticity of $\mathbf{a}$ on $\mathbb{K}_h$ would not be satisfied, thereby preventing the utilization of the Babuška-Brezzi theory to establish the well-posedness of the discrete problem.*

*In turn, it is worth noting that although the analysis in this section was conducted using $\mathbf{BDM}_{k+1}$ elements for approximating the pseudostress and $\mathbf{RT}_k$ elements for the velocity, alternative options are also valid. For example, another viable choice would be to approximate both the pseudostress and velocity using $\mathbf{RT}_k$ elements. In general, the choice of the discrete spaces $\mathbb{X}_h$, $\mathbf{V}_h$, and $\Psi_h$ for $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\varphi$, respectively, must satisfy the inclusions $\mathbf{div}\,(\mathbb{X}_h) \subseteq \mathbf{V}_h\cap\mathbf{H}(\mathrm{div}^0;\Omega)$ and $\mathbf{div}\,(\mathbb{X}_h) \subseteq \nabla\Psi_h$ in order to ensure that the discrete inf-sup condition* (3.6) *holds.*

### 3.3 A priori error estimates

Now we derive the Cea's estimate and the corresponding rates of convergence.

**Theorem 3.5** *Let* $(\boldsymbol{\sigma},(\mathbf{u},0)) \in \mathbb{X}_0 \times \mathbf{M}$ *and* $(\boldsymbol{\sigma}_h,(\mathbf{u}_h,\varphi_h)) \in \mathbb{X}_{h,0}^{k+1} \times \mathbf{M}_h^k$ *be the unique solutions of* (2.6) *and* (3.1)*, respectively. Then, there exist positive constants* $\widehat{c}_1$, $\widehat{c}_2$, $\widehat{c}_3$ *and* $\widehat{c}_4$*, independent of* $\nu$ *and* $h$*, such that,*

$$|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} \,\leq\, \widehat{c}_1 \inf_{\boldsymbol{\tau}_h\in\mathbb{X}_{h,0}^{k+1}} |\boldsymbol{\sigma}-\boldsymbol{\tau}_h|_{\mathbf{div},\Omega} \,+\, \widehat{c}_2\,\nu \inf_{\mathbf{v}_h\in\mathbf{V}_{h,0}^k} \|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega}, \tag{3.9}$$

*and*

$$\|(\mathbf{u}-\mathbf{u}_h,\varphi_h)\|_{\mathbf{M},\nu} \,\leq\, \widehat{c}_3 \inf_{\boldsymbol{\tau}_h\in\mathbb{X}_{h,0}^{k+1}} |\boldsymbol{\sigma}-\boldsymbol{\tau}_h|_{\mathbf{div},\Omega}, + \nu\widehat{c}_4 \inf_{\mathbf{v}_h\in\mathbf{V}_{h,0}^k} \|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega}. \tag{3.10}$$

*Proof.* The proof follows from a direct application of [14, Theorem 2.6] and the fact that

$$\inf_{(\mathbf{v}_h,\psi_h)\in\mathbf{M}_h} \|(\mathbf{u},0)-(\mathbf{v}_h,\psi_h)\|_{\mathbf{M},\nu} \leq \nu \inf_{\mathbf{v}_h\in\mathbf{V}_{h,0}^k} \|\mathbf{u}-\mathbf{v}_h\|_{0,\Omega}.$$

□

Next, to derive the rate of convergence for the Galerkin scheme (3.1), we recall from [5, Section 2.5] that (3.2), (3.3) and (3.4) imply the global estimate

$$\|\tau \,-\, \Pi_h^\star(\tau)\|_{\mathrm{div},\Omega} \leq ch^m\Big\{|\tau|_{m,\Omega} + |\mathrm{div}\,\tau|_{m,\Omega}\Big\} \quad \star\in\{RT,BDM\},$$

for all $1\leq m\leq k+1$, for all $\tau\in\mathbf{H}^m(\Omega)$ with $\mathrm{div}\,\tau\in\mathrm{H}^m(\Omega)$. From this estimate and the error estimates (3.9) and (3.10) we readily obtain the corresponding theoretical rates of convergence. This result is established next.

**Theorem 3.6** *Let* $(\boldsymbol{\sigma},(\mathbf{u},0)) \in \mathbb{X}_0 \times \mathbf{M}$ *and* $(\boldsymbol{\sigma}_h,(\mathbf{u}_h,\varphi_h)) \in \mathbb{X}_{h,0}^{k+1} \times \mathbf{M}_h^k$ *be the unique solutions of* (2.6) *and* (3.1)*, respectively, and assume that* $\boldsymbol{\sigma}\in\mathbb{H}^m(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma}\in\mathbf{H}^m(\Omega)$, $\mathbf{u}\in\mathbf{H}^m(\Omega)$ *for* $1\leq m\leq k+1$*. Then, there exist positive constants* $\hat{c}_1$, $\hat{c}_2$, $\hat{c}_3$ *and* $\hat{c}_4$*, independent of* $\nu$ *and* $h$*, such that,*

$$|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} \,\leq\, \hat{c}_1 h^m\{|\boldsymbol{\sigma}|_{m,\Omega} + |\mathbf{div}\,\boldsymbol{\sigma}|_{m,\Omega}\} \,+\, \nu\hat{c}_2 h^m|\mathbf{u}|_{m,\Omega}, \tag{3.11}$$

*and*

$$\|(\mathbf{u}-\mathbf{u}_h,\varphi_h)\|_{\mathbf{M},\nu} \,\leq\, \hat{c}_3 h^m\{|\boldsymbol{\sigma}|_{m,\Omega} + |\mathbf{div}\,\boldsymbol{\sigma}|_{m,\Omega}\} \,+\, \nu\hat{c}_4 h^m|\mathbf{u}|_{m,\Omega}. \tag{3.12}$$

**Remark 3.7** *Recalling that the exact pressure can be recovered through the post-processing formula* $p=-\frac{1}{d}\mathrm{tr}\,(\boldsymbol{\sigma})$*, it is clear that a suitable approximation for* $p$ *is* $p_h := -\frac{1}{d}\mathrm{tr}\,(\boldsymbol{\sigma}_h)$*. Moreover, from* (2.7) *and* (3.11) *it is easy to see that the following estimate holds*

$$\|p-p_h\|_{0,\Omega} \,\leq\, \widehat{c}_1 h^m\{|\boldsymbol{\sigma}|_{m,\Omega} + |\mathbf{div}\,\boldsymbol{\sigma}|_{m,\Omega}\} \,+\, \nu\widehat{c}_2 h^m|\mathbf{u}|_{m,\Omega},$$

*with* $\widehat{c}_1,\widehat{c}_2$*, independent of* $\nu$ *and* $h$*.*

*On the other hand, from* (3.12) *we observe that*

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \,\leq\, \hat{c}_3 h^m\nu^{-1}\{|\boldsymbol{\sigma}|_{m,\Omega} + |\mathbf{div}\,\boldsymbol{\sigma}|_{m,\Omega}\} \,+\, \hat{c}_4 h^m|\mathbf{u}|_{m,\Omega},$$

*then the error of the velocity is amplified by the inverse of the viscosity, which is corroborated in Example 1 in Section 6 (see Table 6.1).*

## 4 A posteriori error analysis

In this section we apply several well-known results from previous works, in particular from [6], [16] and [18], to derive a reliable and efficient residual-based a posteriori error estimator for our mixed finite element scheme (2.6). We begin as in [8] by introducing some notations that will allow us to derive the analysis in two and three dimensions in a unified framework.

Let $\mathcal{E}_h$ be the set of edges or faces of $\mathcal{T}_h$, whose corresponding diameters are denoted $h_e$, and define

$$\mathcal{E}_h(\Omega) := \left\{ e \in \mathcal{E}_h : \quad e \subseteq \Omega \right\} \quad \text{and} \quad \mathcal{E}_h(\Gamma) := \left\{ e \in \mathcal{E}_h : \quad e \subseteq \Gamma \right\}.$$

For each $T \in \mathcal{T}_h$, we let $\mathcal{E}_{h,T}$ be the set of edges or faces of $T$, and we denote

$$\mathcal{E}_{h,T}(\Omega) = \left\{ e \subseteq \partial T : \quad e \in \mathcal{E}_h(\Omega) \right\} \quad \text{and} \quad \mathcal{E}_{h,T}(\Gamma) = \left\{ e \subseteq \partial T : \quad e \in \mathcal{E}_h(\Gamma) \right\}.$$

We also define the unit normal vector $\mathbf{n}_e$ on each edge or face by

$$\mathbf{n}_e := (n_1, .., n_d)^{\mathrm{t}} \quad \forall\, e \in \mathcal{E}_h .$$

Hence, when $d = 2$, we can define the tangential vector $\mathbf{s}_e$ by

$$\mathbf{s}_e := (-n_2, n_1)^{\mathrm{t}} \quad \forall\, e \in \mathcal{E}_h .$$

However, when no confusion arises, we will simply write $\mathbf{n}$ and $\mathbf{s}$ instead of $\mathbf{n}_e$ and $\mathbf{s}_e$, respectively.

The usual jump operator $[\![\cdot]\!]$ across internal edges or faces are defined for piecewise continuous matrix, vector, or scalar-valued functions $\boldsymbol{\zeta}$ by

$$[\![\boldsymbol{\zeta}]\!] = (\boldsymbol{\zeta}|_{T_+})|_e - (\boldsymbol{\zeta}|_{T_-})|_e \quad \text{with} \quad e = \partial T_+ \cap \partial T_-,$$

where $T_+$ and $T_-$ are the elements of $\mathcal{T}_h$ having $e$ as a common edge or face. Finally, for sufficiently smooth scalar $\psi$, vector $\mathbf{v} := (v_1, .., v_d)^{\mathrm{t}}$, and tensor fields $\boldsymbol{\tau} := (\tau_{ij})_{1\le i,j\le d}$, for $d = 2$ we let

$$\operatorname{curl}(\psi) := \left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1}\right)^{\mathrm{t}}, \quad \operatorname{rot}(\mathbf{v}) := \frac{\partial \mathbf{v}_2}{\partial x_1} - \frac{\partial \mathbf{v}_1}{\partial x_2}, \quad \mathbf{curl}(\mathbf{v}) = \begin{pmatrix} \operatorname{curl}(v_1)^{\mathrm{t}} \\ \operatorname{curl}(v_2)^{\mathrm{t}} \end{pmatrix},$$

$$\underline{\mathbf{curl}}(\boldsymbol{\tau}) = \begin{pmatrix} \operatorname{rot}(\boldsymbol{\tau}_1) \\ \operatorname{rot}(\boldsymbol{\tau}_2) \end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma}_*(\boldsymbol{\tau}) = \boldsymbol{\tau}\mathbf{s}, \tag{4.1}$$

and for $d = 3$ we let

$$\mathbf{curl}(\mathbf{v}) = \nabla \times \mathbf{v}, \quad \underline{\mathbf{curl}}(\boldsymbol{\tau}) = \begin{pmatrix} \mathbf{curl}(\boldsymbol{\tau}_1) \\ \mathbf{curl}(\boldsymbol{\tau}_2) \\ \mathbf{curl}(\boldsymbol{\tau}_3) \end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma}_*(\boldsymbol{\tau}) = \begin{pmatrix} \boldsymbol{\tau}_1 \times \mathbf{n} \\ \boldsymbol{\tau}_2 \times \mathbf{n} \\ \boldsymbol{\tau}_3 \times \mathbf{n} \end{pmatrix}, \tag{4.2}$$

where $\boldsymbol{\tau}_i$ is the $i-$th row of $\boldsymbol{\tau}$ and the derivatives involved are taken in the distributional sense.

Then we let $(\boldsymbol{\sigma}_h, (\mathbf{u}_h, \varphi_h)) \in \mathbb{X}_{h,0}^{k+1} \times \mathbf{M}_h^k$ be the unique solution to (3.1) and introduce the global a posteriori error estimator:

$$\Theta = \left\{ \sum_{T \in \mathcal{T}_h} \Theta_T^2 \right\}^{1/2}, \tag{4.3}$$

where, for each $T \in \mathcal{T}_h$, the local error indicator is defined as follows:

$$
\begin{aligned}
\Theta_T^2 := & \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2 + h_T^2 \left\|\nu\nabla\mathbf{u}_h - \boldsymbol{\sigma}_h^{\mathrm{d}}\right\|_{0,T}^2 + h_T^2 \left\|\underline{\mathbf{curl}}\left(\boldsymbol{\sigma}_h^{\mathrm{d}}\right)\right\|_{0,T}^2 \\
& + \sum_{e\in\mathcal{E}_{h,T}(\Omega)} h_e \left\|\left[\left[\boldsymbol{\gamma}_*\left(\boldsymbol{\sigma}_h^{\mathrm{d}}\right)\right]\right]\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}_{h,T}(\Gamma)} h_e \left\|\boldsymbol{\gamma}_*\left(\boldsymbol{\sigma}_h^{\mathrm{d}} - \nu\nabla\mathbf{u}_D\right)\right\|_{0,e}^2 \\
& + \sum_{e\in\mathcal{E}_{h,T}(\Gamma)} h_e\nu^2 \left\|\mathbf{u}_D - \mathbf{u}_h\right\|_{0,e}^2 ,
\end{aligned} \tag{4.4}
$$

where $\boldsymbol{\gamma}_*$ is the tangential component defined in (4.1) for $d=2$ and (4.2) for $d=3$ and $\mathbf{u}_D$ is assumed to be in $\mathbf{H}^1(\Gamma)$.

In the following sections we prove the reliability and efficiency of $\Theta$. We begin with the reliability estimate.

## 4.1 Reliability of the a posteriori error estimator

In what follows we prove the following result.

**Theorem 4.1** *Assume that $\mathbf{u}_D \in \mathbf{H}^1(\Gamma)$ and let $(\boldsymbol{\sigma},(\mathbf{u},0)) \in \mathbb{X}_0 \times \mathbf{M}$ and $(\boldsymbol{\sigma}_h,(\mathbf{u}_h,\varphi_h)) \in \mathbb{X}_{h,0}^{k+1} \times \mathbf{M}_h^k$ be the unique solutions of (2.6) and (3.1), respectively. Then, there exists $C_{rel} > 0$, independent of $h$ and $\nu$, such that*

$$
|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} + \|(\mathbf{u}-\mathbf{u}_h,\varphi_h)\|_{\mathbf{M},\nu} \;\le\; C_{rel}\Theta \tag{4.5}
$$

To that end, we begin by recalling that the ellipticity of $\mathbf{a}$ and the inf-sup condition of $\mathbf{b}$ (cf. (2.9) and (2.10), respectively) imply the global inf-sup estimate (see [13, Proposition 2.36]):

$$
|\boldsymbol{\zeta}|_{\mathbf{div},\Omega} + \|(\mathbf{w},\phi)\|_{\mathbf{M},\nu} \;\le\; C S(\boldsymbol{\zeta},(\mathbf{w},\phi)), \tag{4.6}
$$

for all $(\boldsymbol{\zeta},(\mathbf{w},\phi)) \in \mathbb{X}_0 \times \mathbf{M}$, where $C > 0$ is a constant independent of $h$ and $\nu$, and

$$
S(\boldsymbol{\zeta},(\mathbf{w},\phi)) := \sup_{\substack{(\boldsymbol{\tau},(\mathbf{v},\psi))\in\mathbb{X}_0\times\mathbf{M}\\ (\boldsymbol{\tau},(\mathbf{v},\psi))\neq\mathbf{0}}} \frac{\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},(\mathbf{w},\phi)) + \mathbf{b}(\boldsymbol{\zeta},(\mathbf{v},\psi))}{|\boldsymbol{\tau}|_{\mathbf{div},\Omega} + \|(\mathbf{v},\psi)\|_{\mathbf{M},\nu}}.
$$

In particular, for $(\boldsymbol{\zeta},(\mathbf{w},\phi)) = (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,(\mathbf{u}-\mathbf{u}_h,-\varphi_h))$, with $(\boldsymbol{\sigma},(\mathbf{u},0))$ and $(\boldsymbol{\sigma}_h,(\mathbf{u}_h,\varphi_h))$ being the unique solutions of (2.6) and (3.1), respectively, it is easy to see that

$$
|S((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,(\mathbf{u}-\mathbf{u}_h,-\varphi_h)))| \le \|\mathcal{R}_1\|_{\mathbb{X}_0'} + \|\mathcal{R}_2\|_{\mathbf{M}'},
$$

where $\mathcal{R}_1 \in \mathbb{X}_0'$ and $\mathcal{R}_2 \in \mathbf{M}'$ are given, respectively, by

$$
\begin{aligned}
\mathcal{R}_1(\boldsymbol{\tau}) := &\;\; \mathbf{a}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},(\mathbf{u}-\mathbf{u}_h,-\varphi_h)) \\
= &\;\; -(\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h,\mathbf{div}\,\boldsymbol{\tau})_\Omega \,+\, \nu\langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle_\Gamma - (\boldsymbol{\sigma}_h^{\mathrm{d}},\boldsymbol{\tau})_\Omega - \nu(\mathbf{div}\,\boldsymbol{\tau},\mathbf{u}_h+\nabla\varphi_h)_\Omega,
\end{aligned}
$$

for all $\boldsymbol{\tau} \in \mathbb{X}_0$, and

$$
\mathcal{R}_2(\mathbf{v},\psi) := \mathbf{b}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,(\mathbf{v},\psi)) = -\nu(\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h,\mathbf{v}+\nabla\psi)_\Omega,
$$

for all $(\mathbf{v}, \psi) \in \mathbf{M}$, and $\|\cdot\|_{\mathbb{X}_0'}$ and $\|\cdot\|_{\mathbf{M}'}$ denote the norms of the dual spaces $\mathbb{X}_0'$ and $\mathbf{M}'$ induced by $|\cdot|_{\mathbf{div},\Omega}$ and $\|\cdot\|_{\mathbf{M},\nu}$, that is

$$\|\mathcal{R}_1\|_{\mathbb{X}_0'} = \sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{X}_0} \frac{|\mathcal{R}_1(\boldsymbol{\tau})|}{|\boldsymbol{\tau}|_{\mathbf{div},\Omega}} \quad \text{and} \quad \|\mathcal{R}_2\|_{\mathbf{M}_0'} = \sup_{\mathbf{0}\neq(\mathbf{v},\psi)\in\mathbf{M}} \frac{|\mathcal{R}_2(\mathbf{v},\psi)|}{|(\mathbf{v},\psi)|_{\mathbf{M},\nu}}.$$

According to the above, to prove (4.5), it suffices to estimate $\|\mathcal{R}_1\|_{\mathbb{X}_0'}$ and $\|\mathcal{R}_2\|_{\mathbf{M}'}$. We begin by observing that the Cauchy-Schwartz inequality implies

$$\|\mathcal{R}_2\|_{\mathbf{M}'} \leq \left\{ \sum_{T\in\mathcal{T}_h} \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2 \right\}^{1/2}. \tag{4.7}$$

In turn, proceeding analogously to [18, Section 4.1], that is, making use of a stable Helmholtz decomposition for $\mathbb{H}_0(\mathbf{div}\,;\Omega)$, and utilizing the local approximation properties of $\Pi_h^{BDM}$ (cf. (3.2) and (3.5)) and the Clément interpolation operator (see [11]), we readily obtain

$$\|\mathcal{R}_1\|_{\mathbb{X}_0'} \leq C \left\{ \sum_{T\in\mathcal{T}_h} \Theta_T^2 \right\}^{1/2}, \tag{4.8}$$

where $C > 0$ is a constant independent of $\nu$ and $h$.

In this way, from (4.6), (4.7) and (4.8) we readily obtain (4.5).

## 4.2 Efficiency of the a posteriori error estimator

The main result of this section is stated as follows

**Theorem 4.2** *Assume that $\mathbf{u}_D$ is a piecewise polynomial. Then there exists $C_{eff} > 0$, independent of $h$, such that*

$$C_{eff}\Theta \leq |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} + \nu\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}.$$

To prove this result, we introduce the following lemma which establishes suitable estimates for each term defining $\Theta$.

**Lemma 4.3** *There exist $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$, independent of $h$ and $\nu$, such that*

a) $\|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T} \leq \|\mathbf{div}\,\boldsymbol{\sigma} - \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T} \leq |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{\mathbf{div},T} \quad \forall\, T \in \mathcal{T}_h$,

b) $h_T^2 \left\|\nu\nabla\mathbf{u}_h - \boldsymbol{\sigma}_h^{\mathrm{d}}\right\|_{0,T}^2$
$\leq C_1 \left\{\nu^2\|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 + h_T^2\|\boldsymbol{\sigma}^{\mathrm{d}} - \boldsymbol{\sigma}_h^{\mathrm{d}}\|_{0,T}^2\right\} \quad \forall\, T \in \mathcal{T}_h$,

c) $h_T^2 \left\|\underline{\mathbf{curl}}\left(\boldsymbol{\sigma}_h^{\mathrm{d}}\right)\right\|_{0,T}^2 \leq C_2 \|\boldsymbol{\sigma}^{\mathrm{d}} - \boldsymbol{\sigma}_h^{\mathrm{d}}\|_{0,T}^2 \quad \forall\, T \in \mathcal{T}_h$,

d) $h_e \left\|\left[\left[\boldsymbol{\gamma}_*\left(\boldsymbol{\sigma}_h^{\mathrm{d}}\right)\right]\right]\right\|_{0,e}^2 \leq C_3 \|\boldsymbol{\sigma}^{\mathrm{d}} - \boldsymbol{\sigma}_h^{\mathrm{d}}\|_{0,w_e}^2$ *for all $e \in \mathcal{E}_h(\Omega)$, where the set $w_e$ is given by* $w_e := \cup\left\{T' \in \mathcal{T}_h \,:\, e \in \mathcal{E}_{h,T'}\right\}$,

e) $h_e \nu^2 \|\mathbf{u}_D - \mathbf{u}_h\|_{0,e}^2 \leq C_4 \left\{ \nu^2 \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 + h_T^2 \|\boldsymbol{\sigma}^{\mathrm{d}} - \boldsymbol{\sigma}_h^{\mathrm{d}}\|_{0,T}^2 \right\}$ *for all* $e \in \mathcal{E}_{h,T}(\Gamma)$.

f) *Additionally, if* $\mathbf{u}_D$ *is piecewise polynomial, there exists* $C_5 > 0$*, independent of* $h$*, such that* $h_e \left\| \boldsymbol{\gamma}_* \left( \boldsymbol{\sigma}_h^{\mathrm{d}} - \nabla \mathbf{u}_D \right) \right\|_{0,e}^2 \leq C_5 \|\boldsymbol{\sigma}^{\mathrm{d}} - \boldsymbol{\sigma}_h^{\mathrm{d}}\|_{0,T_e}^2$ *for all* $e \in \mathcal{E}_h(\Gamma)$*, where* $T_e$ *is the triangle of* $\mathcal{T}_h$ *having* $e$ *as an edge.*

*Proof.* The estimate a) follows from the identity $\mathbf{f} = -\mathbf{div}\,\boldsymbol{\sigma}$ in $\Omega$ (see Theorem 2.5). In turn, after slight modifications of the proofs of lemmas 4.11, 4.13 and 4.14 in [18] one can easily deduce the two-dimensional versions of b)–f). Finally, to deduce the three-dimensional counterparts of b)–f) it suffices to slightly modify the proofs of lemmas 4.12, 4.9, 4.10 and 4.14 and 4.13, respectively in [16]. We omit further details. $\square$

We remark here that if $\mathbf{u}_D$ were not piecewise polynomial but sufficiently smooth, then higher order terms given by the errors arising from suitable polynomial approximations would appear in (4.2). More precisely, estimate (4.2) would become

$$C_{eff}\Theta + h.o.t. \leq |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} + \nu\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}. \tag{4.9}$$

# 5 The Stokes problem with mixed boundary conditions

In this section we briefly explain how to extend the results from the previous sections to obtain a mass conservative stress-based mixed finite element method form the Stokes problem with mixed boundary conditions. More precisely, we consider a domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, with polygonal boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Sigma}$, where $\Gamma, \Sigma \subset \partial\Omega$ are such that $\Gamma \cap \Sigma \neq \emptyset$, and focus on extending the previous results to the problem

$$\begin{gathered}
\boldsymbol{\sigma} = 2\nu\mathbf{e}(\mathbf{u}) - p\mathbb{I}, \qquad -\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\
\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma, \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Sigma,
\end{gathered} \tag{5.1}$$

where $\mathbf{e}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$ and $\mathbf{n}$ is the exterior unit normal on $\partial\Omega$.

## 5.1 Continuous Problem

To introduce the variational system of (5.1) we first recall from [14, Section 2.4.3] that $\mathbf{e}(\mathbf{u})$ can be decomposed as $\mathbf{e}(\mathbf{u}) = \nabla\mathbf{u} - \boldsymbol{\gamma}$ in $\Omega$, with $\boldsymbol{\gamma} := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^t)$ in $\Omega$. Then, using the incompressibility condition $\operatorname{div}\mathbf{u} = 0$ in $\Omega$, and proceeding as in Section 1, we rewrite (5.1) as

$$\begin{gathered}
\boldsymbol{\sigma}^{\mathrm{d}} = 2\nu\nabla\mathbf{u} - 2\nu\boldsymbol{\gamma}, \qquad -\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^t, \\
\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma, \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Sigma.
\end{gathered}$$

In this way, combining the techniques in Section 2.1 and in [14, Section 2.4.3.2], we obtain the variational problem: Find $(\boldsymbol{\sigma}, (\boldsymbol{\gamma}, \mathbf{u}, \varphi)) \in \mathbb{X}_\Sigma \times (\mathbb{Y} \times \mathbf{M})$, such that

$$\begin{aligned}
\widehat{\mathbf{a}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \widehat{\mathbf{b}}(\boldsymbol{\tau}, (\boldsymbol{\gamma}, \mathbf{u}, \varphi)) \quad &= \widehat{F}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_\Sigma, \\
\widehat{\mathbf{b}}(\boldsymbol{\sigma}, (\boldsymbol{\eta}, \mathbf{v}, \psi)) \quad &= \widehat{G}(\boldsymbol{\eta}, \mathbf{v}, \psi) \quad \forall\, (\boldsymbol{\eta}, \mathbf{v}, \psi) \in \mathbb{Y} \times \mathbf{M},
\end{aligned} \tag{5.2}$$

where

$$\mathbb{X}_\Sigma := \{\boldsymbol{\tau} \in \mathbb{X} : \boldsymbol{\tau}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Sigma\}, \qquad \mathbb{Y} := \{\boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \boldsymbol{\eta} + \boldsymbol{\eta}^t = \mathbf{0} \quad \text{in} \quad \Omega\},$$

and the bilinear forms $\widehat{\mathbf{a}} : \mathbb{X}_\Sigma \times \mathbb{X}_\Sigma \to \mathbb{R}$ and $\widehat{\mathbf{b}} : \mathbb{X}_\Sigma \times (\mathbb{Y} \times \mathbf{M}) \to \mathbb{R}$, and the functionals $\widehat{F} \in \mathbb{X}_\Sigma'$ and $\widehat{G} \in (\mathbb{Y} \times \mathbf{M})'$ are given by

$$\widehat{\mathbf{a}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := (\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau})_\Omega, \quad \widehat{\mathbf{b}}(\boldsymbol{\tau}, (\boldsymbol{\eta}, (\mathbf{v}, \psi))) := 2\nu(\boldsymbol{\tau}, \boldsymbol{\eta})_\Omega + 2\nu(\mathbf{div}\ \boldsymbol{\tau}, \mathbf{v} + \nabla\psi)_\Omega,$$

$$\widehat{F}(\boldsymbol{\tau}) \ := \ 2\nu\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma - (\mathbf{f}, \mathbf{div}\,\boldsymbol{\tau})_\Omega \quad \text{and} \quad \widehat{G}(\boldsymbol{\eta}, \mathbf{v}, \psi) \ := \ -2\nu(\mathbf{f}, \mathbf{v} + \nabla\psi)_\Omega,$$

with $\langle \cdot, \cdot \rangle_\Gamma$ being product of duality between the trace space $\mathbf{H}^{1/2}(\Gamma)$ and its dual $\mathbf{H}^{-1/2}(\Gamma)$.

Now, to verify the well-posedness of (5.2), we first observe that thanks to [14, Lemma 2.5], the seminorm $|\cdot|_{\mathbf{div},\Omega}$ is also a norm on $\mathbb{X}_\Sigma$, which implies that $\widehat{\mathbf{a}}$ is $\mathbb{X}_\Sigma$–elliptic. In addition, as for the Dirichlet case, we endow $\mathbb{Y} \times \mathbf{M}$ with the $\nu$-dependent norm

$$\|(\boldsymbol{\eta}, \mathbf{v}, \psi)\|_{\mathbb{Y}\times\mathbf{M},\nu} = \nu\{\|\boldsymbol{\eta}\|_{0,\Omega}^2 + \|\mathbf{v}\|_{\mathrm{div};\Omega}^2 + |\psi|_{1,\Omega}^2\}^{1/2}$$

and similarly to [14, Section 2.4.3.2], given $(\boldsymbol{\eta}, \mathbf{v}, \psi) \in \mathbb{Y} \times \mathbf{M}$, we can obtain that there exists $\widetilde{\boldsymbol{\tau}} \in \mathbb{X}_\Sigma$, satisfying $(\widetilde{\boldsymbol{\tau}}, \boldsymbol{\eta})_\Omega = \nu\|\boldsymbol{\eta}\|_{0,\Omega}^2$, $\mathbf{div}\,\widetilde{\boldsymbol{\tau}} = \nu(\mathbf{v} + \nabla\psi)$ and $|\widetilde{\boldsymbol{\tau}}|_{\mathbf{div},\Omega} \leq C\nu(\|\mathbf{v} + \nabla\psi\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega})$, which can be employed to deduce the inf-sup condition

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{X}_\Sigma} \frac{\widehat{\mathbf{b}}(\boldsymbol{\tau}, (\boldsymbol{\gamma}, \mathbf{v}, \psi))}{|\boldsymbol{\tau}|_{\mathbf{div},\Omega}} \geq \beta\|(\boldsymbol{\gamma}, \mathbf{v}, \psi)\|_{\mathbb{Y}\times\mathbf{M},\nu},$$

with $\beta > 0$, independent of $\nu$.

According to the above, and applying the Babuška–Brezzi theory, we obtain the well-posedness of (5.2). This result is presented in the following theorem, whose proof is omitted since it is analogously to the proof of Theorem 2.4.

**Theorem 5.1** *There exists a unique* $(\boldsymbol{\sigma}, (\boldsymbol{\gamma}, (\mathbf{u}, \varphi))) \in \mathbb{X}_\Sigma \times (\mathbb{Y} \times \mathbf{M})$ *solution to* (5.2) *with* $\varphi = 0$ *in* $\Omega$. *Furthermore, the following a priori estimates hold:*

$$|\boldsymbol{\sigma}|_{\mathbf{div},\Omega} \leq \nu c_1\|\mathbf{u}_D\|_{1/2,\Gamma} + c_2\|\mathbf{f}\|_{0,\Omega} \quad \textit{and} \quad \nu(\|\mathbf{u}\|_{0,\Omega} + \|\boldsymbol{\gamma}\|_{0,\Omega}) \leq c_3\nu\|\mathbf{u}_D\|_{1/2,\Gamma} + c_4\|\mathbf{f}\|_{0,\Omega},$$

*with* $c_1, c_2, c_3, c_4 > 0$*, all of them independent of* $\nu$.

## 5.2 Galerkin scheme

Now, to introduce the Galerkin scheme associated to (5.2), we adopt the notations introduced in Section 3 and additionally introduce the discrete spaces:

$$\begin{aligned} \mathbb{Y}_h^k \ &:= \{\boldsymbol{\eta}_h \in \mathbb{L}^2(\Omega) : \boldsymbol{\eta}_h + \boldsymbol{\eta}_h^t = \mathbf{0} \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{\eta}_h|_T \in [P_k(T)]^{d\times d} \quad \forall T \in \mathcal{T}_h\}, \\ \mathbb{X}_{h,\Sigma}^{k+1} \ &:= \mathbb{X}_h^{k+1} \cap \mathbb{X}_\Sigma. \end{aligned}$$

Then, the discrete problem associated to (5.2) reads: Find $(\boldsymbol{\sigma}_h, (\boldsymbol{\gamma}_h, \mathbf{u}_h, \varphi_h)) \in \mathbb{X}_{h,\Sigma}^{k+1} \times (\mathbb{Y}_h^k \times \mathbf{M}_h^k)$, such that

$$\begin{aligned} \widehat{\mathbf{a}}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \ + \ \widehat{\mathbf{b}}(\boldsymbol{\tau}_h, (\boldsymbol{\gamma}_h, \mathbf{u}_h, \varphi_h)) \quad &= \ \widehat{F}(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{h,\Sigma}^{k+1}, \\ \widehat{\mathbf{b}}(\boldsymbol{\sigma}_h, (\boldsymbol{\eta}_h, \mathbf{v}_h, \psi_h)) \quad &= \ \widehat{G}(\boldsymbol{\eta}_h, \mathbf{v}_h, \psi_h) \quad \forall\, (\boldsymbol{\eta}_h, \mathbf{v}_h, \psi_h) \in \mathbb{Y}_h^k \times \mathbf{M}_h^k. \end{aligned} \tag{5.3}$$

Notice that, since $\mathbf{V}_{h,0}^k \subseteq \mathbf{Q}_h^k$ and $\nabla\Psi_{h,0}^{k+1} \subseteq \mathbf{Q}_h^k$, then $\mathbb{X}_{h,\Sigma}^{k+1} \times \mathbb{Y}_h^k \times \mathbf{V}_{h,0}^k$ and $\mathbb{X}_{h,\Sigma}^{k+1} \times \mathbb{Y}_h^k \times \nabla\Psi_{h,0}^{k+1}$ are subspaces of the well-known Arnold-Falk-Winther elements for the elasticity problem (see [4]).

In the following theorem we summarize the well-posedness and the error estimates for the discrete scheme (5.3)

**Theorem 5.2** *There exists a unique* $(\boldsymbol{\sigma}_h, (\boldsymbol{\gamma}_h, (\mathbf{u}_h, \varphi_h))) \in \mathbb{X}_{h,\Sigma}^{k+1} \times (\mathbb{Y}_h^k \times \mathbf{M}_h^k)$ *solution to* (5.2). *Furthermore, the following a priori estimates hold:*

$$|\boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} \leq \nu \tilde{c}_1 \|\mathbf{u}_D\|_{1/2,\Gamma} + \tilde{c}_2 \|\mathbf{f}\|_{0,\Omega} \quad \text{and} \quad \|(\boldsymbol{\gamma}_h, (\mathbf{u}_h, \varphi_h))\|_{\mathbb{Y}\times\mathbf{M},\nu} \leq \tilde{c}_3 \nu \|\mathbf{u}_D\|_{1/2,\Gamma} + \tilde{c}_4 \|\mathbf{f}\|_{0,\Omega}, \tag{5.4}$$

*with* $c_1, c_2, c_3, c_4 > 0$, *all of them independent of* $\nu$. *In addition, if* $(\boldsymbol{\sigma}, (\boldsymbol{\gamma}, (\mathbf{u}, 0))) \in \mathbb{X}_0 \times (\mathbb{Y} \times \mathbf{M})$ *is the unique solution of* (5.2), *then there exist positive constants* $\widehat{c}_1$, $\widehat{c}_2$, $\widehat{c}_3$, $\widehat{c}_4$, $\widehat{c}_5$ *and* $\widehat{c}_6$, *independent of* $\nu$ *and* $h$, *such that,*

$$|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} \leq \widehat{c}_1 \inf_{\boldsymbol{\tau}_h \in \mathbb{X}_{h,0}^{k+1}} |\boldsymbol{\sigma} - \boldsymbol{\tau}_h|_{\mathbf{div},\Omega} + \nu \widehat{c}_2 \inf_{\mathbf{v}_h \in \mathbf{V}_{h,0}^k} \|\mathbf{u} - \mathbf{v}_h\|_{0,\Omega}, + \nu \widehat{c}_3 \inf_{\boldsymbol{\eta}_h \in \mathbb{Y}_{h,0}^k} \|\boldsymbol{\gamma} - \boldsymbol{\eta}_h\|_{0,\Omega} \tag{5.5}$$

*and*

$$\begin{aligned} \|(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, (\mathbf{u} - \mathbf{u}_h, \varphi_h))\|_{\mathbb{Y}\times\mathbf{M},\nu} \leq\ & \widehat{c}_4 \inf_{\boldsymbol{\tau}_h \in \mathbb{X}_{h,0}^{k+1}} |\boldsymbol{\sigma} - \boldsymbol{\tau}_h|_{\mathbf{div},\Omega} + \nu \widehat{c}_5 \inf_{\mathbf{v}_h \in \mathbf{V}_{h,0}^k} \|\mathbf{u} - \mathbf{v}_h\|_{0,\Omega} \\ & + \nu \widehat{c}_6 \inf_{\boldsymbol{\eta}_h \in \mathbb{Y}_{h,0}^k} \|\boldsymbol{\gamma} - \boldsymbol{\eta}_h\|_{0,\Omega}. \end{aligned} \tag{5.6}$$

*Finally, if* $\boldsymbol{\sigma} \in \mathbb{H}^m(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^m(\Omega)$, $\boldsymbol{\gamma} \in \mathbb{H}^m(\Omega)$ *and* $\mathbf{u} \in \mathbf{H}^m(\Omega)$ *for* $1 \leq m \leq k+1$. *Then, there exist positive constants* $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4 > 0$, *independent of* $\nu$ *and* $h$, *such that,*

$$|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} \leq \hat{c}_1 h^m \{ |\boldsymbol{\sigma}|_{m,\Omega} + |\mathbf{div}\,\boldsymbol{\sigma}|_{m,\Omega} \} + \nu \hat{c}_2 h^m (|\mathbf{u}|_{m,\Omega} + |\boldsymbol{\gamma}|_{m,\Omega}), \tag{5.7}$$

*and*

$$\|(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, (\mathbf{u} - \mathbf{u}_h, \varphi_h))\|_{\mathbb{Y}\times\mathbf{M},\nu} \leq \hat{c}_3 h^m \{ |\boldsymbol{\sigma}|_{m,\Omega} + |\mathbf{div}\,\boldsymbol{\sigma}|_{m,\Omega} \} + \nu \hat{c}_4 h^m (|\mathbf{u}|_{m,\Omega} + |\boldsymbol{\gamma}|_{m,\Omega}). \tag{5.8}$$

*Proof.* It is clear that $\widehat{\mathbf{a}}$ is $\mathbb{X}_{h,\Sigma}^{k+1}$–elliptic. In turn, given $(\boldsymbol{\eta}_h, (\mathbf{v}_h, \psi_h)) \in \mathbb{Y}_h^k \times \mathbf{M}_h^k$, by adapting [3, Theorem 11.9] to our context we can easily deduce that there exists $\widetilde{\boldsymbol{\tau}}_h \in \mathbb{X}_{\Sigma,h}^{k+1}$, satisfying $(\widetilde{\boldsymbol{\tau}}_h, \boldsymbol{\eta}_h)_\Omega = \nu \|\boldsymbol{\eta}_h\|_{0,\Omega}^2$, $\mathbf{div}\,\widetilde{\boldsymbol{\tau}}_h = \nu(\mathbf{v}_h + \nabla\psi_h)$ in $\Omega$ and $|\widetilde{\boldsymbol{\tau}}_h|_{\mathbf{div},\Omega} \leq C\nu(\|\mathbf{v}_h + \nabla\psi_h\|_{0,\Omega} + \|\boldsymbol{\eta}_h\|_{0,\Omega})$, which can be employed to deduce the estimate

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \mathbb{X}_{h,\Sigma}^{k+1}} \frac{\widehat{\mathbf{b}}(\boldsymbol{\tau}_h, (\boldsymbol{\gamma}_h, \mathbf{v}_h, \psi_h))}{|\boldsymbol{\tau}_h|_{\mathbf{div},\Omega}} \geq \beta^* \|(\boldsymbol{\gamma}_h, \mathbf{v}_h, \psi_h)\|_{\mathbb{Y}\times\mathbf{M},\nu}, \qquad \forall (\boldsymbol{\eta}_h, (\mathbf{v}_h, \psi_h)) \in \mathbb{Y}_h^k \times \mathbf{M}_h^k,$$

with $\beta^* > 0$, independent of $h$ and $\nu$. In this way, employing similar arguments to those in applied in Section 3 we can obtain that problem (5.2) is well-posed and the estimates (5.4)–(5.8) hold. We omit further details □

### 5.3 A posteriori error estimator

Let us consider the notations and definitions introduced in Section 4. Then, analogously to Section 4.1 we employ the associated global inf-sup condition, to obtain

$$|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{\mathbf{div},\Omega}+\|(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,(\mathbf{u}-\mathbf{u}_h,\varphi_h))\|_{\mathbb{Y}\times\mathbf{M},\nu}\;\le\;C(\|\widehat{\mathcal{R}}_1\|_{\mathbb{X}_\Sigma'}+\|\widehat{\mathcal{R}}_2\|_{(\mathbb{Y}\times\mathbf{M})'}) \tag{5.9}$$

with

$$\begin{aligned}\widehat{\mathcal{R}}_1(\boldsymbol{\tau}):=&\;\;\widehat{\mathbf{a}}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\tau})+\widehat{\mathbf{b}}(\boldsymbol{\tau},(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,(\mathbf{u}-\mathbf{u}_h,-\varphi_h)))\\ =&\;\;-(\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h,\mathbf{div}\,\boldsymbol{\tau})_\Omega\;+\;2\nu\langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle_\Gamma-(\boldsymbol{\sigma}_h^{\mathrm{d}}+2\nu\boldsymbol{\gamma}_h,\boldsymbol{\tau})_\Omega-2\nu(\mathbf{div}\,\boldsymbol{\tau},\mathbf{u}_h+\nabla\varphi_h)_\Omega,\end{aligned}$$

for all $\boldsymbol{\tau}\in\mathbb{X}_\Sigma$, and

$$\begin{aligned}\widehat{\mathcal{R}}_2(\boldsymbol{\eta},(\mathbf{v},\psi)):=&\;\;\mathbf{b}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,(\boldsymbol{\eta},(\mathbf{v},\psi)))=-2\nu(\boldsymbol{\sigma}_h,\boldsymbol{\eta})_\Omega-2\nu(\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h,\mathbf{v}+\nabla\psi)_\Omega\\ =&\;\;-\nu(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^t,\boldsymbol{\eta})_\Omega-2\nu(\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h,\mathbf{v}+\nabla\psi)_\Omega,\end{aligned}$$

for all $(\boldsymbol{\eta},(\mathbf{v},\psi))\in\mathbb{Y}\times\mathbf{M}$. In this way, we assume that there exists a convex domain $B\subseteq\mathbb{R}^d$, such that $\bar{\Omega}\subseteq B$ and $\Sigma\subseteq\partial B$ to obtain a stable Helmholtz decomposition for $\mathbb{X}_\Sigma$ (see [1, Lemma 3.9] for $d=2$ and [15, Theorem 3.2] for $d=3$) and assume further that $\mathbf{u}_D\in\mathbf{H}^1(\Gamma)$, to proceed similarly to [18] and [16], to obtain

$$\|\widehat{\mathcal{R}}_1\|_{\mathbb{X}_\Sigma'}+\|\widehat{\mathcal{R}}_2\|_{(\mathbb{Y}\times\mathbf{M})'}\le C\widehat{\Theta}, \tag{5.10}$$

with $C>0$, independent of $h$ and $\nu$, and

$$\widehat{\Theta}=\left\{\sum_{T\in\mathcal{T}_h}\widehat{\Theta}_T^2\right\}^{1/2},$$

where for each $T\in\mathcal{T}_h$, the local error indicator $\widehat{\Theta}_T$ is defined by

$$\begin{aligned}\widehat{\Theta}_T^2:=&\;\|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2+h_T^2\left\|2\nu\nabla\mathbf{u}_h-\boldsymbol{\sigma}_h^{\mathrm{d}}-2\nu\boldsymbol{\gamma}_h\right\|_{0,T}^2+h_T^2\left\|\underline{\mathbf{curl}}\left(\boldsymbol{\sigma}_h^{\mathrm{d}}+2\nu\boldsymbol{\gamma}_h\right)\right\|_{0,T}^2\\ &+\sum_{e\in\mathcal{E}_{h,T}(\Omega)}h_e\left\|\left[\left[\gamma_*\left(\boldsymbol{\sigma}_h^{\mathrm{d}}+2\nu\boldsymbol{\gamma}_h\right)\right]\right]\right\|_{0,e}^2+\sum_{e\in\mathcal{E}_{h,T}(\Gamma)}h_e\left\|\gamma_*\left(\boldsymbol{\sigma}_h^{\mathrm{d}}+2\nu\boldsymbol{\gamma}_h-2\nu\nabla\mathbf{u}_D\right)\right\|_{0,e}^2\\ &+\sum_{e\in\mathcal{E}_{h,T}(\Gamma)}h_e\nu^2\left\|\mathbf{u}_D-\mathbf{u}_h\right\|_{0,e}^2.\end{aligned} \tag{5.11}$$

We summarize the main results of this section in the following theorem.

**Theorem 5.3** *Assume that $\mathbf{u}_D\in\mathbf{H}^1(\Gamma)$ and that there exists a convex domain $B\subseteq\mathbb{R}^d$, such that $\bar{\Omega}\subseteq B$ and $\Sigma\subseteq\partial B$. Then, there exists $\widehat{C}_{rel}>0$, independent of $h$ and $\nu$, such that*

$$|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{\mathbf{div},\Omega}+\|(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,(\mathbf{u}-\mathbf{u}_h,\varphi_h))\|_{\mathbb{Y}\times\mathbf{M},\nu}\;\le\;\widehat{C}_{rel}\widehat{\Theta}. \tag{5.12}$$

*In addition, if $\mathbf{u}_D$ is a piecewise polynomial, then there exists $\widehat{C}_{eff}>0$, independent of $h$ and $\nu$, such that*

$$\widehat{C}_{eff}\widehat{\Theta}\le|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{\mathbf{div},\Omega}+\nu(\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{0,\Omega}+\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}) \tag{5.13}$$

*Proof.* We begin by observing that (5.12) follows from (5.9) and (5.10). In turn, analogously to Lemma 4.3 we can bound each one of the terms defining $\widehat{\Theta}$ (cf. (5.11)) by applying estimates already available in the literature (see, for instance, [9, Section 6] and [17, Section 4]) to obtain (5.13). We omit further details.

$\square$

# 6 Numerical results

In this section, for simplicity, we restrict ourselves to the Stokes problem with Dirichlet boundary condition in two-dimensions and report two numerical examples to illustrate the performance of the proposed finite element scheme and confirm the theoretical results. We begin by mentioning that the numerical results below are performed by imposing the divergence-free constraint for the velocity by means of a suitable Lagrange multiplier $r_h \in Q_h$. More precisely, we replace the numerical scheme (3.1) by the system: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \varphi_h, r_h, \lambda_h) \in \mathbb{X}_h^{k+1} \times \mathbf{V}_h^k \times \Psi_{h,0}^{k+1} \times Q_h^k \times \mathbb{R}$, such that

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \,+\, \mathbf{b}(\boldsymbol{\tau}_h, (\mathbf{u}_h, \varphi)) \,+\, \lambda_h(\operatorname{tr}\boldsymbol{\tau}_h, 1)_\Omega &= F(\boldsymbol{\tau}_h),\\
\mathbf{b}(\boldsymbol{\sigma}_h, (\mathbf{v}_h, \psi_h) + (r_h, \operatorname{div}\mathbf{v}_h)_\Omega &= G(\mathbf{v}_h, \psi_h),\\
(s_h, \operatorname{div}\mathbf{u}_h)_\Omega &= 0,\\
\eta_h(\operatorname{tr}\boldsymbol{\sigma}_h, 1)_\Omega &= 0,
\end{aligned}$$

$\forall\, (\boldsymbol{\tau}_h, \mathbf{v}_h, \psi_h, s_h, \eta_h) \in \mathbb{X}_h^{k+1} \times \mathbf{V}_h^k \times \Psi_{h,0}^{k+1} \times Q_h^k \times \mathbb{R}$. Notice that the condition $(\operatorname{tr}\boldsymbol{\sigma}_h, 1)_\Omega = 0$ is imposed through a penalization strategy using a scalar Lagrange multiplier (adding just one row and one column to the system).

We now introduce some additional notations. In what follows, $N$ stands for the total number of degrees of freedom defining $\mathbb{X}_h^{k+1} \times \mathbf{V}_h^k \times \Psi_{h,0}^{k+1} \times Q_h^k \times \mathbb{R}$ associated to the system (3.1) We denote the individual errors by

$$\mathsf{e}(\boldsymbol{\sigma}) \,:=\, |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{\mathbf{div},\Omega}\,, \quad \mathsf{e}(\mathbf{u}) \,:=\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}\,, \quad \mathsf{e}(p) \,:=\, \|p - p_h\|_{0,\Omega}\,,$$

where $p$ is the exact pressure and the approximate pressure $p_h$ is computed through the post-processing formula $p_h = -\frac{1}{d}\operatorname{tr}(\boldsymbol{\sigma}_h)$. In turn, noticing that from (4.5) and (4.9) we have

$$C_{eff}\Theta + h.o.t. \,\leq |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{\mathbf{div},\Omega} + \nu\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \leq\, C_{rel}\Theta,$$

we let

$$\mathsf{e}_\nu(\boldsymbol{\sigma}, \mathbf{u}) := \left\{(\mathsf{e}(\boldsymbol{\sigma}))^2 + (\nu\mathsf{e}(\mathbf{u}))^2\right\}^{1/2},$$

and define the effectivity index with respect to $\Theta$ (cf. (4.3)) by

$$\operatorname{eff}(\Theta) := \mathsf{e}_\nu(\boldsymbol{\sigma}, \mathbf{u})/\Theta.$$

In addition, we let $r(\%)$ be the experimental rate of convergence given by

$$r(\%) \,:=\, \frac{\log(\mathsf{e}(\%)/\mathsf{e}'(\%))}{\log(h/h')}$$

where $\mathsf{e}(\%)$ is any of the errors defined above and $h$ and $h'$ are two consecutive meshsizes with errors $\mathsf{e}$ and $\mathsf{e}'$. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where $N$ and $N'$ denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. Example 1 is used to illustrate the performance of the two dimensional mixed finite element scheme under a quasi-uniform refinement, whereas Examples 2 is utilized to illustrate the behavior of the adaptive algorithm associated to the a posteriori error estimators $\Theta$ defined in (4.4). Here we apply the following adaptive procedure from [29]:

1) Start with a coarse mesh $\mathcal{T}_h$.

2) Solve the discrete problem (3.1) for the current mesh $\mathcal{T}_h$.

3) Compute $\Theta_T := \Theta$ for each triangle $T \in \mathcal{T}_h$.

4) Check the stopping criterion and decide whether to finish or go to next step.

5) Use *blue-green* refinement on those $T' \in \mathcal{T}_h$ whose indicator $\Theta_{T'}$ satisfies

$$\Theta_{T'} \geq \frac{1}{2} \max_{T\in\mathcal{T}_h} \{\Theta_T : T \in \mathcal{T}_h\} .$$

6) Define resulting meshes as current meshes $\mathcal{T}_h$, and go to step 2.

The first example focuses on the performance of our method as a function of the viscosity $\nu$, by considering an exact solution $(\mathbf{u}, p)$ in the domain $\Omega := (0,1)^2$ given by

$$\mathbf{u}(x_1,x_2) = \begin{pmatrix} 2x_1^2x_2(2x_2-1)(x_2-1)(x_1-1)^2 \\ -2x_1x_2^2(x_2-1)^2(2x_1-1)(x_1-1) \end{pmatrix}, \qquad p(x_1,x_2) = x_1^5 + x_2^5 - \frac{1}{3},$$

In Table 6.1, we summarize the convergence history for a sequence of quasi-uniform triangulations, considering the viscosity $\nu = 1$ and $\nu = 1.0e-6$. We see there that the rate of convergence provided by Theorem 3.6 is attained by the unknowns. In addition, the $l^\infty$–norm of $\operatorname{div}\mathbf{u}_h$ in each mesh is close to 0 which shows that this method is mass conserving.

In our second test we consider a singularly perturbed problem where the solution has boundary layers at $x_1 = 1$ and $x_2 = 1$. We consider $\nu = 0.01$, $\Omega = (0,1)^2$ and the exact solution is given by

$$\mathbf{u}(x_1,x_2) = \begin{pmatrix} x_2 - \dfrac{exp(x_2/\nu)-1}{exp(1/\nu)-1} \\ x_1 - \dfrac{exp(x_1/\nu)-1}{exp(1/\nu)-1} \end{pmatrix}, \qquad p(x_1,x_2) = x_2 - x_1,$$

In Table 6.2 we present the convergence history of the method (in its lowest-order configuration), considering firstly a quasi-uniform refinement (at the top) and secondly an adaptive refinement (at the bottom). There, we observe that the a posteriori error estimator clearly improves the performance of the method, that is, the error decays faster and with optimal rate of convergence with the adaptive procedure, which can be also seen in Figure 6.2. In turn, examples of some adapted meshes are collected in Figure 6.1. We can observe there a clear clustering of elements near the boudary layer as expected.

Errors and rates of convergence with $k = 0$ and $\nu = 1.0$

| $N$ | $h$ | $\mathsf{e}(\boldsymbol{\sigma})$ | $r(\boldsymbol{\sigma})$ | $\mathsf{e}(\mathbf{u})$ | $r(\mathbf{u})$ | $\mathsf{e}(p)$ | $r(p)$ | $\|\text{div}\,\mathbf{u}_h\|_{l^\infty}$ | $\|\varphi_h\|_{l^\infty}$ |
|---|---|---|---|---|---|---|---|---|---|
| 374 | 0.373 | 0.606 | – | 0.327e-2 | – | | – | 2.8e-17 | 1.8e-16 |
| 1466 | 0.190 | 0.306 | 1.017 | 0.181e-2 | 0.884 | 0.790e-2 | 2.011 | 6.9e-17 | 3.3e-16 |
| 5630 | 0.095 | 0.145 | 1.078 | 0.093e-2 | 0.962 | 0.176e-2 | 2.165 | 1.1e-16 | 6.7e-16 |
| 22094 | 0.049 | 0.072 | 1.061 | 0.046e-2 | 1.072 | 0.044e-2 | 2.121 | 2.2e-16 | 1.2e-15 |
| 87368 | 0.028 | 0.037 | 1.183 | 0.023e-2 | 1.182 | 0.012e-2 | 2.321 | 5.1e-16 | 5.4e-15 |

Errors and rates of convergence with $k = 0$ and $\nu = 1.0e^{-6}$

| $N$ | $h$ | $\mathsf{e}(\boldsymbol{\sigma})$ | $r(\boldsymbol{\sigma})$ | $\mathsf{e}(\mathbf{u})$ | $r(\mathbf{u})$ | $\mathsf{e}(p)$ | $r(p)$ | $\|\text{div}\,\mathbf{u}_h\|_{l^\infty}$ | $\|\varphi_h\|_{l^\infty}$ |
|---|---|---|---|---|---|---|---|---|---|
| 374 | 0.373 | 0.583 | – | 344.270 | – | | – | 1.8e-12 | 3.0e-10 |
| 1466 | 0.190 | 0.287 | 1.051 | 63.551 | 2.510 | 0.776e-2 | 2.031 | 4.5e-13 | 3.5e-10 |
| 5630 | 0.095 | 0.138 | 1.055 | 8.761 | 2.859 | 0.174e-2 | 2.159 | 1.3e-13 | 1.6e-09 |
| 22094 | 0.049 | 0.068 | 1.067 | 1.864 | 2.345 | 0.429e-3 | 2.120 | 1.1e-13 | 2.3e-09 |
| 87368 | 0.028 | 0.035 | 1.171 | 0.283 | 3.321 | 0.1163-3 | 2.308 | 5.7e-14 | 4.9e-09 |

Table 6.1: Example 1: Degrees of freedom, meshsizes, errors, rates of convergence, $L^\infty$–norm of $\text{div}\,\mathbf{u}_h$ and $\varphi_h$ for the Galerkin scheme (3.1) with $k = 0$, considering $\nu = 1.0$ and $\nu = 1.0e-6$.

Numerical scheme with $k = 0$ and quasi-uniform refinement

| $N$ | $\mathsf{e}_\nu(\boldsymbol{\sigma}, \mathbf{u})$ | $r(\boldsymbol{\sigma}, \mathbf{u})$ | $\Theta$ | eff | $\|\text{div}\,\mathbf{u}_h\|_{l^\infty}$ | $\|\varphi_h\|_{l^\infty}$ |
|---|---|---|---|---|---|---|
| 602 | 5.3246 | – | 5.3310 | 0.9988 | 0.4e-14 | 1.5e-14 |
| 2318 | 6.8262 | -0.3686 | 6.8294 | 0.9995 | 0.7e-14 | 3.3e-8 |
| 9008 | 6.0515 | 0.1775 | 6.0526 | 0.9998 | 1.4e-14 | 3.3e-8 |
| 35474 | 4.1995 | 0.5331 | 4.1999 | 0.9999 | 2.8e-14 | 1.9e-8 |
| 140852 | 2.3912 | 0.8168 | 2.3914 | 0.9999 | 5.7e-14 | 1.2e-10 |

Numerical scheme with $k = 0$ and adaptive refinement

| $N$ | $\mathsf{e}_\nu(\boldsymbol{\sigma}, \mathbf{u})$ | $r(\boldsymbol{\sigma}, \mathbf{u})$ | $\Theta$ | eff | $\|\text{div}\,\mathbf{u}_h\|_{l^\infty}$ | $\|\varphi_h\|_{l^\infty}$ |
|---|---|---|---|---|---|---|
| 602 | 5.3246 | – | 5.3310 | 0.9988 | 3.6e-15 | 1.5e-14 |
| 1496 | 6.7340 | -0.5160 | 6.7365 | 0.9996 | 1.1e-14 | 3.8e-05 |
| 3350 | 4.9944 | 0.7414 | 4.9951 | 0.9999 | 5.7e-14 | 1.4e-06 |
| 7964 | 2.7041 | 1.4171 | 2.7043 | 0.9999 | 5.7e-14 | 1.7e-07 |
| 19520 | 1.5309 | 1.2691 | 1.5310 | 0.9999 | 1.1e-13 | 4.3e-08 |
| 41780 | 1.0389 | 1.0190 | 1.0390 | 0.9999 | 2.8e-13 | 1.3e-08 |
| 89522 | 0.6854 | 1.0917 | 0.6854 | 0.9999 | 3.4e-13 | 6.0e-09 |

Table 6.2: Example 2: convergence history and effectivity index for the Galerkin scheme (3.1) with $k = 0$ under quasi-uniform and adaptive refinements.

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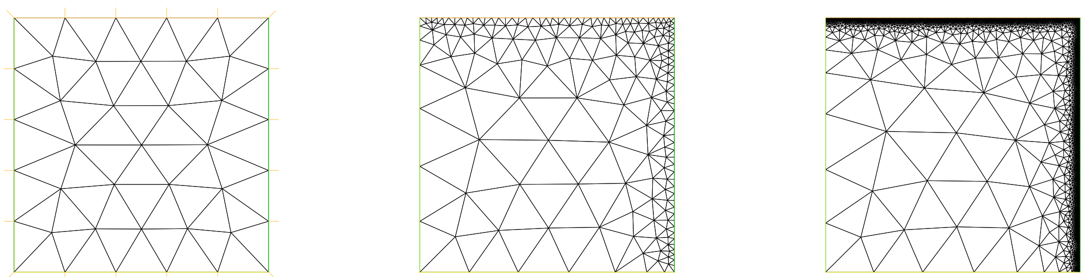

Figure 6.1: Example 2: three snapshots of successively refined meshes according to the indicator $\Theta$.

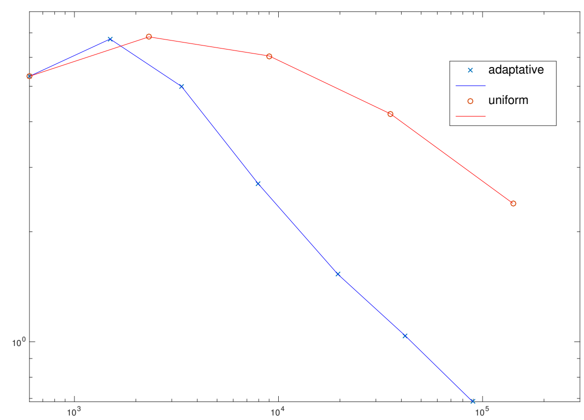


Figure 6.2: Example 2: loglog plot to compare the convergence of the total error under quasi-uniform and adaptive refinements.

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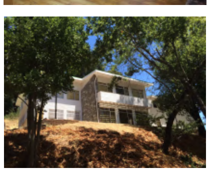


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