UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

Centro de Investigación en Ingeniería Matemática

A mass conservative finite element method for a nonisothermal Navier-Stokes/Darcy coupled system

Jessika Camaño, Ricardo Oyarzúa, Miguel Serón, Manuel Solano

PREPRINT 2023-30

SERIE DE PRE-PUBLICACIONES

A mass conservative finite element method for a nonisothermal Navier-Stokes/Darcy coupled system *

Jessika Camaño[†1,4], Ricardo Oyarzúa[‡2,4], Miguel Serón[§2] and Manuel Solano[¶3,4]

[1]Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Concepción, Chile
[2]GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Concepción, Chile
[3]Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Concepción, Chile
[4]Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, Concepción, Chile

Abstract

We propose and analyze an $\mathbf{H}(\mathrm{div})$-conforming and mass conservative finite element method for the coupling of nonisothermal fluid flow with nonisothermal porous media flow. The governing equations are the Navier-Stokes/heat system, commonly known as the Boussinesq system, in the free-fluid region, and the Darcy-heat coupled model in the membrane. These systems are coupled through buoyancy terms and a set of transmission conditions on the fluid-membrane interface, including mass conservation, balance of normal forces, the Beavers-Joseph-Saffman law, and continuity of heat flux and fluid temperature. We consider a velocity-pressure-temperature variational scheme for the Boussinesq system in the free-fluid region whereas in the membrane region we consider a dual-mixed formulation for the Darcy system coupled with a primal equation for the temperature model. In this way, the unknowns of the resulting formulation are given by the velocity, pressure and temperature in both domains. For the associated Galerkin scheme, we combine an $\mathbf{H}(\mathrm{div})$-conforming scheme for the fluid variables and a conforming Galerkin discretization for the heat equation. Therefore, the resulting numerical scheme produces exactly divergence-free velocities and also allows preserve the law of conservation of mass at a discrete level. The analysis of the continuous and discrete problems is carried out by means of a fixed-point strategy under a sufficiently small data assumption. We derive optimal error estimates under an additional assumption over the data and present numerical results illustrating the performance of the method.

Key words: fluid-membrane, Navier–Stokes, Darcy, heat equation, mass conservative, Discontinuous Galerkin, mixed finite element method.

Mathematics subject classifications (2020): 65N15, 65N30, 35K05, 76D05, 76S05, 74K15, 76R05, 76B03.

*This research was partially supported by ANID-Chile through projects Fondecyt Regular 1200569, 1200666 and 1231336, Centro de Modelamiento Matemático (FB210005), Anillo of Computational Mathematics for Desalination Processes (ACT210087) and Becas Chile Programme for national students 21190766.

† jecamano@ucsc.cl
‡ royarzua@ubiobio.cl
§ mseron@ubiobio.cl
¶ msolano@ing-mat.udec.cl

1 Introduction

For more than two decades, an important part of the numerical analysis community has been actively working on the devising of more efficient and robust numerical methods for simulating the dynamic interaction between a fluid flow, governed by the Stokes or Navier–Stokes equations, and porous media flow, modeled by the Darcy law. The above, motivated by its applicability in different areas of interest, such as medicine, petroleum engineering, and environmental and geophysical sciences, to name a few.

Several contributions can be found in the literature, from conforming to nonconforming schemes, including primal and mixed methods, along with Discontinuous Galerkin (DG) schemes and Hybridazible DG methods, domain decomposition and mortar methods. See, for instance, [4, 13, 14, 21, 22, 23, 25, 27, 28, 29, 30, 31, 32, 43], and the references therein. Nevertheless, the extension to the nonisothermal case remains limited, despite the diversity of applications where the model, recently studied in [44] (also see [45]), can be found, such as in geophysical sciences (see [35, 42]), petroleum engineering ([1, 36]), and desalination processes ([47, 52]). In fact, the first contribution in this direction is [12], where a conforming numerical scheme for the steady-state nonisothermal Navier-Stokes-Darcy coupled system is proposed. More precisely, a velocity-pressure-temperature variational formulation was employed for both the free-fluid and porous medium regions, coupled through terms involving a Lagrange multiplier representing the trace of the porous medium pressure on the interface. The Galerkin scheme utilizes Bernardi-Raugel and Raviart-Thomas elements for velocities in the free-fluid and porous medium domains, respectively, whereas piece-wise constant and linear elements were used for pressure and temperature, respectively, and continuous and piece-wise linear functions for the Lagrange multiplier. Through the integration of the theories presented in [22] and [7], and employing a suitable lifting technique for temperature data (as in [18] and [8]), the study in [12] establishes the existence of a solution using a fixed-point strategy under a smallness assumption on the data. Moreover, with more stringent conditions applied to the temperature solution, the paper demonstrates both uniqueness (for the continuous problem) and convergence.

In this work we proceed similarly to [39] and consider an $\mathbf{H}(\mathrm{div})$-conforming approximation for the velocity in both, the free-fluid region and the porous medium domain, to propose a strongly mass conservative discretization for the nonisothermal Navier-Stokes-Darcy coupled system. With this approach, the continuity of the normal components of the velocity are exactly preserved along the interface, thereby eliminating the need for introducing the aforementioned Lagrange multiplier, and differently from [12], the analysis of the discrete scheme can be conducted without modifying the convective term of the heat equation. In addition, the method has the distinct property that it yields exactly divergence-free velocity approximations. To enforce the H^1 continuity of the velocity in the free-fluid region, we proceed analogously to [17] and make use of an interior penalty discontinuous Galerkin (DG) technique.

For the discretization of the formulation, we utilize Brezzi–Douglas–Marini (BDM) elements of order k for the velocities, discontinuous elements of order $k-1$ for the pressure, and standard continuous elements of order k for the temperature in the free-fluid and porous media domains, respectively.

The analysis of the discrete problem is carried out by means of a sufficiently small data assumption and a fixed-point strategy. More precisely, we rewrite the variational problem as an equivalent fixed-point problem and apply the classical Brouwer fixed-point theorem to conclude existence of solution. In addition, under a smallness assumption on data and on the temperature in the membrane, we obtain the convergence of the Galerkin scheme and the corresponding theoretical rate of convergence.

To validate the theoretical results, we conduct several numerical simulations using manufactured solutions. These simulations demonstrate that the theoretical rates of convergence are consistently achieved in all scenarios, while also confirming the mass conservation property of the method. Furthermore, we employ the proposed numerical scheme in more realistic scenarios, where it successfully captures phenomena documented in the literature. In particular, as highlighted in [45] (also see [44]), it is observed that convection patterns within the free-fluid and porous medium regions undergo changes as the size of the free-fluid region decreases. To be more precise, when the free-fluid region has a shallow depth, the convection patterns extend to encompass both domains, whereas when the free-fluid region has a greater depth, the predominant convection patterns are primarily confined within the free-fluid region. In this context, we conduct simulations for both cases in Section 5 and achieve results that closely align with those reported in [45].

The remainder of the paper is organized as follows. In Section 2, we introduce the model problem, and present the corresponding variational formulation and the stability results together with the corresponding existence and uniqueness of the solution. Next, in Section 3, we introduce the mass-conservative numerical scheme and analyze its well-posedness. Section 4 focuses on error analysis and the derivation of the theoretical rate of convergence and conclude in Section 5 with the aforementioned numerical results.

We conclude this section by introducing some definitions and establishing specific notations. Let $\mathcal{O} \subseteq \mathbb{R}^d$, $d \in \{2,3\}$, denote a domain with Lipschitz boundary Γ. For $s \geq 0$ and $p \in [1,+\infty]$, we denote by $\mathrm{W}^{s,p}(\mathcal{O})$ the usual Sobolev space endowed with the norm $\|\cdot\|_{s,p,\mathcal{O}}$. If $s = 0$, $\mathrm{W}^{0,p}(\mathcal{O})$ corresponds to the usual Lebesgue space $\mathrm{L}^p(\mathcal{O})$, which is endowed with the norm $\|\cdot\|_{0,p,\mathcal{O}}$. If $p = 2$, we write $\mathrm{H}^s(\mathcal{O})$ in place of $\mathrm{W}^{s,2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$, respectively, and the seminorm by $|\cdot|_{s,\mathcal{O}}$. In addition, $\mathrm{H}^1_0(\mathcal{O})$ will denote the space of functions in $\mathrm{H}^1(\mathcal{O})$ with null trace on Γ, and $\mathrm{L}^2_0(\mathcal{O})$ will be the space of $\mathrm{L}^2(\mathcal{O})$ functions with zero mean value over $\mathcal{O}$, that is

$$\mathrm{L}^2_0(\mathcal{O}) := \Big\{ v \in \mathrm{L}^2(\mathcal{O}) : \int_{\mathcal{O}} v = 0 \Big\}.$$

Given $p, q \in (1,+\infty)$ satisfying $1/p + 1/q = 1$, in what follows, we will denote by $\mathrm{W}^{1/q,p}(\Gamma)$ the trace space of $\mathrm{W}^{1,p}(\mathcal{O})$ and by $\mathrm{W}^{-1/q,q}(\Gamma)$ the dual space of $\mathrm{W}^{1/q,p}(\Gamma)$ endowed with the norms $\|\cdot\|_{1/q,p;\Gamma}$ and $\|\cdot\|_{-1/q,q;\Gamma}$, defined respectively by

$$\|\phi\|_{1/q,p,\Gamma} := \inf\big\{ \|\psi\|_{1,p,\mathcal{O}} : \psi \in \mathrm{W}^{1,p}(\mathcal{O}), \psi|_\Gamma = \phi \big\} \qquad \forall \phi \in \mathrm{W}^{1/q,p}(\Gamma),$$

and

$$\|\psi\|_{-1/q,p,\Gamma} = \sup_{\xi \in \mathrm{W}^{1/q,p}(\Gamma)\backslash\{\mathbf{0}\}} \frac{\langle \psi, \xi \rangle_\Gamma}{\|\xi\|_{1/q,p,\Gamma}} \qquad \forall \psi \in \mathrm{W}^{-1/q,q}(\Gamma),$$

where $\langle \cdot,\cdot \rangle_\Gamma$ denotes the duality parity between $\mathrm{W}^{-1/q,q}(\Gamma)$ and $\mathrm{W}^{1/q,p}(\Gamma)$, which coincides with the inner product on $\mathrm{L}^2(\Gamma)$ when restricted to $\mathrm{L}^2(\Gamma)$. When $p = 2$, we will write $\mathrm{H}^{1/2}(\Gamma) := \mathrm{W}^{1/2,2}(\Gamma)$, $\|\cdot\|_{1/2,2,\Gamma} = \|\cdot\|_{1/2,\Gamma}$, $\mathrm{H}^{-1/2}(\Gamma) := \mathrm{W}^{-1/2,2}(\Gamma)$ and $\|\cdot\|_{-1/2,2,\Gamma} = \|\cdot\|_{-1/2,\Gamma}$.

Additionally, we recall that $\mathbf{H}(\mathrm{div};\mathcal{O}) := \Big\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \operatorname{div}\mathbf{w} \in \mathrm{L}^2(\mathcal{O}) \Big\}$, endowed with the norm $\|\mathbf{w}\|_{\mathrm{div};\mathcal{O}} := \Big(\|\mathbf{w}\|^2_{0,\mathcal{O}} + \|\operatorname{div}\mathbf{w}\|^2_{0,\mathcal{O}} \Big)^{1/2}$ is a standard Hilbert space in the realm of mixed problems (see, e.g., [10]). We also define the following subspace of $\mathbf{H}(\mathrm{div};\mathcal{O})$

$$\mathbf{H}_0(\mathrm{div};\mathcal{O}) := \Big\{ \mathbf{v} \in \mathbf{H}(\mathrm{div};\mathcal{O}) : \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O} \Big\},$$

where $\mathbf{n}$ is the exterior unit normal vector on define on the boundary $\partial\mathcal{O}$.

For simplicity, in what follows for any scalar fields v and w, vector fields $\mathbf{v} = (v_i)_{i=1,d}$ and $\mathbf{w} = (w_i)_{i=1,d}$, and tensor fields $\mathbf{A} = (a_{ij})_{i,j=1,d}$ and $\mathbf{B} = (b_{ij})_{i,j=1,d}$, we will denote

$$(v,w)_D := \int_D vw, \quad (\mathbf{v},\mathbf{w})_D := \int_D \mathbf{v}\cdot\mathbf{w} \quad \text{and} \quad (\mathbf{A},\mathbf{B})_D := \int_D \mathbf{A} : \mathbf{B},$$

where $D \in \{\mathcal{O}, \partial\mathcal{O}, e\}$, with $e \subseteq \partial\mathcal{O}$ and $\mathbf{A} : \mathbf{B} := \sum_{i,j=1}^{d} a_{ij} b_{ij}$.

By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, In turn, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$. Furthermore, given a non-negative integer k and a subset S of $\mathbb{R}^d$, $\mathrm{P}_k(S)$ stands for the space of polynomials defined on S of degree $\leq k$.

In the sequel we will employ $\mathbf{0}$ as a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

Finally we recall that the following continuous embeddings hold (see [48, Theorem 1.3.4] and [9, Proposition 1.4.2.]):

$$\mathrm{W}^{q,r}(\mathcal{O}) \hookrightarrow \mathrm{C}^0(\overline{\mathcal{O}}) \quad \text{and} \quad \mathrm{W}^{1,s}(\mathcal{O}) \hookrightarrow \mathrm{W}^{1,t}(\mathcal{O}), \tag{1.1}$$

for all $q > \frac{d}{r}$, and for all $1 \leq t \leq s \leq \infty$, respectively.

2 Continuous problem

In this section, we briefly present the model problem and establish the existence, uniqueness and stability of the associated weak formulation. The corresponding analysis can be found in [12, Section 2].

2.1 The model problem and its variational formulation

In order to describe the geometry of the problem, we let Ω_{f} and Ω_{m} be two bounded and simply connected polygonal domains in $\mathbb{R}^2$, such that $\partial\Omega_{\mathrm{f}} \cap \partial\Omega_{\mathrm{m}} = \Sigma \neq \emptyset$, and $\Omega_{\mathrm{f}} \cap \Omega_{\mathrm{m}} = \emptyset$. Then, we let $\Gamma_{\mathrm{f}} := \partial\Omega_{\mathrm{f}} \setminus \overline{\Sigma}$, $\Gamma_{\mathrm{m}} := \partial\Omega_{\mathrm{m}} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_{\mathrm{f}} \cup \Omega_{\mathrm{m}} \cup \Sigma$ and Ω_{f} (and hence inward to Ω_{m} when seen on Σ). On Σ we also consider a unit tangent vector $\mathbf{t}$ (see Fig. 2.1).

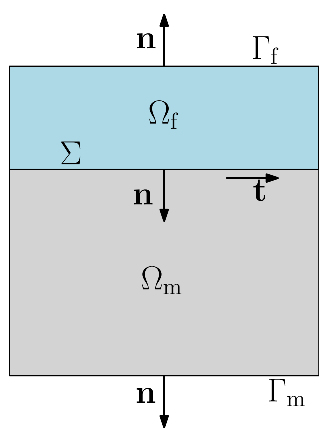


Figure 2.1: Sketch of the geometry of the domains.

The problem we are interested in consists of the movement of an incompressible viscous fluid subject to a heat source occupying Ω_{f} which flows towards and from a porous membrane Ω_{m} through Σ, where

Ω_{m} is saturated with the same fluid (see [44, 47]). The mathematical model is defined by two separate groups of equations and a set of coupling terms. In the free fluid domain Ω_{f}, the motion of the fluid can be described by the following Navier–Stokes/Heat system:

$$\boldsymbol{\sigma}_{\mathrm{f}} = 2\mu\,\mathbf{e}(\mathbf{u}_{\mathrm{f}}) - p_{\mathrm{f}}\,\mathbf{I} \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{2.1a}$$

$$-\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{f}} + (\mathbf{u}_{\mathrm{f}}\cdot\nabla)\,\mathbf{u}_{\mathrm{f}} - \mathbf{g}_{\mathrm{f}}\,\theta_{\mathrm{f}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{2.1b}$$

$$\operatorname{div}\mathbf{u}_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{2.1c}$$

$$-\kappa_{\mathrm{f}}\,\Delta\,\theta_{\mathrm{f}} + \mathbf{u}_{\mathrm{f}}\cdot\nabla\,\theta_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{2.1d}$$

where $\mu > 0$ is the dynamic viscosity of the fluid, $\mathbf{u}_{\mathrm{f}}$ is the fluid velocity, p_{f} is the fluid pressure, $\boldsymbol{\sigma}_{\mathrm{f}}$ is the Cauchy stress tensor, $\mathbf{I}$ is the 2×2 identity matrix, θ_{f} is the fluid temperature, $\kappa_{\mathrm{f}} > 0$ is the fluid thermal conductivity, $\mathbf{g}_{\mathrm{f}} \in \mathbf{L}^2(\Omega_{\mathrm{f}})$ is the external force per unit mass, $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor, and $\mathbf{e}(\mathbf{u}_{\mathrm{f}})$ is the strain rate tensor given by $\mathbf{e}(\mathbf{u}_{\mathrm{f}}) := \frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{f}} + (\nabla\mathbf{u}_{\mathrm{f}})^t\right)$, where the superscript t denotes transposition.

In the porous membrane Ω_{m} the behavior of the fluid can be described by the following Darcy-Heat system,

$$\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{m}} + \nabla\,p_{\mathrm{m}} - \mathbf{g}_{\mathrm{m}}\,\theta_{\mathrm{m}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{2.2a}$$

$$\operatorname{div}\mathbf{u}_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{2.2b}$$

$$-\kappa_{\mathrm{m}}\,\Delta\,\theta_{\mathrm{m}} + \mathbf{u}_{\mathrm{m}}\cdot\nabla\,\theta_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{2.2c}$$

where $\mathbf{u}_{\mathrm{m}}$ represents the fluid velocity, p_{m} the fluid pressure, θ_{m} the fluid temperature, $\mathbf{g}_{\mathrm{m}} \in \mathbf{L}^3(\Omega_{\mathrm{m}})$ a given external force, $\kappa_{\mathrm{m}} > 0$ the thermal conductivity, and $\mathbf{K} \in [\mathrm{L}^\infty(\Omega_{\mathrm{m}})]^{2\times 2}$ is a symmetric and uniformly positive definite tensor in Ω_{m} representing the intrinsic permeability $\boldsymbol{\kappa}$ of the membrane divided by the dynamic viscosity μ of the fluid. Throughout the paper we assume that there exists $C_{\mathbf{K}} > 0$ such that

$$\boldsymbol{\xi}^t\mathbf{K}(x)\,\boldsymbol{\xi} \,\geq\, C_{\mathbf{K}}\,|\boldsymbol{\xi}|^2,$$

for almost all $x \in \Omega_{\mathrm{m}}$, and for all $\boldsymbol{\xi} \in \mathbb{R}^2$.

The transmission conditions that couple the systems (2.1) and (2.2) on the interface Σ are given by

$$\begin{aligned} &\theta_{\mathrm{f}} = \theta_{\mathrm{m}} \quad \text{on} \quad \Sigma, && \kappa_{\mathrm{f}}\,\nabla\,\theta_{\mathrm{f}}\cdot\mathbf{n} = \kappa_{\mathrm{m}}\,\nabla\,\theta_{\mathrm{m}}\cdot\mathbf{n} \quad \text{on} \quad \Sigma, \\ &\mathbf{u}_{\mathrm{f}}\cdot\mathbf{n} = \mathbf{u}_{\mathrm{m}}\cdot\mathbf{n} \quad \text{on} \quad \Sigma, && \boldsymbol{\sigma}_{\mathrm{f}}\,\mathbf{n} + \frac{\alpha_d\,\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\cdot\mathbf{t}}}(\mathbf{u}_{\mathrm{f}}\cdot\mathbf{t})\,\mathbf{t} = -p_{\mathrm{m}}\,\mathbf{n} \quad \text{on} \quad \Sigma, \end{aligned} \tag{2.3}$$

where α_d is a dimensionless constant which depends only on the geometrical characteristics of the membrane (see [6]). In particular, the fourth condition in (2.3) can be decomposed, at least formally, into its normal and tangential components as follows:

$$(\boldsymbol{\sigma}_{\mathrm{f}}\,\mathbf{n})\cdot\mathbf{n} = -p_{\mathrm{m}} \quad \text{and} \quad (\boldsymbol{\sigma}_{\mathrm{f}}\,\mathbf{n})\cdot\mathbf{t} = -\frac{\alpha_d\,\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\cdot\mathbf{t}}}(\mathbf{u}_{\mathrm{f}}\cdot\mathbf{t}) \quad \text{on} \quad \Sigma. \tag{2.4}$$

The first equation in (2.4) corresponds to the balance of normal forces, whereas the second one is known as the Beavers–Joseph–Saffman law, which establishes that the slip velocity along Σ is proportional to the shear stress along Σ (assuming also, based on experimental evidence, that $\mathbf{u}_{\mathrm{m}}\cdot\mathbf{t}$ is negligible). We refer to [6, 37, 50] for further details on this interface condition. Finally, the Navier–Stokes/Darcy/Heat system (2.1), (2.2) and (2.3) is complemented with suitable boundary conditions:

$$\begin{aligned} &\mathbf{u}_{\mathrm{f}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{f}}, && \mathbf{u}_{\mathrm{m}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{m}}, \\ &\theta_{\mathrm{f}} = \theta_{\mathrm{D}}|_{\Gamma_{\mathrm{f}}} \quad \text{on} \quad \Gamma_{\mathrm{f}}, && \theta_{\mathrm{m}} = \theta_{\mathrm{D}}|_{\Gamma_{\mathrm{m}}} \quad \text{on} \quad \Gamma_{\mathrm{m}}, \end{aligned} \tag{2.5}$$

where $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ is a given function defined on $\Gamma := \Gamma_{\mathrm{f}} \cup \Gamma_{\mathrm{m}}$.

2.2 The variational formulation

Here, we derive the variational formulation for the coupled problem given by (2.1), (2.2), (2.3) and (2.5). We observe in advance that the weak problem introduced below is equivalent to the one introduced in [12].

First, we let

$$\mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) := \Big\{ \mathbf{v} \in \mathbf{H}^1(\Omega_{\mathrm{f}}) : \quad \mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{f}} \Big\}, \qquad \Psi_\infty := \{ \psi \in \mathrm{H}^1(\Omega) \,:\, \psi|_{\Omega_{\mathrm{m}}} \in \mathrm{L}^\infty(\Omega_{\mathrm{m}}) \},$$

and define the spaces

$$\mathbf{H} := \{ \mathbf{v} \in \mathbf{H}_0(\mathrm{div};\Omega) \,:\, \mathbf{v}|_{\Omega_{\mathrm{f}}} \in \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \} \quad \text{and} \quad \Psi_{\infty,0} := \Psi_\infty \cap \mathrm{H}^1_0(\Omega),$$

where $\mathbf{H}$ is endowed with the norm

$$\|\mathbf{v}\|_{\mathbf{H}} := \|\mathbf{v}|_{\Omega_{\mathrm{f}}}\|_{1,\Omega_{\mathrm{f}}} + \|\mathbf{v}|_{\Omega_{\mathrm{m}}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} \qquad \forall\, \mathbf{v} \in \mathbf{H}.$$

Then we define the global unknowns

$$\mathbf{u} := \mathbf{u}_{\mathrm{f}}\chi_{\mathrm{f}} + \mathbf{u}_{\mathrm{m}}\chi_{\mathrm{m}}, \qquad p := p_{\mathrm{f}}\chi_{\mathrm{f}} + p_{\mathrm{m}}\chi_{\mathrm{m}} \quad \text{and} \quad \theta := \theta_{\mathrm{f}}\chi_{\mathrm{f}} + \theta_{\mathrm{m}}\chi_{\mathrm{m}},$$

with $\chi_\star$ being the characteristic function:

$$\chi_\star := \begin{cases} 1 & \text{in} \quad \Omega_\star, \\ 0 & \text{in} \quad \Omega \setminus \overline{\Omega}_\star, \end{cases}$$

for $\star \in \{\mathrm{f}, \mathrm{m}\}$, and proceed analogously to [12] to obtain the variational problem: Find $\mathbf{u} \in \mathbf{H}$, $p \in \mathrm{L}^2_0(\Omega)$ and $\theta \in \mathrm{H}^1(\Omega)$, with $\theta|_\Gamma = \theta_{\mathrm{D}}$, such that:

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u};\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) - D(\theta,\mathbf{v}) &= 0 \qquad \forall\, \mathbf{v} \in \mathbf{H}, \\ B(\mathbf{u},q) &= 0 \qquad \forall\, q \in \mathrm{L}^2_0(\Omega), \\ A_{\mathrm{T}}(\theta,\psi) + O_{\mathrm{T}}(\mathbf{u};\theta,\psi) &= 0 \qquad \forall\, \psi \in \Psi_{\infty,0}, \end{aligned} \tag{2.6}$$

where the forms $A_{\mathrm{F}} : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$, $O_{\mathrm{F}} : \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \to \mathbb{R}$, $B : \mathbf{H} \times \mathrm{L}^2_0(\Omega) \to \mathbb{R}$, $D : \mathrm{H}^1(\Omega) \times \mathbf{H} \to \mathbb{R}$, $A_{\mathrm{T}} : \mathrm{H}^1(\Omega) \times \Psi_{\infty,0} \to \mathbb{R}$, and $O_{\mathrm{T}} : \mathbf{H} \times \mathrm{H}^1(\Omega) \times \Psi_{\infty,0} \to \mathbb{R}$, are defined respectively, as

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) &:= a_{\mathrm{F,f}}(\mathbf{u},\mathbf{v}) + a_{\mathrm{F,m}}(\mathbf{u},\mathbf{v}), & A_{\mathrm{T}}(\theta,\psi) &:= \kappa_{\mathrm{f}}(\nabla\theta,\nabla\psi)_{\Omega_{\mathrm{f}}} + \kappa_{\mathrm{m}}(\nabla\theta,\nabla\psi)_{\Omega_{\mathrm{m}}}, \\ B(\mathbf{v},q) &:= -(q,\mathrm{div}\,\mathbf{v})_\Omega, & D(\theta,\mathbf{v}) &:= (\theta\,\mathbf{g}_{\mathrm{f}},\mathbf{v})_{\Omega_{\mathrm{f}}} + (\theta\,\mathbf{g}_{\mathrm{m}},\mathbf{v})_{\Omega_{\mathrm{m}}}, \\ O_{\mathrm{F}}(\mathbf{w};\mathbf{u},\mathbf{v}) &:= ((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v})_{\Omega_{\mathrm{f}}}, & O_{\mathrm{T}}(\mathbf{w};\theta,\psi) &:= (\mathbf{w}\cdot\nabla\theta,\psi)_{\Omega_{\mathrm{f}}} + (\mathbf{w}\cdot\nabla\theta,\psi)_{\Omega_{\mathrm{m}}}, \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} a_{\mathrm{F,f}}(\mathbf{u},\mathbf{v}) &:= 2\mu\,(\mathbf{e}(\mathbf{u}),\mathbf{e}(\mathbf{v}))_{\Omega_{\mathrm{f}}} + \left\langle \tfrac{\alpha_d\,\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\cdot\mathbf{t}}}(\mathbf{u}\cdot\mathbf{t}), \mathbf{v}\cdot\mathbf{t} \right\rangle_\Sigma, \\ a_{\mathrm{F,m}}(\mathbf{u},\mathbf{v}) &:= (\mathbf{K}^{-1}\,\mathbf{u},\mathbf{v})_{\Omega_{\mathrm{m}}}. \end{aligned}$$

We observe that (2.6) is nothing but a slight reformulation of the continuous variational formulation analyzed in [12]. In fact since the following inf-sup condition holds

$$\sup_{(\mathbf{v}_\mathrm{f},\mathbf{v}_\mathrm{m})\in\mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})\times\mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m})}\frac{\langle\mathbf{v}_\mathrm{f}\cdot\mathbf{n}-\mathbf{v}_\mathrm{m}\cdot\mathbf{n},\xi\rangle_\Sigma}{\|\mathbf{v}_\mathrm{f}\|_{1,\Omega_\mathrm{f}}+\|\mathbf{v}_\mathrm{m}\|_{\mathrm{div};\Omega_\mathrm{m}}}\geq C\|\xi\|_{1/2,\Sigma}\qquad\forall\,\xi\in\mathrm{H}^{1/2}(\Sigma),$$

where $\mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m}):=\{\mathbf{v}\in\mathbf{H}(\mathrm{div};\Omega_\mathrm{m})\,:\,\mathbf{v}\cdot\mathbf{n}=0\quad\text{on}\quad\Gamma_\mathrm{m}\}$ (see [22, Lemma 1]), it is easy to see that problem (2.6) is equivalent to [12, eq. (2.8)].

To establish the existence, uniqueness and stability result for (2.6), we need to introduce some further notations and results. We first let $\mathrm{E}:\mathrm{W}^{3/4,4}(\Gamma)\to\mathrm{W}^{1,4}(\Omega)$ be the usual lifting operator (see for instance [24, Corollary B.53]), satisfying

$$\gamma_0(\mathrm{E}(\zeta))=\zeta\quad\text{and}\quad\|\mathrm{E}(\zeta)\|_{1,4,\Omega}\leq c\|\zeta\|_{3/4,4,\Gamma}\quad\forall\,\zeta\in\mathrm{W}^{3/4,4}(\Gamma),$$

where $\gamma_0:\mathrm{W}^{1,4}(\Omega)\to\mathrm{W}^{3/4,4}(\Gamma)$ is the trace operator. In turn, we let $\delta>0$, and similarly to [8, Lemma 2.8], define the function $\beta_\delta:\mathbb{R}^2\to\mathbb{R}$ given by

$$\beta_\delta(\mathbf{x}):=\begin{cases}1 & \text{if } 0\leq\mathrm{dist}(\mathbf{x},\Gamma)\leq\delta,\\ 2-\delta^{-1}\mathrm{dist}(\mathbf{x},\Gamma) & \text{if } \delta\leq\mathrm{dist}(\mathbf{x},\Gamma)\leq 2\delta,\\ 0 & \text{if } \mathrm{dist}(\mathbf{x},\Gamma)\geq 2\delta,\end{cases}$$

where $\mathrm{dist}(\mathbf{x},\Gamma)$ denotes the distance from the point $\mathbf{x}$ to the boundary Γ. Observe that β_δ is continuous and satisfies

$$\beta_\delta\in\mathrm{W}^{1,\infty}(\Omega),\quad 0\leq\beta_\delta\leq 1\quad\text{in}\quad\Omega_\delta,\quad\beta_\delta\equiv 0\quad\text{in}\quad\Omega\setminus\Omega_\delta,\quad\text{and}\quad\|\nabla\beta_\delta\|_{0,4,\Omega_\delta}\leq\delta^{-1}|\Omega_\delta|^{1/4},$$

where $\Omega_\delta:=\{\mathbf{x}\in\Omega:\mathrm{dist}(\mathbf{x},\Gamma)<2\delta\}$. In this way, in order to handle the non-homogeneous Dirichlet boundary condition for the temperature, we introduce the extension operator

$$\mathrm{E}_\delta:=\beta_\delta\mathrm{E}:\mathrm{W}^{3/4,4}(\Gamma)\to\mathrm{W}^{1,4}(\Omega).\tag{2.8}$$

The following theorem establishes the stability of solution of problem (2.6).

Theorem 2.1 *Let $\theta_1=\mathrm{E}_\delta(\theta_\mathrm{D})\in\mathrm{W}^{1,4}(\Omega)$, with $\delta>0$ satisfying*

$$\frac{c_1}{\alpha_\mathrm{F}\alpha_\mathrm{T}}\gamma_\mathbf{g}\,\delta^{1/12}\,\|\theta_\mathrm{D}\|_{3/4,4,\Gamma}\leq 1,\tag{2.9}$$

where $\gamma_\mathbf{g}:=\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}}+\|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}$, $\alpha_\mathrm{F}:=\frac{1}{2}\min\{c_2\mu,C_\mathbf{K}\}$, $\alpha_\mathrm{T}:=c_3\min\{\kappa_\mathrm{f},\kappa_\mathrm{m}\}$. Let $(\mathbf{u},p,\theta)$ be a solution of (2.6) and assume that

$$\|\mathbf{u}\cdot\mathbf{n}\|_{0,\Sigma}\,\leq\,c_4\mu.$$

Assume further that θ_1 satisfies

$$\frac{c_5}{\alpha_\mathrm{F}\,\alpha_\mathrm{T}}\,\gamma_\mathbf{g}\,\|\theta_1\|_{0,\infty,\Omega}\,\leq\,1.\tag{2.10}$$

There hold

$$\|\mathbf{u}\|_\mathbf{H}\,\leq\,C_\mathbf{u}\,\gamma_\mathbf{g}\,\|\theta_1\|_{1,\Omega},\qquad\|\theta_0\|_{1,\Omega}\,\leq\,C_\theta\,\|\theta_1\|_{1,\Omega},\tag{2.11}$$

and

$$\|p\|_{0,\Omega}\,\leq\,c_7\Big(C_{A_\mathrm{F}}C_\mathbf{u}\,+\,C_\mathbf{u}^2\,\gamma_\mathbf{g}\,\|\theta_1\|_{1,\Omega}\,+\,(C_\theta+1)\Big)\gamma_\mathbf{g}\|\theta_1\|_{1,\Omega},\tag{2.12}$$

where $\theta_0=\theta-\theta_1\in\mathrm{H}^1_0(\Omega)$, $C_\mathbf{u}:=c_6\,\alpha_\mathrm{F}^{-1}\,\alpha_\mathrm{T}^{-1}\,(C_{A_\mathrm{T}}+\alpha_\mathrm{T})$ and $C_\theta:=\alpha_\mathrm{T}^{-1}\,(2\,C_{A_\mathrm{T}}+\alpha_\mathrm{T})$, with C_{A_T} and C_{A_F} being the continuity constants of the bilinear forms A_T and A_F, respectively. Above, $c_1,\ldots,c_7>0$ are constants independent of the physical parameters.

Proof. Estimates (2.11) can be found in [12, Theorem 2.6] whereas (2.12) can be obtained analogously to [12, Corollary 2.7]. We omit further details. $\square$

Now we establish the existence and uniqueness result for problem (2.6).

Theorem 2.2 *Let $\delta > 0$ satisfying (2.9) and let $\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$ be such that (2.10) holds. Assume further that*

$$C_\mathbf{u}\,\gamma_\mathbf{g}\,\delta^{-3/4}(1+\delta^2)^{1/2}\,\|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \le C_1\mu,$$

where $\gamma_\mathbf{g} = \|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}$ and C_1 is a positive constant independent of the physical parameters. Then, there exist $\mathbf{u} \in \mathbf{H}$, $\theta_0 \in \mathrm{H}^1_0(\Omega)$ and $p \in \mathrm{L}^2_0(\Omega)$, such that $(\mathbf{u}, p, \theta_0 + \theta_1)$ is a solution to problem (2.6). In addition, if we assume that $\theta_0|_{\Omega_\mathrm{m}} \in \mathrm{L}^\infty(\Omega_\mathrm{m})$, and

$$(M_1\,\gamma_\mathbf{g} + M_2)\,C_{\mathrm{lift},2}\delta^{-3/2}(1+\delta^2)^{1/2}\|\theta_\mathrm{D}\|_{3/4,4,\Gamma} + M_3\|\theta_0\|_{0,\infty,\Omega_\mathrm{m}} + M_3\|\theta_1\|_{0,\infty,\Omega_\mathrm{m}} + M_4\gamma_\mathbf{g} < 1,$$

where

$$M_1 := C_2\alpha_\mathrm{F}^{-1}\,C_\mathbf{u}, \quad M_2 := C_3\alpha_\mathrm{T}^{-1}\,\left(C_\theta + 1\right), \quad M_3 := C_4\alpha_\mathrm{T}^{-1}, \quad \text{and} \quad M_4 := C_5\alpha_\mathrm{F}^{-1},$$

with C_1, C_2, C_3, C_4, C_5 being positive constants independent of the physical parameters, then the solution is unique.

Proof. The existence result is a direct consequence of [12, Theorem 2.12 and Lemma 2.5], whereas the uniqueness result can be found in [12, Theorem 2.13]. $\square$

3 A mass conservative numerical scheme

In this section we combine an $\mathbf{H}(\mathrm{div})$-conforming scheme for the fluid variables and a conforming Galerkin discretization for the temperature, to obtain a mass conservative numerical method to approximate the solution of problem (2.6). As we shall see next in the forthcoming sections, the study of the associated discrete scheme can be easily derived by applying the same arguments employed for the finite-dimensional problem presented in [12, Section 2.3.4].

3.1 Preliminaries

Let $\mathcal{T}_h^\mathrm{f}$ and $\mathcal{T}_h^\mathrm{m}$ be respective triangulations of the domains Ω_f and Ω_m, which are formed by shape-regular triangles of diameter h_T, and assume that they match in Σ so that $\mathcal{T}_h := \mathcal{T}_h^\mathrm{f} \cup \mathcal{T}_h^\mathrm{m}$ is a triangulation of $\Omega = \Omega_\mathrm{f} \cup \Sigma \cup \Omega_\mathrm{m}$. In addition, we let $h = \max\{h_T : T \in \mathcal{T}_h\}$.

For each $T \in \mathcal{T}_h$, we denote by $\mathbf{n}_T$ the unit outward normal vector on the boundary ∂T and let $\mathcal{E}(T)$ be the set of edges of T. We also denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h$, subdivided as follows:

$$\mathcal{E}_h := \mathcal{E}_h(\Gamma_\mathrm{f}) \cup \mathcal{E}_h(\Gamma_\mathrm{m}) \cup \mathcal{E}_h(\Omega_\mathrm{f}) \cup \mathcal{E}_h(\Omega_\mathrm{m}) \cup \mathcal{E}_h(\Sigma)\,,$$

where $\mathcal{E}_h(\Gamma_\star) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Gamma_\star \,\}$, $\mathcal{E}_h(\Omega_\star) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Omega_\star \,\}$ for each $\star \in \{\mathrm{f}, \mathrm{m}\}$, and $\mathcal{E}_h(\Sigma) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Sigma \,\}$. In what follows, h_e stands for the diameter of a given edge $e \in \mathcal{E}_h$.

We will use standard average and jump operators on $\overline{\Omega}_\mathrm{f}$. To define them, let T^+ and T^- be two adjacent elements of $\mathcal{T}_h^\mathrm{f}$, and $e = \partial T^+ \cap \partial T^- \in \mathcal{E}_h(\Omega_\mathrm{f})$. Let $\mathbf{v}$ be a piecewise smooth vector–valued function and let us denote by $\mathbf{v}^\pm$ its trace taken from within the interior of $T^\pm$. Then the jump $[\![\cdot]\!]$ acting on $\mathbf{v}$ is defined as

$$[\![\mathbf{v}]\!] := \begin{cases} \mathbf{v}^+ \otimes \mathbf{n}_{T^+} + \mathbf{v}^- \otimes \mathbf{n}_{T^-}, & e \in \mathcal{E}_h(\Omega_\mathrm{f}), \\ \mathbf{v} \otimes \mathbf{n}, & e \in \mathcal{E}_h(\Gamma_\mathrm{f}) \cup \mathcal{E}_h(\Sigma), \end{cases}$$

where $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega_{\mathrm{f}} = \Gamma_{\mathrm{f}} \cup \Sigma$ and $\mathbf{u} \otimes \mathbf{n}$ is the tensor product matrix $\mathbf{u} \otimes \mathbf{n} = (u_i n_j)_{i,j}$. In turn, for any smooth enough piecewise (vector- or tensor-valued) function η, and denoting by $\eta^{\pm}$ its trace taken from within the interior of $T^{\pm}$, we define its average as

$$\{\!\{\eta\}\!\} := \begin{cases} \frac{1}{2}\left(\eta^{+} + \eta^{-}\right), & e \in \mathcal{E}_h(\Omega_{\mathrm{f}}), \\ \eta, & e \in \mathcal{E}_h(\Gamma_{\mathrm{f}}) \cup \mathcal{E}_h(\Sigma). \end{cases}$$

3.2 Galerkin Scheme

For $k \geq 1$, let $\mathrm{P}_k(T)$ be the space of polynomials functions on T of degree less or equal than k and inspired by [39, 46], define the following finite element subspaces

$$\begin{aligned} &\mathbf{H}_h := \{\mathbf{v} \in \mathbf{H}_0(\mathrm{div};\Omega) : \mathbf{v}|_T \in [\mathrm{P}_k(T)]^d \quad \forall\, T \in \mathcal{T}_h\}, \\ &\mathrm{Q}_h := \{q \in \mathrm{L}^2_0(\Omega) : q|_T \in \mathrm{P}_{k-1}(T) \quad \forall\, T \in \mathcal{T}_h\}, \\ &\Psi_h := \{\psi \in \mathrm{C}^0(\overline{\Omega}) : \psi|_T \in \mathrm{P}_k(T) \quad \forall\, T \in \mathcal{T}_h\}, \quad \text{and} \quad \Psi_{h,0} := \Psi_h \cap \mathrm{H}^1_0(\Omega). \end{aligned} \tag{3.1}$$

Notice that $\mathbf{H}_h$ corresponds to the well-known Brezzi–Douglas–Marini finite element space [5]. In turn, denoting by $\mathbf{H}_h(\Omega_\star)$ the restriction of $\mathbf{H}_h$ to $\Omega_\star$ for $\star \in \{\mathrm{f}, \mathrm{m}\}$, we observe that $\mathbf{H}_h(\Omega_{\mathrm{m}}) \subseteq \mathbf{H}(\mathrm{div};\Omega_{\mathrm{m}})$, whereas $\mathbf{H}_h(\Omega_{\mathrm{f}})$ is not a subspace of $\mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$. Then, to overcome the nonconformity in Ω_{f}, we proceed similarly to [17] and introduce the following discontinuous versions of the forms $a_{\mathrm{F,f}}$ and O_{F}:

$$\begin{aligned} a^h_{\mathrm{F,f}}(\mathbf{u},\mathbf{v}) := \quad & 2\mu \sum_{T \in \mathcal{T}^{\mathrm{f}}_h} (\mathbf{e}(\mathbf{u}), \mathbf{e}(\mathbf{v}))_T - 2\mu \sum_{e \in \mathcal{E}_h(\Omega_{\mathrm{f}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{f}})} (\{\!\{\mathbf{e}(\mathbf{u})\}\!\}, [\![\mathbf{v}]\!])_e - 2\mu \sum_{e \in \mathcal{E}_h(\Omega_{\mathrm{f}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{f}})} (\{\!\{\mathbf{e}(\mathbf{v})\}\!\}, [\![\mathbf{u}]\!])_e \\ & + 2\mu \sum_{e \in \mathcal{E}_h(\Omega_{\mathrm{f}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{f}})} \frac{a^{\mathrm{pen}}}{h_e} ([\![\mathbf{u}]\!], [\![\mathbf{v}]\!])_e + \sum_{e \in \mathcal{E}_h(\Sigma)} \left\langle \frac{\alpha_d\, \mu}{\sqrt{\mathbf{t} \cdot \boldsymbol{\kappa} \cdot \mathbf{t}}} (\mathbf{u} \cdot \mathbf{t}), \mathbf{v} \cdot \mathbf{t} \right\rangle_e, \\ O^h_{\mathrm{F}}(\mathbf{w};\mathbf{u},\mathbf{v}) := \quad & \sum_{T \in \mathcal{T}^{\mathrm{f}}_h} ((\mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{v})_T + \sum_{e \in \mathcal{E}_h(\Omega_{\mathrm{f}})} \frac{1}{2} \left(\mathbf{w} \cdot \mathbf{n}_T - |\mathbf{w} \cdot \mathbf{n}_T|, (\mathbf{u}^{ext} - \mathbf{u}) \cdot \mathbf{v}\right)_e. \end{aligned}$$

Above, $a^{\mathrm{pen}} > 0$ is the well-known interior penalty parameter chosen on each edge to enforce stability (see, e.g., [2, 20]) whereas $\mathbf{u}^{ext}$ is the trace of $\mathbf{u}$ taken from within the exterior of T. $a^h_{\mathrm{F,f}}$, is the well-known Symmetric Interior Penalty Galerkin discrete form (SIPG) (see [2, 32]) and O^h_{F} is the upwind form introduced by Lasaint–Raviart in [41]. Other choices for $a_{\mathrm{F,f}}$ and O_{F} are equally feasible (see, e.g., [3] and [20, Section 6]), provided that the stability properties in Section 3.3 below hold.

Now, to introduce an approximation for the boundary datum θ_{D}, we proceed similarly to [12]. In fact, we let $\mathrm{I}_h : \mathrm{C}^0(\overline{\Omega}) \to \Psi_h$ be the well-known Lagrange interpolation operator and recall that, under the assumption $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ and for a given $\delta > 0$, $\mathrm{E}_\delta(\theta_{\mathrm{D}})$ belongs to $\mathrm{W}^{1,4}(\Omega) \subseteq \mathrm{C}^0(\overline{\Omega})$ (cf. (1.1)). Then, for a fixed $\delta > 0$ (to be specified below), we define the following approximation to θ_{D}:

$$\theta^\delta_{\mathrm{D},h} = \mathrm{I}_h(\mathrm{E}_\delta(\theta_{\mathrm{D}}))|_\Gamma \in \{\psi_{\mathrm{D},h} \in \mathrm{C}^0(\overline{\Gamma}) : \psi_{\mathrm{D},h}|_e \in \mathrm{P}_k(e) \text{ for all } e \in \mathcal{E}_h(\Gamma_{\mathrm{f}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{m}})\}. \tag{3.2}$$

Let us observe that since Ω is a polygonal domain, Ω_δ is also a polygon that can be discretized by shaped-regular triangles. According to this, for the forthcoming analysis we let $\mathcal{T}^\delta_h$ be a triangulation of Ω_δ and assume that $\mathcal{T}^\delta_h \subseteq \mathcal{T}_h$.

In this way, we propose the following numerical scheme to approximate the solution of problem (2.6): Find $(\mathbf{u}_h, p_h, \theta_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$, such that $\theta_h|_\Gamma = \theta^\delta_{\mathrm{D},h}$, and

$$\begin{aligned} A^h_{\mathrm{F}}(\mathbf{u}_h, \mathbf{v}) + O^h_{\mathrm{F}}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) + B(\mathbf{v}, p_h) - D(\theta_h, \mathbf{v}) &= 0 \quad \forall\, \mathbf{v} \in \mathbf{H}_h, \\ B(\mathbf{u}_h, q) &= 0 \quad \forall\, q \in \mathrm{Q}_h, \\ A_{\mathrm{T}}(\theta_h, \psi) + O_{\mathrm{T}}(\mathbf{u}_h; \theta_h, \psi) &= 0 \quad \forall\, \psi \in \Psi_{h,0}, \end{aligned} \tag{3.3}$$

where

$$A^h_{\mathrm{F}}(\mathbf{u}, \mathbf{v}) := a^h_{\mathrm{F,f}}(\mathbf{u}, \mathbf{v}) + a_{\mathrm{F,m}}(\mathbf{u}, \mathbf{v}), \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{H}_h,$$

and $a_{\mathrm{F,m}}$, B, D, A_{T}, and O_{T} are the forms defined in (2.7).

Remark 3.1 *If $\mathbf{u}_h \in \mathbf{H}_h$ is the discrete velocity satisfying (3.3), then the second equation of (3.3) and the fact that $\mathbf{H}_h \subseteq \mathbf{H}_0(\mathrm{div}; \Omega)$, imply that $\mathrm{div}\, \mathbf{u}_h = 0$ in Ω and $\mathbf{u}_h|_{\Omega_{\mathrm{f}}} \cdot \mathbf{n} = \mathbf{u}_h|_{\Omega_{\mathrm{m}}} \cdot \mathbf{n}$ on Σ, thus the method is strongly conservative. In turn, another feasible choice to approximate the velocity is*

$$\mathbf{H}_h := \{\mathbf{v} \in \mathbf{H}_0(\mathrm{div}; \Omega) : \mathbf{v}|_T \in \mathbf{RT}_k(T), \quad \forall\, T \in \mathcal{T}_h\},$$

where $\mathbf{RT}_k(T)$ corresponds to the well-known local Raviart–Thomas space of order k defined by $\mathbf{RT}_k(T) := [\mathrm{P}_k(T)]^2 \oplus \widetilde{\mathrm{P}}_k(T)\mathbf{x}$, with $\widetilde{\mathrm{P}}_k(T)$ being the space of polynomials with total degree equal to k on element T, and $\mathbf{x}$ is the generic vector in $\mathbb{R}^2$.

3.3 Discrete stability properties

Here we discuss the stability properties of the forms involved restricted to the corresponding discrete spaces. To that end, for a given $l \geq 1$, we first define the following broken Sobolev space

$$\mathbf{H}^l_h := \left\{ \mathbf{v} \in \mathbf{H}(\mathrm{div}; \Omega) : \mathbf{v}|_T \in \mathbf{H}^l(T) \quad \forall\, T \in \mathcal{T}^{\mathrm{f}}_h \right\}.$$

On $\mathbf{H}^l_h$, and for each $l = 1, 2$ we define the following norm:

$$\|\mathbf{v}\|_{l,\mathbf{H}_h} := \left(\|\mathbf{v}\|^2_{l,\mathcal{T}^{\mathrm{f}}_h} + \|\mathbf{v}\|^2_{\mathrm{div};\Omega_{\mathrm{m}}} \right)^{1/2} \qquad \forall\, \mathbf{v} \in \mathbf{H}^l_h,$$

where

$$\|\mathbf{v}\|_{1,\mathcal{T}^{\mathrm{f}}_h} := \left(\sum_{T \in \mathcal{T}^{\mathrm{f}}_h} \|\nabla \mathbf{v}\|^2_{0,T} + \frac{\alpha_d\, \mu}{\sqrt{\mathbf{t} \cdot \boldsymbol{\kappa} \cdot \mathbf{t}}} \sum_{e \in \mathcal{E}_h(\Sigma)} \|\mathbf{v} \cdot \mathbf{t}\|^2_{0,e} + \sum_{e \in \mathcal{E}_h(\Omega_{\mathrm{f}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{f}})} \frac{a^{\mathrm{pen}}}{h_e} \|[\![\mathbf{v}]\!]\|^2_{0,e} \right)^{1/2},$$

for all $\mathbf{v} \in \mathbf{H}^1_h$, and

$$\|\mathbf{v}\|_{2,\mathcal{T}^{\mathrm{f}}_h} := \left(\|\mathbf{v}\|^2_{1,\mathcal{T}^{\mathrm{f}}_h} + \sum_{T \in \mathcal{T}^{\mathrm{f}}_h} h^2_T |\mathbf{v}|^2_{2,T} \right)^{1/2},$$

for all $\mathbf{v} \in \mathbf{H}^2_h$.

In turn, for all $T \in \mathcal{T}_h$, we recall the following inverse and trace inequalities (see, e.g., [20])

$$|\eta|_{2,T} \leq c h^{-1}_T |\eta|_{1,T} \qquad \forall\, \eta \in \mathrm{P}_l(T), \tag{3.4}$$

$$\|\eta\|_{0,\partial T} \leq c(h^{-1/2}_T \|\eta\|_{0,T} + h^{1/2}_T |\eta|_{1,T}) \qquad \forall\, \eta \in \mathrm{H}^1(T), \tag{3.5}$$

$$\|\eta\|_{0,q,\partial T} \leq c h^{-1/q}_T \|\eta\|_{0,q,T} \qquad \forall\, \eta \in \mathrm{P}_l(T), \tag{3.6}$$

where c represents a positive constant independent of the mesh-size. In particular, from (3.4) we obtain that there exists a positive constant c, independent of the mesh size, such that (see [46, eq. (3.11)])

$$\|\mathbf{v}\|_{2,\mathcal{T}_h^{\mathrm{f}}} \leq c\|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \qquad \forall\, \mathbf{v} \in \mathbf{H}_h. \tag{3.7}$$

This inequality, and estimates (3.5) and (3.6), the latter with $q = 2$, imply (see [3])

$$|A_{\mathrm{F}}^h(\mathbf{u},\mathbf{v})| \leq \widetilde{C}_{A_{\mathrm{F}}}\|\mathbf{u}\|_{2,\mathbf{H}_h}\|\mathbf{v}\|_{1,\mathbf{H}_h} \qquad \forall\, \mathbf{u} \in \mathbf{H}_h^2,\, \forall\, \mathbf{v} \in \mathbf{H}_h, \tag{3.8a}$$

$$|A_{\mathrm{F}}^h(\mathbf{u},\mathbf{v})| \leq \widehat{C}_{A_{\mathrm{F}}}\|\mathbf{u}\|_{1,\mathbf{H}_h}\|\mathbf{v}\|_{1,\mathbf{H}_h} \qquad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{H}_h. \tag{3.8b}$$

In addition, the following estimates are straightforward

$$|B(\mathbf{v},q)| \leq \widehat{C}_B\|\mathbf{v}\|_{1,\mathbf{H}_h}\|q\|_{0,\Omega} \qquad \forall\, \mathbf{v} \in \mathbf{H}_h, \forall\, q \in \mathrm{Q}_h, \tag{3.9}$$

$$|A_{\mathrm{T}}(\theta,\psi)| \leq C_{A_{\mathrm{T}}}\|\theta\|_{1,\Omega}\|\psi\|_{1,\Omega} \qquad \forall\, \theta,\psi \in \Psi_{h,0}. \tag{3.10}$$

Now, for each $q \geq 1$ if $d = 2$ and $q \in [1, 6]$ if $d = 3$ we recall from [38, Proposition 4.5] that the following inequality holds

$$\|\mathbf{v}\|_{0,q,\Omega_{\mathrm{f}}} \leq c\|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \qquad \forall\, \mathbf{v} \in \mathbf{H}_h^1, \tag{3.11}$$

where $c > 0$ represents a positive constant independent of h. In addition, from [32, Theorem 4.4] we know that there exists $\widehat{C}_{\mathrm{tr},q} > 0$ independent of h, such that

$$\|\mathbf{v}\|_{0,q,\Sigma} \leq \widehat{C}_{\mathrm{tr},q}\|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \qquad \forall\, \mathbf{v} \in \mathbf{H}_h. \tag{3.12}$$

In particular, from (3.11), and after simple computations, we obtain

$$|D(\theta,\mathbf{v})| \leq \widehat{C}_D(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta\|_{1,\Omega}\|\mathbf{v}\|_{1,\mathbf{H}_h} \qquad \forall\, \theta \in \Psi_h,\, \forall\, \mathbf{v} \in \mathbf{H}_h. \tag{3.13}$$

On the other hand, proceeding analogously to the proof of [16, Proposition 4.2] for the two-dimensional case and as in [38, Section 7] for the 3D case (see also [46, Lemma 3.4]), we can obtain the estimate

$$|O_{\mathrm{F}}^h(\mathbf{w}_1;\mathbf{u},\mathbf{v}) - O_{\mathrm{F}}^h(\mathbf{w}_2;\mathbf{u},\mathbf{v})| \leq C_{\mathrm{Lip,F}}\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\mathcal{T}_h^{\mathrm{f}}}\|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}}\|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}}, \tag{3.14}$$

for all $\mathbf{w}_1, \mathbf{w}_2, \mathbf{u} \in \mathbf{H}_h^2$ and for all $\mathbf{v} \in \mathbf{H}_h$, with $C_{\mathrm{Lip,F}} > 0$, being a constant independent of h.

Let us now define the discrete kernel of B as

$$\mathbf{V}_h := \{\mathbf{v} \in \mathbf{H}_h : B(\mathbf{v},q) = 0 \quad \forall\, q \in \mathrm{Q}_h\}.$$

Since the pair $\big(\mathbf{H}_h, \mathrm{Q}_h\big)$ satisfies $\mathrm{div}\,(\mathbf{H}_h) \subseteq \mathrm{Q}_h$, it readily follows that (see [17])

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{H}_h : \mathrm{div}\,\mathbf{v} = 0 \quad \text{in} \quad \Omega\}.$$

We observe that, proceeding similarly to [12, Lemma 2.2], that is, integrating by parts and employing estimate (3.11), we deduce

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \leq C_{O_{\mathrm{T}}}\|\mathbf{w}\|_{1,\mathbf{H}_h}\|\psi\|_{1,\Omega}(\|\theta\|_{0,3,\Omega_{\mathrm{f}}} + \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}) \qquad \forall\, \mathbf{w} \in \mathbf{V}_h,\, \forall\, \theta,\psi \in \Psi_{h,0}, \tag{3.15}$$

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \leq \widetilde{C}_{O_{\mathrm{T}}}\|\mathbf{w}\|_{1,\mathbf{H}_h}\|\psi\|_{1,\Omega}(\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}) \qquad \forall\, \mathbf{w} \in \mathbf{V}_h,\, \forall\, \theta,\psi \in \Psi_{h,0}. \tag{3.16}$$

In turn, noticing that A_{T} is elliptic on $\Psi_{h,0}$ and, for a given $\mathbf{w} \in \mathbf{V}_h$, $O_{\mathrm{T}}(\mathbf{w};\cdot,\cdot)$ is skew-symmetric, that is

$$A_{\mathrm{T}}(\psi,\psi) \geq \alpha_{\mathrm{T}}\|\psi\|_{1,\Omega}^2 \quad \text{and} \quad O_{\mathrm{T}}(\mathbf{w};\psi,\psi) = 0 \qquad \forall\, \psi \in \Psi_{h,0},$$

where α_{T} is a positive constant independent of h, we conclude that for all $\mathbf{w} \in \mathbf{V}_h$, $A_{\mathrm{T}}(\cdot,\cdot)+O_{\mathrm{T}}(\mathbf{w};\cdot,\cdot)$ is elliptic:

$$A_{\mathrm{T}}(\psi,\psi) + O_{\mathrm{T}}(\mathbf{w};\psi,\psi) \geq \alpha_{\mathrm{T}} \|\psi\|_{1,\Omega}^2 \qquad \forall\, \psi \in \Psi_{h,0}. \tag{3.17}$$

Now, we recall from [49, Lemma 2.6] that, for a sufficiently large choice of $a^{\mathrm{pen}} > 0$, there holds

$$a_{\mathrm{F,f}}^h(\mathbf{v},\mathbf{v}) + a_{\mathrm{F,m}}(\mathbf{v},\mathbf{v}) \;\geq\; 2\mu\widehat{\alpha}_{\mathrm{f}} \|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}}^2 + C_{\mathbf{K}} \|\mathbf{v}\|_{\mathrm{div};\Omega_{\mathrm{m}}}^2 \qquad \forall\, \mathbf{v} \in \mathbf{V}_h, \tag{3.18}$$

with $\widehat{\alpha}_{\mathrm{f}} > 0$ independent of h. In addition, we recall from [16] that for any $\mathbf{w} \in \mathbf{V}_h$, O_{F}^h satisfies

$$O_{\mathrm{F}}^h(\mathbf{w};\mathbf{v},\mathbf{v}) = \frac{1}{2} \sum_{T \in \mathcal{T}_h^{\mathrm{f}}} \big(|\mathbf{w}\cdot\mathbf{n}|, |\mathbf{v}|^2\big)_{0,\partial K \setminus \partial\Omega_{\mathrm{f}}} + \frac{1}{2} \sum_{e \in \mathcal{E}_h(\Sigma)} (\mathbf{w}\cdot\mathbf{n}, |\mathbf{v}|^2)_e \qquad \forall\, \mathbf{v} \in \mathbf{H}_h. \tag{3.19}$$

Then, combining (3.18) and (3.19) we readily obtain the following result.

Lemma 3.1 *Let $\mathbf{w} \in \mathbf{V}_h$, be such that*

$$\|\mathbf{w}\cdot\mathbf{n}\|_{0,\Sigma} \leq 2\mu\widehat{\alpha}_{\mathrm{f}} \widehat{C}_{\mathrm{tr},4}^{-2}. \tag{3.20}$$

Then,

$$A_{\mathrm{F}}^h(\mathbf{v},\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{w};\mathbf{v},\mathbf{v}) \geq \widehat{\alpha}_{\mathrm{F}} \|\mathbf{v}\|_{1,\mathbf{H}_h}^2 \qquad \forall\, \mathbf{v} \in \mathbf{V}_h. \tag{3.21}$$

Proof. Given $\mathbf{w} \in \mathbf{V}_h$ satisfying (3.20), from (3.18) and (3.19) we obtain

$$A_{\mathrm{F}}^h(\mathbf{v},\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{w};\mathbf{v},\mathbf{v}) \;\geq\; 2\mu\widehat{\alpha}_{\mathrm{f}} \|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}}^2 + C_{\mathbf{K}} \|\mathbf{v}\|_{\mathrm{div};\Omega_{\mathrm{m}}}^2 - \frac{1}{2} \sum_{e \in \mathcal{E}_h(\Sigma)} |(\mathbf{w}\cdot\mathbf{n}, |\mathbf{v}|^2)_e| \quad \forall\, \mathbf{v} \in \mathbf{V}_h.$$

Then, the result follows by applying (3.12) and proceeding analogously to the proof of [22, Lemma 10]. We omit further details. □

Let us now observe that combining [51, Theorem 6.12] and [11, eq. (7.1.28)], it is possible to prove that B satisfies the discrete inf-sup condition

$$\sup_{\mathbf{v} \in \mathbf{H}_h \setminus \{\mathbf{0}\}} \frac{B(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} \geq \widehat{\beta} \|q\|_{0,\Omega} \qquad \forall\, q \in \mathrm{Q}_h, \tag{3.22}$$

with $\widehat{\beta} > 0$, independent of h.

Finally, analogously to the continuous case, for a given $\delta > 0$ we introduce the discrete extension operator $\mathrm{E}_{\delta,h} : \mathrm{W}^{3/4,4}(\Gamma) \to \Psi_h$ given by $\mathrm{E}_{\delta,h} := \mathrm{I}_h \mathrm{E}_\delta$, where E_δ is the extension operator defined in (2.8) and I_h is the Lagrange interpolation operator. Then, it is clear from (3.2) that there holds

$$\theta_{\mathrm{D},h}^\delta = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})|_\Gamma. \tag{3.23}$$

Moreover, we recall from [12, Lemma 3.3], that the aforementioned operator satisfies:

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,3,\Omega} \leq \widehat{C}_{\mathrm{lift},1} \delta^{1/12} (h\delta^{-1} + h + 1) \|\zeta\|_{3/4,4,\Gamma}, \tag{3.24a}$$

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{1,\Omega} \leq \widehat{C}_{\mathrm{lift},2} \delta^{1/4} (2 + \delta^{-1}) \|\zeta\|_{3/4,4,\Gamma}, \tag{3.24b}$$

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,\infty,\Omega} \leq 3 \|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta}, \tag{3.24c}$$

for all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$, where $\widehat{C}_{\mathrm{lift},1}, \widehat{C}_{\mathrm{lift},2} > 0$ are constants independent of h and δ.

3.4 Existence of solution

In this section we proceed similarly to [12, Section 2.3.4] to establish existence of a discrete solution to problem (3.3) by means of an equivalent fixed-point problem. To that end, we let $\delta > 0$ be fixed (to be specified below in Lemma 3.4) and decompose the discrete temperature θ_h as $\theta_h = \theta_{h,0} + \theta_{h,1}$, with $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_\mathrm{D}) \in \Psi_h$ and $\theta_{h,0} = \theta_h - \theta_{h,1} \in \Psi_{h,0}$. In turn, to simplify the subsequent analysis, we introduce the reduced version of problem (3.3): Find $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ such that

$$\begin{aligned} A_\mathrm{F}^h(\mathbf{u}_h, \mathbf{v}) + O_\mathrm{F}^h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) - D(\theta_{h,0}, \mathbf{v}) &= D(\theta_{h,1}, \mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{V}_h, \\ A_\mathrm{T}(\theta_{h,0}, \psi) + O_\mathrm{T}(\mathbf{u}_h; \theta_{h,0} + \theta_{h,1}, \psi) &= -A_\mathrm{T}(\theta_{h,1}, \psi) && \forall\, \psi \in \Psi_{h,0}. \end{aligned} \tag{3.25}$$

The equivalence between problems (3.3) and (3.25) follow from the discrete inf-sup condition (3.22) and the definition of the lifting $\theta_{h,1}$ (cf. (3.23)). This is established in the following lemma, whose proof is standard and therefore omitted.

Lemma 3.2 *If $(\mathbf{u}_h, p_h, \theta_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ is a solution of (3.3), then $\mathbf{u}_h \in \mathbf{V}_h$ and $(\mathbf{u}_h, \theta_{h,0}) = (\mathbf{u}_h, \theta_h - \theta_{h,1})$ is a solution of (3.25). Conversely, if $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ is a solution of (3.25), then there exists $p_h \in \mathrm{Q}_h$, such that $(\mathbf{u}_h, p_h, \theta_h) = (\mathbf{u}_h, p_h, \theta_{h,0} + \theta_{h,1})$ is a solution of (3.3).*

According to the previous lemma, to prove existence of solution of problem (2.6), it suffices to prove solvability of problem (3.25). To do that, now we define the bounded and convex set

$$\mathbf{X}_h := \left\{ \begin{array}{c} (\mathbf{w}, \phi) \in \mathbf{V}_h \times \Psi_{h,0} : \|\mathbf{w}\|_{1,\mathbf{H}_h} \leq \widehat{C}_\mathbf{u}(\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}})\|\theta_{h,1}\|_{1,\Omega} \\ \text{and } \|\phi\|_{1,\Omega} \leq C_\theta \|\theta_{h,1}\|_{1,\Omega} \end{array} \right\}, \tag{3.26}$$

with

$$\widehat{C}_\mathbf{u} := 2\widehat{\alpha}_\mathrm{F}^{-1}\alpha_\mathrm{T}^{-1}\widehat{C}_D(C_{A_\mathrm{T}} + \alpha_\mathrm{T}) \quad \text{and} \quad C_\theta := \alpha_\mathrm{T}^{-1}(2C_{A_\mathrm{T}} + \alpha_\mathrm{T}),$$

and the discrete operator $\mathcal{J}_h : \mathbf{X}_h \to \mathbf{V}_h \times \Psi_{h,0}$, given by

$$\mathcal{J}_h(\mathbf{w}, \phi) = (\mathbf{u}_h, \theta_{h,0}) \qquad \forall\, (\mathbf{w}, \phi) \in \mathbf{X}_h,$$

where $(\mathbf{u}_h, \theta_{h,0})$ is the solution of the linearized version of problem (3.25): Find $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$, such that

$$\begin{aligned} A_\mathrm{F}^h(\mathbf{u}_h, \mathbf{v}) + O_\mathrm{F}^h(\mathbf{w}; \mathbf{u}_h, \mathbf{v}) &= D(\phi, \mathbf{v}) + D(\theta_{h,1}, \mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{V}_h, \\ A_\mathrm{T}(\theta_{h,0}, \psi) + O_\mathrm{T}(\mathbf{w}; \theta_{h,0}, \psi) &= -A_\mathrm{T}(\theta_{h,1}, \psi) - O_\mathrm{T}(\mathbf{w}; \theta_{h,1}, \psi) && \forall\, \psi \in \Psi_{h,0}. \end{aligned} \tag{3.27}$$

It is clear that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ is a solution of problem (3.25), if and only if, $\mathcal{J}_h(\mathbf{u}_h, \theta_{h,0}) = (\mathbf{u}_h, \theta_{h,0})$. In this way, to prove solvability of (3.25) in what follows we prove equivalently that $\mathcal{J}_h$ has a fixed-point in $\mathbf{X}_h$ by means of the well-known Brouwer's fixed-point theorem written in the following form (see [15, Theorem 9.9-2]):

Theorem 3.3 *Let $f : \mathrm{Y} \to \mathrm{Y}$ be a continuous mapping. If Y is a compact and convex subset of a finite dimensional Banach space X, then f has at least one fixed point.*

The following result establishes existence of a fixed-point of $\mathcal{J}_h$.

Lemma 3.4 *Let $\delta > 0$ be such that $h \leq \delta$ and*

$$\frac{\widehat{C}_D C_{O_\mathrm{T}}}{\widehat{\alpha}_\mathrm{F}\alpha_\mathrm{T}}(\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}})\widehat{C}_{\mathrm{lift},1}\delta^{1/12}(\delta + 2)\|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \leq \frac{1}{4}. \tag{3.28}$$

In addition, assume that $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ and θ_D satisfy, respectively,

$$\frac{\widehat{C}_D C_{O_{\mathrm{T}}}}{\widehat{\alpha}_{\mathrm{F}}\alpha_{\mathrm{T}}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_{h,1}\|_{0,\infty,\Omega} \le \frac{1}{4}, \tag{3.29}$$

and

$$\widehat{C}_{\mathrm{tr},2}\widehat{C}_{\mathbf{u}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\widehat{C}_{\mathrm{lift},2}\delta^{1/4}(2+\delta^{-1})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \le 2\mu\widehat{\alpha}_{\mathrm{f}}\widehat{C}_{\mathrm{tr},4}^{-2}. \tag{3.30}$$

Then, $\mathcal{J}_h$ is well defined and there exists at least one $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$, such that $\mathcal{J}_h(\mathbf{u}_h, \theta_{h,0}) = (\mathbf{u}_h, \theta_{h,0})$.

Proof. Let us start by noting that the assumption $h \le \delta$ implies that estimate (3.24a) with $\zeta = \theta_{\mathrm{D}}$ becomes

$$\|\theta_{h,1}\|_{0,3,\Omega} = \|\mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})\|_{0,3,\Omega} \le \widehat{C}_{\mathrm{lift},1}\delta^{1/12}(\delta+2)\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}.$$

In addition, for a given $(\mathbf{w}, \phi) \in \mathbf{X}_h$, we note that using the trace inequality (3.12) and the estimates (3.24b) and (3.30), we get

$$\|\mathbf{w}\cdot\mathbf{n}\|_{0,\Sigma} \le \widehat{C}_{\mathrm{tr},2}\|\mathbf{w}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \le \widehat{C}_{\mathrm{tr},2}\|\mathbf{w}\|_{1,\mathbf{H}_h} \le \widehat{C}_{\mathrm{tr},2}\widehat{C}_{\mathbf{u}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_{h,1}\|_{1,\Omega} \le 2\mu\widehat{\alpha}_{\mathrm{f}}\widehat{C}_{\mathrm{tr},4}^{-2}.$$

Then, from (3.17) and (3.21) and the Lax-Milgram lemma we conclude the unique solvability of (3.27), which implies that $\mathcal{J}_h$ is well defined. In addition, employing (3.17), (3.21), (3.27), (3.28) and (3.29) we easily obtain that $(\mathbf{u}_h, \theta_{h,0}) = \mathcal{J}_h(\mathbf{w}, \phi) \in \mathbf{X}_h$, thus $\mathcal{J}_h(\mathbf{X}_h) \subseteq \mathbf{X}_h$ (see [12, Lemma 2.9] for details). In turn, proceeding analogously to the proof of [12, Lemma 2.10], is it possible to obtain that $\mathcal{J}_h$ is continuous. In this way, applying Theorem 3.3 it follows that there exists at least one $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$, such that $(\mathbf{u}_h, \theta_{h,0}) = \mathcal{J}_h(\mathbf{u}_h, \theta_{h,0})$. □

We end this section by establishing the existence result for problem (3.3).

Theorem 3.5 *Assume that the hypotheses of Lemma 3.4 hold. Then, there exists at least one $(\mathbf{u}_h, p_h, \theta_h) = (\mathbf{u}_h, p_h, \theta_{h,0} + \theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ solution to (3.3). Moreover, there exists $C > 0$, independent of the solution and h, such that*

$$\|\mathbf{u}_h\|_{1,\mathbf{H}_h} + \|p_h\|_{0,\Omega} + \|\theta_h\|_{1,\Omega} \le C\left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}} + \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}\right). \tag{3.31}$$

Proof. The existence of solution of (3.3) is a direct consequence of Lemmas 3.2 and 3.4, whereas estimate (3.31) follows from the fact that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$ (cf. (3.26)) and the inf-sup condition (3.22). We omit further details and refer the reader to [12, Corollary 2.7]. □

4 Error analysis

In this section, we carry out the error analysis of the finite element scheme (3.3). To that end, from now on we assume that the hypotheses of Theorem 2.2 hold and let $(\mathbf{u}, p, \theta) = (\mathbf{u}, p, \theta_0 + \theta_1) \in \mathbf{H} \times \mathrm{L}_0^2(\Omega) \times \mathrm{H}^1(\Omega)$ be the unique solution of problem (2.6), with $\theta_1 = \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega)$ and $\theta_0 \in \mathrm{H}_0^1(\Omega)$. In addition, we assume that the hypotheses of Lemma 3.4 hold and let $(\mathbf{u}_h, p_h, \theta_h) = (\mathbf{u}_h, p_h, \theta_{h,0} + \theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ be a solution of (3.3), with $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ and $\theta_{h,0} \in \Psi_{h,0}$. In addition, given $k \ge 1$, in what follows we assume that the exact solution satisfies:

$$\begin{gathered}\mathbf{u} \in \left\{\mathbf{v} \in \mathbf{H} \,:\, \mathbf{v}|_{\Omega_{\mathrm{f}}} \in \mathbf{H}^{k+1}(\Omega_{\mathrm{f}}), \quad \mathbf{v}|_{\Omega_{\mathrm{m}}} \in \mathbf{H}^k(\Omega_{\mathrm{m}}) \quad \text{and} \quad \mathrm{div}\,(\mathbf{v}|_{\Omega_{\mathrm{m}}}) \in \mathrm{H}^k(\Omega_{\mathrm{m}})\right\}, \\ p \in \mathrm{H}^k(\Omega), \quad \theta \in \mathrm{H}^{k+1}(\Omega), \quad \theta|_{\Omega_{\mathrm{m}}} \in \mathrm{W}^{k+1,4}(\Omega_{\mathrm{m}}).\end{gathered} \tag{4.1}$$

Then, we let $\mathbf{\Pi}_h : \mathbf{H}^1(\Omega) \to \mathbf{H}_h$ be the BDM interpolation, $\mathcal{P}_h : \mathrm{L}^2(\Omega) \to \mathrm{Q}_h$ the L^2-projection, and $\mathrm{I}_h : \mathrm{C}(\overline{\Omega}) \to \Psi_h$ the nodal projection, and write the corresponding errors as

$$\mathbf{e}_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_h, \qquad \mathrm{e}_p = p - p_h, \quad \text{and} \quad \mathrm{e}_\theta = \theta - \theta_h.$$

then, we decompose these errors as follows

$$\begin{aligned}
&\mathbf{e}_{\mathbf{u}} = \boldsymbol{\varrho}_{\mathbf{u}} + \boldsymbol{\chi}_{\mathbf{u}}, && \boldsymbol{\varrho}_{\mathbf{u}} = \mathbf{u} - \mathbf{\Pi}_h(\mathbf{u}), && \boldsymbol{\chi}_{\mathbf{u}} = \mathbf{\Pi}_h(\mathbf{u}) - \mathbf{u}_h, \\
&\mathrm{e}_p = \varrho_p + \chi_p, && \varrho_p = p - \mathcal{P}_h(p), && \chi_p = \mathcal{P}_h(p) - p_h, \\
&\mathrm{e}_\theta = \varrho_\theta + \chi_\theta, && \varrho_\theta = \theta - \mathrm{I}_h(\theta), && \chi_\theta = \mathrm{I}_h(\theta) - \theta_h.
\end{aligned} \tag{4.2}$$

With the above definitions, we observe from [11, Chapter 2], [26, Chapter 3], and [24, Section 1.5] that the following estimates hold

$$\begin{gathered}
\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\mathrm{f}}} \le c h^k \|\mathbf{u}\|_{k+1,\Omega_{\mathrm{f}}}, \quad \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} \le c h^k (\|\mathbf{u}\|_{k,\Omega_{\mathrm{m}}} + \|\mathrm{div}\,\mathbf{u}\|_{k,\Omega_{\mathrm{m}}}), \\
\|\varrho_p\|_{0,\Omega} \le c h^k \|p\|_{k,\Omega}, \quad \|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} \le c h^k \|\theta\|_{k+1,\Omega_{\mathrm{f}}}, \quad \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}} \le c h^k \|\theta\|_{k+1,4,\Omega_{\mathrm{m}}}.
\end{gathered} \tag{4.3}$$

Finally, the following estimate for O_{T} will be employed next in the error analysis for a given $\mathbf{w} \in \mathbf{H}_h$, whose proof can be obtained proceeding analogously to the proof of [12, Lemma 4.1]: There exists $\bar{C}_{O_{\mathrm{T}}} > 0$, independent of h, such that

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \le \bar{C}_{O_{\mathrm{T}}} \|\mathbf{w}\|_{1,\mathbf{H}_h} (\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{1,4,\Omega_{\mathrm{m}}}) \|\psi\|_{1,\Omega}, \tag{4.4}$$

for all $\theta \in \{\psi \in \mathrm{H}^1(\Omega) : \psi|_{\Omega_{\mathrm{m}}} \in \mathrm{W}^{1,4}(\Omega_{\mathrm{m}})\}$ and $\psi \in \Psi_{h,0}$.

Now we are in position of establishing the main result of this section, namely, the theoretical rate of convergence for the Galerkin scheme (3.3).

Theorem 4.1 *Assume that the hypotheses in Theorem 2.2 and Lemma 3.4 hold. Let* $(\mathbf{u},p,\theta) = (\mathbf{u},p,\theta_0 + \theta_1) \in \mathbf{H} \times \mathrm{L}_0^2(\Omega) \times \mathrm{H}^1(\Omega)$ *be the unique solution to* (2.6)*, with* $\theta_1 = \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega)$ *and* $\theta_0 \in \mathrm{H}_0^1(\Omega)$*, and assume that* (4.1) *holds. In addition, let* $(\mathbf{u}_h,p_h,\theta_h) = (\mathbf{u}_h,p_h,\theta_{h,0}+\theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ *be a solution to* (3.3)*, with* $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ *and* $\theta_{h,0} \in \Psi_{h,0}$*. Finally, let* $\gamma_{\mathbf{g}} = \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$*, and assume further that*

$$C_1 \delta^{-3/2}(1+\delta^2)^{1/2} \gamma_{\mathbf{g}} \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + C_2 \gamma_{\mathbf{g}} \|\theta\|_{1,4,\Omega_{\mathrm{m}}} \le \frac{1}{2}, \tag{4.5}$$

where C_1 *and* C_2 *are positive constants (defined in* (4.12)*) independent of* h*. Then, there exists* $C_{rate} > 0$*, independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned}
\|\mathbf{e}_{\mathbf{u}}\|_{1,\mathbf{H}_h} + \|\mathrm{e}_p\|_{0,\Omega} + \|\mathrm{e}_\theta\|_{1,\Omega} \le C_{rate} h^k \Big\{ & \|\mathbf{u}\|_{k+1,\Omega_{\mathrm{f}}} + \|\mathbf{u}\|_{k,\Omega_{\mathrm{m}}} + \|\mathrm{div}\,\mathbf{u}\|_{k,\Omega_{\mathrm{m}}} \\
& + \|p\|_{k,\Omega} + \|\theta\|_{k+1,\Omega_{\mathrm{f}}} + \|\theta\|_{k+1,4,\Omega_{\mathrm{m}}} \Big\}.
\end{aligned} \tag{4.6}$$

Proof. We begin by noticing that, owing to the extra regularity of the exact solution, we have

$$A_{\mathrm{F}}^h(\mathbf{u},\mathbf{v}) = A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) \quad \text{and} \quad O_{\mathrm{F}}^h(\mathbf{u};\mathbf{u},\mathbf{v}) = O_{\mathrm{F}}(\mathbf{u};\mathbf{u},\mathbf{v}), \qquad \forall\, \mathbf{v} \in \mathbf{H}_h.$$

Then, the following orthogonality property holds:

$$\begin{aligned}
A_{\mathrm{F}}^h(\mathbf{e}_{\mathbf{u}},\mathbf{v}) + \big[O_{\mathrm{F}}^h(\mathbf{u};\mathbf{u},\mathbf{v}) - O_{\mathrm{F}}^h(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v})\big] + B(\mathbf{v},\mathrm{e}_p) - D(\mathrm{e}_\theta,\mathbf{v}) &= 0 \qquad \forall\, \mathbf{v} \in \mathbf{H}_h, \\
B(\mathbf{e}_{\mathbf{u}},q) &= 0 \qquad \forall\, q \in \mathrm{Q}_h, \\
A_{\mathrm{T}}(\mathrm{e}_\theta,\psi) + [O_{\mathrm{T}}(\mathbf{u};\theta,\psi) - O_{\mathrm{T}}(\mathbf{u}_h;\theta_h,\psi)] &= 0 \qquad \forall\, \psi \in \Psi_{h,0}.
\end{aligned} \tag{4.7}$$

In particular, from the first equation of (4.7), adding and subtracting suitable terms and utilizing the decomposition (4.2), we obtain

$$\begin{aligned} A^h_{\mathrm{F}}(\boldsymbol{\chi}_{\mathbf{u}}, \mathbf{v}) + O^h_{\mathrm{F}}(\mathbf{u}_h; \boldsymbol{\chi}_{\mathbf{u}}, \mathbf{v}) = & - A^h_{\mathrm{F}}(\boldsymbol{\varrho}_{\mathbf{u}}, \mathbf{v}) - O^h_{\mathrm{F}}(\mathbf{u}_h; \boldsymbol{\varrho}_{\mathbf{u}}, \mathbf{v}) \\ & + D(\chi_\theta, \mathbf{v}) + D(\varrho_\theta, \mathbf{v}) - \Big[O^h_{\mathrm{F}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) - O^h_{\mathrm{F}}(\mathbf{u}_h; \mathbf{u}, \mathbf{v}) \Big] , \end{aligned}$$

for all $\mathbf{v} \in \mathbf{V}_h$. From this identity with $\mathbf{v} = \boldsymbol{\chi}_{\mathbf{u}}$, and employing the coercivity of $A^h_{\mathrm{F}}(\cdot,\cdot) + O^h_{\mathrm{F}}(\mathbf{u}_h; \cdot, \cdot)$ (cf. (3.21)), the continuity of A^h_{F} (cf. (3.8a)), the continuity of D (cf. (3.13)), and the Lipschitz continuity of O^h_{F} (cf. (3.14)), we obtain

$$\widehat{\alpha}_{\mathrm{F}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \leq \widehat{C}_D \gamma_{\mathbf{g}} \|\chi_\theta\|_{1,\Omega} + C_{\mathrm{Lip,F}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}} + L_1, \tag{4.8}$$

where

$$L_1 := \widetilde{C}_{A_{\mathrm{F}}} \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathbf{H}_h} + C_{\mathrm{Lip,F}} (\|\mathbf{u}_h\|_{1,\mathcal{T}_h^{\mathrm{f}}} + \|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}}) \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{1,\mathcal{T}_h^{\mathrm{f}}} + \widehat{C}_D \gamma_{\mathbf{g}} \|\varrho_\theta\|_{1,\Omega}.$$

Similarly, from the third equation of (4.7), we obtain

$$A_{\mathrm{T}}(\chi_\theta, \psi) + O_{\mathrm{T}}(\mathbf{u}_h; \chi_\theta, \psi) = -A_{\mathrm{T}}(\varrho_\theta, \psi) - O_{\mathrm{T}}(\mathbf{u}_h; \varrho_\theta, \psi) - [O_{\mathrm{T}}(\mathbf{u}; \theta, \psi) - O_{\mathrm{T}}(\mathbf{u}_h; \theta, \psi)] ,$$

for all $\psi \in \Psi_{h,0}$, and in particular taking $\psi = \chi_\theta$, and making use of estimates (3.10), (3.17), (4.4), to obtain

$$\alpha_{\mathrm{T}} \|\chi_\theta\|_{1,\Omega} \leq \bar{C}_{O_{\mathrm{T}}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} (\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{1,4,\Omega_{\mathrm{m}}}) + L_2, \tag{4.9}$$

where

$$L_2 := C_{A_{\mathrm{T}}} \|\varrho_\theta\|_{1,\Omega} + \bar{C}_{O_{\mathrm{T}}} \|\mathbf{u}_h\|_{1,\mathbf{H}_h} (\|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}}) + \bar{C}_{O_{\mathrm{T}}} \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{1,\mathbf{H}_h} (\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{1,4,\Omega_{\mathrm{m}}}) .$$

Then, combining (4.8) and (4.9), it follows that

$$\begin{aligned} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \leq \quad & \widehat{\alpha}_{\mathrm{F}}^{-1} \widehat{C}_D \gamma_{\mathbf{g}} \alpha_{\mathrm{T}}^{-1} \bar{C}_{O_{\mathrm{T}}} \|\theta\|_{1,\Omega_{\mathrm{f}}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} + \widehat{\alpha}_{\mathrm{F}}^{-1} \widehat{C}_D \gamma_{\mathbf{g}} \alpha_{\mathrm{T}}^{-1} \bar{C}_{O_{\mathrm{T}}} \|\theta\|_{1,4,\Omega_{\mathrm{m}}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \\ & + \widehat{\alpha}_{\mathrm{F}}^{-1} C_{\mathrm{Lip,F}} \|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} + \widehat{\alpha}_{\mathrm{F}}^{-1} L_1 + \widehat{\alpha}_{\mathrm{F}}^{-1} \widehat{C}_D \gamma_{\mathbf{g}} \alpha_{\mathrm{T}}^{-1} L_2. \end{aligned} \tag{4.10}$$

Now, using the following estimate for the continuous lifting (see [12, Lemma 2.4])

$$\|\mathrm{E}_\delta \theta_{\mathrm{D}}\|_{1,\Omega} \leq C_{\mathrm{lift},2}\, \delta^{-3/2} (1+\delta^2)^{1/2}\, \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma},$$

from (2.11) we observe that $\mathbf{u}$ and θ satisfy

$$\begin{aligned} & \|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \leq \|\mathbf{u}\|_{1,\Omega_{\mathrm{f}}} \leq \|\mathbf{u}\|_{\mathbf{H}} \leq C_{\mathbf{u}} C_{\mathrm{lift},2} \delta^{-3/2} (1+\delta^2)^{1/2} \gamma_{\mathbf{g}} \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}, \\ & \|\theta\|_{1,\Omega_{\mathrm{f}}} \leq \|\theta\|_{1,\Omega} \leq (C_\theta + 1) C_{\mathrm{lift},2} \delta^{-3/2} (1+\delta^2)^{1/2} \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}. \end{aligned} \tag{4.11}$$

Hence, from (4.10) and (4.11), we obtain

$$\begin{aligned} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \leq \; & (C_1 \delta^{-3/2} (1+\delta^2)^{1/2} \gamma_{\mathbf{g}} \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + C_2 \gamma_{\mathbf{g}} \|\theta\|_{1,4,\Omega_{\mathrm{m}}}) \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \\ & + \widehat{\alpha}_{\mathrm{F}}^{-1} L_1 + \widehat{\alpha}_{\mathrm{F}}^{-1} \widehat{C}_D \gamma_{\mathbf{g}} \widehat{\alpha}_{\mathrm{T}}^{-1} L_2, \end{aligned}$$

with

$$\begin{aligned} C_1 &:= \widehat{\alpha}_{\mathrm{F}}^{-1} \widehat{C}_D \alpha_{\mathrm{T}}^{-1} \bar{C}_{O_{\mathrm{T}}} (C_\theta + 1) C_{\mathrm{lift},2} + \widehat{\alpha}_{\mathrm{F}}^{-1} C_{\mathrm{Lip,F}} C_{\mathbf{u}} C_{\mathrm{lift},2}, \\ C_2 &:= \widehat{\alpha}_{\mathrm{F}}^{-1} \widehat{C}_D \alpha_{\mathrm{T}}^{-1} \bar{C}_{O_{\mathrm{T}}}. \end{aligned} \tag{4.12}$$

Which together with assumption (4.5), implies

$$\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \leq c_1 L_1 + c_2 L_2, \tag{4.13}$$

with $c_1 := 2\widehat{\alpha}_{\mathrm{F}}^{-1}$ and $c_2 := 2\widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{C}_D\gamma_{\mathbf{g}}\alpha_{\mathrm{T}}^{-1}$. Moreover, replacing (4.13) in (4.9), and employing (4.5) and the estimate for θ in (2.11), we deduce

$$\|\chi_\theta\|_{1,\Omega} \leq c_3 L_1 + c_4 L_2, \tag{4.14}$$

with

$$\begin{aligned} c_3 := &\quad 2\widehat{\alpha}_{\mathrm{F}}^{-1}\alpha_{\mathrm{T}}^{-1}\bar{C}_{O_{\mathrm{T}}}[(C_\theta+1)C_{\mathrm{lift},2}\delta^{-3/2}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + (2C_2\gamma_{\mathbf{g}})^{-1}], \\ c_4 := &\quad 2\widehat{\alpha}_{\mathrm{F}}^{-1}\alpha_{\mathrm{T}}^{-2}\widehat{C}_D\gamma_{\mathbf{g}}\bar{C}_{O_{\mathrm{T}}}[(C_\theta+1)C_{\mathrm{lift},2}\delta^{-3/2}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + (2C_2\gamma_{\mathbf{g}})^{-1}] + \alpha_{\mathrm{T}}^{-1}. \end{aligned}$$

In turn, observing that estimate (3.24b) and the fact that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$ (cf. (3.26)) imply that $\mathbf{u}_h$ satisfies

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h^{\mathrm{f}}} \leq \|\mathbf{u}_h\|_{1,\mathbf{H}_h} \leq \widehat{C}_{\mathbf{u}}\gamma_{\mathbf{g}}\|\theta_{h,1}\|_{1,\Omega} \leq \widehat{C}_{\mathbf{u}}\gamma_{\mathbf{g}}\widehat{C}_{\mathrm{lift},2}\delta^{1/4}(2+\delta^{-1})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}, \tag{4.15}$$

from (3.7), (4.5), (4.10), (4.15), and the fact that $\|\varrho_\theta\|_{1,\Omega} \leq \|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + c\|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}}$, we deduce

$$\begin{aligned} L_1 &\leq c_5\big(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathbf{H}_h} + \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\mathrm{f}}} + \|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}}\big), \\ L_2 &\leq c_6\big(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathbf{H}_h} + \|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}}\big), \end{aligned} \tag{4.16}$$

with c_5, c_6 being positive constants independent of h. Therefore, from (4.13), (4.14) and (4.16), the fact $\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathbf{H}_h} \leq \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\mathrm{f}}} + \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathrm{div};\Omega_{\mathrm{m}}}$, and the triangle inequality, it follows that

$$\|\mathbf{e}_{\mathbf{u}}\|_{1,\mathbf{H}_h} + \|\mathrm{e}_\theta\|_{1,\Omega} \leq c_7\left(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\mathrm{f}}} + \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} + \|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}}\right), \tag{4.17}$$

where $c_7 > 0$ is independent of h.

On the other hand, to estimate e_p we first use the discrete inf-sup condition (3.22), to deduce that

$$\begin{aligned} \widehat{\beta}\|\chi_p\|_{0,\Omega} &\leq \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},\chi_p)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} \\ &= \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},\mathrm{e}_p)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} + \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},-\varrho_p)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} \\ &\leq \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},\mathrm{e}_p)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} + \|\varrho_p\|_{0,\Omega}. \end{aligned} \tag{4.18}$$

Then, noticing that adding and subtracting $O_{\mathrm{F}}^h(\mathbf{u}_h;\mathbf{u},\mathbf{v})$ in the first equation of (4.7), yields

$$B(\mathbf{v},\mathrm{e}_p) = -A_{\mathrm{F}}^h(\mathbf{e}_{\mathbf{u}},\mathbf{v}) - \Big[O_{\mathrm{F}}^h(\mathbf{u};\mathbf{u},\mathbf{v}) - O_{\mathrm{F}}^h(\mathbf{u}_h;\mathbf{u},\mathbf{v})\Big] - O_{\mathrm{F}}^h(\mathbf{u}_h;\mathbf{e}_{\mathbf{u}},\mathbf{v}) + D(\mathrm{e}_\theta,\mathbf{v}),$$

for all $\mathbf{v} \in \mathbf{H}_h$, employing estimates (3.8a), (3.13) and (3.14) we obtain

$$|B(\mathbf{v},\mathrm{e}_p)| \leq \Big(\widetilde{C}_{A_{\mathrm{F}}}\|\mathbf{e}_{\mathbf{u}}\|_{2,\mathbf{H}_h} + C_{\mathrm{Lip,F}}\|\mathbf{e}_{\mathbf{u}}\|_{1,\mathcal{T}_h^{\mathrm{f}}}(\|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}} + \|\mathbf{u}_h\|_{1,\mathcal{T}_h^{\mathrm{f}}}) + \widehat{C}_D\gamma_{\mathbf{g}}\|\mathrm{e}_\theta\|_{1,\Omega}\Big)\|\mathbf{v}\|_{1,\mathbf{H}_h},$$

and then, from the latter, estimates (4.11), (4.15), (4.17), (4.18) and the triangle inequality, we obtain

$$\|\mathrm{e}_p\|_{0,\Omega} \leq \|\chi_p\|_{0,\Omega} + \|\varrho_p\|_{0,\Omega} \leq c_8\left(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\mathrm{f}}} + \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} + \|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}} + \|\varrho_p\|_{0,\Omega}\right), \tag{4.19}$$

where $c_8 > 0$ is independent of h.

We conclude the proof by observing that (4.6) is a direct consequence of (4.17), (4.19) and the approximation properties (4.3). $\square$

5 Numerical results

In this section we present some numerical results illustrating the performance of our nonconforming scheme (3.3) analyzed in Section 3 for approximating the solutions of (2.6), and to confirm the theoretical converges rates (4.6) predicted by the theory according to the Theorem 4.1. Our implementation is based on a *FreeFem++* code [34], in conjunction with the direct linear solver *UMFPACK* [19]. The experimental errors and convergence rates for the velocity, pressure, and temperature are the result of iterations based on a Picard-type algorithm over a family of quasi-uniform triangulations of the corresponding domains. This process ends when the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, that is

$$\frac{\|\mathbf{coeff}^{n+1} - \mathbf{coeff}^{n}\|_{l^2}}{\|\mathbf{coeff}^{n+1}\|_{l^2}} \leq tol,$$

where tol is a fixed tolerance and $\|\cdot\|_{l^2}$ stands for the usual euclidean norm in $\mathbb{R}^{\mathrm{dof}}$, with dof denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$, Q_h, and Ψ_h.

Now, we introduce some additional notations. As in Section 4, the individual errors for each variable are denoted by $\mathbf{e_u}$, e_p, and e_θ. In addition, we define the experimental rates of convergence $\mathbf{r_u}$, r_p, and r_θ, as

$$\mathbf{r_u} := \frac{\log(\mathbf{e_u}/\mathbf{e}'_{\mathbf{u}})}{\log(h/h')}, \quad \mathrm{r}_p := \frac{\log(\mathrm{e}_p/\mathrm{e}'_p)}{\log(h/h')}, \quad \text{and} \quad \mathrm{r}_\theta := \frac{\log(\mathrm{e}_\theta/\mathrm{e}'_\theta)}{\log(h/h')},$$

where h and h' denote two consecutive mesh sizes with their respective errors $\mathbf{e}$, $\mathbf{e}'$ (or e, e$'$).

For all examples bellow, we simply take $\mathbf{u}^0 = \mathbf{0}$ and $\theta^0 = 0$ as initial guess, we consider $tol = 1e-6$ and $a^{\mathrm{pen}} = 5$.

Example 1: Robustness and rates of convergence

For our first example, we illustrate the robustness and accuracy of our discontinuous method considering a manufactured exact solution defined on the following computational domain $\Omega = \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$, with $\Omega_{\mathrm{f}} := (-1,1)\times(0,1)$ and $\Omega_{\mathrm{m}} := (-1,1)\times(-1,0)$. We chose the parameters $\mu = 1$, $\mathbf{g}_{\mathrm{f}} = (0,-1)^t$, $\mathbf{g}_{\mathrm{m}} = (0,-1)^t$, $\alpha_d = 1$, $\kappa_{\mathrm{f}} = 1$, $\kappa_{\mathrm{m}} = 1$, $\mathbf{K} = \mathbf{I}$, $\boldsymbol{\kappa} = \mathbf{I}$, and the terms on the right-hand side are adjusted so that the exact solution is given by the functions:

$$\begin{aligned}
\mathbf{u}(x_1,x_2) &:= \begin{pmatrix} -2\sin(\pi x_1)^2(x_2-1) \\ \pi\sin(2\pi x_1)(x_2-1)^2 \end{pmatrix} \quad \text{in} \quad \Omega, \\
p(x_1,x_2) &:= x_1^5 + x_1^3 + x_1x_2 \quad \text{in} \quad \Omega, \\
\theta(x_1,x_2) &:= \exp(-x_1x_2) \quad \text{in} \quad \Omega.
\end{aligned}$$

We notice that $\mathbf{u}_{\mathrm{f}}|_\Sigma = \mathbf{u}_{\mathrm{m}}|_\Sigma$, $\theta_{\mathrm{f}}|_\Sigma = \theta_{\mathrm{m}}|_\Sigma$, and $\kappa_{\mathrm{f}}\nabla\theta_{\mathrm{f}}|_\Sigma = \kappa_{\mathrm{m}}\nabla\theta_{\mathrm{m}}|_\Sigma$. We notice also that these functions do not satisfy the interface conditions (2.3), thus the difference must be incorporated as a functional at the right-hand side of the resulting system.

Furthermore, in order to assess the introduction of other type of discretizations for $a_{\mathrm{F,f}}$, in the following results we consider the bilinear form

$$\begin{aligned}
a^h_{\mathrm{F,f}}(\mathbf{u},\mathbf{v}) := \;& 2\mu\sum_{T\in\mathcal{T}_h^{\mathrm{f}}}(\mathbf{e}(\mathbf{u}),\mathbf{e}(\mathbf{v}))_T - 2\mu\sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{f}})\cup\mathcal{E}_h(\Gamma_{\mathrm{f}})}(\{\!\{\mathbf{e}(\mathbf{u})\}\!\},[\![\mathbf{v}]\!])_e + 2\mu\epsilon\sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{f}})\cup\mathcal{E}_h(\Gamma_{\mathrm{f}})}(\{\!\{\mathbf{e}(\mathbf{v})\}\!\},[\![\mathbf{u}]\!])_e \\
& + 2\mu\sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{f}})\cup\mathcal{E}_h(\Gamma_{\mathrm{f}})}\frac{a^{\mathrm{pen}}}{h_e}([\![\mathbf{u}]\!],[\![\mathbf{v}]\!])_e + \sum_{e\in\mathcal{E}_h(\Sigma)}\left\langle\frac{\alpha_d\,\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\cdot\mathbf{t}}}(\mathbf{u}\cdot\mathbf{t}),\mathbf{v}\cdot\mathbf{t}\right\rangle_e,
\end{aligned}$$

for all $\mathbf{u}$, $\mathbf{v} \in \mathbf{H}_h$, with $\epsilon \in \{1, 0, -1\}$. We note that for $\epsilon = -1$ we recover the SIPG form employed in Section 3.2. The cases $\epsilon = 1$ and $\epsilon = 0$ correspond to the Non-symmetric Interior Penalty Galerkin (NIPG) and Incomplete Interior Penalty Galerkin (IIPG) methods, respectively. We also observe that for any $\epsilon \in \{1, 0, -1\}$, the form $a^h_{\mathrm{F,f}}$ satisfies all properties mentioned for $a^h_{\mathrm{F,f}}$ in Section 3.3 (see [32]).

In Tables 5.1, 5.2, and 5.3, we summarize the convergence history on a sequence of quasi-uniform triangulations for the finite element families presented in (3.1) with $k = 1$, considering the SIPG ($\epsilon = -1$), NIPG ($\epsilon = 1$) and IIPG ($\epsilon = 0$) methods, respectively. We observe there that the rate of convergence $O(h)$ predicted by Theorem 4.1 is attained in all the cases for all unknowns. In addition, in the last two columns of each table we show the l^∞-norm of $\mathrm{div}\mathbf{u}_h$ and the number of iterations required to stop the algorithm. In particular, we observe there that the velocity is practically divergence-free for all refinement steps. Finally, in the first row of Figure 5.1 we display the exact velocity, pressure and temperature (from left to right) and we compare them with their exact counterpart (second row). We observe there that the finite element method provides very accurate approximations to the unknowns.

dof	h	$\mathbf{e}_\mathbf{u}$	$\mathbf{r}_\mathbf{u}$	e_p	r_p	e_θ	r_θ	$\lVert\mathrm{div}\,\mathbf{u}_h\rVert_{l^\infty}$	itt
697	0.375	8.640	–	4.159	–	0.336	–	1.001e-08	8
2815	0.190	4.487	0.964	2.777	0.594	0.162	1.068	1.185e-08	9
10516	0.098	1.971	1.236	1.264	1.183	0.084	0.993	1.314e-08	10
41200	0.052	0.941	1.163	0.621	1.117	0.042	1.067	1.390e-08	10
164713	0.027	0.449	1.152	0.313	1.071	0.021	1.078	1.433e-08	10
672280	0.015	0.224	1.158	0.171	1.005	0.010	1.183	1.455e-08	13

Table 5.1: Example 1: Degrees of Freedom, mesh sizes, errors, rates of convergence, l^∞-norm of $\mathrm{div}\,\mathbf{u}_h$ and number of iterations, considering the SIPG method ($\epsilon = -1$).

dof	h	$\mathbf{e}_\mathbf{u}$	$\mathbf{r}_\mathbf{u}$	e_p	r_p	e_θ	r_θ	$\lVert\mathrm{div}\,\mathbf{u}_h\rVert_{l^\infty}$	itt
697	0.375	6.455	–	4.011	–	0.333	–	9.840e-09	8
2815	0.190	3.821	0.771	2.718	0.572	0.161	1.064	1.176e-08	9
10516	0.098	1.909	1.043	1.229	1.192	0.083	0.984	1.310e-08	10
41200	0.052	0.929	1.132	0.606	1.113	0.042	1.067	1.390e-08	10
164713	0.027	0.447	1.143	0.305	1.072	0.021	1.078	1.433e-08	10
672280	0.015	0.223	1.156	0.168	0.988	0.010	1.183	1.455e-08	10

Table 5.2: Example 1: Degrees of Freedom, mesh sizes, errors, rates of convergence, l^∞-norm of $\mathrm{div}\,\mathbf{u}_h$ and number of iterations, considering the NIPG method ($\epsilon = 1$).

Example 2: Kovasznay's analytical solution

In our second example, we focus on the performance of the iterative method with respect to the viscosity μ. To this end, we consider the domain $\Omega = \Omega_\mathrm{f} \cup \Sigma \cup \Omega_\mathrm{m}$, with $\Omega_\mathrm{f} := (-1/2, 3/2) \times (0, 1/2)$ and $\Omega_\mathrm{m} := (-1/2, 3/2) \times (-1/2, 0)$. In turn, the terms on the right-hand side are adjusted so that the exact solution are given by the functions

$$\mathbf{u}(x_1,x_2) := \begin{pmatrix} 1 - e^{\lambda x_1}\cos(2\pi x_2) \\ \frac{\lambda}{2\pi} e^{\lambda x_1}\sin(2\pi x_2) \end{pmatrix}, \quad p(x_1,x_2) := \tfrac{-1}{2} e^{2\lambda x_1} + c_0, \quad \text{and} \quad \theta(x_1,x_2) := \exp(-x_1 x_2),$$

dof	h	$\mathbf{e_u}$	$\mathbf{r_u}$	e_p	r_p	e_θ	r_θ	$\Vert\mathrm{div}\,\mathbf{u}_h\Vert_{l^\infty}$	itt
697	0.375	6.366	–	4.036	–	0.331	–	9.939e-09	8
2815	0.190	3.789	0.763	2.737	0.571	0.161	1.064	1.180e-08	9
10516	0.098	2.085	0.897	1.246	1.183	0.083	0.981	1.312e-08	10
41200	0.052	1.218	0.845	0.614	1.114	0.042	1.065	1.390e-08	10
164713	0.027	0.761	0.734	0.308	1.073	0.021	1.077	1.433e-08	10
672280	0.015	0.498	0.704	0.169	0.997	0.010	1.183	1.455e-08	10

Table 5.3: EXAMPLE 1: Degrees of Freedom, mesh sizes, errors, rates of convergence, l^∞-norm of $\mathrm{div}\,\mathbf{u}_h$ and number of iterations, considering the IIPG method ($\epsilon = 0$).

in Ω. Above, λ is given by

$$\lambda := \frac{-8\pi^2}{\mu^{-1} + \sqrt{\mu^{-2} + 16\pi^2}},$$

with $\mu > 0$ being the viscosity of the fluid and c_0 is a constant chosen in such a way $(p, 1)_\Omega = 0$. We note that $(\mathbf{u}_\mathrm{f}, p_\mathrm{f})$ is the well known analytical solution for the Navier–Stokes problem obtained by Kovasznay in [40], which presents a boundary layer at $\{-1/2\} \times (0, 2)$.

In Table 5.4 we show the behavior of the iterative method as a function of the viscosity μ, considering different mesh sizes h. As expected, when viscosity is reduced, the method requires a larger number of iterations to achieve convergence. We do not present numerical experiments for smaller values of μ as, in such instances, the maximum iteration limit set in the code (200 iterations) is reached for all meshes. Next, in Table 5.5 we show the convergence history considering the viscosity $\mu = 0.1$. We observe there that the rate of convergence $O(h)$ predicted by Theorem 4.1 is attained by all unknowns.

μ	$h = 0.362$	$h = 0.180$	$h = 0.094$	$h = 0.047$	$h = 0.023$	$h = 0.013$
1	9	9	9	9	9	9
0.1	13	14	14	14	14	14
0.01	14	17	17	17	17	17
0.001	**	31	31	31	31	31

Table 5.4: Number of iterations of the iterative method with respect to μ.

dof	h	$\mathbf{e_u}$	$\mathbf{r_u}$	e_p	r_p	e_θ	r_θ	$\Vert\mathrm{div}\,\mathbf{u}_h\Vert_{l^\infty}$	itt
391	0.362	4.249	–	2.557	–	0.240	–	7.757e-05	13
1636	0.180	2.012	1.066	0.518	2.275	0.108	1.140	9.426e-05	14
6133	0.094	1.064	0.993	0.276	0.983	0.054	1.081	9.240e-05	14
24370	0.047	0.486	1.146	0.138	1.011	0.026	1.018	9.452e-05	14
101101	0.023	0.233	1.054	0.068	1.019	0.013	0.997	9.478e-05	14
388987	0.013	0.121	1.099	0.034	1.137	0.006	1.167	9.482e-05	14

Table 5.5: EXAMPLE 2: Degree of Freedom, mesh sizes, errors, rates of convergence, l^∞-norm of $\mathrm{div}\,\mathbf{u}_h$ for the coupled problem and iterations for the Kovasnay solution with $\mu = 0.1$.

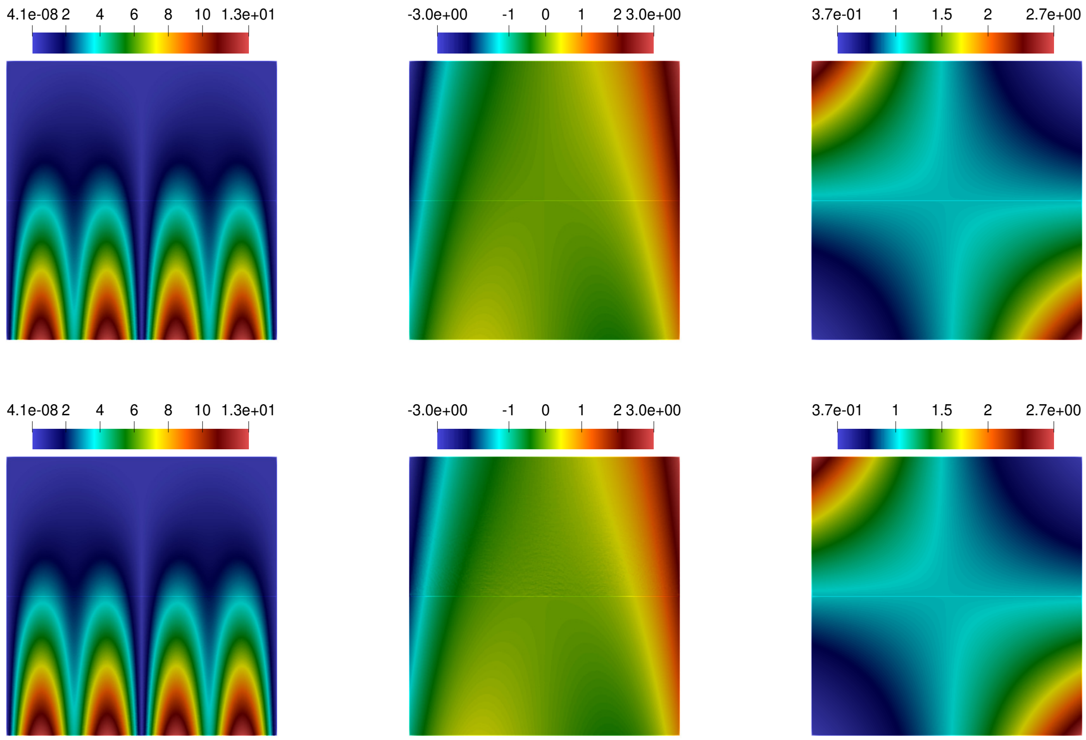


Figure 5.1: Example 1: On top: exact magnitude of the velocity (left), pressure (center) and temperature (right). On bottom: approximate magnitude of the velocity (left), pressure (center) and temperature (right).

Example 3: Dimensionless problem

In our last example, we consider the domain $\Omega = \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$, with $\Omega_{\mathrm{f}} := (-1,1) \times (0, d_{\mathrm{f}})$ and $\Omega_{\mathrm{m}} := (-1,1) \times (-d_{\mathrm{m}}, 0)$ and address the behavior of convection patterns as d_{f} decreases. As documented in [44] and [45], it has been observed that when the free-fluid region has a shallow depth, the convection patterns extend to encompass both domains (full convection). Conversely, when the free-fluid region has greater depth, the dominant convection patterns are mainly localized within the free-fluid region (fluid-dominated convection). To study this behavior, we proceed similarly to [33, 44, 45] and consider the following dimensionless problem

$$
\begin{aligned}
&\widetilde{\boldsymbol{\sigma}}_{\mathrm{f}} = 2\,\mathbf{e}(\widetilde{\mathbf{u}}_{\mathrm{f}}) - \widetilde{p}_{\mathrm{f}}\,\mathbf{I}, \quad -\mathbf{div}(\widetilde{\boldsymbol{\sigma}}_{\mathrm{f}}) + (\widetilde{\mathbf{u}}_{\mathrm{f}} \cdot \nabla)\,\widetilde{\mathbf{u}}_{\mathrm{f}} + \mathrm{Ra}_{\mathrm{f}} \widetilde{\theta}_{\mathrm{f}} \mathbf{g}_{\mathrm{f}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{f}}, \\
&\mathrm{div}\,\widetilde{\mathbf{u}}_{\mathrm{f}} = 0, \quad -\epsilon_T \Delta \widetilde{\theta}_{\mathrm{f}} + \mathrm{Pr}_{\mathrm{f}}\,\widetilde{\mathbf{u}}_{\mathrm{f}} \cdot \nabla\,\widetilde{\theta}_{\mathrm{f}} + \tfrac{1}{\epsilon_T} \widetilde{\mathbf{u}}_{\mathrm{f}} \cdot \mathbf{g}_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \\
&\tfrac{1}{\mathrm{Da}} \widetilde{\mathbf{u}}_{\mathrm{m}} + \nabla \widetilde{p}_{\mathrm{m}} = -\tfrac{\mathrm{Ra}_{\mathrm{m}}}{\mathrm{Da}} \widetilde{\theta}_{\mathrm{m}} \mathbf{g}_{\mathrm{m}}, \quad \mathrm{div}\,\widetilde{\mathbf{u}}_{\mathrm{m}} = 0, \quad -\Delta \widetilde{\theta}_{\mathrm{m}} + \mathrm{Pr}_{\mathrm{m}}\,\widetilde{\mathbf{u}}_{\mathrm{m}} \cdot \nabla \widetilde{\theta}_{\mathrm{m}} = -\widetilde{\mathbf{u}}_{\mathrm{m}} \cdot \mathbf{g}_{\mathrm{m}} \quad \text{in} \quad \Omega_{\mathrm{m}}, \\
&\widetilde{\theta}_{\mathrm{f}} = \widetilde{\theta}_{\mathrm{m}}, \quad \epsilon_T \nabla \widetilde{\theta}_{\mathrm{f}} \cdot \mathbf{n} = \nabla \widetilde{\theta}_{\mathrm{m}} \cdot \mathbf{n}, \quad \widetilde{\mathbf{u}}_{\mathrm{f}} \cdot \mathbf{n} = \widetilde{\mathbf{u}}_{\mathrm{m}} \cdot \mathbf{n}, \quad \widetilde{\boldsymbol{\sigma}}_{\mathrm{f}}\,\mathbf{n} + \tfrac{\alpha_d}{\sqrt{\mathrm{Da}}} (\widetilde{\mathbf{u}}_{\mathrm{f}} \cdot \mathbf{t})\,\mathbf{t} = -\widetilde{p}_{\mathrm{m}}\,\mathbf{n} \quad \text{on} \quad \Sigma, \\
&\widetilde{\theta}_{\mathrm{f}} = 0, \quad \widetilde{\mathbf{u}}_{\mathrm{f}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{f}}, \\
&\widetilde{\theta}_{\mathrm{m}} = 0, \quad \widetilde{\mathbf{u}}_{\mathrm{m}} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{m}},
\end{aligned} \tag{5.1}
$$

where $\mathrm{Pr}_\star$ and $\mathrm{Ra}_\star$ represent the Prandtl and Rayleigh numbers in the domain $\Omega_\star$ for $\star \in \{\mathrm{f}, \mathrm{m}\}$, Da represents the Darcy number, and $\epsilon_T = \frac{\kappa_{\mathrm{f}}}{\kappa_{\mathrm{m}}}$ is the ratio of thermal diffusivities. Additionally, we define the dimensionless number $\widehat{d} = \frac{d_{\mathrm{f}}}{d_{\mathrm{m}}}$ which represents the depth ratio.

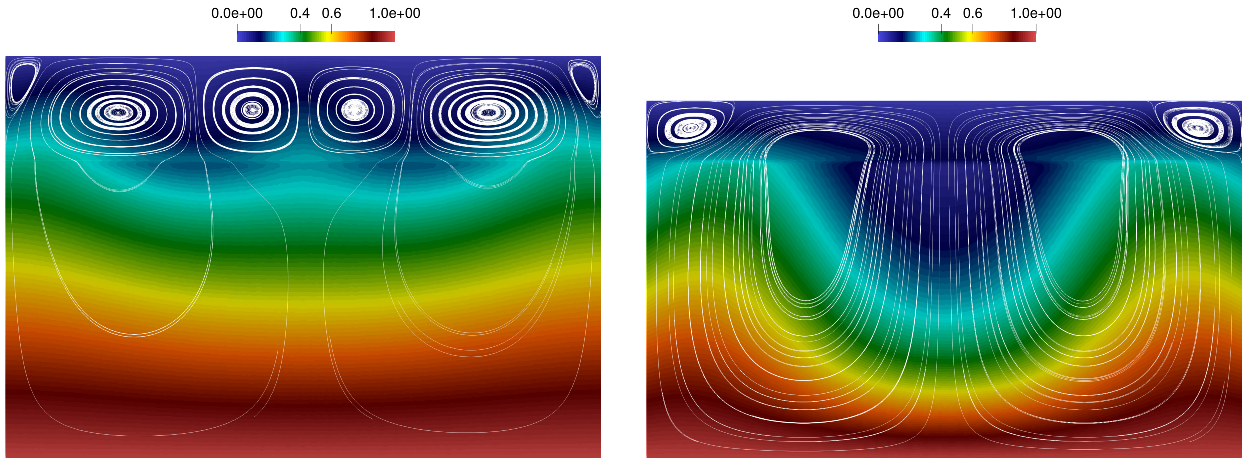


Figure 5.2: Example 3: Fluid-dominated and Full convection with temperature (colour) and streamlines (contour), with $\widehat{d} = 0.35$, $\mathrm{Ra_m} = 10$ (left) and $\widehat{d} = 0.2$, $\mathrm{Ra_m} = 30$ (right). Fixed parameters $\mathrm{Pr_f} = 0.7$, $\mathrm{Pr_m} = 0.7$, $\epsilon_T = 0.7$, $\mathrm{Da} = 1.0 \times 10^{-4}$, and $\alpha_d = 1.0$.

We observe that the temperature profile can be recovered by means of the relationship

$$\theta = (\overline{T}_\mathrm{f} + \widetilde{\theta}_\mathrm{f}(T_U - T_0)\mathrm{Pr_f}\epsilon_T^{-1})\chi_\mathrm{f} + (\overline{T}_\mathrm{m} + \widetilde{\theta}_\mathrm{m}(T_0 - T_B)\mathrm{Pr_m})\chi_\mathrm{m},$$

where $\widetilde{\theta}_\star$ for $\star \in \{\mathrm{f}, \mathrm{m}\}$ is the temperature solution of (5.1), T_U and T_B are constant temperatures in the upper and bottom boundaries, respectively, $\overline{T}_\mathrm{f} := T_0 + y\frac{T_U - T_0}{d_\mathrm{f}}$, $\overline{T}_\mathrm{m} := T_0 + y\frac{T_0 - T_B}{d_\mathrm{m}}$, and $\overline{T}_0 := \frac{\widehat{d}T_B + \epsilon_T T_U}{\widehat{d} + \epsilon_T}$. In [44] and [45] is also established that if $T_U > T_B$, the conductive state is stable, whereas if $T_B > T_U$, buoyancy can destabilize the system.

In Figure 5.2, we present flow configurations (streamlines) and temperature profiles (color) for two cases with different $\widehat{d}$ values, specifically $\widehat{d} = 0.2$ and $\widehat{d} = 0.35$. We set the temperatures as $T_B = 1$ and $T_U = 0$, representing an unstable scenario.

On the right side, similarly to [44] and [45], we observe a situation of full convection with the following parameter values: $d_\mathrm{f} = 0.2$, $d_\mathrm{m} = 1.0$, and $\mathrm{Ra_m} = 30$. On the left side, we witness a fluid-dominated convection scenario characterized by the parameters $d_\mathrm{f} = 0.35$, $d_\mathrm{m} = 1.0$, and $\mathrm{Ra_m} = 10$. For both cases, we consider the following parameter settings: $\mathrm{Pr_f} = 0.7$, $\mathrm{Pr_m} = 0.7$, $\epsilon_T = 0.7$, $\mathrm{Da} = 1.0 \times 10^{-4}$, $\alpha_d = 1.0$, $\mathbf{g}_\mathrm{f} = (0, 1)^t$, $\mathbf{g}_\mathrm{m} = (0, 1)^t$, and the Rayleigh number within Ω_f is calculated as $\mathrm{Ra_f} = \frac{\mathrm{Ra_m}}{\mathrm{Da}}$.

References

[1] Allen, M.B. Collocation Techniques for Modeling Compositional Flows in Oil Reservoirs. Springer, (1984).

[2] Arnold, D. N. *An interior penalty finite element method with discontinuous elements.* SIAM J. Numer. Anal., vol. 19, no. 4, pp. 742–760, (1982).

[3] Arnold, D. N., Brezzi, F., Cockburn, B., & Marini, L. D. *Unified analysis of discontinuous Galerkin methods for elliptic problems.* SIAM J. Numer. Anal., vol. 39(5), pp. 1749–1779, (2001/02).

[4] Badea, L., Discacciati, M., & Quarteroni, A. *Numerical analysis of the Navier–Stokes/Darcy coupling.* Numer. Math., vol. 115, no. 2, pp. 195–227, (2010).

[5] Brezzi, F., Douglas, Jr., J., & Marini, L. D. *Two families of mixed finite elements for second order elliptic problems.* Numer. Math., vol. 47, no. 2, pp. 217–235, (1985).

[6] Beavers, G. & Joseph, D. *Boundary conditions at a naturally permeable wall.* J. Fluid Mech, vol. 30, pp. 197–207, (1967).

[7] Bernardi, C., Dib, S., Girault, V., Hecht, F., Murat, F. & Sayah, T. *Finite element methods for Darcy's problem coupled with the heat equation.* Numer. Math., vol. 139, pp. 315–348, (2018).

[8] Bernardi, C., Mtivet, B. & Pernaud-Thomas, B. *Couplage des équations de Navier–Stokes et de la chaleur le modéle et son approximation par éléments finis.* RAIRO Modél. Math. Anal. Numér, vol. 29, no. 7, pp. 871–921, (1995).

[9] Brenner, S. & Scott, L. The mathematical theory of finite element methods, Third edition. Texts in Applied Mathematics, 15. Springer, New York, (2008).

[10] Brezzi F. & Fortin M. Mixed and Hybrid Finite Element Methods. Springer Series in Computational Mathematics, 15. Springer-Verlag, New York, (1991).

[11] Boffi, D., Brezzi, F. & Fortin, M. Mixed Finite Element Methods and Applications. Springer Series in Computational Mathematics, 44. Springer, (2013).

[12] Camaño, J., Oyarzúa R., Serón, M. & Solano M.E. *A conforming finite element method for a nonisothermal fluid-membrane interaction.* Preprint 2023-14, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), UdeC, (2023).

[13] Chidyagwai, P. & Rivière, B., *On the solution of the coupled Navier–Stokes and Darcy equations.* Comput. Methods Appl. Mech. Engrg. 198, no. 47-48, 3806–3820, (2009).

[14] Çeşmelioğlu, A. & Rhebergen, S. *A hybridizable discontinuous Galerkin method for the coupled Navier-Stokes and Darcy problem*, J. Comput. Appl. Math., vol. 422, 114923, (2023).

[15] Ciarlet, P. *Linear and nonlinear functional analysis with applications.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013. Society for Industrial and Applied Mathematics, Philadelphia, PA, pp. xiv+832, (2013).

[16] Cockburn, B.; Kanschat, G.; Schötzau, D. *A locally conservative LDG method for the incompressible NavierStokes equations.* Math. Comp., vol. 74, pp. 1067–1095, (2005).

[17] Cockburn, B.; Kanschat, G.; Schötzau, D. *A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations.* J. Sci. Comput., vol. 31, no. 1-2, pp. 61–73, (2007).

[18] Colmenares, E. & Neilan, M. *Dual-mixed finite element methods for the stationary Boussinesq problem.* Comput. Math. Appl., vol. 72, no. 7, pp. 1828–1850, (2016).

[19] Davis, T. *Algorithm 832: UMFPACK V4.3-An Unsymmetric-Pattern Multifrontal Method.* ACM Trans. Math. Softw., vol. 30, no. 2, pp. 196–199, (2004).

[20] Di Pietro, D. A., & Ern, A. *Mathematical aspects of discontinuous Galerkin methods.* Mathématiques & Applications, vol. 69. Springer, Heidelberg, (2012).

[21] Discacciati, M. , Miglio, E. & A. Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows.* Appl. Numer. Math. 43, no. 1-2, 57–74, (2002).

[22] Discacciati, M., & Oyarzúa, R. *A conforming mixed finite element method for the Navier–Stokes/Darcy coupled problem.* Numer. Math, vol. 135, pp. 571–606, (2017).

[23] Discacciati, M. & Quarteroni, A., *Navier–Stokes/Darcy coupling: modeling, analysis, and numerical approximation.* Rev. Mat. Complut. 22, no. 2, 315–426, (2009).

[24] Ern, A. & Guermond, J.-L. *Theory and Practice of Finite Elements.* Springer, New York, vol. 159, (2004).

[25] Galvis, J. & Sarkis, M.,*Non matching mortar discretization analysis for the coupling Stokes-Darcy equations.* Electron. Trans. Numer. Anal. 26, 350–384, (2007).

[26] Gatica, G.N. A simple introduction to the mixed finite element method. Theory and applications. Springer Briefs in Mathematics, pp. XII+132, (2014).

[27] Gatica, G.N., Meddahi, S. & Oyarzúa, R. *A conforming mixed finite-element method for the coupling of fluid flow with porous media flow.* IMA J. Numer. Anal, vol. 29, no. 1, pp. 86–108, (2009).

[28] Gatica, G.N., Oyarzúa, R. & Sayas, F.J., *Analysis of fully-mixed finite element methods for the Stokes–Darcy coupled problem.* Math. Comp. 80, no. 276, 1911–1948, (2011).

[29] Gatica, G. N., Oyarzúa, R. & Valenzuela, N. *A five-field augmented fully-mixed finite element method for the Navier-Stokes/Darcy coupled problem.* Comput. Math. with Appl., vol. 80, no. 8, pp. 1944–1963, (2020).

[30] Gatica, G.N. & Sequeira, F., *Analysis of the HDG method for the Stokes-Darcy coupling.* Numer. Methods Partial Differential Equations 33, no. 3, 885–917, (2017).

[31] Girault, V. Kanschat, G. & B. Rivière. *On the coupling of incompressible Stokes or NavierStokes and Darcy flows through porous media.* Modelling and simulation in fluid dynamics in porous media, 1–25, Springer Proc. Math. Stat., 28, Springer, New York, (2013).

[32] Girault, V. & Rivière, B. *DG approximation of coupled Navier–Stokes and Darcy equations by Beaver–Joseph–Saffman interface condition.* SIAM J. Numer. Anal, vol. 47, pp. 2052–2089, (2009).

[33] Gobin, D. & Goyeau, B. *Natural convection in partially porous media: a brief overview.* International Journal of Numerical Methods for Heat & Fluid Flow, Vol. 18 No. 3/4, pp. 465-490, (2008).

[34] Hecht, F. *New development in FreeFem++.* J. Numer. Math, vol. 20, no. 3–4, pp. 251–265, (2012).

[35] Hirata, S.C., Goyeau, B. & Gobin, D. *Onset of convective instabilities in under-ice melt ponds.* Phys. Rev. E vol. 85, no. 6, pp. 066306, (2012)

[36] Huppert, H.E. & Neufeld, J.A. *The fluid mechanics of carbon dioxide sequestration.* Annu. Rev. Fluid Mech. vol. 46, pp. 255–272, (2014).

[37] Jäger, W., & Mikelić, A. *On the interface boundary condition of Beavers, Joseph and Saffman.* SIAM J. Appl. Math, vol. 60, no. 4, pp. 1111–1127, (2000).

[38] Karakashian, O. A., & Jureidini, W. N. *A nonconforming finite element method for the stationary Navier–Stokes equations.* SIAM J. Numer. Anal., vol. 35, no. 1, pp. 93–12, (1998).

[39] Kanschat, G., & Riviére, B. *A strongly conservative finite element method for the coupling of Stokes and Darcy flow.* J. Comput. Phys., vol. 229, no. 17, pp. 5933–5943, (2010).

[40] L. Kovasznay, *Laminar flow behind a two-dimensional grid.* Mathematical Proceedings of the Cambridge Philosophical Society, 44(1), pp. 58–62, (1948).

[41] Lasaint, P., & Raviart, P.-A. *On a finite element method for solving the neutron transport equation.* Mathematical aspects of finite elements in partial differential equations. In: Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974. Publication No. 33, Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, pp. 89–123, (1974).

[42] Lasser, J., Ernst, M. & Goehring, L. *Stability and dynamics of convection in dry salt lakes.* J. Fluid Mech. vol. 917, pp. A14, (2021).

[43] Masud, A., *A stabilized mixed finite element method for Darcy-Stokes flow.* Internat. J. Numer. Methods Fluids 54, no. 6-8, 665–681, (2007).

[44] McCurdy, M., Moore, N., & Wang, X. *Convection in a coupled free flow-porous media system.* SIAM J. Appl. Math., vol. 79, no. 6, pp. 2313–2339, (2019).

[45] McCurdy, M., Moore, N., & Wang, X. *Predicting convection configurations in coupled flui-porous systems.* Journal of Fluid Mechanics, vol. 953, A23, (2022).

[46] Oyarzúa, R., Qin, T., & Schötzau, D. *An exactly divergence-free finite element method for a generalized Boussinesq problem.* IMA J Numer. Anal., vol. 34, no. 3, pp. 1104–1135, (2014).

[47] Perfilov V., Ali A. & Fila V. *A general predictive model for direct contact membrane distillation.* Desalination, vol. 445, pp. 181–196, (2018).

[48] Quarteroni, A. & Valli, A. *Numerical approximation of partial differential equations.* Springer Series in Computational Mathematics, vol. 23, pp. XVI+543, (1994).

[49] Rivière, B. & Yotov, I. *Locally conservative coupling of Stokes and Darcy flows.* SIAM J. Numer. Anal., vol. 42, no. 5, pp. 1959–1977, (2005).

[50] Saffman, P. *On the boundary condition at the interface of a porous medium.* Stud. Appl. Math, vol. 1, pp. 93–101, (1971).

[51] Schötzau, D., Schwab, C., and Toselli, A. *Mixed hp-DGFEM for incompressible flows.* SIAM J. Numer. Anal. vol. 40, pp. 2171–2194, (2003).

[52] Villaluenga, J.P.G. & Cohen, Y. *Numerical model of non-isothermal pervaporation in a rectangular channel* J. Membr. Sci., vol. 260, pp. 119–130, (2005).

Centro de Investigación en Ingeniería Matemática (CI²MA)

PRE-PUBLICACIONES 2023

2023-19 Isaac Bermudez, Claudio I. Correa, Gabriel N. Gatica, Juan P. Silva: *A perturbed twofold saddle point-based mixed finite element method for the Navier-Stokes equations with variable viscosity*

2023-20 Paola Goatin, Daniel Inzunza, Luis M. Villada: *Nonlocal macroscopic models of multi-population pedestrian flows for walking facilities optimization*

2023-21 Boumediene Chentouf, Aissa Guesmia, Mauricio Sepúlveda, Rodrigo Véjar: *Boundary stabilization of the Korteweg-de Vries-Burgers equation with an infinite memory-type control and applications: a qualitative and numerical analysis*

2023-22 Franz Chouly: *A short journey into the realm of numerical methods for contact in elastodynamics*

2023-23 Stéphane P. A. Bordas, Marek Bucki, Huu Phuoc Bui, Franz Chouly, Michel Duprez, Arnaud Lejeune, Pierre-Yves Rohan: *Automatic mesh refinement for soft tissue*

2023-24 Mauricio Sepúlveda, Nicolás Torres, Luis M. Villada: *Well-posedness and numerical analysis of an elapsed time model with strongly coupled neural networks*

2023-25 Franz Chouly, Patrick Hild, Yves Renard: *Lagrangian and Nitsche methods for frictional contact*

2023-26 Sergio Caucao, Gabriel N. Gatica, Juan P. Ortega: *A three-field mixed finite element method for the convective Brinkman–Forchheimer problem with varying porosity*

2023-27 Raimund Bürger, Yessennia Martínez, Luis M. Villada: *Front tracking and parameter identification for a conservation law with a space-dependent coefficient modeling granular segregation*

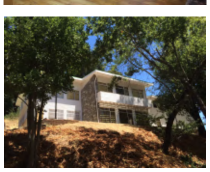
2023-28 Marie Haghebaert, Beatrice Laroche, Mauricio Sepúlveda: *Study of the numerical method for an inverse problem of a simplified intestinal crypt*

2023-29 Rodolfo Araya, Fabrice Jaillet, Diego Paredes, Frederic Valentin: *Generalizing the Multiscale Hybrid-Mixed Method for Reactive-Advective-Diffusive Equations*

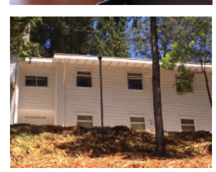
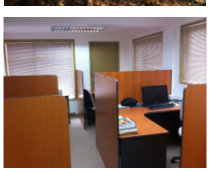
2023-30 Jessika Camaño, Ricardo Oyarzúa, Miguel Serón, Manuel Solano: *A mass conservative finite element method for a nonisothermal Navier-Stokes/Darcy coupled system*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: Director, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, Tel.: 41-2661324, o bien, visitar la página web del centro: `http://www.ci2ma.udec.cl`



CENTRO DE INVESTIGACIÓN EN
INGENIERÍA MATEMÁTICA (CI^{2}MA)
Universidad de Concepción

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl