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A skew-symmetric-based mixed FEM for stationary MHD flows in highly porous media*

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Abstract

We propose and analyze a new mixed variational formulation for the coupling of the convective Brinkman–Forchheimer and Maxwell equations for stationary magnetohydrodynamic flows in highly porous media. Besides the velocity, magnetic field, and a Lagrange multiplier associated with the divergence-free condition of the magnetic field, our approach introduces a convenient translation of the velocity gradient and the pseudostress tensor as additional unknowns. Consequently, we obtain a five-field mixed variational formulation within a Banach space framework, where the aforementioned variables are the main unknowns of the system, exploiting the skew-symmetric property of one of the involved operators. The resulting mixed scheme is then equivalently written as a fixed-point equation, allowing the application of the well-known Banach theorem, combined with classical results on nonlinear monotone operators and a sufficiently small data assumption, to prove the unique solvability of the continuous and discrete systems. In particular, the analysis of the discrete scheme requires a quasi-uniformity assumption on the mesh. The finite element discretization involves Raviart–Thomas elements of order $k \geq 0$ for the pseudostress tensor, discontinuous piecewise polynomial elements of degree k for the velocity and the velocity gradient translation, Nédélec elements of degree k for the magnetic field, and continuous piecewise polynomial elements of degree $k+1$ for the Lagrange multiplier. We establish stability, convergence, and optimal *a priori* error estimates for the corresponding Galerkin scheme. Theoretical results are illustrated by numerical tests.

Key words: convective Brinkman–Forchheimer equations, Maxwell equations, mixed finite element methods, fixed point theory, *a priori* error analysis

Mathematics subject classifications (2020): 65N30, 65N12, 65N15, 76M10

1 Introduction

The study of electrically conducting fluid flow in the presence of magnetic fields falls under the research area known as Magnetohydrodynamics (MHD). In recent years, interest in MHD has surged, owing to its significance in both scientific research and engineering applications, spanning a diverse range of

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physical systems from liquid metals to cosmic plasmas. The mathematical model of classical MHD includes equations that govern fluid motion in magnetic fields and those that describe electromagnetic fields in moving fluids. This results in a coupled system of the Navier–Stokes and Maxwell equations, interconnected through the Lorentz force and Ohm's law. Nevertheless, various physical scenarios often necessitate modifications or simplifications of these equations to accurately capture pertinent phenomena. For instance, when considering fluid flows through porous media rather than free flows, Darcy's law could be used instead of the Navier–Stokes equations. However, for scenarios involving higher velocities or highly porous media, Darcy's law may not provide accurate predictions (see, e.g., [32, 34, 22] and references therein). To overcome this deficiency, the convective Brinkman–Forchheimer equations are employed, incorporating additional terms to accommodate high-velocity flows and high porosity (see, e.g., [13, 39, 31, 38, 10], and [9]). This approach, along with the growing interest in MHD modeling in porous media, has motivated the introduction of the coupled problem between the convective Brinkman–Forchheimer and Maxwell equations.

Concerning literature devoted to studying the coupling of the convective Brinkman–Forchheimer and Maxwell equations, to the best of the authors' knowledge, [37] stands as one of the initial works analyzing the well-posedness of the coupled problem in the transient regime. In particular, a Faedo-Galerkin approximation procedure is employed to prove the short-time existence of solutions. Thanks to an *a priori* bound, this existence is extended globally in time before passing to the limit. Additionally, we refer to [2] and [1] for the analysis of the coupling of the Brinkman–Forchheimer and Maxwell equations. In [2], the authors establish the existence of weak solutions and uniqueness under small data assumptions. Furthermore, a convergence result of the weak solutions to a solution of the system formed by the Darcy–Forchheimer equations and the magnetic induction equation as the Brinkman coefficient tends to zero is also established. Meanwhile, [1] presents a five-field mixed formulation posed in a Banach space framework and a mixed finite element method for coupling the stationary Brinkman–Forchheimer and Maxwell equations. Stability, convergence, and optimal *a priori* error estimates for the associated Galerkin scheme are obtained.

Regarding the design and analysis of numerical schemes for classical MHD, we start mentioning to [27], where the well-posedness and convergence of a conforming FEM for MHD, using inf-sup stable velocity-pressure elements for the hydrodynamic variables and standard H^1-conforming finite elements for the magnetic field was studied. Similarly, [25] and [28] consider the magnetic field in H^1. However, in non-convex polyhedral domains, the magnetic induction may have regularity below H^1, leading to finite element approximations that miss certain singular solution components induced by reentrant vertices or edges (see [17]). To address this, [36] proposes imposing the divergence-free condition of the magnetic field weakly, allowing for the magnetic field to be approximated by curl-conforming Nédélec elements and removing the need for the convex domain assumption. We also highlight recent works related to the convective Brinkman–Forchheimer (CBF) equations, including [13, 39, 31, 38, 10], and [9]. In particular, [13] analyzed the continuous dependence of solutions of the CBF equations on the Forchheimer coefficient in H^1-norm. Later on, an approximation of solutions for the incompressible CBF equations via the artificial compressibility method was proposed and developed in [39]. More recently, an augmented mixed pseudostress-velocity formulation was proposed and analyzed in [9]. Additionally, [10] proposed a Banach space-based mixed formulation for the CBF problem, differing from [9] by not requiring augmentation for the formulation or solvability analysis. This non-augmented scheme was expressed as a fixed-point equation, utilizing recent results from [16] on perturbed saddle-point problems in Banach spaces, along with the Banach–Nečas–Babuška and Banach theorems, to establish well-posedness for both continuous and discrete systems.

This paper aims to advance the development of a new numerical method for the MHD model in highly porous media described by the coupling of the convective Brinkman–Forchheimer and Maxwell equations. To that end, unlike previous works [37, 2] and motivated by [15], [11], and [1], we introduce a convenient translation of the velocity gradient and the pseudostress tensor as additional unknowns, alongside the velocity, magnetic field, and a Lagrange multiplier associated with the divergence-free condition of the magnetic field. This approach provides several advantages, including direct and accurate approximations of the velocity gradient and pseudostress tensor. It also provides optimal theoretical convergence rates, even in non-convex domains, along with suitable postprocessing formulas for pressure, vorticity, and the shear stress tensor. Another significant novelty and advantage of this work is that it generalizes the model studied in [1] by including a nonlinear convective term and spatially varying Darcy and Forchheimer coefficients, thereby addressing viscous flows in highly porous media.

We establish the existence and uniqueness of a solution to the continuous weak formulation written in a Banach space framework by employing techniques from [11], [15], and [36], combined with a fixed-point argument, an abstract result from [11], classical results on nonlinear monotone operators, sufficiently small data assumptions, and the Banach theorem. We emphasize that our formulation exploits the skew-symmetric property of a certain operator involved in both the fluid equations and the coupling terms, allowing us to relax the data assumptions. Additionally, since the formulation shares a similar structure to the one studied in [1], our present analysis certainly makes use of similar arguments employed there. As for the numerical scheme, whose solvability is established similarly to the continuous case, we employ Raviart–Thomas elements of order $k \geq 0$ for the pseudostress tensor, discontinuous piecewise polynomial elements of degree k for the velocity and the velocity gradient translation, Nédélec elements of degree k for the magnetic field, and continuous piecewise polynomial elements of degree $k+1$ for the Lagrange multiplier. We further perform error analysis for the discrete scheme establishing optimal rates of convergence.

Outline. We have organized the contents of this paper as follows. In the remainder of this section we introduce some standard notation and needed functional spaces. In Section 2, we describe the model problem of interest, reformulate it as an equivalent set of equations, and derive our skew-symmetric-based mixed variational formulation. In Section 3 we show that it is well posed using classical results on nonlinear monotone operators and the Banach fixed point theorem. Next, in Section 4 we introduce and analyze the associated Galerkin scheme, provide particular families of stable finite elements, for which well-posedness is attained by mimicking the theory developed for the continuous problem under a quasi-uniformity assumption on the mesh. In Section 5 we establish the corresponding Céa's estimate and the consequent rates of convergence. Finally, in Section 6, we assess the method's performance through two numerical examples, verifying the previously mentioned rates of convergence. These examples illustrate its adaptability in handling spatially varying parameters across convex and non-convex geometries.

Preliminary notations. Let $\Omega \subset \mathrm{R}^3$, denote a bounded domain with polyhedral boundary Γ, and denote by $\mathbf{n}$ the outward unit normal vector on Γ. Standard notations will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$, with $t \in [1, \infty]$ and Sobolev spaces $\mathrm{W}^{s,t}(\Omega)$ with $s \geq 0$, endowed with the norms $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{s,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t} = \mathrm{L}^t(\Omega)$. If $t = 2$, we write $\mathrm{H}^s(\Omega)$ in place of $\mathrm{W}^{s,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{s,\Omega}$. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. With $\langle\cdot,\cdot\rangle_\Gamma$ we denote the corresponding product of duality between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,3}$ and $\mathbf{w} = (w_i)_{i=1,3}$, we define the gradient, curl, divergence, cross and tensor products operators, as $\nabla \mathbf{v} := \left(\dfrac{\partial v_i}{\partial x_j}\right)_{i,j=1,3}$, $\operatorname{curl}(\mathbf{v}) := \left(\dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3}, \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1}, \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}\right)^{\mathrm{t}}$,

$\operatorname{div}(\mathbf{v}) := \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}$, $\mathbf{v} \times \mathbf{w} = (v_2\, w_3 - v_3\, w_2, v_3\, w_1 - v_1\, w_3, v_1\, w_2 - v_2\, w_1)^{\mathrm{t}}$, and $\mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,3}$. In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,3}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,3}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as $\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,3}$, $\operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{3} \tau_{ii}$, $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{3} \tau_{ij}\zeta_{ij}$, and $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\tau})\,\mathbb{I}$, where $\mathbb{I}$ is the identity matrix in $\mathrm{R}^{3\times 3}$. We recall that for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in R^3 and tensor field $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,3}$ in $\mathrm{R}^{3\times 3}$, the following identities there hold

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \;=\; -\,\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) \tag{1.1}$$

$$\text{and} \quad (\boldsymbol{\tau}\,\mathbf{w}) \cdot \mathbf{v} \;=\; \boldsymbol{\tau} : (\mathbf{v} \otimes \mathbf{w})\,. \tag{1.2}$$

Additionally, we recall the Hilbert spaces

$$\mathbf{H}(\mathrm{div};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \; \operatorname{div}(\boldsymbol{\tau}) \in \mathrm{L}^2(\Omega) \Big\} \quad \text{and} \quad \mathbf{H}(\mathrm{curl};\Omega) := \Big\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \; \operatorname{curl}(\mathbf{v}) \in \mathbf{L}^2(\Omega) \Big\}$$

endowed with the norms $\|\boldsymbol{\tau}\|^2_{\mathrm{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\operatorname{div}(\boldsymbol{\tau})\|^2_{0,\Omega}$ and $\|\mathbf{v}\|^2_{\mathrm{curl};\Omega} := \|\mathbf{v}\|^2_{0,\Omega} + \|\operatorname{curl}(\mathbf{v})\|^2_{0,\Omega}$, respectively. Both spaces are standard in mixed and electromagnetism problems, respectively. We denote by $\mathbf{H}(\mathrm{div}^0;\Omega)$ the subspace of $\mathbf{H}(\mathrm{div};\Omega)$ with divergence zero. In addition, in the sequel we will make use of the well-known Hölder inequality given by

$$\int_\Omega |fg| \le \|f\|_{0,t;\Omega} \|g\|_{0,t^*;\Omega} \quad \forall\, f \in \mathrm{L}^t(\Omega),\, \forall\, g \in \mathrm{L}^{t^*}(\Omega), \quad \text{with} \quad \frac{1}{t} + \frac{1}{t^*} = 1\,.$$

Finally, we recall the continuous injection i_t of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^t(\Omega)$ for $t \in [1,6]$ in R^3 (cf. [35, Theorem 1.3.4]). More precisely, we have the following inequality

$$\|w\|_{0,t;\Omega} \;\le\; \|i_t\|\, \|w\|_{1,\Omega} \quad \forall\, w \in \mathrm{H}^1(\Omega)\,, \tag{1.3}$$

with $\|i_t\| > 0$ depending only on $|\Omega|$ and t. We will denote by $\mathbf{i}_t$ the vectorial version of i_t.

2 The model problem and its continuous formulation

In this section, we introduce the model problem and derive its corresponding weak formulation.

2.1 The model problem

We are interested in analyzing the behavior of stationary magnetohydrodynamic flows within a fluid-saturated highly porous medium. These flows can be modeled by coupling the convective Brinkman–Forchheimer and Maxwell equations (see, for instance, [37, 2]). More precisely, assuming that the bounded Lipschitz polyhedral domain $\Omega \subset \mathrm{R}^3$ is simply-connected and with a connected boundary Γ, we focus on finding a velocity field $\mathbf{u}$, a pressure field p, and a magnetic field $\mathbf{b}$, such that

$$-\nu\,\Delta\mathbf{u} + (\nabla\mathbf{u})\mathbf{u} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} + \nabla p - \frac{1}{\mu}\operatorname{curl}(\mathbf{b}) \times \mathbf{b} = \mathbf{f}_f \quad \text{in} \quad \Omega\,, \tag{2.1a}$$

$$\operatorname{div}(\mathbf{u}) = g_f \quad \text{in} \quad \Omega\,, \tag{2.1b}$$

$$\frac{1}{\varrho\,\mu}\,\mathrm{curl}(\mathrm{curl}(\mathbf{b})) + \nabla\lambda - \mathrm{curl}(\mathbf{u}\times\mathbf{b}) = \mathbf{f}_m \quad \text{in} \quad \Omega\,, \tag{2.1c}$$

$$\mathrm{div}(\mathbf{b}) = 0 \quad \text{in} \quad \Omega\,, \tag{2.1d}$$

where, the unknown λ is the corresponding Lagrange multiplier associated with (2.1d) (see [36], [7], and [1] for similar approaches). Notice that to guarantee uniqueness of the pressure (cf. (2.1a)), this unknown will be sought in the space

$$\mathrm{L}^2_0(\Omega) := \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\}.$$

In addition, $g_f \in \mathrm{L}^\infty(\Omega)$ denotes a nonzero mass source, and $\mathbf{f}_f \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{f}_m \in \mathbf{L}^2(\Omega)$ are external forces. In turn, the constant $\nu > 0$ is the Brinkman coefficient, p is a given number in $[3,4]$, $\mu > 0$ is the magnetic permeability, $\varrho > 0$ is the electric conductivity, whereas, $\mathtt{D}$ and $\mathtt{F}$ stand for the Darcy and Forchheimer coefficients, respectively, both being positive and bounded spatially varying functions, i.e., there exist positive constants $\mathtt{D}_0$, $\mathtt{D}_1$, $\mathtt{F}_0$, and $\mathtt{F}_1$, such that

$$0 \,<\, \mathtt{D}_0 \,\leq\, \mathtt{D}(\mathbf{x}) \,\leq\, \mathtt{D}_1 \quad \text{and} \quad 0 \,<\, \mathtt{F}_0 \,\leq\, \mathtt{F}(\mathbf{x}) \,\leq\, \mathtt{F}_1 \quad \forall\,\mathbf{x} = (x_1,x_2,x_3)\in\Omega\,. \tag{2.2}$$

Finally, we consider the following boundary conditions:

$$\mathbf{u} = \mathbf{u}_{\mathrm{D}}, \quad \mathbf{n}\times\mathbf{b} = \mathbf{0}, \quad \text{and} \quad \lambda = 0 \quad \text{on} \quad \Gamma\,, \tag{2.3}$$

where $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$ is the prescribed velocity on Γ satisfying the compatibility condition

$$\int_\Gamma \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \,=\, \int_\Omega g_f\,. \tag{2.4}$$

Next, in order to derive a new skew-symmetric-based mixed formulation for (2.1)–(2.4), we proceed as in [15], [11] and [1], and incorporate as further unknowns a translation of the velocity gradient $\mathbf{t}$ and the pseudostress tensor $\boldsymbol{\sigma}$, which are defined, respectively, by

$$\mathbf{t} := \nabla\mathbf{u} - \frac{1}{3}\,g_f\,\mathbb{I} \quad \text{and} \quad \boldsymbol{\sigma} := \nu\,\nabla\mathbf{u} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u}) - p\,\mathbb{I} \quad \text{in} \quad \Omega\,. \tag{2.5}$$

In this way, applying the matrix trace to the tensors $\mathbf{t}$ and $\boldsymbol{\sigma}$, and utilizing the condition (2.1b), one arrives at $\mathrm{tr}(\mathbf{t}) = 0$ in Ω and

$$p = -\frac{1}{3}\left(\mathrm{tr}(\boldsymbol{\sigma}) + \frac{1}{2}\mathrm{tr}(\mathbf{u}\otimes\mathbf{u}) - \nu\,g_f\right) \quad \text{in} \quad \Omega\,. \tag{2.6}$$

In addition, using again (2.1b), we are able to deduce that

$$\mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = (\nabla\mathbf{u})\mathbf{u} \,+\, g_f\,\mathbf{u} \quad \text{in} \quad \Omega\,. \tag{2.7}$$

Hence, replacing back (2.6) in the second equation of (2.5), employing (2.7) and simple computations, we find that the model problem (2.1)–(2.3) can be rewritten, equivalently, as follows: Find $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma})$ and $(\mathbf{b},\lambda)$, in suitable spaces to be indicated below, such that

$$\nabla\mathbf{u} \,-\, \frac{1}{3}\,g_f\,\mathbb{I} = \mathbf{t} \quad \text{in} \quad \Omega\,, \tag{2.8a}$$

$$\nu\,\mathbf{t} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} = \boldsymbol{\sigma}^{\mathrm{d}} \ \text{in} \quad \Omega\,, \tag{2.8b}$$

$$\frac{1}{2}\mathbf{t}\,\mathbf{u} + \Big(\mathtt{D} - \frac{1}{3}\,g_f\Big)\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) - \frac{1}{\mu}\,\mathrm{curl}(\mathbf{b})\times\mathbf{b} = \mathbf{f}_f \quad \text{in} \quad \Omega\,, \tag{2.8c}$$

$$\int_\Omega \Big(\mathrm{tr}(\boldsymbol{\sigma}) + \frac{1}{2}\mathrm{tr}(\mathbf{u}\otimes\mathbf{u}) - \nu\, g_f\Big) = 0\,, \tag{2.8d}$$

$$\frac{1}{\varrho\,\mu^2}\,\mathrm{curl}(\mathrm{curl}(\mathbf{b})) + \frac{1}{\mu}\nabla\lambda - \frac{1}{\mu}\mathrm{curl}(\mathbf{u}\times\mathbf{b}) = \frac{1}{\mu}\,\mathbf{f}_m \quad \text{in} \quad \Omega\,, \tag{2.8e}$$

$$\mathrm{div}(\mathbf{b}) = 0 \quad \text{in} \quad \Omega\,, \tag{2.8f}$$

$$\mathbf{u} = \mathbf{u}_{\mathrm{D}}, \quad \mathbf{n}\times\mathbf{b} = \mathbf{0}, \quad \text{and} \quad \lambda = 0 \quad \text{on} \quad \Gamma\,. \tag{2.8g}$$

At this point we stress that, as suggested by (2.6), p is eliminated from the present formulation and computed afterwards in terms of $\mathbf{u}$, $\boldsymbol{\sigma}$ and g_f by using that identity. This fact justifies (2.8d), which aims to ensure that the resulting p does belong to $\mathrm{L}^2_0(\Omega)$. Notice also that further variables of interest, such as the velocity gradient $\mathbf{G} = \nabla\mathbf{u}$, the vorticity $\boldsymbol{\omega} = \frac{1}{2}\big(\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{t}}\big)$, and the stress $\widetilde{\boldsymbol{\sigma}} := \nu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{t}}) - p\,\mathbb{I}$ can be computed, respectively, as follows

$$\mathbf{G} \,=\, \mathbf{t} + \frac{1}{3}\,g_f\,\mathbb{I}, \quad \boldsymbol{\omega} \,=\, \frac{1}{2}\big(\mathbf{t} - \mathbf{t}^{\mathrm{t}}\big), \quad \text{and} \quad \widetilde{\boldsymbol{\sigma}} \,=\, \boldsymbol{\sigma} + \nu\,\mathbf{t}^{\mathrm{t}} + \frac{1}{2}(\mathbf{u}\otimes\mathbf{u}) + \frac{\nu}{3}\,g_f\,\mathbb{I}\,. \tag{2.9}$$

2.2 The five-field mixed variational formulation

In this section we derive a new five-field mixed variational formulation for the system (2.8). To that end, we first proceed as in [1, Section 2.2] (see also [15], [11] for similar approaches), extending the analysis derived there to the current convective Brinkman–Forchheimer equations. This includes considering the two nonlinear terms $(\nabla\mathbf{u})\mathbf{u}$ and $|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}$, with $\mathrm{p}\in[3,4]$, as well as spatially varying Darcy and Forchheimer coefficients. In fact, multiplying (2.8a), (2.8b) and (2.8c) by suitable test functions $\boldsymbol{\tau}, \mathbf{s}$, and $\mathbf{v}$, respectively, integrating by parts and using the Dirichlet boundary condition $\mathbf{u} = \mathbf{u}_{\mathrm{D}}$ on Γ (cf. (2.8g)), we get

$$-\int_\Omega \mathbf{t} : \boldsymbol{\tau} - \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \frac{1}{3}\int_\Omega g_f\,\mathrm{tr}(\boldsymbol{\tau}) - \langle\boldsymbol{\tau}\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle_\Gamma\,, \tag{2.10a}$$

$$\nu\int_\Omega \mathbf{t} : \mathbf{s} - \frac{1}{2}\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \mathbf{s} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s} = 0\,, \tag{2.10b}$$

$$\begin{aligned}\frac{1}{2}\int_\Omega (\mathbf{t}\,\mathbf{u})\cdot\mathbf{v} + \int_\Omega \Big(\mathtt{D} - \frac{1}{3}\,g_f\Big)\mathbf{u}\cdot\mathbf{v} + \int_\Omega \mathtt{F}\,|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}\cdot\mathbf{v} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \\ -\frac{1}{\mu}\int_\Omega (\mathrm{curl}(\mathbf{b})\times\mathbf{b})\cdot\mathbf{v} = \int_\Omega \mathbf{f}_f\cdot\mathbf{v}\,,\end{aligned} \tag{2.10c}$$

for all $(\boldsymbol{\tau}, \mathbf{s}, \mathbf{v}) \in \mathbb{X}\times\mathbb{Q}\times\mathbf{M}$, where $\mathbb{X}, \mathbb{Q}$ and $\mathbf{M}$ are spaces to be defined below. On the other hand, for the Maxwell equations (2.8e)–(2.8f), we proceed as in [36] (see also [7] for a similar approach), that is, we introduce the space

$$\mathbf{H}_0(\mathrm{curl};\Omega) := \Big\{\mathbf{d}\in\mathbf{H}(\mathrm{curl};\Omega) : \quad \mathbf{n}\times\mathbf{d} = \mathbf{0} \quad \text{on} \quad \Gamma\Big\}\,,$$

and multiply (2.8e) by $\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega)$, and integrate by parts, to get

$$\frac{1}{\varrho\,\mu^2}\int_\Omega \mathrm{curl}(\mathbf{b})\cdot\mathrm{curl}(\mathbf{d}) + \frac{1}{\mu}\int_\Omega \nabla\lambda\cdot\mathbf{d} - \frac{1}{\mu}\int_\Omega (\mathbf{u}\times\mathbf{b})\cdot\mathrm{curl}(\mathbf{d}) \,=\, \frac{1}{\mu}\int_\Omega \mathbf{f}_m\cdot\mathbf{d}\,.$$

Then, applying the identity (1.1) to $\mathbf{u}$, $\mathbf{b}$, and $\mathrm{curl}(\mathbf{d})$ in the third term of the foregoing equation, and testing (2.8f) by $\xi \in \mathrm{H}^1_0(\Omega)$, integrating by parts, and multiplying the resulting equation by $1/\mu$, we obtain

$$\frac{1}{\varrho\,\mu^2}\int_\Omega \mathrm{curl}(\mathbf{b})\cdot\mathrm{curl}(\mathbf{d}) + \frac{1}{\mu}\int_\Omega \nabla\lambda\cdot\mathbf{d} + \frac{1}{\mu}\int_\Omega \mathbf{u}\cdot(\mathrm{curl}(\mathbf{d})\times\mathbf{b}) \;=\; \frac{1}{\mu}\int_\Omega \mathbf{f}_m\cdot\mathbf{d}\,, \tag{2.11a}$$

$$\frac{1}{\mu}\int_\Omega \mathbf{b}\cdot\nabla\xi \;=\; 0\,, \tag{2.11b}$$

for all $(\mathbf{d},\xi)\in \mathbf{H}_0(\mathrm{curl};\Omega)\times \mathrm{H}^1_0(\Omega)$. In this way, at first we are interested in finding $\boldsymbol{\sigma}\in\mathbb{X}$, $\mathbf{t}\in\mathbb{Q}$, $\mathbf{u}\in\mathbf{M}$, $\mathbf{b}\in\mathbf{H}_0(\mathrm{curl};\Omega)$ and $\lambda\in\mathrm{H}^1_0(\Omega)$ satisfying (2.10)–(2.11) and the condition (2.8d).

Now, we turn to specify the spaces $\mathbb{X}$, $\mathbb{Q}$, and $\mathbf{M}$. We begin by noting that the first term in (2.10b) is well defined for $\mathbf{t},\mathbf{s}\in\mathbb{L}^2(\Omega)$, but due to the condition $\mathrm{tr}(\mathbf{t})=0$ in Ω, it makes sense to look for $\mathbf{t}$, and consequently the test function $\mathbf{s}$, in $\mathbb{Q}=\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, with

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) := \Big\{\mathbf{s}\in\mathbb{L}^2(\Omega): \quad \mathrm{tr}(\mathbf{s})=0 \quad\text{in}\quad \Omega\Big\}.$$

This fact and a direct application of the identity (1.2) to $\mathbf{t},\mathbf{u}$, and $\mathbf{v}$ in the first term of (2.10c), implies that (2.10b) and (2.10c) can be rewritten, equivalently, as

$$\nu\int_\Omega \mathbf{t}:\mathbf{s} - \frac{1}{2}\int_\Omega(\mathbf{u}\otimes\mathbf{u}):\mathbf{s} - \int_\Omega \boldsymbol{\sigma}:\mathbf{s} = 0 \quad \forall\,\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{2.12a}$$

$$\begin{aligned}\frac{1}{2}\int_\Omega \mathbf{t}:(\mathbf{v}\otimes\mathbf{u}) + \int_\Omega\Big(\mathtt{D}-\frac{1}{3}g_f\Big)\mathbf{u}\cdot\mathbf{v} + \int_\Omega \mathtt{F}\,|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}\cdot\mathbf{v} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma})&\\ -\frac{1}{\mu}\int_\Omega(\mathrm{curl}(\mathbf{b})\times\mathbf{b})\cdot\mathbf{v} = \int_\Omega \mathbf{f}_f\cdot\mathbf{v} \quad \forall\,\mathbf{v}\in\mathbf{M}\,.&\end{aligned} \tag{2.12b}$$

In turn, we let

$$\mathbf{C} := \Big\{\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega): \quad \int_\Omega \mathbf{d}\cdot\nabla\xi = 0 \quad \forall\,\xi\in\mathrm{H}^1_0(\Omega)\Big\} = \mathbf{H}_0(\mathrm{curl};\Omega)\cap\mathbf{H}(\mathrm{div}^0;\Omega)\,, \tag{2.13}$$

and observe that, since $\mathbf{b}$ satisfies (2.11b), with constant $\mu>0$, then $\mathbf{b}\in\mathbf{C}$ (see [26, Section I.2.2]). Thus, since $\mathbf{C}$ is continuously embedded into $\mathbf{H}^s(\Omega)$ for some $s>1/2$ (cf. [3, Proposition 3.7]), which in turn is continuously embedded into $\mathbf{L}^{3+\delta}(\Omega)$, for some $\delta>0$ (see [35, Theorem 1.3.4]), we obtain

$$\|\mathbf{b}\|_{0,3+\delta;\Omega} \;\le\; c_1\,\|\mathbf{b}\|_{\mathrm{curl};\Omega} \quad \forall\,\mathbf{b}\in\mathbf{C}\,.$$

Therefore, using the well-known embedding inequality

$$\|\mathbf{v}\|_{0,t;\Omega} \;\le\; c_2\,\|\mathbf{v}\|_{0,6;\Omega} \quad \forall\, t\in[1,6)\,, \tag{2.14}$$

and defining $\delta^* := \frac{4\,\delta}{1+\delta} > 0$, it follows that

$$\left|\int_\Omega(\mathrm{curl}(\mathbf{d})\times\mathbf{b})\cdot\mathbf{v}\right| \le \|\mathrm{curl}(\mathbf{d})\|_{0,\Omega}\,\|\mathbf{b}\|_{0,3+\delta;\Omega}\,\|\mathbf{v}\|_{0,6-\delta^*;\Omega} \le C_s\,\|\mathbf{d}\|_{\mathrm{curl};\Omega}\,\|\mathbf{b}\|_{\mathrm{curl};\Omega}\,\|\mathbf{v}\|_{0,6;\Omega}\,, \tag{2.15}$$

for all $\mathbf{d}\in\mathbf{H}(\mathrm{curl};\Omega), \mathbf{b}\in\mathbf{C}$ and $\mathbf{v}\in\mathbf{L}^6(\Omega)$, with C_s the resulting constant from the aforementioned embedding inequalities. According to the above, the fifth and third terms in (2.12b) (or (2.10c)) and (2.11a), respectively, are well defined if we set $\mathbf{M} := \mathbf{L}^6(\Omega)$, which, thanks to (2.14), is consistent with the first, second, and third terms of (2.12b), and consequently, the second and fourth terms in (2.10a) and (2.12b), respectively, are well defined if $\mathbf{div}(\boldsymbol{\sigma})$ and $\mathbf{div}(\boldsymbol{\tau})$ belong to $\mathbf{L}^{6/5}(\Omega)$. In addition, using the fact that the first and third terms in (2.10a) and (2.12a) (or (2.10b)), respectively, are well defined if $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$, we introduce the Banach space

$$\mathbb{H}(\mathbf{div}_{6/5};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^{6/5}(\Omega) \Big\},$$

equipped with the norm $\|\boldsymbol{\tau}\|^2_{\mathbf{div}_{6/5};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,6/5;\Omega}$, and deduce that (2.10) and (2.12) are well defined if we choose the spaces $\mathbb{Q} := \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbf{M} := \mathbf{L}^6(\Omega)$, and $\mathbb{X} := \mathbb{H}(\mathbf{div}_{6/5};\Omega)$, with their respective norms: $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{0,6;\Omega}$, and $\|\cdot\|_{\mathbf{div}_{6/5};\Omega}$. Now, for convenience of the subsequent analysis and similarly as in [1] (see also [7, 6, 15]) we consider the decomposition

$$\mathbb{H}(\mathbf{div}_{6/5};\Omega) \;=\; \mathbb{H}_0(\mathbf{div}_{6/5};\Omega) \oplus \mathrm{R}\,\mathbb{I}\,,$$

where

$$\mathbb{H}_0(\mathbf{div}_{6/5};\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{6/5};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \right\},$$

that is, $\mathrm{R}\,\mathbb{I}$ is a topological supplement for $\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$. More precisely, each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{6/5};\Omega)$ can be decomposed uniquely as

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbb{I}\,, \quad \text{with} \quad \boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega) \quad \text{and} \quad d := \frac{1}{3\,|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \in \mathrm{R}\,.$$

In particular, using from (2.8d) that $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega \big(2\,\nu g_f - \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\big)$, we obtain

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0\,\mathbb{I} \quad \text{with} \quad \boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega) \quad \text{and} \quad c_0 = \frac{1}{6\,|\Omega|}\int_\Omega \Big(2\,\nu g_f - \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\Big)\,. \tag{2.16}$$

In this way, knowing explicitly c_0 in terms of $\mathbf{u}$ and g_f, it remains to find the $\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$-component $\boldsymbol{\sigma}_0$ of $\boldsymbol{\sigma}$ to fully determine it. In this regard, using the fact that $\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{div}(\boldsymbol{\sigma}_0)$ and $\boldsymbol{\sigma} : \mathbf{s} = \boldsymbol{\sigma}_0 : \mathbf{s}$, for all $\mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, we deduce that (2.10b)–(2.10c) remain unchanged if $\boldsymbol{\sigma}$ is replaced there by $\boldsymbol{\sigma}_0$. Moreover it is easy to see, thanks to the compatibility condition (2.4) satisfied by the Dirichlet datum $\mathbf{u}_{\mathrm{D}}$, that both sides of (2.10a) always holds when $\boldsymbol{\tau} \in \mathrm{R}\,\mathbb{I}$, and hence, testing this equation against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{6/5};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$. According to the above, and redenoting from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$, we arrive to the variational problem: Find $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{b},\lambda) \in \mathbf{H}_0(\mathrm{curl};\Omega) \times \mathrm{H}^1_0(\Omega)$, such that

$$[a_f(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})] + [d_f(\mathbf{u})(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})] + [c_f(\mathbf{b})(\mathbf{b}),\mathbf{v}] + [b_f(\mathbf{v},\mathbf{s}),\boldsymbol{\sigma}] = [F_1,(\mathbf{v},\mathbf{s})]\,, \tag{2.17a}$$

$$[b_f(\mathbf{u},\mathbf{t}),\boldsymbol{\tau}] = [F_2,\boldsymbol{\tau}]\,, \tag{2.17b}$$

$$[a_m(\mathbf{b}),\mathbf{d}] + [c_m(\mathbf{b})(\mathbf{u}),\mathbf{d}] + [b_m(\mathbf{d}),\lambda] = [F_3,\mathbf{d}]\,, \tag{2.17c}$$

$$[b_m(\mathbf{b}),\xi] = 0\,, \tag{2.17d}$$

for all $(\mathbf{v},\mathbf{s},\boldsymbol{\tau}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{d},\xi) \in \mathbf{H}_0(\mathrm{curl};\Omega) \times \mathrm{H}^1_0(\Omega)$, where the operators a_f, b_f, a_m, b_m, $d_f(\mathbf{w})$, $c_f(\widehat{\mathbf{b}})$, $c_m(\widehat{\mathbf{b}})$, for given $\mathbf{w} \in \mathbf{L}^6(\Omega)$ and $\widehat{\mathbf{b}} \in \mathbf{C}$ (cf. (2.13)), are defined, respectively, as

$$[a_f(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})] \;:=\; \int_\Omega \Big(\mathtt{D} - \frac{1}{3}\,g_f\Big)\mathbf{u}\cdot\mathbf{v} + \int_\Omega \mathtt{F}\,|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}\cdot\mathbf{v} + \nu\int_\Omega \mathbf{t} : \mathbf{s}\,, \tag{2.18}$$

$$[b_f(\mathbf{v},\mathbf{s}),\boldsymbol{\tau}] \;:=\; -\int_\Omega \boldsymbol{\tau} : \mathbf{s} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \tag{2.19}$$

$$[a_m(\mathbf{b}),\mathbf{d}] \;:=\; \frac{1}{\varrho\,\mu^2}\int_\Omega \mathrm{curl}(\mathbf{b})\cdot\mathrm{curl}(\mathbf{d})\,, \quad [b_m(\mathbf{d}),\xi] \;:=\; \frac{1}{\mu}\int_\Omega \mathbf{d}\cdot\nabla\xi\,, \tag{2.20}$$

$$[d_f(\mathbf{w})(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})] \;:=\; \frac{1}{2}\left\{\int_\Omega \mathbf{t} : (\mathbf{v}\otimes\mathbf{w}) - \int_\Omega (\mathbf{u}\otimes\mathbf{w}) : \mathbf{s}\right\}, \tag{2.21}$$

and

$$[c_f(\widehat{\mathbf{b}})(\mathbf{b}),\mathbf{v}] \;:=\; -\frac{1}{\mu}\int_\Omega \big(\mathrm{curl}(\mathbf{b})\times\widehat{\mathbf{b}}\big)\cdot\mathbf{v}\,, \quad [c_m(\widehat{\mathbf{b}})(\mathbf{u}),\mathbf{d}] \;:=\; \frac{1}{\mu}\int_\Omega \mathbf{u}\cdot\big(\mathrm{curl}(\mathbf{d})\times\widehat{\mathbf{b}}\big)\,, \tag{2.22}$$

for all $(\mathbf{v},\mathbf{s},\boldsymbol{\tau}) \in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{d},\xi) \in \mathbf{H}_0(\mathrm{curl};\Omega)\times\mathrm{H}^1_0(\Omega)$. In turn, F_1, F_2, and F_3 are the bounded linear functionals defined by

$$[F_1,(\mathbf{v},\mathbf{s})] \;:=\; \int_\Omega \mathbf{f}_f\cdot\mathbf{v}\,, \quad [F_2,\boldsymbol{\tau}] \;:=\; \frac{1}{3}\int_\Omega g_f\,\mathrm{tr}(\boldsymbol{\tau}) \;-\; \langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_\mathrm{D}\rangle_\Gamma\,, \tag{2.23}$$

and

$$[F_3,\mathbf{d}] \;:=\; \frac{1}{\mu}\int_\Omega \mathbf{f}_m\cdot\mathbf{d}\,. \tag{2.24}$$

In all the terms above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators.

2.3 The skew-symmetric-based mixed variational formulation

Now, we derive the reduced problem of (2.17), which features a skew-symmetric-based mixed formulation. To do this, let's first define the global unknowns and space:

$$\vec{\mathbf{u}} := (\mathbf{u},\mathbf{t},\mathbf{b}) \in \mathbf{X} := \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{C}\,, \tag{2.25}$$

where $\mathbf{X}$ is endowed with the norm

$$\|\vec{\mathbf{v}}\|^2_{\mathbf{X}} = \|(\mathbf{v},\mathbf{s},\mathbf{d})\|^2_{\mathbf{X}} = \|\mathbf{v}\|^2_{0,6;\Omega} + \|\mathbf{s}\|^2_{0,\Omega} + \|\mathbf{d}\|^2_{\mathrm{curl};\Omega} \quad \forall\,\vec{\mathbf{v}} := (\mathbf{v},\mathbf{s},\mathbf{d}) \in \mathbf{X}. \tag{2.26}$$

Next, we can introduce the reduced problem: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ such that

$$\begin{aligned}
[\mathbf{A}(\mathbf{u},\mathbf{b})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] + [\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\sigma}] &= [\mathbf{F},\vec{\mathbf{v}}] \quad \forall\,\vec{\mathbf{v}} \in \mathbf{X}\,,\\
[\mathbf{B}(\vec{\mathbf{u}}),\boldsymbol{\tau}] &= [\mathbf{G},\boldsymbol{\tau}] \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,,
\end{aligned} \tag{2.27}$$

where, given $(\mathbf{w},\widehat{\mathbf{b}}) \in \mathbf{L}^6(\Omega)\times\mathbf{C}$, the operator $\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}}) : \mathbf{X}\to\mathbf{X}'$ is defined by

$$[\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] \;:=\; [\mathbf{a}(\vec{\mathbf{u}}),\vec{\mathbf{v}}] + [\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] \tag{2.28}$$

with

$$[\mathbf{a}(\vec{\mathbf{u}}),\vec{\mathbf{v}}] \;:=\; [a_f(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})] + [a_m(\mathbf{b}),\mathbf{d}]\,, \tag{2.29}$$

and

$$[\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] \;:=\; [d_f(\mathbf{w})(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})] \;+\; [c_f(\widehat{\mathbf{b}})(\mathbf{b}),\mathbf{v}] \;+\; [c_m(\widehat{\mathbf{b}})(\mathbf{u}),\mathbf{d}]\,. \tag{2.30}$$

In particular, from (2.21) and (2.22), we deduce the aforementioned skew-symmetric-based nature of the operator $\mathbf{c}$ (cf. (2.30)), namely,

$$\begin{aligned} [\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}] \,&=\, \frac{1}{2}\left\{\int_\Omega \mathbf{t} : (\mathbf{v}\otimes\mathbf{w}) \,-\, \int_\Omega (\mathbf{u}\otimes\mathbf{w}) : \mathbf{s}\right\} \\ &+\, \frac{1}{\mu}\left\{\int_\Omega \mathbf{u}\cdot\big(\mathrm{curl}(\mathbf{d})\times\widehat{\mathbf{b}}\big) \,-\, \int_\Omega \big(\mathrm{curl}(\mathbf{b})\times\widehat{\mathbf{b}}\big)\cdot\mathbf{v}\right\}. \end{aligned} \tag{2.31}$$

In turn, the operator $\mathbf{B} : \mathbf{X} \to \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'$ is given by

$$[\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}] \,:=\, [b_f(\mathbf{v},\mathbf{s}),\boldsymbol{\tau}]\,, \tag{2.32}$$

whereas, the functionals $\mathbf{F}$ and $\mathbf{G}$ are set as

$$[\mathbf{F},\vec{\mathbf{v}}] \,:=\, [F_1,(\mathbf{v},\mathbf{s})] + [F_3,\mathbf{d}] \quad \text{and} \quad [\mathbf{G},\boldsymbol{\tau}] \,:=\, [F_2,\boldsymbol{\tau}]\,. \tag{2.33}$$

According to the definition of $\mathbf{C}$ (cf. (2.13)), owing to the inf-sup condition of the operator b_m (see [36, Section 2.4] or [30, Section 5.4] for details):

$$\sup_{\substack{\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega)\\ \mathbf{d}\neq 0}} \frac{[b_m(\mathbf{d}),\xi]}{\|\mathbf{d}\|_{\mathrm{curl};\Omega}} \,\geq\, \beta_m\,\|\xi\|_{1,\Omega} \quad \forall\,\xi\in\mathrm{H}_0^1(\Omega)\,, \tag{2.34}$$

with $\beta_m > 0$, and using similar arguments to the ones developed in [1, Lemma 2.1] (see also [36]), it is not difficult to show that the problem (2.27) is equivalent to (2.17), reason why its proof is omitted. This result is stated next.

Lemma 2.1 *If* $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma}) \in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ *and* $(\mathbf{b},\lambda) \in \mathbf{H}_0(\mathrm{curl};\Omega)\times\mathrm{H}_0^1(\Omega)$ *is a solution of* (2.17)*, then* $\mathbf{b}\in\mathbf{C}$ *and* $(\vec{\mathbf{u}},\boldsymbol{\sigma}) = ((\mathbf{u},\mathbf{t},\mathbf{b}),\boldsymbol{\sigma}) \in \mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ *is a solution of* (2.27)*. Conversely, if* $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ *is a solution of* (2.27)*, then there exists* $\lambda\in\mathrm{H}_0^1(\Omega)$ *such that* $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma})$ *and* $(\mathbf{b},\lambda)$ *is a solution of* (2.17)*.*

As a consequence of the above, in what follows we focus on analyzing problem (2.27).

3 Analysis of the coupled problem

In this section we apply the abstract result provided by [1, Theorem 3.1] (see also [11, Theorem 3.1]), which establishes sufficient conditions for the well-posedness of a nonlinear saddle-point problem in Banach spaces and the classical Banach fixed-point theorem, to prove the well-posedness of (2.27) (equivalently of (2.17)) under suitable smallness assumptions on the data.

3.1 Preliminaries

Here we establish the stability properties of some of the operators involved in (2.17) and (2.27). We begin by observing that the operators a_m, $\mathbf{B}$ and functionals F_3, $\mathbf{F}$, $\mathbf{G}$ are linear. In turn, from (2.20), (2.32), (2.24), (2.33), and employing Hölder and Cauchy–Schwarz inequalities, there hold

$$|[a_m(\mathbf{b}),\mathbf{d}]| \,\leq\, \frac{1}{\varrho\,\mu^2}\,\|\mathbf{b}\|_{\mathrm{curl};\Omega}\,\|\mathbf{d}\|_{\mathrm{curl};\Omega} \quad \forall\,\mathbf{b},\mathbf{d}\in\mathbf{H}(\mathrm{curl};\Omega)\,, \tag{3.1}$$

$$|[\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}]| \,\leq\, \|\vec{\mathbf{v}}\|_{\mathbf{X}}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5};\Omega} \quad \forall\,\vec{\mathbf{v}}\in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{H}(\mathrm{curl};\Omega),\,\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,, \tag{3.2}$$

$$|[F_3,\mathbf{d}]| \,\leq\, \frac{1}{\mu}\|\mathbf{f}_m\|_{0,\Omega}\|\mathbf{d}\|_{\mathrm{curl};\Omega} \quad \forall\,\mathbf{d}\in\mathbf{H}(\mathrm{curl};\Omega)\,, \tag{3.3}$$

$$|[\mathbf{F},\vec{\mathbf{v}}]| \,\leq\, C_{\mathbf{F}}\left(\|\mathbf{f}_f\|_{0,6/5;\Omega}+\|\mathbf{f}_m\|_{0,\Omega}\right)\|\vec{\mathbf{v}}\|_{\mathbf{X}} \quad \forall\,\vec{\mathbf{v}}\in\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{H}(\mathrm{curl};\Omega)\,, \tag{3.4}$$

and

$$|[\mathbf{G},\boldsymbol{\tau}]| \,\leq\, C_{\mathbf{G}}\left(\|g_f\|_{0,\infty;\Omega}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right)\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5};\Omega} \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,, \tag{3.5}$$

where $C_{\mathbf{F}} := \max\left\{1, 1/\mu\right\}$ and $C_{\mathbf{G}} := \max\left\{|\Omega|^{1/2}/\sqrt{3}, C_\Omega\right\}$, with C_Ω a positive constant depending on $\|\mathbf{i}_6\|$ (for more details see [6, Lemma 3.5] and (1.3)). Notice that (3.2) and (3.4) also hold for all $\vec{\mathbf{v}}\in\mathbf{X}$. We have written (3.2) and (3.4) in a more general form since both inequalities will be used later on to prove well-posedness of the Galerkin scheme proposed in Section 4 and to derive the *a priori* error analysis (cf. Lemma 5.1). In addition, using [15, Lemma 3.4] and the definition of the operators $d_f(\mathbf{w})$ (cf. (2.21)), we observe that for any $\mathbf{w}\in\mathbf{L}^6(\Omega)$ there holds

$$\begin{aligned}\big|[d_f(\mathbf{w})(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})]\big| \,&\leq\, \frac{1}{2}\,|\Omega|^{1/6}\|\mathbf{w}\|_{0,6;\Omega}\big(\|\mathbf{t}\|^2_{0,\Omega}+\|\mathbf{u}\|^2_{0,6;\Omega}\big)^{1/2}\big(\|\mathbf{s}\|^2_{0,\Omega}+\|\mathbf{v}\|^2_{0,6;\Omega}\big)^{1/2}\\ &\leq\, \frac{1}{2}\,|\Omega|^{1/6}\,\|\mathbf{w}\|_{0,6;\Omega}\,\|\vec{\mathbf{u}}\|_{\mathbf{X}}\,\|\vec{\mathbf{v}}\|_{\mathbf{X}} \quad \forall\,\vec{\mathbf{u}}=(\mathbf{u},\mathbf{t},\mathbf{b}),\vec{\mathbf{v}}=(\mathbf{v},\mathbf{s},\mathbf{d})\in\mathbf{X}\,.\end{aligned} \tag{3.6}$$

Finally, given $(\mathbf{w},\widehat{\mathbf{b}})\in\mathbf{L}^6(\Omega)\times\mathbf{C}$, from the definition of the operators $c_f(\widehat{\mathbf{b}})\,,c_m(\widehat{\mathbf{b}})\,,\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})$ (cf. (2.22), (2.30)) and (2.15), we obtain

$$\big|[c_f(\widehat{\mathbf{b}})(\mathbf{b}),\mathbf{v}]\big| \,\leq\, \frac{C_s}{\mu}\,\|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\,\|\mathbf{b}\|_{\mathrm{curl};\Omega}\,\|\mathbf{v}\|_{0,6;\Omega} \quad \forall\,(\mathbf{b},\mathbf{v})\in\mathbf{H}(\mathrm{curl};\Omega)\times\mathbf{L}^6(\Omega)\,, \tag{3.7}$$

$$\big|[c_m(\widehat{\mathbf{b}})(\mathbf{u}),\mathbf{d}]\big| \,\leq\, \frac{C_s}{\mu}\,\|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega}\,\|\mathbf{u}\|_{0,6;\Omega}\,\|\mathbf{d}\|_{\mathrm{curl};\Omega} \quad \forall\,(\mathbf{u},\mathbf{d})\in\mathbf{L}^6(\Omega)\times\mathbf{H}(\mathrm{curl};\Omega)\,, \tag{3.8}$$

and, combining (3.6), (3.7), and (3.8), there holds

$$\big|[\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}]\big| \,\leq\, C_{\mathbf{c}}\,\|(\mathbf{w},\widehat{\mathbf{b}})\|_{\mathtt{MH}}\,\|\vec{\mathbf{u}}\|_{\mathbf{X}}\,\|\vec{\mathbf{v}}\|_{\mathbf{X}} \quad \forall\,\vec{\mathbf{u}}=(\mathbf{u},\mathbf{t},\mathbf{b}),\vec{\mathbf{v}}=(\mathbf{v},\mathbf{s},\mathbf{d})\in\mathbf{X}\,, \tag{3.9}$$

with $C_{\mathbf{c}} := \max\left\{C_s/\mu, |\Omega|^{1/6}/2\right\}$ and the product norm

$$\|(\mathbf{v},\mathbf{d})\|^2_{\mathtt{MH}} := \|\mathbf{v}\|^2_{0,6;\Omega}+\|\mathbf{d}\|^2_{\mathrm{curl};\Omega} \quad \forall\,(\mathbf{v},\mathbf{d})\in\mathbf{L}^6(\Omega)\times\mathbf{H}(\mathrm{curl};\Omega)\,.$$

In addition, it is easy to see from (2.31) the skew-symmetric property of the operator $\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})$:

$$[\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{v}}),\vec{\mathbf{v}}] \,=\, 0 \quad \forall\,\vec{\mathbf{v}}\in\mathbf{X}\,. \tag{3.10}$$

3.2 A fixed point strategy

We begin the solvability analysis of (2.27) (equivalently of (2.17)) by defining the operator $\mathbf{T}:\mathbf{L}^6(\Omega)\times\mathbf{C}\to\mathbf{L}^6(\Omega)\times\mathbf{C}$ by

$$\mathbf{T}(\mathbf{w},\widehat{\mathbf{b}}) \,:=\, (\mathbf{u},\mathbf{b}) \quad \forall\,(\mathbf{w},\widehat{\mathbf{b}})\in\mathbf{L}^6(\Omega)\times\mathbf{C}\,, \tag{3.11}$$

where $(\mathbf{u},\mathbf{b})$ is part of the element $(\vec{\mathbf{u}},\boldsymbol{\sigma})=((\mathbf{u},\mathbf{t},\mathbf{b}),\boldsymbol{\sigma})$ in $\mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ satisfying

$$\begin{aligned}[\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}]+[\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\sigma}] \;&=\; [\mathbf{F},\vec{\mathbf{v}}] \quad \forall\,\vec{\mathbf{v}}\in\mathbf{X}\,,\\ [\mathbf{B}(\vec{\mathbf{u}}),\boldsymbol{\tau}] \;&=\; [\mathbf{G},\boldsymbol{\tau}] \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,.\end{aligned} \tag{3.12}$$

Notice that solving (2.27) is equivalent to finding $(\mathbf{u},\mathbf{b}) \in \mathbf{L}^6(\Omega) \times \mathbf{C}$ such that

$$\mathbf{T}(\mathbf{u},\mathbf{b}) \,=\, (\mathbf{u},\mathbf{b})\,.$$

In this way, in what follows we focus on proving that $\mathbf{T}$ possesses a unique fixed-point. Given $(\mathbf{w},\widehat{\mathbf{b}}) \in \mathbf{L}^6(\Omega) \times \mathbf{C}$, we first show that [1, Theorem 3.1] (see also [11, Theorem 3.1]) can be applied to ensure the well-posed of the coupled problem (3.12), which means, equivalently, that $\mathbf{T}$ (cf. 3.11) is indeed well-defined. More precisely, setting $\mathcal{A} := \mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})$, $\mathcal{B} := \mathbf{B}$, $X_1 := \mathbf{L}^6(\Omega)$, $X_2 := \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $X_3 := \mathbf{C}$, $X := \mathbf{X}$, $Y := \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$, and $V := \mathbf{V}$, being X_1, X_2 and X_3 uniformly convex and $\mathbf{V}$ the kernel of the operator $\mathbf{B}$, the following assumptions need to be satisfied:

(i) there exist constants $L > 0$ and p_1, p_2, $p_3 \geq 2$, such that

$$\|\mathcal{A}(\vec{u}) - \mathcal{A}(\vec{v})\|_{X'} \,\leq\, L \sum_{j=1}^{3} \Big\{ \|u_j - v_j\|_{X_j} + \big(\|u_j\|_{X_j} + \|v_j\|_{X_j}\big)^{p_j-2} \|u_j - v_j\|_{X_j} \Big\} \tag{3.13}$$

for all $\vec{u} = (u_1,u_2,u_3)$, $\vec{v} = (v_1,v_2,v_3) \in X$,

(ii) the family of operators $\Big\{ \mathcal{A}(\,\cdot + \vec{z}) : V \to V' : \quad \vec{z} \in X \Big\}$ is uniformly strongly monotone, that is there exists $\alpha > 0$ such that

$$[\mathcal{A}(\vec{u} + \vec{z}) - \mathcal{A}(\vec{v} + \vec{z}), \vec{u} - \vec{v}] \,\geq\, \alpha\, \|\vec{u} - \vec{v}\|^2_X\,, \tag{3.14}$$

for all $\vec{z} \in X$, and for all $\vec{u}, \vec{v} \in V$, and

(iii) there exists $\beta > 0$ such that

$$\sup_{\substack{\vec{v} \in X \\ \vec{v} \neq 0}} \frac{[\mathcal{B}(\vec{v}), \tau]}{\|\vec{v}\|_{\mathrm{X}}} \,\geq\, \beta\, \|\tau\|_Y \quad \forall\, \tau \in Y\,. \tag{3.15}$$

Indeed, it is clear from the uniform convexity and separability of $\mathrm{L}^t(\Omega)$, for $t \in (1,+\infty)$, all the spaces involved in (3.12), that is, $\mathbf{L}^6(\Omega)$, $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbf{C}$ and $\mathbb{H}_0(\mathbf{div}_{6/5},\Omega)$, share the same properties. Next, we continue by proving that, given $(\mathbf{w},\widehat{\mathbf{b}}) \in \mathbf{L}^6(\Omega) \times \mathbf{C}$, the nonlinear operator $\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})$ (cf. (2.28)) satisfies hypothesis (i) of [1, Theorem 3.1] (cf. (3.13)), with $p_1 = \mathrm{p} \in [3,4]$ and $p_2 = p_3 = 2$.

Lemma 3.1 *Let* $\mathrm{p} \in [3,4]$. *Given* $(\mathbf{w},\widehat{\mathbf{b}}) \in \mathbf{L}^6(\Omega) \times \mathbf{C}$, *there exists* $L_{\mathtt{MH}} > 0$, *depending on* ν, $\mathtt{F}_1$, $\mathtt{D}_1$, ϱ, μ, $|\Omega|$, C_s, *and* $\|g_f\|_{0,\infty;\Omega}$, *such that*

$$\begin{aligned} &\|\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}) - \mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{v}})\|_{\mathbf{X}'} \\ &\quad\leq\, L_{\mathtt{MH}} \Big\{ \big(1 + \|(\mathbf{w},\widehat{\mathbf{b}})\|_{\mathtt{MH}}\big) \|\vec{\mathbf{u}} - \vec{\mathbf{v}}\|_{\mathbf{X}} \,+\, \big(\|\mathbf{u}\|_{0,6;\Omega} + \|\mathbf{v}\|_{0,6;\Omega}\big)^{\mathrm{p}-2} \|\mathbf{u} - \mathbf{v}\|_{0,6;\Omega} \Big\}\,, \end{aligned} \tag{3.16}$$

for all $\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t},\mathbf{b})$, $\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s},\mathbf{d}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{H}(\mathrm{curl};\Omega)$.

Proof. Let $\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t},\mathbf{b})$, $\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s},\mathbf{d})$, and $\vec{\mathbf{z}} = (\mathbf{z},\mathbf{r},\mathbf{e}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{H}(\mathrm{curl};\Omega)$. From the definition of the operator $\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})$ (cf. (2.28)), the Cauchy–Schwarz and Hölder inequalities, the continuity bound of $\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})$ (cf. (3.9)), (2.2) and simple computations, we deduce that

$$\begin{aligned} &[\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}) - \mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{v}}), \vec{\mathbf{z}}] \,\leq\, \mathtt{F}_1\, \||\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - |\mathbf{v}|^{\mathrm{p}-2}\mathbf{v}\|_{0,\mathrm{q};\Omega}\, \|\mathbf{z}\|_{0,\mathrm{p};\Omega} \\ &\quad + \Big(\mathtt{D}_1 + \frac{1}{3}\, \|g_f\|_{0,\infty;\Omega}\Big) |\Omega|^{2/3}\, \|\mathbf{u} - \mathbf{v}\|_{0,6;\Omega} \|\mathbf{z}\|_{0,6;\Omega} + \nu\, \|\mathbf{t} - \mathbf{s}\|_{0,\Omega}\, \|\mathbf{r}\|_{0,\Omega} \\ &\quad + \frac{1}{\varrho\mu^2}\, \|\mathbf{b} - \mathbf{d}\|_{\mathrm{curl};\Omega}\, \|\mathbf{e}\|_{\mathrm{curl};\Omega} \,+\, \frac{C_s}{\mu}\, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega} \big(\|\mathbf{u} - \mathbf{v}\|_{0,6;\Omega} + \|\mathbf{b} - \mathbf{d}\|_{\mathrm{curl};\Omega}\big)\, \|\vec{\mathbf{z}}\|_{\mathbf{X}} \\ &\quad + \frac{1}{2}\, |\Omega|^{1/6}\, \|\mathbf{w}\|_{0,6;\Omega}\, \big(\|\mathbf{t} - \mathbf{s}\|_{0,\Omega} \,+\, \|\mathbf{u} - \mathbf{v}\|_{0,6;\Omega}\big)\, \|\vec{\mathbf{z}}\|_{\mathbf{X}}\,, \end{aligned} \tag{3.17}$$

where $\mathrm{q} \in [4/3, 3/2]$ and $1/\mathrm{p}+1/\mathrm{q} = 1$. In turn, using [4, Lemma 2.1, eq.(2.1a)] to bound the first term on the right hand side of (3.17), and the embedding (2.14) of $\mathbf{L}^6(\Omega)$ into $\mathbf{L}^t(\Omega)$, with $t = \mathrm{p} \in [3, 4]$, we deduce that there exists $c_\mathrm{p} > 0$, depending only on $|\Omega|$ and p, such that

$$
\begin{aligned}
\| |\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - |\mathbf{v}|^{\mathrm{p}-2}\mathbf{v}\|_{0,\mathrm{q};\Omega} \, \|\mathbf{z}\|_{0,\mathrm{p};\Omega} \, &\le \, c_\mathrm{p} \left(\|\mathbf{u}\|_{0,\mathrm{p};\Omega} + \|\mathbf{v}\|_{0,\mathrm{p};\Omega} \right)^{\mathrm{p}-2} \|\mathbf{u} - \mathbf{v}\|_{0,\mathrm{p};\Omega} \, \|\mathbf{z}\|_{0,\mathrm{p};\Omega} \\
&\le \, c_\mathrm{p} \, |\Omega|^{(6-\mathrm{p})/6} \left(\|\mathbf{u}\|_{0,6;\Omega} + \|\mathbf{v}\|_{0,6;\Omega} \right)^{\mathrm{p}-2} \|\mathbf{u} - \mathbf{v}\|_{0,6;\Omega} \, \|\mathbf{z}\|_{0,6;\Omega} \, .
\end{aligned} \tag{3.18}
$$

Thus, replacing back (3.18) into (3.17), and using the explicit expression of $C_\mathbf{c}$ (cf. (3.9)), we obtain (3.16) with

$$
L_{\mathtt{MH}} := \max \left\{ \left(\mathtt{D}_1 \, + \, \frac{1}{3} \|g_f\|_{0,\infty;\Omega} \right) |\Omega|^{2/3}, \mathtt{F}_1 \, c_\mathrm{p} \, |\Omega|^{(6-\mathrm{p})/6}, \nu, \frac{1}{\varrho \mu^2}, \frac{C_s}{\mu}, \frac{1}{2} \, |\Omega|^{1/6} \right\} ,
$$

which completes the proof. □

At this point we observe that since (3.16) holds for all $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in $\mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{H}(\mathrm{curl}; \Omega)$, it is clear that it also holds for all $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in $\mathbf{X}$ (cf. (2.25)). We write (3.16) in the current general form since it will be used later on to derive the *a priori* error analysis (cf. Lemma 5.1).

Now, let us look at the kernel of the operator $\mathbf{B}$ (cf. (2.32)), that is

$$
\mathbf{V} := \left\{ \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}, \mathbf{d}) \in \mathbf{X} : \quad [\mathbf{B}(\vec{\mathbf{v}}), \boldsymbol{\tau}] = 0 \quad \forall \, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5}; \Omega) \right\}
$$

which, proceeding similarly to [15, eq. (3.34)] reduce to

$$
\mathbf{V} := \mathbf{K} \times \mathbf{C} \, , \quad \text{where} \quad \mathbf{K} = \left\{ (\mathbf{v}, \mathbf{s}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \nabla \mathbf{v} = \mathbf{s} \quad \text{and} \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega) \right\} . \tag{3.19}
$$

In addition, we recall from [33, Corollary 3.51] that

$$
\|\mathrm{curl}(\mathbf{d})\|^2_{0,\Omega} \, \ge \, \alpha_m \, \|\mathbf{d}\|^2_{\mathrm{curl};\Omega} \quad \forall \, \mathbf{d} \in \mathbf{C} \, . \tag{3.20}
$$

Thus, the following lemma shows that the operator $\mathbf{A}(\mathbf{w}, \widehat{\mathbf{b}})$ satisfies hypothesis (ii) of [1, Theorem 3.1] (cf. (3.14)) with $p_1 = \mathrm{p} \in [3, 4]$ and $p_2 = p_3 = 2$.

Lemma 3.2 *Given $(\mathbf{w}, \widehat{\mathbf{b}}) \in \mathbf{L}^6(\Omega) \times \mathbf{C}$, and assume that the datum $g_f \in \mathrm{L}^\infty(\Omega)$ satisfy*

$$
\|g_f\|_{0,\infty;\Omega} \, \le \, \frac{3}{2} \, \mathtt{D}_0 \, , \tag{3.21}
$$

with $\mathtt{D}_0$ satisfying (2.2). Then, the family of operators $\left\{ \mathbf{A}(\mathbf{w}, \widehat{\mathbf{b}})(\cdot + \vec{\mathbf{z}}) : \mathbf{V} \to \mathbf{V}' : \vec{\mathbf{z}} \in \mathbf{X} \right\}$ is uniformly strongly monotone, that is, there exists $\alpha_{\mathtt{MH}} > 0$, depending on ν, $\mathtt{D}_0$, ϱ, μ, α_m, and $\|\boldsymbol{i}_6\|$, such that

$$
[\mathbf{A}(\mathbf{w}, \widehat{\mathbf{b}})(\vec{\mathbf{u}} + \vec{\mathbf{z}}) - \mathbf{A}(\mathbf{w}, \widehat{\mathbf{b}})(\vec{\mathbf{v}} + \vec{\mathbf{z}}), \vec{\mathbf{u}} - \vec{\mathbf{v}}] \, \ge \, \alpha_{\mathtt{MH}} \, \|\vec{\mathbf{u}} - \vec{\mathbf{v}}\|^2_{\mathbf{X}} \tag{3.22}
$$

for all $\vec{\mathbf{z}} = (\mathbf{z}, \mathbf{r}, \mathbf{e}) \in \mathbf{X}$, and for all $\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t}, \mathbf{b})$, $\vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}, \mathbf{d}) \in \mathbf{V}$.

Proof. Let $\vec{\mathbf{z}} = (\mathbf{z}, \mathbf{r}, \mathbf{e}) \in \mathbf{X}$ and $\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t}, \mathbf{b}), \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}, \mathbf{d}) \in \mathbf{V}$. Bearing in mind the definition of $\mathbf{A}(\mathbf{w}, \widehat{\mathbf{b}})$, $\mathbf{a}$, and $\mathbf{c}(\mathbf{w}, \widehat{\mathbf{b}})$ (cf. (2.28), (2.29), (2.30)), using (2.2) and the skew-symmetric property (3.10), we get

$$
\begin{aligned}
& [\mathbf{A}(\mathbf{w}, \widehat{\mathbf{b}})(\vec{\mathbf{u}} + \vec{\mathbf{z}}) - \mathbf{A}(\mathbf{w}, \widehat{\mathbf{b}})(\vec{\mathbf{v}} + \vec{\mathbf{z}}), \vec{\mathbf{u}} - \vec{\mathbf{v}}] \\
& \quad = \, [\mathbf{a}(\vec{\mathbf{u}} + \vec{\mathbf{z}}) - \mathbf{a}(\vec{\mathbf{v}} + \vec{\mathbf{z}}), \vec{\mathbf{u}} - \vec{\mathbf{v}}] + [\mathbf{c}(\mathbf{w}, \widehat{\mathbf{b}})(\vec{\mathbf{u}} - \vec{\mathbf{v}}), \vec{\mathbf{u}} - \vec{\mathbf{v}}] \\
& \quad \ge \, \left(\mathtt{D}_0 - \frac{1}{3} \, \|g_f\|_{0,\infty;\Omega} \right) \|\mathbf{u} - \mathbf{v}\|^2_{0,\Omega} \, + \, \nu \, \|\mathbf{t} - \mathbf{s}\|^2_{0,\Omega} + \frac{1}{\varrho \, \mu^2} \, \|\mathrm{curl}(\mathbf{b} - \mathbf{d})\|^2_{0,\Omega} \\
& \qquad + \, \int_\Omega \mathtt{F} \left(|\mathbf{u} + \mathbf{z}|^{\mathrm{p}-2}(\mathbf{u} + \mathbf{z}) - |\mathbf{v} \, + \, \mathbf{z}|^{\mathrm{p}-2}(\mathbf{v} + \mathbf{z}) \right) \cdot (\mathbf{u} - \mathbf{v}) \, .
\end{aligned} \tag{3.23}
$$

In turn, using [4, Lemma 2.1, eq.(2.1b)], there exists $C_{\mathrm{p}} > 0$ depending only on $|\Omega|$ and p, such that

$$\big(|\mathbf{u}+\mathbf{z}|^{\mathrm{p}-2}(\mathbf{u}+\mathbf{z}) - |\mathbf{v}+\mathbf{z}|^{\mathrm{p}-2}(\mathbf{v}+\mathbf{z})\big)\cdot(\mathbf{u}-\mathbf{v}) \,\geq\, C_{\mathrm{p}}\,|\mathbf{u}-\mathbf{v}|^{\mathrm{p}}\,,$$

which, together with the lower bound of $\mathtt{F}(\mathbf{x})$ (cf. (2.2)), yields

$$\int_\Omega \mathtt{F}\,\big(|\mathbf{u}+\mathbf{z}|^{\mathrm{p}-2}(\mathbf{u}+\mathbf{z}) - |\mathbf{v}+\mathbf{z}|^{\mathrm{p}-2}(\mathbf{v}+\mathbf{z})\big)\cdot(\mathbf{u}-\mathbf{v}) \,\geq\, \mathtt{F}_0\,C_{\mathrm{p}}\,\|\mathbf{u}-\mathbf{v}\|^{\mathrm{p}}_{0,\mathrm{p};\Omega} \,\geq\, 0\,,$$

and combining the latter with (3.20), (3.21) and (3.23), we find that

$$[\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}+\vec{\mathbf{z}}) - \mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{v}}+\vec{\mathbf{z}}),\vec{\mathbf{u}}-\vec{\mathbf{v}}] \,\geq\, \frac{\mathtt{D}_0}{2}\,\|\mathbf{u}-\mathbf{v}\|^2_{0,\Omega} + \nu\,\|\mathbf{t}-\mathbf{s}\|^2_{0,\Omega} + \frac{\alpha_m}{\varrho\,\mu^2}\,\|\mathbf{b}-\mathbf{d}\|^2_{\mathrm{curl};\Omega}\,. \quad (3.24)$$

Next, employing the fact that $\mathbf{t}-\mathbf{s} = \nabla(\mathbf{u}-\mathbf{v}) \in \Omega$ and $\mathbf{u}-\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ (cf. (3.19)), and using the continuous injection $\mathbf{i}_6$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^6(\Omega)$ (cf. (1.3)), we deduce that

$$\begin{aligned}&[\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}+\vec{\mathbf{z}}) - \mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{v}}+\vec{\mathbf{z}}),\vec{\mathbf{u}}-\vec{\mathbf{v}}]\\ &\qquad \geq\, \frac{1}{2}\min\{\mathtt{D}_0,\nu\}\,\|\mathbf{i}_6\|^{-2}\,\|\mathbf{u}-\mathbf{v}\|^2_{0,6;\Omega} \,+ \frac{\nu}{2}\,\|\mathbf{t}-\mathbf{s}\|^2_{0,\Omega} + \frac{\alpha_m}{\varrho\,\mu^2}\,\|\mathbf{b}-\mathbf{d}\|^2_{\mathrm{curl};\Omega}\,,\end{aligned}$$

which yields (3.22) with

$$\alpha_{\mathtt{MH}} \,:=\, \frac{1}{2}\,\min\left\{\nu, \|\mathbf{i}_6\|^{-2}\min\{\mathtt{D}_0,\nu\}\,,\frac{2\,\alpha_m}{\varrho\,\mu^2}\right\}\,. \quad (3.25)$$

□

As a corollary of Lemma 3.2, replacing $\vec{\mathbf{u}}$, $\vec{\mathbf{v}} \in \mathbf{V}$ and $\vec{\mathbf{z}} \in \mathbf{X}$ in (3.22) by $\vec{\mathbf{u}}-\vec{\mathbf{v}}$, $\mathbf{0} \in \mathbf{V}$ and $\vec{\mathbf{v}} \in \mathbf{X}$, respectively, we arrive at

$$[\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}) - \mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{v}}),\vec{\mathbf{u}}-\vec{\mathbf{v}}] \,\geq\, \alpha_{\mathtt{MH}}\,\|\vec{\mathbf{u}}-\vec{\mathbf{v}}\|^2_{\mathbf{X}}\,, \quad (3.26)$$

for all $\vec{\mathbf{u}},\vec{\mathbf{v}} \in \mathbf{X}$ such that $\vec{\mathbf{u}}-\vec{\mathbf{v}} \in \mathbf{V}$.

Remark 3.1 *We emphasize that the analysis developed in this paper can be extended to accommodate g_f belonging to $\mathrm{L}^2(\Omega)$ instead of $\mathrm{L}^\infty(\Omega)$. However, in such cases, the data assumption on g_f (cf.* (3.21)*), which is necessary for the strong monotonicity of $\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})$ (cf.* (3.22)*), must be replaced by*

$$\|g_f\|_{0,\Omega} \,\leq\, \frac{3}{|\Omega|^{1/12}}\,\min\left\{\mathtt{D}_0,\frac{\nu}{2}\right\}\,\|\mathbf{i}_6\|^{-2}\,,$$

which involves dependence on both physical parameters $\mathtt{D}_0$ and ν. According to this, and for the sake of simplicity, we keep g_f in $\mathrm{L}^\infty(\Omega)$ throughout the document.

We end the verification of the hypotheses of [1, Theorem 3.1] (cf. (3.15)), with the corresponding inf-sup condition for the operator $\mathbf{B}$ (cf. (2.32), (2.19)).

Lemma 3.3 *There exists a constant $\beta_{\mathtt{MH}} \geq 0$, such that*

$$\sup_{\substack{\vec{\mathbf{v}}\in\mathbf{X}\\ \vec{\mathbf{v}}\neq\mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}]}{\|\vec{\mathbf{v}}\|_{\mathbf{X}}} \,\geq\, \beta_{\mathtt{MH}}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5};\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,. \quad (3.27)$$

Proof. We proceed as in [1, Lemma 3.4]. Indeed, we note that from a slight adaptation of [15, Lemma 3.3] the following inf-sup condition for b_f holds

$$\sup_{\substack{(\mathbf{v},\mathbf{s})\in\mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\\(\mathbf{v},\mathbf{s})\neq\mathbf{0}}} \frac{[b_f(\mathbf{v},\mathbf{s}),\boldsymbol{\tau}]}{\|(\mathbf{v},\mathbf{s})\|_{\mathtt{CBF}}} \,\geq\, \beta_{\mathtt{MH}}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5};\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\,, \tag{3.28}$$

where

$$\|(\mathbf{v},\mathbf{s})\|^2_{\mathtt{CBF}} := \|\mathbf{v}\|^2_{0,6;\Omega} + \|\mathbf{s}\|^2_{0,\Omega}\,. \tag{3.29}$$

Thus, (3.27) follows straightforwardly from (3.28) and the definition of the operator $\mathbf{B}$ (cf. (2.32)). □

Now, we are in a position of establishing the solvability of the nonlinear problem (3.12).

Lemma 3.4 *Assume that the datum $g_f\in\mathrm{L}^\infty(\Omega)$ satisfy* (3.21). *Then for each $(\mathbf{w},\widehat{\mathbf{b}})\in\mathbf{L}^6(\Omega)\times\mathbf{C}$, the problem* (3.12) *has a unique solution $(\vec{\mathbf{u}},\boldsymbol{\sigma}) = ((\mathbf{u},\mathbf{t},\mathbf{b}),\boldsymbol{\sigma})\in\mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$, and hence $\mathbf{T}(\mathbf{w},\widehat{\mathbf{b}}) := (\mathbf{u},\mathbf{b})\in\mathbf{L}^6(\Omega)\times\mathbf{C}$ is well-defined. Moreover, there exists a constant $C_{\mathbf{T}}\geq 0$, independent of $\mathbf{w}$ and $\widehat{\mathbf{b}}$, but depending on C_s, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, α_m, ϱ, μ, $|\Omega|$, and $\beta_{\mathtt{MH}}$, such that*

$$\|\mathbf{T}(\mathbf{w},\widehat{\mathbf{b}})\|_{\mathtt{MH}} \,\leq\, \|\vec{\mathbf{u}}\|_{\mathbf{X}} \,\leq\, C_{\mathbf{T}}\left\{\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right\}. \tag{3.30}$$

Proof. Given $(\mathbf{w},\widehat{\mathbf{b}})\in\mathbf{L}^6(\Omega)\times\mathbf{C}$, we first recall from (3.2), (3.4) and (3.5) that $\mathbf{B}$, $\mathbf{F}$ and $\mathbf{G}$ are all linear and bounded. Thus, thanks to Lemmas 3.1, 3.2, and 3.3, a straightforward application of [1, Theorem 3.1], with $p_1 = \mathrm{p}\in[3,4]$ and $p_2 = p_3 = 2$ to problem (3.12) completes the proof. In particular, given $(\mathbf{w},\widehat{\mathbf{b}})\in\mathbf{L}^6(\Omega)\times\mathbf{C}$, noting from (2.28) that $\mathcal{A}(\mathbf{0}) := \mathbf{A}(\mathbf{w},\widehat{\mathbf{b}})(\mathbf{0})$ is the null functional, and setting $\mathcal{F} := \mathbf{F}$ and $\mathcal{G} := \mathbf{G}$, we obtain from [1, eq. (3.4) in Theorem 3.1] that

$$\mathcal{M}(\mathbf{F},\mathbf{G}) = \|\mathbf{F}\|_{\mathbf{X}'} + 3\,\|\mathbf{G}\|_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'} + \|\mathbf{G}\|^{\mathrm{p}-1}_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'}\,,$$

and hence the *a priori* estimate [1, eq. (3.2) in Theorem 3.1] yields

$$\|\vec{\mathbf{u}}\|_{\mathbf{X}} \,\leq\, C_1\left\{\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{G}\|_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'} + \|\mathbf{G}\|^{\mathrm{p}-1}_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'}\right\},$$

with a positive constant C_1 depending only on $L_{\mathtt{MH}}$, $\alpha_{\mathtt{MH}}$ and $\beta_{\mathtt{MH}}$. The foregoing inequality together with the bounds of $\|\mathbf{F}\|_{\mathbf{X}'}$ and $\|\mathbf{G}\|_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'}$ (cf. (3.4), (3.5)) imply (3.30) with $C_{\mathbf{T}}$ depending on $L_{\mathtt{MH}}, \alpha_{\mathtt{MH}}, \mu$ and $\beta_{\mathtt{MH}}$, which in turn depend on the parameters C_s, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, α_m, $|\Omega|$, and ϱ, thus completing the proof. □

For later use in the paper we note here that, applying [1, eq. (3.3) in Theorem 3.1], and using again the bounds (3.4) and (3.5) for $\|\mathbf{F}\|_{\mathbf{X}'}$ and $\|\mathbf{G}\|_{\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)'}$, respectively, the *a priori* estimate for the second component of the solution to the problem defining $\mathbf{T}$ (*cf.* (3.12)) reduces to

$$\|\boldsymbol{\sigma}\|_{\mathbf{div}_{6/5};\Omega} \,\leq\, C_{\boldsymbol{\sigma}}\sum_{i\in\{\mathrm{p},2\}}\left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right)^{i-1}, \tag{3.31}$$

with $C_{\boldsymbol{\sigma}}$ depending on C_s, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, α_m, $|\Omega|$, ϱ, μ, and $\beta_{\mathtt{MH}}$.

3.3 Well-posedness of the continuous formulation

Having proved the well-posedness of the coupled problem (3.12) which ensures that the operator $\mathbf{T}$ is well defined, we now aim to establish the existence of a unique fixed-point of the operator $\mathbf{T}$. For this purpose, in what follows we will verify the hypothesis of the Banach fixed-point theorem. We begin by providing suitable conditions under which $\mathbf{T}$ maps a ball into itself.

Lemma 3.5 *Given $r > 0$, we let $\mathbf{W}_r$ be the closed ball in $\mathbf{L}^6(\Omega) \times \mathbf{C}$ defined by*

$$\mathbf{W}_r := \Big\{ (\mathbf{w}, \widehat{\mathbf{b}}) \in \mathbf{L}^6(\Omega) \times \mathbf{C} : \quad \|(\mathbf{w}, \widehat{\mathbf{b}})\|_{\mathtt{MH}} \le r \Big\} . \tag{3.32}$$

Assume that the data satisfy (3.21), *and*

$$C_{\mathbf{T}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j \in \{\mathrm{p},2\}} \big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \big)^{j-1} \right\} \le r \, . \tag{3.33}$$

Then, there holds $\mathbf{T}(\mathbf{W}_r) \subseteq \mathbf{W}_r$.

Proof. It is a direct consequence of the *a priori* estimate (3.30) and the assumption (3.33). □

We now aim to prove that the operator $\mathbf{T}$ is Lipschitz continuous.

Lemma 3.6 *Let $C_{\mathbf{c}}$, $\alpha_{\mathtt{MH}}$, and $C_{\mathbf{T}}$ be positive constants satisfying* (3.9), (3.25), *and* (3.30), *respectively, and assume that the datum $g_f \in \mathrm{L}^\infty(\Omega)$ satisfy* (3.21). *Then, there holds*

$$\begin{aligned} & \|\mathbf{T}(\mathbf{w}, \widehat{\mathbf{b}}) - \mathbf{T}(\mathbf{w}_0, \widehat{\mathbf{b}}_0)\|_{\mathtt{MH}} \\ & \le \frac{C_{\mathbf{c}}\, C_{\mathbf{T}}}{\alpha_{\mathtt{MH}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j \in \{\mathrm{p},2\}} \big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \big)^{j-1} \right\} \|(\mathbf{w}, \widehat{\mathbf{b}}) - (\mathbf{w}_0, \widehat{\mathbf{b}}_0)\|_{\mathtt{MH}} \, , \end{aligned} \tag{3.34}$$

for all $(\mathbf{w}, \widehat{\mathbf{b}})$, $(\mathbf{w}_0, \widehat{\mathbf{b}}_0) \in \mathbf{L}^6(\Omega) \times \mathbf{C}$.

Proof. Given $(\mathbf{w}, \widehat{\mathbf{b}})$, $(\mathbf{w}_0, \widehat{\mathbf{b}}_0) \in \mathbf{L}^6(\Omega) \times \mathbf{C}$, we let $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) := ((\mathbf{u}, \mathbf{t}, \mathbf{b}), \boldsymbol{\sigma})$ and $(\vec{\mathbf{u}}_0, \boldsymbol{\sigma}_0) := ((\mathbf{u}_0, \mathbf{t}_0, \mathbf{b}_0), \boldsymbol{\sigma}_0) \in \mathbf{X} \times \mathbb{H}_0(\mathbf{div}_{6/5}; \Omega)$ be the corresponding solutions of (3.12) so that $(\mathbf{u}, \mathbf{b}) := \mathbf{T}(\mathbf{w}, \widehat{\mathbf{b}})$ and $(\mathbf{u}_0, \mathbf{b}_0) := \mathbf{T}(\mathbf{w}_0, \widehat{\mathbf{b}}_0)$. Then, subtracting the corresponding problems from (3.12), and using the definition of the operator $\mathbf{A}(\mathbf{w}, \widehat{\mathbf{b}})$ (cf. (2.28)), we obtain

$$\begin{aligned} [\mathbf{A}(\mathbf{w}_0, \widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}) - \mathbf{A}(\mathbf{w}_0, \widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}_0), \vec{\mathbf{v}}] + [\mathbf{B}(\vec{\mathbf{v}}), \boldsymbol{\sigma} - \boldsymbol{\sigma}_0] &= -[\mathbf{c}(\mathbf{w} - \mathbf{w}_0, \widehat{\mathbf{b}} - \widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}), \vec{\mathbf{v}}] \, , \\ [\mathbf{B}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0), \boldsymbol{\tau}] &= 0 \, , \end{aligned} \tag{3.35}$$

for all $\vec{\mathbf{v}} \in \mathbf{X}$ and $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5}; \Omega)$. We note from the second equation of (3.35) that $\vec{\mathbf{u}} - \vec{\mathbf{u}}_0 \in \mathbf{V}$ (cf. (3.19)). Hence, taking $\vec{\mathbf{v}} := \vec{\mathbf{u}} - \vec{\mathbf{u}}_0 \in \mathbf{V}$ in the first equation of (3.35), considering (3.21), applying (3.26) with $\vec{\mathbf{u}}$, $\vec{\mathbf{u}}_0 \in \mathbf{X}$, and using the continuity bound of $\mathbf{c}(\mathbf{w}, \widehat{\mathbf{b}})$ (cf. (3.9)), we obtain

$$\begin{aligned} \alpha_{\mathtt{MH}} \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{X}}^2 &\le [\mathbf{A}(\mathbf{w}_0, \widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}) - \mathbf{A}(\mathbf{w}_0, \widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}_0), \vec{\mathbf{u}} - \vec{\mathbf{u}}_0] = -[\mathbf{c}(\mathbf{w} - \mathbf{w}_0, \widehat{\mathbf{b}} - \widehat{\mathbf{b}}_0)(\vec{\mathbf{u}}), \vec{\mathbf{v}}] \\ &\le C_{\mathbf{c}} \|\vec{\mathbf{u}}\|_{\mathbf{X}} \|(\mathbf{w}, \widehat{\mathbf{b}}) - (\mathbf{w}_0, \widehat{\mathbf{b}}_0)\|_{\mathtt{MH}} \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{X}} \, , \end{aligned}$$

which, together with (3.30) to bound $\|\vec{\mathbf{u}}\|_{\mathbf{X}}$, implies (3.34), completing the proof. □

We are now in position to establish the main result concerning the solvability of (2.27)

Theorem 3.7 *Given $r > 0$, let $\mathbf{W}_r$ as in* (3.32)*, and assume that the data satisfy* (3.21)*,* (3.33)*, and*

$$\frac{C_{\mathbf{c}}\, C_{\mathbf{T}}}{\alpha_{\mathtt{MH}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right\} < 1\,. \tag{3.36}$$

Then the operator $\mathbf{T}$ has a unique fixed point $(\mathbf{u},\mathbf{b}) \in \mathbf{W}_r$. Equivalently, the coupled problem (2.27) *has a unique solution $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{X} \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$, with $(\mathbf{u},\mathbf{b}) \in \mathbf{W}_r$. Moreover, there exist positive constants $C_{\mathbf{T}}, C_{\boldsymbol{\sigma}}$, depending on C_s, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, α_m, $|\Omega|$, ϱ, μ, and $\beta_{\mathtt{MH}}$, such that the following a priori estimates hold*

$$\|\vec{\mathbf{u}}\|_{\mathbf{X}} \le C_{\mathbf{T}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right\}, \tag{3.37}$$

$$\|\boldsymbol{\sigma}\|_{\mathbf{div}_{6/5};\Omega} \le C_{\boldsymbol{\sigma}} \sum_{i\in\{\mathrm{p},2\}} \left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right)^{i-1}. \tag{3.38}$$

Proof. We begin by recalling from Lemma 3.5 that, under the assumption (3.33), $\mathbf{T}$ maps the ball $\mathbf{W}_r$ into itself, and hence, for each $(\mathbf{u},\mathbf{b}) \in \mathbf{W}_r$ we have that both $\|(\mathbf{u},\mathbf{b})\|_{\mathtt{MH}}$ and $\|\mathbf{T}(\mathbf{u},\mathbf{b})\|_{\mathtt{MH}}$ are bounded by r. In turn, it is clear from (3.34) in Lemma 3.6 and Hypotheses (3.36) that $\mathbf{T}$ is a contraction. Therefore, the Banach fixed-point theorem provides the existence of a unique fixed point $(\mathbf{u},\mathbf{b}) \in \mathbf{W}_r$ of $\mathbf{T}$, equivalently, the existence of a unique solution $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{X} \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$, of the coupled problem (2.27), with $(\mathbf{u},\mathbf{b}) \in \mathbf{W}_r$. In addition, it is clear that the estimates (3.37) and (3.38) follow straightforwardly from (3.30) and (3.31), respectively, which finishes the proof. $\square$

We end this section by establishing the well-posedness of (2.17). We observe that employing Lemma 2.1, the inf-sup condition of the operator b_m (cf. (2.34)), the continuity of a_m, F_3 (cf. (3.1), (3.3)) and similar arguments to the ones developed in [1, Corollary 3.9], its proof can be derived. The result is stated next.

Corollary 3.8 *Let $\mathbf{f}_f \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{f}_m \in \mathbf{L}^2(\Omega)$, $g_f \in \mathrm{L}^\infty(\Omega)$, and $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$, such that* (3.21)*,* (3.33) *and* (3.36) *hold. Then, there exists a unique $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma}) \in \mathbf{L}^6(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{b},\lambda) \in \mathbf{H}_0(\mathrm{curl};\Omega) \times \mathrm{H}^1_0(\Omega)$ solution to* (2.17)*. In addition, $(\mathbf{u},\mathbf{t},\mathbf{b})$ and $\boldsymbol{\sigma}$ satisfy* (3.37) *and* (3.38)*, respectively, and for λ, there exists a positive constant C_λ depending on C_s, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, α_m, $|\Omega|$, ϱ, μ, $\beta_{\mathtt{MH}}$, and β_m, such that*

$$\|\lambda\|_{1,\Omega} \le C_\lambda \sum_{i=1}^{2} \left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1} \right)^{i}.$$

4 The Galerkin scheme

In this section we introduce and analyze the corresponding Galerkin scheme for the mixed formulation (2.17) (and also for (2.27)). We mention in advance that, as we will see in the forthcoming subsections, the analysis of well-posedness straightforwardly follows by adapting the results derived for the continuous problem to the discrete case, hence most of the details will be omitted.

4.1 Discrete setting

We first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polyhedral region $\overline{\Omega}$ made up of tetrahedra T in R^3 of diameter h_T such that $\overline{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$ and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. Given an integer $l \geq 0$ and a subset S of R^3, we denote by $P_l(S)$ the space of polynomials of total degree at most l defined on S, $\widetilde{P}_l(S)$ the space of homogeneous polynomials of degree exactly l on S and $\mathbf{M}_l(S)$ the space of polynomials $\mathbf{p}$ in $\widetilde{\mathbf{P}}_l(S)$ satisfying $\mathbf{p}(\mathbf{x}) \cdot \mathbf{x} = 0$ on S, where $\mathbf{x} := (x_1, x_2, x_3)^t$ is a generic vector of R^3. Hence, for each integer $k \geq 0$ and for each $T \in \mathcal{T}_h$, we define the local Raviart–Thomas and Nédélec elements of order k (see for instance [5] and [33]), respectively, by

$$\mathbf{RT}_k(T) := \mathbf{P}_k(T) \oplus \widetilde{P}_k(T)\mathbf{x} \quad \text{and} \quad \mathbf{ND}_k(T) := \mathbf{P}_k(T) \oplus \mathbf{M}_{k+1}(T)\,.$$

Then, denoting by $\boldsymbol{\tau}_{h,i}$ the i-th row of a tensor $\boldsymbol{\tau}_h$, the finite element subspaces on Ω are defined as

$$\begin{aligned}
\mathbf{H}_h^{\mathbf{u}} &:= \Big\{\mathbf{v}_h \in \mathbf{L}^6(\Omega) : \quad \mathbf{v}_h|_T \in \mathbf{P}_k(T) \quad \forall\, T \in \mathcal{T}_h\Big\}\,, \\
\mathbb{H}_h^{\mathbf{t}} &:= \Big\{\mathbf{r}_h \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \mathbf{r}_h|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}_h\Big\}\,, \\
\mathbb{H}_h^{\boldsymbol{\sigma}} &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega) : \quad \boldsymbol{\tau}_{h,i}|_T \in \mathbf{RT}_k(T) \quad \forall\, i \in \{1,\dots,n\}\,, \quad \forall\, T \in \mathcal{T}_h\Big\}\,, \\
\mathbf{H}_h^{\mathbf{b}} &:= \Big\{\mathbf{d}_h \in \mathbf{H}_0(\mathrm{curl};\Omega) : \quad \mathbf{d}_h|_T \in \mathbf{ND}_k(T) \quad \forall\, T \in \mathcal{T}_h\Big\}\,, \\
\mathrm{H}_h^{\lambda} &:= \Big\{\xi_h \in \mathrm{H}_0^1(\Omega) : \quad \xi_h|_T \in P_{k+1}(T) \quad \forall\, T \in \mathcal{T}_h\Big\}\,,
\end{aligned} \tag{4.1}$$

the Galerkin scheme for (2.17) reads: Find $(\mathbf{u}_h, \mathbf{t}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\mathbf{b}_h, \lambda_h) \in \mathbf{H}_h^{\mathbf{b}} \times \mathrm{H}_h^{\lambda}$, such that

$$\begin{aligned}
&[a_f(\mathbf{u}_h, \mathbf{t}_h), (\mathbf{v}_h, \mathbf{s}_h)] + [d_f(\mathbf{u}_h)(\mathbf{u}_h, \mathbf{t}_h), (\mathbf{v}_h, \mathbf{s}_h)] \\
&\qquad + [c_f(\mathbf{b}_h)(\mathbf{b}_h), \mathbf{v}_h] + [b_f(\mathbf{v}_h, \mathbf{s}_h), \boldsymbol{\sigma}_h] = [F_1, (\mathbf{v}_h, \mathbf{s}_h)]\,,
\end{aligned} \tag{4.2a}$$

$$[b_f(\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\tau}_h] = [F_2, \boldsymbol{\tau}_h]\,, \tag{4.2b}$$

$$[a_m(\mathbf{b}_h), \mathbf{d}_h] + [c_m(\mathbf{b}_h)(\mathbf{u}_h), \mathbf{d}_h] + [b_m(\mathbf{d}_h), \lambda_h] = [F_3, \mathbf{d}_h]\,, \tag{4.2c}$$

$$[b_m(\mathbf{b}_h), \xi_h] = 0\,, \tag{4.2d}$$

for all $(\mathbf{v}_h, \mathbf{s}_h, \boldsymbol{\tau}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and for all $(\mathbf{d}_h, \xi_h) \in \mathbf{H}_h^{\mathbf{b}} \times \mathrm{H}_h^{\lambda}$.

Now, analogously to the continuous case, defining the discrete version of $\mathbf{C}$ (cf. (2.13)) as

$$\mathbf{C}_h := \left\{\mathbf{d}_h \in \mathbf{H}_h^{\mathbf{b}} : \quad \int_\Omega \mathbf{d}_h \cdot \nabla \xi_h = 0 \quad \forall\, \xi_h \in \mathrm{H}_h^{\lambda}\right\}\,, \tag{4.3}$$

and denoting from now on

$$\vec{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h)\,,\ \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h, \mathbf{d}_h) \in \mathbf{X}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbf{C}_h\,,$$

the discrete version of (2.27) reads: Find $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ such that:

$$\begin{aligned}
[\mathbf{A}(\mathbf{u}_h, \mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h] &= [\mathbf{F}, \vec{\mathbf{v}}_h] \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{X}_h\,, \\
[\mathbf{B}(\vec{\mathbf{u}}_h), \boldsymbol{\tau}_h] &= [\mathbf{G}, \boldsymbol{\tau}_h] \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,,
\end{aligned} \tag{4.4}$$

where, as in the continuous case, given $(\mathbf{w}_h, \widehat{\mathbf{b}}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h$, the operator $\mathbf{A}(\mathbf{w}_h, \widehat{\mathbf{b}}_h) : \mathbf{X}_h \to \mathbf{X}_h'$ is defined by

$$[\mathbf{A}(\mathbf{w}_h, \widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] \,:=\, [\mathbf{a}(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{c}(\mathbf{w}_h, \widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] \tag{4.5}$$

where $\mathbf{X}_h$ is endowed with the norm defined in (2.26).

At this point, we recall from [30, Section 5.4] that the operator b_m (cf. (2.20)) satisfies the discrete inf-sup condition:

$$\sup_{\substack{\mathbf{d}_h \in \mathbf{H}_h^{\mathbf{b}} \\ \mathbf{d}_h \neq \mathbf{0}}} \frac{[b_m(\mathbf{d}_h), \xi_h]}{\|\mathbf{d}_h\|_{\mathrm{curl};\Omega}} \,\geq\, \beta_m \, \|\xi_h\|_{1,\Omega} \quad \forall\, \xi_h \in \mathrm{H}_h^{\lambda}\,, \tag{4.6}$$

with $\beta_m > 0$ being the same constant satisfying (2.34), which certainly is independent of h. Thus, using (4.6) and similar arguments to the ones employed in [1, Lemma 2.1], the discrete problems (4.2) and (4.4) are equivalent. According to this, in what follows we focus on analyzing (4.4).

We now develop the discrete analogue of the fixed-point approach utilized in Section 3.2. To this end, we introduce the operator $\mathbf{T}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h \to \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h$ defined by

$$\mathbf{T}_{\mathtt{d}}(\mathbf{w}_h, \widehat{\mathbf{b}}_h) \,:=\, (\mathbf{u}_h, \mathbf{b}_h) \quad \forall\, (\mathbf{w}_h, \widehat{\mathbf{b}}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h\,, \tag{4.7}$$

where $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) = ((\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h), \boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ is the unique solution (to be confirmed below) of the problem

$$\begin{aligned} [\mathbf{A}(\mathbf{w}_h, \widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h] \;&=\; [\mathbf{F}, \vec{\mathbf{v}}_h] \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{X}_h\,, \\ [\mathbf{B}(\vec{\mathbf{u}}_h), \boldsymbol{\tau}_h] \;&=\; [\mathbf{G}, \boldsymbol{\tau}_h] \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,. \end{aligned} \tag{4.8}$$

Therefore solving (4.4) is equivalent to seeking a fixed point of the operator $\mathbf{T}_{\mathtt{d}}$, that is: Find $(\mathbf{u}_h, \mathbf{b}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h$ such that

$$\mathbf{T}_{\mathtt{d}}(\mathbf{u}_h, \mathbf{b}_h) \,=\, (\mathbf{u}_h, \mathbf{b}_h)\,. \tag{4.9}$$

4.2 Solvability analysis

We begin by proving that (4.8) is well-posed, or equivalently that $\mathbf{T}_{\mathtt{d}}$ (cf. (4.7)) is well defined. We remark in advance that the respective proof, being the discrete analogue of the one of Lemma 3.4, makes use again of the abstract result given by [1, Theorem 3.1]. We note also that the discrete kernel of b_m, namely $\mathbf{C}_h$ (cf. (4.3)), is not included in its continuous counterpart $\mathbf{C}$ (cf. (2.13)), and consequently, we can not employ the embedding $\mathbf{C} \subseteq \mathbf{H}^s(\Omega)$ for some $s > 1/2$. In order to overcome this drawback, and similarly as in [1, Section 4.2], from now on we need to assume that the mesh is quasi-uniform. Then, recalling the inverse inequality (see [14, Theorem 3.2.6]):

$$\|\xi\|_{0,t;\Omega} \,\leq\, C_I\, h^{3\,(1/t - 1/t^*)} \|\xi\|_{0,t^*;\Omega}, \quad 1 \,\leq\, t^* \,\leq\, t \,\leq\, \infty\,, \tag{4.10}$$

for all piecewise polynomial functions ξ and $C_I > 0$ independent of h, we are able to establish general versions of (3.7), (3.8), and (3.9).

Lemma 4.1 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Given $(\mathbf{w}, \widehat{\mathbf{b}}) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C} + \mathbf{C}_h$, there exist constants $C_{s,\mathtt{d}}, C_{\mathbf{c},\mathtt{d}} > 0$, independent of h and the physical parameters, such that*

$$\big|[c_f(\widehat{\mathbf{b}})(\mathbf{b}), \mathbf{v}]\big| \,\leq\, \frac{C_{s,\mathtt{d}}}{\mu} \, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega} \|\mathbf{b}\|_{\mathrm{curl};\Omega} \, \|\mathbf{v}\|_{0,6;\Omega} \quad \forall\, (\mathbf{b}, \mathbf{v}) \in \mathbf{H}_h^{\mathbf{b}} \times \mathbf{L}^6(\Omega)\,, \tag{4.11}$$

$$\big|[c_m(\widehat{\mathbf{b}})(\mathbf{u}), \mathbf{d}]\big| \,\leq\, \frac{C_{s,\mathtt{d}}}{\mu} \, \|\widehat{\mathbf{b}}\|_{\mathrm{curl};\Omega} \, \|\mathbf{u}\|_{0,6;\Omega} \, \|\mathbf{d}\|_{\mathrm{curl};\Omega} \quad \forall\, (\mathbf{u}, \mathbf{d}) \in \mathbf{L}^6(\Omega) \times \mathbf{H}_h^{\mathbf{b}}\,, \tag{4.12}$$

and

$$\left|[\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})(\vec{\mathbf{u}}),\vec{\mathbf{v}}]\right| \,\le\, C_{\mathbf{c},\mathtt{d}}\,\|(\mathbf{w},\widehat{\mathbf{b}})\|_{\mathtt{MH}}\,\|\vec{\mathbf{u}}\|_{\mathbf{X}}\,\|\vec{\mathbf{v}}\|_{\mathbf{X}}\,, \tag{4.13}$$

for all $\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t},\mathbf{b}), \vec{\mathbf{v}} = (\mathbf{v},\mathbf{s},\mathbf{d}) \in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{H}^{\mathbf{b}}_h$.

Proof. First, the proof of (4.11)–(4.12) follows from [1, eqs. (4.11)–(4.12) in Lemma 4.1], with positive constant $C_{s,\mathtt{d}} := C_{\mathbf{S}}\,C_I + C_s/\alpha_m^{1/2}$, where $C_{\mathbf{S}}$ and C_I satisfy [1, eq. (4.15)] and (4.10), respectively, all them independent of h and physical parameters. On the other hand, from the definition of the operator $\mathbf{c}(\mathbf{w},\widehat{\mathbf{b}})$ (cf. (2.30)) and using (4.11), (4.12), and (3.6), we are able to obtain (4.13), with

$$C_{\mathbf{c},\mathtt{d}} := \max\left\{\frac{C_{s,\mathtt{d}}}{\mu},\frac{|\Omega|^{1/6}}{2}\right\}, \tag{4.14}$$

completing the proof. □

The following result establishes that the nonlinear operator $\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)$ (cf. (4.5)) satisfies hypothesis (i) of [1, Theorem 3.1] (cf. (3.13)) with $p_1 = \mathrm{p}\in[3,4]$ and $p_2 = p_3 = 2$.

Lemma 4.2 *Assume that* $\{\mathcal{T}_h\}_{h>0}$ *is a family of quasi-uniform triangulations. Let* $\mathrm{p}\in[3,4]$*. Given* $(\mathbf{w}_h,\widehat{\mathbf{b}}_h)\in\mathbf{H}^{\mathbf{u}}_h\times\mathbf{C}_h$*, there exists* $L_{\mathtt{MH,d}}>0$*, depending on* ν, $\mathtt{F}_1$, $\mathtt{D}_1$, $|\Omega|$, C_s, ϱ, $\|g_f\|_{0,\infty;\Omega}$*, and* μ*, such that*

$$\begin{aligned}&\|\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h)-\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h)\|_{\mathbf{X}_h'}\\&\quad\le L_{\mathtt{MH,d}}\left\{\big(1+\|(\mathbf{w}_h,\widehat{\mathbf{b}}_h)\|_{\mathtt{MH}}\big)\,\|\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h\|_{\mathbf{X}_h} \,+\, \big(\|\mathbf{u}_h\|_{0,6;\Omega}+\|\mathbf{v}_h\|_{0,6;\Omega}\big)^{\mathrm{p}-2}\|\mathbf{u}_h-\mathbf{v}_h\|_{0,6;\Omega}\right\},\end{aligned} \tag{4.15}$$

for all $\vec{\mathbf{u}}_h = (\mathbf{u}_h,\mathbf{t}_h,\mathbf{b}_h)$, $\vec{\mathbf{v}}_h = (\mathbf{v}_h,\mathbf{s}_h,\mathbf{d}_h)\in\mathbf{X}_h$.

Proof. First, given $\widehat{\mathbf{b}}_h\in\mathbf{C}_h$, we observe from the definition of the operator $\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)$ (cf. (4.5)) that for $\vec{\mathbf{u}}_h = (\mathbf{u}_h,\mathbf{t}_h,\mathbf{b}_h)$, $\vec{\mathbf{v}}_h = (\mathbf{v}_h,\mathbf{s}_h,\mathbf{d}_h)\in\mathbf{X}_h$ there certainly holds

$$\|\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h)-\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h)\|_{\mathbf{X}_h'} \,\le\, \|\mathbf{a}(\vec{\mathbf{u}}_h)-\mathbf{a}(\vec{\mathbf{v}}_h)\|_{\mathbf{X}_h'} \,+\, \|\mathbf{c}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h)\|_{\mathbf{X}_h'}\,.$$

Then, employing similar arguments to (3.16) and considering (4.13) with the explicit expression of $C_{\mathbf{c},\mathtt{d}}$ defined in (4.14), we obtain (4.15), with

$$L_{\mathtt{MH,d}} := \max\left\{\Big(\mathtt{D}_1 \,+\, \frac{1}{3}\|g_f\|_{0,\infty;\Omega}\Big)\,|\Omega|^{2/3}, \mathtt{F}_1\,c_{\mathrm{p}}\,|\Omega|^{(6-\mathrm{p})/6}, \nu, \frac{1}{\varrho\mu^2}, \frac{C_{s,\mathtt{d}}}{\mu}, \frac{1}{2}\,|\Omega|^{1/6}\right\}.$$

□

Next, in order to prove the hypotheses (ii) and (iii) of [1, Theorem 3.1] (cf. (3.14), (3.15)), we set the discrete kernel of the operator $\mathbf{B}$, which is given by $\mathbf{V}_h := \mathbf{K}_h\times\mathbf{C}_h$, with

$$\mathbf{K}_h \,:=\, \left\{(\mathbf{v}_h,\mathbf{s}_h)\in\mathbf{H}^{\mathbf{u}}_h\times\mathbb{H}^{\mathbf{t}}_h : \quad -\int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_h - \int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\,\boldsymbol{\tau}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_h\right\}. \tag{4.16}$$

Then, from a slight adaptation of [11, Lemma 4.1], which in turn follows by using similar arguments to the ones developed in [15, Section 5], we now provide the discrete inf-sup condition for the operator b_f (cf. (2.19)) and an intermediate result that will be used to show later on the strong monotonicity of $\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)$ on $\mathbf{V}_h$.

Lemma 4.3 *There exist positive constants* $\beta_{\mathtt{MH,d}}$ *and* $C_{\mathtt{d}}$ *such that*

$$\sup_{\substack{(\mathbf{v}_h,\mathbf{s}_h)\in\mathbf{H}_h^{\mathbf{u}}\times\mathbb{H}_h^{\mathbf{t}}\\(\mathbf{v}_h,\mathbf{s}_h)\neq\mathbf{0}}}\frac{[b_f(\mathbf{v}_h,\mathbf{s}_h),\boldsymbol{\tau}_h]}{\|(\mathbf{v}_h,\mathbf{s}_h)\|_{\mathtt{CBF}}}\,\geq\,\beta_{\mathtt{MH,d}}\,\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}\qquad\forall\,\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\,,\tag{4.17}$$

with the norm $\|(\cdot,\cdot)\|_{\mathtt{CBF}}$ *defined in* (3.29)*, and*

$$\|\mathbf{s}_h\|_{0,\Omega}\,\geq\,C_{\mathtt{d}}\|\mathbf{v}_h\|_{0,6;\Omega}\qquad\forall\,(\mathbf{v}_h,\mathbf{s}_h)\in\mathbf{K}_h\,.\tag{4.18}$$

In addition, we recall from [30, Theorem 4.7] that

$$\|\mathrm{curl}(\mathbf{d}_h)\|_{0,\Omega}^2\,\geq\,\alpha_{m,\mathtt{d}}\,\|\mathbf{d}_h\|_{\mathrm{curl};\Omega}^2\quad\forall\,\mathbf{d}_h\in\mathbf{C}_h\,.\tag{4.19}$$

We now establish the discrete strong monotonicity property of $\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)$ (cf. (4.5)).

Lemma 4.4 *Given* $(\mathbf{w}_h,\widehat{\mathbf{b}}_h)\in\mathbf{H}_h^{\mathbf{u}}\times\mathbf{C}_h$*, and assume that the datum* $g_f\in\mathrm{L}^\infty(\Omega)$ *satisfy* (3.21)*. Then, the family of operators* $\big\{\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\,\cdot+\vec{\mathbf{z}})\,:\,\mathbf{V}_h\to\mathbf{V}_h'\,:\,\vec{\mathbf{z}}_h\in\mathbf{X}_h\big\}$ *is uniformly strongly monotone, that is, there exists* $\alpha_{\mathtt{MH,d}}>0$*, depending on* ν*,* $\alpha_{m,\mathtt{d}}$*,* $C_{\mathtt{d}}$*,* ϱ*, and* μ *such that*

$$[\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h+\vec{\mathbf{z}}_h)-\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h+\vec{\mathbf{z}}_h),\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h]\,\geq\,\alpha_{\mathtt{MH.d}}\,\|\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h\|_{\mathbf{X}}^2\tag{4.20}$$

for all $\vec{\mathbf{z}}_h=(\mathbf{z}_h,\mathbf{r}_h,\mathbf{e}_h)\in\mathbf{X}_h$*, and for all* $\vec{\mathbf{u}}_h=(\mathbf{u}_h,\mathbf{t}_h,\mathbf{b}_h)$*,* $\vec{\mathbf{v}}_h=(\mathbf{v}_h,\mathbf{s}_h,\mathbf{d}_h)\in\mathbf{V}_h$*.*

Proof. We follow an analogous reasoning to the proof of Lemma 3.2. In fact, let $\vec{\mathbf{z}}_h=(\mathbf{z}_h,\mathbf{r}_h,\mathbf{e}_h)\in\mathbf{X}_h$ and $\vec{\mathbf{u}}_h=(\mathbf{u}_h,\mathbf{t}_h,\mathbf{b}_h),\vec{\mathbf{v}}_h=(\mathbf{v}_h,\mathbf{s}_h,\mathbf{d}_h)\in\mathbf{V}_h=\mathbf{K}_h\times\mathbf{C}_h$ (cf. (4.16)). Then, according to the definition of $\mathbf{A}(\mathbf{w},\widehat{\mathbf{b}}_h)$ (cf. (4.5)), and using the identity (3.10) (which is also true when $(\mathbf{w},\widehat{\mathbf{b}})\in\mathbf{H}_h^{\mathbf{u}}\times\mathbf{C}_h$ and $\vec{\mathbf{v}}\in\mathbf{X}_h$), [4, Lemma 2.1, eq.(2.1b)], (3.21) and (4.19), we get, similarly to (3.24) that

$$\begin{aligned}&[\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h+\vec{\mathbf{z}}_h)-\mathbf{A}(\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h+\vec{\mathbf{z}}_h),\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h]\\&\qquad\geq\,\frac{\mathtt{D}_0}{2}\,\|\mathbf{u}_h-\mathbf{v}_h\|_{0,\Omega}^2+\nu\,\|\mathbf{t}_h-\mathbf{s}_h\|_{0,\Omega}^2+\frac{\alpha_{m,\mathtt{d}}}{\varrho\,\mu^2}\,\|\mathbf{b}_h-\mathbf{d}_h\|_{\mathrm{curl};\Omega}^2\,.\end{aligned}\tag{4.21}$$

Next, bounding below the first term on the right hand side of (4.21) by 0, and using the fact that $\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h:=((\mathbf{u}_h-\mathbf{v}_h,\mathbf{t}_h-\mathbf{s}_h),\mathbf{b}_h-\mathbf{d}_h)\in\mathbf{K}_h\times\mathbf{C}_h$ in combination with the estimate (4.18), we are able to deduce (4.20) with

$$\alpha_{\mathtt{MH,d}}\,:=\,\frac{1}{2}\min\left\{\nu,\nu\,C_{\mathtt{d}}^2,\frac{2\,\alpha_{m,\mathtt{d}}}{\varrho\,\mu^2}\right\}\,.\tag{4.22}$$

□

Similar to the continuous case, replacing $\vec{\mathbf{u}}_h$, $\vec{\mathbf{v}}_h\in\mathbf{V}_h$ and $\vec{\mathbf{z}}_h\in\mathbf{X}_h$ by $\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h$, $\mathbf{0}\in\mathbf{V}_h$ and $\vec{\mathbf{v}}_h\in\mathbf{X}_h$ in (4.20), we arrive at

$$[\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\vec{\mathbf{u}}_h)-\mathbf{A}(\mathbf{w}_h,\widehat{\mathbf{b}}_h)(\vec{\mathbf{v}}_h),\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h]\,\geq\,\alpha_{\mathtt{MH,d}}\,\|\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h\|_{\mathbf{X}}^2\,,\tag{4.23}$$

for all $\vec{\mathbf{u}}_h,\vec{\mathbf{v}}_h\in\mathbf{X}_h$ such that $\vec{\mathbf{u}}_h-\vec{\mathbf{v}}_h\in\mathbf{V}_h$.

We continue with the discrete inf-sup condition for the operator $\mathbf{B}$ (cf. (2.32), (2.19)).

Lemma 4.5 *There exists a positive constant* $\beta_{\mathtt{MH,d}}$*, such that*

$$\sup_{\substack{\vec{\mathbf{v}}_h\in\mathbf{X}_h\\\vec{\mathbf{v}}_h\neq\mathbf{0}}}\frac{[\mathbf{B}(\vec{\mathbf{v}}_h),\boldsymbol{\tau}_h]}{\|\vec{\mathbf{v}}_h\|_{\mathbf{X}}}\,\geq\,\beta_{\mathtt{MH,d}}\,\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}\qquad\forall\,\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\,.\tag{4.24}$$

Proof. The statement follows directly from the definition of the operator $\mathbf{B}$ (cf. (2.32)) and (4.17). $\square$

We are now in position of establishing the discrete analogue of Lemma 3.4.

Lemma 4.6 *Assume that* $\{\mathcal{T}_h\}_{h>0}$ *is a family of quasi-uniform triangulations and that* $g_f \in \mathrm{L}^\infty(\Omega)$ *satisfies* (3.21)*. Then, for each* $(\mathbf{w}_h, \widehat{\mathbf{b}}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h$*, the problem* (4.8) *has a unique solution* $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) = ((\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h), \boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$*, and hence* $\mathbf{T}_{\mathtt{d}}(\mathbf{w}_h, \widehat{\mathbf{b}}_h) := (\mathbf{u}_h, \mathbf{b}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h$ *is well-defined. Moreover, there exists a positive constant* $C_{\mathbf{T}_{\mathtt{d}}}$*, independent of* $(\mathbf{w}_h, \widehat{\mathbf{b}}_h)$*, but depending on* $C_{s,\mathtt{d}}$*,* ν*,* $\mathtt{F}_1$*,* $\mathtt{D}_0$*,* $\mathtt{D}_1$*,* $\alpha_{m,\mathtt{d}}$*,* $|\Omega|$*,* $C_{\mathtt{d}}$*,* ϱ*,* μ*, and* $\beta_{\mathtt{MH},\mathtt{d}}$*, such that*

$$\|\mathbf{T}_{\mathtt{d}}(\mathbf{w}_h, \widehat{\mathbf{b}}_h)\|_{\mathtt{MH}} \le \|\vec{\mathbf{u}}_h\|_{\mathbf{X}} \le C_{\mathbf{T}_{\mathtt{d}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \left(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right)^{j-1} \right\}. \tag{4.25}$$

Proof. According to Lemmas 4.2 and 4.4 and the discrete inf-sup condition for $\mathbf{B}$ provided by (4.24) (cf. Lemma 4.5), the proof follows from a direct application of [1, Theorem 3.1], with $p_1 = \mathrm{p} \in [3,4]$ and $p_2 = p_3 = 2$, to the discrete setting represented by (4.8). In particular, the *a priori* bound (4.25) is consequence of the abstract estimate [1, eq. (3.2) in Theorem 3.1] applied to (4.8), which makes use of the bounds for $\|\mathbf{F}\|_{\mathbf{X}_h'}$ and $\|\mathbf{G}\|_{\mathbb{H}_h^{\boldsymbol{\sigma}\prime}}$ (cf. (3.4), (3.5)). $\square$

We remark here that, proceeding similarly to the derivation of (3.31), we obtain

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \le C_{\boldsymbol{\sigma}_{\mathtt{d}}} \sum_{i\in\{\mathrm{p},2\}} \left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \left(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right)^{j-1} \right)^{i-1}, \tag{4.26}$$

with $C_{\boldsymbol{\sigma}_{\mathtt{d}}}$ depending only on $C_{s,\mathtt{d}}$, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, $\alpha_{m,\mathtt{d}}$, $|\Omega|$, $C_{\mathtt{d}}$, ϱ, μ, and $\beta_{\mathtt{MH},\mathtt{d}}$.

We now proceed to analyze the fixed-point equation (4.9). We begin with the discrete version of Lemma 3.5, whose proof, follows straightforwardly from Lemma 4.6.

Lemma 4.7 *Given* $r > 0$*, let* $\mathbf{W}_{r,\mathtt{d}}$ *be the closed ball in* $\mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h$ *defined by*

$$\mathbf{W}_{r,\mathtt{d}} := \left\{ (\mathbf{w}_h, \widehat{\mathbf{b}}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h : \quad \|(\mathbf{w}_h, \widehat{\mathbf{b}}_h)\|_{\mathtt{MH}} \le r \right\}. \tag{4.27}$$

Assume that the data satisfy (3.21) *and*

$$C_{\mathbf{T}_{\mathtt{d}}} \left\{ \|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}} \left(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right)^{j-1} \right\} \le r. \tag{4.28}$$

Then, there holds $\mathbf{T}_{\mathtt{d}}(\mathbf{W}_{r,\mathtt{d}}) \subseteq \mathbf{W}_{r,\mathtt{d}}$.

Next, we address the discrete counterpart of Lemma 3.6, whose proof, being almost verbatim of the continuous ones, is omitted. We just remark that Lemma 4.8 below is derived using the strong monotonicity of $\mathbf{A}(\mathbf{w}_h, \widehat{\mathbf{b}}_h)$ on $\mathbf{V}_h$ (cf. (4.20)) and the continuity bound of $\mathbf{c}(\mathbf{w}_h, \widehat{\mathbf{b}}_h)$ (cf. (4.13)). Thus, we simply state the corresponding result as follow.

Lemma 4.8 *Assume that* $\{\mathcal{T}_h\}_{h>0}$ *is a family of quasi-uniform triangulations and that* $g_f \in \mathrm{L}^\infty(\Omega)$ *satisfies* (3.21)*. Let* $C_{\mathbf{c},\mathtt{d}}$*,* $\alpha_{\mathtt{MH},\mathtt{d}}$*, and* $C_{\mathbf{T}_{\mathtt{d}}}$ *be positive constants satisfying* (4.13)*,* (4.22)*, and* (4.25)*,* *respectively. Then, there holds*

$$
\begin{aligned}
\|\mathbf{T}_{\mathtt{d}}(\mathbf{w}_h,\widehat{\mathbf{b}}_h) - \mathbf{T}_{\mathtt{d}}(\mathbf{w}_{0,h},\widehat{\mathbf{b}}_{0,h})\|_{\mathtt{MH}} \;\le\; & \frac{C_{\mathbf{c},\mathtt{d}}\,C_{\mathbf{T}_{\mathtt{d}}}}{\alpha_{\mathtt{MH},\mathtt{d}}}\Bigg\{\|\mathbf{f}_m\|_{0,\Omega} + \|\mathbf{f}_f\|_{0,6/5;\Omega} \\
& + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\Bigg\}\|(\mathbf{w}_h,\widehat{\mathbf{b}}_h) - (\mathbf{w}_{0,h},\widehat{\mathbf{b}}_{0,h})\|_{\mathtt{MH}}\,,
\end{aligned}
\tag{4.29}
$$

for all $(\mathbf{w}_h,\widehat{\mathbf{b}}_h), (\mathbf{w}_{0,h},\widehat{\mathbf{b}}_{0,h}) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{C}_h$.

We are now in position of establishing the well-posedness of (4.4)

Theorem 4.9 *Assume that* $\{\mathcal{T}_h\}_{h>0}$ *is a family of quasi-uniform triangulations. Given* $r > 0$, *let* $\mathbf{W}_{r,\mathtt{d}}$ *as in* (4.27), *and assume that the data satisfy* (3.21), (4.28), *and*

$$
\frac{C_{\mathbf{c},\mathtt{d}}\,C_{\mathbf{T}_{\mathtt{d}}}}{\alpha_{\mathtt{MH},\mathtt{d}}}\left\{\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right\} < 1\,. \tag{4.30}
$$

Then the operator $\mathbf{T}_{\mathtt{d}}$ *has a unique fixed point* $(\mathbf{u}_h,\mathbf{b}_h) \in \mathbf{W}_{r,\mathtt{d}}$. *Equivalently, the problem* (4.4) *has a unique solution* $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$, *with* $(\mathbf{u}_h,\mathbf{b}_h) \in \mathbf{W}_{r,\mathtt{d}}$. *Moreover, there exist positive constants* $C_{\mathbf{T}_{\mathtt{d}}}, C_{\boldsymbol{\sigma}_{\mathtt{d}}}$, *depending on* $C_{s,\mathtt{d}}$, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, $\alpha_{m,\mathtt{d}}$, $|\Omega|$, $C_{\mathtt{d}}$, ϱ, μ, *and* $\beta_{\mathtt{MH},\mathtt{d}}$, *such that the following a priori estimates hold*

$$
\|\vec{\mathbf{u}}_h\|_{\mathbf{X}} \;\le\; C_{\mathbf{T}_{\mathtt{d}}}\left\{\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right\}, \tag{4.31}
$$

$$
\|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \;\le\; C_{\boldsymbol{\sigma}_{\mathtt{d}}}\sum_{i\in\{\mathrm{p},2\}}\left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right)^{i-1}. \tag{4.32}
$$

Proof. It follows similarly to the proof of Theorem 3.7. Indeed, we first notice from Lemma 4.7 that $\mathbf{T}_{\mathtt{d}}$ maps the ball $\mathbf{W}_{r,\mathtt{d}}$ into itself. Next, it is easy to see from (4.29) (cf. Lemma 4.8) and (4.30) that $\mathbf{T}_{\mathtt{d}}$ is a contraction, and hence the existence and uniqueness results follow from the Banach fixed-point theorem. In addition, it is clear that the estimates (4.31) and (4.32) follow straightforwardly from (4.25) and (4.26), which ends the proof. □

We end this section by establishing the well-posedness of (4.2), whose proof follows from similar arguments to the ones employed in [1, Corollary 3.9], the discrete inf-sup condition of b_m (cf. (4.6)) and the continuity bound of $c_m(\widehat{\mathbf{b}}_h)$ (cf. (4.12)). This result is stated next.

Corollary 4.10 *Assume that* $\{\mathcal{T}_h\}_{h>0}$ *is a family of quasi-uniform triangulations. Let* $\mathbf{f}_f \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{f}_m \in \mathbf{L}^2(\Omega)$, $g_f \in \mathrm{L}^\infty(\Omega)$, *and* $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$, *such that* (3.21), (4.28) *and* (4.30) *hold. Then, there exists a unique* $(\mathbf{u}_h,\mathbf{t}_h,\boldsymbol{\sigma}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ *and* $(\mathbf{b}_h,\lambda_h) \in \mathbf{H}_h^{\mathbf{b}} \times \mathrm{H}_h^{\lambda}$ *solution to* (4.2). *In addition,* $(\mathbf{u}_h,\mathbf{t}_h,\mathbf{b}_h)$ *and* $\boldsymbol{\sigma}_h$ *satisfy* (4.31) *and* (4.32), *respectively, and for* λ_h, *there exists a positive constant* $C_{\lambda_{\mathtt{d}}}$ *depending on* $C_{s,\mathtt{d}}$, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, $|\Omega|$, $C_{\mathtt{d}}$, ϱ, μ, $\beta_{\mathtt{MH},\mathtt{d}}$, *and* β_m, *such that*

$$
\|\lambda_h\|_{1,\Omega} \;\le\; C_{\lambda_{\mathtt{d}}}\sum_{i=1}^{2}\left(\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\big(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)^{j-1}\right)^{i}.
$$

5 A priori error analysis

In this section we derive Céa's estimate for the Galerkin scheme (4.2) with the finite element subspaces given by (4.1) (cf. Section 4.1), and then use the approximation properties of the latter to establish the corresponding rates of convergence. In fact, let $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma}) \in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{b},\lambda)\in\mathbf{H}_0(\mathrm{curl};\Omega)\times\mathrm{H}^1_0(\Omega)$, with $(\mathbf{u},\mathbf{b})\in\mathbf{W}_r$, be the unique solution of the problem (2.17) and let $(\mathbf{u}_h,\mathbf{t}_h,\boldsymbol{\sigma}_h)\in\mathbf{H}^{\mathbf{u}}_h\times\mathbb{H}^{\mathbf{t}}_h\times\mathbb{H}^{\boldsymbol{\sigma}}_h$ and $(\mathbf{b}_h,\lambda_h)\in\mathbf{H}^{\mathbf{b}}_h\times\mathrm{H}^{\lambda}_h$, with $(\mathbf{u}_h,\mathbf{b}_h)\in\mathbf{W}_{r,\mathbf{d}}$, be the unique solution of the discrete problem (4.2). Then, we are interested in obtaining an *a priori* estimate for the global error

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,6;\Omega} + \|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\mathbf{b}-\mathbf{b}_h\|_{\mathrm{curl};\Omega} + \|\lambda-\lambda_h\|_{1,\Omega}\,. \tag{5.1}$$

For this purpose, in what follows we introduce some definitions. Hereafter, given a subspace X_h of a generic Banach space $(X,\|\cdot\|_X)$, we set as usual

$$\mathrm{dist}\,(x,X_h) := \inf_{x_h\in X_h}\|x-x_h\|_X \qquad \forall\, x\in X\,.$$

We stress here that in order to derive an *a priori* bound for the global error (5.1), similar to [1, Section 5], we first separately bound the terms $\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\|_{\mathbf{X}}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$ and $\|\lambda-\lambda_h\|_{1,\Omega}$, being $(\vec{\mathbf{u}},\boldsymbol{\sigma}) = ((\mathbf{u},\mathbf{t},\mathbf{b}),\boldsymbol{\sigma})\in\mathbf{X}\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ the unique solution of the problem (2.27), and $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h) = ((\mathbf{u}_h,\mathbf{t}_h,\mathbf{b}_h),\boldsymbol{\sigma}_h)\in\mathbf{X}_h\times\mathbb{H}^{\boldsymbol{\sigma}}_h$ the unique solution of the discrete problem (4.4). This is proved below in Lemmas 5.1 and 5.2, respectively. We begin by bounding $\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\|_{\mathbf{X}}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$. To that end, since $\mathbf{C}_h$ is not included in its continuous counterpart $\mathbf{C}$, we note that we cannot directly apply a Strang-type lemma as the one derived in [11, Lemma 5.1]. Nevertheless, most of the arguments used to prove [11, Lemma 5.1] are employed below in Lemma 5.1 for the context given by (2.27) and (4.4), namely, discrete strong monotonicity of $\mathbf{A}(\mathbf{u}_h,\mathbf{b}_h)$ (cf. (4.23)), continuity of the operator $\mathbf{c}(\mathbf{u}_h,\mathbf{b}_h)$ (cf. (4.13)), and discrete inf-sup condition of $\mathbf{B}$ (cf. (4.24)).

Next, we define the set

$$\mathbf{V}^{\mathbf{G}}_h := \Big\{\vec{\mathbf{z}}_h\in\mathbf{X}_h :\quad [\mathbf{B}(\vec{\mathbf{z}}_h),\boldsymbol{\tau}_h] = [\mathbf{G},\boldsymbol{\tau}_h]\quad\forall\,\boldsymbol{\tau}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_h\Big\}\,, \tag{5.2}$$

which is clearly nonempty, since (4.24) holds. Note from the second equation of (4.4) that $\vec{\mathbf{u}}_h\in\mathbf{V}^{\mathbf{G}}_h$ and then $\vec{\mathbf{u}}_h-\vec{\mathbf{z}}_h\in\mathbf{V}_h$ for all $\vec{\mathbf{z}}_h\in\mathbf{V}^{\mathbf{G}}_h$. In addition, we recall that the discrete inf-sup conditions (4.24) and (4.6), and a classical result on mixed methods (see, for instance, [24, eq. (2.89) in Theorem 2.6]) ensure the existence of $C_1, C_2>0$, independent of h, such that:

$$\mathrm{dist}\,(\vec{\mathbf{u}},\mathbf{V}^{\mathbf{G}}_h) \,\le\, C_1\,\mathrm{dist}\,(\vec{\mathbf{u}},\mathbf{X}_h) \,\le\, C_1\Big(\mathrm{dist}\,(\mathbf{u},\mathbf{H}^{\mathbf{u}}_h)+\mathrm{dist}\,(\mathbf{t},\mathbb{H}^{\mathbf{t}}_h)+\mathrm{dist}\,(\mathbf{b},\mathbf{C}_h)\Big) \tag{5.3}$$

and

$$\mathrm{dist}\,(\mathbf{b},\mathbf{C}_h) \,\le\, C_2\,\mathrm{dist}\,(\mathbf{b},\mathbf{H}^{\mathbf{b}}_h)\,. \tag{5.4}$$

Throughout the rest of the paper, given any $r>0$, both $c(r)$ and $C(r)$, with or without sub-indexes, denote positive constants depending on r, and eventually on other constants or parameters.

The announced preliminary result regarding $\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\|_{\mathbf{X}}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$ is established as follows.

Lemma 5.1 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Let $\mathbf{f}_f\in\mathbf{L}^{6/5}(\Omega)$, $\mathbf{f}_m\in\mathbf{L}^2(\Omega)$, $g_f\in\mathrm{L}^\infty(\Omega)$, and $\mathbf{u}_{\mathrm{D}}\in\mathbf{H}^{1/2}(\Gamma)$, satisfying* (3.21) *and*

$$\frac{C_{\mathbf{c},\mathbf{d}}\,C_{\mathbf{T}}}{\alpha_{\mathtt{MH},\mathbf{d}}}\left\{\|\mathbf{f}_f\|_{0,6/5;\Omega}+\|\mathbf{f}_m\|_{0,\Omega}+\sum_{j\in\{\mathrm{p},2\}}\Big(\|g_f\|_{0,\infty;\Omega}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big)^{j-1}\right\} \,\le\, \frac{1}{2}\,. \tag{5.5}$$

Then, there exists a constant $C_1(r) > 0$, independent of h, such that

$$\begin{aligned}
&\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \\
&\quad \leq C_1(r) \left\{ \sum_{j \in \{\mathrm{p},2\}} \left(\mathrm{dist}\,(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) + \mathrm{dist}\,(\mathbf{t}, \mathbb{H}_h^{\mathbf{t}}) + \mathrm{dist}\,(\mathbf{b}, \mathbf{H}_h^{\mathbf{b}})\right)^{j-1} + \mathrm{dist}\,(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) \right\}.
\end{aligned} \tag{5.6}$$

Proof. We follow an analogous reasoning to the proof of [1, Lemma 5.1]. In fact, let $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}, \mathbf{b}), \boldsymbol{\sigma}) \in \mathbf{X} \times \mathbb{H}_0(\mathbf{div}_{6/5}; \Omega)$ and $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) = ((\mathbf{u}_h, \mathbf{t}_h, \mathbf{b}_h), \boldsymbol{\sigma}_h) \in \mathbf{X}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ be the unique solutions of the problems (2.27) and (4.4), respectively. We begin by noting that the first equation in (2.27) is well-defined even dough for test functions in $\mathbf{X}_h$. Then, we subtract the first equations of (2.27) and (4.4), to obtain

$$[\mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{v}}_h] - [\mathbf{A}(\mathbf{u}_h, \mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma} - \boldsymbol{\sigma}_h] = 0 \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{X}_h\,. \tag{5.7}$$

Next, let $\vec{\mathbf{z}}_h = (\mathbf{z}_h, \mathbf{r}_h, \mathbf{e}_h)$ be an arbitrary element in $\mathbf{V}_h^{\mathbf{G}}$ (cf. (5.2)), adding and subtracting suitable terms in (5.7), we arrive at

$$\begin{aligned}
[\mathbf{A}(\mathbf{u}_h, \mathbf{b}_h)(\vec{\mathbf{z}}_h) - \mathbf{A}(\mathbf{u}_h, \mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] \;&=\; [\mathbf{A}(\mathbf{u}_h, \mathbf{b}_h)(\vec{\mathbf{z}}_h), \vec{\mathbf{v}}_h] - [\mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{z}}_h), \vec{\mathbf{v}}_h] \\
&\quad + [\mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{z}}_h) - \mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{v}}_h] - [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\,,
\end{aligned} \tag{5.8}$$

for all $\vec{\mathbf{v}}_h \in \mathbf{X}_h$. Testing (5.8) with $\vec{\mathbf{v}}_h = \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h \in \mathbf{V}_h$, using (4.23) (cf. Lemma 4.4 and (3.21)) and the fact that $[\mathbf{B}(\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h), \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h] = 0$ for all $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$, we get

$$\begin{aligned}
\alpha_{\mathtt{MH,d}}\, \|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}^2 \;&\leq\; \left|[\mathbf{A}(\mathbf{u}_h, \mathbf{b}_h)(\vec{\mathbf{z}}_h), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h] - [\mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{z}}_h), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h]\right| \\
&\quad + \left|[\mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{z}}_h) - \mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h]\right| \;+\; \left|[\mathbf{B}(\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h), \boldsymbol{\sigma} - \boldsymbol{\tau}_h]\right|,
\end{aligned} \tag{5.9}$$

where, using the definitions of $\mathbf{A}(\mathbf{u}, \mathbf{b})$ (cf. (2.28)) and $\mathbf{A}(\mathbf{u}_h, \mathbf{b}_h)$ (cf. (4.5)), and employing Lemma 4.1 and triangle inequality, we first deduce that

$$\begin{aligned}
&\left|[\mathbf{A}(\mathbf{u}_h, \mathbf{b}_h)(\vec{\mathbf{z}}_h), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h] - [\mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{z}}_h), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h]\right| \;=\; \left|[\mathbf{c}(\mathbf{u}_h - \mathbf{u}, \mathbf{b}_h - \mathbf{b})(\vec{\mathbf{z}}_h), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h]\right| \\
&\quad\leq\; C_{\mathbf{c},\mathtt{d}}\, \|(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\|_{\mathtt{MH}} \left(\|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{u}}\|_{\mathbf{X}}\right) \|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \\
&\quad\leq\; C_{\mathbf{c},\mathtt{d}} \Big\{ \Big(\|(\mathbf{u}, \mathbf{b})\|_{\mathtt{MH}} + \|(\mathbf{u}_h, \mathbf{b}_h)\|_{\mathtt{MH}}\Big) \|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}} \;+\; \|\vec{\mathbf{u}}\|_{\mathbf{X}}\, \|(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\|_{\mathtt{MH}} \Big\} \|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,.
\end{aligned}$$

Then, using the fact that $(\mathbf{u}, \mathbf{b}) \in \mathbf{W}_r$, $(\mathbf{u}_h, \mathbf{b}_h) \in \mathbf{W}_{r,\mathtt{d}}$, and bounding $\|(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\|_{\mathtt{MH}}$ by $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}$, we arrive at

$$\begin{aligned}
&\left|[\mathbf{A}(\mathbf{u}_h, \mathbf{b}_h)(\vec{\mathbf{z}}_h), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h] - [\mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{z}}_h), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h]\right| \\
&\quad\leq\; \Big(c_1(r) \|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}} + C_{\mathbf{c},\mathtt{d}}\, \|\vec{\mathbf{u}}\|_{\mathbf{X}}\, \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\Big) \|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,,
\end{aligned} \tag{5.10}$$

with $c_1(r)$ depending on $C_{\mathbf{c},\mathtt{d}}$ and r. In turn, using Lemma 3.1, and simple computations, we get

$$\begin{aligned}
&\left|[\mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{z}}_h) - \mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h]\right| \;\leq\; \|\mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{z}}_h) - \mathbf{A}(\mathbf{u}, \mathbf{b})(\vec{\mathbf{u}})\|_{\mathbf{X}'} \|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \\
&\quad\leq\; L_{\mathtt{MH}} \Big\{ \left(1 + \|(\mathbf{u}, \mathbf{b})\|_{\mathtt{MH}}\right) \|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}} \;+\; \left(2\, \|\mathbf{u}\|_{0,6;\Omega} + \|\mathbf{u} - \mathbf{z}_h\|_{0,6;\Omega}\right)^{\mathrm{p}-2} \|\mathbf{u} - \mathbf{z}_h\|_{0,6;\Omega} \Big\} \|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,,
\end{aligned}$$

which combined with the property $(a+b)^m \le 2^{m-1}(a^m+b^m)$ for $a,b>0$ and $m>1$, the fact that $(\mathbf{u},\mathbf{b}) \in \mathbf{W}_r$, and similar arguments to the ones employed in (5.10), yields

$$|[\mathbf{A}(\mathbf{u},\mathbf{b})(\vec{\mathbf{z}}_h) - \mathbf{A}(\mathbf{u},\mathbf{b})(\vec{\mathbf{u}}), \vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h]| \le c_2(r)\left\{\|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}}^{\mathrm{p}-1}\right\}\|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}, \tag{5.11}$$

with $c_2(r)$ depending on $L_{\mathtt{MH}}$, p, and r. In addition, we observe from (3.2), that

$$\left|[\mathbf{B}(\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h), \boldsymbol{\sigma} - \boldsymbol{\tau}_h]\right| \le \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}\|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}. \tag{5.12}$$

Thus, replacing back (5.10), (5.11) and (5.12) into (5.9), and bounding $\|\vec{\mathbf{u}}\|_{\mathbf{X}}$ by (3.37), we obtain

$$\begin{aligned}\|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}} &\le c_3(r)\left\{\|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}}^{\mathrm{p}-1} + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}\right\}\\ &+ \frac{C_{\mathbf{c},\mathtt{d}}\,C_{\mathbf{T}}}{\alpha_{\mathtt{MH},\mathtt{d}}}\left\{\|\mathbf{f}_f\|_{0,6/5;\Omega} + \|\mathbf{f}_m\|_{0,\Omega} + \sum_{j\in\{\mathrm{p},2\}}\left(\|g_f\|_{0,\infty;\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right)^{j-1}\right\}\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}},\end{aligned}$$

with $c_3(r)$ depending on $\alpha_{\mathtt{MH},\mathtt{d}}$, $C_{\mathbf{c},\mathtt{d}}$, $L_{\mathtt{MH}}$, p, and r. Hence, triangle inequality $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \le \|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{z}}_h - \vec{\mathbf{u}}_h\|_{\mathbf{X}}$, and the assumption (5.5), yields

$$\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}} \le c_4(r)\left\{\|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{u}} - \vec{\mathbf{z}}_h\|_{\mathbf{X}}^{\mathrm{p}-1} + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}\right\}, \tag{5.13}$$

with $c_4(r)$ only depending on $\alpha_{\mathtt{MH},\mathtt{d}}$, $C_{\mathbf{c},\mathtt{d}}$, $L_{\mathtt{MH}}$, p, and r.

On the other hand, to estimate the term $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$, we consider an arbitrary element $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$ and use the discrete inf-sup condition (4.24), to get

$$\beta_{\mathtt{MH},\mathtt{d}}\,\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \le \sup_{\substack{\vec{\mathbf{v}}_h\in\mathbf{X}_h\\ \vec{\mathbf{v}}_h\neq\mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h - \boldsymbol{\sigma}] + [\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma} - \boldsymbol{\tau}_h]}{\|\vec{\mathbf{v}}_h\|}, \tag{5.14}$$

where, using again (5.7) and adding and subtracting suitable terms, we obtain

$$[\mathbf{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h - \boldsymbol{\sigma}] = [\mathbf{A}(\mathbf{u},\mathbf{b})(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] - [\mathbf{A}(\mathbf{u}_h,\mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] + [\mathbf{A}(\mathbf{u},\mathbf{b})(\vec{\mathbf{u}}) - \mathbf{A}(\mathbf{u},\mathbf{b})(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h].$$

In turn, similarly to (5.10) and (5.11), using the fact that $(\mathbf{u},\mathbf{b}) \in \mathbf{W}_r$ and $(\mathbf{u}_h,\mathbf{b}_h) \in \mathbf{W}_{r,\mathtt{d}}$, and combining (4.31) with (4.28) to bound $\|\vec{\mathbf{u}}_h\|_{\mathbf{X}}$ by r, we deduce, respectively, that

$$\begin{aligned}&\left|[\mathbf{A}(\mathbf{u},\mathbf{b})(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h] - [\mathbf{A}(\mathbf{u}_h,\mathbf{b}_h)(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h]\right|\\ &\quad\le C_{\mathbf{c},\mathtt{d}}\,\|\vec{\mathbf{u}}_h\|_{\mathbf{X}}\,\|(\mathbf{u}-\mathbf{u}_h,\mathbf{b}-\mathbf{b}_h)\|_{\mathtt{MH}}\,\|\vec{\mathbf{v}}_h\|_{\mathbf{X}} \le c_5(r)\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\,\|\vec{\mathbf{v}}_h\|_{\mathbf{X}}\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}&\left|[\mathbf{A}(\mathbf{u},\mathbf{b})(\vec{\mathbf{u}}) - \mathbf{A}(\mathbf{u},\mathbf{b})(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h]\right|\\ &\quad\le L_{\mathtt{MH}}\left\{1 + \|(\mathbf{u},\mathbf{b})\|_{\mathtt{MH}} + \left(\|\mathbf{u}\|_{0,6;\Omega} + \|\mathbf{u}_h\|_{0,6;\Omega}\right)^{\mathrm{p}-2}\right\}\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\|\vec{\mathbf{v}}_h\|_{\mathbf{X}}\\ &\quad\le c_6(r)\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\|\vec{\mathbf{v}}_h\|_{\mathbf{X}},\end{aligned} \tag{5.16}$$

with $c_5(r)$ and $c_6(r)$ only depending on $C_{\mathbf{c},\mathtt{d}}, L_{\mathtt{MH}}$, p, and r. Thus, replacing back (5.15) and (5.16) into (5.14), using (3.2), triangle inequality, and some algebraic manipulations, we obtain

$$\begin{aligned}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} &\le \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}\\ &\le c_7(r)\left\{\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{X}}\right\},\end{aligned} \tag{5.17}$$

with $c_7(r)$ only depending on $C_{\mathbf{c},\mathtt{d}}$, $L_{\mathtt{MH}}$, p, $\beta_{\mathtt{MH},\mathtt{d}}$ and r. Finally, combining (5.13) and (5.17), using the fact that $\vec{\mathbf{z}}_h \in \mathbf{V}_h^{\mathbf{G}}$ and $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$ are arbitrary, taking infimum over the corresponding discrete subspaces $\mathbf{V}_h^{\mathbf{G}}$ and $\mathbb{H}_h^{\boldsymbol{\sigma}}$, and applying (5.3)–(5.4), we conclude (5.6) completing the proof. □

The aforementioned result regarding $\|\lambda - \lambda_h\|_{1,\Omega}$, the proof of which is omitted as it follows exactly as in [1, Lemma 5.2], is stated next.

Lemma 5.2 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Assume further that the data satisfy* (3.21) *and* (5.5). *Then, there exists a constant $C_2(r) > 0$, independent of h, such that*

$$\|\lambda - \lambda_h\|_{1,\Omega} \,\leq\, C_2(r)\left\{\sum_{j\in\{\mathrm{p},2\}} \big(\mathrm{dist}\,(\mathbf{u},\mathbf{H}_h^{\mathbf{u}}) + \mathrm{dist}\,(\mathbf{t},\mathbb{H}_h^{\mathbf{t}}) + \mathrm{dist}\,(\mathbf{b},\mathbf{H}_h^{\mathbf{b}})\big)^{j-1} \right.$$
$$\left. + \,\mathrm{dist}\,(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}) + \mathrm{dist}\,(\lambda,\mathrm{H}_h^{\lambda})\right\}.$$

We are now in position of establishing the Céa estimate of (4.2). The aforementioned result follows straightforwardly from Lemmas 5.1 and 5.2.

Theorem 5.3 *Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations. Assume further that the data satisfy* (3.21) *and* (5.5). *Then, there exists a constant $C(r) > 0$, independent of h, but depending on r, C_s, $C_{s,\mathtt{d}}$, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, $\alpha_{m,\mathtt{d}}$, $|\Omega|$, $C_{\mathtt{d}}$, ϱ, μ, $\beta_{\mathtt{MH},\mathtt{d}}$, and β_m, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,6;\Omega} + \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\mathbf{b} - \mathbf{b}_h\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\lambda - \lambda_h\|_{1,\Omega}$$
$$\leq C(r)\left\{\sum_{j\in\{\mathrm{p},2\}} \big(\mathrm{dist}\,(\mathbf{u},\mathbf{H}_h^{\mathbf{u}}) + \mathrm{dist}\,(\mathbf{t},\mathbb{H}_h^{\mathbf{t}}) + \mathrm{dist}\,(\mathbf{b},\mathbf{H}_h^{\mathbf{b}})\big)^{j-1} + \mathrm{dist}\,(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}) + \mathrm{dist}\,(\lambda,\mathrm{H}_h^{\lambda})\right\}.$$

In order to establish the rate of convergence of the Galerkin scheme (4.2), we recall next the approximation properties of the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{b}}$ and H_h^{λ} (cf. (4.1)), whose derivations can be found in [5], [23], [24], [26], [33, Theorem 5.41] and [8, Section 3.1] (see also [15, Section 5]):

$(\mathbf{AP})_{\mathtt{CBF}}$: there exist positive constants C_1, C_2, and C_3, independent of h, such that for each $\mathbf{v} \in \mathbf{W}^{k+1,6}(\Omega)$, $\mathbf{s} \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and $\boldsymbol{\tau} \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{k+1,6/5}(\Omega)$, there hold

$$\mathrm{dist}\,(\mathbf{v},\mathbf{H}_h^{\mathbf{u}}) := \inf_{\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{0,6;\Omega} \,\leq\, C_1\, h^{k+1}\, \|\mathbf{v}\|_{k+1,6;\Omega}\,,$$

$$\mathrm{dist}\,(\mathbf{s},\mathbb{H}_h^{\mathbf{t}}) := \inf_{\mathbf{s}_h\in\mathbb{H}_h^{\mathbf{t}}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \,\leq\, C_2\, h^{k+1}\, \|\mathbf{s}\|_{k+1,\Omega}\,,$$

and

$$\mathrm{dist}\,(\boldsymbol{\tau},\mathbb{H}_h^{\boldsymbol{\sigma}}) := \inf_{\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \,\leq\, C_3\, h^{k+1}\left\{\|\boldsymbol{\tau}\|_{k+1,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{k+1,6/5;\Omega}\right\}.$$

$(\mathbf{AP})_{\mathtt{M}}$: there exist positive constants C_4 and C_5, independent of h, such that for each $\mathbf{d} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0(\mathrm{curl};\Omega)$ with $\mathrm{curl}(\mathbf{d}) \in \mathbf{H}^{k+1}(\Omega)$, and $\xi \in \mathrm{H}^{k+2}(\Omega) \cap \mathrm{H}^1_0(\Omega)$, there hold

$$\mathrm{dist}\,(\mathbf{d},\mathbf{H}_h^{\mathbf{b}}) := \inf_{\mathbf{d}_h\in\mathbf{H}_h^{\mathbf{b}}} \|\mathbf{d} - \mathbf{d}_h\|_{\mathrm{curl};\Omega} \,\leq\, C_4\, h^{k+1}\left\{\|\mathbf{d}\|_{k+1,\Omega} + \|\mathrm{curl}(\mathbf{d})\|_{k+1,\Omega}\right\},$$

and

$$\mathrm{dist}\,(\xi, \mathrm{H}_h^\lambda) := \inf_{\xi_h \in \mathrm{H}_h^\lambda} \|\xi - \xi_h\|_{1,\Omega} \,\leq\, C_5\, h^{k+1}\, \|\xi\|_{k+2,\Omega}\,.$$

Now we are in a position to provide the theoretical rate of convergence of the Galerkin scheme (4.2).

Theorem 5.4 *In addition to the hypotheses of Theorems 3.7, 4.9, and 5.3, given an integer $k \geq 0$, assume that $\mathbf{u} \in \mathbf{W}^{k+1,6}(\Omega)$, $\mathbf{t} \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ with $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{k+1,6/5}(\Omega)$, $\mathbf{b} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0(\mathrm{curl};\Omega)$ with $\mathrm{curl}(\mathbf{b}) \in \mathbf{H}^{k+1}(\Omega)$, and $\lambda \in \mathrm{H}^{k+2}(\Omega) \cap \mathrm{H}^1_0(\Omega)$. Then, there exists a constant $C_{\mathtt{rate}} > 0$, independent of h, but depending on r, C_s, $C_{s,\mathtt{d}}$, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, $\alpha_{m,\mathtt{d}}$, $|\Omega|$, $C_{\mathtt{d}}$, ϱ, μ, $\beta_{\mathtt{MH,d}}$, and β_m, such that*

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|_{0,6;\Omega} + \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\mathbf{b} - \mathbf{b}_h\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\lambda - \lambda_h\|_{1,\Omega} \\
&\quad \leq\, C_{\mathtt{rate}}\, h^{k+1} \left\{ \sum_{j \in \{\mathrm{p},2\}} \Big(\|\mathbf{u}\|_{k+1,6;\Omega} + \|\mathbf{t}\|_{k+1,\Omega} + \|\mathbf{b}\|_{k+1,\Omega} + \|\mathrm{curl}(\mathbf{b})\|_{k+1,\Omega} \Big)^{j-1} \right. \\
&\qquad \left. +\, \|\boldsymbol{\sigma}\|_{k+1,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{k+1,6/5;\Omega} + \|\lambda\|_{k+2,\Omega} \right\}.
\end{aligned}$$

Proof. The result follows from a direct application of Theorem 5.3 and the approximation properties provided by $(\mathbf{AP})_{\mathtt{CBF}}$ and $(\mathbf{AP})_{\mathtt{M}}$. Further details are omitted. □

We end this section by introducing suitable approximations for other variables of interest, such as the pressure p, the velocity gradient $\mathbf{G} = \nabla\mathbf{u}$, the vorticity $\boldsymbol{\omega} = \frac{1}{2}\left(\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{t}}\right)$, and the stress $\widetilde{\boldsymbol{\sigma}} := \nu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{t}}) - p\,\mathbb{I}$, are all them written in terms of the solution of the discrete problem (4.2a)–(4.2b). In fact, using (2.6), (2.9), and (2.16), and after simple computations, we deduce that at the continuous level, there hold

$$\begin{aligned}
&p = -\frac{1}{3}\Big(\mathrm{tr}(\boldsymbol{\sigma}) + \frac{1}{2}\,\mathrm{tr}(\mathbf{u}\otimes\mathbf{u}) - \nu\, g_f\Big) - c_0\,, \quad \mathbf{G} = \mathbf{t} + \frac{1}{3}\, g_f\,\mathbb{I}\,, \quad \boldsymbol{\omega} = \frac{1}{2}\left(\mathbf{t} - \mathbf{t}^t\right), \text{ and} \\
&\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \frac{1}{2}\,(\mathbf{u}\otimes\mathbf{u}) + \nu\,\mathbf{t}^t + \Big(\frac{\nu}{3} g_f + c_0\Big)\,\mathbb{I}, \text{ with } \quad c_0 = \frac{1}{6|\Omega|}\int_\Omega \Big(2\nu g_f - \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\Big)\,,
\end{aligned} \tag{5.18}$$

provided the discrete solution $(\mathbf{u}_h, \mathbf{t}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ of problem (4.2a)–(4.2b), we propose the following approximations for the aforementioned variables:

$$\begin{aligned}
&p_h = -\frac{1}{3}\Big(\mathrm{tr}(\boldsymbol{\sigma}_h) + \frac{1}{2}\mathrm{tr}(\mathbf{u}_h\otimes\mathbf{u}_h) - \nu\, g_f\Big) - c_{0,h}\,, \quad \mathbf{G}_h = \mathbf{t}_h + \frac{1}{3}\, g_f\mathbb{I}\,, \quad \boldsymbol{\omega}_h = \frac{1}{2}\Big(\mathbf{t}_h - \mathbf{t}_h^t\Big), \text{ and} \\
&\widetilde{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h + \frac{1}{2}(\mathbf{u}_h\otimes\mathbf{u}_h) + \nu\,\mathbf{t}_h^t + \Big(\frac{\nu}{3} g_f + c_{0,h}\Big)\,\mathbb{I}, \text{ with } \quad c_{0,h} = \frac{1}{6|\Omega|}\int_\Omega \Big(2\nu g_f - \mathrm{tr}(\mathbf{u}_h\otimes\mathbf{u}_h)\Big).
\end{aligned} \tag{5.19}$$

The following result, whose proof follows directly from Theorem 5.4, establishes the corresponding approximation result for this post-processing procedure.

Corollary 5.5 *Let $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma}) \in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)$ and $(\mathbf{b},\lambda) \in \mathbf{H}_0(\mathrm{curl};\Omega)\times\mathrm{H}^1_0(\Omega)$ be the unique solution of the continuous problem (2.17), and let $p, \mathbf{G}, \boldsymbol{\omega}$ and $\widetilde{\boldsymbol{\sigma}}$ given by (5.18). In addition, let $p_h, \mathbf{G}_h, \boldsymbol{\omega}_h$ and $\widetilde{\boldsymbol{\sigma}}_h$ be the discrete counterparts introduced in (5.19). Let an integer $k \geq 0$ and assume that the hypotheses of the Theorem 5.4 be hold. Then, there exists a constant $C_{\mathtt{post}} > 0$,* *independent of h, but depending on r, C_s, $C_{s,\mathtt{d}}$, ν, $\mathtt{F}_1$, $\mathtt{D}_0$, $\mathtt{D}_1$, $\alpha_{m,\mathtt{d}}$, $|\Omega|$, $C_{\mathtt{d}}$, ϱ, μ, $\beta_{\mathtt{MH},\mathtt{d}}$, and β_m, such that*

$$\|p - p_h\|_{0,\Omega} + \|\mathbf{G} - \mathbf{G}_h\|_{0,\Omega} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}$$
$$\le C_{\mathtt{post}}\, h^{k+1} \Bigg\{ \sum_{j\in\{\mathrm{p},2\}} \Big(\|\mathbf{u}\|_{k+1,6;\Omega} + \|\mathbf{t}\|_{k+1,\Omega} + \|\mathbf{b}\|_{k+1,\Omega} + \|\mathrm{curl}(\mathbf{b})\|_{k+1,\Omega} \Big)^{j-1}$$
$$+ \|\boldsymbol{\sigma}\|_{k+1,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{k+1,6/5;\Omega} + \|\lambda\|_{k+2,\Omega} \Bigg\}.$$

Proof. First, from (5.18) and (5.19), the triangle and Cauchy–Schwarz inequalities, it is not difficult to show that there exists $C > 0$, independent of h, such that the following estimate holds

$$\begin{aligned} &\|p - p_h\|_{0,\Omega} + \|\mathbf{G} - \mathbf{G}_h\|_{0,\Omega} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \\ &\quad \le C \Big\{ \|(\mathbf{u}\otimes\mathbf{u}) - (\mathbf{u}_h\otimes\mathbf{u}_h)\|_{0,\Omega} + \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \Big\}, \end{aligned} \tag{5.20}$$

where, adding and subtracting $\mathbf{u}\otimes\mathbf{u}_h$ (it also works with $\mathbf{u}_h\otimes\mathbf{u}$), using Hölder's inequality, the Sobolev embedding (2.14) and the fact that $(\mathbf{u},\mathbf{b}) \in \mathbf{W}_r$ and $(\mathbf{u}_h,\mathbf{b}_h) \in \mathbf{W}_{r,\mathtt{d}}$, we find that

$$\|(\mathbf{u}\otimes\mathbf{u}) - (\mathbf{u}_h\otimes\mathbf{u}_h)\|_{0,\Omega} \le |\Omega|^{1/6} \big(\|\mathbf{u}\|_{0,6;\Omega} + \|\mathbf{u}_h\|_{0,6;\Omega} \big) \|\mathbf{u} - \mathbf{u}_h\|_{0,6;\Omega} \le C\, \|\mathbf{u} - \mathbf{u}_h\|_{0,6;\Omega}\,. \tag{5.21}$$

Then, replacing back (5.21) into (5.20) the result follows straightforwardly from Theorem 5.4. Further details are omitted. □

6 Numerical results

In this section we report two examples illustrating the performance of the mixed finite element method (4.2), on a set of quasi-uniform triangulations of the respective 3D domains, and considering the finite element subspaces defined by (4.1) (cf. Section 4.1). In what follows, we refer to the corresponding sets of finite element subspaces generated by $k = 0$ and $k = 1$, as simply $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ and $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{ND}_1 - \mathrm{P}_2$, respectively. The implementation is based on a `FreeFem++` code [29]. In order to solve the nonlinear problem (4.2), given $\mathbf{0} \neq \mathbf{w} \in \mathbf{L}^6(\Omega)$ and $\mathbf{r} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$ we introduce the Gâteaux dirivatives and functional associated, respectively, to a_f, d_f, and F_1 (cf. (2.18), (2.21), (2.23)), that is

$$[\mathcal{D}a_f(\mathbf{w})(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})] := \int_\Omega \Big(\mathtt{D} - \frac{1}{3} g_f\Big)\mathbf{u}\cdot\mathbf{v} + \int_\Omega \mathtt{F}\,|\mathbf{w}|^{\mathrm{p}-2}\mathbf{u}\cdot\mathbf{v} + (\mathrm{p}-2)\int_\Omega \mathtt{F}\,|\mathbf{w}|^{\mathrm{p}-4}(\mathbf{w}\cdot\mathbf{u})(\mathbf{w}\cdot\mathbf{v}) + \nu \int_\Omega \mathbf{t} : \mathbf{s}\,,$$

$$[\mathcal{D}d_f(\mathbf{w},\mathbf{r})(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s})] := \frac{1}{2}\int_\Omega \mathbf{t} : (\mathbf{v}\otimes\mathbf{w}) + \frac{1}{2}\int_\Omega \mathbf{r} : (\mathbf{v}\otimes\mathbf{u}) - \frac{1}{2}\int_\Omega \Big\{ (\mathbf{u}\otimes\mathbf{w}) + (\mathbf{w}\otimes\mathbf{u}) \Big\} : \mathbf{s}\,,$$

and

$$[F_1(\mathbf{w},\mathbf{r}),(\mathbf{v},\mathbf{s})] := [F_1,(\mathbf{v},\mathbf{s})] + (\mathrm{p}-2)\int_\Omega \mathtt{F}\,|\mathbf{w}|^{\mathrm{p}-2}\mathbf{w}\cdot\mathbf{v} + \frac{1}{2}\int_\Omega \mathbf{r} : (\mathbf{v}\otimes\mathbf{w}) - \frac{1}{2}\int_\Omega (\mathbf{w}\otimes\mathbf{w}) : \mathbf{s}\,,$$

for all $(\mathbf{u},\mathbf{t}),(\mathbf{v},\mathbf{s}) \in \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)$. In this way, we propose the Newton-type strategy: Given $\mathbf{0} \neq \mathbf{u}_h^0 \in \mathbf{H}_h^{\mathbf{u}}$ and $\mathbf{t}_h^0 \in \mathbb{H}_h^{\mathbf{t}}$, for $i \ge 1$, solve

$$\begin{aligned} [a_m(\mathbf{b}_h^i),\mathbf{d}_h] + [c_m(\mathbf{b}_h^i)(\mathbf{u}_h^{i-1}),\mathbf{d}_h] + [b_m(\mathbf{d}_h),\lambda_h^i] &= [F_3,\mathbf{d}_h]\,, \\ [b_m(\mathbf{b}_h^i),\xi_h] &= 0\,, \end{aligned} \tag{6.1}$$

for all $\mathbf{d}_h \in \mathbf{H}_h^{\mathbf{b}}$ and $\xi_h \in \mathrm{H}_h^{\lambda}$, and

$$[\mathcal{D}a_f(\mathbf{u}_h^{i-1})(\mathbf{u}_h^i,\mathbf{t}_h^i),(\mathbf{v}_h,\mathbf{s}_h)] + [\mathcal{D}d_f(\mathbf{u}_h^{i-1},\mathbf{t}_h^{i-1})(\mathbf{u}_h^i,\mathbf{t}_h^i),(\mathbf{v}_h,\mathbf{s}_h)] \\ + [b_f(\mathbf{v}_h,\mathbf{s}_h),\boldsymbol{\sigma}_h^i] = [F_1(\mathbf{u}_h^{i-1},\mathbf{t}_h^{i-1}),(\mathbf{v}_h,\mathbf{s}_h)] - [c_f(\mathbf{b}_h^i)(\mathbf{b}_h^i),\mathbf{v}_h]\,, \\ [b_f(\mathbf{u}_h^i,\mathbf{t}_h^i),\boldsymbol{\tau}_h] = [F_2,\boldsymbol{\tau}_h]\,, \tag{6.2}$$

for all $(\mathbf{v}_h,\mathbf{s}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}}$ and $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$. More precisely, we first solve the linear system (6.1) with the given $\mathbf{u}_h^0$, whose solution is denoted $(\mathbf{b}_h^1,\lambda_h^1)$. Next, we solve (6.2) with the given $(\mathbf{u}_h^0,\mathbf{t}_h^0,\mathbf{b}_h^1)$, so that, starting from $\mathbf{u}_h^0 := (0, 1\mathrm{E}-6, 0)^{\mathrm{t}}$ and $\mathbf{t}_h^0 = \mathbf{0}$, we perform just one Newton iteration to obtain $(\mathbf{u}_h^1,\mathbf{t}_h^1,\boldsymbol{\sigma}_h^1)$ as an approximate solution of it. Then, the process continues with $(\mathbf{u}_h^i,\mathbf{t}_h^i)$ for each $i \geq 1$. In this way, for a fixed tolerance $\mathsf{tol} = 1\mathrm{E}-6$, the above iterations are terminated, which yields the number of Newton iterations reported in the tables below, once the relative error between two consecutive iterates, say $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\mathtt{DoF}}}{\|\mathbf{coeff}^{m+1}\|_{\mathtt{DoF}}} \leq \mathsf{tol}\,,$$

where $\|\cdot\|_{\mathtt{DoF}}$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DoF}}$ with $\mathtt{DoF}$ denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{b}}$ and H_h^{λ}.

We now introduce some additional notations. The individual errors are denoted by:

$$\mathrm{e}(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{0,6;\Omega}\,,\quad \mathrm{e}(\mathbf{t}) := \|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega}\,,\quad \mathrm{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}\,,$$
$$\mathrm{e}(\mathbf{b}) := \|\mathbf{b}-\mathbf{b}_h\|_{\mathrm{curl};\Omega}\,,\quad \mathrm{e}(\lambda) := \|\lambda-\lambda_h\|_{1,\Omega}\,,\quad \mathrm{e}(p) := \|p-p_h\|_{0,\Omega}\,,$$
$$\mathrm{e}(\mathbf{G}) := \|\mathbf{G}-\mathbf{G}_h\|_{0,\Omega}\,,\quad \mathrm{e}(\boldsymbol{\omega}) := \|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{0,\Omega}\,,\quad \mathrm{e}(\widetilde{\boldsymbol{\sigma}}) := \|\widetilde{\boldsymbol{\sigma}}-\widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,,$$

where the pressure p, the velocity gradient $\mathbf{G}$, the vorticity $\boldsymbol{\omega}$, and the shear stress tensor $\widetilde{\boldsymbol{\sigma}}$ are further variables of physical interest that are recovered by using the corresponding postprocessing formulae p_h, $\mathbf{G}_h$, $\boldsymbol{\omega}_h$, and $\widetilde{\boldsymbol{\sigma}}_h$ detailed in (5.18)–(5.19). Next, as usual, for each $\star \in \big\{\mathbf{u},\mathbf{t},\boldsymbol{\sigma},\mathbf{b},\lambda,p,\mathbf{G},\boldsymbol{\omega},\widetilde{\boldsymbol{\sigma}}\big\}$ we let $\mathsf{r}(\star)$ be the experimental rate of convergence given by $\mathsf{r}(\star) := \log(\mathrm{e}(\star)/\mathrm{e}'(\star))/\log(h/h')$, where h and h' denote two consecutive meshsizes with errors e and e' respectively.

The examples to be considered in this section are described next. In all them, for sake of simplicity, we take $\nu = 1$, $\mu = 1$, $\varrho = 1$, set the vector $\mathbf{1} := (1,1,1)^{\mathrm{t}} \in \mathrm{R}^3$, and choose the Darcy and Forchheimer coefficients, by

$$\mathtt{D}(\mathbf{x}) = \exp(-(x_1+x_2+x_3)) \quad \text{and} \quad \mathtt{F}(\mathbf{x}) = \exp(x_1+x_2+x_3) \quad \forall\,\mathbf{x} = (x_1,x_2,x_3)^{\mathrm{t}} \in \Omega\,,$$

respectively, which satisfy (2.2). In addition, the mean value of $\mathrm{tr}(\boldsymbol{\sigma}_h^i)$ over Ω, with $i \geq 1$, is fixed via a Lagrange multiplier strategy (adding one row and one column to the matrix system that solves (6.2) for $\mathbf{u}_h^i$, $\mathbf{t}_h^i$, and $\boldsymbol{\sigma}_h^i$).

Example 1: Non-convex domain with different values of the parameter p.

In this test we corroborate the rates of convergence and also study the performance of the numerical method (6.1)–(6.2) (cf. (4.2)) in a non-convex domain with respect to the total error and different values values of the power p in the inertial term $|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}$ (cf. (2.8c)). We consider the Fichera's corner domain $\Omega := (-1,1)^3 \setminus [0,1)^3$, where, due to the regularity of the Neumann problem (see [18] and [19] for details), there holds $\mathbf{H}_0(\mathrm{curl};\Omega) \cap \mathbf{H}(\mathrm{div}^0;\Omega) \subseteq \mathbf{H}^s(\Omega)$ for $s \in \big(1/2, 2/3\big)$. First, we choose $\mathrm{p} = 4$ and the data $\mathbf{f}_f$, $\mathbf{f}_m$, g_f and $\mathbf{u}_\mathrm{D}$ so that the exact solution is given by

$$\mathbf{u} = \begin{pmatrix} -\,x_1\,(x_2 - x_3)\,(x_2 + x_3) \\ 2x_2\,(x_1 - x_3)\,(x_1 + x_3) \\ -\,x_3\,(x_1 - x_2)\,(x_1 + x_2) \end{pmatrix}, \quad p = x_1\,x_2\,x_3 - c_p\,,$$

$$\mathbf{b} = \mathrm{curl}\big(\sin^2(\pi x_1)\,\sin^2(\pi x_2)\,\sin^2(\pi x_3)\mathbf{1}\big)\,, \quad \lambda = \sin(\pi x_1)\,\sin(\pi x_2)\,\sin(\pi x_3)\,,$$

where $c_p \in \mathrm{R}$ is chosen in such a way $p \in \mathrm{L}^2_0(\Omega)$. Table 6.1 shows the convergence history for $k \in \{0,1\}$ and a set of quasi-uniform mesh refinements, including the number of Newton iterations when $\mathrm{p} = 4$. Notice that we are able not only to approximate the original unknowns but also the pressure field, the velocity gradient tensor, the vorticity, and the shear stress tensor through the formulae (5.19). Note also that, due to computational limitation, we display the results for only three meshes when $k = 1$. Nevertheless, we observe that in both cases, the mixed finite element method converges optimally with order $\mathcal{O}(h^{k+1})$, as it was proved by Theorem 5.4 and Corollary 5.5. Notice that $\mathrm{e}(\mathbf{t}) = \mathrm{e}(\mathbf{G})$ since $\mathbf{t}$ (resp. $\mathbf{t}_h$) is just a translation of $\mathbf{G}$ (resp. $\mathbf{G}_h$). In addition, some components of the numerical solution are displayed in Figure 6.1, which were built using the mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ approximation with meshsize $h = 0.1414$ and 42,000 tetrahedra elements (actually representing 782,121 DoF). On the other hand, in Table 6.2, we show the behavior of the method with respect to the total error

$$\mathrm{e}_{\mathtt{total}} := \Big(\mathrm{e}(\mathbf{u})^2 + \mathrm{e}(\mathbf{t})^2 + \mathrm{e}(\boldsymbol{\sigma})^2 + \mathrm{e}(\mathbf{b})^2 + \mathrm{e}(\lambda)^2\Big)^{1/2},$$

considering different powers $\mathrm{p} \in \{3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0\}$ in the inertial term $|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}$ (cf. (2.8c)), polynomial degree $k = 0$, and different mesh sizes h. Here we observe that Newton's method demonstrates robustness concerning both h and p, even for values of p outside the interval $[3,4]$. This observation is justified by the fact that this range is not a mathematical limitation but rather the most commonly used in the literature (see [21] for $\mathrm{p} = 3$ and [20] for $\mathrm{p} = 4$). We stress that the theory developed in this work can be readily extended to values of p within the range $[3,6]$.

Example 2: Convergence against smooth exact solutions in a transient regime.

In the second example, we study numerically the rates of convergence and performance of the numerical method (6.1)–(6.2) for a fluid in transient regime. To that end, we consider the domain $\Omega := (0,1) \times (0,0.5) \times (0,0.5)$, the final time $T = 0.01\,s$, and the unsteady version of the problem (2.1) (cf. (2.8)), that is, we replace (2.1a) and (2.1c), respectively, by

$$\frac{\partial\,\mathbf{u}}{\partial\,t} - \nu\,\Delta\mathbf{u} + (\nabla\mathbf{u})\mathbf{u} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} + \nabla p - \frac{1}{\mu}\,\mathrm{curl}(\mathbf{b}) \times \mathbf{b} = \mathbf{f}_f \quad \text{in} \quad \Omega \times (0,T]$$

and

$$\frac{\partial\,\mathbf{b}}{\partial\,t} + \frac{1}{\varrho\,\mu}\,\mathrm{curl}(\mathrm{curl}(\mathbf{b})) + \nabla\lambda - \mathrm{curl}(\mathbf{u} \times \mathbf{b}) = \mathbf{f}_m \quad \text{in} \quad \Omega \times (0,T]\,.$$

We consider $\mathrm{p} = 3$ and the data $\mathbf{f}_f$, $\mathbf{f}_m$, and g_f are adjusted so that the exact solution is given by the smooth functions

$$\mathbf{u} = \exp(t) \begin{pmatrix} \sin(\pi x_1)\,\cos(\pi x_2)\,\cos(\pi x_3) \\ -2\cos(\pi x_1)\,\sin(\pi x_2)\,\cos(\pi x_3) \\ \cos(\pi x_1)\,\cos(\pi x_2)\,\sin(3\pi x_3) \end{pmatrix}, \quad p = \exp(t)\,(x_1 - 0.5)\,x_2\,x_3\,,$$

$$\mathbf{b} = \sin(t)\,\mathrm{curl}\big(x_1^2\,(x_2 - 0.5)^2 x_3^2 \cos(\pi x_3)^2\mathbf{1}\big)\,, \quad \lambda = t\,x_1\,x_2\,x_3\,(x_1 - 1)\,(x_2 - 0.5)\,(x_3 - 0.5)\,.$$

The model problem is then complemented with the appropriate Dirichlet boundary condition and initial data. We employ a suitable backward Euler time discretization, with time step $\Delta t = 10^{-3}\, s$, and let $t_m = m\,\Delta t$, $m = 0, \ldots, N$, with $N = 10$. In particular, the errors for the velocity are computed by using the discrete-in-time norm:

$$\mathrm{e}(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_h\|_{\ell^2(0,T;\mathbf{L}^6(\Omega))} := \left(\Delta t \sum_{k=1}^{N} \|\mathbf{u}(t_m) - \mathbf{u}_{h,m}\|_{0,6;\Omega}^2 \right)^{1/2},$$

where, $\mathbf{u}_{h,m}$ represents the approximation of $\mathbf{u}$ obtained with the numerical method at time t_m. Similar norms are used to compute the errors for the others unknowns. We observe that at each time step we are solving a slight adaptation of the discrete stationary problem (6.1)–(6.2). Note also that the time step is sufficiently small, so that the time discretization error does not affect the convergence rates.

Table 6.3 shows the convergence history for a sequence of quasi-uniform mesh refinements, including the average number of Newton iterations. The results illustrate numerically that optimal rates of convergence $\mathcal{O}(h^{k+1})$ are attained for $k = 0$ also for a transient regime. The well-posedness analysis for the unsteady version of (2.8) can be addressed by following similar arguments to the ones developed in [12]. This is a topic of current research. The Newton method exhibits a behavior independent of the mesh size, converging in average of 2.1 iterations in almost all cases. In Figure 6.2 we display some solutions obtained with the mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ approximation with mesh size $h = 0.0505$ and $32,928$ tetrahedra elements (actually representing $613,593$ `DoF`) at time $T = 0.01$.

$\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ approximation									
DoF	h	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{b}$)	r($\mathbf{b}$)
6665	0.7071	6.8E-01	–	4.7E-00	–	8.5E+01	–	3.9E+01	–
51249	0.3536	3.5E-01	0.952	1.8E-00	1.388	4.7E+01	0.854	1.8E+01	1.103
170713	0.2357	2.4E-01	0.980	9.8E-01	1.488	3.0E+01	1.091	1.3E+01	0.933
402017	0.1768	1.8E-01	0.989	6.8E-01	1.269	2.3E+01	0.982	9.5E-00	0.967
782121	0.1414	1.4E-01	0.993	5.2E-01	1.183	1.8E+01	0.996	7.6E-00	0.980

e(λ)	r(λ)	e(p)	r(p)	e($\mathbf{G}$)	r($\mathbf{G}$)	e($\boldsymbol{\omega}$)	r($\boldsymbol{\omega}$)	e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	it
4.1E-00	–	4.0E-00	–	4.7E-00	–	2.6E-00	–	1.1E+01	–	4
2.4E-00	0.750	1.2E-00	1.745	1.8E-00	1.388	1.1E-00	1.196	3.5E-00	1.611	6
1.7E-00	0.911	6.7E-01	1.464	9.8E-01	1.488	6.6E-01	1.383	1.9E-00	1.547	6
1.3E-00	0.955	4.2E-01	1.623	6.8E-01	1.268	4.6E-01	1.216	1.2E-00	1.445	6
1.0E-00	0.973	2.9E-01	1.708	5.2E-01	1.183	3.6E-01	1.156	9.2E-01	1.384	6

$\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{ND}_1 - \mathrm{P}_2$ approximation									
DoF	h	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{b}$)	r($\mathbf{b}$)
28017	0.7071	1.0E-01	–	1.2E-00	–	4.0E+01	–	1.7E+01	–
217697	0.3536	2.3E-02	2.160	1.6E-01	2.916	8.5E-00	2.219	4.7E-00	1.836
727633	0.2357	1.0E-02	2.013	5.4E-02	2.593	3.9E-00	1.930	2.1E-00	1.911

e(λ)	r(λ)	e(p)	r(p)	e($\mathbf{G}$)	r($\mathbf{G}$)	e($\boldsymbol{\omega}$)	r($\boldsymbol{\omega}$)	e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	it
1.5E-00	–	1.2E-00	–	1.2E-00	–	7.2E-01	–	2.9E-00	–	6
4.5E-01	1.738	1.4E-01	3.057	1.6E-01	2.916	1.0E-01	2.835	3.6E-01	3.014	6
2.1E-01	1.897	3.5E-02	3.480	5.4E-02	2.593	3.7E-02	2.457	1.0E-01	3.019	6

Table 6.1: [Example 1] Number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the mixed approximations with p = 4.

DoF	h	p = 3.0			p = 3.5			p = 4.0			p = 4.5			p = 5.0			p = 5.5			p = 6.0		
		$\mathtt{e}_{\mathtt{total}}$	rate	it	$\mathtt{e}_{\mathtt{total}}$	rate	it	$\mathtt{e}_{\mathtt{total}}$	rate	it	$\mathtt{e}_{\mathtt{total}}$	rate	it	$\mathtt{e}_{\mathtt{total}}$	rate	it	$\mathtt{e}_{\mathtt{total}}$	rate	it	$\mathtt{e}_{\mathtt{total}}$	rate	it
6665	0.7071	9.37E+01	–	4	9.37E+01	–	4	9.39E+01	–	4	9.39E+01	–	4	9.40E+01	–	4	9.41E+01	–	4	9.43E+01	–	4
51249	0.3536	5.05E+01	0.8925	6	5.05E+01	0.8925	6	5.06E+01	0.8929	6	5.06E+01	0.8925	6	5.07E+01	0.8917	6	5.08E+01	0.8904	6	5.09E+01	0.8889	6
170713	0.2357	3.28E+01	1.0677	6	3.28E+01	1.0677	6	3.28E+01	1.0686	6	3.28E+01	1.0692	6	3.28E+01	1.0700	6	3.29E+01	1.0712	6	3.30E+01	1.0728	6
402017	0.1768	2.47E+01	0.9792	6	2.47E+01	0.9792	6	2.47E+01	0.9800	6	2.47E+01	0.9806	6	2.48E+01	0.9815	6	2.48E+01	0.9829	6	2.48E+01	0.9847	6
782121	0.1414	1.98E+01	0.9935	6	1.98E+01	0.9935	6	1.98E+01	0.9941	6	1.98E+01	0.9946	6	1.98E+01	0.9953	6	1.98E+01	0.9964	6	1.99E+01	0.9980	6

Table 6.2: [Example 1] Number of degrees of freedom, meshsizes, total errors, rates of convergence, and Newton iteration count for the mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ approximation considering $\mathrm{p} \in \{3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0\}$.

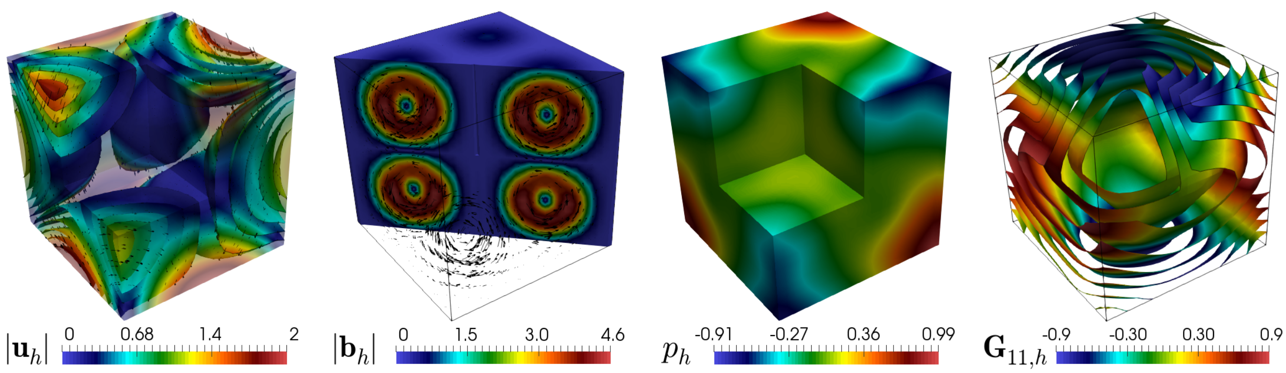

Figure 6.1: [Example 1] Computed magnitude of the velocity and magnetic field, pressure field, and velocity gradient component.

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$\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{ND}_0 - \mathrm{P}_1$ approximation																				
DoF	h	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{b}$)	r($\mathbf{b}$)	e(λ)	r(λ)	e(p)	r(p)	e($\mathbf{G}$)	r($\mathbf{G}$)	e($\boldsymbol{\omega}$)	r($\boldsymbol{\omega}$)	e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	it
1977	0.3536	3.2E-02	–	1.1E-01	–	6.9E-01	–	3.2E-05	–	8.4E-07	–	7.0E-02	–	1.1E-01	–	5.8E-02	–	2.2E-01	–	2.4
28791	0.1414	1.4E-02	0.884	4.8E-02	0.936	2.3E-01	1.173	1.5E-05	0.871	3.9E-07	0.853	2.7E-02	1.043	4.8E-02	0.936	2.5E-02	0.919	8.9E-02	0.972	2.1
115905	0.0884	8.9E-03	0.975	3.1E-02	0.961	1.4E-01	1.067	9.2E-06	0.980	2.5E-07	0.964	1.5E-02	1.201	3.1E-02	0.961	1.6E-02	0.967	5.5E-02	1.016	2.1
298959	0.0643	6.5E-03	0.990	2.3E-02	0.974	1.0E-01	1.047	6.7E-06	0.992	1.8E-07	0.984	1.0E-02	1.270	2.3E-02	0.974	1.2E-02	0.980	4.0E-02	1.033	2.1
613593	0.0505	5.1E-03	0.995	1.8E-02	0.982	7.9E-02	1.036	5.3E-06	0.996	1.4E-07	0.991	7.5E-03	1.275	1.8E-02	0.982	9.1E-03	0.987	3.1E-02	1.033	2.1

Table 6.3: [Example 2] Number of degrees of freedom, meshsizes, errors, rates of convergence, and average number of Newton iterations for the mixed approximation with p = 3.

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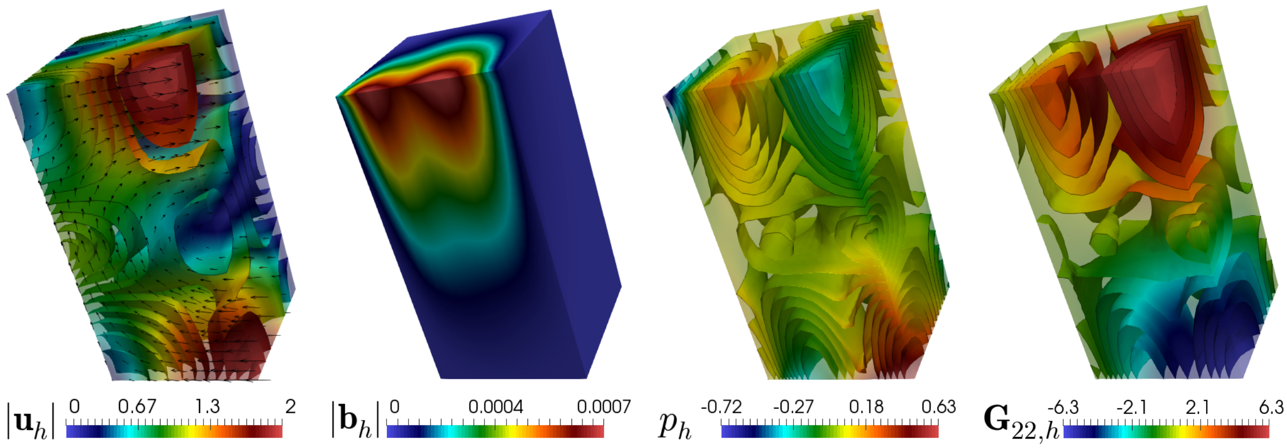

Figure 6.2: [Example 2] Computed magnitude of the velocity and magnetic field, pressure field, and velocity gradient component at time $T = 0.01$.

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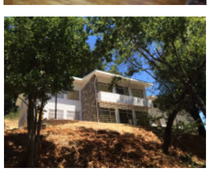

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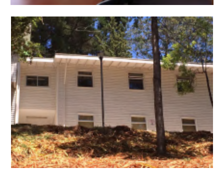
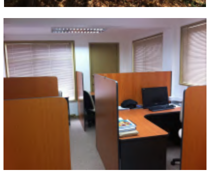

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