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A note on the generalized Babuška-Brezzi theory: revisiting the proof of the associated Strang error estimates*

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Abstract

In this note we simplify the derivation of the error estimates for the generalized Babuška-Brezzi theory with Galerkin schemes defined in terms of approximate bilinear forms and functionals. More precisely, we provide a straight proof that makes no use of any translated continuous or discrete kernel nor of the distance between them, but of suitable upper bounds of the distances of each component of the Galerkin solution to any other member of the respective finite element subspace. In this way, the Strang error estimates are obtained simply by applying the aforementioned bounds along with the triangle inequality, so that they become cleaner and with fully explicit constants. The case in which the discrete bilinear forms can be evaluated at the continuous solution is also considered, which yields the consistency terms to appear separately from the distances to the subspaces, thus allowing the former to be handled independently from the latter.

Key words: Babuška-Brezzi theory, Galerkin scheme, Strang error estimates

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1 Introduction

The generalized Babuška-Brezzi theory establishes necessary and sufficient conditions for the well-posedness of problems of the type: Find $(\sigma, u) \in X_2 \times M_1$ such that

$$\begin{aligned} a(\sigma,\tau) + b_1(\tau,u) &= F(\tau) \qquad \forall\, \tau \in X_1 \\ b_2(\sigma,v) &= G(v) \qquad \forall\, v \in M_2\,, \end{aligned} \tag{1.1}$$

where $(X_2, \|\cdot\|_{X_2})$, $(M_1, \cdot\|_{M_1})$, $(X_1, \|\cdot\|_{X_1})$, and $(M_2, \cdot\|_{M_2})$ are real Banach spaces, $a : X_2 \times X_1 \to R$ and $b_i : X_i \times M_i \to R$, $i \in \{1,2\}$, are bounded bilinear forms, and $F : X_1 \to R$ and $G : M_2 \to R$ are bounded linear functionals. The corresponding result reads as follows (cf. [1]).

Theorem 1.1. *Let $(X_2, \|\cdot\|_{X_2})$, $(M_1, \cdot\|_{M_1})$, $(X_1, \|\cdot\|_{X_1})$, and $(M_2, \cdot\|_{M_2})$ be real Banach spaces, such that M_1, X_1, and M_2 are reflexive, and let $a : X_2 \times X_1 \to R$ and $b_i : X_i \times M_i \to R$, $i \in \{1,2\}$, be bounded bilinear forms with induced operators given by $\mathcal{A} \in \mathcal{L}(X_2, X_1')$ and $\mathcal{B}_i \in \mathcal{L}(X_i, M_i')$, respectively. In addition, for each $i \in \{1,2\}$ let*

$$\mathcal{K}_i := N(\mathcal{B}_i) = \Big\{ \tau \in X_i : \quad b_i(\tau, v) = 0 \quad \forall\, v \in M_i \Big\}\,, \tag{1.2}$$

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and assume that

i) *one of the following equivalent pairs of hypotheses on a is satisfied*

i-1) $\sup_{\zeta\in\mathcal{K}_1} a(\tau,\zeta) \,>\, 0 \quad \forall\,\tau\in\mathcal{K}_2\backslash\{0\}\,,$

i-2) *there exists $\alpha>0$ such that*

$$\sup_{\substack{\tau\in\mathcal{K}_2\\ \tau\neq 0}} \frac{a(\tau,\zeta)}{\|\tau\|} \,\geq\, \alpha\,\|\zeta\| \quad \forall\,\zeta\in\mathcal{K}_1\,,$$

i-1)' $\sup_{\tau\in\mathcal{K}_2} a(\tau,\zeta) \,>\, 0 \quad \forall\,\zeta\in\mathcal{K}_1\backslash\{0\}\,,$

i-2)' *there exists $\alpha>0$ such that*

$$\sup_{\substack{\zeta\in\mathcal{K}_1\\ \zeta\neq 0}} \frac{a(\tau,\zeta)}{\|\zeta\|} \,\geq\, \alpha\,\|\tau\| \quad \forall\,\tau\in\mathcal{K}_2\,,$$

ii) *for each $i\in\{1,2\}$ there exists a constant $\beta_i>0$ such that*

$$\sup_{\substack{\tau\in X_i\\ \tau\neq 0}} \frac{b_i(\tau,v)}{\|\tau\|} \,\geq\, \beta_i\,\|v\| \qquad \forall\,v\in M_i\,.$$

Then, for each pair $(F,G)\in X_1'\times M_2'$ there exists a unique $(\sigma,u)\in X_2\times M_1$ solution to problem (1.1), *and there hold the following a priori bounds*:

$$\|\sigma\| \,\leq\, \frac{1}{\alpha}\,\|F\| \,+\, \frac{1}{\beta_2}\Big(1+\frac{\|\mathcal{A}\|}{\alpha}\Big)\|G\|\,, \tag{1.3}$$

$$\|u\| \,\leq\, \frac{1}{\beta_1}\Big(1+\frac{\|\mathcal{A}\|}{\alpha}\Big)\|F\| \,+\, \frac{\|\mathcal{A}\|}{\beta_1\,\beta_2}\,\Big(1+\frac{\|\mathcal{A}\|}{\alpha}\Big)\|G\|\,. \tag{1.4}$$

Moreover, the hypotheses i) *and* ii) *are also necessary.*

Proof. See [1, Section 2.1, Theorem 2.1, Corollary 2.1].

□

On the other hand, in order to introduce a general Galerkin scheme associated with (1.1), we now let $\{X_{2,h}\}_{h>0}$, $\{M_{1,h}\}_{h>0}$, $\{X_{1,h}\}_{h>0}$, and $\{M_{2,h}\}_{h>0}$ be families of finite dimensional subspaces of X_2, M_1, X_1, and M_2, respectively, and let $a_h: X_{2,h}\times X_{1,h}\to R$ and $b_{i,h}: X_{i,h}\times M_{i,h}\to R$, $i\in\{1,2\}$, be bounded bilinear forms approximating a and b_i, with induced operators $\mathcal{A}_h\in\mathcal{L}(X_{2,h},X_{1,h}')$ and $\mathcal{B}_{i,h}\in\mathcal{L}(X_{i,h},M_{i,h}')$, respectively. Then, given $F_h\in X_{1,h}'$ and $G_h\in M_{2,h}'$ approximating F and G, respectively, we consider the discrete system: Find $(\sigma_h,u_h)\in X_{2,h}\times M_{1,h}$ such that

$$\begin{aligned} a_h(\sigma_h,\tau_h) \,+\, b_{1,h}(\tau_h,u_h) &\;=\; F_h(\tau_h) \qquad \forall\,\tau_h\,\in\,X_{1,h}\\ b_{2,h}(\sigma_h,v_h) &\;=\; G_h(v_h) \qquad \forall\,v_h\,\in\,M_{2,h}\,. \end{aligned} \tag{1.5}$$

Thus, as a direct application of Theorem 1.1 to the present discrete context, the well-posedness of (1.5) reads as follows (see also [1, Section 2.2]).

Theorem 1.2. *In addition to the previous notations and definitions, for each $i\in\{1,2\}$ let*

$$\mathcal{K}_{i,h} \,:=\, N(\mathcal{B}_{i,h}) \,=\, \Big\{\tau_h\in X_{i,h}: \quad b_{i,h}(\tau_h,v_h)\,=\,0 \quad \forall\,v_h\in M_{i,h}\Big\}\,, \tag{1.6}$$

and assume that:

i) *one of the following equivalent pairs of hypotheses on* a_h *is satisfied*

i-1) $\displaystyle\sup_{\zeta_h \in \mathcal{K}_{1,h}} a_h(\tau_h, \zeta_h) \,>\, 0 \quad \forall\, \tau_h \,\in\, \mathcal{K}_{2,h} \backslash \{0\}\,,$

i-2) *there exists* $\alpha_{\mathtt{d}} > 0$ *such that*

$$\sup_{\substack{\tau_h \in \mathcal{K}_{2,h}\\ \tau_h \neq 0}} \frac{a_h(\tau_h, \zeta_h)}{\|\tau_h\|} \geq \alpha_{\mathtt{d}}\, \|\zeta_h\| \quad \forall\, \zeta_h \in \mathcal{K}_{1,h},$$

i-1)' $\displaystyle\sup_{\tau_h \in \mathcal{K}_{2,h}} a_h(\tau_h, \zeta_h) \,>\, 0 \quad \forall\, \zeta_h \,\in\, \mathcal{K}_{1,h} \backslash \{0\}\,,$

i-2)' *there exists* $\alpha_{\mathtt{d}} > 0$ *such that*

$$\sup_{\substack{\zeta_h \in \mathcal{K}_{1,h}\\ \zeta_h \neq 0}} \frac{a_h(\tau_h, \zeta_h)}{\|\zeta_h\|} \geq \alpha_{\mathtt{d}}\, \|\tau_h\| \quad \forall\, \tau_h \in \mathcal{K}_{2,h},$$

ii) *for each* $i \in \{1, 2\}$ *there exists a constant* $\beta_{i,\mathtt{d}} > 0$ *such that*

$$\sup_{\substack{\tau_h \in X_{i,h}\\ \tau_h \neq 0}} \frac{b_{i,h}(\tau_h, v_h)}{\|\tau_h\|} \,\geq\, \beta_{i,\mathtt{d}}\, \|v_h\| \qquad \forall\, v_h \in M_{i,h}\,.$$

Then, for each pair $(F_h, G_h) \in X'_{1,h} \times M'_{2,h}$ *there exists a unique* $(\sigma_h, u_h) \in X_{2,h} \times M_{1,h}$ *solution to problem* (1.5)*, and there hold the following a priori bounds*:

$$\|\sigma_h\| \,\leq\, \frac{1}{\alpha_{\mathtt{d}}}\, \|F_h\| \,+\, \frac{1}{\beta_{2,\mathtt{d}}}\Big(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\Big)\|G_h\|\,, \tag{1.7}$$

$$\|u_h\| \,\leq\, \frac{1}{\beta_{1,\mathtt{d}}}\Big(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\Big)\|F_h\| \,+\, \frac{\|\mathcal{A}_h\|}{\beta_{1,\mathtt{d}}\,\beta_{2,\mathtt{d}}}\,\Big(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\Big)\|G_h\|\,. \tag{1.8}$$

Moreover, the hypotheses i) *and* ii) *are also necessary.*

We remark here that in the case that $\dim\big(\mathcal{K}_{1,h}\big) \,=\, \dim\big(\mathcal{K}_{2,h}\big)$, the Fredholm alternative in finite dimensional spaces allows us to conclude that in Theorem 1.2 the assumptions i-1) and i-1)' are redundant with respect to i-2) and i-2)', respectively, and hence the former can be dropped.

Furthermore, the a priori estimates for the Galerkin error $\|\sigma - \sigma_h\| + \|u - u_h\|$ were originally established in [1, Section 2.3] by introducing the translated continuous and discrete kernels $\mathcal{K}_2(G)$ and $\mathcal{K}_{2,h}(G_h)$ defined, respectively, by

$$\mathcal{K}_2(G) \,:=\, \Big\{\tau \in X_2 : \quad b_2(\tau, v) = G(v) \quad \forall\, v \in M_2\Big\} \quad \text{and}$$

$$\mathcal{K}_{2,h}(G_h) \,:=\, \Big\{\tau_h \in X_{2,h} : \quad b_{2,h}(\tau_h, v_h) \,=\, G_h(v_h) \quad \forall\, v_h \in M_{2,h}\Big\}\,,$$

and by previously estimating $\mathrm{dist}\big(\tau, \mathcal{K}_{2,h}(G_h)\big)$ for each $\tau \in \mathcal{K}_2(G)$. Hereafter, given a subspace S of a generic Banach space $(X, \|\cdot\|_X)$, we set $\mathrm{dist}(x, S) \,:=\, \inf\limits_{s \in S} \|x - s\|_X$ for all $x \in X$. Note from the second equations of (1.1) and (1.5) that, in particular, $\sigma \in \mathcal{K}_2(G)$ and $\sigma_h \in \mathcal{K}_{2,h}(G_h)$, which explains the dependence on $\mathrm{dist}\big(\sigma, \mathcal{K}_{2,h}(G_h)\big)$ of the Strang error estimates provided in [1, Section 2.3].

At this point, we find it important to stress that the aforementioned results from [1] extend those established in [3], where Hilbert spaces were considered, the same bilinear forms and linear functionals were utilized for introducing the continuous and Galerkin formulations, and only the sufficiency of the hypotheses was proved.

The main purpose of this note, to be addressed in the next section, is to provide a more direct proof of the a priori error estimates arising from (1.1) and (1.5). To this end, and instead of employing any translated kernel, we base our analysis on the derivation of suitable upper bounds of $\|\sigma_h - \tau_h\|$ and $\|u_h - v_h\|$ for each $(\tau_h, v_h) \in X_{2,h} \times M_{1,h}$. To some extent, our approach can be seen as a generalization of the technique employed in the proof of [2, Lemma 2.44], where the particular case in which $X_2 = X_1$, $M_2 = M_1$, and the continuous and discrete bilinear forms and functionals coincide, was considered.

2 The a priori error estimates

We begin with the following two lemmas establishing the results referred to at the end of Section 1.

Lemma 2.1. *Assume the hypotheses of Theorems* 1.1 *and* 1.2, *and let* $(\sigma, u) \in X_2 \times M_1$ *and* $(\sigma_h, u_h) \in X_{2,h} \times M_{1,h}$ *be the unique solutions of* (1.1) *and* (1.5), *respectively. Then, for each* $(\tau_h, v_h) \in X_{2,h} \times M_{1,h}$ *there holds*

$$\begin{aligned} \|\sigma_h - \tau_h\| \le & \left(\frac{\|\mathcal{A}\|}{\alpha_{\mathtt{d}}} + \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right) \frac{\|\mathcal{B}_2\|}{\beta_{2,\mathtt{d}}} \right) \|\sigma - \tau_h\| \\ + & \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right) \frac{1}{\beta_{2,\mathtt{d}}} \|(b_2 - b_{2,h})(\tau_h, \cdot)\|_{M'_{2,h}} + \frac{1}{\alpha_{\mathtt{d}}} \|(a - a_h)(\tau_h, \cdot)\|_{\mathcal{K}'_{1,h}} \\ + & \frac{\|\mathcal{B}_1\|}{\alpha_{\mathtt{d}}} \|u - v_h\| + \frac{1}{\alpha_{\mathtt{d}}} \|(b_1 - b_{1,h})(\cdot, v_h)\|_{\mathcal{K}'_{1,h}} \\ + & \frac{1}{\alpha_{\mathtt{d}}} \|F - F_h\|_{\mathcal{K}'_{1,h}} + \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right) \frac{1}{\beta_{2,\mathtt{d}}} \|G - G_h\|_{M'_{2,h}} \end{aligned} \tag{2.1}$$

where the consistency terms in (2.1) *are defined as*

$$\|(b_2 - b_{2,h})(\tau_h, \cdot)\|_{M'_{2,h}} := \sup_{\substack{w_h \in M_{2,h} \\ w_h \neq 0}} \frac{(b_2 - b_{2,h})(\tau_h, w_h)}{\|w_h\|}, \tag{2.2}$$

$$\|(a - a_h)(\tau_h, \cdot)\|_{\mathcal{K}'_{1,h}} := \sup_{\substack{\zeta_h \in \mathcal{K}_{1,h} \\ \zeta_h \neq 0}} \frac{(a - a_h)(\tau_h, \zeta_h)}{\|\zeta_h\|}, \tag{2.3}$$

$$\|(b_1 - b_{1,h})(\cdot, v_h)\|_{\mathcal{K}'_{1,h}} := \sup_{\substack{\zeta_h \in \mathcal{K}_{1,h} \\ \zeta_h \neq 0}} \frac{(b_1 - b_{1,h})(\zeta_h, v_h)}{\|\zeta_h\|}, \tag{2.4}$$

$$\|F - F_h\|_{\mathcal{K}'_{1,h}} := \sup_{\substack{\zeta_h \in \mathcal{K}_{1,h} \\ \zeta_h \neq 0}} \frac{(F - F_h)(\zeta_h)}{\|\zeta_h\|}, \quad \textit{and} \tag{2.5}$$

$$\|G - G_h\|_{M'_{2,h}} := \sup_{\substack{w_h \in M_{2,h} \\ w_h \neq 0}} \frac{(G - G_h)(w_h)}{\|w_h\|}. \tag{2.6}$$

Proof. We first notice, thanks to the obvious reflexivity of $M_{2,h}$, that the inf-sup condition for $b_{2,h}$ (cf. ii)) is equivalent to stating

$$\|\mathcal{B}'_{2,h}(\mathcal{G}_h)\| \ \ge\ \beta_{2,\mathtt{d}} \, \|\mathcal{G}_h\| \qquad \forall \, \mathcal{G}_h \in M''_{2,h} .$$

Hence, applying the converse of [2, Lemma A.42], we deduce that for each $\widetilde{G}_h \in M'_{2,h}$ there exists $\widetilde{\tau}_h \in X_{2,h}$ such that

$$\mathcal{B}_{2,h}(\widetilde{\tau}_h) \,=\, \widetilde{G}_h \quad \text{and} \quad \|\widetilde{\tau}_h\| \,\le\, \frac{1}{\beta_{2,\mathtt{d}}}\,\|\widetilde{G}_h\|_{M'_{2,h}}\,. \tag{2.7}$$

In particular, given $\tau_h \in X_{2,h}$, there exists $\widetilde{\tau}_h \in X_{2,h}$ such that

$$\mathcal{B}_{2,h}(\widetilde{\tau}_h) \,=\, b_{2,h}(\tau_h,\cdot) - G_h \in M'_{2,h}\,,$$

that is, such that

$$b_{2,h}(\widetilde{\tau}_h, w_h) \,=\, b_{2,h}(\tau_h, w_h) - G_h(w_h) \qquad \forall\, w_h \in M_{2,h}\,, \tag{2.8}$$

and

$$\|\widetilde{\tau}_h\| \,\le\, \frac{1}{\beta_{2,\mathtt{d}}}\,\|b_{2,h}(\tau_h,\cdot) - G_h\|_{M'_{2,h}} \,:=\, \frac{1}{\beta_{2,\mathtt{d}}}\sup_{\substack{w_h\in M_{2,h}\\ w_h\neq 0}} \frac{b_{2,h}(\tau_h, w_h) - G_h(w_h)}{\|w_h\|}\,. \tag{2.9}$$

Now, using from the second equation of (1.1) that $b_2(\sigma, w_h) = G(w_h)$, and then subtracting and adding τ_h in the first component of $b_2(\sigma, w_h)$, we find

$$b_{2,h}(\tau_h, w_h) - G_h(w_h) \,=\, -b_2(\sigma - \tau_h, w_h) \,+\, (b_{2,h} - b_2)(\tau_h, w_h) \,+\, (G - G_h)(w_h)\,,$$

which, replaced back into (2.9), yields

$$\|\widetilde{\tau}_h\| \,\le\, \frac{\|\mathcal{B}_2\|}{\beta_{2,\mathtt{d}}}\,\|\sigma - \tau_h\| \,+\, \frac{1}{\beta_{2,\mathtt{d}}}\,\|(b_2 - b_{2,h})(\tau_h,\cdot)\|_{M'_{2,h}} \,+\, \frac{1}{\beta_{2,\mathtt{d}}}\,\|G - G_h\|_{M'_{2,h}}\,. \tag{2.10}$$

In turn, it is clear from (2.8) and the second equation of (1.5) that $\sigma_h - (\tau_h - \widetilde{\tau}_h) \in \mathcal{K}_{2,h}$, whence, according to i-2)' from Theorem 1.2, we have

$$\alpha_{\mathtt{d}}\,\|\sigma_h - (\tau_h - \widetilde{\tau}_h)\| \,\le\, \sup_{\substack{\zeta_h\in \mathcal{K}_{1,h}\\ \zeta_h\neq 0}} \frac{a_h\big(\sigma_h - (\tau_h - \widetilde{\tau}_h), \zeta_h\big)}{\|\zeta_h\|}\,. \tag{2.11}$$

Then, employing the first rows of (1.5) and (1.1), we obtain for each $\zeta_h \in \mathcal{K}_{1,h}$

$$\begin{aligned} a_h\big(\sigma_h - (\tau_h - \widetilde{\tau}_h), \zeta_h\big) &= F_h(\zeta_h) - a_h(\tau_h - \widetilde{\tau}_h, \zeta_h) + a(\sigma, \zeta_h) + b_1(\zeta_h, u) - F(\zeta_h)\\ &=\, a(\sigma - \tau_h, \zeta_h) \,+\, (a - a_h)(\tau_h, \zeta_h) \,+\, a_h(\widetilde{\tau}_h, \zeta_h)\\ &\quad+\, b_1(\zeta_h, u - v_h) \,+\, (b_1 - b_{1,h})(\zeta_h, v_h) \,-\, (F - F_h)(\zeta_h)\,, \end{aligned} \tag{2.12}$$

where we have subtracted and added an arbitrary $v_h \in M_{1,h}$ in the second component of $b_1(\zeta_h, u)$. We remark here that there is no reason to assume in general that $\mathcal{K}_{1,h}$ is contained in $\mathcal{K}_1$, but if this were the case, then $b_1(\zeta_h, u)$ would vanish, there would be no need to consider the aforementioned $v_h \in M_{1,h}$, and the final estimate (2.1) would not include its third row. We will go back to this remark later on. Also, note that while $b_{1,h}(\zeta_h, v_h)$ is certainly null, we maintain it in the writing of the foregoing equation in order to emphasize that $(b_1 - b_{1,h})(\zeta_h, v_h)$ is a consistency term. In this way, replacing (2.12) back into (2.11), and performing the corresponding bounding procedures, we arrive at

$$\begin{aligned} \|\sigma_h - (\tau_h - \widetilde{\tau}_h)\| \,\le\, &\frac{\|\mathcal{A}\|}{\alpha_{\mathtt{d}}}\,\|\sigma - \tau_h\| + \frac{1}{\alpha_{\mathtt{d}}}\,\|(a - a_h)(\tau_h,\cdot)\|_{\mathcal{K}'_{1,h}} + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\,\|\widetilde{\tau}_h\|\\ &+ \frac{\|\mathcal{B}_1\|}{\alpha_{\mathtt{d}}}\,\|u - v_h\| + \frac{1}{\alpha_{\mathtt{d}}}\,\|(b_1 - b_{1,h})(\cdot, v_h)\|_{\mathcal{K}'_{1,h}} + \frac{1}{\alpha_{\mathtt{d}}}\,\|F - F_h\|_{\mathcal{K}'_{1,h}}\,. \end{aligned} \tag{2.13}$$

Therefore, as a consequence of the triangle inequality and (2.13) we get

$$\begin{aligned} \|\sigma_h - \tau_h\| \,&\le\, \frac{\|\mathcal{A}\|}{\alpha_{\mathtt{d}}}\,\|\sigma - \tau_h\| + \frac{1}{\alpha_{\mathtt{d}}}\,\|(a-a_h)(\tau_h,\cdot)\|_{\mathcal{K}_{1,h}'} + \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right)\,\|\widetilde{\tau}_h\| \\ &+\, \frac{\|\mathcal{B}_1\|}{\alpha_{\mathtt{d}}}\,\|u - v_h\| + \frac{1}{\alpha_{\mathtt{d}}}\,\|(b_1 - b_{1,h})(\cdot, v_h)\|_{\mathcal{K}_{1,h}'} + \frac{1}{\alpha_{\mathtt{d}}}\,\|F - F_h\|_{\mathcal{K}_{1,h}'}\,, \end{aligned} \tag{2.14}$$

which, along with the estimate (2.10), leads to (2.1) and concludes the proof.

□

Lemma 2.2. *Assume the hypotheses of Theorems* 1.1 *and* 1.2*, and let* $(\sigma, u) \in X_2 \times M_1$ *and* $(\sigma_h, u_h) \in X_{2,h} \times M_{1,h}$ *be the unique solutions of* (1.1) *and* (1.5)*, respectively. Then, denoting*

$$C_{\mathtt{d}} \,:=\, \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right)\,\frac{1}{\beta_{1,\mathtt{d}}}\,, \tag{2.15}$$

for each $(\tau_h, v_h) \in X_{2,h} \times M_{1,h}$ *there holds*

$$\begin{aligned} \|u_h - v_h\| \,&\le\, C_{\mathtt{d}}\,\left(\|\mathcal{A}\| + \frac{\|\mathcal{A}_h\|\,\|\mathcal{B}_2\|}{\beta_{2,\mathtt{d}}}\right)\,\|\sigma - \tau_h\| \\ &+\, C_{\mathtt{d}}\,\left(\frac{\|\mathcal{A}_h\|}{\beta_{2,\mathtt{d}}}\,\|(b_2 - b_{2,h})(\tau_h,\cdot)\|_{M_{2,h}'} \,+\, \|(a-a_h)(\tau_h,\cdot)\|_{X_{1,h}'}\right) \\ &+\, C_{\mathtt{d}}\,\Big(\|\mathcal{B}_1\|\,\|u - v_h\| \,+\, \|(b_1 - b_{1,h})(\cdot, v_h)\|_{X_{1,h}'}\Big) \\ &+\, C_{\mathtt{d}}\,\left(\|F - F_h\|_{X_{1,h}'} \,+\, \frac{\|\mathcal{A}_h\|}{\beta_{2,\mathtt{d}}}\,\|G - G_h\|_{M_{2,h}'}\right)\,, \end{aligned} \tag{2.16}$$

where the consistency terms in (2.16) *are defined as in the statement of Lemma* 2.1 (*cf.* (2.2) - (2.6)), *except that in* (2.3) - (2.5) $\mathcal{K}_{1,h}$ *is replaced by* $X_{1,h}$.

Proof. We first observe from the inf-sup condition for $b_{1,h}$ (cf. ii) in Theorem 1.2) that

$$\beta_{1,\mathtt{d}}\,\|u_h - v_h\| \,\le\, \sup_{\substack{\zeta_h \in X_{1,h} \\ \zeta_h \neq 0}} \frac{b_{1,h}(\zeta_h, u_h - v_h)}{\|\zeta_h\|}\,. \tag{2.17}$$

Then, according to the first equations of (1.5) and (1.1), we obtain

$$\begin{aligned} b_{1,h}(\zeta_h, u_h - v_h) \,&=\, F_h(\zeta_h) - a_h(\sigma_h,\zeta_h) - b_{1,h}(\zeta_h, v_h) \\ =\, \big(F_h - F\big)(\zeta_h) &+ a(\sigma,\zeta_h) + b_1(\zeta_h, u) - a_h(\sigma_h,\zeta_h) - b_{1,h}(\zeta_h, v_h)\,, \end{aligned}$$

from which, subtracting and adding $\tau_h \in X_{2,h}$ (resp. $v_h \in M_{1,h}$) in the first component of $a(\sigma,\zeta_h)$ (resp. second component of $b_1(\zeta_h, u)$), we get

$$\begin{aligned} b_{1,h}(\zeta_h, u_h - v_h) \,&=\, \big(F_h - F\big)(\zeta_h) + a(\sigma - \tau_h,\zeta_h) + (a - a_h)(\tau_h,\zeta_h) \\ &-\, a_h(\sigma_h - \tau_h,\zeta_h) + b_1(\zeta_h, u - v_h) + (b_1 - b_{1,h})(\zeta_h, v_h)\,. \end{aligned} \tag{2.18}$$

Thus, replacing (2.18) back into (2.17), and proceeding similarly to the proof of Lemma 2.1, we readily deduce that

$$\begin{aligned} \|u_h - v_h\| \,\le\, & \frac{\|\mathcal{A}\|}{\beta_{1,\mathtt{d}}}\,\|\sigma - \tau_h\| \,+\, \frac{1}{\beta_{1,\mathtt{d}}}\,\|(a-a_h)(\tau_h,\cdot)\|_{X'_{1,h}} \,+\, \frac{\|\mathcal{A}_h\|}{\beta_{1,\mathtt{d}}}\,\|\sigma_h - \tau_h\| \\ & +\, \frac{\|\mathcal{B}_1\|}{\beta_{1,\mathtt{d}}}\,\|u - v_h\| \,+\, \frac{1}{\beta_{1,\mathtt{d}}}\,\|(b_1 - b_{1,h})(\cdot, v_h)\|_{X'_{1,h}} \,+\, \frac{1}{\beta_{1,\mathtt{d}}}\,\|F - F_h\|_{X'_{1,h}} \,. \end{aligned} \tag{2.19}$$

Finally, employing (2.1) (cf. Lemma 2.1) in the foregoing inequality, performing minor algebraic manipulations, and bounding $\|(a-a_h)(\tau_h,\cdot)\|_{\mathcal{K}'_{1,h}}$, $\|(b_1 - b_{1,h})(\cdot, v_h)\|_{\mathcal{K}'_{1,h}}$, and $\|F - F_h\|_{\mathcal{K}'_{1,h}}$, by the terms $\|(a-a_h)(\tau_h,\cdot)\|_{X'_{1,h}}$, $\|(b_1 - b_{1,h})(\cdot, v_h)\|_{X'_{1,h}}$, and $\|F - F_h\|_{X'_{1,h}}$, respectively, we arrive at (2.16) and conclude the proof.

□

We are now in position to state the main result of this section.

Theorem 2.1. *Assume the hypotheses of Theorems* 1.1 *and* 1.2*, and let* $(\sigma, u) \in X_2 \times M_1$ *and* $(\sigma_h, u_h) \in X_{2,h} \times M_{1,h}$ *be the unique solutions of* (1.1) *and* (1.5)*, respectively. Then, there holds*

$$\begin{aligned} \|\sigma - \sigma_h\| \le & \inf_{\tau_h \in X_{2,h}} \left\{ \left(1 + \frac{\|\mathcal{A}\|}{\alpha_{\mathtt{d}}} + \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}} \right) \frac{\|\mathcal{B}_2\|}{\beta_{2,\mathtt{d}}} \right) \|\sigma - \tau_h\| \right. \\ & + \left. \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}} \right) \frac{1}{\beta_{2,\mathtt{d}}} \|(b_2 - b_{2,h})(\tau_h,\cdot)\|_{M'_{2,h}} + \frac{1}{\alpha_{\mathtt{d}}}\,\|(a-a_h)(\tau_h,\cdot)\|_{\mathcal{K}'_{1,h}} \right\} \\ & + \inf_{v_h \in M_{1,h}} \left\{ \frac{\|\mathcal{B}_1\|}{\alpha_{\mathtt{d}}}\,\|u - v_h\| + \frac{1}{\alpha_{\mathtt{d}}}\,\|(b_1 - b_{1,h})(\cdot, v_h)\|_{\mathcal{K}'_{1,h}} \right\} \\ & + \frac{1}{\alpha_{\mathtt{d}}}\,\|F - F_h\|_{\mathcal{K}'_{1,h}} + \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}} \right) \frac{1}{\beta_{2,\mathtt{d}}}\,\|G - G_h\|_{M'_{2,h}} \,, \end{aligned} \tag{2.20}$$

and, with the constant $C_{\mathtt{d}}$ *from* (2.15)*,*

$$\begin{aligned} \|u - u_h\| \,\le\, C_{\mathtt{d}} & \inf_{\tau_h \in X_{2,h}} \left\{ \left(\|\mathcal{A}\| + \frac{\|\mathcal{A}_h\|\,\|\mathcal{B}_2\|}{\beta_{2,\mathtt{d}}} \right) \|\sigma - \tau_h\| \right. \\ & + \left. \frac{\|\mathcal{A}_h\|}{\beta_{2,\mathtt{d}}}\,\|(b_2 - b_{2,h})(\tau_h,\cdot)\|_{M'_{2,h}} \,+\, \|(a-a_h)(\tau_h,\cdot)\|_{X'_{1,h}} \right\} \\ & + \inf_{v_h \in M_{1,h}} \left\{ \left(1 + C_{\mathtt{d}}\,\|\mathcal{B}_1\| \right) \|u - v_h\| \,+\, C_{\mathtt{d}}\,\|(b_1 - b_{1,h})(\cdot, v_h)\|_{X'_{1,h}} \right\} \\ & + C_{\mathtt{d}} \left(\|F - F_h\|_{X'_{1,h}} \,+\, \frac{\|\mathcal{A}_h\|}{\beta_{2,\mathtt{d}}}\,\|G - G_h\|_{M'_{2,h}} \right) \,, \end{aligned} \tag{2.21}$$

Proof. Applying the triangle inequality we have

$$\|\sigma - \sigma_h\| \,\le\, \|\sigma - \tau_h\| \,+\, \|\sigma_h - \tau_h\| \qquad \forall\, \tau_h \in X_{2,h} \,,$$

and

$$\|u - u_h\| \,\le\, \|u - v_h\| \,+\, \|u_h - v_h\| \qquad \forall\, v_h \in M_{1,h} \,,$$

which, along with the estimates for $\|\sigma_h - \tau_h\|$ and $\|u_h - v_h\|$ provided by (2.1) and (2.16), respectively, and after taking infimum with respect to both $\tau_h \in X_{2,h}$ and $v_h \in M_{1,h}$, leads to (2.20) and (2.21).

□

As announced within the proof of Lemma 2.1, we now stress that when $\mathcal{K}_{1,h} \subseteq \mathcal{K}_1$, the Strang estimate (2.20) reduces to

$$\begin{aligned}
\|\sigma - \sigma_h\| \le \inf_{\tau_h \in X_{2,h}} \Bigg\{ & \left(1 + \frac{\|\mathcal{A}\|}{\alpha_{\mathtt{d}}} + \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right) \frac{\|\mathcal{B}_2\|}{\beta_{2,\mathtt{d}}}\right) \|\sigma - \tau_h\| \\
+ & \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right) \frac{1}{\beta_{2,\mathtt{d}}} \|(b_2 - b_{2,h})(\tau_h, \cdot)\|_{M'_{2,h}} + \frac{1}{\alpha_{\mathtt{d}}} \|(a - a_h)(\tau_h, \cdot)\|_{\mathcal{K}'_{1,h}} \Bigg\} \\
+ & \frac{1}{\alpha_{\mathtt{d}}} \|F - F_h\|_{\mathcal{K}'_{1,h}} + \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right) \frac{1}{\beta_{2,\mathtt{d}}} \|G - G_h\|_{M'_{2,h}} ,
\end{aligned} \tag{2.22}$$

whereas (2.21) remains unaltered.

On the other hand, a significant simplification arises when $a_h(\sigma, \zeta_h)$, $b_{1,h}(\zeta_h, u)$, and $b_{2,h}(\sigma, w_h)$ are computable for all $(\zeta_h, w_h) \in X_{1,h} \times M_{2,h}$. Indeed, in this case it is possible to show that, instead of (2.10), (2.13), and (2.19), there hold, respectively

$$\|\widetilde{\tau}_h\| \le \frac{\|\mathcal{B}_{2,h}\|}{\beta_{2,\mathtt{d}}} \|\sigma - \tau_h\| + \frac{1}{\beta_{2,\mathtt{d}}} \|(b_2 - b_{2,h})(\sigma, \cdot)\|_{M'_{2,h}} + \frac{1}{\beta_{2,\mathtt{d}}} \|G - G_h\|_{M'_{2,h}} , \tag{2.23}$$

$$\begin{aligned}
\|\sigma_h - (\tau_h - \widetilde{\tau}_h)\| \le & \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}} \|\sigma - \tau_h\| + \frac{1}{\alpha_{\mathtt{d}}} \|(a - a_h)(\sigma, \cdot)\|_{\mathcal{K}'_{1,h}} + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}} \|\widetilde{\tau}_h\| \\
+ & \frac{\|\mathcal{B}_{1,h}\|}{\alpha_{\mathtt{d}}} \|u - v_h\| + \frac{1}{\alpha_{\mathtt{d}}} \|(b_1 - b_{1,h})(\cdot, u)\|_{\mathcal{K}'_{1,h}} + \frac{1}{\alpha_{\mathtt{d}}} \|F - F_h\|_{\mathcal{K}'_{1,h}} ,
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
\|u_h - v_h\| \le & \frac{\|\mathcal{A}_h\|}{\beta_{1,\mathtt{d}}} \|\sigma - \tau_h\| + \frac{1}{\beta_{1,\mathtt{d}}} \|(a - a_h)(\sigma, \cdot)\|_{X'_{1,h}} + \frac{\|\mathcal{A}_h\|}{\beta_{1,\mathtt{d}}} \|\sigma_h - \tau_h\| \\
+ & \frac{\|\mathcal{B}_{1,h}\|}{\beta_{1,\mathtt{d}}} \|u - v_h\| + \frac{1}{\beta_{1,\mathtt{d}}} \|(b_1 - b_{1,h})(\cdot, u)\|_{X'_{1,h}} + \frac{1}{\beta_{1,\mathtt{d}}} \|F - F_h\|_{X'_{1,h}} .
\end{aligned} \tag{2.25}$$

As a consequence of the foregoing estimates we obtain the following result, in which the new consistency terms appear separately from the respective infima, and hence can be handled independently from them.

Theorem 2.2. *Assume the hypotheses of Theorems* 1.1 *and* 1.2*, and let* $(\sigma, u) \in X_2 \times M_1$ *and* $(\sigma_h, u_h) \in X_{2,h} \times M_{1,h}$ *be the unique solutions of* (1.1) *and* (1.5)*, respectively. In addition, assume that* $a_h(\sigma, \zeta_h)$*,* $b_{1,h}(\zeta_h, u)$*, and* $b_{2,h}(\sigma, w_h)$ *are computable for all* $(\zeta_h, w_h) \in X_{1,h} \times M_{2,h}$*. Then, there holds*

$$\begin{aligned}
\|\sigma - \sigma_h\| \le & \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right) \left(1 + \frac{\|\mathcal{B}_{2,h}\|}{\beta_{2,\mathtt{d}}}\right) \operatorname{dist}(\sigma, X_{2,h}) \\
+ & \frac{\|\mathcal{B}_{1,h}\|}{\alpha_{\mathtt{d}}} \operatorname{dist}(u, M_{1,h}) + \frac{1}{\alpha_{\mathtt{d}}} \|(b_1 - b_{1,h})(\cdot, u)\|_{\mathcal{K}'_{1,h}} \\
+ & \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right) \frac{1}{\beta_{2,\mathtt{d}}} \|(b_2 - b_{2,h})(\sigma, \cdot)\|_{M'_{2,h}} + \frac{1}{\alpha_{\mathtt{d}}} \|(a - a_h)(\sigma, \cdot)\|_{\mathcal{K}'_{1,h}} \\
+ & \frac{1}{\alpha_{\mathtt{d}}} \|F - F_h\|_{\mathcal{K}'_{1,h}} + \left(1 + \frac{\|\mathcal{A}_h\|}{\alpha_{\mathtt{d}}}\right) \frac{1}{\beta_{2,\mathtt{d}}} \|G - G_h\|_{M'_{2,h}} ,
\end{aligned} \tag{2.26}$$

and

$$
\begin{aligned}
\|u - u_h\| \;\le\; & C_{\mathtt{d}}\, \|\mathcal{A}_h\| \left(1 + \frac{\|\mathcal{B}_{2,h}\|}{\beta_{2,\mathtt{d}}}\right) \operatorname{dist}\big(\sigma, X_{2,h}\big) \\
& + \Big(1 + C_{\mathtt{d}}\, \|\mathcal{B}_{1,h}\|\Big) \operatorname{dist}\big(u, M_{1,h}\big) \;+\; C_{\mathtt{d}}\, \|(b_1 - b_{1,h})(\cdot, u)\|_{X'_{1,h}} \\
& + C_{\mathtt{d}} \left(\frac{\|\mathcal{A}_h\|}{\beta_{2,\mathtt{d}}}\, \|(b_2 - b_{2,h})(\sigma, \cdot)\|_{M'_{2,h}} \;+\; \|(a - a_h)(\sigma, \cdot)\|_{X'_{1,h}} \right) \\
& + C_{\mathtt{d}} \left(\|F - F_h\|_{X'_{1,h}} \;+\; \frac{\|\mathcal{A}_h\|}{\beta_{2,\mathtt{d}}}\, \|G - G_h\|_{M'_{2,h}} \right) .
\end{aligned} \tag{2.27}
$$

Proof. It follows analogously to the proof of Theorem 2.1, by utilizing now the estimates for $\|\sigma_h - \tau_h\|$ and $\|u_h - v_h\|$ arising from (2.23), (2.24), and (2.25). $\square$

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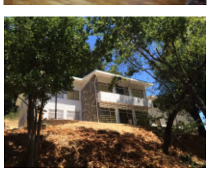

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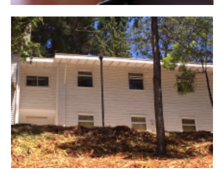

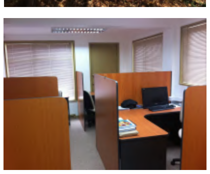

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