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An a posteriori error estimator for an augmented variational formulation of the Brinkman problem with mixed boundary conditions and non-null source terms

Dedicated to Prof. R. Rodríguez, on the occasion of his 70th birthday

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Abstract

The aim of this work is the development of an a posteriori error analysis for the Brinkman problem with non-homogeneous mixed boundary conditions. In order to clarify the analysis, we first study, for simplicity, the model problem with null mixed boundary conditions. Then, we derive a suitable augmented variational formulation, based on the pseudo-stress and the velocity unknowns. This process involves the elimination of the pressure, which can be recovered once the system is solved. Applying known arguments, we can prove the unique solvability of the referred formulation, as well as of the corresponding Galerkin scheme. Moreover, we can establish the convergence of the method, when consider row-wise Raviart-Thomas elements to approximate the pseudo-stress in $H(\mathbf{div};\Omega)$, and continuous piecewise polynomials for the velocity. Then, we proceed to deduce an a posteriori error estimator, which results to be reliable and local efficient. To obtain this, we basically take into account the ellipticity of the bilinear form that defines our scheme. Since we do not require any type of Helmholtz decomposition of functions living in $H(\mathbf{div};\Omega)$, the analysis is valid for 2D and 3D. The novelty of the current work relies on how we deal with the case we have non homogeneous mixed boundary conditions. The strategy is to perform first a suitable lifting for the Neumann and Dirichlet data, respectively, in order to homogenize them, and apply the procedure introduced at the beginning. Then, we also can establish the well posedness of the augmented variational formulation, at continuous and discrete levels, as well as the convergence of the method and the derivation of an a posteriori error estimator. We point out that in this case, the corresponding estimator consists of two residual terms, and two oscillation terms related to the boundary data, which are not present when the boundary data are piecewise polynomials. In that situation, the a posteriori error estimator results to be reliable, and locally efficient. We include some numerical experiments, which are in agreement with the theoretical results we have obtained here.

Mathematics Subject Classifications (1991): 65N15, 65N30, 65N50

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1 Introduction

When one deals with a fluid, confined in a region, that shares viscous and porous media, it is usual to consider the so called linear Brinkman problem. This lets us to study the fluid's behavior, from Darcy to Stokes regime, by introducing a zeroth order term for the velocity in the corresponding momentum equation. This kind of equation can be also derived after discretization in time the non-steady Stokes problem. In this framework, it is known as the generalized Stokes problem.

These facts have motivated the design of several numerical methods for solving these equations (see for e.g. [3, 4, 16, 18, 29, 31] and the references therin), with applications in oil recovery, filtering porous layers, subsurface water treatment, etc. In all these papers, the corresponding variational formulation is based on the natural unknowns: velocity and pressure.

In what follows, we focus our attention on solving the generalized Stokes problem with non-homogeneous Dirichlet condition, by designing mixed finite element schemes, based on the introduction of suitable additional unknowns. For example, in [15] the authors propose a two-fold saddle point formulation in terms of the pseudostress and the gradient of the velocity as additional unknowns. Years later, the same problem is treated in [10], by proposing and studying a variational formulation in pseudostress and velocity only, following the ideas given in [17]. The corresponding a posteriori error analysis has been described in [8], establishing the reliability and local efficiency of the estimator. This is proved with the help of a suitable quasi-Helmholtz decomposition of functions belonging to $H(\mathrm{div})$.

With the purpose of circumventing the inf-sup condition, which usually enforces a restriction on the choice of the approximation spaces, the mixed formulation is augmented with kind of least-squares terms. This leads to the well known *Augmented Mixed Finite Element Methods*, which gives us more freedom to choose the discrete spaces. Moreover, this procedure in general is supported by the fact that the velocity unknown, seeking in L^2 in the non-augmented mixed variational formulation, belongs indeed to H^1. Then, in [10] the authors take into account this and propose an augmented variational formulation based on pseudostress-velocity, but with the velocity belonging to the functional space H^1.

On the other hand, in [33] the authors propose a mixed variational formulation for the generalized Stokes problem, in velocity, vorticity and pressure, considering only pure Dirichlet or pure Neumann boundary conditions. Inspired in this work, in [1] the authors develop an augmented variational formulation based on these three fields, for the Brinkman problem with homogeneous data for the vorticity and normal component of the velocity in some part of the boundary of the domain, and null pressure and given tangential component of the velocity on the other part of that boundary. The corresponding analysis includes the derivation of an a posteriori error estimate, which results to be reliable and locally efficient, with respect to the natural norms. Additionally, in [2] the authors present a non-augmented mixed variational formulation for the Brinkman problem, but measuring the vorticity in a norm that depends on the viscosity constant.

In this article, we are interested in the linear Brinkman problem, formulated in pseudostress-velocity, and subject to mixed boundary conditions. We can mention that in [26], the authors deal with this kind of problem. After eliminating the pressure and the velocity, they derive a non-augmented mixed variational formulation, based on pseudostress and the trace of velocity on Neumann boundary. This formulation results to be well-posed, at continuous as well as discrete levels. However, the proof of the convergence of the method is just available for 2D, since it needs a suitable independent mesh of the Neumann boundary and to the best of our knowledge, there is no criterion for building this independent mesh for higher dimensions. At least this could be very expensive to perform for the 3D case. Our proposal consists of a homogenization of the Neumann datum first, which allows us not to introduce any independent mesh of the Neumann boundary. As a result, the convergence of the method is established in 2D and 3D.

The outline of the paper is organized as follows. In Section 2, we introduce the model problem, and discuss the well-posedness of the mixed variational formulation in pseudostress-velocity, at continuous and discrete levels. Next, we propose an augmented formulation in Section 3, considering for simplicity homogeneous mixed boundary conditions. As a remark, we describe what happens in a more general case. In Section 4, some numerical examples are included, whose results are in agreement with the theory we have developed. Finally, we give some conclusions and final comments in Section 6.

We end this section with some notations to be used throughout the paper. Given a Hilbert space W, we denote by W^n and $W^{n\times n}$ the space of vectors and square tensors of order n with entries in W, respectively, where $n \in \{2,3\}$. Given $\boldsymbol{\rho} := (\rho_{ij})\,, \boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{n\times n}$, we set $\boldsymbol{\rho}^{\mathtt{t}} := (\rho_{ji})$, $\mathrm{tr}(\boldsymbol{\rho}) := \sum_{j=1}^{n} \rho_{jj}$, and $\boldsymbol{\rho} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \rho_{ij}\zeta_{ij}$. Moreover, we introduce the deviator of $\boldsymbol{\rho}$ by $\boldsymbol{\rho}^{\mathtt{d}} := \boldsymbol{\rho} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\rho})\,\boldsymbol{I}$, with $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ being the identity tensor.

2 The model problem and its mixed form

Let $\Omega \subset \mathbb{R}^n$ be a bounded and simply connected domain with a Lipschitz-continuous boundary $\Gamma := \partial\Omega$, which is decomposed in two disjoint parts: Γ_D and Γ_N, with $|\Gamma_D|, |\Gamma_N| > 0$. In order to clarify the presentation, we first consider the case with both boundary conditions homogeneous. Later we focus on studying the most general case, taking into account the analysis that will be studied in this section. Then, given the external forces $\boldsymbol{f} \in [L^2(\Omega)]^n$, $\tilde{f} \in L^2(\Omega)$, we look for the velocity $\boldsymbol{u}$ and the pressure p of a fluid in Ω, such that

$$\begin{aligned} \alpha\,\boldsymbol{u} - \nu\,\boldsymbol{\Delta}\,\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in} \quad \Omega\,, \\ \mathrm{div}(\boldsymbol{u}) &= \tilde{f} \quad \text{in} \quad \Omega\,, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on} \quad \Gamma_D\,, \\ \Big(-\nu\,\nabla\boldsymbol{u} + p\,\boldsymbol{I}\Big)\,\boldsymbol{n} &= \boldsymbol{0} \quad \text{on} \quad \Gamma_N\,, \end{aligned} \tag{1}$$

where α is a positive parameter, while $\nu > 0$ corresponds to the kinematic viscosity of the fluid. Hereafter, $\boldsymbol{n}$ denotes the unit outward normal to Γ.

We introduce now the pseudo-stress $\boldsymbol{\sigma} := \nu \nabla \boldsymbol{u} - p \boldsymbol{I}$, which allows us to deduce $p = \frac{\nu}{n}\tilde{f} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\sigma})$. As a consequence, we can eliminate p from (1), leading to the following first order system in $(\boldsymbol{\sigma}, \boldsymbol{u})$:

$$\begin{cases} \dfrac{1}{\nu}\boldsymbol{\sigma}^{\mathrm{d}} - \nabla \boldsymbol{u} = -\dfrac{1}{n}\tilde{f}\boldsymbol{I} & \text{in} \quad \Omega, \\ \alpha \boldsymbol{u} - \mathbf{div}(\boldsymbol{\sigma}) = \boldsymbol{f} & \text{in} \quad \Omega, \\ \boldsymbol{u} = \mathbf{0} & \text{on} \quad \Gamma_D, \\ \boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0} & \text{on} \quad \Gamma_N. \end{cases} \tag{2}$$

Next, we consider the mixed variational formulation: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in H_{\mathtt{N}}(\mathbf{div};\Omega) \times [L^2(\Omega)]^n$ such that

$$\begin{cases} \dfrac{1}{\nu}\displaystyle\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = -\dfrac{1}{n}\int_\Omega \tilde{f}\,\mathrm{tr}(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in H_{\mathtt{N}}(\mathbf{div};\Omega), \\ \displaystyle\int_\Omega \boldsymbol{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) - \alpha\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} = -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} & \forall\, \boldsymbol{v} \in [L^2(\Omega)]^n, \end{cases} \tag{3}$$

where $H_{\mathtt{N}}(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \boldsymbol{\tau}\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_N\}$.

Remark 1 *Thanks to Theorem 4.3.2 in [12], we can ensure the unique solvability of* (3)*. By choosing appropriate stable pair of discrete subspaces of* $H_{\mathtt{N}}(\mathbf{div};\Omega)$ *and* $[L^2(\Omega)]^n$*, we can invoke Theorem 5.5.2 in [12], and establish that the corresponding discrete scheme to* (3) *has also a unique solution.*

Remark 2 *Adapting the ideas given in [7], we can obtain a reliable and efficient a posteriori error estimator. For establishing the local efficiency property, we invoke a quasi Helmholtz decomposition of functions living in* $H_{\mathtt{N}}(\mathbf{div};\Omega)$*, whose stability property is ensured only in 2D. In order to circumvent this difficulty, in the next section we proceed to stabilize the scheme.*

Remark 3 *The theory of distributions allows us to show that* (3) *implies that the velocity* $\boldsymbol{u}$ *belongs to* $[H^1(\Omega)]^n$*. This motivates us to apply a stabilization technique, by summing suitable terms of least squares type to the variational formulation* (3)*. The resulting augmented scheme is described in next section.*

3 An augmented formulation

At this point, we define the space $[H^1_{\Gamma_D}(\Omega)]^n := \{v \in [H^1(\Omega)]^n; \quad v = 0 \quad \text{on} \quad \Gamma_D\}$ and we denote $\boldsymbol{H} := H(\mathbf{div};\Omega) \times [H^1(\Omega)]^n$. We provide $\boldsymbol{H}$ with the usual norm

$$||(\boldsymbol{\tau}, \boldsymbol{v})||_{\boldsymbol{H}} := (||\boldsymbol{\tau}||^2_{\mathrm{div};\Omega} + ||\boldsymbol{v}||^2_{1,\Omega})^{1/2} \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}.$$

Now, following the ideas given in Section 3 in [5], it is not difficult to derive the augmented variational formulation, at continuous level: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}_0 := H_{\mathtt{N}}(\mathbf{div};\Omega) \times [H^1_{\Gamma_D}(\Omega)]^n$, such that

$$A_{\mathtt{s}}((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) = F_{\mathtt{s}}(\boldsymbol{\tau}, \boldsymbol{v}) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}_0, \tag{4}$$

where the bilinear form $A_{\mathtt{s}} : \boldsymbol{H} \times \boldsymbol{H} \to \mathbb{R}$ and the linear functional $F_{\mathtt{s}} : \boldsymbol{H} \to \mathbb{R}$, are given by

$$\begin{aligned} A_{\mathtt{s}}((\boldsymbol{\theta}, \boldsymbol{z}), (\boldsymbol{\tau}, \boldsymbol{v})) &:= \frac{1}{\nu} \int_\Omega \boldsymbol{\theta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \int_\Omega \boldsymbol{z} \cdot \mathbf{div}(\boldsymbol{\tau}) - \int_\Omega \boldsymbol{v} \cdot \mathbf{div}(\boldsymbol{\theta}) + \alpha \int_\Omega \boldsymbol{z} \cdot \boldsymbol{v} \\ &\quad + \kappa_0 \int_\Omega (\mathbf{div}(\boldsymbol{\theta}) - \alpha\, \boldsymbol{z}) \cdot (\mathbf{div}(\boldsymbol{\tau}) + \alpha\, \boldsymbol{v}) + \kappa_1 \int_\Omega \left(\nabla \boldsymbol{z} - \frac{1}{\nu} \boldsymbol{\theta}^{\mathtt{d}} \right) : \left(\nabla \boldsymbol{v} + \frac{1}{\nu} \boldsymbol{\tau}^{\mathtt{d}} \right), \\ F_{\mathtt{s}}(\boldsymbol{\tau}, \boldsymbol{v}) &:= -\frac{1}{n} \int_\Omega \tilde{f}\, \boldsymbol{I} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} - \kappa_0 \int_\Omega \boldsymbol{f} \cdot (\mathbf{div}(\boldsymbol{\tau}) + \alpha\, \boldsymbol{v}) \\ &\quad + \frac{\kappa_1}{n} \int_\Omega \tilde{f}\, \boldsymbol{I} : \left(\nabla \boldsymbol{v} + \frac{1}{\nu} \boldsymbol{\tau}^{\mathtt{d}} \right). \end{aligned}$$

We remark that the positive parameters κ_0, κ_1 are at our disposal, to ensure the unique solvability of (4). In the current situation, we choose them in order to establish the ellipticity of $A_{\mathtt{s}}$ on $\boldsymbol{H}_0$. Keeping that in mind, we can obtain, after some algebraic manipulations, that for any $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}$ there holds

$$A_{\mathtt{s}}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{v})) = \nu^{-1} (1 - \kappa_1 \nu^{-1}) \, ||\boldsymbol{\tau}^{\mathtt{d}}||_{0,\Omega}^2 + \kappa_0 \, ||\mathbf{div}(\boldsymbol{\tau})||_{0,\Omega}^2 + \alpha\, (1 - \alpha\, \kappa_0) \, ||\boldsymbol{v}||_{0,\Omega}^2 + \kappa_1 \, |\boldsymbol{v}|_{1,\Omega}^2 . \quad (5)$$

As $\kappa_0, \kappa_1 > 0$, we just require that $1 - \kappa_1 \nu^{-1} > 0$ and $1 - \alpha\, \kappa_0 > 0$, for ensuring that each of the four terms on the right hand side of (5) is positive. On the other hand, we point out that any $\boldsymbol{\tau} \in H_{\mathtt{N}}(\mathbf{div}; \Omega)$, can be rewritten as $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \lambda \boldsymbol{I}$, where $\boldsymbol{\tau}_0 \in H_0(\mathbf{div}; \Omega) := \{\boldsymbol{\rho} \in H(\mathbf{div}; \Omega) : \int_\Omega \mathrm{tr}(\boldsymbol{\rho}) = 0\}$, and $\lambda \in \mathbb{R}$ is chosen such that $\boldsymbol{\tau}\, \boldsymbol{n} = \mathbf{0}$ on Γ_N. Now, in order to give a self-contained manuscript, we recall the following technical result, whose proof can be found in [25] (cf. Lemma 2.2).

Lemma 4 *There exists $C_2 > 0$, depending only on Ω and Γ_N, such that*

$$\forall\, \boldsymbol{\tau} \in H_N(\mathbf{div}; \Omega) : C_2 \, ||\boldsymbol{\tau}||_{\mathrm{div};\Omega} \leq ||\boldsymbol{\tau}_0||_{\mathrm{div};\Omega} . \quad (6)$$

Boundedness and coercivity of $A_{\mathtt{s}}$ on $\boldsymbol{H}_0$ are established in next result.

Lemma 5 *For any $\kappa_0 \in (0, \alpha^{-1})$ and $\kappa_1 \in (0, \nu)$, there exists $C_{\mathtt{b}} > 0$, $C_{\mathtt{ell}} > 0$, depending on α and ν, such that*

$$|A_{\mathtt{s}}((\boldsymbol{\theta}, \boldsymbol{z}), (\boldsymbol{\tau}, \boldsymbol{v}))| \leq C_{\mathtt{b}}(\alpha, \nu) \, ||(\boldsymbol{\theta}, \boldsymbol{z})||_{\boldsymbol{H}} \, ||(\boldsymbol{\tau}, \boldsymbol{v})||_{\boldsymbol{H}} \quad \forall\, (\boldsymbol{\theta}, \boldsymbol{z}), (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H} , \quad (7)$$

$$A_{\mathtt{s}}((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{v})) \geq C_{\mathtt{ell}}(\alpha, \nu) \, ||(\boldsymbol{\tau}, \boldsymbol{v})||_{\boldsymbol{H}}^2 \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}_0 , \quad (8)$$

where

$$\begin{aligned} C_{\mathtt{b}}(\alpha, \nu) &:= \max\{\nu^{-1}(1 - \kappa_1 \nu^{-1}) + \kappa_0, 1 - \alpha\kappa_0, \alpha(1 - \alpha\kappa_0) + \kappa_1, \kappa_1 \nu^{-1}\} , \\ C_{\mathtt{ell}}(\alpha, \nu) &:= \min\{1, C_2\} \, \min\{\nu^{-1}(1 - \kappa_1 \nu^{-1}), \kappa_0, \alpha(1 - \alpha\kappa_0), \kappa_1\} . \end{aligned}$$

Proof. It relies on Cauchy-Schwarz inequality and some algebraic manipulations. To establish the ellipticity of $A_{\mathtt{s}}$ on $\boldsymbol{H}_0$ we need, in addition, to invoke Lemma 4. We omit further details. □

Similarly, the following lemma ensures that $F_{\mathtt{s}} \in \boldsymbol{H}_0'$.

Lemma 6 *Under the same assumptions as in Lemma 5, there exists $C_{\mathtt{F}} > 0$, depending on α and ν, such that*

$$|F_{\mathtt{s}}(\boldsymbol{\tau}, \boldsymbol{v})| \,\leq C_{\mathtt{F}}(\alpha, \nu)\, (||\tilde{f}||_{0,\Omega} \,+\, ||\boldsymbol{f}||_{0,\Omega})\, ||(\boldsymbol{\tau}, \boldsymbol{v})||_{\boldsymbol{H}} \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in\, \boldsymbol{H}\,, \tag{9}$$

where

$$C_{\mathtt{F}}(\alpha, \nu) \,:=\, \max\{1, C_1\}\, \max\{\nu^{-1}(1 - \kappa_1 \nu^{-1}), \kappa_1 \nu^{-1}, 1 - \alpha\kappa_0, \kappa_0, \kappa_1, 1\}\,.$$

By Lax-Milgram theorem, it is proved that the augmented variational formulation (4) has a unique solution $(\boldsymbol{\sigma}, \boldsymbol{u}) \,\in\, \boldsymbol{H}_0$, and there exists a positive constant C_*, depending on α and ν, such that

$$\|(\boldsymbol{\sigma}, \boldsymbol{u})\|_{\boldsymbol{H}} \,\leq\, C_* \left(||\tilde{f}||_{0,\Omega} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\right)\,, \tag{10}$$

with $C_* \,:=\, \dfrac{C_{\mathtt{F}}(\alpha, \nu)}{C_{\mathtt{ell}}(\alpha, \nu)}$.

On the other hand, for the discretization, we assume that Ω is a polyhedral / polygonal region and let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ such that $\overline{\Omega} \,=\, \cup\{\, T \,:\, T \,\in\, \mathcal{T}_h\,\}$. Given a tetrahedra / triangle $T \,\in\, \mathcal{T}_h$, we denote by h_T its diameter and define the mesh size $h := \max\{\, h_T \,:\, T \in \mathcal{T}_h\,\}$. In addition, given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^n$, we denote by $\mathcal{P}_\ell(S)$ the space of polynomials in n variables defined in S of total degree at most ℓ, and for each $T \in \mathcal{T}_h$, we define the local Raviart-Thomas space of order ℓ, as $\mathcal{RT}_\ell(T) \,:=\, [\mathcal{P}_\ell(T)]^n \,\oplus\, \mathcal{P}_\ell(T)\, \boldsymbol{x} \,\subseteq\, [\mathcal{P}_{\ell+1}(T)]^n$, for all $\boldsymbol{x} \,\in\, T$. On the other hand, we set $\mathcal{F}_h$ as the skeleton of $\mathcal{T}_h$, i.e. the list of all the faces (counted once) induced by the triangulation $\mathcal{T}_h$. We can decomposed $\mathcal{F}_h \,=\, \mathcal{F}_h^{\mathtt{I}} \cup \mathcal{F}_h^{\partial}$, with $\mathcal{F}_h^{\mathtt{I}} \,:=\, \{F \,\in\, \mathcal{F}_h \,:\, F \,\subset\, \Omega\}$ and $\mathcal{F}_h^{\partial} \,:=\, \{F \,\in\, \mathcal{F}_h \,:\, F \,\subset\, \partial\Omega\}$. We have also that $\mathcal{F}_h^{\partial} \,:=\, \mathcal{F}_h^{\mathtt{D}} \cup \mathcal{F}_h^{\mathtt{N}}$, with $\mathcal{F}_h^{\mathtt{D}} := \{F \in \mathcal{F}_h^{\partial} \,:\, F \,\subset\, \Gamma_D\}$ and $\mathcal{F}_h^{\mathtt{N}}, := \{F \in \mathcal{F}_h^{\partial} \,:\, F \,\subset\, \Gamma_N\}$. As a consequence, given a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, its skeleton can be characterized as $\mathcal{F}_h \,=\, \mathcal{F}_h^{\mathtt{I}} \cup \mathcal{F}_h^{\mathtt{D}} \cup \mathcal{F}_h^{\mathtt{N}}$.

From now on, in order to define the discrete variational formulation associated to problem (4), we first introduce the finite element subspaces:

$$\begin{aligned}
H_h^{\boldsymbol{\sigma}} \,&:=\, \left\{\, \boldsymbol{\tau}_h \in H(\mathbf{div};\Omega) \,:\, \boldsymbol{\tau}_h|_T \,\in\, [\mathcal{RT}_\ell(T)^{\mathtt{t}}]^n\,, \quad \forall\, T \,\in\, \mathcal{T}_h \,\right\}\,,\\
H_{\mathtt{N},h}^{\boldsymbol{\sigma}} \,&:=\, \left\{\, \boldsymbol{\tau}_h \in H_h^{\boldsymbol{\sigma}} \,:\, \boldsymbol{\tau}_h\, \boldsymbol{n} \,=\, 0 \quad \text{on} \quad \mathcal{F}_h^{\mathtt{N}} \,\right\}\,,\\
H_h^{\boldsymbol{u}} \,&:=\, \left\{\, \boldsymbol{v}_h \,\in\, [\mathcal{C}(\overline{\Omega})]^n \,:\, \boldsymbol{v}_h|_T \,\in\, [\mathcal{P}_{\ell+1}(T)]^n \quad \forall\, T \,\in\, \mathcal{T}_h \right\}\,,\\
H_{0,h}^{\boldsymbol{u}} \,&:=\, \left\{\, \boldsymbol{v}_h \,\in\, H_h^{\boldsymbol{u}} \,: \quad \boldsymbol{v}_h \,=\, 0 \quad \text{on} \quad \mathcal{F}_h^{\mathtt{D}} \right\}\,.
\end{aligned}$$

Then, we propose the finite element subspace $\boldsymbol{H}_{0,h} \,:=\, H_{\mathtt{N},h}^{\boldsymbol{\sigma}} \times H_{0,h}^{\boldsymbol{u}} \,\subset\, \boldsymbol{H}_0$. As a result, we derive the conforming discrete variational formulation: *Find* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \,\in\, \boldsymbol{H}_{0,h}$ *such that*

$$A_{\mathtt{s}}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) \,=\, F_{\mathtt{s}}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \qquad \forall\, (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in\, \boldsymbol{H}_{0,h}\,. \tag{11}$$

We remark that the Galerkin scheme (11) is well-posed and a Céa's estimate can be obtained. In addition, the corresponding rate of convergence of the Galerkin scheme (11) for this particular choice of finite element subspaces, is presented in the next theorem.

Theorem 7 *Let $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}_{\mathbf{0}}$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{H}_{0,h}$ be the unique solutions to problems* (4) *and* (11)*, respectively. In addition, assume that $\boldsymbol{\sigma} \in [H^r(\Omega)]^{n\times n}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^r(\Omega)]^n$ and $\boldsymbol{u} \in [H^{r+1}(\Omega)]^n$ for some $r \in (0, \ell+1]$. Then, there exists $\tilde{C}_* > 0$, independent of h, but depending on C_* (introduced in* (10)*), such that there holds*

$$\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_{\boldsymbol{H}} \,\leq\, \tilde{C}_*\, h^r \,\left(\|\boldsymbol{\sigma}\|_{r,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{r,\Omega} \,+\, \|\boldsymbol{u}\|_{r+1,\Omega} \right) .$$

Proof. It is a consequence of Céa's estimate, and the corresponding approximation properties. We omit further details. □

4 An a posteriori error estimator

In the framework of a posteriori error analysis for the Brinkman model, we would like to comment on some previous works in this direction, which lead us to propose the new alternative described in this article. Specifically, we refer to [9] and [8]. In [9] an a posteriori error analysis based on the Ritz projection of the error is presented. There, in order to localize the $H^{1/2}(\Gamma)$-norm, the authors introduce an auxiliary discrete function and deduce an a posteriori error estimator that is reliable and locally quasi-efficient (see Theorem 3 in [9]). On the other hand, in [8] using a quasi Helmholtz decomposition, an estimator error estimator is obtained, which is reliable and local efficiency, circumventing the need of any auxiliary discrete function. Unfortunately, this decomposition is not available for 3D with mixed boundary conditions, which is one of the reason for presenting this alternative analysis.

The aim here is the derivation of a reliable and locally efficient, residual a posteriori error estimator. First, we take into account the ellipticity of $A_{\mathtt{s}}$ on $\boldsymbol{H}_0$, and establish that

$$C_{\mathtt{ell}}\, ||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h)||_{\boldsymbol{H}} \,\leq\, \sup_{(\boldsymbol{\tau},\boldsymbol{v})\in \boldsymbol{H}_0\backslash\{\mathbf{0}\}} \frac{A_{\mathtt{s}}((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h), (\boldsymbol{\tau}, \boldsymbol{v}))}{||(\boldsymbol{\tau}, \boldsymbol{v})||_{\boldsymbol{H}}} \,. \tag{12}$$

We emphasize that the right hand side of the inequality (12) corresponds to the norm of the Ritz projection of the error (for more details see Subsection 4.2 in [5]).

Our next goal is to bound the supremum in (12). To this end, we require the following result.

Lemma 8 *For any $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}_0$, there holds*

$$A_{\mathtt{s}}((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h), (\boldsymbol{\tau}, \boldsymbol{v})) \,=\, R_1(\boldsymbol{\tau}) \,+\, R_2(\boldsymbol{v})\,, \tag{13}$$

where

$$R_1(\boldsymbol{\tau}) \,:=\, (1 - \nu^{-1}\,\kappa_1) \int_\Omega \left(\nabla \boldsymbol{u}_h \,-\, \frac{1}{\nu} \boldsymbol{\sigma}_h^{\mathtt{d}} - \frac{1}{n} \tilde{f}\, \boldsymbol{I} \right) \,:\, \boldsymbol{\tau}$$
$$-\, \kappa_0 \int_\Omega (\boldsymbol{f} \,+\, \mathbf{div}(\boldsymbol{\sigma}_h) \,-\, \alpha\, \boldsymbol{u}_h) \cdot \mathbf{div}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in H_{\mathtt{N}}(\mathrm{div};\Omega)\,, \tag{14}$$

$$R_2(\boldsymbol{v}) \,:=\, (1 - \alpha\, \kappa_0) \int_\Omega (\boldsymbol{f} \,+\, \mathbf{div}(\boldsymbol{\sigma}_h) \,-\, \alpha\, \boldsymbol{u}_h) \cdot \boldsymbol{v}$$
$$+\, \kappa_1 \int_\Omega \left(\nabla \boldsymbol{u}_h \,-\, \frac{1}{\nu} \boldsymbol{\sigma}_h^{\mathtt{d}} - \frac{1}{n} \tilde{f}\, \boldsymbol{I} \right) \,:\, \nabla \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in [H^1_{\Gamma_D}(\Omega)]^n\,. \tag{15}$$

Proof. First, given $(\boldsymbol{\tau},\boldsymbol{v}) \in \boldsymbol{H}_0$, we invoke (4) to have

$$A_{\mathtt{s}}((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{u}-\boldsymbol{u}_h),(\boldsymbol{\tau},\boldsymbol{v})) \,=\, F_{\mathtt{s}}(\boldsymbol{\tau},\boldsymbol{v}) \,-\, A_{\mathtt{s}}((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\tau},\boldsymbol{v}))\,.$$

The rest of the proof relies on the definition of $F_{\mathtt{s}}$ and $A_{\mathtt{s}}$, and some algebraic manipulations. We omit further details. □

As an immediate consequence of Lemma 8, we deduce an a posteriori error estimator, which is given in the next theorem.

Theorem 9 *There exists $C_{\mathtt{rel}} > 0$, depending on α and ν, such that*

$$||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{u}-\boldsymbol{u}_h)||_{\boldsymbol{H}} \,\leq\, C_{\mathtt{rel}}\,\eta\,, \tag{16}$$

with $\eta := \left(\sum_{T\in\mathcal{T}_h}\eta_T^2\right)^{1/2}$*, where for any $T\in\mathcal{T}_h$*

$$\begin{aligned}\eta_T^2 \,:=\, & \max\{1-\alpha\,\kappa_0,\kappa_0\}^2\,||\boldsymbol{f} \,+\, \mathbf{div}(\boldsymbol{\sigma}_h) \,-\, \alpha\,\boldsymbol{u}_h||_{0,T}^2 \\ & +\,\max\{1-\nu^{-1}\,\kappa_1,\kappa_1\}^2\,\left\|\nabla\boldsymbol{u}_h \,-\, \frac{1}{\nu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \frac{1}{n}\tilde{f}\,\boldsymbol{I}\right\|_{0,T}^2\,.\end{aligned} \tag{17}$$

For proving the local efficiency of our estimator, we first introduce, for any $T\in\mathcal{T}_h$

$$||(\boldsymbol{\tau},\boldsymbol{v})||_{\boldsymbol{H}_T} \,:=\, \Big(||\boldsymbol{\tau}||_{H(\mathrm{div};T)}^2 \,+\, ||\boldsymbol{v}||_{1,T}^2\Big)^{1/2} \quad \forall\,(\boldsymbol{\tau},\boldsymbol{v})\in\boldsymbol{H}\,.$$

Theorem 10 *There exists $C_{\mathtt{eff}} > 0$, depending on α and ν, such that for any $T\in\mathcal{T}_h$*

$$\eta_T \,\leq\, C_{\mathtt{eff}}\,||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{u}-\boldsymbol{u}_h)||_{\boldsymbol{H}_T}\,.$$

Remark 11 *For $(\kappa_0,\kappa_1) := (\frac{1}{2\,\alpha},\frac{\nu}{2})$, we deduce*

$$\begin{cases} C_* \,=\, \tilde{C}_* \,=\, \mathcal{O}(\alpha^2)\,, \quad C_{\mathtt{rel}} \,=\, C_{\mathtt{eff}} \,=\, \mathcal{O}(\alpha) & \alpha \,\geq\, \nu \,\geq\, 1\,,\\ C_* \,=\, \tilde{C}_* \,=\, \mathcal{O}(\max\{\alpha,\nu^{-1}\}^2)\,, \quad C_{\mathtt{rel}} \,=\, C_{\mathtt{eff}} \,=\, \mathcal{O}(\max\{\alpha,\nu^{-1}\}) & \alpha \,>\, 1 \,>\, \nu\,.\end{cases}$$

5 An a posteriori error analysis for Brinkman model with inhomogeneous boundary data

In this section, we extend the analysis described in Sections 2 and 3 to a quite more general model problem (1), with nonhomogeneous boundary data. In order to do that, we consider the same geometry and notations as provided in Section 2. Next, we introduce the spaces $[H_{00}^{1/2}(\Gamma_D)]^n := \{v|_{\Gamma_D} \,:\, v\in[H_{\Gamma_N}^1(\Omega)]^n\}$ and $[H_{00}^{1/2}(\Gamma_N)]^n := \{v|_{\Gamma_N} \,:\, v\in[H_{\Gamma_D}^1(\Omega)]^n\}$. By $[H_{00}^{-1/2}(\Gamma_N)]^n$ we denote the dual space of $[H_{00}^{1/2}(\Gamma_N)]^n$, with respect to $L^2(\Gamma_N)$-inner product. Its norm will be represented by $||\cdot||_{-1/2,00,\Gamma_N}$.

Now, given $\boldsymbol{u}_D \in [H_{00}^{1/2}(\Gamma_D)]^n$, $\boldsymbol{g} \in [H_{00}^{-1/2}(\Gamma_N)]^n$, $\boldsymbol{f} \in [L^2(\Omega)]^n$, and $\tilde{f} \in L^2(\Omega)$, we seek for the velocity $\boldsymbol{u}$ and the pressure p of a fluid in Ω, such that

$$\begin{aligned} \alpha\,\boldsymbol{u} \,-\, \nu\,\boldsymbol{\Delta}\,\boldsymbol{u} \,+\, \nabla\,p \,=\, \boldsymbol{f} & \quad \text{in} \quad \Omega\,, \\ \operatorname{div}(\boldsymbol{u}) \,=\, \tilde{f} & \quad \text{in} \quad \Omega\,, \\ \boldsymbol{u} \,=\, \boldsymbol{u}_D & \quad \text{on} \quad \Gamma_D\,, \\ \Big(-\,\nu\,\nabla\boldsymbol{u} \,+\, p\,\boldsymbol{I}\Big)\,\boldsymbol{n} \,=\, \boldsymbol{g} & \quad \text{on} \quad \Gamma_N\,, \end{aligned} \tag{18}$$

The strategy relies on performing a homogenization of the Neumann datum, in the same spirit as described in [7] and [6]. First, we introduce the pseudo-stress $\tilde{\boldsymbol{\sigma}} \,:=\, \nu\,\nabla\,\boldsymbol{u} \,-\, p\,\boldsymbol{I}$, which allows us to deduce $p \,=\, \frac{\nu}{n}\tilde{f} \,-\, \frac{1}{n}\operatorname{tr}(\tilde{\boldsymbol{\sigma}})$. As a consequence, we can eliminate p from (18), leading to the following first order system:

$$\begin{cases} \dfrac{1}{\nu}\,\tilde{\boldsymbol{\sigma}}^{\mathtt{d}} \,-\, \nabla\,\boldsymbol{u} \,=\, -\dfrac{1}{n}\tilde{f}\,\boldsymbol{I} & \text{in} \quad \Omega\,, \\ \alpha\,\boldsymbol{u} \,-\, \mathbf{div}(\tilde{\boldsymbol{\sigma}}) \,=\, \boldsymbol{f} & \text{in} \quad \Omega\,, \\ \boldsymbol{u} \,=\, \boldsymbol{u}_D & \text{on} \quad \Gamma_D\,, \\ \tilde{\boldsymbol{\sigma}}\,\boldsymbol{n} \,=\, -\boldsymbol{g} & \text{on} \quad \Gamma_N\,. \end{cases} \tag{19}$$

Next, we set $\tilde{\boldsymbol{\sigma}}_{\boldsymbol{g}} \,:=\, -\nabla\boldsymbol{z}$, with $\boldsymbol{z} \in [H^1_{\Gamma_D}(\Omega)]^n$ being the unique weak solution of the auxiliary problem

$$-\Delta\boldsymbol{z} \,=\, 0 \quad \text{in} \quad \Omega\,, \qquad \boldsymbol{z} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_D\,, \qquad \partial_{\boldsymbol{n}}\boldsymbol{z} \,=\, \boldsymbol{g} \quad \text{on} \quad \Gamma_N\,. \tag{20}$$

As a result, we notice that $\tilde{\boldsymbol{\sigma}}_{\boldsymbol{g}} \in H(\mathbf{div};\Omega)$, is divergence-free, and satisfies $\tilde{\boldsymbol{\sigma}}_{\boldsymbol{g}}\,\boldsymbol{n} \,=\, -\boldsymbol{g}$ on Γ_N. Moreover, there exists $C_1 \,>\, 0$, such that $||\tilde{\boldsymbol{\sigma}}_{\boldsymbol{g}}||_{H(\mathrm{div};\Omega)} \,\leq\, C_1\,||\boldsymbol{g}||_{-1/2,00,\Gamma_N}$. Introducing $\boldsymbol{\sigma} \,:=\, \tilde{\boldsymbol{\sigma}} \,-\, \tilde{\boldsymbol{\sigma}}_{\boldsymbol{g}}$, (19) can be rewritten in terms of $(\boldsymbol{\sigma},\boldsymbol{u})$ as

$$\begin{cases} \dfrac{1}{\nu}\,\boldsymbol{\sigma}^{\mathtt{d}} \,-\, \nabla\,\boldsymbol{u} \,=\, -\dfrac{1}{n}\tilde{f}\,\boldsymbol{I} \,-\, \dfrac{1}{\nu}\tilde{\boldsymbol{\sigma}}^{\mathtt{d}}_{\boldsymbol{g}} & \text{in} \quad \Omega\,, \\ \alpha\,\boldsymbol{u} \,-\, \mathbf{div}(\boldsymbol{\sigma}) \,=\, \boldsymbol{f} & \text{in} \quad \Omega\,, \\ \boldsymbol{u} \,=\, \boldsymbol{u}_D & \text{on} \quad \Gamma_D\,, \\ \boldsymbol{\sigma}\,\boldsymbol{n} \,=\, \mathbf{0} & \text{on} \quad \Gamma_N\,. \end{cases} \tag{21}$$

Proceeding as in Section 2, we deduce the mixed variational formulation: Find $(\boldsymbol{\sigma},\boldsymbol{u}) \in H_{\mathtt{N}}(\mathbf{div};\Omega) \times [L^2(\Omega)]^n$ such that

$$\begin{cases} \dfrac{1}{\nu}\displaystyle\int_\Omega \boldsymbol{\sigma}^{\mathtt{d}} \,:\, \boldsymbol{\tau}^{\mathtt{d}} \,+\, \int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,=\, \langle\boldsymbol{\tau n},\boldsymbol{u}_D\rangle_{\Gamma_D} - \dfrac{1}{n}\int_\Omega \tilde{f}\operatorname{tr}(\boldsymbol{\tau}) \,-\, \dfrac{1}{\nu}\int_\Omega \tilde{\boldsymbol{\sigma}}^{\mathtt{d}}_{\boldsymbol{g}} \,:\, \boldsymbol{\tau} & \forall\,\boldsymbol{\tau} \in H_{\mathtt{N}}(\mathbf{div};\Omega)\,, \\ \displaystyle\int_\Omega \boldsymbol{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \,-\, \alpha\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} \,=\, -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} & \forall\,\boldsymbol{v} \in [L^2(\Omega)]^n\,. \end{cases} \tag{22}$$

We note that the difference with (3) lies only on the right side of (22). Therefore, as pointed out in Remark 1, the existence and uniqueness of solution of (22) as well as of its corresponding conforming discrete scheme, are guaranteed thanks to Theorems 4.3.2 and 5.5.2 in [12], respectively. We can also establish the convergence of the method in this situation. Of course, as in Remark 3, we can prove that $\boldsymbol{u} \in [H^1(\Omega)]^n$.

5.1 An a priori error analysis of the augmented formulation

As in Section 3, we consider the augmented formulation: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}_N := H_{\mathtt{N}}(\mathbf{div};\Omega) \times [H^1(\Omega)]^n$, such that

$$A_{\mathtt{s}}((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) = \tilde{F}_{\mathtt{s}}(\boldsymbol{\tau}, \boldsymbol{v}) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}_N\,, \tag{23}$$

where the bilinear form $A_{\mathtt{s}} : \boldsymbol{H} \times \boldsymbol{H} \to \mathbb{R}$ is defined as in Section 3, while the linear functional $\tilde{F}_{\mathtt{s}} : \boldsymbol{H} \to \mathbb{R}$, is given by

$$\tilde{F}_{\mathtt{s}}(\boldsymbol{\tau}, \boldsymbol{v}) := \langle \boldsymbol{\tau n}, \boldsymbol{u}_D\rangle_{\Gamma_D} - \int_\Omega \left(\frac{1}{\nu}\tilde{\boldsymbol{\sigma}}_{\boldsymbol{g}}^{\mathtt{d}} + \frac{1}{n}\tilde{f}\,\boldsymbol{I}\right) : \boldsymbol{\tau} + \int_\Omega \boldsymbol{f}\cdot \boldsymbol{v} - \kappa_0 \int_\Omega \boldsymbol{f}\cdot(\mathbf{div}(\boldsymbol{\tau}) + \alpha\,\boldsymbol{v})$$
$$+ \kappa_1 \int_\Omega \left(\frac{1}{\nu}\tilde{\boldsymbol{\sigma}}_{\boldsymbol{g}}^{\mathtt{d}} + \frac{1}{n}\tilde{f}\,\boldsymbol{I}\right) : \left(\nabla \boldsymbol{v} + \frac{1}{\nu}\boldsymbol{\tau}^{\mathtt{d}}\right).$$

In what follows, we assume that $\kappa_0 \in (0, \alpha^{-1})$ and $\kappa_1 \in (0, \nu)$. Then, thanks to Lemma 5, $A_{\mathtt{s}}$ is bounded and strongly elliptic on $\boldsymbol{H}_N$. In addition, there exists $C_{\tilde{\mathtt{F}}} > 0$, depending on α and ν, such that

$$|\tilde{F}_{\mathtt{s}}(\boldsymbol{\tau}, \boldsymbol{v})| \le C_{\tilde{\mathtt{F}}}(\alpha, \nu)\left(||\boldsymbol{u}_D||_{1/2,00,\Gamma_D} + ||\boldsymbol{g}||_{-1/2,00,\Gamma_N} + ||\tilde{f}||_{0,\Omega} + ||\boldsymbol{f}||_{0,\Omega}\right) ||(\boldsymbol{\tau}, \boldsymbol{v})||_{\boldsymbol{H}} \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}\,, \tag{24}$$

with

$$C_{\tilde{\mathtt{F}}}(\alpha, \nu) := \max\{1, C_1\}\, \max\{\nu^{-1}(1 - \kappa_1\nu^{-1}), \kappa_1\nu^{-1}, 1 - \alpha\kappa_0, \kappa_0, \kappa_1, 1\}\,.$$

By Lax-Milgram theorem, it is proved that the augmented variational formulation (23) has a unique solution $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}_N$, and there exists a positive constant $\hat{C}_*$, depending on α and ν, such that

$$\|(\boldsymbol{\sigma}, \boldsymbol{u})\|_{\boldsymbol{H}} \le \hat{C}_*\left(||\tilde{f}||_{0,\Omega} + \|\boldsymbol{f}\|_{0,\Omega} + ||\boldsymbol{u}_D||_{1/2,00,\Gamma_D} + ||\boldsymbol{g}||_{-1/2,00,\Gamma_N}\right), \tag{25}$$

with $\hat{C}_* := \dfrac{C_{\tilde{\mathtt{F}}}(\alpha, \nu)}{C_{\mathtt{ell}}(\alpha, \nu)}$. The classic discrete conforming variational formulation read as: *Find* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{H}_{N,h} := H_{\mathtt{N},h}^{\boldsymbol{\sigma}} \times H_h^{\boldsymbol{u}}$ *such that*

$$A_{\mathtt{s}}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) = \tilde{F}_{\mathtt{s}}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \qquad \forall\, (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \boldsymbol{H}_{N,h}\,. \tag{26}$$

We remark that the Galerkin scheme (26) is well-posed and a Céa's estimate can be obtained. In addition, the corresponding rate of convergence of the Galerkin scheme (26) for this particular choice of finite element subspaces, is presented in the next theorem.

Theorem 12 *Let* $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}_N$ *and* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{H}_{N,h}$ *be the unique solutions to problems* (23) *and* (26)*, respectively. In addition, assume that* $\boldsymbol{\sigma} \in [H^r(\Omega)]^{n\times n}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^r(\Omega)]^n$ *and* $\boldsymbol{u} \in [H^{r+1}(\Omega)]^n$ *for some* $r \in (0, \ell + 1]$. *Then, there exists* $\bar{C}_* > 0$*, independent of* h*, but depending on* $\hat{C}_*$ *(introduced in* (25)*), such that there holds*

$$\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_{\boldsymbol{H}} \le \bar{C}_*\, h^r\, \left(\|\boldsymbol{\sigma}\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{r,\Omega} + \|\boldsymbol{u}\|_{r+1,\Omega}\right).$$

Proof. It is a consequence of Céa's estimate, and the corresponding approximation properties. We omit further details. □

Remark 13 *We note that this discrete scheme entails difficulties inherent to boundary condition treatments. For example, the way to treat the Dirichlet datum involves the introduction of the nodes on this part of the boundary as unknowns into the system, that is, where the solution is known. In addition, the treatment of the Neumann datum involves higher computational cost (more details in remark below). One way to avoid these problems is discussed in the next subsection.*

Remark 14 *At first glance, we propose $\tilde{\boldsymbol{\sigma}}_h := \boldsymbol{\sigma}_h - \nabla \boldsymbol{z}_h$, with $\boldsymbol{z}_h \in H_h^{\boldsymbol{u}}$ being the solution of discrete primal formulation associated to (20). Then, assuming that $\boldsymbol{z} \in [H^1_{\Gamma_D}(\Omega) \cap H^{s+1}(\Omega)]^n$, for some $s \in (0, \ell+1]$, and invoking Theorem 12, there exists $C > 0$, independent of h, but depending on α and ν, such that*

$$\|(\tilde{\boldsymbol{\sigma}}, \boldsymbol{u}) - (\tilde{\boldsymbol{\sigma}}_h, \boldsymbol{u}_h)\|_{\boldsymbol{H}} \leq C \left\{ h^r \left(\|\boldsymbol{\sigma}\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{r,\Omega} + \|\boldsymbol{u}\|_{r+1,\Omega} \right) + h^s \, ||\boldsymbol{z}||_{s+1,\Omega} \right\} .$$

At this point, we point out that in general $\nabla \boldsymbol{z}_h$ does not belong to $H^{\boldsymbol{\sigma}}_{\mathtt{N},h}$, and requires to compute first $\boldsymbol{z}_h \in H_h^{\boldsymbol{u}}$, in order to compute it. Clearly, from a computational point of view, this could be expensive, which motivates us to present an alternative procedure.

5.2 An a posteriori error analysis of the augmented formulation

To circumvent the difficulties associated with the treatment of non-homogeneous boundary conditions, we take into account [11] for the Neumann type, which suggest approximate the data as follow: we set $\boldsymbol{g}_h := \pi_h^\ell(\boldsymbol{g})$ on $\mathcal{F}_h^{\mathtt{N}}$, with $\pi_h^\ell(\boldsymbol{g})$ representing the L^2-orthogonal projection of $\boldsymbol{g}$ onto $\mathcal{P}_\ell(\mathcal{F}_h^{\mathtt{N}})$, component-wise. In addition, for the Dirichlet type we denote by $\boldsymbol{u}_{D,h}$ the quasi-interpolated of the function $\boldsymbol{u}_D$ of degree one (obtained first by considering the Clement type operator and then modify it to take into account the boundary condition as in [27], Section 5-2, see also [21, 32]). Now, we define $\boldsymbol{m}_h$ as the piecewise linear continuous function such that $\boldsymbol{m}_h(\boldsymbol{x}) = \mathbf{0}$ for each node $\boldsymbol{x} \in \Omega \cup \Gamma_N$ and $\boldsymbol{m}_h = \boldsymbol{u}_{D,h}$ on Γ_D. Clearly, $\boldsymbol{m}_h \in [H^1_{\Gamma_N}(\Omega)]^n$. Next, we consider the problem:

$$\begin{cases} \tilde{\boldsymbol{\rho}} & = \nu \nabla \tilde{\boldsymbol{w}} - q \, \boldsymbol{I} \quad \text{in} \quad \Omega \,, \\ \alpha \, \tilde{\boldsymbol{w}} - \mathbf{div}(\tilde{\boldsymbol{\rho}}) & = \boldsymbol{f} \quad \text{in} \quad \Omega \,, \\ \mathrm{div}(\tilde{\boldsymbol{w}}) & = \tilde{f} \quad \text{in} \quad \Omega \,, \\ \tilde{\boldsymbol{w}} & = \boldsymbol{u}_{D,h} \quad \text{on} \quad \Gamma_D \,, \\ \tilde{\boldsymbol{\rho}} \boldsymbol{\nu} & = -\boldsymbol{g}_h \quad \text{on} \quad \Gamma_N \,, \end{cases} \tag{27}$$

which, by virtue of what has been described at the beginning of the section, it can be established that has a unique solution $(\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{w}}, q)$. In addition, we notice that $q = \frac{\nu}{n} \tilde{f} - \frac{1}{n} \mathrm{tr}(\tilde{\boldsymbol{\rho}})$.

This allows us to establish the existence and uniqueness of $(\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{w}}) \in \boldsymbol{H}$ and $(\tilde{\boldsymbol{\rho}}_h, \tilde{\boldsymbol{w}}_h) \in \boldsymbol{H}_h$, solution of the analog problems (23) and (26), respectively. It is not difficult to check that

$$||(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\rho}}_h, \boldsymbol{u} - \tilde{\boldsymbol{w}}_h)||_{\boldsymbol{H}} \leq ||(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\rho}}, \boldsymbol{u} - \tilde{\boldsymbol{w}})||_{\boldsymbol{H}} + ||(\tilde{\boldsymbol{\rho}} - \tilde{\boldsymbol{\rho}}_h, \tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h)||_{\boldsymbol{H}} \,. \tag{28}$$

Now, we set $\mathcal{T}_h^{\mathtt{N}} := \{T \in \mathcal{T}_h : |E(T) \cap \mathcal{F}_h^{\mathtt{N}}| > 0\}$, $\mathcal{T}_h^{\mathtt{D}} := \{T \in \mathcal{T}_h : |E(T) \cap \mathcal{F}_h^{\mathtt{D}}| > 0\}$, and we assume that $\boldsymbol{g} \in [L^2(\Gamma_N)]^n$ and $\boldsymbol{u}_D \in [H^1(\Gamma_D)]^n \cap [H_{00}^{1/2}(\Gamma_D)]^n$. Then, proceeding similarly as in [11], we can ensure that there exists $C > 0$, independent of h, but depending on $\bar{C}_*$, such that

$$||(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\rho}}, \boldsymbol{u} - \boldsymbol{w})||_{\boldsymbol{H}} \leq C \left(\sum_{T \in \mathcal{T}_h^{\mathtt{N}}} \operatorname{osc}(\boldsymbol{g}, T)^2 + \sum_{T \in \mathcal{T}_h^{\mathtt{D}}} \operatorname{osc}(\boldsymbol{u}_D, T)^2 \right)^{1/2}, \tag{29}$$

where for each $T \in \mathcal{T}_h^{\mathtt{N}}$

$$\operatorname{osc}(\boldsymbol{g}, T)^2 := \sum_{F \in E(T) \cap \mathcal{F}_h^{\mathtt{N}}} h_F \, ||\boldsymbol{g} - \pi_h^{\ell}(\boldsymbol{g})||_{0,F}^2 . \tag{30}$$

Concerning Dirichlet datum, in 2D case, under the assumption that $\boldsymbol{u}_D \in [H^1(\Gamma_D)]^2$, it is possible to define $\boldsymbol{u}_{D,h}$ as the linear Lagrange interpolation of the function $\boldsymbol{u}_D$, i.e., we have $\boldsymbol{u}_{D,h}(\boldsymbol{x}) = \boldsymbol{u}_D(\boldsymbol{x})$ for each node $\boldsymbol{x} \in \Gamma_D$. We notice that the function $\boldsymbol{u}_D - \boldsymbol{u}_{D,h}$ vanishes at the nodes of $\mathcal{T}_h$ lying on Γ_D, allowing us to estimate its $[H_{00}^{1/2}(\Gamma_D)]^2$-norm in terms of L^2-local norms on the edges of Γ_D. More precisely, according to Theorem 1 in [19], there holds

$$\|\boldsymbol{u}_D - \boldsymbol{u}_{D,h}\|_{[H_{00}^{1/2}(\Gamma_D)]^2}^2 \leq C \left\{ \log[1 + C_h(\Gamma)] \sum_{j=1}^{m} h_j \left\| \frac{\partial \boldsymbol{u}_D}{\partial \boldsymbol{t}} - \frac{\partial \boldsymbol{u}_{D,h}}{\partial \boldsymbol{t}} \right\|_{[L^2(\Gamma_j)]^2}^2 \right\},$$

where $h_j = |\Gamma_j|$, $j = 1, \ldots, m$, $C_h(\Gamma) := \max\left\{ \frac{h_i}{h_j} : \Gamma_i \text{ is a neighbor of } \Gamma_j , \; i, j \in \{1, ..., m\} \right\}$, and $\{\Gamma_1, \ldots, \Gamma_m\}$ is the partition of Γ_D induced by $\mathcal{T}_h$. We remind that given an edge e induced by $\mathcal{T}_h$, and lying on Γ, we set $\boldsymbol{t}$ as the tangential vector associated to e. In addition, $\frac{\partial \boldsymbol{u}_D}{\partial \boldsymbol{t}}$ represents the tangential derivative of $\boldsymbol{u}_D$ along e. Similar meaning is given to $\frac{\partial \boldsymbol{u}_{D,h}}{\partial \boldsymbol{t}}$. This result allows us to establish

$$\operatorname{osc}(\boldsymbol{u}_D, T)^2 := \sum_{F \in E(T) \cap \mathcal{F}_h^{\mathtt{D}}} h_F \left\| \frac{\partial \boldsymbol{u}_D}{\partial \boldsymbol{t}} - \frac{\partial \boldsymbol{u}_{D,h}}{\partial \boldsymbol{t}} \right\|_{[L^2(F)]^2}^2 . \tag{31}$$

For the 3D case, the Sobolev interpolation theorem implies

$$\|\boldsymbol{u}_D - \boldsymbol{u}_{D,h}\|_{[H_{00}^{1/2}(\Gamma_D)]^3}^2 \leq C \|\boldsymbol{u}_D - \boldsymbol{u}_{D,h}\|_{[L^2(\Gamma_D)]^3} \|\boldsymbol{u}_D - \boldsymbol{u}_{D,h}\|_{[H^1(\Gamma_D)]^3} \tag{32}$$

which allows us to deduce

$$\operatorname{osc}(\boldsymbol{u}_D, T)^2 := h_{\mathtt{D}} \sum_{F \in E(T) \cap \mathcal{F}_h^{\mathtt{D}}} \|\boldsymbol{u}_D - \boldsymbol{u}_{D,h}\|_{[H^1(F)]^3}^2 , \tag{33}$$

where $h_{\mathtt{D}} := \max\{h_F : \; F \in \mathcal{F}_h^{\mathtt{D}}\}$.

Now, invoking the *Raviart-Thomas local lifting operator on the normal trace* (cf. Proposition 2.4 in [22]), we can build $\tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h} \in H(\mathbf{div};\Omega)\cap[\mathcal{RT}_\ell(\mathcal{T}_h)]^n$, such that $\tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h}\boldsymbol{\nu} = -\boldsymbol{g}_h$ on Γ_N. Then, we consider $\hat{\boldsymbol{\rho}} := \tilde{\boldsymbol{\rho}} - \tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h}$ in Ω and $\hat{\boldsymbol{w}} := \tilde{\boldsymbol{w}} - \boldsymbol{m}_h$ in Ω. This helps us to rewrite (27) as:

$$\begin{cases} \alpha\,\hat{\boldsymbol{w}} - \mathbf{div}(\hat{\boldsymbol{\rho}}) & = \hat{\boldsymbol{f}} \quad \text{in} \quad \Omega\,, \\ \frac{1}{\nu}\,\hat{\boldsymbol{\rho}}^{\mathtt{d}} - \nabla\hat{\boldsymbol{w}} & = \nabla\boldsymbol{m}_h - \frac{1}{\nu}\tilde{\boldsymbol{\rho}}^{\mathtt{d}}_{\boldsymbol{g}_h} - \frac{1}{n}\tilde{f}\,\boldsymbol{I} \quad \text{in} \quad \Omega\,, \\ \hat{\boldsymbol{w}} & = \mathbf{0} \quad \text{on} \quad \Gamma_D\,, \\ \hat{\boldsymbol{\rho}}\,\boldsymbol{\nu} & = \mathbf{0} \quad \text{on} \quad \Gamma_N\,, \end{cases} \tag{34}$$

where $\hat{\boldsymbol{f}} := \boldsymbol{f} + \mathbf{div}(\tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h}) - \alpha\,\boldsymbol{m}_h$. We emphasize that $(\hat{\boldsymbol{\rho}},\hat{\boldsymbol{w}}) \in \boldsymbol{H}_0$ and $(\hat{\boldsymbol{\rho}}_h,\hat{\boldsymbol{w}}_h) \in \boldsymbol{H}_{0,h}$ satisfy

$$\forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in \boldsymbol{H}_0 : A_{\mathtt{s}}((\hat{\boldsymbol{\rho}},\hat{\boldsymbol{w}}),(\boldsymbol{\tau},\boldsymbol{v})) = \hat{F}_{\mathtt{s}}(\boldsymbol{\tau},\boldsymbol{v})\,, \tag{35}$$

$$\forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in \boldsymbol{H}_{0,h} : A_{\mathtt{s}}((\hat{\boldsymbol{\rho}}_h,\hat{\boldsymbol{w}}_h),(\boldsymbol{\tau},\boldsymbol{v})) = \hat{F}_{\mathtt{s}}(\boldsymbol{\tau},\boldsymbol{v})\,, \tag{36}$$

respectively, with $\hat{F}_{\mathtt{s}} : \boldsymbol{H} \to \mathbb{R}$ being the linear functional, given for any $(\boldsymbol{\tau},\boldsymbol{v}) \in \boldsymbol{H}$, by

$$\begin{aligned} \hat{F}_{\mathtt{s}}(\boldsymbol{\tau},\boldsymbol{v}) := & \int_\Omega \hat{\boldsymbol{f}}\cdot\boldsymbol{v} - \int_\Omega \left(\frac{1}{n}\tilde{f}\,\boldsymbol{I} + \frac{1}{\nu}\tilde{\boldsymbol{\rho}}^{\mathtt{d}}_{\boldsymbol{g}_h} - \nabla\boldsymbol{m}_h\right) : \boldsymbol{\tau} \\ & - \kappa_0 \int_\Omega \hat{\boldsymbol{f}}\cdot(\alpha\,\boldsymbol{v} + \mathbf{div}(\boldsymbol{\tau})) + \kappa_1 \int_\Omega \left(\frac{1}{n}\tilde{f}\,\boldsymbol{I} + \frac{1}{\nu}\tilde{\boldsymbol{\rho}}^{\mathtt{d}}_{\boldsymbol{g}_h} - \nabla\boldsymbol{m}_h\right) : \left(\nabla\boldsymbol{v} + \frac{1}{\nu}\,\boldsymbol{\tau}^{\mathtt{d}}\right). \end{aligned}$$

Now, taking into account the ellipticity of $A_{\mathtt{s}}$ on $\boldsymbol{H}_0$, we can establish that

$$C_{\mathtt{ell}}\,||(\hat{\boldsymbol{\rho}}-\hat{\boldsymbol{\rho}}_h,\hat{\boldsymbol{w}}-\hat{\boldsymbol{w}}_h)||_{\boldsymbol{H}} \le \sup_{(\boldsymbol{\tau},\boldsymbol{v})\in\boldsymbol{H}_0\setminus\{\mathbf{0}\}} \frac{A_{\mathtt{s}}((\hat{\boldsymbol{\rho}}-\hat{\boldsymbol{\rho}}_h,\hat{\boldsymbol{w}}-\hat{\boldsymbol{w}}_h),(\boldsymbol{\tau},\boldsymbol{v}))}{||(\boldsymbol{\tau},\boldsymbol{v})||_{\boldsymbol{H}}}\,. \tag{37}$$

Our next aim, is to bound the supremum in (37). To this end, we require the following result.

Lemma 15 *For any $(\boldsymbol{\tau},\boldsymbol{v}) \in \boldsymbol{H}_0$, there holds*

$$A_{\mathtt{s}}((\hat{\boldsymbol{\rho}}-\hat{\boldsymbol{\rho}}_h,\hat{\boldsymbol{w}}-\hat{\boldsymbol{w}}_h),(\boldsymbol{\tau},\boldsymbol{v})) = R_1(\boldsymbol{\tau}) + R_2(\boldsymbol{v})\,, \tag{38}$$

where

$$\begin{aligned} R_1(\boldsymbol{\tau}) := & -\int_\Omega \left(\nabla(\hat{\boldsymbol{w}}_h + \boldsymbol{m}_h) - \frac{1}{\nu}(\hat{\boldsymbol{\rho}}_h + \tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h})^{\mathtt{d}} - \frac{1}{n}\tilde{f}\,\boldsymbol{I}\right) : \boldsymbol{\tau} \\ & - \nu^{-1}\,\kappa_1 \int_\Omega \left(\nabla(\hat{\boldsymbol{w}}_h + \boldsymbol{m}_h) - \frac{1}{\nu}(\hat{\boldsymbol{\rho}}_h + \tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h})^{\mathtt{d}} - \frac{1}{n}\tilde{f}\,\boldsymbol{I}\right) : \boldsymbol{\tau}^{\mathtt{d}} \\ & - \kappa_0 \int_\Omega (\hat{\boldsymbol{f}} + \mathbf{div}(\hat{\boldsymbol{\rho}}_h) - \alpha\,\hat{\boldsymbol{w}}_h)\cdot\mathbf{div}(\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\tau} \in H_{\mathtt{N}}(\mathrm{div};\Omega)\,, \end{aligned} \tag{39}$$

$$\begin{aligned} R_2(\boldsymbol{v}) := & (1-\alpha\,\kappa_0) \int_\Omega (\hat{\boldsymbol{f}} + \mathbf{div}(\hat{\boldsymbol{\rho}}_h) - \alpha\,\hat{\boldsymbol{w}}_h)\cdot\boldsymbol{v} \\ & - \kappa_1 \int_\Omega \left(\nabla(\hat{\boldsymbol{w}}_h + \boldsymbol{m}_h) - \frac{1}{\nu}(\hat{\boldsymbol{\rho}}_h + \tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h})^{\mathtt{d}} - \frac{1}{n}\tilde{f}\,\boldsymbol{I}\right) : \nabla\boldsymbol{v} \quad \forall\,\boldsymbol{v} \in [H^1_{\Gamma_D}(\Omega)]^n\,. \end{aligned} \tag{40}$$

Proof. First, given $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}_0$, we invoke (35) to have

$$A_{\mathtt{s}}((\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\rho}}_h, \hat{\boldsymbol{w}} - \hat{\boldsymbol{w}}_h), (\boldsymbol{\tau}, \boldsymbol{v})) \,=\, \hat{F}_{\mathtt{s}}(\boldsymbol{\tau}, \boldsymbol{v}) \,-\, A_{\mathtt{s}}((\hat{\boldsymbol{\rho}}_h, \hat{\boldsymbol{w}}_h), (\boldsymbol{\tau}, \boldsymbol{v}))\,.$$

The rest of the proof relies on the definition of $\hat{F}_{\mathtt{s}}$ and $A_{\mathtt{s}}$, and some algebraic manipulations. We omit further details. □

As an immediate consequence of Lemma 15, we deduce an a posteriori error estimator, which is given in the next theorem.

Theorem 16 *There exists $C_{\mathtt{rel}} > 0$, depending on α and ν, such that*

$$||(\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\rho}}_h, \hat{\boldsymbol{w}} - \hat{\boldsymbol{w}}_h)||_{\boldsymbol{H}} \,\leq\, C_{\mathtt{rel}}\, \eta\,, \tag{41}$$

with $\eta := \left(\sum_{T \in \mathcal{T}_h} \eta_T^2\right)^{1/2}$, *where for any $T \in \mathcal{T}_h$*

$$\begin{aligned} \eta_T^2 \,:=\, & \max\{1 - \alpha\, \kappa_0, \kappa_0\}^2\, ||\hat{\boldsymbol{f}} \,+\, \mathbf{div}(\hat{\boldsymbol{\rho}}_h) \,-\, \alpha\, \hat{\boldsymbol{w}}_h||_{0,T}^2 \\ & + \max\{1, \nu^{-1}\, \kappa_1, \kappa_1\}^2 \left\|\nabla(\hat{\boldsymbol{w}}_h \,+\, \boldsymbol{m}_h) \,-\, \frac{1}{\nu}(\hat{\boldsymbol{\rho}}_h \,+\, \tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h})^{\mathtt{d}} - \frac{1}{n}\tilde{f}\, \boldsymbol{I}\right\|_{0,T}^2 . \end{aligned} \tag{42}$$

Remark 17 *Since $\tilde{\boldsymbol{\rho}} \,=\, \hat{\boldsymbol{\rho}} \,+\, \tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h}$ and $\tilde{\boldsymbol{w}} \,=\, \hat{\boldsymbol{w}} \,+\, \boldsymbol{m}_h$, we introduce $\tilde{\boldsymbol{\rho}}_h \,:=\, \hat{\boldsymbol{\rho}}_h \,+\, \tilde{\boldsymbol{\rho}}_{\boldsymbol{g}_h}$ and $\tilde{\boldsymbol{w}}_h \,=\, \hat{\boldsymbol{w}}_h \,+\, \boldsymbol{m}_h$ as its approximations. Then, we have*

$$||(\tilde{\boldsymbol{\rho}} - \tilde{\boldsymbol{\rho}}_h, \tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h)||_{\boldsymbol{H}} \,=\, ||(\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\rho}}_h, \hat{\boldsymbol{w}} - \hat{\boldsymbol{w}}_h)||_{\boldsymbol{H}} \,\leq\, C_{\mathtt{rel}}\, \eta\,.$$

Theorem 18 *There exists $C_{\mathtt{eff}} > 0$, depending on α and ν, such that for any $T \in \mathcal{T}_h$*

$$\eta_T \,\leq\, C_{\mathtt{eff}}\, ||(\tilde{\boldsymbol{\rho}} - \tilde{\boldsymbol{\rho}}_h, \tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}_h)||_{\boldsymbol{H}_T}\,.$$

Remark 19 *Invoking a suitable triangle inequality, we can establish that*

$$\eta \,\leq\, \bar{C}_{\mathtt{eff}} \left(||(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\rho}}_h, \boldsymbol{u} - \boldsymbol{w}_h)||_{\boldsymbol{H}}^2 + \sum_{T \in \mathcal{T}_h^{\mathtt{N}}} \mathrm{osc}(\boldsymbol{g}, T)^2 + \sum_{T \in \mathcal{T}_h^{\mathtt{D}}} \mathrm{osc}(\boldsymbol{u}_{\boldsymbol{D}}, T)^2 \right)^{1/2}.$$

Remark 20 *For $(\kappa_0, \kappa_1) \,:=\, (\frac{1}{2\,\alpha}, \frac{\nu}{2})$, we deduce*

$$\begin{cases} C_* \,=\, \tilde{C}_* \,=\, \mathcal{O}(\alpha^2)\,, \quad C_{\mathtt{rel}} \,=\, C_{\mathtt{eff}} \,=\, \mathcal{O}(\alpha) & \alpha \,\geq\, \nu \,\geq\, 1\,, \\ C_* \,=\, \tilde{C}_* \,=\, \mathcal{O}(\max\{\alpha, \nu^{-1}\}^2)\,, \quad C_{\mathtt{rel}} \,=\, C_{\mathtt{eff}} \,=\, \mathcal{O}(\max\{\alpha, \nu^{-1}\}) & \alpha \,>\, 1 \,>\, \nu\,. \end{cases}$$

6 Numerical experiments

Numerical experiments illustrating the performance of the augmented schemes (36) and the properties of the a posteriori error estimators η are reported in this section. We present three examples, by simplicity, all of them in 2D. The aim of the first one is to confirm the a priori error estimate (see Theorem 12 and Remark 14), and to verify the efficiency and reliability of the a posteriori error estimator derived in Section 4. We also analyze the influence of relatively small values of the viscosity ν, and for moderate large values of α, in the numerical method. The second and third examples allow us to show the effectivity of the adaptive method. In what follows, in all examples we fixed the parameters as $(\kappa_0, \kappa_1) := (\frac{1}{2\,\alpha}, \frac{\nu}{2})$ and we consider the finite element subspaces of lowest order of approximation (i.e. for $\ell = 0$).

The adaptive refinement algorithm we consider can be found in [35], and reads as follows:

Algorithm 1: Adaptive Refinement Algorithm

Result: Improvement of quality of approximation

Input: tolerance `tol`, initial / coarse mesh $\mathcal{T}_h^0$;

`Step 1:` Solve the Galerkin scheme for the current mesh $\mathcal{T}_h^0$. Then compute $\{\eta_T\}_{T\in\mathcal{T}_h^0}$.

while η > `tol` **do**

`Mark each element` $T' \in \mathcal{T}_h$ `such that`

$$\eta_{T'} \;\geq\; \frac{1}{2}\,\max\{\eta_T \,:\, T \in \mathcal{T}_h\}\,.$$

`Refine marked elements and remove hanging nodes if corresponds`;

`This generates an adapted mesh` $\mathcal{T}_h$;

$\mathcal{T}_h^0 \leftarrow \mathcal{T}_h$ `and go to Step 1.`

end

In what follows, $\mathcal{N}$ denotes the total number of degrees of freedom (dof) for the corresponding discrete scheme. We define the error for each unknown, $e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)}$, $e(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{[H^1(\Omega)]^2}$. In addition, we include the computation of the error for pressure when it is approximated by $p_h = \frac{\nu}{n}\tilde{f} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\sigma}_h)$. This is defined by $e(p) := \|p - p_h\|_{[L^2(\Omega)]^2}$. The total error is given by $E := \Big(\, e(\boldsymbol{\sigma})^2 \,+\, e(\boldsymbol{u})^2 \,\Big)^{1/2}$. The efficiency and effectivity indexes, with respect to the error indicator η, are defined as E/η and η/E, respectively.

6.1 A smooth example

Let $\Omega =]0,1[^2$, we considere mixed boundary condition with $\Gamma_D := 0 \times (0,1) \cup (0,1) \times 0$. $\Gamma_N := \partial\Omega \setminus \Gamma_D$ and we choose the data so that the exact solution of problem (18) is:

$$\boldsymbol{u}(x,y) = (x^2 + 2x^2(1-x)^2 y(1-y)(1-2y), y^2 - 2x(1-x)(1-2x)y^2(1-y)^2)$$
$$p = \exp\left(-10\left((x-1/2)^2 + (y-1/2)^2\right)\right),$$

We solved the problem over a sequence of uniform meshes (i.e., in each step the elements of the mesh are bisected twice). In Figure 1 we show the total error versus the DOFs for $\alpha = 1$ and different values of ν, whereas in Figure 2 we display the total error versus DOF for $\nu = 1$ and different values of α. We observe that optimal convergence rates are attained in all cases. In Figure 3 we exhibit the corresponding effectivity indexes for the numerical solutions obtained for the different values of ν (left) and for different values of α (right). We observe in all cases, the effectivity indexes remain bounded, which is in accordance with the theory.

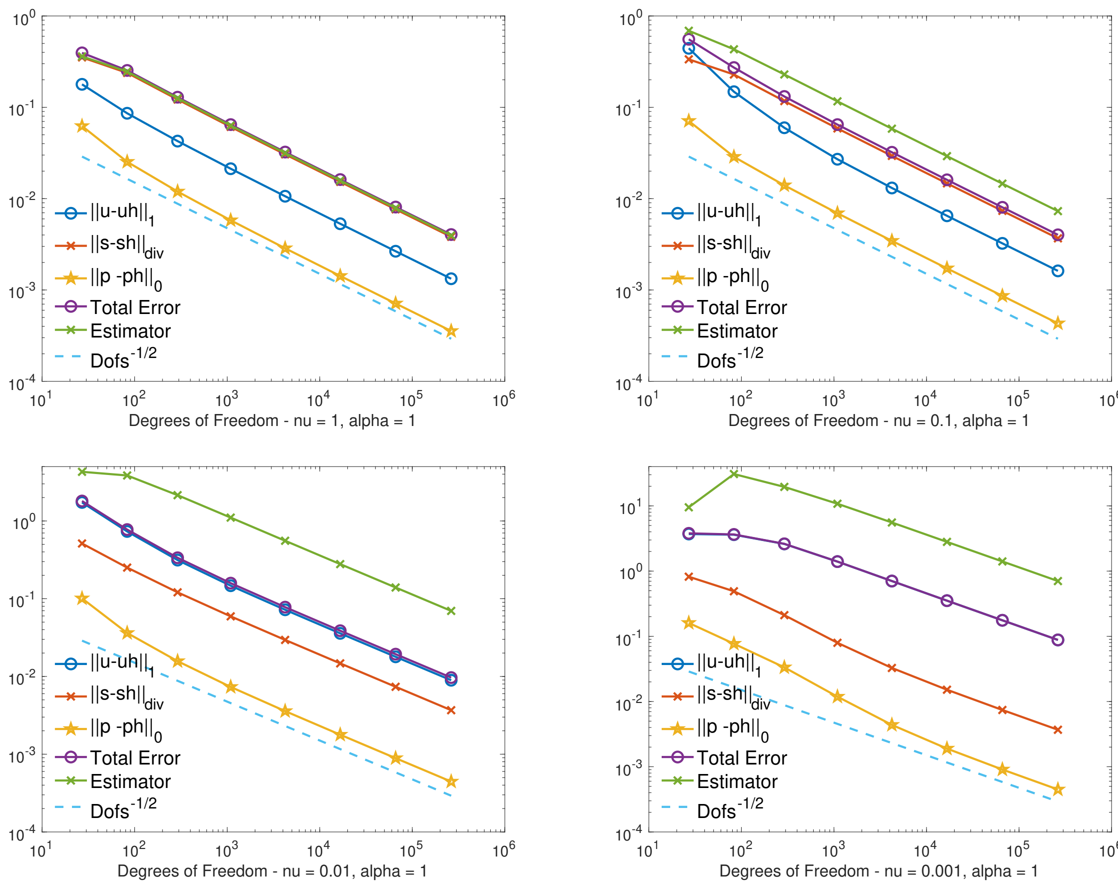

Figure 1: Example 1: From top-left to bottom-right, Individual, Total errors and a posteriori error estimator for $\alpha = 1$ and $\nu \in \{1, 10^{-1}, 10^{-2}, 10^{-3}\}$, obtained with uniform refinement for Example 1.

6.2 A non smooth benchmark solution

In this example, we are inspired by [35] (cf. page 113) and define $\Omega = \{|x| + |y| < 1\} \cap \{x < 0 \text{ or } y > 0\}$, which clearly is a non-convex Lipschitz domain. We assume $\Gamma_D = \partial\Omega$, then $\Gamma_N = \emptyset$. The data of this

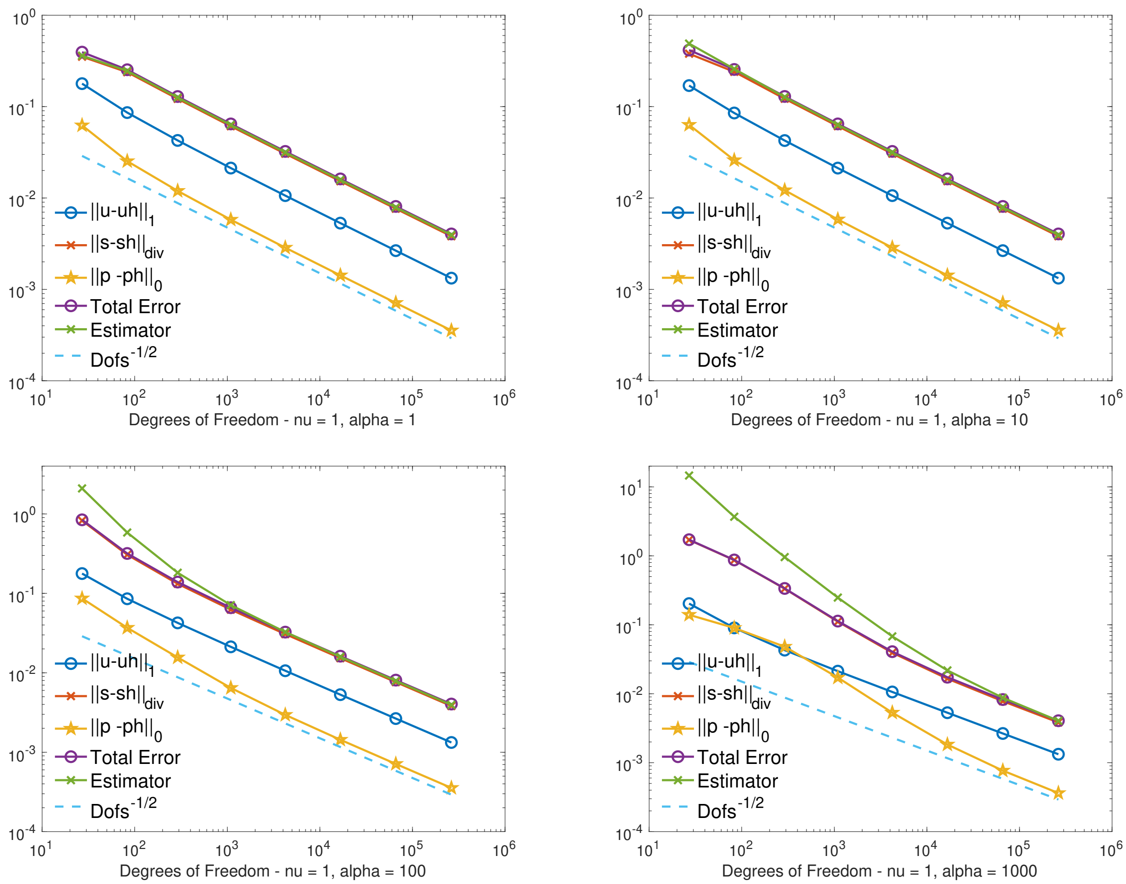

Figure 2: Example 1: From top-left to bottom-right, Individual, Total errors and estimator for $\alpha \in \{1, 10^1, 10^2, 10^3\}$, obtained with uniform refinement (UR) for Example 1 with $\nu = 1$.

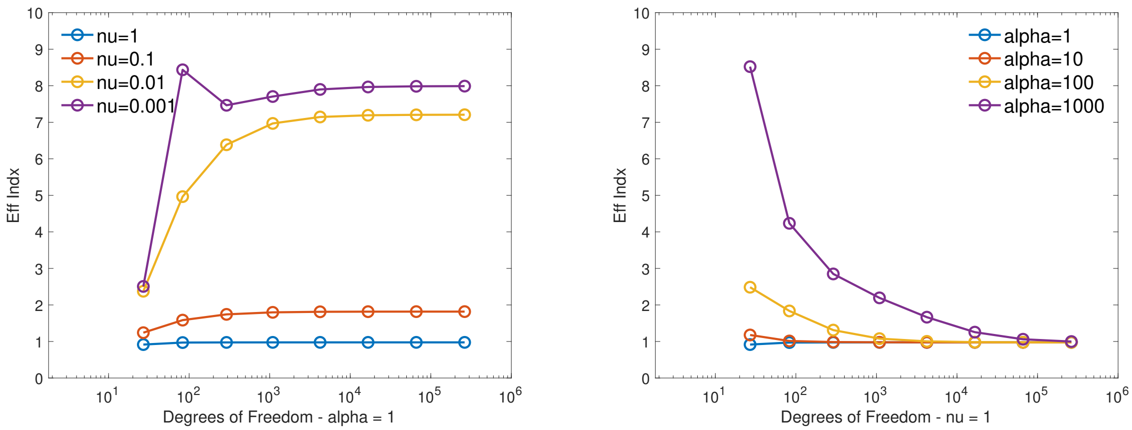

Figure 3: Example 1: Effectivity indexes vs. DOFs. Left: fixed $\alpha = 1$ and different values of ν. Righ: fixed $\nu = 1$ and different values of α.

problem is given such that the exact solution, in polar coordinates, is

$$\boldsymbol{u}(r,\theta) := \begin{pmatrix} r^{\lambda}[(1+\lambda)\,\sin(\theta)\,\psi(\theta) + \cos(\theta)\,\psi'(\theta)] \\ r^{\lambda}[-(1+\lambda)\,\cos(\theta)\,\psi(\theta) + \sin(\theta)\,\psi'(\theta)] \end{pmatrix}, \quad \text{and}$$

$$p(r,\theta) := -\frac{r^{\lambda-1}}{1-\lambda}[(1+\lambda)^2\,\psi'(\theta) + \psi'''(\theta)]\,,$$

with

$$\psi(\theta) := \frac{1}{1+\lambda}\sin((1+\lambda)\theta)\,\cos(\lambda\,\omega) - \cos((1+\lambda)\theta) - \frac{1}{1-\lambda}\sin((1-\lambda)\theta)\,\cos(\lambda\,\omega) + \cos((1-\lambda)\theta)\,,$$

$$\lambda := 0.54448373678246\,, \quad \omega := \frac{3}{2}\pi\,.$$

Moreover, in this case the exact solution $(\boldsymbol{u}, p)$ lives in $[H^{1+\lambda}(\Omega)]^2 \times H^{\lambda}(\Omega)$. The history of convergence of the method is displayed in Figure 4 (Left), considering sequences of uniform and adaptive refined meshes generated according to the proposed Algorithm 1. We notice that due to the low regularity of the exact solution, the total error, when applying uniform refinement, behaves as $\mathcal{O}(h^{\lambda})$. On the other hand, when performing the adaptive refinement Algorithm 1, based on our a posteriori error estimator η, the quality of approximation is improved, recovering the optimal rate of convergence as if the solution were smooth. In addition, this adaptive procedure is able to identify the singularity of $\boldsymbol{u}$ and p at origin. This is shown in Figure 5, which contains some of the adapted meshes obtained in this process. Concerning the index of efficiency, we observe that their values are bounded, when considering uniformly refined meshes and the

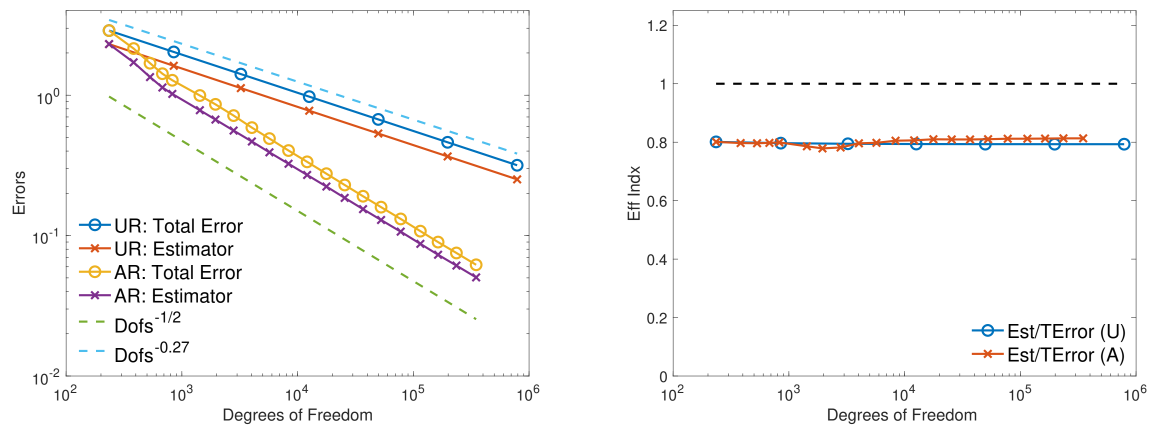

Figure 4: Example 2: Left: Total errors and Estimators vs. DOFs (Uniform and Adaptive refinements). Right: efficiency indexes.

sequence of meshes obtained by applying the adaptive refinement algorithm, as is displayed in Figure 4 (Right). These let us to state that our a posteriori error estimator η, is reliable and locally efficient, in agreement with our theoretical findings.

6.3 A flow of a fluid in a porous media with obstacles

As it is known, the Brinkman equation can be used to describe the flow of a viscous fluid in a porous medium. This motivates us to simulate the flow of a fluid in a porous medium with obstacles. As the domain Ω, we consider a tube with three holes, which represent the obstacles. The tube is given by $T = [0, 8] \times [0, 1]$. On the other hand, we introduce three obstacles O_1, O_2 and O_3, that are defined as the circles of ratio 1/3 and center at the points (15/6,1/2), (23/6,1/2) and (31/6,1/2), respectively. Then, we set $\Omega := T \setminus (O_1 \cup O_2 \cup O_3)$, $\Gamma_N = \{8\} \times [0, 1]$, $\Gamma_D = \partial\Omega \setminus \Gamma_N$. For simplicity, we choose $\alpha = 1.0$, $\nu = 1.0$, and the data $\tilde{f} = 0$, $\boldsymbol{f} = \mathbf{0}$ in Ω, while the Dirichlet boundary condition is set as $\boldsymbol{u}_D(x, y) = (20y(1-y), 0)^{\mathtt{t}}$ on $\{0\} \times [0, 1]$ and $\boldsymbol{u}_D = \mathbf{0}$ on $\Gamma_D \setminus \{0\} \times [0, 1]$. On Neumann boundary, we impose the so-called do-nothing boundary condition, that is, $\boldsymbol{g} = \mathbf{0}$ on Γ_N.

Based on the a posteriori error estimator η, in Figure 6 we display the behavior of η for uniform and adaptive refinements. There we can notice the optimal order. In addition, in Figures 7 and 8, we show the velocity and pressure for an intermediate and final iteration. Finally, some adapted meshes are shown in Figure 9, where we can observe that the estimator recognizes the places where the error is more dominant.

7 Concluding remarks

In this paper, we have been able to obtain a reliable and (quasi) local efficient a posteriori error estimator for the Brinkman problem with non trivial mixed boundary conditions. To this aim, we consider an augmented Galerkin variational formulation based on the pseudo-stress and velocity unknowns. This allows us to

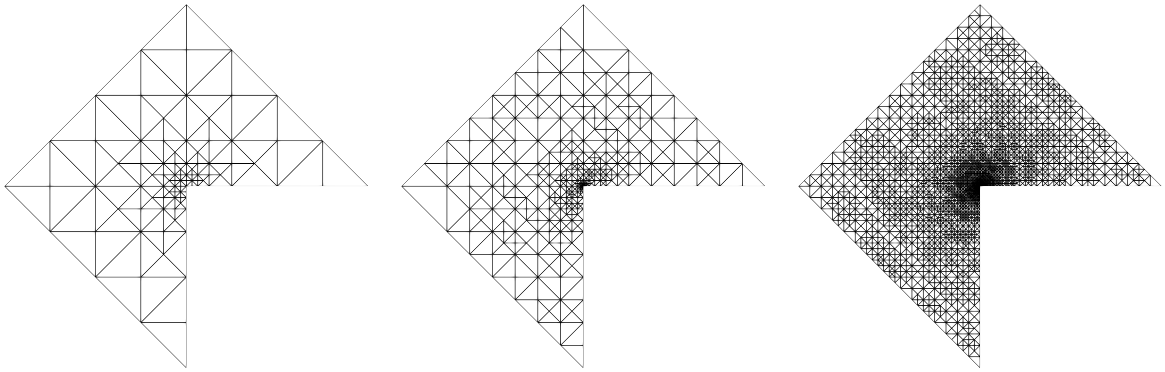

Figure 5: Example 2: Intermediate meshes, from iterations 3, 7 and 13.

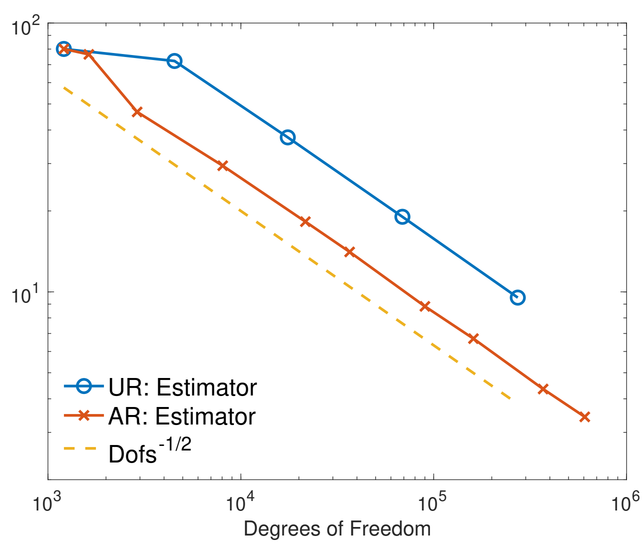

Figure 6: Example 3: Estimators vs. DOFs (Uniform and Adaptive refinements).

have more freedom to choose the approximation spaces. We point out that the resulting a posteriori error estimator is obtained after performing an homogenization procedure to the Dirichlet and Neumann boundary data, which let us to ensure the well posedness of the variational formulation. In fact, the homogenization of Dirichlet datum helps us not to increase the size of the linear system induced by the discrete scheme. This means that it is not necessary to take the degrees of freedom associated to the Dirichlet boundary as unknowns, since we already know how the solution behaves on it. In other words, the analysis of the Brinkman problem with non null mixed boundary conditions relies on the analysis of the same model problem, but with homogeneous mixed boundary conditions and a *corrected* source term. This is the aim of introducing suitable lifting mechanisms for the Dirichlet and Neumann data. On the other hand, this

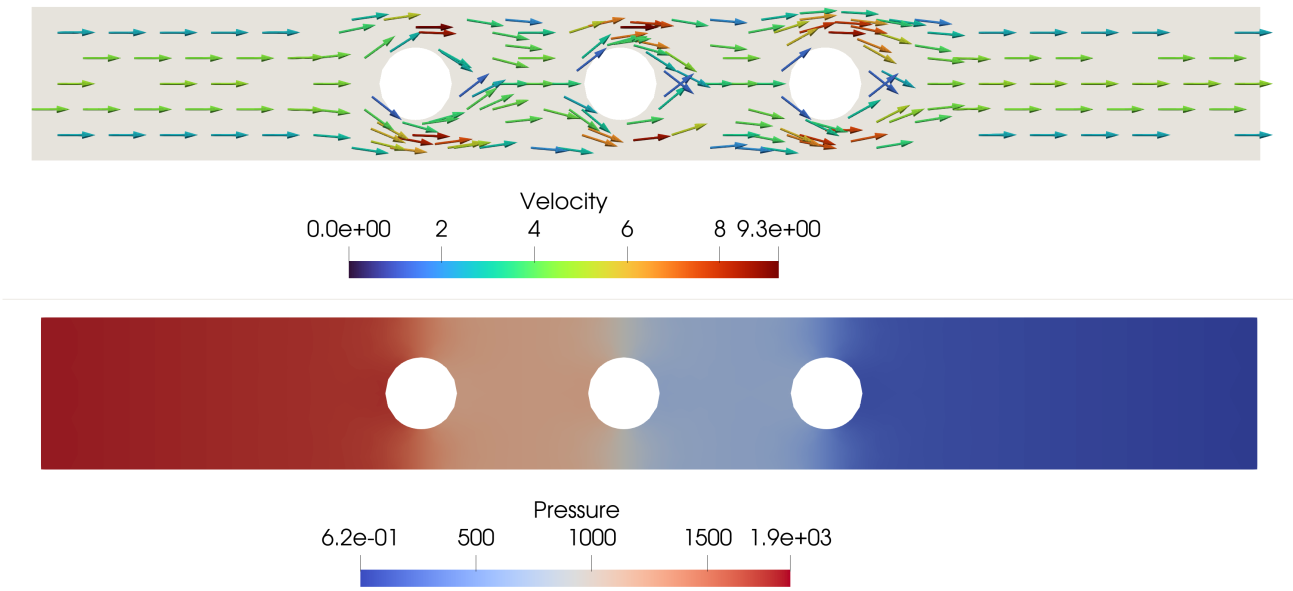

Figure 7: Example 3: Vector field of the velocity and pressure (Intermediate mesh from iteration 3).

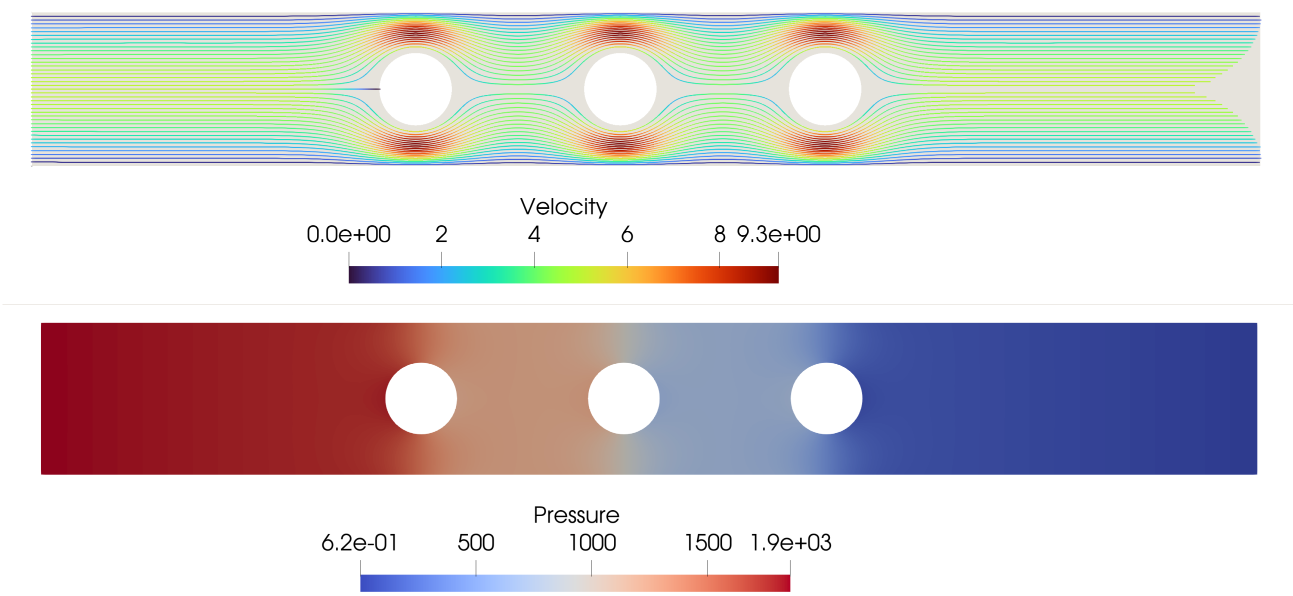

Figure 8: Example 3: Streamlines of the velocity and pressure (final iteration).

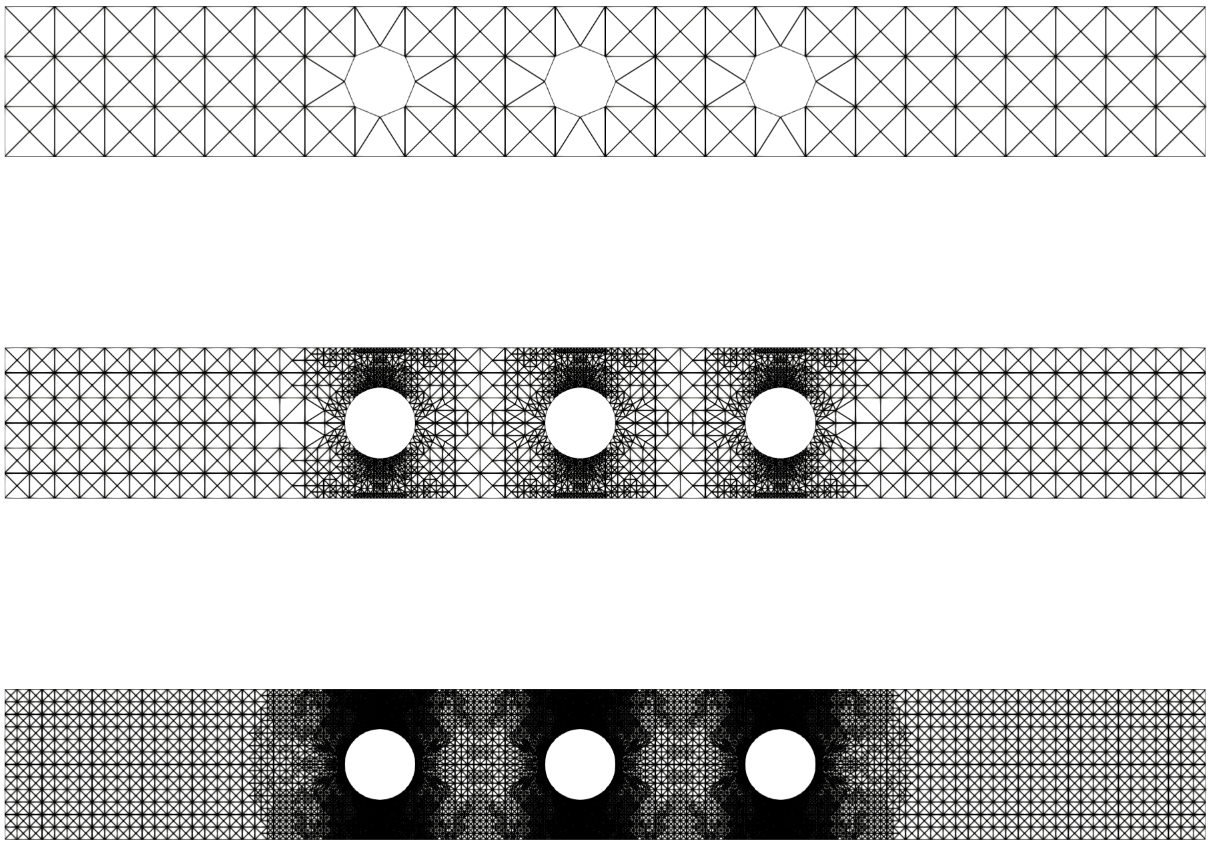

Figure 9: Example 3: Initial and Intermediate adapted meshes (Iterations 4 and 7).

strategy lets us to ensure that the resulting a posteriori error analysis is valid in 2D and in 3D. As a result, our estimator consists of just two residual terms, and eventually two oscillation terms, related to the Dirichlet and Neumann data. This means that the estimator has, from a computational point of view, low cost. We notice that the results of numerical simulations presented in this work, are in agreement with our theoretical findings (cf. Theorem 16, Remark 17 and Theorem 18).

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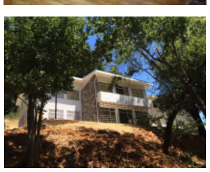

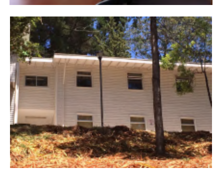

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