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A conforming mixed finite element method for a coupled Navier–Stokes/transport system modelling reverse osmosis processes *

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Abstract

We consider the coupled Navier–Stokes/transport equations with nonlinear transmission conditions, which constitute one of the most common models utilized to simulate a reverse osmosis effect in water desalination processes when considering feed and permeate channels coupled through a semi-permeate membrane. The variational formulation consists of a set of equations where the velocities, the concentrations, along with tensors and vector fields introduced as auxiliary unknowns and two Lagrange multipliers are the main unknowns of the system. The latter are introduced to deal with the trace of functions that do not have enough regularity to be restricted to the boundary. In addition, the pressures can be recovered afterwards by a postprocessing formula. As a consequence, we obtain a nonlinear Banach spaces-based mixed formulation, which has a perturbed saddle point structure. We analyze the continuous and discrete solvability of this problem by linearizing the perturbation and applying the classical Banach fixed point theorem along with the Banach–Nečas–Babuška result. Regarding the discrete scheme, feasible choices of finite element subspaces that can be used include Raviart–Thomas spaces for the auxiliary tensor and vector unknowns, piecewise polynomials for the velocities and the concentrations, and continuous polynomial space of lowest order for the traces, yielding stable discrete schemes. An optimal *a priori* error estimate is derived, and numerical results illustrating both, the performance of the scheme confirming the theoretical rates of convergence, and its applicability, are reported.

Key words: Navier–Stokes, transport, nonlinear interface, mixed finite element methods, a priori error analysis, reverse osmosis, water desalination

Mathematics subject classifications (2020): 35J66, 65J15, 65N12, 65N15, 65N30, 47J26, 76D07.

1 Introduction

Membrane-based seawater desalination processes have received special attention during the last decade due to their notable advantages, which include relatively low energy consumption compared to thermal-based techniques like multi-stage flash [30], as well as their capability to use renewable or low-grade energy sources [2, 33]. In this regard, the reverse osmosis (RO) process takes a prominent position, being employed in 69% of industrial desalination plants globally [17]. Its mathematical modeling is usually based on the Navier–Stokes and convection-diffusion equations, although the Brinkman equations can also be considered in some scenarios. In particular, for some detailed review of different mathematical models we refer to [26, 27, 32].

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In the reverse osmosis processes water flows at high pressure through a membrane module [30], which consists of several channels separated by semi-permeate membranes. The domain of simulation is a representative rectangular section of these channels. Most of the numerical methods developed in the literature consider only a single desalination channel, due to the high computational cost of simulating several channels. They are usually based on finite differences and finite volumes methods. Regarding the finite element method, most of the simulations are performed using commercial softwares and, to the best of our knowledge, a proper mathematical framework has not been developed. Recently in [10] a finite element method using the Nitsche technique on an RO model was addressed, on which the resulting discrete model is easily implemented since the linearization of the model depends only on the linearization of the flow.

One of the novelties of our work is the consideration of two channels coupled by a semi-permeate membrane for which, as far as we know, no numerical method has been developed in the literature and, we propose and analyze a mixed variational formulation for a RO model by coupling of the Navier–Stokes and transport equations. In this regard, the consideration of nonlinear transmission conditions on the membrane represents the major difficulty to address the problem, from theoretical and computational points of view, and constitutes another novelty of our work. In the formulation, we introduce two Lagrange multipliers associated to the concentration to deal with the trace of functions that do not have enough regularity to be restricted to the boundary. In this way, the non-linearity at the interface is handled with these new unknowns. Subsequently, an appropriate linearization allows us to overcome this difficulty. Another complexity in analyzing the problem arises from the presence of convective terms in the equations, which can be approached in two ways. The first method involves an augmentation procedure (see, e.g. [8, 9]), offering more flexibility in choosing finite element subspaces, but increasing the complexity and computational cost significantly. Alternatively, one can consider the approach of Banach space-based mixed finite element methods to solve perturbed saddle point formulations. We point out that the motivation of employing the latter approach has the advantages of not requiring an augmentation procedure (see, e.g. [6, 7, 13]) and the spaces where the unknowns are sought are the natural ones that result from applying the Cauchy–Schwarz and Hölder inequalities to the terms obtained from testing and integrating by parts the equations of the model. Consequently, simpler formulations closely aligned with the original physical model are achieved.

The manuscript is organized as follows. In the rest of this section, we provide an overview of the standard notation and functional spaces that will be utilized throughout the paper, introduce the model problem of interest and define the unknowns to be considered in the variational formulation. Subsequently, in Section 2 we identify the saddle point structure of the corresponding variational system. Section 3 analyzes the continuous solvability and the equivalent fixed point setting, presenting the well-posedness result under the assumption of sufficiently small data. In Section 4, we investigate the associated Galerkin scheme by utilizing a discrete version of the fixed point strategy developed in Section 3 for the continuous case. Additionally, we derive the associated *a priori* error estimate in the same section. Furthermore, in Section 5 we specify particular choices of discrete subspaces that satisfy the hypotheses from Section 4 and show the theoretical behaviour of the errors. Next, in Section 6, numerical examples illustrate the performance of the numerical scheme. We end with conclusions in Section 7.

1.1 Preliminaries

Sobolev and Banach spaces. Given a Lipschitz-continuous domain $\mathcal{O}$ of $\mathbb{R}^2$ with boundary Γ, we adopt standard notations for Lebesgue spaces $\mathrm{L}^t(\mathcal{O})$ and Sobolev spaces $\mathrm{W}^{l,t}(\mathcal{O})$, with $l \geqslant 0$ and $t \in [1,+\infty)$, whose corresponding norms, either for the scalar- and vector-valued case, are denoted by $\|\cdot\|_{0,t;\mathcal{O}}$ and $\|\cdot\|_{l,t;\mathcal{O}}$, respectively. Note that $\mathrm{W}^{0,t}(\mathcal{O}) = \mathrm{L}^t(\mathcal{O})$, and if $t = 2$ we write $\mathrm{H}^l(\mathcal{O})$ instead of $\mathrm{W}^{l,2}(\mathcal{O})$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{l,\mathcal{O}}$ and $|\cdot|_{l,\mathcal{O}}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ denotes the space of traces of $\mathrm{H}^1(\mathcal{O})$ and $\mathrm{H}^{-1/2}(\Gamma)$ its dual space, provided with the duality pairing $\langle\cdot,\cdot\rangle_\Gamma$. Also, given $\widetilde{\Gamma} \subseteq \Gamma$, $\mathrm{H}^{1/2}(\widetilde{\Gamma})$ denotes the restriction to $\widetilde{\Gamma}$ of $\mathrm{H}^1(\mathcal{O})$-functions.

On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vector- and tensor-valued counterparts. Furthermore, as usual, $\mathbb{I}$ stands for the identity tensor in $\mathbb{R}^{2\times 2}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$. On the other hand, given $t \in (1,+\infty)$, we introduce the Banach space $\mathbf{H}(\mathrm{div}_t;\mathcal{O}) := \big\{\mathbf{v} \in \mathbf{L}^2(\mathcal{O}) : \mathrm{div}(\mathbf{v}) \in \mathrm{L}^t(\mathcal{O})\big\}$, endowed with the natural norm $\|\mathbf{v}\|_{\mathrm{div}_t;\mathcal{O}} := \|\mathbf{v}\|_{0,\mathcal{O}} + \|\mathrm{div}(\mathbf{v})\|_{0,t;\mathcal{O}}$. The space of matrix-valued functions whose rows belong to $\mathbf{H}(\mathrm{div}_t;\mathcal{O})$ will be denoted by $\mathbb{H}(\mathbf{div}_t;\mathcal{O})$, endowed with the norm $\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\mathcal{O}}$. Here, we let $\mathbf{div}$ be the usual divergence operator div acting row-wise on each tensor.

Additionally, for any tensor fields $\boldsymbol{\tau}$, we let the deviatoric tensor as $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}$.

We recall some definitions and technical results concerning boundary conditions and extension operators [16, 19, 24]. Let $\widetilde{\Gamma} \subseteq \Gamma$, denote by $\widetilde{\Gamma}^{\mathrm{c}}$ its complement and $\mathbf{n}$ the unit outward normal vector on Γ.

Restriction to $\widetilde{\Gamma}$ of functionals in $\mathrm{H}^{-1/2}(\Gamma)$. Let $E_{0,\widetilde{\Gamma}} : \mathrm{H}^{1/2}(\widetilde{\Gamma}) \to \mathrm{L}^2(\Gamma)$ be the extension operator defined as follows: given $\eta \in \mathrm{H}^{1/2}(\widetilde{\Gamma})$, by $E_{0,\widetilde{\Gamma}}(\eta) = \eta$ on $\widetilde{\Gamma}$, and $E_{0,\widetilde{\Gamma}}(\eta) = 0$, otherwise. We define $\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma}) := \Big\{\eta \in \mathrm{H}^{1/2}(\widetilde{\Gamma}) :\ E_{0,\widetilde{\Gamma}}(\eta) \in \mathrm{H}^{1/2}(\Gamma)\Big\}$, endowed with the norm $\|\eta\|_{1/2,00,\widetilde{\Gamma}} := \|E_{0,\widetilde{\Gamma}}(\eta)\|_{1/2,\Gamma}$, and denote by $\mathrm{H}^{-1/2}_{00}(\widetilde{\Gamma})$ its dual space. Now, given $\mu \in \mathrm{H}^{-1/2}(\Gamma)$, its restriction to $\widetilde{\Gamma}^{\mathrm{c}}$, say $\mu|_{\widetilde{\Gamma}}$, is defined as

$$\langle \mu|_{\widetilde{\Gamma}}, \eta\rangle_{\widetilde{\Gamma}} := \langle \mu, E_{0,\widetilde{\Gamma}}(\eta)\rangle_{\Gamma} \quad \forall\, \eta \in \mathrm{H}^{1/2}_{00}(\widetilde{\Gamma}), \tag{1.1}$$

where $\langle\cdot,\cdot\rangle_{\widetilde{\Gamma}}$ stands for the duality pairing of the spaces $\mathrm{H}^{-1/2}_{00}(\widetilde{\Gamma})$ and $\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma})$ with respect to the $\mathrm{L}^2(\widetilde{\Gamma})$ inner product. Then, it is clear that $\mu|_{\widetilde{\Gamma}} \in \mathrm{H}^{-1/2}_{00}(\widetilde{\Gamma})$. On the other hand, the boundary condition $\mu = 0$ on $\widetilde{\Gamma}$ means

$$\langle \mu, E_{0,\widetilde{\Gamma}}(\eta)\rangle_{\Gamma} = 0 \quad \forall\, \eta \in \mathrm{H}^{1/2}_{00}(\widetilde{\Gamma}). \tag{1.2}$$

A continuous extension of $\mathrm{H}^{1/2}(\widetilde{\Gamma})$-functions. According to [19, Section 2], the restriction of $\mu \in \mathrm{H}^{-1/2}(\Gamma)$ to $\widetilde{\Gamma}$ can be identified with an element of $\mathrm{H}^{-1/2}(\widetilde{\Gamma})$, namely

$$\langle \mu, \eta\rangle_{\widetilde{\Gamma}} := \langle \mu, E_{\widetilde{\Gamma}}(\eta)\rangle_{\Gamma} \quad \forall\, \eta \in \mathrm{H}^{1/2}(\widetilde{\Gamma}), \tag{1.3}$$

where $E_{\widetilde{\Gamma}} : \mathrm{H}^{1/2}(\widetilde{\Gamma}) \to \mathrm{H}^{1/2}(\Gamma)$ is any bounded extension operator. In particular, given $\eta \in \mathrm{H}^{1/2}(\widetilde{\Gamma})$, we consider the extension $E_{\widetilde{\Gamma}}(\eta) := z|_{\Gamma}$, where $z \in \mathrm{H}^1(\mathcal{O})$ is the unique solution to the boundary value problem

$$\triangle z = 0 \quad \text{in} \quad \mathcal{O}, \quad z = \eta \quad \text{on} \quad \widetilde{\Gamma}, \quad \text{and} \quad \nabla z \cdot \mathbf{n} = 0 \quad \text{on} \quad \widetilde{\Gamma}^{\mathrm{c}}, \tag{1.4}$$

satisfying $\|z\|_{1,\mathcal{O}} \leqslant C\|\eta\|_{1/2,\widetilde{\Gamma}}$, where C is a positive constant. The latter implies $\|E_{\widetilde{\Gamma}}(\eta)\|_{1/2,\Gamma} \leqslant C\|\eta\|_{1/2,\widetilde{\Gamma}}$.

Decomposition of $\mathrm{H}^{1/2}(\Gamma)$-functions. Given $\zeta \in \mathrm{H}^{1/2}(\Gamma)$, it is not difficult to prove (see, [19, Lemma 2.2]) that there exist unique elements $\zeta_{\widetilde{\Gamma}} \in \mathrm{H}^{1/2}(\widetilde{\Gamma})$ and $\zeta_{\widetilde{\Gamma}^{\mathrm{c}}} \in \mathrm{H}^{1/2}_{00}(\widetilde{\Gamma}^{\mathrm{c}})$ such that $\zeta = E_{\widetilde{\Gamma}}(\zeta_{\widetilde{\Gamma}}) + E_{0,\widetilde{\Gamma}^{\mathrm{c}}}(\zeta_{\widetilde{\Gamma}^{\mathrm{c}}})$, and hence

$$\langle \mu, \zeta\rangle_{\Gamma} := \langle \mu, E_{\widetilde{\Gamma}}(\zeta_{\widetilde{\Gamma}})\rangle_{\Gamma} + \langle \mu, E_{0,\widetilde{\Gamma}^{\mathrm{c}}}(\zeta_{\widetilde{\Gamma}^{\mathrm{c}}})\rangle_{\Gamma} \quad \forall\, \mu \in \mathrm{H}^{-1/2}(\Gamma). \tag{1.5}$$

Remark 1. We denote by $\boldsymbol{E}_{\widetilde{\Gamma}}$ and $\boldsymbol{E}_{0,\widetilde{\Gamma}^{\mathrm{c}}}$ the extension operators acting on vector-valued functions, and (1.5) also holds in this case. They are defined as the element-wise application of the extension operator specified above.

1.2 The model problem

In order to describe the geometry, we let Ω_{f} and Ω_{p} be two open bounded and simply connected polygonal domains in $\mathbb{R}^2$ such that $\partial\Omega_{\mathrm{f}} \cap \partial\Omega_{\mathrm{p}} = \Sigma \neq \emptyset$ and $\Omega_{\mathrm{f}} \cap \Omega_{\mathrm{p}} = \emptyset$, and set $\Omega := \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{p}}$. In turn, for each $\star \in \{\mathrm{f}, \mathrm{p}\}$, $\partial\Omega_\star \backslash \overline{\Sigma}$ is divided in three parts: $\Gamma_{\mathrm{in},\star}$ (inlet), $\Gamma_{\mathrm{out},\star}$ (outlet) and $\Gamma_{\mathrm{w},\star}$ (wall), such that $\partial\Omega_\star \backslash \overline{\Sigma} = \Gamma_{\mathrm{in},\star} \cup \overline{\Gamma}_{\mathrm{w},\star} \cup \Gamma_{\mathrm{out},\star}$ as depicted in Figure 1.1.The unit normal vector, $\mathbf{n}_\star$, is chosen pointing outward from $\Omega_\star$, thus $\mathbf{n}_{\mathrm{f}} = -\mathbf{n}_{\mathrm{p}}$ on the interior points of Σ. We also consider a unit tangent vector $\boldsymbol{m}_\Sigma$ on Σ as drawn Figure 1.1. We are interested in the Navier–Stokes/transport coupled problem, which is formulated in what follows in terms of the fluid velocity $\mathbf{u}_\star$, the fluid pressure $p_\star$, and the salt concentration $\widetilde{\phi}_\star$ occupying the region $\Omega_\star$, for each $\star \in \{\mathrm{f}, \mathrm{p}\}$. More precisely, the corresponding system of equations is given by

$$\begin{gathered}
-\nu\,\triangle\mathbf{u}_\star + \rho\,\mathbf{div}(\mathbf{u}_\star \otimes \mathbf{u}_\star) + \nabla p_\star = \mathbf{0} \quad \text{in} \quad \Omega_\star, \quad \mathrm{div}(\mathbf{u}_\star) = 0 \quad \text{in} \quad \Omega_\star,\\
\mathbf{u}_\star = \mathbf{u}_{\mathrm{in},\star} \quad \text{on} \quad \Gamma_{\mathrm{in},\star}, \quad \mathbf{u}_\star = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{w},\star}, \quad (\nu\nabla\mathbf{u}_\star - p_\star\mathbb{I})\mathbf{n}_\star = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{out},\star},\\
-\kappa\,\triangle\widetilde{\phi}_\star + \mathbf{u}_\star \cdot \nabla\widetilde{\phi}_\star = 0 \quad \text{in} \quad \Omega_\star,\\
\widetilde{\phi}_\star = \widetilde{\phi}_{\mathrm{in},\star} \quad \text{on} \quad \Gamma_{\mathrm{in},\star}, \quad \kappa\,\nabla\widetilde{\phi}_\star \cdot \mathbf{n}_\star = 0 \quad \text{on} \quad \Gamma_{\mathrm{w},\star} \cup \Gamma_{\mathrm{out},\star},
\end{gathered} \tag{1.6}$$

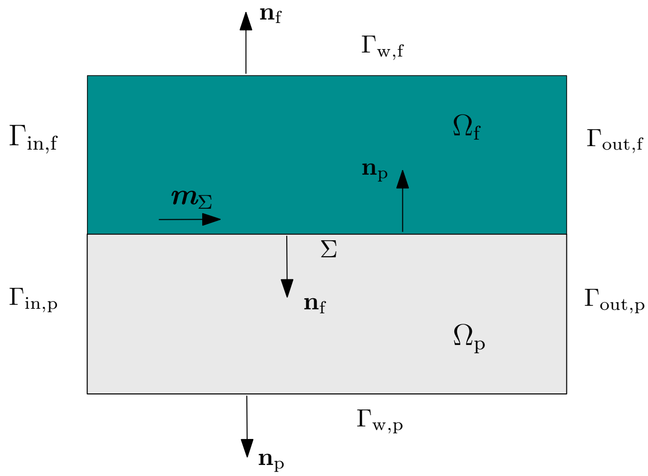

Figure 1.1: Sketch of the geometry.

where ν is the fluid dynamic viscosity, ρ is the fluid density and κ is the solute diffusivity through the solvent. All these parameters are assumed to be positive constants. In addition, for each $\star \in \{\rm f, p\}$, $\mathbf{u}_{{\rm in},\star} \in \mathbf{H}^{1/2}(\Gamma_{{\rm in},\star})$ is a given inlet velocity profile, and $\widetilde{\phi}_{{\rm in},\star} \in \mathbb{R}$. The corresponding transmission conditions are given by

$$\begin{gathered}\mathbf{u}_\star \cdot \boldsymbol{m}_\Sigma = 0, \quad \mathbf{u}_{\rm f} \cdot \mathbf{n}_{\rm f} \;=\; -\mathbf{u}_{\rm p} \cdot \mathbf{n}_{\rm p}, \quad \mathbf{u}_{\rm f} \cdot \mathbf{n}_{\rm f} \;=\; A\Delta P - AiRT(\widetilde{\phi}_{\rm f} - \widetilde{\phi}_{\rm p}) \quad \text{on} \quad \Sigma, \\ (\widetilde{\phi}_{\rm f}\mathbf{u}_{\rm f} - \kappa\,\nabla\widetilde{\phi}_{\rm f}) \cdot \mathbf{n}_{\rm f} \;=\; -(\widetilde{\phi}_{\rm p}\mathbf{u}_{\rm p} - \kappa\,\nabla\widetilde{\phi}_{\rm p}) \cdot \mathbf{n}_{\rm p}, \quad (\widetilde{\phi}_{\rm f}\mathbf{u}_{\rm f} - \kappa\,\nabla\widetilde{\phi}_{\rm f}) \cdot \mathbf{n}_{\rm f} \;=\; B(\widetilde{\phi}_{\rm f} - \widetilde{\phi}_{\rm p}) \quad \text{on} \quad \Sigma.\end{gathered} \tag{1.7}$$

Here, A, ΔP, i, R, T and B are physical parameters assumed to be positive constants. Specific values of the parameters can be found in Table 6.2. In turn, denoting $a_0 := A\Delta P$, $a_1 := AiRT$ and $a_2 := B$, we realize that they are also positive constants satisfying the following conditions:

$$2a_1\widetilde{\phi}_{\rm in,f} + a_2 \;\geqslant\; a_0 + a_1\widetilde{\phi}_{\rm in,p}, \quad \text{and} \quad 2a_1\widetilde{\phi}_{\rm in,p} + a_0 + a_2 \;\geqslant\; a_1\widetilde{\phi}_{\rm in,f}. \tag{1.8}$$

Next, since we are interested in a mixed variational formulation, and in order to employ the integration by parts formula typically required by this approach, motivated by the Neumann-type boundary conditions, we introduce the auxiliary unknowns:

$$\boldsymbol{\sigma}_\star := \nu\nabla\mathbf{u}_\star - p_\star\mathbb{I} \quad \text{and} \quad \mathbf{t}_\star := \kappa\,\nabla\phi_\star \quad \text{in} \quad \Omega_\star. \tag{1.9}$$

In this way, noting that $\mathbf{div}(\mathbf{u}_\star \otimes \mathbf{u}_\star) = (\nabla\mathbf{u}_\star)\mathbf{u}_\star$, which makes use of the fact that $\mathrm{div}(\mathbf{u}_\star) = 0$ in $\Omega_\star$, we find that the first equation of (1.6) can be rewritten as $\mathbf{div}(\boldsymbol{\sigma}_\star) = \rho(\nabla\mathbf{u}_\star)\mathbf{u}_\star$, whereas the third row of (1.6) becomes $k\,\mathrm{div}(\mathbf{t}_\star) = \mathbf{u}_\star \cdot \mathbf{t}_\star$.

In turn, it is straightforward to see, taking matrix trace, that the first equation of (1.9) together with the incompressibility condition $\mathrm{div}(\mathbf{u}_\star) = 0$, are equivalent to the pair

$$\frac{1}{\nu}\boldsymbol{\sigma}_\star^{\rm d} \;=\; \nabla\mathbf{u}_\star \quad \text{in} \quad \Omega_\star, \quad \text{and} \quad p_\star \;=\; -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_\star) \quad \text{in} \quad \Omega_\star. \tag{1.10}$$

On the other hand, considering the new variables, the transmission conditions (1.7) become

$$\begin{gathered}\mathbf{u}_\star \cdot \boldsymbol{m}_\Sigma = 0, \quad \mathbf{u}_{\rm f} \cdot \mathbf{n}_{\rm f} \;=\; -\mathbf{u}_{\rm p} \cdot \mathbf{n}_{\rm p}, \quad \mathbf{u}_{\rm f} \cdot \mathbf{n}_{\rm f} \;=\; a_0 - a_1(\widetilde{\phi}_{\rm f} - \widetilde{\phi}_{\rm p}) \quad \text{on} \quad \Sigma, \\ (\widetilde{\phi}_{\rm f}\mathbf{u}_{\rm f} - \mathbf{t}_{\rm f}) \cdot \mathbf{n}_{\rm f} \;=\; -(\widetilde{\phi}_{\rm p}\mathbf{u}_{\rm p} - \mathbf{t}_{\rm p}) \cdot \mathbf{n}_{\rm p}, \quad (\widetilde{\phi}_{\rm f}\mathbf{u}_{\rm f} - \mathbf{t}_{\rm f}) \cdot \mathbf{n}_{\rm f} \;=\; a_2(\widetilde{\phi}_{\rm f} - \widetilde{\phi}_{\rm p}) \quad \text{on} \quad \Sigma.\end{gathered} \tag{1.11}$$

Moreover, for the sake of the subsequent analysis, in order to obtain a homogeneous Dirichlet condition for the concentration unknown, we consider the change of variable $\phi_\star := \widetilde{\phi}_\star - \widetilde{\phi}_{{\rm in},\star}$ in $\Omega_\star$, and find that the third equation of the first row of (1.11) can be rewritten as

$$\mathbf{u}_{\rm f} \cdot \mathbf{n}_{\rm f} = -a_1(\phi_{\rm f} - \phi_{\rm p}) + \widetilde{a}_0 \quad \text{on} \quad \Sigma, \quad \text{where } \widetilde{a}_0 = a_0 - a_1(\widetilde{\phi}_{\rm in,f} - \widetilde{\phi}_{\rm in,p}), \tag{1.12}$$

whereas the second row of (1.11), becomes

$$\begin{aligned}(\{\phi_{\rm f} + \widetilde{\phi}_{\rm in,f}\}\mathbf{u}_{\rm f} - \mathbf{t}_{\rm f}) \cdot \mathbf{n}_{\rm f} \;&=\; -(\{\phi_{\rm p} + \widetilde{\phi}_{\rm in,p}\}\mathbf{u}_{\rm p} - \mathbf{t}_{\rm p}) \cdot \mathbf{n}_{\rm p} \quad \text{on} \quad \Sigma, \\ (\{\phi_{\rm f} + \widetilde{\phi}_{\rm in,f}\}\mathbf{u}_{\rm f} - \mathbf{t}_{\rm f}) \cdot \mathbf{n}_{\rm f} \;&=\; a_2(\phi_{\rm f} - \phi_{\rm p} + \widetilde{\phi}_{\rm in,f} - \widetilde{\phi}_{\rm in,p}) \quad \text{on} \quad \Sigma.\end{aligned} \tag{1.13}$$

In this way, replacing (1.12) back into (1.13) and utilizing the second equation of the first row of (1.11), some algebraic manipulations allow us to arrive at the following system of equations:

$$\begin{gathered}\frac{1}{\nu}\boldsymbol{\sigma}_\star^{\mathrm{d}} = \nabla\mathbf{u}_\star \quad \text{in} \quad \Omega_\star, \quad \mathbf{div}(\boldsymbol{\sigma}_\star) = \frac{\rho}{\nu}\boldsymbol{\sigma}_\star^{\mathrm{d}}\mathbf{u}_\star \quad \text{in} \quad \Omega_\star, \\ \mathbf{u}_\star = \mathbf{u}_{\mathrm{in},\star} \quad \text{on} \quad \Gamma_{\mathrm{in},\star}, \quad \mathbf{u}_\star = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{w},\star}, \quad \boldsymbol{\sigma}_\star\mathbf{n}_\star = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{out},\star}, \\ \mathbf{t}_\star = \kappa\nabla\phi_\star \quad \text{in} \quad \Omega_\star, \quad \kappa\,\mathrm{div}(\mathbf{t}_\star) = \mathbf{u}_\star\cdot\mathbf{t}_\star \quad \text{in} \quad \Omega_\star, \\ \phi_\star = 0 \quad \text{on} \quad \Gamma_{\mathrm{in},\star}, \quad \text{and} \quad \mathbf{t}_\star\cdot\mathbf{n}_\star = 0 \quad \text{on} \quad \Gamma_{\mathrm{out},\star}\cup\Gamma_{\mathrm{w},\star},\end{gathered} \tag{1.14}$$

with transmission conditions

$$\begin{aligned}\mathbf{u}_\star\cdot\boldsymbol{m}_\Sigma = 0, \quad \mathbf{u}_\mathrm{f}\cdot\mathbf{n}_\mathrm{f} &= -\mathbf{u}_\mathrm{p}\cdot\mathbf{n}_\mathrm{p}, \quad \mathbf{u}_\mathrm{f}\cdot\mathbf{n}_\mathrm{f} = -a_1(\phi_\mathrm{f}-\phi_\mathrm{p}) + \widetilde{a}_0 \quad \text{on} \quad \Sigma, \\ \mathbf{t}_\mathrm{f}\cdot\mathbf{n}_\mathrm{f} &= -a_1(\phi_\mathrm{f}-\phi_\mathrm{p})\phi_\mathrm{f} - (\widetilde{a}_{1,\mathrm{f}}-\widetilde{a}_0)\phi_\mathrm{f} + \widetilde{a}_{1,\mathrm{f}}\phi_\mathrm{p} + \widetilde{a}_{2,\mathrm{f}} \quad \text{on} \quad \Sigma, \quad \text{and} \\ \mathbf{t}_\mathrm{p}\cdot\mathbf{n}_\mathrm{p} &= a_1(\phi_\mathrm{f}-\phi_\mathrm{p})\phi_\mathrm{p} - (\widetilde{a}_0+\widetilde{a}_{1,\mathrm{p}})\phi_\mathrm{p} + \widetilde{a}_{1,\mathrm{p}}\phi_\mathrm{f} - \widetilde{a}_{2,\mathrm{p}} \quad \text{on} \quad \Sigma,\end{aligned} \tag{1.15}$$

where $\widetilde{a}_0 = a_0 - a_1(\widetilde{\phi}_{\mathrm{in,f}} - \widetilde{\phi}_{\mathrm{in,p}})$, $\widetilde{a}_{1,\star} = a_1\widetilde{\phi}_{\mathrm{in},\star} + a_2$ and $\widetilde{a}_{2,\star} = \widetilde{a}_0\widetilde{\phi}_{\mathrm{in},\star} - a_2(\widetilde{\phi}_{\mathrm{in,f}} - \widetilde{\phi}_{\mathrm{in,p}})$.

Note that the pressure has been eliminated from the system, but can be recovered by (1.10). Also, we observe that, thanks to the above change of variable, condition (1.8) becomes:

$$\widetilde{a}_{1,\mathrm{f}} - \widetilde{a}_0 \geqslant 0 \quad \text{and} \quad \widetilde{a}_{1,\mathrm{p}} + \widetilde{a}_0 \geqslant 0, \tag{1.16}$$

which will be used later to guarantee the result in (3.20).

2 The mixed formulation

Given $\star \in \{\mathrm{f},\mathrm{p}\}$, we first set $\Gamma_{\mathrm{in},\star}^{\mathrm{c}} := \partial\Omega_\star\backslash\Gamma_{\mathrm{in},\star}$ and $\Gamma_{\mathrm{out},\star}^{\mathrm{c}} := \partial\Omega_\star\backslash\Gamma_{\mathrm{out},\star}$. We begin by testing the first equations of the first and third rows of (1.14) against tensor- and vector-valued functions $\boldsymbol{\tau}_\star$ and $\mathbf{s}_\star$, yielding

$$\frac{1}{\nu}\int_{\Omega_\star}\boldsymbol{\sigma}_\star^{\mathrm{d}} : \boldsymbol{\tau}_\star - \int_{\Omega_\star}\nabla\mathbf{u}_\star : \boldsymbol{\tau}_\star = 0 \quad \text{and} \quad \frac{1}{\kappa}\int_{\Omega_\star}\mathbf{t}_\star\cdot\mathbf{s}_\star - \int_{\Omega_\star}\nabla\phi_\star\cdot\mathbf{s}_\star = 0, \tag{2.1}$$

respectively. It is clear that the first terms in (2.1) are well-defined if $\boldsymbol{\sigma}_\star, \boldsymbol{\tau}_\star \in \mathbb{L}^2(\Omega_\star)$ and $\mathbf{t}_\star, \mathbf{s}_\star \in \mathbf{L}^2(\Omega_\star)$. In turn, multiplying the second equations of the first and third rows of (1.14) by a vector- and scalar-valued functions $\mathbf{v}_\star$ and $\psi_\star$, respectively, we notice that

$$\int_{\Omega_\star}\mathbf{v}_\star\cdot\mathbf{div}(\boldsymbol{\sigma}_\star) - \frac{\rho}{\nu}\int_{\Omega_\star}\boldsymbol{\sigma}_\star^{\mathrm{d}}\mathbf{u}_\star\cdot\mathbf{v}_\star = 0 \quad \text{and} \quad \int_{\Omega_\star}\mathrm{div}(\mathbf{t}_\star)\psi_\star - \frac{1}{\kappa}\int_{\Omega_\star}\mathbf{u}_\star\cdot\mathbf{t}_\star\psi_\star = 0. \tag{2.2}$$

Then, knowing that $\boldsymbol{\sigma}_\star$ and $\mathbf{t}_\star$ are L^2-functions, using the Cauchy–Schwarz and Hölder inequalities, we find that for all $s,t \in (1,+\infty)$, such that $\frac{1}{s}+\frac{1}{t}=1$, there hold

$$\left|\int_{\Omega_\star}\boldsymbol{\sigma}_\star^{\mathrm{d}}\mathbf{u}_\star\cdot\mathbf{v}_\star\right| \leqslant \|\boldsymbol{\sigma}_\star\|_{0;\Omega_\star}\|\mathbf{u}_\star\|_{0,2s;\Omega_\star}\|\mathbf{v}_\star\|_{0,2t;\Omega_\star} \quad \text{and} \quad \left|\int_{\Omega_\star}\mathbf{u}_\star\cdot\mathbf{t}_\star\psi_\star\right| \leqslant \|\mathbf{u}_\star\|_{0,2s;\Omega_\star}\|\mathbf{t}_\star\|_{0,\Omega_\star}\|\psi_\star\|_{0,2t;\Omega_\star},$$

which show that the second terms of the left-hand sides in the equations of (2.2) make sense for $\mathbf{u}_\star \in \mathbf{L}^{2s}(\Omega_\star)$, $\mathbf{v}_\star \in \mathbf{L}^{2t}(\Omega_\star)$ and $\psi_\star \in \mathrm{L}^{2t}(\Omega_\star)$. Thus, forcing their first terms to require that $\mathbf{div}(\boldsymbol{\sigma}_\star) \in \mathbf{L}^{(2t)'}(\Omega_\star)$, and $\mathrm{div}(\mathbf{t}_\star) \in \mathrm{L}^{(2t)'}(\Omega_\star)$, where $(2t)' := \frac{2t}{2t-1}$ is the conjugate of $2t$.

Now, we go back to the second equation of (2.1) to deal with the second term. After applying the integration by parts formula and considering the test function $\mathbf{s}_\star$ in the same space of $\mathbf{t}_\star$, we realize that the volumetric terms are well defined if $\mathrm{div}(\mathbf{s}_\star) \in \mathrm{L}^{(2t)'}(\Omega_\star)$ and $\phi_\star \in \mathrm{L}^{2t}(\Omega_\star)$. Since traces of $\mathrm{L}^{2t}(\Omega_\star)$-functions are not defined, we introduce a Lagrange multiplier $\xi_\star := -\phi_\star|_{\Gamma_{\mathrm{in},\star}^{\mathrm{c}}} \in \mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{in},\star}^{\mathrm{c}})$, and realize that the second equation of (2.1), becomes

$$\frac{1}{\kappa}\int_{\Omega_\star}\mathbf{t}_\star\cdot\mathbf{s}_\star + \int_{\Omega_\star}\phi_\star\mathrm{div}(\mathbf{s}_\star) + \langle\mathbf{s}_\star\cdot\mathbf{n}_\star, \xi_\star\rangle_{\Gamma_{\mathrm{in},\star}^{\mathrm{c}}} = 0.$$

Here, we have used the facts that $\phi_\star = 0$ on $\Gamma_{\text{in},\star}$ and $\mathbf{s}_\star \cdot \mathbf{n}_\star$ is well defined if $\mathrm{H}^1(\Omega_\star)$ is continuously embedded in $\mathrm{L}^{2t}(\Omega_\star)$ (see [13, Section 3.1] for details). The latter is guaranteed in two dimensions for $2t \in [1, +\infty)$.

Similarly, to apply the integration by parts formula to the first equation of (2.1) and obtain

$$\frac{1}{\nu}\int_{\Omega_\star} \boldsymbol{\sigma}^{\mathrm{d}}_\star : \boldsymbol{\tau}_\star + \int_{\Omega_\star} \mathbf{u}_\star \cdot \mathbf{div}(\boldsymbol{\tau}_\star) \;=\; \langle \boldsymbol{\tau}_\star \mathbf{n}_\star, \mathbf{u}_\star \rangle_{\partial\Omega_\star}, \tag{2.3}$$

it suffices to assume that $\mathbf{div}(\boldsymbol{\tau}_\star) \in \mathbf{L}^{(2s)'}(\Omega_\star)$, and that $\mathbf{H}^1(\Omega_\star)$ is continuously embedded in $\mathbf{L}^{2s}(\Omega_\star)$, where $(2s)' := \frac{2s}{2s-1}$ is the conjugate of $2s$, so that $\boldsymbol{\tau}_\star \mathbf{n}_\star$ is well defined. This is guaranteed in two dimensions for $2s \in [1, +\infty)$. It remains to properly handle the term $\langle \boldsymbol{\tau}_\star \mathbf{n}_\star, \mathbf{u}_\star \rangle_{\partial\Omega_\star}$ since, *a priori*, the trace of $\mathbf{u}_\star$ is not in $\mathbf{H}^{1/2}(\partial\Omega_\star)$. To that end, we will make use of the boundary and transmission conditions specified in (1.14) and (1.15). More precisely, let $\star \in \{\mathrm{f}, \mathrm{p}\}$ and for the sake of convenience we define the following auxiliary functions

$$\mathbf{g}_\star := \begin{cases} \mathbf{0} & \text{on} \quad \Gamma_{\mathrm{w},\star}, \\ \mathbf{u}_{\mathrm{in},\star} & \text{on} \quad \Gamma_{\mathrm{in},\star}, \\ \widetilde{a}_0 \mathbf{n}_\mathrm{f} & \text{on} \quad \Sigma, \end{cases} \quad \text{and} \quad \mathbf{g}^\xi_\star := \begin{cases} \mathbf{0} & \text{on} \quad \Gamma_{\mathrm{w},\star}, \\ \mathbf{0} & \text{on} \quad \Gamma_{\mathrm{in},\star}, \\ a_1(\xi_\mathrm{f} - \xi_\mathrm{p})\mathbf{n}_\mathrm{f} & \text{on} \quad \Sigma. \end{cases} \tag{2.4}$$

We observe that $\mathbf{g}_\star \in \mathbf{H}^{1/2}(\Gamma^{\mathrm{c}}_{\mathrm{out},\star})$ if $\mathbf{u}_{\mathrm{in},\star}$ fulfills the following compatibility conditions:

$$\begin{aligned} \mathbf{u}_{\mathrm{in,f}} &= \mathbf{0} \quad \text{on} \quad \overline{\Gamma}_{\mathrm{in,f}} \cap \overline{\Gamma}_{\mathrm{w,f}}, \qquad & \mathbf{u}_{\mathrm{in,p}} &= \mathbf{0} \quad \text{on} \quad \overline{\Gamma}_{\mathrm{in,p}} \cap \overline{\Gamma}_{\mathrm{w,p}}, \\ \mathbf{u}_{\mathrm{in,f}} &= \widetilde{a}_0 \mathbf{n}_\mathrm{f} \quad \text{on} \quad \overline{\Gamma}_{\mathrm{in,f}} \cap \overline{\Sigma}, \qquad & \mathbf{u}_{\mathrm{in,p}} &= -\widetilde{a}_0 \mathbf{n}_\mathrm{p} \quad \text{on} \quad \overline{\Gamma}_{\mathrm{in,p}} \cap \overline{\Sigma}, \end{aligned} \tag{2.5}$$

where, for $\boldsymbol{x} \in \overline{\Gamma}_{\mathrm{in},\star} \cap \overline{\Sigma}$, $\mathbf{n}_\star(\boldsymbol{x})$ is taking as $\mathbf{n}_\star(\boldsymbol{x}) = \lim_{\varepsilon \to 0^+} \mathbf{n}_\star(\boldsymbol{x} - \varepsilon \boldsymbol{m}_\Sigma)$. Moreover $\mathbf{g}^\xi_\star$ is also in $\mathbf{H}^{1/2}(\Gamma^{\mathrm{c}}_{\mathrm{out},\star})$ since we recall that $\xi_\star \in \mathrm{H}^{1/2}_{00}(\Gamma^{\mathrm{c}}_{\mathrm{in},\star})$. In this way, bearing in mind the boundary and transmission conditions (1.14) and (1.15), and considering the test function $\boldsymbol{\tau}_\star$ in the space $\mathbb{H}_{\Gamma_{\mathrm{out}}}(\mathbf{div}_{(2s)'}; \Omega_\star) := \Big\{ \boldsymbol{\tau}_\star \in \mathbb{H}(\mathbf{div}_{(2s)'}; \Omega_\star) :$ $\boldsymbol{\tau}_\star \mathbf{n}_\star = \mathbf{0}$ on $\Gamma_{\mathrm{out},\star} \Big\}$, where $\boldsymbol{\tau}_\star \mathbf{n}_\star = \mathbf{0}$ on $\Gamma_{\mathrm{out},\star}$ is understood in the sense of (1.2), from (2.3) we find that

$$\frac{1}{\nu}\int_{\Omega_\star} \boldsymbol{\sigma}^{\mathrm{d}}_\star : \boldsymbol{\tau}_\star + \int_{\Omega_\star} \mathbf{u}_\star \cdot \mathbf{div}(\boldsymbol{\tau}_\star) \;=\; \langle \boldsymbol{\tau}_\star \mathbf{n}_\star, \mathbf{g}_\star + \mathbf{g}^\xi_\star \rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out},\star}} \quad \forall\, \boldsymbol{\tau}_\star \in \mathbb{H}_{\Gamma_{\mathrm{out}}}(\mathbf{div}_{(2s)'}; \Omega_\star).$$

If we would also like to seek $\boldsymbol{\sigma}_\star$ and $\boldsymbol{\tau}_\star$ in the same function space, it follows immediately that $s = t = 2$ and $(2s)' = (2t)' = 4/3$, which we will be considered from now on.

Remark 2. The compatibility conditions (2.5) are satisfied, for example, if we consider a domain $\Omega = \Omega_\mathrm{f} \cup \Omega_\mathrm{p} \cup \Sigma$, where

$$\Omega_\mathrm{f} \;=\; (0, L) \times (d, 2d), \quad \Omega_\mathrm{p} \;=\; (0, L) \times (0, d), \quad \text{and} \quad \Sigma \;=\; (0, L) \times \{d\},$$

with the following inlet velocities:

$$\mathbf{u}_{\mathrm{in,f}} := \begin{pmatrix} 6\mathrm{u}_{\mathrm{in,f}}\Big(1 - \frac{y}{d}\Big)\left(\frac{y}{d} - 2\right) \\ \widetilde{a}_0\Big(\frac{y}{d} - 2\Big) \end{pmatrix}, \quad y \in [d, 2d], \quad \text{and} \quad \mathbf{u}_{\mathrm{in,p}} := \begin{pmatrix} 6\mathrm{u}_{\mathrm{in,p}}\frac{y}{d}\Big(1 - \frac{y}{d}\Big) \\ -\widetilde{a}_0 \frac{y}{d} \end{pmatrix}, \quad y \in [0, d],$$

where $\mathrm{u}_{\mathrm{in,f}}$ and $\mathrm{u}_{\mathrm{in,p}}$ stand for the inlet mean feed and permeate fluid velocities, respectively. We observe that, when the second component of any of the above velocities is zero, the first component is similar to that of the Poiseuille flux, which is a parabolic profile with unit mean velocity. On the other hand, when the second component of the inlet velocity does not vanish, the profile is similar to the Berman flow. The latter is commonly used to model a constant permeate flux through a membrane as shown in [4, 5, 10, 26, 32] to name a few references.

In turn, thanks to the last equation in (1.14), the last two equations of (1.15), and the fact that $\xi_\star \in \mathrm{H}^{1/2}_{00}(\Gamma^{\mathrm{c}}_{\mathrm{in},\star})$, we deduce that

$$\langle \mathbf{t}_\mathrm{f} \cdot \mathbf{n}_\mathrm{f}\,, \eta_\mathrm{f} \rangle_{\Gamma^{\mathrm{c}}_{\mathrm{in,f}}} \;=\; a_1 \int_\Sigma (\xi_\mathrm{p} - \xi_\mathrm{f})\, \xi_\mathrm{f}\, \eta_\mathrm{f} + \big(\widetilde{a}_{1,\mathrm{f}} - \widetilde{a}_0\big) \int_\Sigma \xi_\mathrm{f}\, \eta_\mathrm{f} \;-\; \widetilde{a}_{1,\mathrm{f}} \int_\Sigma \xi_\mathrm{p} \eta_\mathrm{f} + \widetilde{a}_{2,\mathrm{f}} \int_\Sigma \eta_\mathrm{f},$$

$$\langle \mathbf{t}_{\rm p}\cdot\mathbf{n}_{\rm p}\,,\,\eta_{\rm p}\rangle_{\Gamma^{\rm c}_{\rm in,p}} \,=\, -\,a_1\int_\Sigma(\xi_{\rm p}-\xi_{\rm f})\,\xi_{\rm p}\,\eta_{\rm p} + \big(\widetilde{a}_{1,\rm p}+\widetilde{a}_0\big)\int_\Sigma \xi_{\rm p}\,\eta_{\rm p} - \widetilde{a}_{1,\rm p}\int_\Sigma \xi_{\rm f}\eta_{\rm p} - \widetilde{a}_{2,\rm p}\int_\Sigma \eta_{\rm p}$$

for all $\eta_\star \in \mathrm{H}^{1/2}_{00}(\Gamma^{\rm c}_{\rm in,\star})$, with $\star\in\{\rm f,p\}$. Consequently, introducing the spaces

$$\mathbb{H}^\star_{\rm S} := \mathbb{H}_{\Gamma_{\rm out}}(\mathbf{div}_{4/3};\Omega_\star), \qquad \mathbf{H}^\star_{\rm T} := \mathbf{H}(\mathrm{div}_{4/3};\Omega_\star), \qquad \mathrm{M}^\star_{\rm T} := \mathrm{H}^{1/2}_{00}(\Gamma^{\rm c}_{\rm in,\star}),$$
$$\mathbf{Q}^\star_{\rm S} := \mathbf{L}^4(\Omega_\star), \qquad \mathrm{X}^\star_{\rm T} := \mathrm{L}^4(\Omega_\star),$$

defining the global spaces

$$\mathbb{H} := \mathbb{H}^{\rm f}_{\rm S}\times\mathbb{H}^{\rm p}_{\rm S}, \qquad \mathbf{H} := \mathbf{H}^{\rm f}_{\rm T}\times\mathbf{H}^{\rm p}_{\rm T}, \qquad \mathrm{M} := \mathrm{M}^{\rm f}_{\rm T}\times\mathrm{M}^{\rm p}_{\rm T},$$
$$\mathbf{Q} := \mathbf{Q}^{\rm f}_{\rm S}\times\mathbf{Q}^{\rm p}_{\rm S}, \qquad \mathrm{X} := \mathrm{X}^{\rm f}_{\rm T}\times\mathrm{X}^{\rm p}_{\rm T}, \qquad \mathrm{Q} := \mathrm{X}\times\mathrm{M},$$

setting the notation

$$\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\rm f},\boldsymbol{\tau}_{\rm p})\in\mathbb{H}, \qquad \vec{\mathbf{s}} := (\mathbf{s}_{\rm f},\mathbf{s}_{\rm p})\in\mathbf{H}, \qquad \vec{\eta} := (\eta_{\rm f},\eta_{\rm p})\in\mathrm{M},$$
$$\vec{\mathbf{w}} := (\mathbf{w}_{\rm f},\mathbf{w}_{\rm p})\in\mathbf{Q}, \qquad \vec{\psi} := (\psi_{\rm f},\psi_{\rm p})\in\mathrm{X}, \qquad (\vec{\psi},\vec{\eta}) := (\psi_{\rm f},\psi_{\rm p},\eta_{\rm f},\eta_{\rm p})\in\mathrm{Q},$$

and equipping the above global spaces with the norms

$$\begin{aligned}
\|\vec{\boldsymbol{\tau}}\|_{\mathbb{H}} &:= \|\boldsymbol{\tau}_{\rm f}\|_{\mathbf{div}_{4/3};\Omega_{\rm f}} + \|\boldsymbol{\tau}_{\rm p}\|_{\mathbf{div}_{4/3};\Omega_{\rm p}} && \forall\,\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\rm f},\boldsymbol{\tau}_{\rm p})\in\mathbb{H},\\
\|\vec{\mathbf{w}}\|_{\mathbf{Q}} &:= \|\mathbf{w}_{\rm f}\|_{0,4;\Omega_{\rm f}} + \|\mathbf{w}_{\rm p}\|_{0,4;\Omega_{\rm p}} && \forall\,\vec{\mathbf{w}} := (\mathbf{w}_{\rm f},\mathbf{w}_{\rm p})\in\mathbf{Q},\\
\|\vec{\mathbf{s}}\|_{\mathbf{H}} &:= \|\mathbf{s}_{\rm f}\|_{\mathrm{div}_{4/3};\Omega_{\rm f}} + \|\mathbf{s}_{\rm p}\|_{\mathrm{div}_{4/3};\Omega_{\rm p}} && \forall\,\vec{\mathbf{s}} := (\mathbf{s}_{\rm f},\mathbf{s}_{\rm p})\in\mathbf{H},\\
\|\vec{\psi}\|_{\mathrm{X}} &:= \|\psi_{\rm f}\|_{0,4;\Omega_{\rm f}} + \|\psi_{\rm p}\|_{0,4;\Omega_{\rm p}} && \forall\,\vec{\psi} := (\psi_{\rm f},\psi_{\rm p})\in\mathrm{X},\\
\|\vec{\eta}\|_{\mathrm{M}} &:= \|\eta_{\rm f}\|_{1/2,00,\Gamma^{\rm c}_{\rm in,f}} + \|\eta_{\rm p}\|_{1/2,00,\Gamma^{\rm c}_{\rm in,p}} && \forall\,\vec{\eta} := (\eta_{\rm f},\eta_{\rm p})\in\mathrm{M},\\
\|(\vec{\psi},\vec{\eta})\|_{\mathrm{Q}} &:= \|\vec{\psi}\|_{\mathrm{X}} + \|\vec{\eta}\|_{\mathrm{M}} && \forall\,(\vec{\psi},\vec{\eta}) := (\psi_{\rm f},\psi_{\rm p},\eta_{\rm f},\eta_{\rm p})\in\mathrm{Q},
\end{aligned}$$

we arrive at the following variational formulation of (1.14): Find $\big(\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}\big)\in\mathbb{H}\times\mathbf{Q}$ and $\big(\vec{\mathbf{t}},(\vec{\phi},\vec{\xi})\big)\in\mathbf{H}\times\mathrm{Q}$ such that

$$\begin{array}{rlllll}
\mathbf{a}_{\rm S}(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) & +\,\mathbf{b}_{\rm S}(\vec{\boldsymbol{\tau}},\vec{\mathbf{u}}) & & & = & \mathbf{F}_{\vec{\xi}}(\vec{\boldsymbol{\tau}}),\\
\mathbf{b}_{\rm S}(\vec{\boldsymbol{\sigma}},\vec{\mathbf{v}}) & & -\,\mathbf{O}_{\rm S}(\vec{\mathbf{u}};\vec{\boldsymbol{\sigma}},\vec{\mathbf{v}}) & & = & 0,\\
\mathbf{a}_{\rm T}(\vec{\mathbf{t}},\vec{\mathbf{s}}) & +\,\mathbf{b}_{\rm T}(\vec{\mathbf{s}},(\vec{\phi},\vec{\xi})) & & & = & 0,\\
\mathbf{b}_{\rm T}(\vec{\mathbf{t}},(\vec{\psi},\vec{\eta})) & -\,\mathbf{d}_{\rm T}((\vec{\phi},\vec{\xi}),(\vec{\psi},\vec{\eta})) & -\,\mathbf{O}_{\rm T}(\vec{\mathbf{u}};\vec{\mathbf{t}},\vec{\psi}) & -\,\mathbf{C}_{\rm T}(\vec{\xi};\vec{\xi},\vec{\eta}) & = & \mathcal{F}_{\vec{\xi}}(\vec{\psi},\vec{\eta}),
\end{array} \tag{2.6}$$

for all $(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q}$ and $(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta}))\in\mathbf{H}\times\mathrm{Q}$, where, $\mathbf{a}_{\rm S}:\mathbb{H}\times\mathbb{H}\to\mathbb{R}$, $\mathbf{a}_{\rm T}:\mathbf{H}\times\mathbf{H}\to\mathbb{R}$, $\mathbf{b}_{\rm S}:\mathbb{H}\times\mathbf{Q}\to\mathbb{R}$, $\mathbf{b}_{\rm T}:\mathbf{H}\times\mathrm{Q}\to\mathbb{R}$, and $\mathbf{d}_{\rm T}:\mathrm{Q}\times\mathrm{Q}\to\mathbb{R}$, are the bilinear forms defined by

$$\mathbf{a}_{\rm S}(\vec{\boldsymbol{\sigma}},\,\vec{\boldsymbol{\tau}}) := \frac{1}{\nu}\sum_{\star\in\{\rm f,p\}}\int_{\Omega_\star}\boldsymbol{\sigma}^{\rm d}_\star:\boldsymbol{\tau}^{\rm d}_\star \qquad \forall\,(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}})\in\mathbb{H}\times\mathbb{H}, \tag{2.7a}$$

$$\mathbf{a}_{\rm T}(\vec{\mathbf{t}},\,\vec{\mathbf{s}}) := \frac{1}{\kappa}\sum_{\star\in\{\rm f,p\}}\int_{\Omega_\star}\mathbf{t}_\star\cdot\mathbf{s}_\star \qquad \forall\,(\vec{\mathbf{t}},\vec{\mathbf{s}})\in\mathbf{H}\times\mathbf{H}, \tag{2.7b}$$

$$\mathbf{b}_{\rm S}(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}) := \sum_{\star\in\{\rm f,p\}}\int_{\Omega_\star}\mathbf{u}_\star\cdot\mathbf{div}(\boldsymbol{\tau}_\star) \qquad \forall\,(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q}, \tag{2.7c}$$

$$\mathbf{b}_{\rm T}(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta})) := \sum_{\star\in\{\rm f,p\}}\left\{\int_{\Omega_\star}\psi_\star\mathrm{div}(\mathbf{s}_\star) + \langle\mathbf{s}_\star\cdot\mathbf{n}_\star,\,\eta_\star\rangle_{\Gamma^{\rm c}_{\rm in,\star}}\right\} \qquad \forall\,(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta}))\in\mathbf{H}\times\mathrm{Q}, \tag{2.7d}$$

$$\mathbf{d}_{\rm T}((\vec{\phi},\vec{\xi}),(\vec{\psi},\vec{\eta})) := (\widetilde{a}_{1,\rm f}-\widetilde{a}_0)\int_\Sigma\xi_{\rm f}\eta_{\rm f} + \big(\widetilde{a}_{1,\rm p}+\widetilde{a}_0\big)\int_\Sigma\xi_{\rm p}\,\eta_{\rm p} \qquad \forall\,((\vec{\phi},\vec{\xi}),(\vec{\psi},\vec{\eta}))\in\mathrm{Q}\times\mathrm{Q}, \tag{2.7e}$$

whereas for each $(\vec{\mathbf{w}},\vec{\chi})\in\mathbf{Q}\times\mathrm{M}$, $\mathbf{O}_{\rm S}(\vec{\mathbf{w}};\cdot,\cdot):\mathbb{H}\times\mathbf{Q}\to\mathbb{R}$, $\mathbf{O}_{\rm T}(\vec{\mathbf{w}};\cdot,\cdot):\mathbf{H}\times\mathrm{X}\to\mathbb{R}$, and $\mathbf{C}_{\rm T}(\vec{\chi};\cdot,\cdot):\mathrm{M}\times\mathrm{Q}\to\mathbb{R}$

are the bilinear forms given by

$$
\begin{aligned}
\mathbf{O}_{\mathrm{S}}(\vec{\mathbf{w}};\vec{\boldsymbol{\sigma}},\vec{\mathbf{v}}) &:= \frac{\rho}{\nu}\sum_{\star\in\{\mathrm{f},\mathrm{p}\}}\int_{\Omega_\star}\boldsymbol{\sigma}_\star^{\mathrm{d}}\,\mathbf{w}_\star\cdot\mathbf{v}_\star && \forall\,(\vec{\boldsymbol{\sigma}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q},\\
\mathbf{O}_{\mathrm{T}}(\vec{\mathbf{w}};\vec{\mathbf{t}},\vec{\psi}) &:= \frac{1}{\kappa}\sum_{\star\in\{\mathrm{f},\mathrm{p}\}}\int_{\Omega_\star}\mathbf{w}_\star\cdot\mathbf{t}_\star\psi_\star && \forall\,(\vec{\mathbf{t}},\vec{\psi})\in\mathbf{H}\times\mathrm{X},\\
\mathbf{C}_{\mathrm{T}}(\vec{\chi};\vec{\xi},\vec{\eta}) &:= a_1\int_\Sigma\xi_{\mathrm{f}}(\chi_{\mathrm{p}}-\chi_{\mathrm{f}})\eta_{\mathrm{f}} - a_1\int_\Sigma\xi_{\mathrm{p}}(\chi_{\mathrm{p}}-\chi_{\mathrm{f}})\eta_{\mathrm{p}} && \forall\,(\vec{\xi},\vec{\eta})\in\mathrm{M}\times\mathrm{M}.
\end{aligned}
\tag{2.8}
$$

Finally, $\mathbf{F}_{\vec{\chi}}\in\mathbf{Q}'$ and $\mathcal{F}_{\vec{\chi}}\in\mathrm{Q}'$ are the functionals defined by

$$
\begin{aligned}
\mathbf{F}_{\vec{\chi}}(\vec{\boldsymbol{\tau}}) &:= \langle\boldsymbol{\tau}_{\mathrm{f}}\,\mathbf{n}_{\mathrm{f}},\,\mathbf{g}_{\mathrm{f}}+\mathbf{g}_{\mathrm{f}}^{\chi}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} + \langle\boldsymbol{\tau}_{\mathrm{p}}\,\mathbf{n}_{\mathrm{p}},\,\mathbf{g}_{\mathrm{p}}+\mathbf{g}_{\mathrm{p}}^{\chi}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,p}}} && \forall\,\vec{\boldsymbol{\tau}}\in\mathbb{H},\\
\mathcal{F}_{\vec{\chi}}(\vec{\psi},\vec{\eta}) &:= -\widetilde{a}_{1,\mathrm{f}}\int_\Sigma\chi_{\mathrm{p}}\eta_{\mathrm{f}} - \widetilde{a}_{1,\mathrm{p}}\int_\Sigma\chi_{\mathrm{f}}\eta_{\mathrm{p}} + \widetilde{a}_{2,\mathrm{f}}\int_\Sigma\eta_{\mathrm{f}} - \widetilde{a}_{2,\mathrm{p}}\int_\Sigma\eta_{\mathrm{p}} && \forall\,(\vec{\psi},\vec{\eta})\in\mathrm{Q}.
\end{aligned}
\tag{2.9}
$$

3 The continuous solvability analysis

We analyze the solvability of (2.6) by applying the results provided by [18, Theorem 2.34] and [14, Theorem 3.4], along with the Banach–Nečas–Babuška Theorem (cf. [18, Theorem 2.6]).

3.1 The fixed-point strategy

We begin by rewriting (2.6) as an equivalent fixed point equation. Indeed, we first let $\widetilde{\mathcal{J}}:\mathbf{Q}\times\mathrm{M}\to\mathbb{H}\times\mathbf{Q}$ be the operator defined for each $\vec{\mathbf{w}}:=(\mathbf{w}_{\mathrm{f}},\mathbf{w}_{\mathrm{p}})$ and $\vec{\chi}:=(\chi_{\mathrm{f}},\chi_{\mathrm{p}})$, with $(\vec{\mathbf{w}},\vec{\chi})\in\mathbf{Q}\times\mathrm{M}$ as

$$
\widetilde{\mathcal{J}}(\vec{\mathbf{w}},\vec{\chi}) \;=\; \big(\widetilde{\mathcal{J}}_1(\vec{\mathbf{w}},\vec{\chi}),\widetilde{\mathcal{J}}_2(\vec{\mathbf{w}},\vec{\chi})\big) := (\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),
\tag{3.1}
$$

where $(\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}})\in\mathbb{H}\times\mathbf{Q}$ is the unique solution (to be confirmed in Theorem 3.3) of the following problem:

$$
\begin{aligned}
\mathbf{a}_{\mathrm{S}}(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) \quad + \mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\tau}},\vec{\mathbf{u}}) \quad &= \mathbf{F}_{\vec{\chi}}(\vec{\boldsymbol{\tau}}) && \forall\,\vec{\boldsymbol{\tau}}\in\mathbb{H},\\
\mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\sigma}},\vec{\mathbf{v}}) \quad - \mathbf{O}_{\mathrm{S}}(\vec{\mathbf{w}};\vec{\boldsymbol{\sigma}},\vec{\mathbf{v}}) &= 0 && \forall\,\vec{\mathbf{v}}\in\mathbf{Q}.
\end{aligned}
\tag{3.2}
$$

In turn, we let $\widehat{\mathcal{J}}:\mathbf{Q}\times\mathrm{M}\to\mathbf{H}\times\mathrm{Q}$ be the operator given by

$$
\widehat{\mathcal{J}}(\vec{\mathbf{w}},\vec{\chi}) \;=\; \big(\widehat{\mathcal{J}}_1(\vec{\mathbf{w}},\vec{\chi}),(\widehat{\mathcal{J}}_2(\vec{\mathbf{w}},\vec{\chi}),\widehat{\mathcal{J}}_3(\vec{\mathbf{w}},\vec{\chi}))\big) := (\vec{\mathbf{t}},(\vec{\phi},\vec{\xi})) \quad \forall\,(\vec{\mathbf{w}},\vec{\chi})\in\mathbf{Q}\times\mathrm{M},
\tag{3.3}
$$

where $(\vec{\mathbf{t}},(\vec{\phi},\vec{\xi}))\in\mathbf{H}\times\mathrm{Q}$ is the unique solution (to be confirmed in Theorem 3.5) of the following system of equations:

$$
\begin{aligned}
\mathbf{a}_{\mathrm{T}}(\vec{\mathbf{t}},\vec{\mathbf{s}}) + \mathbf{b}_{\mathrm{T}}(\vec{\mathbf{s}},(\vec{\phi},\vec{\xi})) &= 0 && \forall\,\vec{\mathbf{s}}\in\mathbf{H},\\
\mathbf{b}_{\mathrm{T}}(\vec{\mathbf{t}},(\vec{\psi},\vec{\eta})) - \mathbf{d}_{\mathrm{T}}((\vec{\phi},\vec{\xi}),(\vec{\psi},\vec{\eta})) - \mathbf{O}_{\mathrm{T}}(\vec{\mathbf{w}};\vec{\mathbf{t}},\vec{\psi}) - \mathbf{C}_{\mathrm{T}}\big(\vec{\chi};\vec{\xi},\vec{\eta}\big) &= \mathcal{F}_{\vec{\chi}}(\vec{\psi},\vec{\eta}) && \forall\,(\vec{\psi},\vec{\eta})\in\mathrm{Q}.
\end{aligned}
\tag{3.4}
$$

Finally, defining the operator $\mathcal{J}:\mathbf{Q}\times\mathrm{M}\to\mathbf{Q}\times\mathrm{M}$ as

$$
\mathcal{J}(\vec{\mathbf{w}},\vec{\chi}) \;=\; \big(\widetilde{\mathcal{J}}_2(\vec{\mathbf{w}},\vec{\chi}),\widehat{\mathcal{J}}_3(\vec{\mathbf{w}},\vec{\chi})\big) = (\vec{\mathbf{u}},\vec{\xi}) \quad \forall\,(\vec{\mathbf{w}},\vec{\chi})\in\mathbf{Q}\times\mathrm{M},
\tag{3.5}
$$

we observe that solving (2.6) is equivalent to seeking a fixed point of $\mathcal{J}$, that is: Find $(\vec{\mathbf{u}},\vec{\xi})\in\mathbf{Q}\times\mathrm{M}$ such that $\mathcal{J}(\vec{\mathbf{u}},\vec{\xi})=(\vec{\mathbf{u}},\vec{\xi})$.

We now aim to prove that the operator $\mathcal{J}$ is well-defined. To do that we first state the boundedness of all the variational forms involved (cf. (2.7), (2.8) and (2.9)). First, it is easy to see through a direct application of Hölder's inequality that

$$
\begin{aligned}
|\mathbf{a}_{\mathrm{S}}(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}})| &\leqslant \|\mathbf{a}_{\mathrm{S}}\|\|\vec{\boldsymbol{\sigma}}\|_{\mathbb{H}}\,\|\vec{\boldsymbol{\tau}}\|_{\mathbb{H}} && \forall\,\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}\in\mathbb{H},\\
|\mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})| &\leqslant \|\mathbf{b}_{\mathrm{S}}\|\|\vec{\boldsymbol{\tau}}\|_{\mathbb{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} && \forall\,\big(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}\big)\in\mathbb{H}\times\mathbf{Q},\\
|\mathbf{a}_{\mathrm{T}}(\vec{\mathbf{t}},\vec{\mathbf{s}})| &\leqslant \|\mathbf{a}_{\mathrm{T}}\|\|\vec{\mathbf{t}}\|_{\mathbf{H}}\,\|\vec{\mathbf{s}}\|_{\mathbf{H}} && \forall\,\vec{\mathbf{t}},\vec{\mathbf{s}}\in\mathbf{H},
\end{aligned}
\tag{3.6}
$$

with $\|\mathbf{a}_{\mathrm{S}}\| := \nu^{-1}$, $\|\mathbf{b}_{\mathrm{S}}\| := 1$ and $\|\mathbf{a}_{\mathrm{T}}\| := \kappa^{-1}$. In turn, employing again the Cauchy–Schwarz and Hölder inequalities, we find that for each $\vec{\mathbf{w}} \in \mathbf{Q}$, there hold

$$|\mathbf{O}_{\mathrm{S}}(\vec{\mathbf{w}};\vec{\boldsymbol{\zeta}},\vec{\mathbf{v}})| \leqslant \frac{\rho}{\nu}\,\|\vec{\mathbf{w}}\|_{\mathbf{Q}}\,\|\vec{\boldsymbol{\zeta}}\|_{\mathbb{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} \qquad \forall\,(\vec{\boldsymbol{\zeta}},\vec{\mathbf{v}}) \in \mathbb{H}\times\mathbf{Q}, \tag{3.7a}$$

$$|\mathbf{O}_{\mathrm{T}}(\vec{\mathbf{w}};\vec{\mathbf{r}},\vec{\psi})| \leqslant \frac{1}{\kappa}\,\|\vec{\mathbf{w}}\|_{\mathbf{Q}}\,\|\vec{\mathbf{r}}\|_{\mathbf{H}}\,\|\vec{\psi}\|_{\mathrm{X}} \qquad \forall\,(\vec{\mathbf{r}},\vec{\psi}) \in \mathbf{H}\times\mathrm{X}. \tag{3.7b}$$

On the other hand, the following bounds for $\mathbf{b}_{\mathrm{T}}$ and $\mathbf{d}_{\mathrm{T}}$ will be proved in Appendix A:

$$|\mathbf{b}_{\mathrm{T}}(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta}))| \leqslant \|\mathbf{b}_{\mathrm{T}}\|\,\|\vec{\mathbf{s}}\|_{\mathbf{H}}\,\|(\vec{\psi},\vec{\eta})\|_{\mathrm{Q}} \qquad \forall\,\big(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta})\big) \in \mathbf{H}\times\mathrm{Q}, \tag{3.8a}$$

$$|\mathbf{d}_{\mathrm{T}}((\vec{\phi},\vec{\xi}),(\vec{\psi},\vec{\eta}))| \leqslant \|\mathbf{d}_{\mathrm{T}}\|\,\|(\vec{\phi},\vec{\xi})\|_{\mathrm{Q}}\,\|(\vec{\psi},\vec{\eta})\|_{\mathrm{Q}} \qquad \forall\,(\vec{\phi},\vec{\xi}),\,(\vec{\psi},\vec{\eta}) \in \mathrm{Q}, \tag{3.8b}$$

where $\|\mathbf{b}_{\mathrm{T}}\| := \max\big\{1,\|i_4^{\mathrm{f}}\|,\|i_4^{\mathrm{p}}\|\big\}$, $\|\mathbf{d}_{\mathrm{T}}\| := (\widetilde{a}_{1,\mathrm{f}}+\widetilde{a}_{1,\mathrm{p}})\max\big\{(c_2^{\mathrm{f}})^2,(c_2^{\mathrm{p}})^2\big\}$, $i_4^\star$ is the continuous injection from $\mathrm{H}^1(\Omega_\star)$ to $\mathrm{L}^4(\Omega_\star)$ and $c_2^\star$ the Sobolev constant defined in (A.1). In turn, denoting $\vec{\mathbf{g}} := (\mathbf{g}_{\mathrm{f}},\mathbf{g}_{\mathrm{p}})$ with $\mathbf{g}_\star$ as in (2.4), in Appendix A we will show that given $\vec{\chi} \in \mathrm{M}$, we have

$$|\mathbf{F}_{\vec{\chi}}(\vec{\boldsymbol{\tau}})| \;\leqslant\; C_{\mathbf{F}}\big\{\|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}} + a_1\|\vec{\chi}\|_{\mathrm{M}}\big\}\,\|\vec{\boldsymbol{\tau}}\|_{\mathbb{H}} \qquad \forall\,\vec{\boldsymbol{\tau}} \in \mathbb{H}, \tag{3.9}$$

where $\|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}} := \|\mathbf{g}_{\mathrm{f}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} + \|\mathbf{g}_{\mathrm{p}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out,p}}}$, $C_{\mathbf{F}} := \max\big\{1,\|\mathbf{i}_4^{\mathrm{f}}\|,\|\mathbf{i}_4^{\mathrm{p}}\|\big\}\,\max\big\{C_{\mathrm{f}},C_{\Sigma,\mathrm{f}},C_{\mathrm{p}},C_{\Sigma,\mathrm{p}},1\big\}$, and $\mathbf{i}_4^\star$ is the vector-valued version of $i_4^\star$. Finally, employing the same tools to bound $\mathbf{d}_{\mathrm{T}}$, given $\vec{\chi} \in \mathrm{M}$, it readily follows that

$$\begin{aligned} |\mathbf{C}_{\mathrm{T}}(\vec{\chi};\vec{\xi},\vec{\eta})| \;&\leqslant\; \widetilde{a}_1\,\|\vec{\chi}\|_{\mathrm{M}}\,\|\vec{\xi}\|_{\mathrm{M}}\,\|\vec{\eta}\|_{\mathrm{M}} \qquad && \forall\,(\vec{\xi},\,\vec{\eta}) \in \mathrm{M}\times\mathrm{M}, \quad \text{and} \\ |\mathcal{F}_{\vec{\chi}}(\vec{\psi},\vec{\eta})| \;&\leqslant\; \widetilde{a}_3\,(1+\|\vec{\chi}\|_{\mathrm{M}})\|(\vec{\psi},\vec{\eta})\|_{\mathrm{Q}} && \forall\,(\vec{\psi},\vec{\eta}) \in \mathrm{Q},\ \text{with} \end{aligned} \tag{3.10}$$

$$\widetilde{a}_1 := a_1\max\big\{(c_3^{\mathrm{f}})^2,(c_3^{\mathrm{p}})^2\big\}\max\big\{c_3^{\mathrm{f}},c_3^{\mathrm{p}}\big\} \quad \text{and} \quad \widetilde{a}_3 := \max\{|\widetilde{a}_{1,\mathrm{f}}|,|\widetilde{a}_{1,\mathrm{p}}|,|\widetilde{a}_{2,\mathrm{f}}|,|\widetilde{a}_{2,\mathrm{p}}|\}\max\big\{c_2^{\mathrm{f}}c_2^{\mathrm{p}},c_2^{\mathrm{f}},c_2^{\mathrm{p}}\big\}. \tag{3.11}$$

Further details of the previous bounds and the involved constants can be found in Appendix A.

3.1.1 Well-definedness of the operator $\widetilde{\mathcal{J}}$

We will show that (3.2) is well-posed and therefore the operator $\widetilde{\mathcal{J}}$ (cf. (3.1)) is well-defined. For that, given $\star \in \{\mathrm{f},\mathrm{p}\}$, we prove that $\mathbf{a}_{\mathrm{S}}$ and $\mathbf{b}_{\mathrm{S}}$ satisfy the corresponding hypotheses from [18, Theorem 2.34]. Let

$$\mathcal{K}_{\mathrm{S}} = \big\{\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{f}},\,\boldsymbol{\tau}_{\mathrm{p}}) \in \mathbb{H}: \quad \mathbf{div}(\boldsymbol{\tau}_\star) = \mathbf{0} \ \text{in}\ \Omega_\star\big\},$$

which corresponds to the kernel of the operator induced by the bilinear form $\mathbf{b}_{\mathrm{S}}$ (cf. (2.7c)). We proceed in a similar way to [20, Section 2.2] to show that $\mathbf{a}_{\mathrm{S}}$ is $\mathcal{K}_{\mathrm{S}}$-elliptic. To do that, it suffices to consider the decomposition $\mathbb{H}(\mathbf{div}_{4/3};\Omega_\star) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_\star) \oplus \mathbb{R}\,\mathbb{I}$, where $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_\star) := \Big\{\boldsymbol{\tau}_\star \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_\star): \quad \int_{\Omega_\star}\mathrm{tr}(\boldsymbol{\tau}_\star) = 0\Big\}$, and recall two useful estimates. First, by suitably modifying the proof of [21, Lemma 2.3], it can be shown (see, e.g. [6, Lemma 3.1]) that there exists a positive constant $c_{1,\star}$, depending only on $\Omega_\star$, such that

$$c_{1,\star}\,\|\boldsymbol{\tau}_\star\|_{0,\Omega_\star} \;\leqslant\; \|\boldsymbol{\tau}_\star^{\mathrm{d}}\|_{0,\Omega_\star} + \|\mathbf{div}(\boldsymbol{\tau}_\star)\|_{0,4/3;\Omega_\star} \quad \forall\,\boldsymbol{\tau}_\star \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_\star). \tag{3.12}$$

Similarly, following the proof of [20, Lemma 2.2] (see also [21, Lemma 2.5]), one can show that there exists a positive constant $c_{2,\star}$, depending only on $\Gamma_{\mathrm{out},\star}$ and $\Omega_\star$, such that

$$c_{2,\star}\,\|\boldsymbol{\tau}_\star\|_{\mathbf{div}_{4/3};\Omega_\star} \;\leqslant\; \|\boldsymbol{\tau}_{\star,0}\|_{\mathbf{div}_{4/3};\Omega_\star} \quad \forall\,\boldsymbol{\tau}_\star := \boldsymbol{\tau}_{\star,0} + d_\star\mathbb{I} \in \mathbb{H}_{\Gamma_{\mathrm{out}}}(\mathbf{div}_{4/3};\Omega_\star), \tag{3.13}$$

with $\boldsymbol{\tau}_{\star,0} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_\star)$ and $d_\star \in \mathbb{R}$.

Lemma 3.1. *There exists $\alpha_{\mathbf{s}} > 0$ such that $\mathbf{a}_{\mathrm{S}}(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\tau}}) \geqslant \alpha_{\mathrm{S}}\,\|\vec{\boldsymbol{\tau}}\|^2_{\mathbb{H}} \quad \forall\,\vec{\boldsymbol{\tau}} \in \mathcal{K}_{\mathrm{S}}$.*

Proof. Given $\vec{\boldsymbol{\tau}} = (\boldsymbol{\tau}_{\mathrm{f}},\boldsymbol{\tau}_{\mathrm{p}}) \in \mathcal{K}_{\mathrm{S}}$, applying (3.12) and (3.13), it is straightforward to see from the definition of $\mathbf{a}_{\mathrm{S}}$ (cf. (2.7a)), that

$$\mathbf{a}_{\mathrm{S}}(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\tau}}) \;\geqslant\; \alpha_{\mathbf{s}}\|\vec{\boldsymbol{\tau}}\|^2_{\mathbb{H}},$$

with constant $\alpha_{\mathbf{s}} = \dfrac{1}{2\nu}\min\{c_{1,\mathrm{f}}c_{2,\mathrm{f}},\,c_{1,\mathrm{p}}c_{2,\mathrm{p}}\}^2$, depending only on $\Gamma_{\mathrm{out},\star}$ and $\Omega_\star$, with $\star \in \{\mathrm{f},\mathrm{p}\}$. □

We now establish the continuous inf-sup condition for the bilinear form $\mathbf{b}_{\mathrm{S}}$, whose proof is an adaptation of [6, Lemma 3.3] to our context.

Lemma 3.2. *There exists a positive constant* β_{S} *such that*

$$\sup_{\substack{\vec{\boldsymbol{\tau}}\in\mathbb{H}\\ \vec{\boldsymbol{\tau}}\neq\mathbf{0}}} \frac{\mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})}{\|\vec{\boldsymbol{\tau}}\|_{\mathbb{H}}} \geqslant \beta_{\mathrm{S}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} \qquad \forall\,\vec{\mathbf{v}}\in\mathbf{Q}. \tag{3.14}$$

Proof. Given $\vec{\mathbf{v}} = (\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{p}}) \in \mathbf{Q}$, we set $\tilde{\mathbf{v}}_\star := |\mathbf{v}_\star|^2\mathbf{v}_\star$ with $\star\in\{\mathrm{f},\mathrm{p}\}$ and let $\mathbf{z}_\star \in \mathbf{H}^1(\Omega_\star)$ be the unique solution to the boundary value problem

$$-\triangle\mathbf{z}_\star = \tilde{\mathbf{v}}_\star \quad\text{in}\quad \Omega_\star,\quad \mathbf{z}_\star = \mathbf{0} \quad\text{on}\quad \Gamma^{\mathrm{c}}_{\mathrm{out},\star},\quad\text{and}\quad \nabla\mathbf{z}_\star\mathbf{n}_\star = \mathbf{0}\quad\text{on}\quad\Gamma_{\mathrm{out},\star}.$$

Thus, defining $\tilde{\boldsymbol{\tau}}_\star := -\nabla\mathbf{z}_\star \in \mathbb{L}^2(\Omega_\star)$, noticing that $\tilde{\boldsymbol{\tau}}_\star \in \mathbb{H}^\star_{\mathrm{S}}$ and proceeding similarly to the proof of [6, Lemma 3.3], it follows that

$$\sup_{\substack{\vec{\boldsymbol{\tau}}\in\mathbb{H}\\ \vec{\boldsymbol{\tau}}\neq\mathbf{0}}} \frac{\mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})}{\|\vec{\boldsymbol{\tau}}\|_{\mathbb{H}}} \geqslant \beta_{\mathrm{S},\star}\|\mathbf{v}_\star\|_{0,4;\Omega_\star},$$

where $\beta_{\mathrm{S},\star} = (c_{\mathrm{p},\star}\|\mathbf{i}^\star_4\|+1)^{-1}$, $c_{\mathrm{p},\star}$ is a Poincaré's constant depending only on $|\Omega_\star|$ and $\mathbf{i}^\star_4$ is the continuous injection of $\mathbf{H}^1(\Omega_\star)$ into $\mathbf{L}^4(\Omega_\star)$. Thus, (3.14) is satisfied with $\beta_{\mathrm{S}} = \dfrac{1}{2}\min\big\{\beta_{\mathrm{S,f}},\,\beta_{\mathrm{S,p}}\big\}$. □

Letting now $\mathbf{A} : (\mathbb{H}\times\mathbf{Q})\times(\mathbb{H}\times\mathbf{Q}) \to \mathbb{R}$ be the bilinear form given by

$$\mathbf{A}\big((\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}}),(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\big) := \mathbf{a}_{\mathrm{S}}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) + \mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\tau}},\vec{\mathbf{z}}) + \mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\zeta}},\vec{\mathbf{v}}) \qquad \forall\,(\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}}),(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q},$$

we deduce that (3.2), can be stated, equivalently as: Find $(\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}})\in\mathbb{H}\times\mathbf{Q}$ such that

$$\mathbf{A}_{\vec{\mathbf{w}}}\big((\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\big) := \mathbf{A}\left((\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\right) - \mathbf{O}_{\mathrm{S}}(\vec{\mathbf{w}};\vec{\boldsymbol{\sigma}},\vec{\mathbf{v}}) = \mathbf{F}_{\vec{\chi}}(\vec{\boldsymbol{\tau}}) \qquad \forall\,(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q}. \tag{3.15}$$

Consequently, knowing from Lemmas 3.1 and 3.2 that $\mathbf{a}_{\mathrm{S}}$ and $\mathbf{b}_{\mathrm{S}}$ satisfy the hipotheses of [18, Theorem 2.34], a direct application of this result yields the existence of a positive constant $\alpha_{\mathbf{A}}$, depending on $\|\mathbf{a}_{\mathrm{S}}\|$, α_{S}, and β_{S}, such that

$$\sup_{\substack{(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q}\\ (\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathbf{A}\big((\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}}),(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\big)}{\|(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \alpha_{\mathbf{A}}\,\|(\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}})\|_{\mathbb{H}\times\mathbf{Q}} \qquad \forall\,(\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}})\in\mathbb{H}\times\mathbf{Q}. \tag{3.16}$$

Then, it follows from (3.7a), (3.15) and (3.16) that

$$\sup_{\substack{(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q}\\ (\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathbf{A}_{\vec{\mathbf{w}}}\big((\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}}),(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\big)}{\|(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \left\{\alpha_{\mathbf{A}} - \frac{\rho}{\nu}\|\vec{\mathbf{w}}\|_{\mathbf{Q}}\right\}\|(\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}})\|_{\mathbb{H}\times\mathbf{Q}} \qquad \forall\,(\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}})\in\mathbb{H}\times\mathbf{Q}.$$

Hence, assuming that $\|\vec{\mathbf{w}}\|_{\mathbf{Q}} \leqslant \dfrac{\nu\,\alpha_{\mathbf{A}}}{2\,\rho}$, we arrive at

$$\sup_{\substack{(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q}\\ (\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathbf{A}_{\vec{\mathbf{w}}}\big((\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}}),(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\big)}{\|(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \frac{\alpha_{\mathbf{A}}}{2}\,\|(\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}})\|_{\mathbb{H}\times\mathbf{Q}} \qquad \forall\,(\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}})\in\mathbb{H}\times\mathbf{Q}. \tag{3.17}$$

Similarly, noting that $\mathbf{A}$ is symmetric, employing (3.7a) and (3.16), and assuming again that $\|\vec{\mathbf{w}}\|_{\mathbf{Q}} \leqslant \dfrac{\nu\,\alpha_{\mathbf{A}}}{2\,\rho}$, we obtain

$$\sup_{\substack{(\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}})\in\mathbb{H}\times\mathbf{Q}\\ (\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}})\neq\mathbf{0}}} \frac{\mathbf{A}_{\vec{\mathbf{w}}}\big((\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}}),(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\big)}{\|(\vec{\boldsymbol{\zeta}},\vec{\mathbf{z}})\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \frac{\alpha_{\mathbf{A}}}{2}\,\|(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\|_{\mathbb{H}\times\mathbf{Q}} > 0 \qquad \forall\,(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q},\ (\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\neq\mathbf{0}. \tag{3.18}$$

Then, we are now in position to prove that the operator $\widetilde{\mathcal{J}}$ (cf. (3.1)) is well-defined.

Theorem 3.3. *For each $\vec{\chi} \in \mathrm{M}$ and $\vec{\mathbf{w}} \in \mathbf{Q}$ such that $\|\vec{\mathbf{w}}\|_{\mathbf{Q}} \leqslant \dfrac{\nu\,\alpha_{\mathbf{A}}}{2\,\rho}$, there exists a unique $(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) \in \mathbb{H} \times \mathbf{Q}$ solution to (3.2) (equivalently (3.15)). Moreover, there holds*

$$\|\widetilde{\mathcal{J}}(\vec{\mathbf{w}}, \vec{\chi})\|_{\mathbb{H}\times\mathbf{Q}} \,=\, \|(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}})\|_{\mathbb{H}\times\mathbf{Q}} \,\leqslant\, \frac{2C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\Big\{\|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}} + a_1\|\vec{\chi}\|_{\mathrm{M}}\Big\}. \tag{3.19}$$

Proof. We first recall from (3.17) and (3.18), that $\mathbf{A}_{\vec{\mathbf{w}}}$ satisfies the hypotheses of the Banach–Nečas–Babuška Theorem [18, Theorem 2.6], which allows us to conclude the well-posedness of (3.15). In turn, the estimate (3.19) is a direct consequence of (3.17), (3.15) and (3.9). □

3.1.2 Well-definedness of the operator $\widehat{\mathcal{J}}$

We will prove now that (3.4) is well-posed, equivalently, that $\widehat{\mathcal{J}}$ is well-defined. Indeed, it is clear that $\mathbf{a}_{\mathrm{T}}$ and $\mathbf{d}_{\mathrm{T}}$ are symmetric, and the former is positive semidefinite. In addition, thanks to (1.16), it follows that $\mathbf{d}_{\mathrm{T}}$ (cf. (2.7e)), satisfies

$$\mathbf{d}_{\mathrm{T}}((\vec{\psi}, \vec{\eta}), (\vec{\psi}, \vec{\eta})) \,=\, (\widetilde{a}_{1,\mathrm{f}} - \widetilde{a}_0)\|\eta_{\mathrm{f}}\|^2_{0,\Sigma} + (\widetilde{a}_{1,\mathrm{p}} + \widetilde{a}_0)\|\eta_{\mathrm{p}}\|^2_{0,\Sigma} \geqslant 0 \qquad \forall\, (\vec{\psi}, \vec{\eta}) \in \mathrm{Q}, \tag{3.20}$$

which confirm hypothesis **i)** of [14, Theorem 3.4]). On the other hand, given $\star \in \{\mathrm{f}, \mathrm{p}\}$, we let $\mathcal{K}_{\mathrm{T}}$ be the kernel of the operator induced by the bilinear form $\mathbf{b}_{\mathrm{T}}$ (cf. (2.7d)), that is

$$\mathcal{K}_{\mathrm{T}} := \Big\{\vec{\mathbf{s}} := (\mathbf{s}_{\mathrm{f}}, \mathbf{s}_{\mathrm{p}}) \in \mathbf{H} : \quad \mathrm{div}(\mathbf{s}_\star) \,=\, 0 \quad \text{in} \quad \Omega_\star \quad \text{and} \quad \mathbf{s}_\star \cdot \mathbf{n}_\star \,=\, 0 \quad \text{on} \quad \Gamma^{\mathrm{c}}_{\mathrm{in},\star}\Big\}.$$

Then, it is straightforward to see from the definition of $\mathbf{a}_{\mathrm{T}}$ (cf. (2.7b)), that for each $\vec{\mathbf{s}} := (\mathbf{s}_{\mathrm{f}}, \mathbf{s}_{\mathrm{p}}) \in \mathcal{K}_{\mathrm{T}}$, there holds

$$\mathbf{a}_{\mathrm{T}}(\vec{\mathbf{s}}, \vec{\mathbf{s}}) \,\geqslant\, \frac{1}{2\kappa}\,\|\vec{\mathbf{s}}\|^2_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{s}} \in \mathcal{K}_{\mathrm{T}}, \tag{3.21}$$

which proves that $\mathbf{a}_{\mathrm{T}}$ is $\mathcal{K}_{\mathrm{T}}$-elliptic with constant $\alpha_{\mathrm{T}} \,=\, \dfrac{1}{2\kappa}$, and hence that $\mathbf{a}_{\mathrm{S}}$ verify the continuous inf-sup condition required by the hypothesis **ii)** of [14, Theorem 3.4]. Now, we provide the corresponding inf-sup condition of the bilinear form $\mathbf{b}_{\mathrm{T}}$ (cf. (2.7d)) and its proof is basically an adaptation of the version in [3, Theorem 2.1].

Lemma 3.4. *There exists a positive constant β_{T} such that*

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{H}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathbf{b}_{\mathrm{T}}(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geqslant\, \beta_{\mathrm{T}}\,\|(\vec{\psi}, \vec{\eta})\|_{\mathrm{Q}} \qquad \forall\, (\vec{\psi}, \vec{\eta}) \in \mathrm{Q}. \tag{3.22}$$

Proof. Let $(\vec{\psi}, \vec{\eta}) \in \mathrm{Q}$. Then, similarly as done in Lemma 3.2, we deduce that

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{H}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathbf{b}_{\mathrm{T}}(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geqslant\, \widetilde{\beta}_{\mathrm{T}}\|\vec{\psi}\|_{\mathrm{X}}, \tag{3.23}$$

with $\widetilde{\beta}_{\mathrm{T}} \,=\, \dfrac{1}{2}\min\Big\{\,\big(c_{\mathrm{p,f}}\|i^{\mathrm{f}}_4\| + 1\big)^{-1},\, (c_{\mathrm{p,p}}\|i^{\mathrm{p}}_4\| + 1)^{-1}\Big\}$, where $i^\star_4$ is the continuous injection of $\mathrm{H}^1(\Omega_\star)$ into $\mathrm{L}^4(\Omega_\star)$ and $c_{\mathrm{p},\star}$ is the Poincare's constant, for each $\star \in \{\mathrm{f}, \mathrm{p}\}$. On the other hand, given $\mu_\star \in \mathrm{H}^{-1/2}_{00}(\Gamma^{\mathrm{c}}_{\mathrm{in},\star})$, we let $\widehat{z}_\star \in \mathrm{H}^1(\Omega_\star)$ be the solution to

$$\triangle\widehat{z}_\star \,=\, 0 \quad \text{in} \quad \Omega_\star, \quad \widehat{z}_\star \,=\, 0 \quad \text{on} \quad \Gamma_{\mathrm{in},\star}, \quad \text{and} \quad \nabla\widehat{z}_\star \cdot \mathbf{n}_\star \,=\, \mu_\star \quad \text{on} \quad \Gamma^{\mathrm{c}}_{\mathrm{in},\star},$$

and set $\widehat{\mathbf{s}}_\star := \nabla\widehat{z}_\star \in \mathbf{L}^2(\Omega_\star)$. Noticing that $\widehat{\mathbf{s}}_\star \in \mathbf{H}^\star_{\mathrm{T}}$, proceeding analogously to the proof of [3, Theorem 2.1], we conclude that there exists a constant $\widehat{\beta}_{\mathrm{T},\star} > 0$ such that

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{H}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathbf{b}_{\mathrm{T}}(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geqslant\, \widehat{\beta}_{\mathrm{T},\star}\|\eta_\star\|_{1/2,00,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}.$$

The above, along with (3.23), yields (3.22) with $\beta_{\mathrm{T}} \,=\, \dfrac{1}{2}\min\Big\{\widetilde{\beta}_{\mathrm{T}}, \dfrac{1}{2}\min\big\{\widehat{\beta}_{\mathrm{T,f}}, \widehat{\beta}_{\mathrm{T,p}}\big\}\Big\}$. □

Now, we let $\mathcal{A} : (\mathbf{H} \times \mathrm{Q}) \times (\mathbf{H} \times \mathrm{Q}) \to \mathbb{R}$ be the bounded bilinear form defined by

$$\mathcal{A}\big((\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})), (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\big) := \mathbf{a}_{\mathrm{T}}\big(\vec{\mathbf{r}}, \vec{\mathbf{s}}\big) + \mathbf{b}_{\mathrm{T}}\big(\vec{\mathbf{s}}, (\vec{\varphi}, \vec{\lambda})\big) + \mathbf{b}_{\mathrm{T}}\big(\vec{\mathbf{r}}, (\vec{\psi}, \vec{\eta})\big) - \mathbf{d}_{\mathrm{T}}\big((\vec{\varphi}, \vec{\lambda}), (\vec{\psi}, \vec{\eta})\big) \tag{3.24}$$

for all $(\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda}))$, $(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \in \mathbf{H} \times \mathrm{Q}$. Next, letting now $\mathcal{A}_{\vec{\mathbf{w}},\vec{\chi}} : (\mathbf{H} \times \mathrm{Q}) \times (\mathbf{H} \times \mathrm{Q}) \to \mathbb{R}$ be the bilinear form such that

$$\mathcal{A}_{\vec{\mathbf{w}},\vec{\chi}}\big((\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})), (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\big) := \mathcal{A}\big((\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})), (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\big) - \mathbf{O}_{\mathrm{T}}(\vec{\mathbf{w}}; \vec{\mathbf{r}}, \vec{\psi}) - \mathbf{C}_{\mathrm{T}}(\vec{\chi}; \vec{\lambda}, \vec{\eta}) \tag{3.25}$$

for all $(\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda}))$, $(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \in \mathbf{H} \times \mathrm{Q}$. We realize that (3.4) can be rewritten, equivalently, as: Find $(\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})) \in \mathbf{H} \times \mathrm{Q}$ such that

$$\mathcal{A}_{\vec{\mathbf{w}},\vec{\chi}}\big((\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})), (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\big) \;=\; \mathcal{F}_{\vec{\chi}}(\vec{\psi}, \vec{\eta}) \qquad \forall\, (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \in \mathbf{H} \times \mathrm{Q}. \tag{3.26}$$

Then, thanks to (3.21), (3.20) and (3.22), the hypotheses of [14, Theorem 3.4] are satisfied, and hence the *a priori* estimates given by [14, Theorem 3.4, eq. (3.51)] imply the existence of a positive constant $\alpha_{\mathcal{A}}$, depending on $\|\mathbf{a}_{\mathrm{T}}\|$, α_{T}, $\|\mathbf{d}_{\mathrm{T}}\|$, and β_{T}, such that

$$\sup_{\substack{(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \in \mathbf{H} \times \mathrm{Q} \\ (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \neq \mathbf{0}}} \frac{\mathcal{A}\big((\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})), (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\big)}{\|(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\|_{\mathbf{H} \times \mathrm{Q}}} \;\geqslant\; \alpha_{\mathcal{A}}\, \|(\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda}))\|_{\mathbf{H} \times \mathrm{Q}} \qquad \forall\, (\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})) \in \mathbf{H} \times \mathrm{Q}. \tag{3.27}$$

Thus, from (3.25), (3.7b), (3.10) and (3.27), it follows that

$$\sup_{\substack{(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \in \mathbf{H} \times \mathrm{Q} \\ (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \neq \mathbf{0}}} \frac{\mathcal{A}_{\vec{\mathbf{w}},\vec{\chi}}\big((\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})), (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\big)}{\|(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\|_{\mathbf{H} \times \mathrm{Q}}} \;\geqslant\; \Big\{\alpha_{\mathcal{A}} - \widetilde{a}_1 \|\vec{\chi}\|_{\mathrm{M}} - \kappa^{-1} \|\vec{\mathbf{w}}\|_{\mathbf{Q}}\Big\}\, \|(\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda}))\|_{\mathbf{H} \times \mathrm{Q}}$$

for all $(\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})) \in \mathbf{H} \times \mathrm{Q}$. Hence, under the assumption $\|(\vec{\mathbf{w}}, \vec{\chi})\|_{\mathbf{Q} \times \mathrm{M}} \leqslant \dfrac{\kappa \alpha_{\mathcal{A}}}{2(1 + \kappa \widetilde{a}_1)}$, we arrive at

$$\sup_{\substack{(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \in \mathbf{H} \times \mathrm{Q} \\ (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \neq \mathbf{0}}} \frac{\mathcal{A}_{\vec{\mathbf{w}},\vec{\chi}}\big((\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})), (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\big)}{\|(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\|_{\mathbf{H} \times \mathrm{Q}}} \;\geqslant\; \frac{\alpha_{\mathcal{A}}}{2}\, \|(\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda}))\|_{\mathbf{H} \times \mathrm{Q}} \qquad \forall\, (\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})) \in \mathbf{H} \times \mathrm{Q}. \tag{3.28}$$

Similarly, using the fact that $\mathcal{A}$ is symmetric, employing the same boundedness estimates for $\mathbf{O}_{\mathrm{T}}$ (cf. (3.7b)) and $\mathbf{C}_{\mathrm{T}}$ (cf. (3.10)), and assuming again that $\|(\vec{\mathbf{w}}, \vec{\chi})\|_{\mathbf{Q} \times \mathrm{M}} \leqslant \dfrac{\kappa \alpha_{\mathcal{A}}}{2(1 + \kappa \widetilde{a}_1)}$, we are able to prove the companion inf-sup condition to (3.28), that is

$$\sup_{\substack{(\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})) \in \mathbf{H} \times \mathrm{Q} \\ (\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})) \neq \mathbf{0}}} \frac{\mathcal{A}_{\vec{\mathbf{w}},\vec{\chi}}\big((\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda})), (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\big)}{\|(\vec{\mathbf{r}}, (\vec{\varphi}, \vec{\lambda}))\|_{\mathbf{H} \times \mathrm{Q}}} \;\geqslant\; \frac{\alpha_{\mathcal{A}}}{2}\, \|(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\|_{\mathbf{H} \times \mathrm{Q}} > 0$$

for all $(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \in \mathbf{H} \times \mathrm{Q}$, $(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \neq \mathbf{0}$. As a consequence, we are in position to establish that $\widehat{\mathcal{J}}$ (cf. (3.3)) is well defined.

Theorem 3.5. *For each* $(\vec{\mathbf{w}}, \vec{\chi}) \in \mathbf{Q} \times \mathrm{Q}$ *such that* $\|(\vec{\mathbf{w}}, \vec{\chi})\|_{\mathbf{Q} \times \mathrm{M}} \leqslant \dfrac{\kappa \alpha_{\mathcal{A}}}{2(1 + \kappa \widetilde{a}_1)}$, *there exists a unique* $(\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})) = \widehat{\mathcal{J}}(\vec{\mathbf{w}}, \vec{\chi}) \in \mathbf{H} \times \mathrm{Q}$ *solution to* (3.4) (*equivalently* (3.26)). *Moreover, there holds*

$$\|\widehat{\mathcal{J}}(\vec{\mathbf{w}}, \vec{\chi})\|_{\mathbf{H} \times \mathrm{Q}} \;=\; \|(\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi}))\|_{\mathbf{H} \times \mathrm{Q}} \;\leqslant\; \frac{2}{\alpha_{\mathcal{A}}} \widetilde{a}_3 \big(1 + \|\vec{\chi}\|_{\mathrm{M}}\big), \tag{3.29}$$

where $\widetilde{a}_1$ *and* $\widetilde{a}_3$ *have been defined in* (3.11).

Proof. It follows from a straightforward application of [18, Theorem 2.6]. In particular, the *a priori* estimate (3.29) follows from (3.28) and the fact that, according to (3.10), there holds $\|\mathcal{F}_{\vec{\chi}}\| \leqslant \widetilde{a}_3 \left(1 + \|\vec{\chi}\|_{\mathrm{M}}\right)$. □

3.2 Solvability analysis of the fixed-point scheme

Knowing from the previous sections that the operators $\widetilde{\mathcal{J}}$, $\widehat{\mathcal{J}}$, and hence $\mathcal{J}$ are well-defined, we now focus on the solvability of the fixed-point equation (3.5). To this end, in what follows we first derive sufficient conditions on $\mathcal{J}$ to map a closed ball of $\mathbf{Q} \times \mathrm{M}$ into itself. Indeed, from now on setting $\delta := \min\left\{ \dfrac{\nu\,\alpha_{\mathbf{A}}}{2\,\rho}, \dfrac{\kappa\alpha_{\mathcal{A}}}{2(1+\kappa\widetilde{a}_1)} \right\}$, we let

$$\mathrm{W}(\delta) := \Big\{ (\vec{\mathbf{w}}, \vec{\chi}) \in \mathbf{Q} \times \mathrm{M} : \quad \|(\vec{\mathbf{w}}, \vec{\chi})\|_{\mathbf{Q}\times\mathrm{M}} \leqslant \delta \Big\}. \tag{3.30}$$

Lemma 3.6. *If*

$$C_{\mathbf{F}} \Big\{ \|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}} + a_1 \delta \Big\} \leqslant \frac{\alpha_{\mathbf{A}}}{4} \delta, \quad \text{and} \quad \widetilde{a}_3 (1+\delta) \leqslant \frac{\alpha_{\mathcal{A}}}{4} \delta, \tag{3.31}$$

then $\mathcal{J}\big(\mathrm{W}(\delta)\big) \subseteq \mathrm{W}(\delta)$.

Proof. Given $(\vec{\mathbf{w}}, \vec{\chi}) \in \mathrm{W}(\delta)$, we first recall from (3.5) that $\mathcal{J}(\vec{\mathbf{w}}, \vec{\chi}) = \big(\widetilde{\mathcal{J}}_2(\vec{\mathbf{w}}, \vec{\chi}), \widehat{\mathcal{J}}_3(\vec{\mathbf{w}}, \vec{\chi})\big)$. In this way, the choice of δ along with assumption (3.31) allows to conclude from (3.19) and (3.29) that $\|\widetilde{\mathcal{J}}_2(\vec{\mathbf{w}}, \vec{\chi})\|_{\mathbf{Q}}$ and $\|\widehat{\mathcal{J}}_3(\vec{\mathbf{w}}, \vec{\chi})\|_{\mathrm{M}}$ are bounded each by $\delta/2$, which implies that $\|\mathcal{J}(\vec{\mathbf{w}}, \vec{\chi})\|_{\mathbf{Q}\times\mathrm{M}} \leqslant \delta$, and hence $\mathcal{J}\big(\mathrm{W}(\delta)\big) \subseteq \mathrm{W}(\delta)$. □

We continue the analysis with the Lipschitz-continuity properties of $\widetilde{\mathcal{J}}$ and $\widehat{\mathcal{J}}$.

Lemma 3.7. *There exist positive constants* $\mathrm{L}_{\mathrm{S},1}$ *and* $\mathrm{L}_{\mathrm{S},2}$*, depending only on* ν*,* ρ*,* $\alpha_{\mathbf{A}}$*,* a_1*, and* $C_{\mathbf{F}}$ *such that*

$$\|\widetilde{\mathcal{J}}(\vec{\mathbf{w}}, \vec{\chi}) - \widetilde{\mathcal{J}}(\vec{\mathbf{w}}_0, \vec{\chi}_0)\|_{\mathbb{H}\times\mathbf{Q}} \leqslant \Big\{ \mathrm{L}_{\mathrm{S},1}\big(\|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}} + a_1\delta \big) + \mathrm{L}_{\mathrm{S},2} \Big\} \|(\vec{\mathbf{w}}, \vec{\chi}) - (\vec{\mathbf{w}}_0, \vec{\chi}_0)\|_{\mathbf{Q}\times\mathrm{M}} \tag{3.32}$$

for all $(\vec{\mathbf{w}}, \vec{\chi}), (\vec{\mathbf{w}}_0, \vec{\chi}_0) \in \mathrm{W}(\delta)$.

Proof. Given $(\vec{\mathbf{w}}, \vec{\chi})$, $(\vec{\mathbf{w}}_0, \vec{\chi}_0) \in \mathrm{W}(\delta)$, we set $\widetilde{\mathcal{J}}(\vec{\mathbf{w}}, \vec{\chi}) = (\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) \in \mathbb{H} \times \mathbf{Q}$ and $\widetilde{\mathcal{J}}(\vec{\mathbf{w}}_0, \vec{\chi}_0) = (\vec{\boldsymbol{\sigma}}_0, \vec{\mathbf{u}}_0) \in \mathbb{H} \times \mathbf{Q}$ as the unique solutions of the formulations (3.2) (equivalently (3.15)), that is

$$\mathbf{A}_{\vec{\mathbf{w}}}\big((\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), (\vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\big) = \mathbf{F}_{\vec{\chi}}(\vec{\boldsymbol{\tau}}) \quad \text{and} \quad \mathbf{A}_{\vec{\mathbf{w}}_0}\big((\vec{\boldsymbol{\sigma}}_0, \vec{\mathbf{u}}_0), (\vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\big) = \mathbf{F}_{\vec{\chi}_0}(\vec{\boldsymbol{\tau}}),$$

respectively, both for all $(\vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H} \times \mathbf{Q}$. Then, applying the global inf-sup condition (3.17) to $(\vec{\boldsymbol{\zeta}}, \vec{\mathbf{z}}) := \big(\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0\big)$, using the above identities, (3.15), (3.7a) and (2.9), we find that

$$\begin{aligned}
\frac{\alpha_{\mathbf{A}}}{2} \,\|\big(\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0\big)\|_{\mathbb{H}\times\mathbf{Q}} &\leqslant \sup_{\substack{(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q}\\ (\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathbf{A}_{\vec{\mathbf{w}}}\big((\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0), (\vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\big)}{\|(\vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}\times\mathbf{Q}}} \\
= \sup_{\substack{(\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\times\mathbf{Q}\\ (\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathbf{O}_{\mathrm{S}}(\vec{\mathbf{w}} - \vec{\mathbf{w}}_0; \vec{\boldsymbol{\sigma}}_0, \vec{\mathbf{v}}) + \big(\mathbf{F}_{\vec{\chi}} - \mathbf{F}_{\vec{\chi}_0}\big)(\vec{\boldsymbol{\tau}})}{\|(\vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}\times\mathbf{Q}}} &\leqslant \frac{\rho}{\nu} \|\vec{\mathbf{w}} - \vec{\mathbf{w}}_0\|_{\mathbf{Q}} \|\vec{\boldsymbol{\sigma}}_0\|_{\mathbb{H}} + a_1 C_{\mathbf{F}}\, \|\vec{\chi} - \vec{\chi}_0\|_{\mathrm{M}},
\end{aligned}$$

which, along with the fact that $\|\widetilde{\mathcal{J}}(\vec{\mathbf{w}}, \vec{\chi}) - \widetilde{\mathcal{J}}(\vec{\mathbf{w}}_0, \vec{\chi}_0)\|_{\mathbb{H}\times\mathbf{Q}} = \|(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) - (\vec{\boldsymbol{\sigma}}_0, \vec{\mathbf{u}}_0)\|_{\mathbb{H}\times\mathbf{Q}}$ and (3.19), yields (3.32), with $\mathrm{L}_{\mathrm{S},1} := \dfrac{4\rho\, C_{\mathbf{F}}}{\nu\alpha_{\mathbf{A}}^2}$ and $\mathrm{L}_{\mathrm{S},2} := \dfrac{2a_1 C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}$, thus completing the proof. □

Lemma 3.8. *There exists a positive constant* L_{T}*, depending only on* $\alpha_{\mathcal{A}}$*,* $\widetilde{a}_1$*,* $\widetilde{a}_3$ *and* κ*, such that*

$$\|\widehat{\mathcal{J}}(\vec{\mathbf{w}}, \vec{\chi}) - \widehat{\mathcal{J}}(\vec{\mathbf{w}}_0, \vec{\chi}_0)\|_{\mathbf{H}\times\mathrm{Q}} \leqslant \mathrm{L}_{\mathrm{T}}\, \big(1+\delta\big)\, \|(\vec{\mathbf{w}}, \vec{\chi}) - (\vec{\mathbf{w}}_0, \vec{\chi}_0)\|_{\mathbf{Q}\times\mathrm{M}} \tag{3.33}$$

for all $(\vec{\mathbf{w}}, \vec{\chi})$, $(\vec{\mathbf{w}}_0, \vec{\chi}_0) \in \mathrm{W}(\delta)$.

Proof. Given $(\vec{\mathbf{w}},\vec{\chi}),\,(\vec{\mathbf{w}}_0,\vec{\chi}_0)\in \mathrm{W}(\delta)$, we let $\widehat{\mathcal{J}}(\vec{\mathbf{w}},\vec{\chi}) = (\vec{\mathbf{t}},(\vec{\phi},\vec{\xi}))\in \mathbf{H}\times \mathrm{Q}$ and $\widehat{\mathcal{J}}(\vec{\mathbf{w}}_0,\vec{\chi}_0) = (\vec{\mathbf{t}}_0,(\vec{\phi}_0,\vec{\xi}_0))\in \mathbf{H}\times\mathrm{Q}$ as the unique solutions of (3.4), Then, subtracting both systems and rearranging the terms appropriately, we find that

$$\mathcal{A}_{\vec{\mathbf{w}},\vec{\chi}}\big((\vec{\mathbf{t}}-\vec{\mathbf{t}}_0,(\vec{\phi}-\vec{\phi}_0,\,\vec{\xi}-\vec{\xi}_0)),\,(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta}))\big) = \mathbf{O}_{\mathrm{T}}(\vec{\mathbf{w}}-\vec{\mathbf{w}}_0;\vec{\mathbf{t}}_0,\vec{\psi}) + \mathbf{C}_{\mathrm{T}}\big(\vec{\chi}-\vec{\chi}_0;\vec{\xi}_0,\vec{\eta}\big) + \big(\mathcal{F}_{\vec{\chi}}-\mathcal{F}_{\vec{\chi}_0}\big)(\vec{\psi},\vec{\eta})$$

for all $(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta}))\in \mathbf{H}\times\mathrm{Q}$. In this way, applying the global inf-sup condition (3.28) to $(\vec{\mathbf{r}},(\vec{\varphi},\vec{\lambda})) := (\vec{\mathbf{t}},(\vec{\phi},\vec{\xi})) - (\vec{\mathbf{t}}_0,(\vec{\phi}_0,\vec{\xi}_0))$, and then employing the foregoing identity along with (3.7b) and (3.10), we obtain

$$\begin{aligned}\frac{\alpha_{\mathcal{A}}}{2}\,\|(\vec{\mathbf{t}},(\vec{\phi},\vec{\xi}))-(\vec{\mathbf{t}}_0,(\vec{\phi}_0,\vec{\xi}_0))\|_{\mathbf{H}\times\mathrm{Q}} &\leqslant \sup_{\substack{(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta}))\in\mathbf{H}\times\mathrm{Q}\\ (\vec{\mathbf{s}},(\vec{\psi},\vec{\eta}))\neq\mathbf{0}}} \frac{\mathcal{A}_{\vec{\mathbf{w}},\vec{\chi}}\big((\vec{\mathbf{t}},(\vec{\phi},\vec{\xi}))-(\vec{\mathbf{t}}_0,(\vec{\phi}_0,\vec{\xi}_0)),(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta}))\big)}{\|(\vec{\mathbf{s}},(\vec{\psi},\vec{\eta}))\|_{\mathbf{H}\times\mathrm{Q}}}\\ &\leqslant \kappa^{-1}\|\vec{\mathbf{w}}-\vec{\mathbf{w}}_0\|_{\mathbf{Q}}\,\|\vec{\mathbf{t}}_0\|_{\mathbf{H}} + \widetilde{a}_1\|\vec{\chi}-\vec{\chi}_0\|_{\mathrm{M}}\|\,\|\vec{\xi}_0\|_{\mathrm{M}} + \widetilde{a}_3\|\vec{\chi}-\vec{\chi}_0\|_{\mathrm{M}},\end{aligned}$$

from which, using the bounds for $\|\vec{\mathbf{t}}_0\|_{\mathbf{H}} = \|\widehat{\mathcal{J}}_1(\vec{\mathbf{w}},\vec{\chi})\|_{\mathbf{H}}$ and $\|\vec{\xi}_0\|_{\mathrm{M}} = \|\widehat{\mathcal{J}}_3(\vec{\mathbf{w}}_0,\vec{\chi}_0)\|_{\mathrm{M}}$ provided by (3.29), we arrive at (3.33) with $\mathrm{L}_{\mathrm{T}} := \dfrac{2}{\alpha_{\mathcal{A}}}\widetilde{a}_3\max\left\{\dfrac{2}{\alpha_{\mathcal{A}}}\big(\kappa^{-1}+\widetilde{a}_1\big),1\right\}$. □

Having proved Lemmas 3.7 and 3.8, we are able to establish now the Lipschitz-continuity of our fixed point operator $\mathcal{J}$ in the closed ball $\mathrm{W}(\delta)$.

Lemma 3.9. *Let $\mathrm{L}_{\mathrm{S},1}$, $\mathrm{L}_{\mathrm{S},2}$ and L_{T} be the constants provided by Lemmas 3.7 and 3.8. There holds*

$$\|\mathcal{J}(\vec{\mathbf{w}},\vec{\chi})-\mathcal{J}(\vec{\mathbf{w}}_0,\vec{\chi}_0)\|_{\mathbf{Q}\times\mathrm{M}} \leqslant \Big\{\mathrm{L}_{\mathrm{S},1}\,\big(\|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}}+a_1\delta\big)+\mathrm{L}_{\mathrm{S},2}+\mathrm{L}_{\mathrm{T}}\,\big(1+\delta\big)\Big\}\,\|(\vec{\mathbf{w}},\vec{\chi})-(\vec{\mathbf{w}}_0,\vec{\chi}_0)\|_{\mathbf{Q}\times\mathrm{M}}$$

for all $(\vec{\mathbf{w}},\vec{\chi}),(\vec{\mathbf{w}}_0,\vec{\chi}_0)\in\mathrm{W}(\delta)$.

Proof. It follows from the definition of $\mathcal{J}$ (cf. (3.5)) and the estimates (3.32) and (3.33). □

Owing to the above analysis, we now establish the main result of this section.

Theorem 3.10. *Let us assume that the given data are sufficiently small to satisfy (3.31), and*

$$\mathrm{L}_{\mathrm{S},1}\,\big(\|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}}+a_1\delta\big)+\mathrm{L}_{\mathrm{S},2}+\mathrm{L}_{\mathrm{T}}\big(1+\delta\big) < 1. \tag{3.34}$$

The problem (2.6) has a unique solution $(\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}})\in\mathbb{H}\times\mathbf{Q}$ and $\big(\vec{\mathbf{t}},(\vec{\phi},\vec{\xi})\big)\in\mathbf{H}\times\mathrm{Q}$ with $(\vec{\mathbf{u}},\vec{\xi})\in\mathrm{W}(\delta)$. Moreover, there hold

$$\|(\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}})\|_{\mathbb{H}\times\mathbf{Q}} \leqslant \frac{2C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\Big\{\|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}}+a_1\delta\Big\},\quad \textit{and} \tag{3.35}$$

$$\|(\vec{\mathbf{t}},(\vec{\phi},\vec{\xi}))\|_{\mathbf{H}\times\mathrm{Q}} \leqslant \frac{2}{\alpha_{\mathcal{A}}}\widetilde{a}_3\big(1+\delta\big). \tag{3.36}$$

Proof. We first recall that the choice of δ and the assumptions of Lemma 3.6 guarantee that $\mathcal{J}$ maps $\mathrm{W}(\delta)$ into itself. Then, bearing in mind the Lipschitz-continuity of $\mathcal{J}:\mathrm{W}(\delta)\to\mathrm{W}(\delta)$ given by Lemma 3.9 and the hypotheses (3.34), a straightforward application of the classical Banach fixed-point Theorem yields the existence of a unique fixed point $(\vec{\mathbf{u}},\vec{\xi})\in\mathrm{W}(\delta)$ of this operator, and hence a unique solution of (2.6). In addition, the *a priori* estimates provided by (3.19) and (3.29), yield (3.35) and (3.36), which completes the proof. □

4 The Galerkin scheme

In order to approximate the solution of our mixed variational formulation (2.6), we now introduce the associated Galerkin scheme, analyze its solvability by applying a discrete version of the fixed-point approach adopted in the previous section, and derive the corresponding *a priori* error estimate.

4.1 Preliminaries

Let $\mathcal{T}_h^{\mathrm{f}}$ and $\mathcal{T}_h^{\mathrm{p}}$ be the respective triangulations of the domains Ω_{f} and Ω_{p} formed by shape-regular triangles of diameter h_K and denote by h_{f} and h_{p} their corresponding meshsizes. Assume that they match on Σ so that $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{f}} \cup \mathcal{T}_h^{\mathrm{p}}$ is a conforming triangulation of $\Omega := \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{p}}$. Hereafter, $h := \max\{h_{\mathrm{f}}, h_{\mathrm{p}}\}$. Now, for each $\star \in \{\mathrm{f}, \mathrm{p}\}$, letting $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}_\star} \subseteq \mathbb{H}(\mathbf{div}_{4/3};\Omega_\star)$, selecting a set of arbitrary discrete spaces, namely

$$
\begin{aligned}
&\mathbb{H}_h^{\boldsymbol{\sigma}_\star} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}_\star} \cap \mathbb{H}_{\Gamma_{\mathrm{out}}}(\mathbf{div}_{4/3};\Omega_\star), \quad \mathbf{H}_h^{\mathbf{t}_\star} \subseteq \mathbf{H}(\mathrm{div}_{4/3};\Omega_\star), \quad \mathrm{M}_h^{\xi_\star} \subseteq \mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{in},\star}^{\mathrm{c}}),\\
&\mathbf{Q}_h^{\mathbf{u}_\star} \subseteq \mathbf{L}^4(\Omega_\star), \qquad\qquad\qquad\qquad\quad \mathrm{X}_h^{\phi_\star} \subseteq \mathrm{L}^4(\Omega_\star),
\end{aligned}
\tag{4.1}
$$

defining the global spaces

$$
\begin{aligned}
&\mathbb{H}_h := \mathbb{H}_h^{\boldsymbol{\sigma}_{\mathrm{f}}} \times \mathbb{H}_h^{\boldsymbol{\sigma}_{\mathrm{p}}}, && \mathbf{H}_h := \mathbf{H}_h^{\mathbf{t}_{\mathrm{f}}} \times \mathbf{H}_h^{\mathbf{t}_{\mathrm{p}}}, && \mathrm{M}_h := \mathrm{M}_h^{\xi_{\mathrm{f}}} \times \mathrm{M}_h^{\xi_{\mathrm{p}}},\\
&\mathbf{Q}_h := \mathbf{Q}_h^{\mathbf{u}_{\mathrm{f}}} \times \mathbf{Q}_h^{\mathbf{u}_{\mathrm{p}}}, && \mathrm{X}_h := \mathrm{X}_h^{\phi_{\mathrm{f}}} \times \mathrm{X}_h^{\phi_{\mathrm{p}}}, && \mathrm{Q}_h := \mathrm{X}_h \times \mathrm{M}_h,
\end{aligned}
$$

and setting the notation

$$
\begin{aligned}
&\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h^{\mathrm{f}}, \boldsymbol{\tau}_h^{\mathrm{p}}) \in \mathbb{H}_h, && \vec{\mathbf{s}}_h := (\mathbf{s}_h^{\mathrm{f}}, \mathbf{s}_h^{\mathrm{p}}) \in \mathbf{H}_h, && \vec{\xi}_h := (\xi_h^{\mathrm{f}}, \xi_h^{\mathrm{p}}) \in \mathrm{M}_h,\\
&\vec{\mathbf{w}}_h := (\mathbf{w}_h^{\mathrm{f}}, \mathbf{w}_h^{\mathrm{p}}) \in \mathbf{Q}_h, && \vec{\psi}_h := (\psi_h^{\mathrm{f}}, \psi_h^{\mathrm{p}}) \in \mathrm{X}_h, && (\vec{\phi}_h, \vec{\xi}_h) := (\phi_h^{\mathrm{f}}, \phi_h^{\mathrm{p}}, \xi_h^{\mathrm{f}}, \xi_h^{\mathrm{p}}) \in \mathrm{Q}_h,
\end{aligned}
$$

the Galerkin scheme associated to (2.6) reads: Find $\big(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h\big) \in \mathbb{H}_h \times \mathbf{Q}_h$ and $\big(\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h)\big) \in \mathbf{H}_h \times \mathrm{Q}_h$ such that

$$
\begin{array}{llll}
\mathbf{a}_{\mathrm{S}}(\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\tau}}_h) & + \mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{u}}_h) & & = \mathbf{F}_{\vec{\xi}_h}(\vec{\boldsymbol{\tau}}_h),\\
\mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{v}}_h) & & - \mathbf{O}_{\mathrm{S}}(\vec{\mathbf{u}}_h; \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{v}}_h) & = 0,\\
\mathbf{a}_{\mathrm{T}}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) & + \mathbf{b}_{\mathrm{T}}(\vec{\mathbf{s}}_h, (\vec{\phi}_h, \vec{\xi}_h)) & & = 0,\\
\mathbf{b}_{\mathrm{T}}(\vec{\mathbf{t}}_h, (\vec{\psi}_h, \vec{\eta}_h)) & - \mathbf{d}_{\mathrm{T}}((\vec{\phi}_h, \vec{\xi}_h), (\vec{\psi}_h, \vec{\eta}_h)) & - \mathbf{O}_{\mathrm{T}}(\vec{\mathbf{u}}_h; \vec{\mathbf{t}}_h, \vec{\psi}_h) - \mathbf{C}_{\mathrm{T}}(\vec{\xi}_h; \vec{\xi}_h, \vec{\eta}_h) & = \mathcal{F}_{\vec{\xi}_h}(\vec{\psi}_h, \vec{\eta}_h),
\end{array}
\tag{4.2}
$$

for all $(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ and $(\vec{\mathbf{s}}_h, (\vec{\psi}_h, \vec{\eta}_h)) \in \mathbf{H}_h \times \mathrm{Q}_h$.

Then, we adopt the discrete version of the strategy employed in Section 3.1 to analyze the solvability of (4.2). To this end, we let $\widetilde{\mathcal{J}}_h : \mathbf{Q}_h \times \mathrm{M}_h \to \mathbb{H}_h \times \mathbf{Q}_h$ be the discrete operator defined by

$$
\widetilde{\mathcal{J}}_h(\vec{\mathbf{w}}_h, \vec{\chi}_h) = \big(\widetilde{\mathcal{J}}_{1,h}(\vec{\mathbf{w}}_h, \vec{\chi}_h), \widetilde{\mathcal{J}}_{2,h}(\vec{\mathbf{w}}_h, \vec{\chi}_h)\big) := (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h)
$$

for all $(\vec{\mathbf{w}}_h, \vec{\chi}_h) \in \mathbf{Q}_h \times \mathrm{M}_h$, where $(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ is the unique solution (to be confirmed in Theorem 4.1) of the following problem:

$$
\begin{array}{lll}
\mathbf{a}_{\mathrm{S}}(\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\tau}}_h) + \mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{u}}_h) & & = \mathbf{F}_{\vec{\chi}_h}(\vec{\boldsymbol{\tau}}_h),\\
\mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{v}}_h) & - \mathbf{O}_{\mathrm{S}}(\vec{\mathbf{w}}_h; \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{v}}_h) & = 0,
\end{array}
\tag{4.3}
$$

for all $(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$. In addition, we also let $\widehat{\mathcal{J}}_h : \mathbf{Q}_h \times \mathrm{M}_h \to \mathbf{H}_h \times \mathrm{Q}_h$ be the discrete operator given by

$$
\widehat{\mathcal{J}}_h(\vec{\mathbf{w}}_h, \vec{\chi}_h) = \big(\widehat{\mathcal{J}}_{1,h}(\vec{\mathbf{w}}_h, \vec{\chi}_h), (\widehat{\mathcal{J}}_{2,h}(\vec{\mathbf{w}}_h, \vec{\chi}_h), \widehat{\mathcal{J}}_{3,h}(\vec{\mathbf{w}}_h, \vec{\chi}_h))\big) := (\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h))
$$

for all $(\vec{\mathbf{w}}_h, \vec{\chi}_h) \in \mathbf{Q}_h \times \mathrm{M}_h$, where $(\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h)) \in \mathbf{H}_h \times \mathrm{Q}_h$ is the unique solution (to be confirmed in Theorem 4.2) of the following system of equations:

$$
\begin{array}{ll}
\mathbf{a}_{\mathrm{T}}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) + \mathbf{b}_{\mathrm{T}}(\vec{\mathbf{s}}_h, (\vec{\phi}_h, \vec{\xi}_h)) & = 0,\\
\mathbf{b}_{\mathrm{T}}(\vec{\mathbf{t}}_h, (\vec{\psi}_h, \vec{\eta}_h)) - \mathbf{d}_{\mathrm{T}}((\vec{\phi}_h, \vec{\xi}_h), (\vec{\psi}_h, \vec{\eta}_h)) - \mathbf{O}_{\mathrm{T}}(\vec{\mathbf{w}}_h; \vec{\mathbf{t}}_h, \vec{\psi}_h) - \mathbf{C}_{\mathrm{T}}\big(\vec{\chi}_h; \vec{\xi}_h, \vec{\eta}_h\big) & = \mathcal{F}_{\vec{\chi}_h}(\vec{\psi}_h, \vec{\eta}_h),
\end{array}
\tag{4.4}
$$

for all $(\vec{\mathbf{s}}_h, (\vec{\psi}_h, \vec{\eta}_h)) \in \mathrm{Q}_h \times \mathbf{H}_h$.

Finally, we define the operator $\mathcal{J}_h : \mathbf{Q}_h \times \mathrm{M}_h \to \mathbf{Q}_h \times \mathrm{M}_h$ as

$$
\mathcal{J}_h(\vec{\mathbf{w}}_h, \vec{\chi}_h) = \big(\widetilde{\mathcal{J}}_{2,h}(\vec{\mathbf{w}}_h, \vec{\chi}_h), \widehat{\mathcal{J}}_{3,h}(\vec{\mathbf{w}}_h, \vec{\chi}_h)\big) = (\vec{\mathbf{u}}_h, \vec{\xi}_h) \quad \forall\, (\vec{\mathbf{w}}_h, \vec{\chi}_h) \in \mathbf{Q}_h \times \mathrm{M}_h,
\tag{4.5}
$$

and notice that solving (4.2) is equivalent to seeking a unique fixed point of $\mathcal{J}_h$, that is $(\vec{\mathbf{u}}_h, \vec{\xi}_h) \in \mathbf{Q}_h \times \mathrm{M}_h$ such that $\mathcal{J}_h(\vec{\mathbf{u}}_h, \vec{\xi}_h) = (\vec{\mathbf{u}}_h, \vec{\xi}_h)$.

4.2 Well-definedness of the operators $\widetilde{\mathcal{J}}_h$ and $\widehat{\mathcal{J}}_h$

In this section we proceed analogously to Sections 3.1.1 and 3.1.2 and establish the well-posedness of the discrete systems (4.3) and (4.4), equivalently that the discrete operators $\widetilde{\mathcal{J}}_h$ and $\widehat{\mathcal{J}}_h$ are well-defined. To this end, given $\star \in \{\mathrm{f}, \mathrm{p}\}$, we introduce certain hypotheses on the finite element subspaces defined above, and the discrete kernels associated with the bilinear forms $\mathbf{b}_{\mathrm{S}}$, and $\mathbf{b}_{\mathrm{T}}$, respectively, that is

$$\mathcal{K}_{\mathrm{S},h} := \Big\{ \vec{\boldsymbol{\tau}}_h \in \mathbb{H}_h : \quad \int_{\Omega_\star} \mathbf{v}_h^\star \cdot \mathbf{div}(\boldsymbol{\tau}_h^\star) \;=\; 0 \quad \forall\, \mathbf{v}_h^\star \in \mathbf{Q}_h^{\mathbf{u}_\star} \Big\},$$
$$\mathcal{K}_{\mathrm{T},h} := \Big\{ \vec{\mathbf{s}}_h \in \mathbf{H}_h : \quad \int_{\Omega_\star} \psi_h^\star \mathrm{div}(\mathbf{s}_h^\star) \;=\; 0 \quad \forall\, \psi_h^\star \in \mathrm{X}_h^{\phi_\star} \quad \text{and} \quad \langle \mathbf{s}_h^\star \cdot \mathbf{n}_\star, \eta_h^\star \rangle_{\Gamma_{\mathrm{in},\star}^{\mathrm{c}}} \;=\; 0 \quad \forall\, \eta_h^\star \in \mathrm{M}_h^{\xi_\star} \Big\}.$$

More precisely, from now on we assume that

(H.1) $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}_\star}$ contains the multiplies of the identity tensor $\mathbb{I}$,

(H.2) $\mathbf{div}(\mathbb{H}_h^{\boldsymbol{\sigma}_\star}) \subseteq \mathbf{Q}_h^{\mathbf{u}_\star}$,

(H.3) there exists a positive constant $\beta_{\mathrm{S},\mathbf{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\boldsymbol{\tau}}_h \in \mathbb{H}_h \\ \vec{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{\mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbb{H}}} \;\geqslant\; \beta_{\mathrm{S},\mathbf{d}}\, \|\vec{\mathbf{v}}_h\|_{\mathbf{Q}} \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{Q}_h, \tag{4.6}$$

(H.4) $\mathrm{div}(\mathbf{H}_h^{\mathbf{t}_\star}) \subseteq \mathrm{X}_h^{\phi_\star}$,

(H.5) there exists a positive constant $\beta_{\mathrm{T},\mathbf{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{s}}_h \in \mathbf{H}_h \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\mathbf{b}_{\mathrm{T}}(\vec{\mathbf{s}}_h, (\vec{\psi}_h, \vec{\eta}_h))}{\|\vec{\mathbf{s}}_h\|_{\mathbf{H}}} \;\geqslant\; \beta_{\mathrm{T},\mathbf{d}}\, \|(\vec{\psi}_h, \vec{\eta}_h)\|_{\mathrm{Q}} \quad \forall\, (\vec{\psi}_h, \vec{\eta}_h) \in \mathrm{Q}_h. \tag{4.7}$$

Then, bearing in mind the assumption **(H.2)**, we find that $\mathcal{K}_{\mathrm{S},h} := \Big\{ \vec{\boldsymbol{\tau}}_h \in \mathbb{H}_h : \mathbf{div}(\boldsymbol{\tau}_h^\star) = \mathbf{0} \quad \text{in} \quad \Omega_\star \Big\}$.

In this regard, it is worth noting that the $\mathcal{K}_{\mathrm{S}}$-ellipticity of the bilinear form $\mathbf{a}_{\mathrm{S}}$, as shown in Lemma 3.1, relies solely on the divergence-free property of the tensors in $\mathcal{K}_{\mathrm{S}}$ along with the estimate (3.13). Therefore, by selecting our discrete space $\mathbb{H}_h^{\boldsymbol{\sigma}_\star} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}_\star} \cap \mathbb{H}_{\Gamma_{\mathrm{out}}}(\mathbf{div}_{4/3}; \Omega_\star)$, it follows that $\mathbf{a}_{\mathrm{S}}$ is also $\mathcal{K}_{\mathrm{S},h}$-elliptic with the same positive constant α_{S}, that is

$$\mathbf{a}_{\mathrm{S}}(\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\tau}}_h) \;\geqslant\; \alpha_{\mathbf{s}}\, \|\vec{\boldsymbol{\tau}}_h\|_{\mathbb{H}}^2 \qquad \forall\, \vec{\boldsymbol{\tau}}_h \in \mathcal{K}_{\mathrm{S},h}. \tag{4.8}$$

In this way, **(H.3)** and (4.8), guarantee, thanks to [18, Proposition 2.42], the discrete global inf-sup condition for $\mathbf{A}$ (cf. (3.16)) with a positive constant $\alpha_{\mathbf{A},\mathbf{d}}$, depending only on α_{S}, $\beta_{\mathrm{S},\mathbf{d}}$, and $\|\mathbf{a}_{\mathrm{S}}\|$, and hence independent of h, that is

$$\sup_{\substack{(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h \\ (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \neq \mathbf{0}}} \frac{\mathbf{A}\big((\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h), (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\big)}{\|(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\|_{\mathbb{H} \times \mathbf{Q}}} \;\geqslant\; \alpha_{\mathbf{A},\mathbf{d}}\, \|(\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h)\|_{\mathbb{H} \times \mathbf{Q}} \qquad \forall\, (\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h, \tag{4.9}$$

so that, for each $\vec{\mathbf{w}}_h \in \mathbf{Q}_h$ such that $\|\vec{\mathbf{w}}_h\|_{\mathbf{Q}} \leqslant \dfrac{\nu\, \alpha_{\mathbf{A},\mathbf{d}}}{2\, \rho}$, there holds

$$\sup_{\substack{(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h \\ (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \neq \mathbf{0}}} \frac{\mathbf{A}_{\vec{\mathbf{w}}_h}\big((\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h), (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\big)}{\|(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\|_{\mathbb{H} \times \mathbf{Q}}} \;\geqslant\; \frac{\alpha_{\mathbf{A},\mathbf{d}}}{2}\, \|(\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h)\|_{\mathbb{H} \times \mathbf{Q}} \qquad \forall\, (\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h. \tag{4.10}$$

Consequently, we are now in position to establish the discrete analogue of Theorem 3.3.

Theorem 4.1. *For each* $\vec{\mathbf{w}}_h \in \mathbf{Q}_h$, *and* $\vec{\chi}_h \in \mathrm{M}_h$ *such that* $\|\vec{\mathbf{w}}_h\|_{\mathbf{Q}} \leqslant \dfrac{\nu\, \alpha_{\mathbf{A},\mathbf{d}}}{2\, \rho}$, *there exists a unique* $(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ *solution to* (4.3), *equivalently*

$$\mathbf{A}_{\vec{\mathbf{w}}_h}\big((\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\big) \;=\; \mathbf{F}_{\vec{\chi}_h}(\vec{\boldsymbol{\tau}}_h) \qquad \forall\, (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h. \tag{4.11}$$

Moreover, there holds

$$\|\widetilde{\mathcal{J}}_h(\vec{\mathbf{w}}_h,\,\vec{\chi}_h)\|_{\mathbb{H}\times\mathbf{Q}} \;=\; \|(\vec{\boldsymbol{\sigma}}_h,\vec{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}} \;\leqslant\; \frac{2C_{\mathbf{F}}}{\alpha_{\mathbf{A},\mathbf{d}}}\Big\{\|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}} + a_1\|\vec{\chi}_h\|_{\mathrm{M}}\Big\}. \tag{4.12}$$

Proof. Let $(\vec{\mathbf{w}}_h,\vec{\chi}_h)\in\mathbf{Q}_h\times\mathrm{M}_h$, such that $\|\vec{\mathbf{w}}_h\|_{\mathbf{Q}}\leqslant\dfrac{\nu\,\alpha_{\mathbf{A},\mathbf{d}}}{2\,\rho}$. Analogously to the continuous case, we find that $\mathbf{A}_{\vec{\mathbf{w}}_h}$ (cf. (4.10)) satisfies the hypotheses of the discrete Banach–Nečas–Babuška theorem [18, Theorem 2.22], and then we conclude the existence of a unique $(\vec{\boldsymbol{\sigma}}_h,\vec{\mathbf{u}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$ solution to (4.11). In addition, the *a priori* error estimates (4.12) is consequence of (4.11), (4.10) and the boundedness of $\mathbf{F}_{\vec{\chi}_h}$ (cf. (3.9)). □

Similarly, thanks to **(H.4)**, it is easy to see that $\mathcal{K}_{\mathrm{T},h}$ can be characterized as

$$\mathcal{K}_{\mathrm{T},h} := \Big\{\vec{\mathbf{s}}_h := (\mathbf{s}^{\mathrm{f}}_h,\mathbf{s}^{\mathrm{p}}_h)\in\mathbf{H}_h:\quad \mathrm{div}(\mathbf{s}^\star_h)=0\quad\text{in}\quad\Omega_\star\quad\text{and}\quad\langle\mathbf{s}^\star_h\cdot\mathbf{n}_\star,\eta^\star_h\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}=0\quad\forall\,\eta^\star_h\in\mathrm{M}^{\xi_\star}_h\Big\},$$

and hence $\mathbf{a}_{\mathrm{T}}(\vec{\mathbf{s}}_h,\vec{\mathbf{s}}_h)\geqslant\frac{1}{2\kappa}\|\vec{\mathbf{s}}_h\|^2_{\mathbf{H}}$ for all $\vec{\mathbf{s}}_h\in\mathcal{K}_{\mathrm{T},h}$. In turn, knowing from the continuous analysis that $\mathbf{d}_{\mathrm{T}}$ is positive semi-definite in Q (cf. (3.20)), this property is also true in Q_h. Hence, bearing in mind **(H.5)**, a straightforward application of [14, Theorem 3.5] implies the discrete global inf-sup condition for $\mathcal{A}$ (cf. (3.24)) with a positive constant $\alpha_{\mathcal{A},\mathbf{d}}$, depending on $\|\mathbf{a}_{\mathrm{T}}\|$, α_{T}, $\|\mathbf{d}_{\mathrm{T}}\|$, and $\beta_{\mathrm{T},\mathbf{d}}$, and thus the same property is shared by $\mathcal{A}_{\vec{\mathbf{w}}_h,\vec{\chi}_h}$ (cf. (3.25)) for each $(\vec{\mathbf{w}}_h,\vec{\chi}_h)\in\mathbf{Q}_h\times\mathrm{M}_h$ such that $\|(\vec{\mathbf{w}}_h,\vec{\chi}_h)\|_{\mathbf{Q}\times\mathrm{M}}\leqslant\dfrac{\kappa\alpha_{\mathcal{A},\mathbf{d}}}{2(1+\kappa\widetilde{a}_1)}$, that is

$$\sup_{\substack{(\vec{\mathbf{s}}_h,(\vec{\psi}_h,\vec{\eta}_h))\in\mathbf{H}_h\times\mathrm{Q}_h\\(\vec{\mathbf{s}}_h,(\vec{\psi}_h,\vec{\eta}_h))\neq\mathbf{0}}}\frac{\mathcal{A}_{\vec{\mathbf{w}}_h,\vec{\chi}_h}\big((\vec{\mathbf{r}}_h,(\vec{\varphi}_h,\vec{\lambda}_h)),(\vec{\mathbf{s}}_h,(\vec{\psi}_h,\vec{\eta}_h))\big)}{\|(\vec{\mathbf{s}}_h,(\vec{\psi}_h,\vec{\eta}_h))\|_{\mathbf{H}\times\mathrm{Q}}}\;\geqslant\;\frac{\alpha_{\mathcal{A},\mathbf{d}}}{2}\,\|(\vec{\mathbf{r}}_h,(\vec{\varphi}_h,\vec{\lambda}_h))\|_{\mathbf{H}\times\mathrm{Q}} \tag{4.13}$$

for all $(\vec{\mathbf{r}}_h,(\vec{\varphi}_h,\vec{\lambda}_h))\in\mathbf{H}_h\times\mathrm{Q}_h$.

In this way, we are now in a position to establish the discrete analogue of Theorem 3.5.

Theorem 4.2. *For each* $(\vec{\mathbf{w}}_h,\vec{\chi}_h)\in\mathbf{Q}_h\times\mathrm{M}_h$ *such that* $\|(\vec{\mathbf{w}}_h,\vec{\chi}_h)\|_{\mathbf{Q}\times\mathrm{M}}\leqslant\dfrac{\kappa\alpha_{\mathcal{A},\mathbf{d}}}{2(1+\kappa\widetilde{a}_1)}$, *there exists a unique* $(\vec{\mathbf{t}}_h,(\vec{\phi}_h,\vec{\xi}_h))=\widehat{\mathcal{J}}_h(\vec{\mathbf{w}}_h,\vec{\chi}_h)\in\mathbf{H}_h\times\mathrm{Q}_h$ *solution to* (4.4)*. Moreover, there holds*

$$\|\widehat{\mathcal{J}}_h(\vec{\mathbf{w}}_h,\vec{\chi}_h)\|_{\mathbf{H}\times\mathrm{Q}} \;=\; \|(\vec{\mathbf{t}}_h,(\vec{\phi}_h,\vec{\xi}_h))\|_{\mathbf{H}\times\mathrm{Q}} \;\leqslant\; \frac{2}{\alpha_{\mathcal{A},\mathbf{d}}}\widetilde{a}_3\big(1+\|\vec{\chi}_h\|_{\mathrm{M}}\big). \tag{4.14}$$

Proof. It reduces to a direct application of [18, Theorem 2.22]. In particular, the *a priori* estimate (4.14) follows from (4.13) and the fact that, according to (3.10), there holds $\|\mathcal{F}_{\vec{\chi}_h}\|\leqslant\widetilde{a}_3\big(1+\|\vec{\chi}_h\|_{\mathrm{M}}\big)$. □

4.3 Discrete solvability analysis

We now address the solvability of the fixed-point equation (4.5). For that, we set $\delta_{\mathbf{d}} := \min\left\{\dfrac{\nu\alpha_{\mathbf{A},\mathbf{d}}}{2\,\rho},\dfrac{\kappa\alpha_{\mathcal{A},\mathbf{d}}}{2(1+\kappa\widetilde{a}_1)}\right\}$, and define

$$\mathrm{W}(\delta_{\mathbf{d}}) := \Big\{(\vec{\mathbf{w}}_h,\vec{\chi}_h)\in\mathbf{Q}_h\times\mathrm{M}_h:\quad\|(\vec{\mathbf{w}}_h,\vec{\chi}_h)\|_{\mathbf{Q}\times\mathrm{M}}\leqslant\delta_{\mathbf{d}}\Big\}, \tag{4.15}$$

to provide sufficient conditions under which $\mathcal{J}_h$ maps $\mathrm{W}(\delta_{\mathbf{d}})$ into itself. More precisely, we have the following result.

Lemma 4.3. *If*

$$C_{\mathbf{F}}\Big\{\|\vec{\mathbf{g}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out}}}+a_1\delta_{\mathbf{d}}\Big\}\;\leqslant\;\frac{\alpha_{\mathbf{A},\mathbf{d}}}{4}\delta_{\mathbf{d}},\quad\textit{and}\quad\widetilde{a}_3\big(1+\delta_{\mathbf{d}}\big)\;\leqslant\;\frac{\alpha_{\mathcal{A},\mathbf{d}}}{4}\delta_{\mathbf{d}}, \tag{4.16}$$

then $\mathcal{J}_h\big(\mathrm{W}(\delta_{\mathbf{d}})\big)\subseteq\mathrm{W}(\delta_{\mathbf{d}})$.

Proof. It follows analogously to the proof of Lemma 3.6, but now using the well-posedness and associated *a priori* estimates of $\widetilde{\mathcal{J}}_h$ and $\widehat{\mathcal{J}}_h$ provided by Theorems 4.1 and 4.2, respectively. □

Next, we establish the Lipschitz-continuity properties of $\widetilde{\mathcal{J}}_h$ and $\widehat{\mathcal{J}}_h$.

Lemma 4.4. *There exist positive constants* $\mathrm{L}^1_{\mathrm{S},\mathbf{d}}$ *and* $\mathrm{L}^2_{\mathrm{S},\mathbf{d}}$ *depending only on* ν, ρ, $\alpha_{\mathbf{A},\mathbf{d}}$, a_1, *and* $C_{\mathbf{F}}$ *such that*

$$\|\widetilde{\mathcal{J}}_h(\vec{\mathbf{w}}_h, \vec{\chi}_h) - \widetilde{\mathcal{J}}_h(\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h})\|_{\mathbb{H}\times\mathbf{Q}} \leqslant \Big\{ \mathrm{L}^1_{\mathrm{S},\mathbf{d}} \left(\|\vec{\mathbf{g}}\|_{1/2,\Gamma^c_{\mathrm{out}}} + a_1 \delta_{\mathbf{d}} \right) + \mathrm{L}^2_{\mathrm{S},\mathbf{d}} \Big\} \, \|(\vec{\mathbf{w}}_h, \vec{\chi}_h) - (\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h})\|_{\mathbf{Q}\times\mathrm{M}}$$

for all $(\vec{\mathbf{w}}_h, \vec{\chi}_h), (\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h}) \in \mathrm{W}(\delta_{\mathbf{d}})$.

Proof. It follows analogously to the proof of Lemma 3.7, using now the discrete inf-sup condition satisfied by $\mathbf{A}_{\vec{\mathbf{w}}_h}$ (cf. (4.10)) with constant $\dfrac{\alpha_{\mathbf{A},\mathbf{d}}}{2}$. □

Lemma 4.5. *There exists a positive constant* $\mathrm{L}_{\mathrm{T},\mathbf{d}}$, *depending only on* $\alpha_{\mathcal{A},\mathbf{d}}$, $\widetilde{a}_1$, $\widetilde{a}_3$ *and* κ, *such that*

$$\|\widehat{\mathcal{J}}_h(\vec{\mathbf{w}}_h, \vec{\chi}_h) - \widehat{\mathcal{J}}_h(\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h})\|_{\mathbf{H}\times\mathrm{Q}} \leqslant \mathrm{L}_{\mathrm{T},\mathbf{d}} \left(1 + \delta_{\mathbf{d}}\right) \|(\vec{\mathbf{w}}_h, \vec{\chi}_h) - (\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h})\|_{\mathbf{Q}\times\mathrm{M}}$$

for all $(\vec{\mathbf{w}}_h, \vec{\chi}_h), (\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h}) \in \mathrm{W}(\delta_{\mathbf{d}})$.

Proof. It follows very closely to the arguments from the proof of Lemma 3.8, employing now the discrete inf-sup condition satisfied by $\mathcal{A}_{\vec{\mathbf{w}}_h,\vec{\chi}_h}$ (cf. (4.13)) with constant $\dfrac{\alpha_{\mathcal{A},\mathbf{d}}}{2}$. □

As a consequence, we are able to establish the Lipschitz-continuity of the operator $\mathcal{J}_h$.

Lemma 4.6. *Let* $\mathrm{L}^1_{\mathrm{S},\mathbf{d}}$, $\mathrm{L}^2_{\mathrm{S},\mathbf{d}}$ *and* $\mathrm{L}_{\mathrm{T},\mathbf{d}}$ *be the constants provided by Lemmas 4.4 and 4.5. There holds*

$$\begin{aligned} &\|\mathcal{J}_h(\vec{\mathbf{w}}_h, \vec{\chi}_h) - \mathcal{J}_h(\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h})\|_{\mathbf{Q}\times\mathrm{M}} \\ &\quad \leqslant \Big\{ \mathrm{L}^1_{\mathrm{S},\mathbf{d}}\big(\|\vec{\mathbf{g}}\|_{1/2,\Gamma^c_{\mathrm{out}}} + a_1\delta_{\mathbf{d}}\big) + \mathrm{L}^2_{\mathrm{S},\mathbf{d}} + \mathrm{L}_{\mathrm{T},\mathbf{d}}\big(1+\delta_{\mathbf{d}}\big)\Big\} \|(\vec{\mathbf{w}}_h, \vec{\chi}_h) - (\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h})\|_{\mathbf{Q}\times\mathrm{M}} \end{aligned} \tag{4.17}$$

for all $(\vec{\mathbf{w}}_h, \vec{\chi}_h), (\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h}) \in \mathrm{W}(\delta_{\mathbf{d}})$.

Proof. Given $(\vec{\mathbf{w}}_h, \vec{\chi}_h), (\vec{\mathbf{w}}_{0,h}, \vec{\chi}_{0,h}) \in \mathrm{W}(\delta_{\mathbf{d}})$, it suffices to employ the definition of $\mathcal{J}_h$ (cf. (4.5)), and the upper bounds of Lemmas 4.4 and 4.5. □

According to the above, the main result of this section is establish as follows.

Theorem 4.7. *If the given data are sufficiently small to satisfy* (4.16)*, then the problem* (4.2) *has at least one solution* $(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ *and* $(\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h)) \in \mathbf{H}_h \times \mathrm{Q}_h$ *with* $(\vec{\mathbf{u}}_h, \vec{\xi}_h) \in \mathrm{W}(\delta_{\mathbf{d}})$. *Moreover, under the further assumption*

$$\mathrm{L}^1_{\mathrm{S},\mathbf{d}} \left(\|\vec{\mathbf{g}}\|_{1/2,\Gamma^c_{\mathrm{out}}} + a_1\delta_{\mathbf{d}}\right) + \mathrm{L}^2_{\mathrm{S},\mathbf{d}} + \mathrm{L}_{\mathrm{T},\mathbf{d}} \left(1 + \delta_{\mathbf{d}}\right) < 1, \tag{4.18}$$

this solution is unique. In addition, in both cases there hold

$$\|(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}} \leqslant \frac{2C_{\mathbf{F}}}{\alpha_{\mathbf{A},\mathbf{d}}} \Big\{ \|\vec{\mathbf{g}}\|_{1/2,\Gamma^c_{\mathrm{out}}} + a_1\delta_{\mathbf{d}} \Big\}, \quad \textit{and} \tag{4.19}$$

$$\|(\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h))\|_{\mathbf{H}\times\mathrm{Q}} \leqslant \frac{2}{\alpha_{\mathcal{A},\mathbf{d}}} \widetilde{a}_3 \left(1 + \delta_{\mathbf{d}}\right). \tag{4.20}$$

Proof. We first notice that the assumptions of Lemma 4.3 guarantee that $\mathcal{J}_h$ maps $\mathrm{W}(\delta_{\mathbf{d}})$ into itself. Then, the continuity of $\mathcal{J}_h : \mathrm{W}(\delta_{\mathbf{d}}) \to \mathrm{W}(\delta_{\mathbf{d}})$ (cf. (4.17)) and a straightforward application of the Brouwer Theorem (cf. [12, Theorem 9.9-2]) implies the existence of at least one solution $(\vec{\mathbf{u}}_h, \vec{\xi}_h) \in \mathrm{W}(\delta_{\mathbf{d}})$ to (4.2). Next, the uniqueness of solution is a consequence of the Banach fixed-point Theorem and assumption (4.18). Finally, thanks to the *a priori* estimates (4.12) and (4.14), we obtain (4.19) and (4.20). □

4.4 *A priori* error analysis

We consider finite element subspaces satisfying the assumptions specified in Section 4.2, and derive the Céa estimate for the Galerkin error $\mathtt{E} := \mathtt{E}_{\mathrm{S}} + \mathtt{E}_{\mathrm{T}}$, where

$$\mathtt{E}_{\mathrm{S}} := \|(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) - (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}} \quad \text{and} \quad \mathtt{E}_{\mathrm{T}} := \|(\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})) - (\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h))\|_{\mathbf{H}\times\mathrm{Q}},$$

and $(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) \in \mathbb{H} \times \mathbf{Q}$, $(\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})) \in \mathbf{H} \times \mathrm{Q}$ and $(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$, $(\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h)) \in \mathbf{H}_h \times \mathrm{Q}_h$ are the unique solutions to (2.6) and (4.2), respectively, with $(\vec{\mathbf{u}}, \vec{\xi}) \in \mathrm{W}(\delta)$ (cf. (3.30)) and $(\vec{\mathbf{u}}_h, \vec{\xi}_h) \in \mathrm{W}(\delta_{\mathbf{d}})$ (cf. (4.15)). In what follows, given a subspace Z_h of an arbitrary Banach space $\big(Z, \|\cdot\|_Z\big)$, we set

$$\operatorname{dist}\big(z, Z_h\big) := \inf_{z_h \in Z_h} \|z - z_h\|_Z \qquad \forall\, z \in Z.$$

Now, using (3.15) and (3.26), we observe that (2.6) and (4.2) can be rewritten as the following pairs of continuous formulations and their associated discrete counterparts

$$\mathbf{A}\big((\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), (\vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\big) = \mathbf{O}_{\mathrm{S}}(\vec{\mathbf{u}}; \vec{\boldsymbol{\sigma}}, \vec{\mathbf{v}}) + \mathbf{F}_{\vec{\xi}}(\vec{\boldsymbol{\tau}}), \quad \mathbf{A}\big((\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\big) = \mathbf{O}_{\mathrm{S}}(\vec{\mathbf{u}}_h; \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{v}}_h) + \mathbf{F}_{\vec{\xi}_h}(\vec{\boldsymbol{\tau}}_h) \tag{4.21}$$

for all $(\vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H} \times \mathbf{Q}$ and $(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$, and

$$\begin{aligned} \mathcal{A}\big((\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})), (\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta}))\big) &= \mathbf{O}_{\mathrm{T}}(\vec{\mathbf{u}}; \vec{\mathbf{t}}, \vec{\psi}) + \mathbf{C}_{\mathrm{T}}(\vec{\xi}; \vec{\xi}, \vec{\eta}) + \mathcal{F}_{\vec{\xi}}(\vec{\psi}, \vec{\eta}), \\ \mathcal{A}\big((\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h)), (\vec{\mathbf{s}}_h, (\vec{\psi}_h, \vec{\eta}_h))\big) &= \mathbf{O}_{\mathrm{T}}(\vec{\mathbf{u}}_h; \vec{\mathbf{t}}_h, \vec{\psi}_h) + \mathbf{C}_{\mathrm{T}}(\vec{\xi}_h; \vec{\xi}_h, \vec{\eta}_h) + \mathcal{F}_{\vec{\xi}_h}(\vec{\psi}_h, \vec{\eta}_h) \end{aligned} \tag{4.22}$$

for all $(\vec{\mathbf{s}}, (\vec{\psi}, \vec{\eta})) \in \mathbf{H} \times \mathrm{Q}$ and $(\vec{\mathbf{s}}_h, (\vec{\psi}_h, \vec{\eta}_h)) \in \mathbf{H}_h \times \mathrm{Q}_h$.

Then, from (4.21), it is easy to see that for each $(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$, there holds

$$\mathbf{A}\left((\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) - (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\right) = \mathbf{O}_{\mathrm{S}}(\vec{\mathbf{u}}; \vec{\boldsymbol{\sigma}}, \vec{\mathbf{v}}_h) - \mathbf{O}_{\mathrm{S}}(\vec{\mathbf{u}}_h; \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{v}}_h) + \big(\mathbf{F}_{\vec{\xi}} - \mathbf{F}_{\vec{\xi}_h}\big)(\vec{\boldsymbol{\tau}}_h),$$

whence, subtracting and adding $\vec{\boldsymbol{\sigma}}_h$ in the second component of the first term, invoking the boundedness properties of $\mathbf{O}_{\mathrm{S}}$ (cf. (3.7a)), $\mathbf{F}_{\vec{\xi}} - \mathbf{F}_{\vec{\xi}_h}$ (cf. (3.9)), and the *a priori* estimates (cf. (3.35) and (4.19)) for $\|\vec{\mathbf{u}}\|_{\mathbf{Q}}$ and $\|\vec{\boldsymbol{\sigma}}_h\|_{\mathbb{H}}$, respectively, we obtain for each $(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$

$$\begin{aligned} &\mathbf{A}\left((\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) - (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\right) \\ &\quad \leqslant \left\{\frac{\rho}{\nu}\|\vec{\mathbf{u}}\|_{\mathbf{Q}}\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbb{H}} + \frac{\rho}{\nu}\|\vec{\boldsymbol{\sigma}}\|_{\mathbb{H}}\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}} + a_1 C_{\mathbf{F}}\|\vec{\xi} - \vec{\xi}_h\|_{\mathrm{M}}\right\}\|(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\|_{\mathbb{H}\times\mathbf{Q}}. \end{aligned} \tag{4.23}$$

Now, the triangle inequality gives for each $(\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$

$$\mathtt{E}_{\mathrm{S}} \leqslant \|(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) - (\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h)\|_{\mathbb{H}\times\mathbf{Q}} + \|(\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h) - (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}}, \tag{4.24}$$

and then, applying (4.9), subtracting and adding $(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}})$ in the first component of $\mathbf{A}$, and using the boundedness of $\mathbf{A}$ with constant $\|\mathbf{A}\|$, which depends on $\|\mathbf{a}_{\mathrm{S}}\|$, and $\|\mathbf{b}_{\mathrm{S}}\|$ (cf. (3.6)), we find that

$$\begin{aligned} \alpha_{\mathbf{A},\mathbf{d}}\|(\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h) - (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}} &\leqslant \sup_{\substack{(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h \\ (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \neq \mathbf{0}}} \frac{\mathbf{A}\big((\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h) - (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\big)}{\|(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\|_{\mathbb{H}\times\mathbf{Q}}} \\ &\leqslant \|\mathbf{A}\|\|(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) - (\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h)\|_{\mathbb{H}\times\mathbf{Q}} + \sup_{\substack{(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h \\ (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h) \neq \mathbf{0}}} \frac{\mathbf{A}\big((\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) - (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), (\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\big)}{\|(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)\|_{\mathbb{H}\times\mathbf{Q}}}. \end{aligned}$$

In this way, inserting (4.23) into the supremum and replacing the resulting estimate in (4.24), and the fact that $(\vec{\boldsymbol{\zeta}}_h, \vec{\mathbf{z}}_h)$ is arbitrary, we conclude that there exists a positive constant $C_{\mathtt{ST}}$ depending only on ρ, ν, $\alpha_{\mathbf{A},\mathbf{d}}$, and hence independent of h, such that

$$\mathtt{E}_{\mathrm{S}} \leqslant C_{\mathtt{ST}}\Big\{\operatorname{dist}\big((\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \mathbb{H}_h \times \mathbf{Q}_h\big) + \|\vec{\mathbf{u}}\|_{\mathbf{Q}}\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbb{H}} + \|\vec{\boldsymbol{\sigma}}\|_{\mathbb{H}}\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}} + a_1 C_{\mathbf{F}}\|\vec{\xi} - \vec{\xi}_h\|_{\mathrm{M}}\Big\}. \tag{4.25}$$

Similarly, for each $(\vec{\mathbf{s}}_h, (\vec{\psi}_h, \vec{\eta}_h)) \in \mathbf{H}_h \times \mathrm{Q}_h$, from (4.22), we deduce that

$$\begin{aligned}&\mathcal{A}\big((\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})) - (\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h)), (\vec{\mathbf{s}}_h, (\vec{\psi}_h, \vec{\eta}_h))\big) \\ &\quad = \mathbf{O}_{\mathrm{T}}(\vec{\mathbf{u}}; \vec{\mathbf{t}}, \vec{\psi}_h) + \mathbf{C}_{\mathrm{T}}(\vec{\xi}; \vec{\xi}, \vec{\eta}_h) - \mathbf{O}_{\mathrm{T}}(\vec{\mathbf{u}}_h; \vec{\mathbf{t}}_h, \vec{\psi}_h) - \mathbf{C}_{\mathrm{T}}(\vec{\xi}_h; \vec{\xi}_h, \vec{\eta}_h) + \big(\mathcal{F}_{\vec{\xi}} - \mathcal{F}_{\vec{\xi}_h}\big)(\vec{\psi}_h, \vec{\eta}_h),\end{aligned}$$

from which, subtracting and adding $\vec{\mathbf{t}}_h$ and $\vec{\xi}_h$ in the second component of the first and second terms, respectively, invoking the boundedness properties of $\mathbf{O}_{\mathrm{T}}$, $\mathbf{C}_{\mathrm{T}}$, and $\mathcal{F}_{\vec{\xi}} - \mathcal{F}_{\vec{\xi}_h}$, the a priori estimates (cf. (3.36) and (4.19)), we proceed exactly as in the previous case for $\mathtt{E}_{\mathrm{S}}$, and realize that there exists a positive constant $C_{\mathtt{TT}}$, depending only on $\|\mathcal{A}\|$, $\alpha_{\mathcal{A},\mathbf{d}}$, $\widetilde{a}_1$ and κ, and hence independent of h, such that

$$\begin{aligned}\mathtt{E}_{\mathrm{T}} \leqslant C_{\mathtt{TT}} \Big\{&\mathrm{dist}\big((\vec{\mathbf{t}}, \vec{\phi}, \vec{\xi}), \mathbf{H}_h \times \mathrm{Q}_h\big) + \|\vec{\mathbf{t}}_h\|_{\mathbf{H}} \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}} + \|\vec{\mathbf{t}} - \mathbf{t}_h\|_{\mathbf{H}} \|\vec{\mathbf{u}}\|_{\mathbf{Q}} \\ &+ \big(\|\xi\|_{\mathrm{M}} + \|\xi_h\|_{\mathrm{M}} + \widetilde{a}_3\big) \|\vec{\xi} - \vec{\xi}_h\|_{\mathrm{M}}\Big\}.\end{aligned} \tag{4.26}$$

Consequently, we are in position to establish the announced Céa estimate.

Theorem 4.8. *In addition to the hipotheses of Theorems 3.10 and 4.7, assume that*

$$\begin{aligned}\frac{2C_{\mathbf{F}}}{\alpha_{\mathbf{A}}} \Big\{\|\vec{\mathbf{g}}\|_{1/2,\Gamma_{\mathrm{out}}^{\mathrm{c}}} + a_1 \delta\Big\} + \frac{2}{\alpha_{\mathcal{A},\mathbf{d}}} \widetilde{a}_3 (1 + \delta_{\mathbf{d}}) &\leqslant \frac{1}{2C_{\mathtt{ST}}}, \quad \text{and} \\ \frac{2C_{\mathbf{F}}}{\alpha_{\mathbf{A}}} \Big\{\|\vec{\mathbf{g}}\|_{1/2,\Gamma_{\mathrm{out}}^{\mathrm{c}}} + a_1 \delta\Big\} + \frac{4}{\widehat{\alpha}_{\mathcal{A}}} \widetilde{a}_3 (1 + \widehat{\delta}) + \widetilde{a}_3 + a_1 C_{\mathbf{F}} &\leqslant \frac{1}{2C_{\mathtt{TT}}},\end{aligned} \tag{4.27}$$

where $\widehat{\alpha}_{\mathcal{A}} := \min\{\alpha_{\mathcal{A}}, \alpha_{\mathcal{A},\mathbf{d}}\}$, *and* $\widehat{\delta} := \max\{\delta, \delta_{\mathbf{d}}\}$. *There exists a positive constant* $C_{\mathtt{e}}$, *independent of* h, *such that*

$$\begin{aligned}&\|(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) - (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h)\|_{\mathbb{H} \times \mathbf{Q}} + \|(\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})) - (\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h))\|_{\mathbf{H} \times \mathrm{Q}} \\ &\quad \leqslant 2C_{\mathtt{e}} \Big\{\mathrm{dist}\big((\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \mathbb{H}_h \times \mathbf{Q}_h\big) + \mathrm{dist}\big((\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})), \mathbf{H}_h \times \mathrm{Q}_h\big)\Big\}.\end{aligned} \tag{4.28}$$

Proof. By combining (4.25) and (4.26), it suffices to use the bounds given by Theorems 3.10 and 4.7 along with (4.27), which yield (4.28). □

We end this section by remarking that (1.10) suggests the following postprocessed approximation for the pressure $p_\star$

$$p_h^\star = -\frac{1}{n} \mathrm{tr}(\boldsymbol{\sigma}_h^\star) \quad \text{in} \quad \Omega_\star, \tag{4.29}$$

so that, it is easy to show that

$$\|p_\star - p_h^\star\|_{0,\Omega_\star} \leqslant \frac{1}{\sqrt{n}} \|\boldsymbol{\sigma}_\star - \boldsymbol{\sigma}_h^\star\|_{0,\Omega_\star}. \tag{4.30}$$

Thus, combining (4.28) and (4.30), we conclude the existence of $\widehat{C}_{\mathtt{e}} \geqslant 0$, independent of h, such that

$$\begin{aligned}&\|(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}) - (\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h)\|_{\mathbb{H} \times \mathbf{Q}} + \|(\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})) - (\vec{\mathbf{t}}_h, (\vec{\phi}_h, \vec{\xi}_h))\|_{\mathbf{H} \times \mathrm{Q}} + \sum_{\star \in \{\mathrm{f},\mathrm{p}\}} \|p_\star - p_h^\star\|_{0,\Omega_\star} \\ &\quad \leqslant \widehat{C}_{\mathtt{e}} \Big\{\mathrm{dist}\big((\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \mathbb{H}_h \times \mathbf{Q}_h\big) + \mathrm{dist}((\vec{\mathbf{t}}, (\vec{\phi}, \vec{\xi})), \mathbf{H}_h \times \mathrm{Q}_h)\Big\}.\end{aligned} \tag{4.31}$$

5 Specific finite element subspaces

In what follows, given $K \in \mathcal{T}_h^\star$, with $\star \in \{\mathrm{f}, \mathrm{p}\}$, we let $\mathrm{P}_0(K)$ be the space of polinomials of degree 0 defined on K, whose vector version is denoted by $\mathbf{P}_0(K) := [\mathrm{P}_0(K)]^2$. Next, we define the corresponding local Raviart–Thomas spaces of order 0 as (see [21, Chapter 3] for further details) $\mathbf{RT}_0(K) := \mathbf{P}_0(K) \oplus \mathrm{P}_0(K)\boldsymbol{x}$, where $\boldsymbol{x} := (x_1, x_2)^t$ is a generic vector in $\mathbb{R}^2$. Then, we introduce, respectively, the following finite element spaces for the variables $\mathbf{t}_\star$, $\boldsymbol{\sigma}_\star$, $\phi_\star$ and $\mathbf{u}_\star$:

$$\mathbf{H}_h^{\mathbf{t}_\star} := \Big\{\mathbf{s}_h^\star \in \mathbf{H}(\mathrm{div}; \Omega_\star) : \quad \mathbf{s}_h^\star|_K \in \mathbf{RT}_0(K) \quad \forall\, K \in \mathcal{T}_h^\star\Big\}, \quad \mathbb{H}_h^{\boldsymbol{\sigma}_\star} := \mathbb{H}_{\Gamma_{\mathrm{out}}}(\mathbf{div}_{4/3}; \Omega_\star) \cap \widetilde{\mathbb{H}}_h^\star, \text{ with}$$

$$
\begin{aligned}
\widetilde{\mathbb{H}}_h^{\star} &:= \Big\{ \boldsymbol{\tau}_h^{\star} \in \mathbb{H}(\mathbf{div};\Omega_\star) : \quad \boldsymbol{\tau}_{h,i}^{\star}|_K \in \mathbf{RT}_0(K), \quad \forall\, i \in \{1,2\}, \quad \forall\, K \in \mathcal{T}_h^{\star} \Big\}, \\
\mathrm{X}_h^{\phi_\star} &:= \Big\{ \psi_h^{\star} \in \mathrm{L}^2(\Omega_\star) : \quad \psi_h^{\star}|_K \in \mathrm{P}_0(K) \quad \forall\, K \in \mathcal{T}_h^{\star} \Big\}, \quad \text{and} \\
\mathbf{Q}_h^{\mathbf{u}_\star} &:= \Big\{ \mathbf{v}_h^{\star} \in \mathbf{L}^2(\Omega_\star) : \quad \mathbf{v}_h^{\star}|_K \in \mathbf{P}_0(K) \quad \forall\, K \in \mathcal{T}_h^{\star} \Big\},
\end{aligned}
$$

where $\boldsymbol{\tau}_{h,i}^{\star}$ denotes the ith-row of $\boldsymbol{\tau}_h^{\star}$. It remains to introduce the finite element space for the variable $\xi_\star$. To that aim, we proceed similarly to [23] and denote by $\Gamma_{\mathrm{in},\star,h}^{\mathrm{c}}$ the partition of $\Gamma_{\mathrm{in},\star}^{\mathrm{c}}$ inherited from $\mathcal{T}_h^{\star}$. Let us assume, without loss of generality, that the number of edges of $\Gamma_{\mathrm{in},\star,h}^{\mathrm{c}}$ is even. Then, we let $\Gamma_{\mathrm{in},\star,2h}^{\mathrm{c}}$ be the partition of $\Gamma_{\mathrm{in},\star}^{\mathrm{c}}$ arising by joining pairs of adjacent edges of $\Gamma_{\mathrm{in},\star,h}^{\mathrm{c}}$. If the number of edges of $\Gamma_{\mathrm{in},\star,h}^{\mathrm{c}}$ is odd, we simply reduce it to the even case by adding one node to the discretization of the boundary $\Gamma_{\mathrm{in},\star}^{\mathrm{c}}$ and locally modify the triangulation to keep the mesh conformity and regularity. In this way, denoting by $\boldsymbol{x}_0$ and $\boldsymbol{x}_N$ the extreme points of $\overline{\Gamma}_{\mathrm{in},\star}^{\mathrm{c}}$, we define the following finite element space:

$$
\mathrm{M}_h^{\xi_\star} := \big\{ \eta_h^{\star} \in \mathcal{C}(\Gamma_{\mathrm{in},\star}^{\mathrm{c}}) : \quad \eta_h^{\star}|_e \in \mathrm{P}_1(e) \quad \forall \text{ edge } e \in \Gamma_{\mathrm{in},\star,2h}^{\mathrm{c}}, \quad \eta_h^{\star}(\boldsymbol{x}_0) = \eta_h^{\star}(\boldsymbol{x}_N) = 0 \big\}\,.
$$

We stress here that the above particular subspaces satisfy the inclusions (4.1). We now verify that these subspaces satisfy the hypotheses **(H.1)**-**(H.5)**.

First, it is easy to show (see, [6, Section 4.2] for details) that $\widetilde{\mathbb{H}}_h^{\star}$ satisfy **(H.1)** and **(H.2)**. Next, in order to check **(H.3)**, we require the following result.

Lemma 5.1. *Assume that there exists a polygonal convex domain $\widehat{B}_\star$ such that $\Omega_\star \subseteq \widehat{B}_\star$ and $\Gamma_{\mathrm{out},\star} \subseteq \partial\widehat{B}_\star$. There exists a positive constant $\beta_{\mathrm{S},\mathbf{d}}^{\star}$, independent of h, such that*

$$
\sup_{\substack{\boldsymbol{\tau}_h^{\star} \in \mathbb{H}_h^{\boldsymbol{\sigma}_\star} \\ \boldsymbol{\tau}_h^{\star} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_\star} \mathbf{v}_h^{\star} \cdot \mathbf{div}(\boldsymbol{\tau}_h^{\star})}{\|\boldsymbol{\tau}_h^{\star}\|_{\mathbf{div}_{4/3};\Omega_\star}} \geqslant \beta_{\mathrm{S},\mathbf{d}}^{\star} \|\mathbf{v}_h^{\star}\|_{0,4;\Omega_\star} \quad \forall\, \mathbf{v}_h^{\star} \in \mathbf{Q}_h^{\mathbf{u}_\star}.
$$

Proof. We proceed similarly to the proof of [6, Lemma 4.3], employing the arguments utilized in [11, Lemma 4.1]. □

In this way, since for each $\star \in \{\mathrm{f}, \mathrm{p}\}$ we have

$$
\sup_{\substack{\vec{\boldsymbol{\tau}}_h \in \mathbb{H}_h \\ \vec{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{\mathbf{b}_{\mathrm{S}}(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbb{H}}} \geqslant \sup_{\substack{\boldsymbol{\tau}_h^{\star} \in \mathbb{H}_h^{\boldsymbol{\sigma}_\star} \\ \boldsymbol{\tau}_h^{\star} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_\star} \mathbf{v}_h^{\star} \cdot \mathbf{div}(\boldsymbol{\tau}_h^{\star})}{\|\boldsymbol{\tau}_h^{\star}\|_{\mathbf{div}_{4/3};\Omega_\star}} \geqslant \beta_{\mathrm{S},\mathbf{d}}^{\star} \|\mathbf{v}_h^{\star}\|_{0,4;\Omega_\star} \quad \forall\, \mathbf{v}_h^{\star} \in \mathbf{Q}_h^{\mathbf{u}_\star}, \tag{5.1}
$$

it is straightforward to see that $\mathbb{H}_h$ and $\mathbf{Q}_h$, satisfy **(H.3)** (cf. (4.6)) with a positive constant $\beta_{\mathrm{S},\mathbf{d}}$, independent of h.

On the other hand, we observe that **(H.4)** holds according to [21, Lemma 3.7]. It only remains to verify **(H.5)**. To that aim, we follow the simplest approach suggested in [24]. More precisely, for each $\star \in \{\mathrm{f}, \mathrm{p}\}$, we recall (cf. [24, Lemma 5.2]) that there exists a positive constant $\beta^{\star}$, independent of h, such that

$$
\sup_{\substack{\mathbf{s}_h^{\star} \in \mathbf{H}_h^{\mathbf{t}_\star} \\ \mathbf{s}_h^{\star} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_\star} \psi_h^{\star} \mathrm{div}(\mathbf{s}_h^{\star}) + \langle \mathbf{s}_h^{\star} \cdot \mathbf{n}_\star, \eta_h^{\star} \rangle_{\Gamma_{\mathrm{in},\star}^{\mathrm{c}}}}{\|\mathbf{s}_h^{\star}\|_{\mathrm{div};\Omega_\star}} \geqslant \beta^{\star} \Big\{ \|\psi_h^{\star}\|_{0;\Omega_\star} + \|\eta_h^{\star}\|_{1/2,00,\Gamma_{\mathrm{in},\star}^{\mathrm{c}}} \Big\} \tag{5.2}
$$

for all $(\psi_h^{\star}, \eta_h^{\star}) \in \mathrm{X}_h^{\phi_\star} \times \mathrm{M}_h^{\xi_\star}$. Then, proceeding similarly to [22, Section 4.4.1], we use the fact that there exists a positive constant $c(\Omega_\star)$, depending only on $|\Omega_\star|$, such that $\|\mathbf{s}_\star\|_{\mathrm{div}_{4/3};\Omega_\star} \leqslant c(\Omega_\star)\, \|\mathbf{s}_\star\|_{\mathrm{div};\Omega_\star}$ for all $\mathbf{s}_\star \in \mathbf{H}(\mathrm{div};\Omega_\star)$, along with (5.2), to deduce that

$$
\sup_{\substack{\mathbf{s}_h^{\star} \in \mathbf{H}_h^{\mathbf{t}_\star} \\ \mathbf{s}_h^{\star} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_\star} \psi_h^{\star} \mathrm{div}(\mathbf{s}_h^{\star}) + \langle \mathbf{s}_h^{\star} \cdot \mathbf{n}_\star, \eta_h^{\star} \rangle_{\Gamma_{\mathrm{in},\star}^{\mathrm{c}}}}{\|\mathbf{s}_h^{\star}\|_{\mathrm{div}_{4/3};\Omega_\star}} \geqslant \frac{\beta^{\star}}{c(\Omega_\star)} \|\eta_h^{\star}\|_{1/2,00,\Gamma_{\mathrm{in},\star}^{\mathrm{c}}} \quad \forall\, (\psi_h^{\star}, \eta_h^{\star}) \in \mathrm{X}_h^{\phi_\star} \times \mathrm{M}_h^{\xi_\star}. \tag{5.3}
$$

In addition, we have the following result.

Lemma 5.2. *Assume that there exists a polygonal convex domain $\widetilde{B}_\star$ such that $\Omega_\star \subseteq \widetilde{B}_\star$ and $\Gamma^{\rm c}_{{\rm in},\star} \subseteq \partial\widetilde{B}_\star$. Then there exists a positive constant $\widetilde{\beta}^\star_{\mathrm{T},\mathbf{d}}$, independent of h, such that*

$$\sup_{\substack{\mathbf{s}^\star_h\in\mathbf{RT}_0(\mathcal{T}^\star_h)\\ \mathbf{s}^\star_h\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_\star}\psi^\star_h \mathrm{div}(\mathbf{s}^\star_h) + \langle \mathbf{s}^\star_h\cdot\mathbf{n}_\star, \eta^\star_h\rangle_{\Gamma^{\rm c}_{{\rm in},\star}}}{\|\mathbf{s}^\star_h\|_{\mathrm{div}_{4/3};\Omega_\star}} \geqslant \widetilde{\beta}^\star_{\mathrm{T},\mathbf{d}}\,\|\psi^\star_h\|_{0,4;\Omega_\star} \quad \forall\,(\psi^\star_h,\eta^\star_h)\in \mathrm{X}^{\phi_\star}_h\times \mathrm{M}^{\xi_\star}_h. \tag{5.4}$$

Proof. Similarly as in Lemma 5.1, the result follows directly from [11, Lemma 4.1]. □

Hence, a straightforward linear combination of (5.3) and (5.4) implies that

$$\sup_{\substack{\mathbf{s}^\star_h\in\mathbf{H}^{\mathbf{t}_\star}_h\\ \mathbf{s}^\star_h\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_\star}\psi^\star_h \mathrm{div}(\mathbf{s}^\star_h) + \langle \mathbf{s}^\star_h\cdot\mathbf{n}_\star, \eta^\star_h\rangle_{\Gamma^{\rm c}_{{\rm in},\star}}}{\|\mathbf{s}^\star_h\|_{\mathrm{div}_{4/3};\Omega_\star}} \geqslant \widehat{\beta}^\star_{\mathrm{T},\mathbf{d}}\left\{\|\psi^\star_h\|_{0,4;\Omega_\star} + \|\eta^\star_h\|_{1/2,00,\Gamma^{\rm c}_{{\rm in},\star}}\right\} \quad \forall\,(\psi^\star_h,\eta^\star_h)\in \mathrm{X}^{\phi_\star}_h\times \mathrm{M}^{\xi_\star}_h,$$

where $\widehat{\beta}^\star_{\mathrm{T},\mathbf{d}}$ is a positive constant depending only on $\beta^\star$, $\widetilde{\beta}^\star_{\mathrm{T},\mathbf{d}}$ and $c(\Omega_\star)$.

Finally, proceeding in the same way as we did in (5.1), we deduce that $\mathbf{H}_h$ and Q_h, satisfy **(H.5)** (cf. (4.7)) with a positive constant $\beta_{\mathrm{T},\mathbf{d}}$, independent of h.

We end this section by providing the rates of convergence of the Galerkin scheme (4.2) with the specific finite element subspaces introduced in Section 5. More precisely, we can state the following main theorem.

Theorem 5.3. *Let us consider the hypothesis of Theorem 4.8. In addition, let $p_\star$ and $p^\star_h$ be the exact and approximate pressures defined by (1.10) and (4.29), respectively. Assume that $\boldsymbol{\sigma}_\star \in \mathbb{H}^1(\Omega_\star)\cap\mathbb{H}_{\Gamma_{\rm out}}(\mathbf{div}_{4/3};\Omega_\star)$, $\mathbf{div}(\boldsymbol{\sigma}_\star) \in \mathbf{W}^{1,4/3}(\Omega_\star)$, $\mathbf{t}_\star \in \mathbf{H}^1(\Omega_\star)$, $\mathrm{div}(\mathbf{t}_\star) \in \mathrm{W}^{1,4/3}(\Omega_\star)$, $\mathbf{u}_\star \in \mathbf{W}^{1,4}(\Omega_\star)$, $\phi_\star \in \mathrm{W}^{1,4}(\Omega_\star)$ and $\xi_\star \in \mathrm{H}^{3/2}_{00}(\Gamma^{\rm c}_{{\rm in},\star})$. Then, there exists a constant $C>0$, independent of h, such that*

$$\begin{aligned}
&\|(\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}) - (\vec{\boldsymbol{\sigma}}_h,\vec{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}} + \|(\vec{\mathbf{t}},(\vec{\phi},\vec{\xi})) - (\vec{\mathbf{t}}_h,(\vec{\phi}_h,\vec{\xi}_h))\|_{\mathbf{H}\times\mathrm{Q}} + \sum_{\star\in\{\mathrm{f},\mathrm{p}\}}\|p_\star - p^\star_h\|_{0,\Omega_\star}\\
&\leqslant C\sum_{\star\in\{\mathrm{f},\mathrm{p}\}} h\Big\{\|\boldsymbol{\sigma}_\star\|_{1,\Omega_\star} + \|\mathbf{div}(\boldsymbol{\sigma}_\star)\|_{1,4/3;\Omega_\star} + \|\mathbf{u}_\star\|_{1,4;\Omega} + \|\mathbf{t}_\star\|_{1,\Omega_\star} + \|\mathrm{div}(\mathbf{t}_\star)\|_{1,4/3;\Omega_\star} + \|\phi_\star\|_{1,4;\Omega_\star}\\
&\quad + \|\xi_\star\|_{3/2,00,\Gamma^{\rm c}_{{\rm in},\star}}\Big\}.
\end{aligned}$$

Proof. The proof follows from the corresponding Céa estimate (4.31), the approximation properties of the subspaces involved. In particular, for $\mathbb{H}^{\boldsymbol{\sigma}_\star}_h$ and $\mathbf{H}^{\mathbf{t}_\star}_h$ we refer, respectively, to [6, eq. (4.7)] and [7, eq. (3.8)], which in turn are consequences of [18, Lemma B.67, Lemma 1.101] and [21, Section 3.4.4]. For $\mathbf{Q}^{\mathbf{u}_\star}_h$ and $\mathrm{X}^{\phi_\star}_h$ we refer to [18, Proposition 1.135, Section 1.6.3], whereas for $\mathrm{M}^{\xi_\star}_h$ we refer to [23, Section 5.4, **(AP3)**]. □

6 Computational results

To illustrate the performance of the method, we present two computational simulations. The first one consists of a manufactured solution to verify numerically the rates of convergence anticipated by Theorem 5.3. The second one is a realistic situation in a reverse osmosis processes, with no analitical solution available. The implementation is done in `FreeFem++` (cf. [25]). In both, the iterative method comes straightforward from the uncoupling strategy presented in Section 4.1 and we use the following stopping criteria. Let us denote by **coeff** the vector that contains all the degrees of freedom associates to $\vec{\boldsymbol{\sigma}}_h,\vec{\mathbf{u}}_h,\vec{\mathbf{t}}_h,\vec{\phi}_h$ and $\vec{\xi}_h$. Given a tolerance `tol`, we stop the fixed point iteration when the relative error between two consecutive iterations (m and $m+1$), satisfies

$$\frac{\|\mathbf{coeff}^{m+1}-\mathbf{coeff}^{m}\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \leqslant \texttt{tol}. \tag{6.1}$$

Here, $\|\cdot\|_{l^2}$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{dof}}$ and `dof` denoting the total number of degrees of freedom. Subsequently, errors are defined as follows:

$$e(\boldsymbol{\sigma}_\star) = \|\boldsymbol{\sigma}_\star - \boldsymbol{\sigma}_h^\star\|_{\mathbf{div}_{4/3};\Omega_\star}, \qquad e(\mathbf{u}_\star) = \|\mathbf{u}_\star - \mathbf{u}_h^\star\|_{0,4;\Omega_\star}, \qquad e(p_\star) = \|p_\star - p_h^\star\|_{0,\Omega_\star},$$

$$e(\mathbf{t}_\star) = \|\mathbf{t}_\star - \mathbf{t}_h^\star\|_{\mathrm{div}_{4/3};\Omega_\star}, \qquad e(\phi_\star) = \|\phi_\star - \phi_h^\star\|_{0,4;\Omega_\star}, \qquad e(\xi_\star) = \|\xi_\star - \xi_h^\star\|_{1/2,00,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}},$$

where we recall, thanks to Sobolev interpolation results (cf. [1, Chapter 5], [28, Appendix B]) that

$$\|\xi_\star - \xi_h^\star\|_{1/2,00,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}} \leqslant C\|\xi_\star - \xi_h^\star\|^{1/2}_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}} \|\xi_\star - \xi_h^\star\|^{1/2}_{1,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}.$$

This relation suggests that the norm in $\mathrm{H}^{1/2}_{00}(\Gamma^{\mathrm{c}}_{\mathrm{in},\star})$ can be estimated by the norms in $\mathrm{L}^2(\Gamma^{\mathrm{c}}_{\mathrm{in},\star})$ and $\mathrm{H}^1(\Gamma^{\mathrm{c}}_{\mathrm{in},\star})$. In turn, the experimental order of convergence, is set as

$$r(*) = \frac{\log(e(*)/e'(*))}{\log(h/h')} \qquad \forall * \in \{\boldsymbol{\sigma}_\star, \mathbf{u}_\star, p_\star, \mathbf{t}_\star, \phi_\star, \xi_\star\},$$

where e and e' denote the errors computed on two consecutive meshes of sizes h and h'. In addition, we refer to the number of iterations as `iter`. Next, in our two examples, we consider the computational domain $\Omega = \Omega_\mathrm{f} \cup \Omega_\mathrm{p} \cup \Sigma$, where $\Omega_\mathrm{f} = (0, L) \times (d, 2d)$, $\Omega_\mathrm{p} = (0, L) \times (0, d)$, and $\Sigma = (0, L) \times \{d\}$.

Example 1: A manufactured smooth solution. In our first numerical test, we consider the computational domain Ω with $L = 1$ and $d = 0.5$, and set the parameters $\nu = 2$, $\rho = 0.1$, $\kappa = 1.6$, $a_0 = 10^{-8}$, $a_1 = 0.01$ and $a_2 = 2.5 \times 10^{-6}$. In addition, we define the manufactured solution:

$$p_\star = \cos(\pi x)\exp(y), \quad \mathbf{u}_\star = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}, \quad \phi_\star = \cos(\pi x)\sin(\pi y),$$

We notice that the only external sources of the physical model (1.14) are due to the velocity profile and concentration at the inlet. So, in order to be able to use the above manufactured solution, we introduce artificial volumetric, boundary and interface sources that make this solution to satisfy the equations. Table 6 shows the history of convergence for a sequence of quasi-uniform mesh refinements. The experiment confirm the theoretical rate of convergence $\mathcal{O}(h)$, provided by Theorem 5.3. In addition, as initial guess to start the iteration, we consider zero velocity and concentration. The number of iterations required to reach the stopping criterion (6.1) with a tolerance of $1e-6$, was less than or equal to 4.

Example 2: Coupled channels. We set $L = 15\,mm$, $d = 0.74\,mm$. The inlet velocity profiles are consider as in Remark 2, and the physical parameters specified in Section 1.2 are in Table 6.2. In Figure 6.1 we display the computed velocity magnitudes, pressure and salt concentration fields, which were built using the fully-mixed $\mathbb{RT}_0 - \mathbf{P}_0$ and $\mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_1$ schemes on a mesh with $h = 0.02$ and 348,140 triangular elements (actually representing 2,616,096 `dof`). In addition, as initial guess to start the iteration, we consider zero velocity and concentration. The number of iterations required to reach the stopping criterion (6.1) with a tolerance of $1e-6$, was equal to 12. We see that the parabolic profile remains in both channels (top panel). The velocity magnitude in the feed channel decreases along the axial axis, while the opposite occurs in the permeate channel. This is expected because water flows from the feed channel to the permeate across the membrane. Also, a pressure loss is observed along the entire channels (center panel), which, for laminar flows, is related to the friction between the fluid and the channel walls, as well as to the accumulation of salt near the membrane [31]. This accumulation is typical in reverse osmosis processes, as shown in the bottom panel.

7 Conclusions

We have developed and analyzed a mixed finite element method for the coupled Navier–Stokes and transport equations with nonlinear transmission conditions. We proved well-posedness of both, the continuous and discrete formulations, specified finite element subspaces and show the convergence properties of the proposed numerical scheme. To the best of our knowledge, this is the first contribution that provides the mathematical framework

$\mathbb{RT}_0 - \mathbf{P}_0$ and $\mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_1$ approximation								
dof	$e(\boldsymbol{\sigma}_\mathrm{f})$	$r(\boldsymbol{\sigma}_\mathrm{f})$	$e(\mathbf{u}_\mathrm{f})$	$r(\mathbf{u}_\mathrm{f})$	$e(p_\mathrm{f})$	$r(p_\mathrm{f})$	$e(\mathbf{t}_\mathrm{f})$	$r(\mathbf{t}_\mathrm{f})$
1161	$1.62e+00$	$*$	$9.43e-02$	$*$	$2.06e-01$	$*$	$5.16e-01$	$*$
4527	$7.74e-01$	0.99	$4.59e-02$	0.96	$9.30e-02$	1.07	$2.50e-01$	0.97
17559	$3.86e-01$	1.17	$2.30e-02$	1.17	$4.60e-02$	1.19	$1.27e-01$	1.14
68913	$1.94e-01$	1.06	$1.16e-02$	1.05	$2.29e-02$	1.08	$6.35e-02$	1.07
280476	$9.55e-02$	1.05	$5.74e-03$	1.04	$1.13e-02$	1.04	$3.14e-02$	1.04
1106742	$4.79e-02$	1.08	$2.89e-03$	1.08	$5.64e-03$	1.10	$1.58e-02$	1.08
h	$e(\phi_\mathrm{f})$	$r(\phi_\mathrm{f})$	$e(\xi_\mathrm{f})$	$r(\xi_\mathrm{f})$	$e(\boldsymbol{\sigma}_\mathrm{p})$	$r(\boldsymbol{\sigma}_\mathrm{p})$	$e(\mathbf{u}_\mathrm{p})$	$r(\mathbf{u}_\mathrm{p})$
0.1863	$7.46e-02$	$*$	$2.51e-01$	$*$	$1.63e+00$	$*$	$9.60e-02$	$*$
0.0884	$3.50e-02$	1.01	$1.13e-01$	1.08	$7.79e-01$	1.19	$4.65e-02$	1.17
0.0488	$1.77e-02$	1.15	$5.62e-02$	1.17	$3.80e-01$	0.96	$2.31e-02$	0.93
0.0255	$8.79e-03$	1.07	$2.86e-02$	1.04	$1.90e-01$	1.10	$1.16e-02$	1.08
0.0130	$4.33e-03$	1.05	$1.41e-02$	1.05	$9.36e-02$	1.09	$5.72e-03$	1.09
0.0069	$2.18e-03$	1.08	$7.13e-03$	1.07	$4.71e-02$	1.22	$2.88e-03$	1.21
iter	$e(p_\mathrm{p})$	$r(p_\mathrm{p})$	$e(\mathbf{t}_\mathrm{p})$	$r(\mathbf{t}_\mathrm{p})$	$e(\phi_\mathrm{p})$	$r(\phi_\mathrm{p})$	$e(\xi_\mathrm{p})$	$r(\xi_\mathrm{p})$
4	$3.07e-01$	$*$	$5.12e-01$	$*$	$7.10e-02$	$*$	$2.15e-01$	$*$
3	$1.26e-01$	1.43	$2.53e-01$	1.13	$3.52e-02$	1.13	$9.55e-02$	1.31
3	$5.65e-02$	1.07	$1.27e-01$	0.92	$1.74e-02$	0.94	$4.62e-02$	0.97
3	$2.76e-02$	1.13	$6.34e-02$	1.09	$8.76e-03$	1.08	$2.30e-02$	1.10
3	$1.35e-02$	1.10	$3.14e-02$	1.08	$4.31e-03$	1.09	$1.14e-02$	1.08
3	$6.80e-03$	1.22	$1.58e-02$	1.22	$2.18e-03$	1.21	$5.72e-03$	1.23

Table 6.1: Example 1, number of degrees of freedom, meshsizes, iterations, errors, and rates of convergence for the $\mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_1$ approximations of the Navier–Stokes/transport model, and convergence of the P_0-approximation of the postprocessed pressures field.

Parameter	Meaning	Value	Units
T	System temperature	298	K
R	Ideal gas constant	8.314	$Jmol^{-1}K^{-1}$
i	Number of ions from salt solution	2	$-$
$\mathrm{u}_{\mathrm{in,f}}/\mathrm{u}_{\mathrm{in,p}}$	Inlet mean feed/permeate fluid velocity	0.01/0.001	$m\,s^{-1}$
$\widetilde{\phi}_{\mathrm{in,f}}/\widetilde{\phi}_{\mathrm{in,p}}$	Inlet feed/permeate salt molar concentration	600/6	$mol\,m^{-3}$
ΔP	Hydrostatic transmembrane pressure	5575875	Pa
ρ	Feed/permeate fluid density	1027.2	$kg\,m^{-3}$
κ	Feed/permeate diffusivity of salt in water	1.611×10^{-9}	m^2s^{-1}
ν	Feed/permeate fluid dynamic viscosity	8.9×10^{-4}	$kg\,m^{-1}s^{-1}$
A	Membrane water permeability	2.5×10^{-12}	$ms^{-1}\,Pa^{-1}$
B	Membrane salt permeability	2.5×10^{-8}	$m\,s^{-1}$

Table 6.2: Physical parameters [4, 10, 29].

to handle nonlinear transmission conditions in domains with mixed boundary conditions. The method proposed here is the main background for modeling other types of configurations. In fact, in different physical models utilized to simulate a reverse osmosis effect in water desalination processes, the coupled Navier–Stokes/transport equations are used in two situations, one where explicit spacers (small obstacles) are located inside the channel and another where the channel does not include spacers (usually called empty channel). The latter is what we have addressed here. However, the former can be easily extended with minor modifications to our work. Moreover, the framework developed in this paper can also be employed to analyze models that couple Brinkman–Forchheimer/tranport equations instead of the Navier-Stokes/tranport equations. This type of models are also present in reverse osmosis for desalination processes where the effect of spacers is implicit. This circumvents the high computational cost of including the spacers and instead handles them as an homogeneous porous medium in the entire domain. This perspective can also be addressed by making minor modifications to the work done

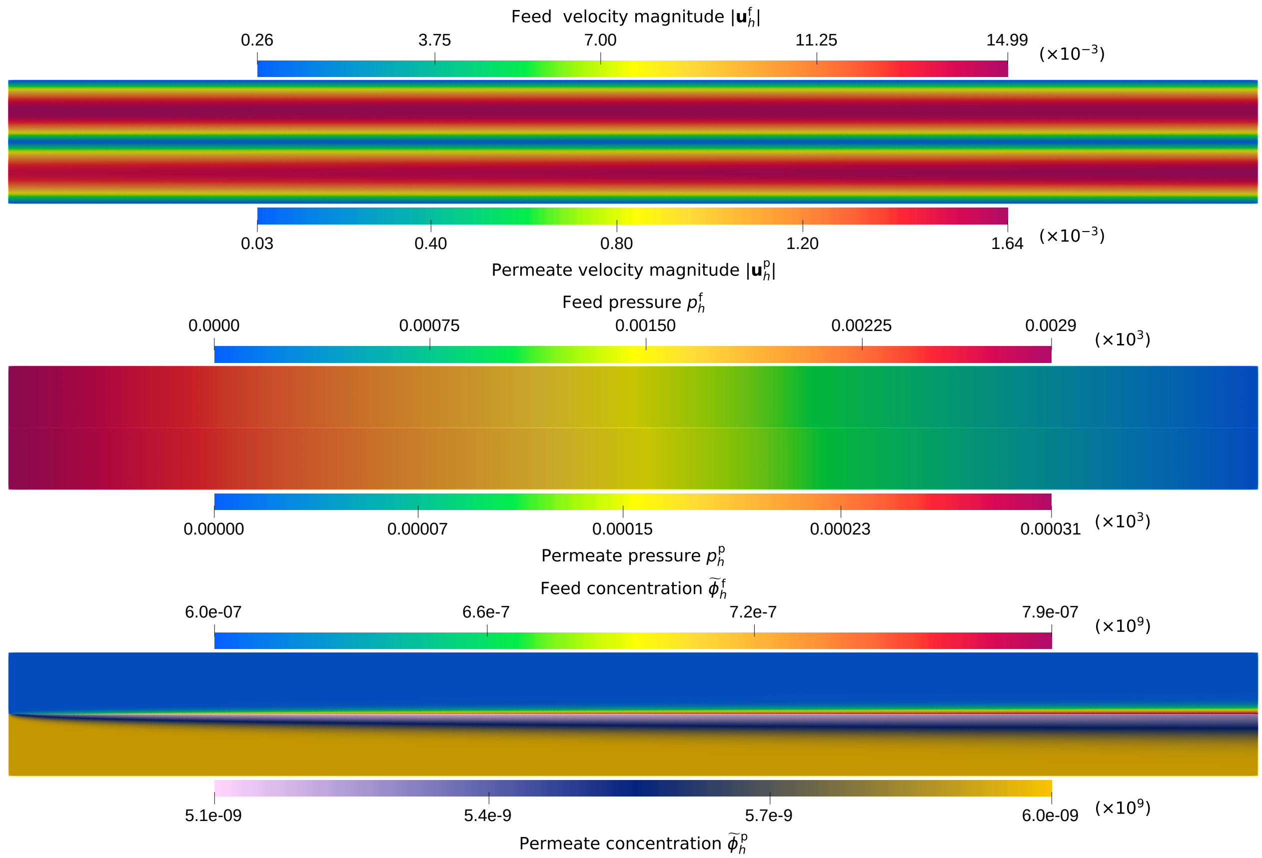

Figure 6.1: Example 2, approximations with `dof` $= 2,616,096$ for the velocity magnitudes of the fluid, pressure fields and concentration levels in the whole domain.

here. Finally, we point out that this work provides the theoretical foundations of the numerical simulations that we have performed in [4] for different models.

A Appendix

A.1. Boundedness of $\mathbf{b}_{\mathrm{T}}$. We proceed similarly to [7, Section 4.1]. Given $\star \in \{\mathrm{f}, \mathrm{p}\}$, we let $\mathbf{s}_\star \in \mathbb{H}_{\mathrm{S}}^\star$ and $\eta_\star \in \mathrm{M}_{\mathrm{T}}^\star$. Thus, by (1.1) we have

$$\langle \mathbf{s}_\star \cdot \mathbf{n}_\star, \eta_\star \rangle_{\Gamma^{\mathrm{c}}_{\mathrm{in},\star}} = \langle \mathbf{s}_\star \cdot \mathbf{n}_\star, E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star) \rangle_{\partial\Omega_\star} = \int_{\Omega_\star} \mathbf{s}_\star \cdot \nabla \widetilde{\gamma}_0^{-1}(E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star)) + \int_{\Omega_\star} \widetilde{\gamma}_0^{-1}(E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star)) \mathrm{div}(\mathbf{s}_\star),$$

where $\widetilde{\gamma}_0^{-1} : \mathrm{H}^{1/2}(\partial\Omega_\star) \to [\mathrm{H}_0^1(\Omega_\star)]^\perp$ is the right inverse of the trace operator $\gamma_0 : \mathrm{H}^1(\Omega_\star) \to \mathrm{H}^{1/2}(\partial\Omega_\star)$ (see, [21, Section 1.3.4]). Thus, applying Hölder's inequality, we obtain

$$|\langle \mathbf{s}_\star \cdot \mathbf{n}_\star, \eta_\star \rangle_{\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}| \leqslant \|\mathbf{s}_\star\|_{0,\Omega_\star} \|\nabla \widetilde{\gamma}_0^{-1}(E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star))\|_{0,\Omega_\star} + \|\widetilde{\gamma}_0^{-1}(E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star))\|_{0,4;\Omega_\star} \|\mathrm{div}(\mathbf{s}_\star)\|_{0,4/3;\Omega_\star}.$$

Next, thanks to the continuous injection $i_4^\star : \mathrm{H}^1(\Omega_\star) \to \mathrm{L}^4(\Omega_\star)$, we have that $\|\widetilde{\gamma}_0^{-1}(E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star))\|_{0,4;\Omega_\star} \leqslant \|i_4^\star\| \, \|\widetilde{\gamma}_0^{-1}(E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star))\|_{1,\Omega_\star}$. Moreover, since $E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star)$ belongs to $\mathrm{H}^{1/2}(\partial\Omega_\star)$, we get (cf. [21, Lemma 1.3])

$$\|\widetilde{\gamma}_0^{-1}(E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star))\|_{0,4;\Omega_\star} \leqslant \|i_4^\star\| \|E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star)\|_{1/2,\partial\Omega_\star} = \|i_4^\star\| \|\eta_\star\|_{1/2,00,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}},$$

whence, $|\langle \mathbf{s}_\star \cdot \mathbf{n}_\star, \eta_\star \rangle_{\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}| \leqslant \max\{1, \|i_4^\star\|\} \|\mathbf{s}_\star\|_{\mathrm{div}_{4/3};\Omega_\star} \|\eta_\star\|_{1/2,00,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}$.
As a consequence of the latter and Hölder's inequality, we deduce (3.8a).

A.2. Boundedness of $\mathbf{d}_{\mathrm{T}}$. We recall from [18, Theorem B.46] (see also [15, Theorem 6.10]) that $\mathrm{H}^{1/2}(\partial\Omega_\star)$ is continuously embedded in $\mathrm{L}^t(\partial\Omega_\star)$ for any $t \in [1, +\infty)$. In other words, for any $\zeta_\star \in \mathrm{H}^{1/2}(\partial\Omega_\star)$, there exists a positive constant $c_t^\star$, depending only on $\partial\Omega_\star$, such that $\|\zeta_\star\|_{0;t,\partial\Omega_\star} \leqslant c_t^\star \|\zeta_\star\|_{1/2,\partial\Omega_\star}$, $\forall\, t \in [1, +\infty)$. In this way, given $\eta_\star \in \mathrm{H}^{1/2}_{00}(\Gamma^{\mathrm{c}}_{\mathrm{in},\star})$ it follows that

$$\|\eta_\star\|_{0,t;\Gamma^{\mathrm{c}}_{\mathrm{in},\star}} = \|E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star)\|_{0,t;\partial\Omega_\star} \leqslant c_t^\star \|E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}(\eta_\star)\|_{1/2,\partial\Omega_\star} = c_t^\star \|\eta_\star\|_{1/2,00,\Gamma^{\mathrm{c}}_{\mathrm{in},\star}}. \tag{A.1}$$

Thus, by the Cauchy–Schwarz inequality and (A.1), we obtain (3.8b).

A.3. Boundedness of $\mathbf{F}_{\vec{\chi}}$. Given $\vec{\chi} = (\chi_{\mathrm{f}}, \chi_{\mathrm{p}}) \in \mathrm{M}$, we stress that to bound the functional $\mathbf{F}_{\vec{\chi}}$, one cannot proceed in a direct way as when applying the Cauchy–Schwarz or Hölder inequalities since the dual parity defining the functional $\mathbf{F}_{\vec{\chi}}$ involves the data $\mathbf{g}_\star$ and $\mathbf{g}_\star^\chi$ (cf. (2.4)), which are defined by parts on the boundary $\Gamma^{\mathrm{c}}_{\mathrm{out},\star}$ and belong to $\mathbf{H}^{1/2}(\Gamma^{\mathrm{c}}_{\mathrm{out},\star})$, for $\star \in \{\mathrm{f}, \mathrm{p}\}$. Therefore, we will make use of the results of Section 1.1 to properly bound each one of these data providing details for $\star = \mathrm{f}$ since $\star = \mathrm{p}$ is analogous. For that, we write

$$\langle \boldsymbol{\tau}_{\mathrm{f}}\, \mathbf{n}_{\mathrm{f}},\, \mathbf{g}_{\mathrm{f}} + \mathbf{g}_{\mathrm{f}}^\chi \rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} = \langle \boldsymbol{\tau}_{\mathrm{f}}\, \mathbf{n}_{\mathrm{f}},\, \mathbf{g}_{\mathrm{f}} \rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} + \langle \boldsymbol{\tau}_{\mathrm{f}}\, \mathbf{n}_{\mathrm{f}},\, \mathbf{g}_{\mathrm{f}}^\chi \rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}, \tag{A.2}$$

For the first term on the right-hand side of (A.2), one can define the extension $\boldsymbol{E}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}(\mathbf{g}_{\mathrm{f}}) := \mathbf{z}_{\mathrm{f}}|_{\partial\Omega_{\mathrm{f}}}$, where $\mathbf{z}_{\mathrm{f}} \in \mathbf{H}^1(\Omega_{\mathrm{f}})$ is the unique solution to (1.4), with $\mathbf{g}_{\mathrm{f}}$, $\mathbf{n}_{\mathrm{f}}$, $\Gamma^{\mathrm{c}}_{\mathrm{out,f}}$ and $\Gamma_{\mathrm{out,f}}$ instead of η, $\mathbf{n}$, $\widetilde{\Gamma}$ and $\widetilde{\Gamma}^{\mathrm{c}}$, respectively. Moreover, there exists a constant $C_{\mathrm{f}} > 0$, such that

$$\|\boldsymbol{E}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}(\mathbf{g}_{\mathrm{f}})\|_{1/2,\partial\Omega_{\mathrm{f}}} \leqslant C_{\mathrm{f}}\, \|\mathbf{g}_{\mathrm{f}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}. \tag{A.3}$$

Next, we recall from the last part of Section 1.1 that since $\boldsymbol{E}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}(\mathbf{g}_{\mathrm{f}}) \in \mathbf{H}^{1/2}(\partial\Omega_{\mathrm{f}})$, there exist unique elements $\boldsymbol{\zeta}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} \in \mathbf{H}^{1/2}(\Gamma^{\mathrm{c}}_{\mathrm{out,f}})$ and $\boldsymbol{\zeta}_{\Gamma_{\mathrm{out,f}}} \in \mathbf{H}^{1/2}_{00}(\Gamma_{\mathrm{out,f}})$ such that

$$\langle \boldsymbol{\tau}_{\mathrm{f}}\mathbf{n}_{\mathrm{f}}, \boldsymbol{E}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}(\mathbf{g}_{\mathrm{f}})\rangle_{\partial\Omega_{\mathrm{f}}} := \langle \boldsymbol{\tau}_{\mathrm{f}}\mathbf{n}_{\mathrm{f}}, \boldsymbol{E}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}(\boldsymbol{\zeta}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}})\rangle_{\partial\Omega_{\mathrm{f}}} + \langle \boldsymbol{\tau}_{\mathrm{f}}\mathbf{n}_{\mathrm{f}}, \boldsymbol{E}_{0,\Gamma_{\mathrm{out,f}}}(\boldsymbol{\zeta}_{\Gamma_{\mathrm{out,f}}})\rangle_{\partial\Omega_{\mathrm{f}}},$$

$\boldsymbol{E}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}(\mathbf{g}_{\mathrm{f}})|_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} = \boldsymbol{\zeta}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}$ and $\boldsymbol{E}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}(\mathbf{g}_{\mathrm{f}})|_{\Gamma_{\mathrm{out,f}}} = \boldsymbol{\zeta}_{\Gamma_{\mathrm{out,f}}}$. Moreover, by uniqueness we have that $\boldsymbol{\zeta}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} = \mathbf{g}_{\mathrm{f}}$. This means that (cf. (1.3))

$$\langle \boldsymbol{\tau}_{\mathrm{f}}\mathbf{n}_{\mathrm{f}}, \boldsymbol{E}_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}(\mathbf{g}_{\mathrm{f}})\rangle_{\partial\Omega_{\mathrm{f}}} = \langle \boldsymbol{\tau}_{\mathrm{f}}\mathbf{n}_{\mathrm{f}}, \mathbf{g}_{\mathrm{f}}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} \quad \forall\, \boldsymbol{\tau}_{\mathrm{f}} \in \mathbb{H}_{\Gamma_{\mathrm{out}}}(\mathbf{div}_{4/3};\Omega_{\mathrm{f}}). \tag{A.4}$$

As a consequence, employing the identity (A.4) and same arguments for bounding $\mathbf{b}_{\mathrm{T}}$, but now with the continuous injection $\mathbf{i}_4^{\mathrm{f}} : \mathbf{H}^1(\Omega_{\mathrm{f}}) \to \mathbf{L}^4(\Omega_{\mathrm{f}})$ (see, [13, Section 3.1]) and (A.3), we obtain

$$|\langle \boldsymbol{\tau}_{\mathrm{f}}\, \mathbf{n}_{\mathrm{f}},\, \mathbf{g}_{\mathrm{f}}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}| \leqslant \max\big\{1, \|\mathbf{i}_4^{\mathrm{f}}\|\big\} C_{\mathrm{f}} \|\boldsymbol{\tau}_{\mathrm{f}}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{f}}} \|\mathbf{g}_{\mathrm{f}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}. \tag{A.5}$$

It remains to deal with the second term on the right-hand side of (A.2). Therefore, in what follows, we make use of a convenient extension operator to define an appropriated $\mathbf{G}_{\mathrm{f},\vec{\chi}} \in \mathbf{H}^{1/2}(\partial\Omega_{\mathrm{f}})$ such that its restriction to $\Gamma^{\mathrm{c}}_{\mathrm{out,f}}$ coincides precisely with $\mathbf{g}_{\mathrm{f}}^\chi$. Since $\chi_{\mathrm{p}} \in \mathrm{H}^{1/2}_{00}(\Gamma^{\mathrm{c}}_{\mathrm{in,p}})$, that is $E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in,p}}}(\chi_{\mathrm{p}}) \in \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{p}})$ we note that $E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in,p}}}(\chi_{\mathrm{p}})|_\Sigma = \chi_{\mathrm{p}} \in \mathrm{H}^{1/2}(\Sigma)$ and $\chi_{\mathrm{p}} = 0$ on $\Gamma_{\mathrm{in,p}}$. Thus, one could define $E_{\Sigma,\mathrm{f}}(\chi_{\mathrm{p}}) := z_{\mathrm{f}}|_{\partial\Omega_{\mathrm{f}}}$, where $z_{\mathrm{f}} \in \mathrm{H}^1(\Omega_{\mathrm{f}})$ is the unique solution to the boundary value problem

$$\triangle z_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \quad z_{\mathrm{f}} = \chi_{\mathrm{p}} \quad \text{on} \quad \Sigma, \quad z_{\mathrm{f}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{in,f}} \quad \text{and} \quad \nabla z_{\mathrm{f}} \cdot \mathbf{n}_{\mathrm{f}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{w,f}} \cup \Gamma_{\mathrm{out,f}},$$

and show that there exists a constant $C_{\Sigma,\mathrm{f}} > 0$, such that $\|E_{\Sigma,\mathrm{f}}(\chi_{\mathrm{p}})\|_{1/2,\partial\Omega_{\mathrm{f}}} \leqslant C_{\Sigma,\mathrm{f}} \|\chi_{\mathrm{p}}\|_{1/2,\Sigma}$. On the other hand, from the fact that

$$\|\chi_{\mathrm{p}}\|_{1/2,\Sigma} = \|E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in,p}}}(\chi_{\mathrm{p}})|_\Sigma\|_{1/2,\Sigma} \leqslant \|E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in,p}}}(\chi_{\mathrm{p}})\|_{1/2,\partial\Omega_{\mathrm{p}}} = \|\chi_{\mathrm{p}}\|_{1/2,00,\Gamma^{\mathrm{c}}_{\mathrm{in,p}}},$$

we get

$$\|E_{\Sigma,\mathrm{f}}(\chi_{\mathrm{p}})\|_{1/2,\partial\Omega_{\mathrm{f}}} \leqslant C_{\Sigma,\mathrm{f}} \|\chi_{\mathrm{p}}\|_{1/2,00,\Gamma^{\mathrm{c}}_{\mathrm{in,p}}}. \tag{A.6}$$

Now, we define

$$\mathbf{G}_{\mathrm{f},\chi} := (E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in,f}}}(\chi_{\mathrm{f}}) - E_{\Sigma,\mathrm{f}}(\chi_{\mathrm{p}}))\mathcal{Q}_{\mathrm{f}}\, \mathbf{n}_{\mathrm{f}} \in \mathbf{H}^{1/2}(\partial\Omega_{\mathrm{f}}),$$

where $\mathcal{Q}_{\mathrm{f}}$ can be any function in $C^\infty(\Omega_{\mathrm{f}})$, such that $\mathcal{Q}_{\mathrm{f}} = a_1$ an Σ, $\mathcal{Q}_{\mathrm{f}} = 0$ on $\Gamma_{\mathrm{w,f}}$ and $\|\mathcal{Q}_{\mathrm{f}}\|_{0,\infty;\Omega_{\mathrm{f}}} \leqslant a_1$. For example, we can consider $\mathcal{Q}_{\mathrm{f}}(x,y) = \dfrac{a_1}{d}(d-y)$, where $\Omega_{\mathrm{f}} = (0,L)\times(0,d)$, $\Sigma = (0,L)\times\{0\}$ and $\Gamma_{\mathrm{w,f}} = (0,L)\times\{d\}$. In turn, noticing that $\mathbf{G}_{\mathrm{f},\chi}|_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} = \mathbf{g}^\chi_{\mathrm{f}} \in \mathrm{H}^{1/2}(\Gamma^{\mathrm{c}}_{\mathrm{out,f}})$, and proceeding similarly as we did for (A.4), it is easy to see that

$$\langle \boldsymbol{\tau}_{\mathrm{f}}\mathbf{n}_{\mathrm{f}}, \mathbf{G}_{\mathrm{f},\chi}\rangle_{\partial\Omega_{\mathrm{f}}} = \langle \boldsymbol{\tau}_{\mathrm{f}}\mathbf{n}_{\mathrm{f}}, \mathbf{g}^\chi_{\mathrm{f}}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}} \quad \forall\, \boldsymbol{\tau}_{\mathrm{f}} \in \mathbb{H}_{\Gamma_{\mathrm{out}}}(\mathbf{div}_{4/3};\Omega_{\mathrm{f}}).$$

Thus, knowing that $\|E_{0,\Gamma^{\mathrm{c}}_{\mathrm{in,f}}}(\chi_{\mathrm{f}})\|_{1/2,\partial\Omega_{\mathrm{f}}} = \|\chi_{\mathrm{f}}\|_{1/2,00,\Gamma^{\mathrm{c}}_{\mathrm{in,f}}}$, and employing (A.6), as well as applying similar arguments to those used in (A.5), we find that

$$|\langle \boldsymbol{\tau}_{\mathrm{f}}\,\mathbf{n}_{\mathrm{f}},\, \mathbf{g}^\chi_{\mathrm{f}}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,f}}}| \leqslant a_1 \max\left\{1, \|\mathbf{i}^{\mathrm{f}}_4\|\right\} \max\left\{C_{\Sigma,\mathrm{f}}, 1\right\} \|\boldsymbol{\tau}_{\mathrm{f}}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{f}}} \|\vec{\chi}\|_{\mathrm{M}}. \tag{A.7}$$

The analogous conclusion is obtained by setting $\star = \mathrm{p}$, in order to write

$$\langle \boldsymbol{\tau}_{\mathrm{p}}\,\mathbf{n}_{\mathrm{p}},\, \mathbf{g}_{\mathrm{p}} + \mathbf{g}^\chi_{\mathrm{p}}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,p}}} = \langle \boldsymbol{\tau}_{\mathrm{p}}\,\mathbf{n}_{\mathrm{p}},\, \mathbf{g}_{\mathrm{p}}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,p}}} + \langle \boldsymbol{\tau}_{\mathrm{p}}\,\mathbf{n}_{\mathrm{p}},\, \mathbf{g}^\chi_{\mathrm{p}}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,p}}},$$

and obtain

$$|\langle \boldsymbol{\tau}_{\mathrm{p}}\,\mathbf{n}_{\mathrm{p}},\, \mathbf{g}_{\mathrm{p}}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,p}}}| \leqslant \max\left\{1, \|\mathbf{i}^{\mathrm{p}}_4\|\right\} C_{\mathrm{p}} \|\boldsymbol{\tau}_{\mathrm{p}}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{p}}} \|\mathbf{g}_{\mathrm{p}}\|_{1/2,\Gamma^{\mathrm{c}}_{\mathrm{out,p}}}, \quad \text{and} \tag{A.8}$$

$$|\langle \boldsymbol{\tau}_{\mathrm{p}}\,\mathbf{n}_{\mathrm{p}},\, \mathbf{g}^\chi_{\mathrm{p}}\rangle_{\Gamma^{\mathrm{c}}_{\mathrm{out,p}}}| \leqslant a_1 \max\left\{1, \|\mathbf{i}^{\mathrm{p}}_4\|\right\} \max\left\{C_{\Sigma,\mathrm{p}}, 1\right\} \|\boldsymbol{\tau}_{\mathrm{p}}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{p}}} \|\vec{\chi}\|_{\mathrm{M}}. \tag{A.9}$$

As a consequence, employing the bounds (A.5), (A.7), (A.8) and (A.9), we obtain (3.9).

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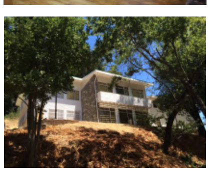

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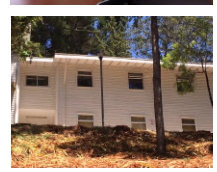

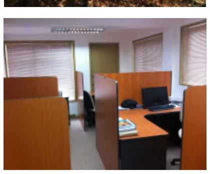

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