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Analysis of a FEM with exactly divergence-free magnetic field for the stationary MHD problem *

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Abstract

In this work we analyze a mixed finite element method for the stationary incompressible magneto-hydrodynamic problem providing an exactly divergence-free approximation of the magnetic field and a direct approximation of the electric field. The method is based on the introduction of the electric field as a further unknown leading to a mixed formulation where the primary magnetic variables consist of the electric and the magnetic fields, and a Lagrange multiplier included to enforce the divergence-free constraint of the magnetic field, whereas the hydrodynamic unknowns are the velocity and pressure. Then the associated Galerkin scheme can be defined by employing Nédélec and Raviart–Thomas elements of lowest order for the electric and magnetic fields, respectively, discontinuous piecewise constants for the Lagrange multiplier and any inf-sup stable pair of elements for the velocity and pressure, such as the Mini-element. The analysis of the continuous and discrete problems are carried out by means of the Banach–Nečas–Babuška theorem and the Banach fixed-point theorem, under a sufficiently small data assumption and quasi-uniformity of the mesh, the latter for the discrete scheme. Finally, we derive the corresponding Cea's estimate and provide the theoretical rate of convergence.

Key words: Incompressible magnetohydrodynamics, mixed finite element method, Banach spaces, Raviart–Thomas elements, Nédélec elements, exactly divergence-free magnetic field.

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1 Introduction

Magnetohydrodynamics (MHD) is a field that studies the dynamics of electrically conducting fluids in the presence of magnetic fields. This interdisciplinary area merges principles from both

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fluid dynamics and electromagnetism, with wide-ranging applications in engineering and physics, such as in astrophysical phenomena, nuclear fusion reactors, and industrial processes involving liquid metals.

The stationary MHD problem, focusing on the steady-state behavior of these conducting fluids, is of particular interest due to its implications in the design and optimization of various technological systems. Unlike its transient counterpart, the stationary MHD problem involves time-independent equations, simplifying the analysis while still capturing essential physical behaviors. However, solving these equations presents significant challenges due to their nonlinear and coupled nature.

To begin the bibliographical discussion, we start by mentioning one of the first works devoted to the analysis of finite element methods (FEM) for MHD by Gunzburger et al. [19]. In this foundational work, the authors developed the well-posedness and convergence analysis for a conforming FEM for MHD, considering inf-sup stable velocity-pressure elements for the hydrodynamic variables and standard nodal finite elements, i.e., H^1-conforming elements, for the magnetic field. An extension to this work can be found in Gerbeau [17], where a stabilized method for the three-field formulation considered in [19] is proposed. Both contributions assume the magnetic field is in $H^1(\Omega)^3$, which is feasible only if the domain is convex.

To address the limitations in non-convex domains, Hasler, Schneebeli, and Schötzau [20] introduced a mixed finite element method based on weighted regularization for the incompressible MHD system, which can be used even in non-convex domains (see also Costabel and Dauge [10]). Another approach to circumvent this problem is presented by Schötzau [28], where the author imposes the divergence-free condition of the magnetic field weakly through the introduction of a Lagrange multiplier. This allows the magnetic field to be approximated by curl-conforming Nédélec elements, eliminating the need for a convex domain assumption. Later, Houston, Schötzau, and Wei [23] introduced a fully discontinuous Galerkin (DG) method for a linearized incompressible MHD model problem based on the mixed method from [28]. While this approach uses discontinuous finite element spaces for all variables, it requires a large number of degrees of freedom. This drawback is addressed by Greif, Li, Schötzau, and Wei [18], who introduced a finite element discretization using divergence-conforming Brezzi-Douglas-Marini (BDM) elements for the velocity and curl-conforming Nédélec elements for the magnetic field, reducing the number of degrees of freedom required. More recently, Camaño et al. [6] introduced a new mixed finite element discretization for MHD that enables the approximation of additional fluid variables of interest, such as the fluid gradient and fluid vorticity, by postprocessing the primary unknowns of the system,. without using numerical differentiation, thereby eliminating extra sources of error. The approach employs the pseudostress-based method introduced in [5] for the fluid variables, while for the magnetic variables, it follows the approach described in [28]. Similar to [5], the fluid variables are sought in nonstandard Banach spaces, which allows for the derivation of optimal rates of convergence.

Recently, various works have focused on developing new numerical methods that exactly satisfy the divergence-free constraint of the magnetic field at the discrete level. For instance, Hiptmair et al. [22] proposed a numerical scheme to preserve the divergence-free constraint for both the velocity and the magnetic field. Similarly, Hu, Ma, and Xu [24] achieved the divergence-free condition for the magnetic field by introducing the electric field as an additional, allowing the magnetic field to be sought in the space $\mathbf{H}(\mathrm{div})$. These contributions do not concentrate on error analysis.

Later, Hu and Xu [25] developed a similar method to approximate the solution of the stationary MHD problem, ensuring an exactly divergence-free magnetic field. Their approach introduces the volume current density as an additional unknown and includes an extra unknown to manage the curl of the volume current density. The discrete system is proven to be well-posed and the corresponding error estimate is achieved under the assumption of quasi-uniformity of the mesh, with constants that depend on the L^∞-norm of the velocity.

While each of these methods has contributed significantly to solving the MHD equations, challenges remain in achieving accurate, stable, and efficient solutions. This paper builds on these advancements by proposing a mixed finite element method tailored for the stationary incompressible MHD problem, aiming to provide an exactly divergence-free approximation of the magnetic field and a direct approximation of the electric field with rigorous theoretical support.

Unlike the previously mentioned contributions, our approach introduces the electric field as an additional unknown. This leads to a mixed formulation where the primary magnetic variables are the electric and magnetic fields, and a Lagrange multiplier to enforce the divergence-free constraint, whereas the hydrodynamic unknowns are the velocity and pressure.

The associated Galerkin scheme employs Nédélec and Raviart-Thomas elements of the lowest order for the electric and magnetic fields, respectively, discontinuous piecewise constants for the Lagrange multiplier, and any inf-sup stable pair of elements for the velocity and pressure, such as the Mini-element. The analysis of both the continuous and discrete problems is carried out using the Banach–Necas–Babuška theorem and the Banach fixed-point theorem, under the assumptions of sufficiently small data and quasi-uniformity of the mesh for the discrete scheme. This rigorous mathematical framework, based on the introduction of suitable Banach spaces (see [5, 6, 7]) where the unknowns and test functions naturally belong, ensures the stability and optimal convergence of the method, where the constants in the corresponding estimates depend solely on the problem data. The most challenging part of the analysis, which we believe can be applied or adapted to other contexts, is deriving the inf-sup conditions for the bilinear forms involving the curl operator. This requires, among other technical results, the application of L^p-theory for vector potentials on non-smooth domains.

The rest of this paper is organized as follows. In Section 2 we present the main aspects of the continuous problem. We reformulate the problem as an equivalent set of equations and derive the mixed variational formulation. In Section 3 we introduce the fixed–point strategy and apply, firstly, the classical Banach–Nečas–Babuška theorem, and secondly, the Banach's fixed–point theorems, to prove that the associated fixed–point operator is well defined and that the continuous problem is uniquely solvable, respectively. Next, in Section 4 we introduce and analyze the associated Galerkin scheme by mimicking the theory developed for the continuous problem. In Section 5 we establish the corresponding Cea's estimate and prove optimal convergence of the method. Finally, in Section 6 we present numerical results for a test problem with a smooth solution to corroborate the theoretical rate of convergence of the method.

We conclude this section by noting that in the following discussions, we will utilize C and c, with or without subscripts, bars, tildes, or hats, to represent generic positive constants independent to represent generic positive constants independent of the discretization parameters. These constants may assume varying values at different locations within our analysis.

2 Continuous problem

In this section we present the model problem and derive the variational formulation. We begin by introducing some notations and definitions.

2.1 Preliminaries

Let us denote by $\Omega \subseteq \mathbb{R}^3$ a given bounded domain with polyhedral boundary Γ. Then, for any $p \in [1, \infty]$ and $r \geq 0$ we let $\mathrm{L}^p(\Omega)$ and $\mathrm{W}^{r,p}(\Omega)$ be the well-known Lebesgue and Sobolev spaces, respectively, endowed with the respective norms $\|\cdot\|_{\mathrm{L}^p(\Omega)}$ and $\|\cdot\|_{\mathrm{W}^{r,p}(\Omega)}$. Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$ and if $p = 2$, we write $\mathrm{H}^r(\Omega)$ in place of $\mathrm{W}^{r,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{r,\Omega}$, respectively. We also write $|\cdot|_{r,\Omega}$ for the H^r-seminorm. In addition, we will denote by $\mathrm{H}^{1/2}(\Gamma)$ the trace space of $\mathrm{H}^1(\Omega)$ and by $\mathrm{H}^{-1/2}(\Gamma)$ its dual. With $\langle\cdot,\cdot\rangle$ we denote the corresponding product of duality between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. In the sequel, for a generic scalar functional space S, we will denote the corresponding vectorial and tensorial counterparts by $\mathbf{S}$ and $\mathbb{S}$, respectively, and $\mathbf{0}$ will denote a generic null vector. When no confusion arises, we will also denote by $\|(u,v)\| := \|(u,v)\|_{U\times V} := \|u\|_U + \|v\|_V$ the norm on the product space $U \times V$ and $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^3$ or $\mathbb{R}^{3\times 3}$

For any vector fields $\mathbf{v} = (v_i)_{i=1,3}$ and $\mathbf{w} = (w_i)_{i=1,3}$ we set the cross product and the curl, gradient and divergence operators, respectively, as

$$\mathbf{w}\times\mathbf{v} := \begin{pmatrix} w_2v_3 - w_3v_2 \\ w_3v_1 - w_1v_3 \\ w_1v_2 - w_2v_1 \end{pmatrix}, \quad \mathrm{curl}\,\mathbf{v} := \nabla\times\mathbf{v}, \quad \nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,2,3}, \quad \mathrm{div}\,\mathbf{v} := \sum_{j=1}^{3}\frac{\partial v_j}{\partial x_j}.$$

For any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,3}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,3}$, we also define the tensor inner product as

$$\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{3} \tau_{ij}\zeta_{ij}.$$

For simplicity, in what follows we denote

$$(v,w)_\Omega := \int_\Omega vw, \quad (\mathbf{v},\mathbf{w})_\Omega := \int_\Omega \mathbf{v}\cdot\mathbf{w}, \quad (\mathbf{v},\mathbf{w})_\Gamma := \int_\Gamma \mathbf{u}\cdot\mathbf{v} \quad \text{and} \quad (\boldsymbol{\tau},\boldsymbol{\zeta})_\Omega := \int_\Omega \boldsymbol{\tau}:\boldsymbol{\zeta}.$$

Furthermore, for given $p, q > 1$ we define the Banach spaces $\mathbf{H}^p(\mathrm{div};\Omega)$ and $\mathbf{H}(\mathrm{curl}_q;\Omega)$, as

$$\mathbf{H}^p(\mathrm{div};\Omega) := \{\mathbf{v}\in\mathbf{L}^p(\Omega) : \mathrm{div}\,\mathbf{v}\in\mathrm{L}^2(\Omega)\} \text{ and } \mathbf{H}(\mathrm{curl}_q;\Omega) := \left\{\boldsymbol{\phi}\in\mathbf{L}^2(\Omega) : \mathrm{curl}\,\boldsymbol{\phi}\in\mathbf{L}^q(\Omega)\right\},$$

endowed, respectively, with the norms

$$\|\mathbf{v}\|_{p,\mathrm{div};\Omega} := \left(\|\mathbf{v}\|^2_{\mathbf{L}^p(\Omega)} + \|\mathrm{div}\,\mathbf{v}\|^2_{0,\Omega}\right)^{1/2} \quad \text{and} \quad \|\boldsymbol{\phi}\|_{\mathrm{curl}_q;\Omega} := \left(\|\boldsymbol{\phi}\|^2_{0,\Omega} + \|\mathrm{curl}\,\boldsymbol{\phi}\|^2_{\mathbf{L}^q(\Omega)}\right)^{1/2}.$$

For the particular case $p = q = 2$, we simply denote $\mathbf{H}(\mathrm{div};\Omega) = \mathbf{H}^2(\mathrm{div};\Omega)$ and $\mathbf{H}(\mathrm{curl};\Omega) = \mathbf{H}(\mathrm{curl}_2;\Omega)$, and for the forthcoming analysis we define the subspace

$$\mathbf{H}^p(\mathrm{div}^0;\Omega) := \{\mathbf{d}\in\mathbf{H}^p(\mathrm{div};\Omega) \,:\, \mathrm{div}\,\mathbf{d} = 0 \quad \text{in} \quad \Omega\},$$

for $p > 1$.

In addition, in the sequel we will make use of the well-known Hölder and Poincaré inequalities, given respectively by

$$|(f,g)_\Omega| \le \|f\|_{\mathrm{L}^p(\Omega)}\|g\|_{\mathrm{L}^q(\Omega)} \quad \forall\, f\in \mathrm{L}^p(\Omega), \forall g\in \mathrm{L}^q(\Omega), \quad \text{with} \quad \frac{1}{p}+\frac{1}{q}=1, \tag{2.1}$$

$$C_P\|v\|_{1,\Omega}^2 \le |v|_{1,\Omega}^2 \qquad \forall\, v\in \mathrm{H}_0^1(\Omega). \tag{2.2}$$

Finally, we recall that $\mathrm{H}^1(\Omega)$ is continuously embedded into $\mathrm{L}^q(\Omega)$ for $q\in[1,\,6]$. More precisely, the following inequality holds

$$\|w\|_{\mathrm{L}^q(\Omega)} \le C_{Sob}(q)\|w\|_{1,\Omega} \qquad \forall w\in \mathrm{H}^1(\Omega), \tag{2.3}$$

with $C_{Sob}(q)$ a positive constant depending only on $|\Omega|$ and q (see [27, Theorem 1.3.4]).

2.2 The stationary incompressible MHD problem and its variational formulation

Let $\Omega\subseteq\mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ. For simplicity, we assume that Ω is simply-connected and that its boundary Γ is connected. We consider the following stationary incompressible magneto-hydrodynamic model (see, e.g. [18, 25, 28]):

$$\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S_c(\mathrm{curl}\ \mathbf{b})\times\mathbf{b} &= \mathbf{f} \quad \text{in} \quad \Omega,\\
S_c\nu_m\mathrm{curl}\ (\mathrm{curl}\ \mathbf{b}) + \nabla r - S_c\mathrm{curl}\ (\mathbf{u}\times\mathbf{b}) &= \mathbf{g} \quad \text{in} \quad \Omega,\\
\mathrm{div}\,\mathbf{u} &= 0 \quad \text{in} \quad \Omega,\\
\mathrm{div}\,\mathbf{b} &= 0 \quad \text{in} \quad \Omega,\\
(p,1)_\Omega &= 0,
\end{aligned} \tag{2.4}$$

where, $\mathbf{u}$ and p represent the velocity and pressure, respectively, of a viscous incompressible fluid occupying Ω, exposed to a magnetic field $\mathbf{b}$, r is a Lagrange multiplier associated with the divergence constraint on the magnetic field $\mathbf{b}$ and $\mathbf{f}$ and $\mathbf{g}$ are given source terms. These equations are characterized by three dimensionless parameters: the hydrodynamic Reynolds number $R_e=\nu^{-1}$, the magnetic Reynolds number $R_m=\nu_m^{-1}$, and the coupling number S_c. In addition, we consider the following boundary conditions:

$$\mathbf{u}=\mathbf{0} \quad \text{on} \quad \Gamma, \qquad \mathbf{n}\times\mathbf{b}=\mathbf{0} \quad \text{on} \quad \Gamma \quad \text{and} \quad r=0 \quad \text{on} \quad \Gamma,$$

where $\mathbf{n}$ is the outward unit normal vector on Γ. Notice that if $\mathrm{div}\,\mathbf{g}=0$, then from the second equation of (2.4) and the boundary condition $r=0$ on Γ, it follows that $r=0$ in Ω.

In order to introduce a finite element scheme providing an exactly divergence-free approximation of the magnetic field $\mathbf{b}$, similarly to [24] we introduce the electric field as an additional unknown, namely

$$\boldsymbol{\varepsilon} := \nu_m\mathrm{curl}\,\mathbf{b} \,-\, \mathbf{u}\times\mathbf{b} \quad \text{in} \quad \Omega,$$

and rewrite the MHD system (2.4) as

$$\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S_c\nu_m^{-1}\boldsymbol{\varepsilon}\times\mathbf{b} - S_c\nu_m^{-1}(\mathbf{u}\times\mathbf{b})\times\mathbf{b} &= \mathbf{f} \quad \text{in} \quad \Omega,\\
\boldsymbol{\varepsilon} \,-\, \nu_m \operatorname{curl}\mathbf{b} \,+\, \mathbf{u}\times\mathbf{b} &= \mathbf{0} \quad \text{in} \quad \Omega,\\
S_c \operatorname{curl}\boldsymbol{\varepsilon} + \nabla r &= \mathbf{g} \quad \text{in} \quad \Omega,\\
\operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega,\\
\operatorname{div}\mathbf{b} &= 0 \quad \text{in} \quad \Omega,\\
(p,1)_\Omega &= 0.
\end{aligned} \tag{2.5}$$

In what follows we employ the latter system to derive the associated variational formulation. To that end we multiply the first equation of (2.5) by $\mathbf{v}\in\mathbf{H}_0^1(\Omega)$, integrate by parts, and make use of the identity $[(\mathbf{u}\times\mathbf{b})\times\mathbf{b}]\cdot\mathbf{v} = -(\mathbf{u}\times\mathbf{b})\cdot(\mathbf{v}\times\mathbf{b})$, to obtain

$$\nu(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega + ((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})_\Omega + \frac{S_c}{\nu_m}\,(\mathbf{u}\times\mathbf{b},\mathbf{v}\times\mathbf{b})_\Omega - (p,\operatorname{div}\mathbf{v})_\Omega - \frac{S_c}{\nu_m}(\boldsymbol{\varepsilon}\times\mathbf{b},\mathbf{v})_\Omega = (\mathbf{f},\mathbf{v})_\Omega, \tag{2.6}$$

for all $\mathbf{v}\in\mathbf{H}_0^1(\Omega)$. In turn, multiplying the second equation of (2.5) by $\boldsymbol{\phi}\in\boldsymbol{\Phi}$, where $\boldsymbol{\Phi}$ is a Banach space to be specified next, integrating by parts and using the boundary condition $\mathbf{n}\times\mathbf{b}=\mathbf{0}$ on Γ and the fact that $(\mathbf{u}\times\mathbf{b})\cdot\boldsymbol{\phi} = -(\boldsymbol{\phi}\times\mathbf{b})\cdot\mathbf{u}$, we arrive at

$$S_c\nu_m^{-1}\,(\boldsymbol{\varepsilon},\boldsymbol{\phi})_\Omega \,-\, S_c\,(\mathbf{b},\operatorname{curl}\boldsymbol{\phi})_\Omega \,-\, S_c\nu_m^{-1}\,((\boldsymbol{\phi}\times\mathbf{b}),\mathbf{u})_\Omega = 0 \quad \forall\,\boldsymbol{\phi}\in\boldsymbol{\Phi}. \tag{2.7}$$

In addition, we multiply the third equation of (2.5) by $\mathbf{d}\in\mathbf{C}$ (to be specified next), integrate by parts and employ the boundary condition $r=0$ on Γ, to obtain

$$S_c(\operatorname{curl}\boldsymbol{\varepsilon},\mathbf{d})_\Omega \,-\, (r,\operatorname{div}\mathbf{d})_\Omega = (\mathbf{g},\mathbf{d})_\Omega, \quad \forall\,\mathbf{d}\in\mathbf{C}. \tag{2.8}$$

Finally, the fourth and fifth equations of (2.5) are imposed weakly as follows:

$$(\operatorname{div}\mathbf{u},q)_\Omega = 0 \quad \forall\, q\in L_0^2(\Omega) \quad \text{and} \quad (\operatorname{div}\mathbf{b},s)_\Omega = 0 \quad \forall\, s\in L^2(\Omega). \tag{2.9}$$

In this way, and according to the above, we arrive at the weak problem: *Find* $\mathbf{u}\in\mathbf{H}_0^1(\Omega)$*,* $\rho\in\boldsymbol{\Phi}$*,* $\mathbf{b}\in\mathbf{C}$*,* $p\in \mathrm{L}_0^2(\Omega)$ *and* $r\in\mathrm{L}^2(\Omega)$*, such that* (2.6)–(2.9) *hold.*

Now we turn to specify the spaces $\mathbf{C}$ and $\boldsymbol{\Phi}$. We begin by observing that the second equation of (2.9) is well defined if $\operatorname{div}\mathbf{b}\in\mathrm{L}^2(\Omega)$. In turn, the third term at the left-hand side of (2.6) is well-defined if $\mathbf{u}\times\mathbf{b}$ and $\mathbf{v}\times\mathbf{b}$ belong to $\mathrm{L}^2(\Omega)$. However, since $\mathbf{u},\mathbf{v}\in\mathbf{H}_0^1(\Omega)$, according to the Sobolev embedding $\mathbf{H}^1(\Omega)\hookrightarrow\mathbf{L}^\lambda(\Omega)$, with $\lambda\in[1,6]$, we conclude that the third term at the left-hand side of (2.6) is well-defined if $\mathbf{b}\in\mathbf{L}^\mu(\Omega)$, with $\mu\geq 3$ satisfying $\frac{1}{\lambda}+\frac{1}{\mu}=\frac{1}{2}$. Consequently, from now on we choose $\lambda=6$, which yields $\mu=3$, and set

$$\mathbf{C} = \mathbf{H}^3(\mathrm{div};\Omega).$$

With this choice for $\mathbf{C}$ we also observe that the last terms at the left-hand side of (2.6) and (2.7) are well defined if $\boldsymbol{\varepsilon}$ and $\boldsymbol{\phi}$ are both in $\mathbf{L}^2(\Omega)$. Nevertheless, since $\mathbf{b},\mathbf{d}\in\mathbf{L}^3(\Omega)$, the second and first terms of (2.7) and (2.8), respectively, force $\operatorname{curl}\boldsymbol{\varepsilon}$ and $\operatorname{curl}\boldsymbol{\phi}$ to be in $\mathbf{L}^{\frac{3}{2}}(\Omega)$. In this way, a suitable choice for the space $\boldsymbol{\Phi}$ is

$$\boldsymbol{\Phi} = \mathbf{H}(\mathrm{curl}_{\frac{3}{2}};\Omega).$$

According to the above, defining the following global unknowns and their corresponding spaces:

$$\boldsymbol{\sigma} := (\mathbf{u}, \boldsymbol{\varepsilon}, \mathbf{b}) \in \mathcal{H} := \mathbf{H}_0^1(\Omega) \times \mathbf{H}(\mathrm{curl}_{\frac{3}{2}}; \Omega) \times \mathbf{H}^3(\mathrm{div}; \Omega),$$
$$\mathbf{p} := (p, r) \in \mathcal{Q} := \mathrm{L}_0^2(\Omega) \times \mathrm{L}^2(\Omega),$$

the system of equations (2.6)–(2.7) can be written as the following nonlinear dual-mixed problem: Find $\boldsymbol{\sigma} = (\mathbf{u}, \boldsymbol{\varepsilon}, \mathbf{b}) \in \mathcal{H}$ and $\mathbf{p} = (p, r) \in \mathcal{Q}$, such that

$$\begin{aligned} \mathcal{A}_{\mathbf{u},\mathbf{b}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathcal{B}(\boldsymbol{\tau}, \mathbf{p}) \quad &= \mathcal{F}(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathcal{H}, \\ \mathcal{B}(\boldsymbol{\sigma}, \mathbf{q}) \quad &= 0 \quad \forall \mathbf{q} \in \mathcal{Q}, \end{aligned} \tag{2.10}$$

where for fixed $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ and $\mathbf{c} \in \mathbf{H}^3(\mathrm{div}; \Omega)$, the bilinear forms $\mathcal{A}_{\mathbf{w},\mathbf{c}} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and $\mathcal{B} : \mathcal{H} \times \mathcal{Q} \to \mathbb{R}$, are given by

$$\begin{aligned} \mathcal{A}_{\mathbf{w},\mathbf{c}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) :=& \quad A_{\mathbf{w},\mathbf{c}}((\mathbf{u}, \boldsymbol{\varepsilon}), (\mathbf{v}, \boldsymbol{\phi})) + B((\mathbf{u}, \boldsymbol{\varepsilon}), \mathbf{d}) + B((\mathbf{v}, \boldsymbol{\phi}), \mathbf{b}), \\ \mathcal{B}(\boldsymbol{\tau}, \mathbf{q}) :=& \quad -(q, \mathrm{div}\, \mathbf{v})_\Omega + (s, \mathrm{div}\, \mathbf{d})_\Omega, \end{aligned} \tag{2.11}$$

for all $\boldsymbol{\sigma} = (\mathbf{u}, \boldsymbol{\varepsilon}, \mathbf{b}), \boldsymbol{\tau} = (\mathbf{v}, \boldsymbol{\phi}, \mathbf{d}) \in \mathcal{H}$ and $\mathbf{q} = (q, s) \in \mathcal{Q}$, with

$$\begin{aligned} A_{\mathbf{w},\mathbf{c}}((\mathbf{u}, \boldsymbol{\varepsilon}), (\mathbf{v}, \boldsymbol{\phi})) :=& \quad \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega + ((\mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{v})_\Omega + S_c \nu_m^{-1} (\mathbf{u} \times \mathbf{c}, \mathbf{v} \times \mathbf{c})_\Omega \\ & + S_c \nu_m^{-1} (\boldsymbol{\varepsilon}, \boldsymbol{\phi})_\Omega - S_c \nu_m^{-1} (\boldsymbol{\varepsilon} \times \mathbf{c}, \mathbf{v})_\Omega - S_c \nu_m^{-1} ((\boldsymbol{\phi} \times \mathbf{c}), \mathbf{u})_\Omega, \\ B((\mathbf{v}, \boldsymbol{\phi}), \mathbf{d}) :=& \quad - S_c (\mathbf{d}, \mathrm{curl}\, \boldsymbol{\phi})_\Omega, \end{aligned} \tag{2.12}$$

and the linear functional $\mathcal{F}$ is given by

$$\mathcal{F}(\boldsymbol{\tau}) := (\mathbf{f}, \mathbf{v})_\Omega - (\mathbf{g}, \mathbf{d})_\Omega, \quad \forall \boldsymbol{\tau} = (\mathbf{v}, \boldsymbol{\phi}, \mathbf{d}) \in \mathcal{H}.$$

3 Analysis of the continuous problem

In this section, we undertake the well-posedness analysis of (2.10) by means of a fixed-point strategy. More precisely, we let $\mathcal{J}$ be the operator defined by

$$\mathcal{J} : \mathbf{H}_0^1(\Omega) \times \mathbf{H}^3(\mathrm{div}; \Omega) \to \mathbf{H}_0^1(\Omega) \times \mathbf{H}^3(\mathrm{div}; \Omega), \quad (\mathbf{w}, \mathbf{c}) \to \mathcal{J}(\mathbf{w}, \mathbf{c}) := (\mathbf{u}, \mathbf{b}), \tag{3.1}$$

where, $\mathbf{u}$ and $\mathbf{b}$ are the first and last components of the solution of the linearized version of problem (2.10): Find $\boldsymbol{\sigma} = (\mathbf{u}, \boldsymbol{\varepsilon}, \mathbf{b}) \in \mathcal{H}$ and $\mathbf{p} \in \mathcal{Q}$, such that

$$\begin{aligned} \mathcal{A}_{\mathbf{w},\mathbf{c}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathcal{B}(\boldsymbol{\tau}, \mathbf{p}) \quad &= \mathcal{F}(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathcal{H}, \\ \mathcal{B}(\boldsymbol{\sigma}, \mathbf{q}) \quad &= 0 \quad \forall \mathbf{q} \in \mathcal{Q}, \end{aligned} \tag{3.2}$$

and observe that proving that (2.10) is well-posed is equivalent to prove existence and uniqueness of solution of the following fixed-point problem: Find $(\mathbf{u}, \mathbf{b}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}^3(\mathrm{div}; \Omega)$, such that

$$\mathcal{J}(\mathbf{u}, \mathbf{b}) = (\mathbf{u}, \mathbf{b}).$$

According to the above, in the subsequent sections, our focus will be on deriving suitable conditions under which the operator $\mathcal{J}$ possesses a unique fixed-point. However, before delving into that, we must first establish the well-definiteness of the fixed-point operator $\mathcal{J}$. This is established next.

3.1 Well-definiteness of $\mathcal{J}$

Based on the definition of $\mathcal{J}$ (as shown in (3.1)), it is evident that in order to establish the well-defined nature of operator $\mathcal{J}$, it is sufficient to demonstrate the well-posedness of problem (3.2) for given $(\mathbf{w},\mathbf{c}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}^3(\mathrm{div};\Omega)$. Consequently, considering the mixed structure of (3.2), in what follows we employ the Banach–Nečas–Babuška theorem and the Babuška–Brezzi theory to prove its well-posedness (eg. [13, Theorem 2.6 and Theorem 2.34]). We begin by establishing the stability properties of the forms involved.

Let us start by observing that the bilinear forms $A_{\mathbf{w},\mathbf{c}}$, B, $\mathcal{A}_{\mathbf{w},\mathbf{c}}$, $\mathcal{B}$ and the functional $\mathcal{F}$, satisfy the following estimates:

$$|A_{\mathbf{w},\mathbf{c}}((\mathbf{u},\boldsymbol{\varepsilon}),(\mathbf{v},\boldsymbol{\phi}))| \leq C_{A_{\mathbf{w},\mathbf{c}}}\|(\mathbf{u},\boldsymbol{\varepsilon})\|\|(\mathbf{v},\boldsymbol{\phi})\|, \quad \forall\,(\mathbf{u},\boldsymbol{\varepsilon}),\,(\mathbf{v},\boldsymbol{\phi}) \in \mathbf{H}_0^1(\Omega)\times\mathbf{H}(\mathrm{curl}_{\frac{3}{2}};\Omega), \tag{3.3}$$

$$|B((\mathbf{v},\boldsymbol{\phi}),\mathbf{d})| \leq S_c\|\mathbf{d}\|_{3,\mathrm{div};\Omega}\|(\mathbf{v},\boldsymbol{\phi})\|, \;\forall\,((\mathbf{v},\boldsymbol{\phi}),\mathbf{d}) \in \mathbf{H}_0^1(\Omega)\times\mathbf{H}(\mathrm{curl}_{\frac{3}{2}};\Omega)\times\mathbf{H}^3(\mathrm{div};\Omega), \tag{3.4}$$

$$|\mathcal{A}_{\mathbf{w},\mathbf{c}}(\boldsymbol{\sigma},\boldsymbol{\tau})| \leq C_{\mathcal{A}_{\mathbf{w},\mathbf{c}}}\|\boldsymbol{\sigma}\|_{\mathcal{H}}\|\boldsymbol{\tau}\|_{\mathcal{H}}, \quad \forall\,\boldsymbol{\sigma},\,\boldsymbol{\tau}\in\mathcal{H}, \tag{3.5}$$

$$|\mathcal{B}(\boldsymbol{\tau},\mathbf{q})| \leq \|\boldsymbol{\tau}\|_{\mathcal{H}}\|\mathbf{q}\|_{\mathcal{Q}}, \quad \forall\,\boldsymbol{\tau}\in\mathcal{H},\,\forall\,\mathbf{q}\in\mathcal{Q}, \tag{3.6}$$

$$|\mathcal{F}(\boldsymbol{\tau})| \leq (\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)})\|\boldsymbol{\tau}\|_{\mathcal{H}}, \quad \forall\,\boldsymbol{\tau}\in\mathcal{H}, \tag{3.7}$$

where $C_{A_{\mathbf{w},\mathbf{c}}}$ and $C_{\mathcal{A}_{\mathbf{w},\mathbf{c}}}$ are given by

$$C_{A_{\mathbf{w},\mathbf{c}}} := C\left(\nu + S_c\nu_m^{-1} + \|\mathbf{w}\|_{1,\Omega} + S_c\nu_m^{-1}\|\mathbf{c}\|_{3,\mathrm{div};\Omega}^2 + S_c\nu_m^{-1}\|\mathbf{c}\|_{3,\mathrm{div};\Omega}\right), \tag{3.8}$$

and

$$C_{\mathcal{A}_{\mathbf{w},\mathbf{c}}} := C_{A_{\mathbf{w},\mathbf{c}}} + 2S_c, \tag{3.9}$$

with $C > 0$, independent of the physical parameters.

Next, we establish the inf-sup condition of the bilinear form $\mathcal{B}$ through the following lemma:

Lemma 3.1 *The exists $\beta > 0$, independent of the physical parameters, such that*

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{H}} \frac{\mathcal{B}(\boldsymbol{\tau},\mathbf{q})}{\|\boldsymbol{\tau}\|_{\mathcal{H}}} \geq \beta\|\mathbf{q}\|_{\mathcal{Q}}, \quad \forall\,\mathbf{q}\in\mathcal{Q}. \tag{3.10}$$

Proof. Owing to the surjectivity of the operator $\mathrm{div}\,:\mathbf{H}_0^1(\Omega)\to\mathrm{L}_0^2(\Omega)$, we have that there exists $c_1 > 0$, such that

$$\sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{(q,\mathrm{div}\,\mathbf{v})_\Omega}{\|\mathbf{v}\|_{1,\Omega}} \geq c_1\|q\|_{0,\Omega} \quad \forall\,q\in\mathrm{L}_0^2(\Omega). \tag{3.11}$$

In turn, given $s\in\mathrm{L}^2(\Omega)$ and $\mathcal{O}\subseteq\mathbb{R}^3$ an open ball satisfying $\Omega\subset\mathcal{O}$, we let $\varphi\in H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$ be the unique weak solution of the boundary value problem

$$-\Delta\varphi = K(s) \quad \text{in} \quad \mathcal{O}, \qquad \varphi = 0 \quad \text{on} \quad \partial\mathcal{O}, \quad \text{with} \quad K(s) = \begin{cases} -s & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \mathcal{O}\backslash\overline{\Omega} \end{cases}.$$

It is well known that the solution satisfies

$$\|\varphi\|_{2,\mathcal{O}} \leq c\|K(s)\|_{0,\mathcal{O}} = c\|s\|_{0,\Omega}, \tag{3.12}$$

with $c > 0$. Then, we let $\widehat{\mathbf{d}} = \nabla\varphi|_\Omega \in \mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$ and observe that $\operatorname{div}\widehat{\mathbf{d}} = s \in \mathrm{L}^2(\Omega)$, which implies that $\widehat{\mathbf{d}} \in \mathbf{H}^3(\operatorname{div};\Omega)$. In addition, from (2.3) and (3.12), we obtain

$$\|\widehat{\mathbf{d}}\|_{\mathbf{L}^3(\Omega)} \le C_{Sob}(3)\|\widehat{\mathbf{d}}\|_{1,\Omega} \le c\|\varphi\|_{2,\mathcal{O}} \le \hat{c}\|s\|_{0,\Omega},$$

which combined with the fact that $s = \operatorname{div}\widehat{\mathbf{d}}$, implies $\|\widehat{\mathbf{d}}\|_{3,\operatorname{div};\Omega} \le \tilde{c}\|s\|_{0,\Omega}$. Consequently, from the above it is easy to see that the following estimate holds:

$$\sup_{\mathbf{0}\neq\mathbf{d}\in\mathbf{H}^3(\operatorname{div};\Omega)} \frac{(s,\operatorname{div}\mathbf{d})_\Omega}{\|\mathbf{d}\|_{3,\operatorname{div};\Omega}} \ge \frac{(s,\operatorname{div}\widehat{\mathbf{d}})_\Omega}{\|\widehat{\mathbf{d}}\|_{3,\operatorname{div};\Omega}} \ge \tilde{c}^{-1}\|s\|_{0,\Omega}. \tag{3.13}$$

In this way, combining (3.11) and (3.13) we readily obtain the desired result. □

Now, we let $\mathcal{K}$ be the null space of the bilinear form $\mathcal{B}$, that is

$$\mathcal{K} := \{\boldsymbol{\tau} \in \mathcal{H} \,:\, \mathcal{B}(\boldsymbol{\tau},\mathbf{q}) = 0 \quad \forall\,\mathbf{q} \in \mathcal{Q}\}, \tag{3.14}$$

which, in accordance with the definition of $\mathcal{B}$, becomes:

$$\mathcal{K} = \mathcal{N} \times \mathcal{M},$$

with

$$\begin{aligned}\mathcal{N} :=&\ \{(\mathbf{v},\boldsymbol{\phi}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}(\operatorname{curl}_{\frac{3}{2}};\Omega) \,:\, \operatorname{div}\mathbf{v} = 0 \quad \text{in} \quad \Omega\},\\ \mathcal{M} :=&\ \mathbf{H}^3(\operatorname{div}^0;\Omega).\end{aligned}$$

Now we turn to prove that for suitable choices of $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ and $\mathbf{c} \in \mathbf{H}^3(\operatorname{div};\Omega)$, the bilinear form $\mathcal{A}_{\mathbf{w},\mathbf{c}}$ satisfies the Banach–Nečas–Babuška conditions on $\mathcal{K}$ (see [13, Theorem 2.6]):

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{K}} \frac{\mathcal{A}_{\mathbf{w},\mathbf{c}}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathcal{H}}} \ge \gamma\|\boldsymbol{\zeta}\|_{\mathcal{H}}, \quad \forall\,\boldsymbol{\zeta} \in \mathcal{K}. \tag{3.15}$$

and

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{K}} \mathcal{A}_{\mathbf{w},\mathbf{c}}(\boldsymbol{\tau},\boldsymbol{\zeta}) > 0, \quad \forall\,\boldsymbol{\zeta} \in \mathcal{K}, \quad \boldsymbol{\zeta} \neq \mathbf{0}. \tag{3.16}$$

However, since $\mathcal{A}_{\mathbf{w},\mathbf{c}}$ has itself a mixed structure (see (2.11)), according to [13, Proposition 2.36], to prove (3.15) and (3.16), it suffices to prove that there exist $\beta_1 > 0$ and $\alpha_1 > 0$, such that

$$\sup_{\mathbf{0}\neq(\mathbf{v},\boldsymbol{\phi})\in\mathcal{N}} \frac{B((\mathbf{v},\boldsymbol{\phi}),\mathbf{d})}{\|\mathbf{v}\|_{1,\Omega} + \|\boldsymbol{\phi}\|_{\operatorname{curl}_{\frac{3}{2}};\Omega}} \ge \beta_1\|\mathbf{d}\|_{\mathbf{L}^3(\Omega)}, \quad \forall\,\mathbf{d} \in \mathcal{M}, \tag{3.17}$$

and

$$A_{\mathbf{w},\mathbf{c}}((\mathbf{v},\boldsymbol{\phi}),(\mathbf{v},\boldsymbol{\phi})) \ge \alpha_1\left(\|\mathbf{v}\|_{1,\Omega}^2 + \|\boldsymbol{\phi}\|_{\operatorname{curl}_{\frac{3}{2}};\Omega}^2\right) \quad \forall\,(\mathbf{v},\phi) \in \mathcal{K}_0,$$

with

$$\begin{aligned}\mathcal{K}_0 :=&\ \{(\mathbf{v},\boldsymbol{\phi}) \in \mathcal{N} \,:\, B((\mathbf{v},\boldsymbol{\phi}),\mathbf{d}) = 0 \quad \forall\,\mathbf{d} \in \mathcal{M}\}\\ =&\ \{(\mathbf{v},\boldsymbol{\phi}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}(\operatorname{curl}_{\frac{3}{2}};\Omega) \,:\, \operatorname{div}\mathbf{v} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad (\mathbf{d},\operatorname{curl}\boldsymbol{\phi})_\Omega = 0 \quad \forall\,\mathbf{d} \in \mathcal{M}\}.\end{aligned}$$

Notice that, owing to the fact that

$$(\nabla w, \operatorname{curl}\boldsymbol{\phi})_\Omega = 0 \quad \forall w \in W_0^{1,3}(\Omega) \quad \text{and} \quad \boldsymbol{\phi} \in \mathbf{H}(\operatorname{curl}_{\frac{3}{2}};\Omega),$$

and the Helmholtz decomposition (see [15, Theorem 11.2]):

$$\mathbf{L}^3(\Omega) = \nabla W_0^{1,3}(\Omega) \oplus \mathbf{H}^3(\operatorname{div}^0;\Omega),$$

$\mathcal{K}_0$ becomes

$$\mathcal{K}_0 = \{(\mathbf{v},\boldsymbol{\phi}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}(\operatorname{curl}_{\frac{3}{2}};\Omega) \,:\, \operatorname{div}\mathbf{v} = 0 \quad \text{and} \quad \operatorname{curl}\boldsymbol{\phi} = \mathbf{0} \quad \text{in} \quad \Omega\}. \tag{3.18}$$

We begin by proving the ellipticity of $A_{\mathbf{w},\mathbf{c}}$ on $\mathcal{K}_0$.

Lemma 3.2 *Let* $(\mathbf{w},\mathbf{c}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}^3(\operatorname{div}^0;\Omega)$ *be such that* $\operatorname{div}\mathbf{w} = 0$ *in* Ω *and*

$$\frac{S_c}{\nu_m \nu} C_1(\Omega)\|\mathbf{c}\|^2_{3,\mathrm{div};\Omega} \le 1, \tag{3.19}$$

where $C_1(\Omega)$ *is a positive constant, independent of physical parameters, as defined in* (3.21). *Then, there exists* $\alpha_1 > 0$ *such that*

$$A_{\mathbf{w},\mathbf{c}}((\mathbf{v},\boldsymbol{\phi}),(\mathbf{v},\boldsymbol{\phi})) \ge \alpha_1 \left(\|\mathbf{v}\|^2_{1,\Omega} + \|\boldsymbol{\phi}\|^2_{\operatorname{curl}_{\frac{3}{2}};\Omega} \right) \quad \forall\, (\mathbf{v},\phi) \in \mathcal{K}_0. \tag{3.20}$$

Proof. Given $(\mathbf{v},\boldsymbol{\phi}) \in \mathcal{K}_0$, from the definition of $A_{\mathbf{w},\mathbf{c}}$, estimates (2.1), (2.2) and the fact that $((\mathbf{w}\cdot\nabla)\mathbf{v},\mathbf{v})_\Omega = 0$, we have

$$\begin{aligned}
A_{\mathbf{w},\mathbf{c}}((\mathbf{v},\boldsymbol{\phi}),(\mathbf{v},\boldsymbol{\phi})) &= \nu|\mathbf{v}|^2_{1,\Omega} + S_c\nu_m^{-1}\left(\|\mathbf{v}\times\mathbf{c}\|^2_{0,\Omega} + \|\boldsymbol{\phi}\|^2_{0,\Omega}\right) - 2S_c\nu_m^{-1}(\boldsymbol{\phi}\times\mathbf{c},\mathbf{v})_\Omega,\\
&\ge \nu C_P\|\mathbf{v}\|^2_{1,\Omega} + S_c\nu_m^{-1}\left(\|\mathbf{v}\times\mathbf{c}\|^2_{0,\Omega} + \|\boldsymbol{\phi}\|^2_{0,\Omega}\right) + 2S_c\nu_m^{-1}(\mathbf{v}\times\mathbf{c},\boldsymbol{\phi})_\Omega,\\
&\ge \nu C_P\|\mathbf{v}\|^2_{1,\Omega} + S_c\nu_m^{-1}\left(\|\mathbf{v}\times\mathbf{c}\|^2_{0,\Omega} + \|\boldsymbol{\phi}\|^2_{0,\Omega} - 2\|\mathbf{v}\times\mathbf{c}\|_{0,\Omega}\|\boldsymbol{\phi}\|_{0,\Omega}\right).
\end{aligned}$$

Then, using the inequality $2ab \le 2a^2 + \dfrac{b^2}{2}$ for all a, $b > 0$, and employing estimates (2.1) and (2.3), the latter with $q = \dfrac{3}{2}$, we obtain

$$\begin{aligned}
A_{\mathbf{w},\mathbf{c}}((\mathbf{v},\boldsymbol{\phi}),(\mathbf{v},\boldsymbol{\phi})) &\ge \nu C_P\|\mathbf{v}\|^2_{1,\Omega} - S_c\nu_m^{-1}\|\mathbf{v}\times\mathbf{c}\|^2_{0,\Omega} + S_c\nu_m^{-1}\frac{1}{2}\|\boldsymbol{\phi}\|^2_{0,\Omega},\\
&\ge \nu C_P\|\mathbf{v}\|^2_{1,\Omega} - S_c\nu_m^{-1}C^2_{Sob}(\tfrac{3}{2})\|\mathbf{v}\|^2_{1,\Omega}\|\mathbf{c}\|^2_{3,\mathrm{div};\Omega} + S_c\nu_m^{-1}\frac{1}{2}\|\boldsymbol{\phi}\|^2_{0,\Omega},\\
&= \nu C_P\left(1 - \frac{S_c}{\nu_m\nu}\frac{C_1(\Omega)}{2}\|\mathbf{c}\|^2_{3,\mathrm{div};\Omega}\right)\|\mathbf{v}\|^2_{1,\Omega} + S_c\nu_m^{-1}\frac{1}{2}\|\boldsymbol{\phi}\|^2_{0,\Omega},
\end{aligned}$$

where

$$C_1(\Omega) := \frac{2C^2_{Sob}(\frac{3}{2})}{C_P}. \tag{3.21}$$

Then, from (3.19) and recalling that $\operatorname{curl} \boldsymbol{\phi} = \mathbf{0}$ in Ω, we obtain (3.20) with

$$\alpha_1 := \frac{1}{2} \min\left\{\nu C_P,\, S_c \nu_m^{-1}\right\}. \tag{3.22}$$

□

Now we turn to prove the inf-sup condition (3.17). To that end, we first introduce the following preliminary result.

Lemma 3.3 *Let $t \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$, where $\varepsilon > 0$ represents a positive constant that depends on Ω, as specified in [15, Corollary 9.3]. Then, for any function $\mathbf{u}$ in $\mathbf{H}^t(\operatorname{div}^0;\Omega)$ there exists a vector potential $\boldsymbol{\psi} \in \mathbf{W}^{1,t}(\Omega)$, such that*

$$\mathbf{u} = \operatorname{curl} \boldsymbol{\psi} \quad \text{in} \quad \Omega \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega. \tag{3.23}$$

Moreover, there exists $C > 0$, such that

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,t}(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{L}^t(\Omega)}, \tag{3.24}$$

Conversely, for any function $\boldsymbol{\psi} \in \mathbf{W}^{1,t}(\Omega)$, the function $\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}$ belongs to $\mathbf{H}^t(\operatorname{div}^0;\Omega)$.

Proof. Given $t \in (\frac{3}{2} - \varepsilon, 3+\varepsilon)$, we let $\mathbf{u} \in \mathbf{H}^t(\operatorname{div}^0;\Omega)$, and define $\mathcal{O} \subseteq \mathbb{R}^3$ an open ball satisfying $\overline{\Omega} \subset \mathcal{O}$. Then, according to [15, Corollary 9.3], there exists a unique $\chi \in W^{1,t}(\mathcal{O} \setminus \overline{\Omega})$, up to an additive constant, such that

$$\Delta\chi = 0 \quad \text{in} \quad \mathcal{O} \setminus \overline{\Omega}, \qquad \nabla\chi \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \quad \text{on} \quad \Gamma, \qquad \nabla\chi \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O}, \tag{3.25}$$

and

$$\|\nabla\chi\|_{\mathbf{L}^t(\mathcal{O}\setminus\overline{\Omega})} \leq C \|\mathbf{u}\|_{\mathbf{L}^t(\Omega)}.$$

Then we let $\mathcal{E}_t : \mathbf{H}^t(\operatorname{div};\Omega) \to \mathbf{H}^t(\operatorname{div};\mathcal{O})$ be the extension operator defined by

$$\mathcal{E}_t(\mathbf{u}) := \begin{cases} \mathbf{u} & \text{in} \quad \Omega, \\ \nabla\chi & \text{in} \quad \mathcal{O} \setminus \overline{\Omega} \end{cases}. \tag{3.26}$$

Observe that $\mathcal{E}_t$ and satisfies

$$\|\mathcal{E}_t(\mathbf{u})\|_{\mathbf{L}^t(\mathcal{O})} \leq C \|\mathbf{u}\|_{\mathbf{L}^t(\Omega)}, \quad \operatorname{div} \mathcal{E}_t(\mathbf{u}) = 0 \quad \text{in} \quad \mathcal{O} \quad \text{and} \quad \mathcal{E}_t(\mathbf{u}) \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O}.$$

In addition, since $\mathcal{O}$ has boundary of class $\mathcal{C}^{1,1}$, owing to [3, Lemma 4.1], it follows that there exists a vector potential $\boldsymbol{\psi}_0 \in \mathbf{W}^{1,t}(\mathcal{O})$, such that

$$\mathcal{E}_t(\mathbf{u}) = \operatorname{curl} \boldsymbol{\psi}_0 \quad \text{in} \quad \mathcal{O}, \quad \operatorname{div} \boldsymbol{\psi}_0 = 0 \quad \text{in} \quad \mathcal{O} \quad \text{and} \quad \|\boldsymbol{\psi}_0\|_{\mathbf{W}^{1,t}(\mathcal{O})} \leq C \|\mathcal{E}_t(\mathbf{u})\|_{\mathbf{L}^t(\mathcal{O})}, \tag{3.27}$$

with $C > 0$ depending only on t and Ω. In this way, it is clear that $\boldsymbol{\psi} := \boldsymbol{\psi}_0|_\Omega \in \mathbf{W}^{1,t}(\Omega)$ satisfies (3.23) and (3.24).

Conversely, if $\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}$, with $\boldsymbol{\psi} \in \mathbf{W}^{1,t}(\Omega)$, it readily follows that $\mathbf{u} \in \mathbf{H}^t(\operatorname{div}^0;\Omega)$, which concludes the proof. □

Remark 3.4 *Let $s, t \in (3/2 - \varepsilon, 3 + \varepsilon)$ with $s \le t$, and $\mathbf{u} \in \mathbf{H}^t(\mathrm{div}^0; \Omega)$. Observing that $\mathbf{H}^t(\mathrm{div}^0; \Omega) \subseteq \mathbf{H}^s(\mathrm{div}^0; \Omega)$ and $W^{1,t}(\mathcal{O} \setminus \overline{\Omega}) \subseteq W^{1,s}(\mathcal{O} \setminus \overline{\Omega})$, we first note that the function χ satisfying (3.25) also belongs to $W^{1,s}(\mathcal{O} \setminus \overline{\Omega})$ and satisfies*

$$\|\nabla \chi\|_{\mathbf{L}^s(\mathcal{O} \setminus \overline{\Omega})} \le C \|\mathbf{u}\|_{\mathbf{L}^s(\Omega)}.$$

Then, $\mathcal{E}_t(\mathbf{u})$ defined by (3.26) satisfies

$$\mathcal{E}_t(\mathbf{u}) \in \mathbf{L}^s(\mathcal{O}) \quad \text{and} \quad \|\mathcal{E}_t(\mathbf{u})\|_{\mathbf{L}^s(\mathcal{O})} \le C \|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} \quad \forall s \le t,$$

implying that $\boldsymbol{\psi}_0$ satisfying (3.27) is in $\mathbf{W}^{1,s}(\mathcal{O})$ and satisfies

$$\|\boldsymbol{\psi}_0\|_{\mathbf{W}^{1,s}(\mathcal{O})} \le c \|\mathcal{E}_t(\mathbf{u})\|_{\mathbf{L}^s(\mathcal{O})} \le \hat{c} \|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} \quad \forall s \le t.$$

Given the above, the vector potential $\boldsymbol{\psi} = \boldsymbol{\psi}_0|_\Omega$ satisfying (3.23) also belongs to $\mathbf{W}^{1,s}(\Omega)$ and satisfies

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,s}(\Omega)} \le \hat{c} \|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} \quad \forall s \le t.$$

Now we are in position of establishing the inf-sup condition of B.

Lemma 3.5 *There exists $\beta_1 > 0$, such that (3.17) holds.*

Proof. Given $\mathbf{d} \in \mathcal{M} = \mathbf{H}^3(\mathrm{div}^0; \Omega)$, we let $\mathbf{f}(\mathbf{d}) := \mathbf{d}|\mathbf{d}|$ and notice that

$$(|\mathbf{f}(\mathbf{d})|^{\frac{3}{2}}, 1)_\Omega = (|\mathbf{d}|^3, 1)_\Omega < +\infty,$$

which implies that $\mathbf{f}(\mathbf{d}) \in \mathbf{L}^{\frac{3}{2}}(\Omega)$ and

$$\|\mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} = \|\mathbf{d}\|^2_{\mathbf{L}^3(\Omega)}. \tag{3.28}$$

On the other hand, since the Helmholtz decomposition

$$\mathbf{L}^{\frac{3}{2}}(\Omega) = \nabla W_0^{1,\frac{3}{2}}(\Omega) \oplus \mathbf{H}^{\frac{3}{2}}(\mathrm{div}^0; \Omega),$$

holds true and is stable (see, [15, Theorem 11.2]), it follows that there exist $\chi \in W_0^{1,\frac{3}{2}}(\Omega)$ and $\mathbf{z} \in \mathbf{H}^{\frac{3}{2}}(\mathrm{div}^0; \Omega)$, such that

$$\mathbf{f}(\mathbf{d}) = \nabla \chi + \mathbf{z} \qquad \text{and} \qquad \|\nabla \chi\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} + \|\mathbf{z}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \le \widehat{C} \|\mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}. \tag{3.29}$$

In turn, owing to Lemma 3.3, we known that there exists a vector potential $\boldsymbol{\psi} \in \mathbf{W}^{1,\frac{3}{2}}(\Omega)$, satisfying

$$\mathbf{z} = \mathrm{curl}\, \boldsymbol{\psi} \quad \text{in} \quad \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{\mathbf{W}^{1,\frac{3}{2}}(\Omega)} \le C \|\mathbf{z}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}.$$

Then, recalling that $\mathbf{W}^{1,\frac{3}{2}}(\Omega)$ is continuously embedded into $\mathbf{L}^2(\Omega)$ (see [13, Corollary B.43]), form the latter we deduce that (3.29) becomes

$$\mathbf{f}(\mathbf{d}) = \nabla \chi + \mathrm{curl}\, \boldsymbol{\psi} \qquad \text{and} \qquad \|\nabla \chi\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{\psi}\|_{\mathrm{curl}_{\frac{3}{2}};\Omega} \le \widetilde{C} \|\mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}. \tag{3.30}$$

In this way, from (3.30) and (3.28) we obtain

$$\begin{aligned}\sup_{\mathbf{0}\neq(\mathbf{v},\boldsymbol{\phi})\in\mathcal{N}}\frac{B((\mathbf{v},\boldsymbol{\phi}),\mathbf{d})}{\|\mathbf{v}\|_{1,\Omega}\,+\,\|\boldsymbol{\phi}\|_{\mathrm{curl}_{\frac{3}{2}};\Omega}} &\geq \frac{B((\mathbf{0},\boldsymbol{\psi}),\mathbf{d})}{\|\boldsymbol{\psi}\|_{\mathrm{curl}_{\frac{3}{2}};\Omega}} = \frac{S_c(\mathbf{d},\mathrm{curl}\,\boldsymbol{\psi})_\Omega}{\|\boldsymbol{\psi}\|_{\mathrm{curl}_{\frac{3}{2}};\Omega}},\\ &\geq \frac{S_c}{\widetilde{C}}\frac{(\mathbf{d},\mathbf{f}(\mathbf{d})-\nabla\chi)_\Omega}{\|\mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}} = \frac{S_c}{\widetilde{C}}\frac{(\mathbf{d},\mathbf{f}(\mathbf{d}))_\Omega}{\|\mathbf{d}\|^2_{\mathbf{L}^3(\Omega)}},\\ &= \beta_1\|\mathbf{d}\|_{\mathbf{L}^3(\Omega)},\end{aligned}$$

which implies (3.17), with $\beta_1 = S_c/\widetilde{C}$. □

Finally, we establish suitable hypotheses under which the bilinear form $\mathcal{A}_{\mathbf{w},\mathbf{c}}$ (cf. (2.11)) satisfies (3.15) and (3.16).

Lemma 3.6 *Let* $\mathbf{w}\in\mathbf{H}_0^1(\Omega)$ *satisfying* $\mathrm{div}\,\mathbf{w}=0$ *in* Ω *and let* $\mathbf{c}\in\mathbf{H}^3(\mathrm{div};\Omega)$ *be such that estimate* (3.19) *holds. Then, estimates* (3.15) *and* (3.16) *hold true, the former with* γ *given by*

$$\gamma(\mathbf{w}) := (\gamma_1(\mathbf{w})+\gamma_2(\mathbf{w}))^{-1},$$

where $\gamma_1(\mathbf{w})$ *and* $\gamma_2(\mathbf{w})$ *are positive constants that depend on* ν, ν_m, *and* S_c, *as given below in* (3.33).

Proof. Owing to Lemmas 3.2, 3.5 and [13, Proposition 2.36] it readily follows that (3.15) and (3.16) hold. In particular, to deduce (3.15) and characterize γ, we firstly observe that from (3.8) and (3.19), there holds

$$C_{A_{\mathbf{w},\mathbf{c}}} \leq C_{A_{\mathbf{w}}} := C\left(\kappa(\nu,\nu_m,S_c)+\|\mathbf{w}\|_{1,\Omega}\right), \tag{3.31}$$

with

$$\kappa(\nu,\nu_m,S_c) := \nu + S_c\nu_m^{-1} + S_c^{1/2}\nu_m^{-1/2}, \tag{3.32}$$

and $C>0$, independent of the physical parameters. Then, given $((\mathbf{u},\boldsymbol{\varepsilon}),\mathbf{b})\in\mathcal{K}$, proceeding analogously to the proof of [13, Proposition 2.36] it is possible to obtain

$$\gamma_1(\mathbf{w})\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{K}}\frac{\mathcal{A}_{\mathbf{w},\mathbf{c}}(((\mathbf{u},\boldsymbol{\varepsilon}),\mathbf{b}),\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathcal{H}}} \geq \|(\mathbf{u},\boldsymbol{\varepsilon})\| \quad\text{and}\quad \gamma_2(\mathbf{w})\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{K}}\frac{\mathcal{A}_{\mathbf{w},\mathbf{c}}(((\mathbf{u},\boldsymbol{\varepsilon}),\mathbf{b}),\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathcal{H}}} \geq \|\mathbf{b}\|_{\mathbf{L}^3(\Omega)},$$

with

$$\gamma_1(\mathbf{w}) := \frac{1}{\alpha_1\beta_1}\left(\alpha_1+\beta_1+C_{A_{\mathbf{w}}}\right)>0 \quad\text{and}\quad \gamma_2(\mathbf{w}) := \frac{1}{\alpha_1\beta_1^2}\left(\alpha_1\beta_1+C_{A_{\mathbf{w}}}(\alpha_1+\beta_1+C_{A_{\mathbf{w}}})\right)>0, \tag{3.33}$$

which combined yield (3.15). □

Remark 3.7 *We notice that, from to* (3.31) *and* (3.33), *given* $\mathbf{w}\in\mathbf{H}_0^1(\Omega)$, $\gamma_1(\mathbf{w})$ *and* $\gamma_2(\mathbf{w})$ *can be bounded in terms of* $\|\mathbf{w}\|_{1,\Omega}$ *as follows*

$$\gamma_1(\mathbf{w}) \leq c_1\left(S_c^{-1}+\alpha_1^{-1}+S_c^{-1}\alpha_1^{-1}\kappa(\nu,\nu_m,S_c)+\|\mathbf{w}\|_{1,\Omega}\right)$$

and

$$\gamma_2(\mathbf{w}) \le c_2 S_c^{-1} \left(1 + \kappa(\nu,\nu_m,S_c) + \alpha_1^{-1} S_c^{-1} \kappa^2(\nu,\nu_m,S_c)\right.$$

$$\left. + (1+\kappa(\nu,\nu_m,S_c))\|\mathbf{w}\|_{1,\Omega} + \alpha_1^{-1} S_c^{-1} \|\mathbf{w}\|_{1,\Omega}^2\right),$$

where $c_1, c_2 > 0$ are positive constants independent of the physical parameters. In this way, combining the estimates above, we deduce that

$$\gamma^{-1}(\mathbf{w}) = \gamma_1(\mathbf{w}) + \gamma_2(\mathbf{w}) \le \delta_1 + \delta_2 \|\mathbf{w}\|_{1,\Omega} + \delta_3 \|\mathbf{w}\|_{1,\Omega}^2, \tag{3.34}$$

where δ_1, δ_2 and δ_3 are positive constants that depend on ν, ν_m and S_c.

Now we are in position of establishing the well-definiteness of operator $\mathcal{J}$.

Theorem 3.8 *Let $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ satisfying $\operatorname{div}\mathbf{w} = 0$ in Ω and let $\mathbf{c} \in \mathbf{H}^3(\operatorname{div};\Omega)$ be such that estimate (3.19) holds. Then, there exists a unique $(\mathbf{u},\mathbf{b}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}^3(\operatorname{div};\Omega)$, such that $\mathcal{J}(\mathbf{w},\mathbf{c}) = (\mathbf{u},\mathbf{b})$. In addition, the following estimate holds:*

$$\|\mathcal{J}(\mathbf{w},\mathbf{c})\| \le \left(\delta_1 + \delta_2\|\mathbf{w}\|_{1,\Omega} + \delta_3\|\mathbf{w}\|_{1,\Omega}^2\right)\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\right), \tag{3.35}$$

where δ_1, δ_2 and δ_3 are the positive constants satisfying (3.34).

Proof. Let $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ and $\mathbf{c} \in \mathbf{H}^3(\operatorname{div};\Omega)$ satisfying $\operatorname{div}\mathbf{w} = 0$ in Ω and (3.19), respectively. Then, we apply Lemmas 3.6, 3.1, and the Babuska–Brezzi theory in Banach spaces ([13, Theorem 2.34]) to deduce that there exists a unique $(\boldsymbol{\sigma},\mathbf{p}) = ((\mathbf{u},\boldsymbol{\varepsilon},\mathbf{b}),\mathbf{p}) \in \mathcal{H}\times\mathcal{Q}$, solution to (3.2). Then, according to the definition of $\mathcal{J}$ (cf.(3.1)), $(\mathbf{u},\mathbf{b})$ is the unique element in $\mathbf{H}_0^1(\Omega) \times \mathbf{H}^3(\operatorname{div};\Omega)$ satisfying $\mathcal{J}(\mathbf{w},\mathbf{c}) = (\mathbf{u},\mathbf{b})$, which implies that $\mathcal{J}$ is well-defined.

Now, to deduce (3.35) we first observe that from the first equation of (3.2), there holds

$$\mathcal{A}_{\mathbf{w},\mathbf{c}}(\boldsymbol{\sigma},\boldsymbol{\tau}) = \mathcal{F}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{K},$$

where $\mathcal{K}$ is the kernel of $\mathcal{B}$ (cf. (3.14)). Then, employing (3.7) and (3.15), we obtain that

$$\|\mathcal{J}(\mathbf{w},\mathbf{c})\| = \|(\mathbf{u},\mathbf{b})\| \le \|\boldsymbol{\sigma}\|_{\mathcal{H}} = \|(\mathbf{u},\boldsymbol{\varepsilon},\mathbf{b})\|_{\mathcal{H}} \le \gamma^{-1}(\mathbf{w})(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}}), \tag{3.36}$$

which together with (3.34) implies (3.35). □

3.2 Well-posedness of the continuous problem

In what follows we prove that, under suitable hypotheses on the data, problem (2.4) is well-posed. This result is established in the next theorem.

Theorem 3.9 *Given $t_0 \le \left(\dfrac{\nu_m \nu}{C_1(\Omega) S_c}\right)^{1/2}$, with $C_1(\Omega)$ being the positive defined in (3.21), let $t \in (0,t_0)$, and assume that $\mathbf{f}$ and $\mathbf{g}$ satisfy*

$$\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \le \frac{t}{\delta_1 + \delta_2 t + \delta_3 t^2}, \tag{3.37}$$

where δ_1, δ_2 and δ_3 are the positive constants satisfying (3.34)*. Assume further that t satisfies*

$$C_2(\Omega)t\,(\delta_1+\delta_2 t+\delta_3 t^2)(1+S_c\nu_m^{-1}(t+1))<1, \tag{3.38}$$

where $C_2(\Omega)$ is the positive constant satisfying (3.44)*. Then, there exists a unique $(\boldsymbol{\sigma},\mathbf{p})\in\mathcal{H}\times\mathcal{Q}$ solution to* (2.10)*. In addition, the solution satisfies*

$$\|\boldsymbol{\sigma}\|_{\mathcal{H}}\le\left(\delta_1+\delta_2 t+\delta_3 t^2\right)\left(\|\mathbf{f}\|_{0,\Omega}+\|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\right), \tag{3.39}$$

$$\|\mathbf{p}\|_{\mathcal{Q}}\le C\left(1+\left(\kappa(\nu,\nu_m,S_c)+S_c+t\right)\left(\delta_1+\delta_2 t+\delta_3 t^2\right)\right)\left(\|\mathbf{f}\|_{0,\Omega}+\|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\right), \tag{3.40}$$

with $C>0$, independet of the physical parameters.

Proof. According to the definition of $\mathcal{J}$ (cf. (3.1)), to prove the well-posedness of (2.10), in what follows we prove equivalently that there exists a unique fixed-point for $\mathcal{J}$. To that end, we let $t\in(0,t_0)$ and define the convex and bounded set

$$\mathbf{K}:=\Big\{(\mathbf{v},\mathbf{b})\in\mathbf{H}^1_0(\Omega)\times\mathbf{H}^3(\mathrm{div};\Omega)\,:\,\mathrm{div}\,\mathbf{v}\,=\,0\quad\text{in}\quad\Omega\quad\text{and}\\ \|(\mathbf{v},\mathbf{b})\|\le\left(\delta_1+\delta_2 t+\delta_3 t^2\right)\left(\|\mathbf{f}\|_{0,\Omega}+\|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\right)\Big\}.$$

Notice that owing to (3.35), it is clear that $\mathcal{J}(\mathbf{K})\subseteq\mathbf{K}$. Then, we let $(\mathbf{w}_1,\mathbf{c}_1)$, $(\mathbf{w}_2,\mathbf{c}_2)\in\mathbf{K}$, and observe from assumption (3.37) that

$$\|(\mathbf{w}_1,\mathbf{c}_1)\|\le t\quad\text{and}\quad\|(\mathbf{w}_2,\mathbf{c}_2)\|\le t, \tag{3.41}$$

which in particular implies that $\mathbf{c}_1$ and $\mathbf{c}_2$ satisfy the inequality in (3.19). Hence, employing Theorem 3.8 we obtain that there exist uniques $(\mathbf{u}_1,\mathbf{b}_1),(\mathbf{u}_2,\mathbf{b}_2)\in\mathbf{H}^1_0(\Omega)\times\mathbf{H}^3(\mathrm{div};\Omega)$, such that

$$(\mathbf{u}_1,\mathbf{b}_1)=\mathcal{J}(\mathbf{w}_1,\mathbf{c}_1)\quad\text{and}\quad(\mathbf{u}_2,\mathbf{b}_2)=\mathcal{J}(\mathbf{w}_2,\mathbf{c}_2),$$

which according to the definition of $\mathcal{J}$ implies that for each $i=1,2$, there exist uniques $\boldsymbol{\varepsilon}_i\in\mathbf{H}(\mathrm{curl}_{\frac{3}{2}};\Omega)$ and $\mathbf{p}_i:=(p_i,r_i)\in\mathcal{Q}$, such that $(\boldsymbol{\sigma}_i,\mathbf{p}_i)=((\mathbf{u}_i,\boldsymbol{\varepsilon}_i,\mathbf{b}_i),\mathbf{p}_i)\in\mathcal{H}\times\mathcal{Q}$, satisfies

$$\begin{aligned}\mathcal{A}_{\mathbf{w}_i,\mathbf{c}_i}(\boldsymbol{\sigma}_i,\boldsymbol{\tau})+\mathcal{B}(\boldsymbol{\tau},\mathbf{p}_i)\quad&=\mathcal{F}(\boldsymbol{\tau}),\quad\forall\,\boldsymbol{\tau}=(\mathbf{v},\boldsymbol{\phi},\mathbf{d})\in\mathcal{H},\\ \mathcal{B}(\boldsymbol{\sigma}_i,\mathbf{q})\quad&=\mathcal{G}(\mathbf{q}),\quad\forall\,\mathbf{q}=(q,s)\in\mathcal{Q}.\end{aligned}$$

Then, we subtract both systems and add and subtract suitable terms to obtain

$$\begin{aligned}\mathcal{A}_{\mathbf{w}_1,\mathbf{c}_1}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2,\boldsymbol{\tau})+\mathcal{B}(\boldsymbol{\tau},\mathbf{p}_1-\mathbf{p}_2)\quad&=\mathcal{A}_{\mathbf{w}_2,\mathbf{c}_2}(\boldsymbol{\sigma}_2,\boldsymbol{\tau})-\mathcal{A}_{\mathbf{w}_1,\mathbf{c}_1}(\boldsymbol{\sigma}_2,\boldsymbol{\tau}),\quad\forall\,\boldsymbol{\tau}=(\mathbf{v},\boldsymbol{\phi},\mathbf{d})\in\mathcal{H},\\ \mathcal{B}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2,\mathbf{q})\quad&=0,\quad\forall\,\mathbf{q}=(q,s)\in\mathcal{Q}.\end{aligned} \tag{3.42}$$

Now, for the right-hand side of the first equation in (3.42), it becomes evident that by applying the definitions of $\mathcal{A}_{\mathbf{w},\mathbf{c}}$ (cf. (2.11)) and $A_{\mathbf{w},\mathbf{c}}$ (cf. (2.12)), and by adding and subtracting appropriate terms, we can establish the following relationship

$$\begin{aligned}\mathcal{A}_{\mathbf{w}_2,\mathbf{c}_2}(\boldsymbol{\sigma}_2,\boldsymbol{\tau}) - \mathcal{A}_{\mathbf{w}_1,\mathbf{c}_1}(\boldsymbol{\sigma}_2,\boldsymbol{\tau}) &= A_{\mathbf{w}_2,\mathbf{c}_2}((\mathbf{u}_2,\boldsymbol{\varepsilon}_2),(\mathbf{v},\boldsymbol{\phi})) - A_{\mathbf{w}_1,\mathbf{c}_1}((\mathbf{u}_2,\boldsymbol{\varepsilon}_2),(\mathbf{v},\boldsymbol{\phi}))\\ &= (((\mathbf{w}_2-\mathbf{w}_1)\cdot\nabla)\mathbf{u}_2,\mathbf{v})_\Omega - S_c\nu_m^{-1}\left(\mathbf{u}_2\times\mathbf{c}_2,\mathbf{v}\times(\mathbf{c}_1-\mathbf{c}_2)\right)_\Omega\\ &-S_c\nu_m^{-1}\left(\mathbf{u}_2\times(\mathbf{c}_1-\mathbf{c}_2),\mathbf{v}\times\mathbf{c}_1\right)_\Omega\\ &-S_c\nu_m^{-1}(\boldsymbol{\varepsilon}_2\times(\mathbf{c}_2-\mathbf{c}_1),\mathbf{v})_\Omega \;-\; S_c\nu_m^{-1}\left((\boldsymbol{\phi}\times(\mathbf{c}_2-\mathbf{c}_1)),\mathbf{u}_2\right)_\Omega .\end{aligned} \tag{3.43}$$

Hence, noticing that (3.34), (3.36), (3.37) and the fact that $\|\mathbf{w}_i\|_{1,\Omega}\leq t$, for $i\in\{1,2\}$, imply

$$\|\mathbf{u}_i\|_{1,\Omega} + \|\boldsymbol{\varepsilon}_i\|_{\mathrm{curl}_{\frac{3}{2}},\Omega} + \|\mathbf{b}_i\|_{3,\mathrm{div},\Omega}\leq t,$$

from (2.3) and (3.41), (3.43) and the Hölder's inequality, we readily obtain

$$\begin{aligned}&|\mathcal{A}_{\mathbf{w}_2,\mathbf{c}_2}(\boldsymbol{\sigma}_2,\boldsymbol{\tau}) - \mathcal{A}_{\mathbf{w}_1,\mathbf{c}_1}(\boldsymbol{\sigma}_2,\boldsymbol{\tau})|\\ &\leq c_1\Big(\|\mathbf{w}_1-\mathbf{w}_2\|_{1,\Omega}\|\mathbf{u}_2\|_{1,\Omega} + S_c\nu_m^{-1}\|\mathbf{u}_2\|_{1,\Omega}\|\mathbf{c}_2\|_{3,\mathrm{div};\Omega}\|\mathbf{c}_1-\mathbf{c}_2\|_{3,\mathrm{div};\Omega}\\ &+S_c\nu_m^{-1}\|\mathbf{u}_2\|_{1,\Omega}\|\mathbf{c}_1-\mathbf{c}_2\|_{3,\mathrm{div};\Omega}\|\mathbf{c}_1\|_{3,\mathrm{div};\Omega} + S_c\nu_m^{-1}\|\boldsymbol{\varepsilon}_2\|_{\mathrm{curl}_{\frac{3}{2}};\Omega}\|\mathbf{c}_1-\mathbf{c}_2\|_{3,\mathrm{div};\Omega}\Big)\|\mathbf{v}\|_{1,\Omega}\\ &+c_2 S_c\nu_m^{-1}\|\mathbf{c}_1-\mathbf{c}_2\|_{3,\mathrm{div};\Omega}\|\mathbf{u}_2\|_{1,\Omega}\|\boldsymbol{\phi}\|_{\mathrm{curl}_{\frac{3}{2}};\Omega}\\ &\leq c_1\left(t\|\mathbf{w}_1-\mathbf{w}_2\|_{1,\Omega} \;+\; S_c\nu_m^{-1}t(2t+1)\|\mathbf{c}_1-\mathbf{c}_2\|_{3,\mathrm{div};\Omega}\right)\|\mathbf{v}\|_{1,\Omega}\\ &+c_2 S_c\nu_m^{-1}t\|\mathbf{c}_1-\mathbf{c}_2\|_{3,\mathrm{div};\Omega}\|\boldsymbol{\phi}\|_{\mathrm{curl}_{\frac{3}{2}};\Omega},\end{aligned}$$

which implies

$$\begin{aligned}&\mathcal{A}_{\mathbf{w}_2,\mathbf{c}_2}(\boldsymbol{\sigma}_2,\boldsymbol{\tau}) - \mathcal{A}_{\mathbf{w}_1,\mathbf{c}_1}(\boldsymbol{\sigma}_2,\boldsymbol{\tau})\\ &\qquad\leq C_2(\Omega)t(1+S_c\nu_m^{-1}(t+1))\left(\|\mathbf{w}_1-\mathbf{w}_2\|_{1,\Omega}+\|\mathbf{c}_1-\mathbf{c}_2\|_{3,\mathrm{div};\Omega}\right)\|\boldsymbol{\tau}\|_{\mathcal{H}},\end{aligned} \tag{3.44}$$

with $C_2(\Omega)>0$, independent of the physical parameters. In this way, from (3.15), (3.34), (3.42) and (3.44) , and recalling that $\|\mathbf{w}_1\|_{1,\Omega}\leq t$, it follows that

$$\begin{aligned}\|\mathcal{J}(\mathbf{w}_1,\mathbf{c}_1)-\mathcal{J}(\mathbf{w}_2,\mathbf{c}_2)\| = \|(\mathbf{u}_1-\mathbf{u}_2,\mathbf{b}_1-\mathbf{b}_2)\| &\leq \gamma^{-1}(\mathbf{w}_1)\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{q})\in\mathcal{K}}\frac{\mathcal{A}_{\mathbf{w}_1,\mathbf{c}_1}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2,\boldsymbol{\tau})}{\|(\boldsymbol{\tau},\mathbf{q})\|}\\ &\leq C_2(\Omega)t(\delta_1+\delta_2 t+\delta_3 t^2)(1+S_c\nu_m^{-1}(t+1))\|(\mathbf{w}_1-\mathbf{w}_2,\mathbf{c}_1-\mathbf{c}_2)\|.\end{aligned}$$

This, together with assumption (3.38), implies that $\mathcal{J}$ is a contraction mapping. Therefore, applying Banach's fixed-point Theorem, we conclude that there exists a unique $(\mathbf{u},\mathbf{b})\in\mathbf{K}$, such that $(\mathbf{u},\mathbf{b})=\mathcal{J}((\mathbf{u},\mathbf{b}))$. In this way, using the definition of $\mathcal{J}$, we obtain that there exist unique $\boldsymbol{\varepsilon}\in\mathbf{H}(\mathrm{curl}_{\frac{3}{2}};\Omega)$ and $\mathbf{p}:=(p,r)\in\mathcal{Q}$, such that $(\boldsymbol{\sigma},\mathbf{p})=((\mathbf{u},\boldsymbol{\varepsilon},\mathbf{b}),\mathbf{p})\in\mathcal{H}\times\mathcal{Q}$, is the unique solution of (2.10).

Now, we proceed to demonstrate that $(\boldsymbol{\sigma},\mathbf{p})$ satisfies estimate (3.39)–(3.40). To establish this, we first observe that $(\mathbf{u},\mathbf{b})=\mathcal{J}(\mathbf{u},\mathbf{b})\in\mathbf{K}$. This, together with (3.36) and (3.37), implies that:

$$\|\mathbf{u}\|_{1,\Omega}+\|\mathbf{b}\|_{3,\mathrm{div},\Omega}\leq t. \tag{3.45}$$

In particular, $\mathbf{b}$ satisfies (3.19). Consequently, the inf-sup condition (3.15) holds true for $(\mathbf{w},\mathbf{c}) = (\mathbf{u},\mathbf{b})$. Furthermore, as $\boldsymbol{\sigma}$ satisfies the second equation of (2.10), we have $\boldsymbol{\sigma} \in \mathcal{K}$. This, and the first equation of (2.10), imply:

$$\|\boldsymbol{\sigma}\|_{\mathcal{H}} \leq \gamma^{-1}(\mathbf{u}) \sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{K}} \frac{\mathcal{A}_{\mathbf{u},\mathbf{b}}(\boldsymbol{\sigma},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathcal{H}}} = \gamma^{-1}(\mathbf{u}) \sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{K}} \frac{\mathcal{F}(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathcal{H}}} \leq \gamma^{-1}(\mathbf{u})(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}),$$

which together with (3.34) and (3.45), implies (3.39). Moreover, from (3.5), (3.7), (3.10), and the first equation of (2.10), we obtain:

$$\begin{aligned}\|\mathbf{p}\|_{\mathcal{Q}} &\leq \beta^{-1} \sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{H}} \frac{\mathcal{B}(\boldsymbol{\tau},\mathbf{p})}{\|\boldsymbol{\tau}\|_{\mathcal{H}}} = \beta^{-1} \sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathcal{H}} \frac{|\mathcal{F}(\boldsymbol{\tau}) - \mathcal{A}_{\mathbf{u},\mathbf{b}}(\boldsymbol{\sigma},\boldsymbol{\tau})|}{\|\boldsymbol{\tau}\|_{\mathcal{H}}}, \\ &\leq \beta^{-1}(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}) + \beta^{-1} C_{\mathcal{A}_{\mathbf{u},\mathbf{b}}} \|\boldsymbol{\sigma}\|_{\mathcal{H}}. \end{aligned} \tag{3.46}$$

Hence, since $\mathbf{b}$ satisfies (3.19), which implies that $C_{\mathcal{A}_{\mathbf{u},\mathbf{b}}}$ (cf. (3.9)) can be bounded using (3.31) as follows

$$C_{\mathcal{A}_{\mathbf{u},\mathbf{b}}} = C_{A_{\mathbf{u},\mathbf{b}}} + 2S_c \leq C\left(\kappa(\nu,\nu_m,S_c) + S_c + \|\mathbf{u}\|_{1,\Omega}\right) \leq C\left(\kappa(\nu,\nu_m,S_c) + S_c + t\right).$$

with $C > 0$ independent of the physical parameters, we combine (3.39) and (3.46) to deduce (3.40), which concludes the proof. □

4 Galerkin scheme

In this section we introduce the Galerkin scheme associated to problem (2.10), analyze its solvability by employing a discrete version of the fixed point strategy developed in Section 3, and finally prove its convergence and derive the corresponding theoretical rate of convergence. We begin by introducing the Galerkin scheme.

4.1 Discrete scheme

Let $\mathcal{T}_h$ be a regular family of triangulations of the polyhedral region $\overline{\Omega}$ made up of tetrahedrons T in $\mathbb{R}^3$ of diameter h_T such that $\overline{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$ and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. Given an integer $l \geq 0$ and a subset S of $\mathbb{R}^3$, we denote by $P_l(S)$ the space of polynomials of total degree at most l defined on S. Hence, for each $T \in \mathcal{T}_h$, we define the local Raviart–Thomas and Nédélec elements of lowest order (see for instance [4] and [26]), respectively by

$$\mathbf{RT}_0(T) := \mathbf{P}_0(T) \oplus P_0(T)\mathbf{x} \quad \text{and} \quad \mathbf{N}_0(T) := \mathbf{P}_0(T) \oplus \mathbf{P}_0(T) \times \mathbf{x},$$

where $\mathbf{x} := (x_1,x_2,x_3)^t$ is a generic vector of $\mathbb{R}^3$. In addition, for each $T \in \mathcal{T}_h$, we define the local MINI-element space

$$\mathbf{B}(T) := \mathbf{P}_1(T) \oplus \langle\{\varphi_1\varphi_2\varphi_3\varphi_4\}\rangle^3,$$

where $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ are the baricentric coordinates of T. Then, we let

$$\begin{aligned}
\mathbf{H}_h &:= \left\{ \mathbf{v}_h \in \mathbf{H}^1(\Omega) \,:\, \mathbf{v}_h|_T \in \mathbf{B}(T), \quad \forall T \in \mathcal{T}_h \right\}, & \mathbf{H}_{h,0} &:= \mathbf{H}_h \cap \mathbf{H}^1_0(\Omega), \\
Q_h &:= \left\{ q_h \in C(\bar{\Omega}) \,:\, q_h|_T \in P_1(T), \quad \forall T \in \mathcal{T}_h \right\}, & Q_{h,0} &:= Q_h \cap L^2_0(\Omega), \\
\mathbf{C}_h &:= \left\{ \mathbf{d}_h \in \mathbf{H}^3(\mathrm{div};\Omega) \,:\, \mathbf{d}_h|_T \in \mathbf{RT}_0(T), \quad \forall T \in \mathcal{T}_h \right\}, & \mathbf{C}^0_h &:= \mathbf{C}_h \cap \mathbf{H}^3(\mathrm{div}^0;\Omega), \\
S_h &:= \left\{ s_h : \Omega \to \mathbb{R} \,:\, s_h|_T \in P_0(T), \quad \forall T \in \mathcal{T}_h \right\}, \\
\mathbf{\Phi}_h &:= \left\{ \boldsymbol{\phi}_h \in \mathbf{H}(\mathrm{curl}_{\frac{3}{2}};\Omega) \,:\, \boldsymbol{\phi}_h|_T \in \mathbf{N}_0(T), \quad \forall T \in \mathcal{T}_h \right\},
\end{aligned}$$

and define the global spaces

$$\mathcal{H}_h := \mathbf{H}_{h,0} \times \mathbf{\Phi}_h \times \mathbf{C}_h \quad \text{and} \quad \mathcal{Q}_h := Q_{h,0} \times S_h, \tag{4.1}$$

to propose the following Galerkin scheme for (2.10): Find $\boldsymbol{\sigma}_h = (\mathbf{u}_h, \boldsymbol{\varepsilon}_h, \mathbf{b}_h) \in \mathcal{H}_h$ and $\mathbf{p}_h = (p_h, r_h) \in \mathcal{Q}_h$, such that

$$\begin{aligned}
\mathcal{A}^h_{\mathbf{u}_h,\mathbf{b}_h}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathcal{B}(\boldsymbol{\tau}_h, \mathbf{p}_h) &= \mathcal{F}(\boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \in \mathcal{H}_h, \\
\mathcal{B}(\boldsymbol{\sigma}_h, \mathbf{q}_h) &= \mathcal{G}(\mathbf{q}_h) \quad \forall\, \mathbf{q}_h \in \mathcal{Q}_h,
\end{aligned} \tag{4.2}$$

where, for given $\mathbf{z}_h \in \mathbf{H}_{h,0}$ and $\mathbf{c}_h \in \mathbf{C}_h$, $\mathcal{A}^h_{\mathbf{z}_h,\mathbf{c}_h}$ is defined by

$$\mathcal{A}^h_{\mathbf{z}_h,\mathbf{c}_h}((\mathbf{u}_h, \boldsymbol{\varepsilon}_h, \mathbf{b}_h), (\mathbf{v}_h, \boldsymbol{\phi}_h, \mathbf{d}_h)) := A^h_{\mathbf{z}_h,\mathbf{c}_h}((\mathbf{u}_h, \boldsymbol{\varepsilon}_h), (\mathbf{v}_h, \boldsymbol{\phi}_h)) + B((\mathbf{u}_h, \boldsymbol{\varepsilon}_h), \mathbf{d}_h) + B((\mathbf{v}_h, \boldsymbol{\phi}_h), \mathbf{b}_h), \tag{4.3}$$

for all $(\mathbf{u}_h, \boldsymbol{\varepsilon}_h, \mathbf{b}_h), (\mathbf{v}_h, \boldsymbol{\phi}_h, \mathbf{d}_h) \in \mathcal{H}_h$, with B being the bilinear form defined in (2.12) and $A^h_{\mathbf{z}_h,\mathbf{c}_h}$ is given by

$$A^h_{\mathbf{z}_h,\mathbf{c}_h}((\mathbf{u}_h, \boldsymbol{\varepsilon}_h), (\mathbf{v}_h, \boldsymbol{\phi}_h)) := A_{\mathbf{z}_h,\mathbf{c}_h}((\mathbf{u}_h, \boldsymbol{\varepsilon}_h), (\mathbf{v}_h, \boldsymbol{\phi}_h)) + \frac{1}{2}(\mathrm{div}\, \mathbf{z}_h, \mathbf{u}_h \cdot \mathbf{v}_h), \tag{4.4}$$

where $A_{\mathbf{z}_h,\mathbf{c}_h}$ is also defined in (2.12).

Remark 4.1 *It is well-established that the pair* $(\mathbf{H}_{h,0}, Q_{h,0})$ *is inf-sup stable, meaning that the discrete counterpart of the inf-sup condition* (3.11) *holds. This condition is the primary criterion for selecting the discrete spaces for velocity and pressure. However, it's worth noting that other pairs that fulfill the inf-sup stability requirement can also be used to approximate these unknowns. For instance, both the Taylor–Hood element and the Bernardi–Raugel element are viable alternatives, and the subsequent analysis remains applicable.*

On the other hand, the inclusion of the term $\frac{1}{2}(\mathrm{div}\, \mathbf{z}_h, \mathbf{u}_h \cdot \mathbf{v}_h)$ *in the bilinear form* $\mathcal{A}^h_{\mathbf{z}_h,\mathbf{c}_h}$ *is motivated by the fact that the pair* $(\mathbf{H}_{h,0}, Q_{h,0})$ *does not exactly yield divergence-free velocities. As in [29], this term is introduced to enable us to derive the discrete counterpart of the estimate presented in* (3.20)*. Alternatively, we could adopt the approach introduced in [11] to approximate the fluid variables and adapt the subsequent analysis to study a fully divergence-free finite element method, similar to the one presented in [22]. This will be addressed in a forthcoming contribution.*

Finally, noticing that $\mathbf{C}_h$ *is nothing but the classical lowest order Raviart–Thomas space for* $\mathbf{H}(\mathrm{div};\Omega)$ *(see, e.g., [4, 16]), we note that an alternative approach could involve using the* *methods proposed in [1, 2] to approximate the magnetic field directly in $\mathbf{C}_h^0$ by employing suitable divergence-free basis functions within $\mathbf{C}_h^0$. This approach would eliminate the need to introduce the discrete Lagrange multiplier r_h. However, we keep the Lagrange multiplier in the scheme (4.2) since the numerical results presented in Section 6 are implemented in FreeFem++, where the aforementioned divergence-free basis functions are not yet available. The computational implementation of the method considering the magnetic field in $\mathbf{C}_h^0$ is subject for future work.*

In the following section we address the unique solvability of (4.2) by adapting the fixed-point strategy developed in Section 3. To that end, and analogously to the continuous case, we introduce the operator $\mathcal{J}_h : \mathbf{H}_{h,0} \times \mathbf{C}_h \to \mathbf{H}_{h,0} \times \mathbf{C}_h, \quad (\mathbf{w}_h, \mathbf{c}_h) \to \mathcal{J}_h(\mathbf{w}_h, \mathbf{c}_h) := (\mathbf{u}_h, \mathbf{b}_h)$, where, $\mathbf{u}_h$ and $\mathbf{b}_h$ are the first and last components of the solution of the linearized version of problem (4.2): Find $\boldsymbol{\sigma}_h = (\mathbf{u}_h, \boldsymbol{\varepsilon}_h, \mathbf{b}_h) \in \mathcal{H}_h$ and $\mathbf{p}_h \in \mathcal{Q}_h$, such that

$$\begin{aligned} \mathcal{A}^h_{\mathbf{w}_h,\mathbf{c}_h}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathcal{B}(\boldsymbol{\tau}_h, \mathbf{p}_h) &= \mathcal{F}(\boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \in \mathcal{H}_h, \\ \mathcal{B}(\boldsymbol{\sigma}_h, \mathbf{q}_h) &= \mathcal{G}(\mathbf{q}_h) \quad \forall\, \mathbf{q}_h \in \mathcal{Q}_h. \end{aligned} \tag{4.5}$$

In this way, to prove the well-posedness of (4.2), in what follows we prove equivalently the existence of a unique $(\mathbf{u}_h, \mathbf{b}_h) \in \mathbf{H}_{h,0} \times \mathbf{C}_h$, such that $\mathcal{J}_h(\mathbf{u}_h, \mathbf{b}_h) = (\mathbf{u}_h, \mathbf{b}_h)$ by means of the Banach-fixed point theorem. Before doing that, as for the continuous case we begin by proving that $\mathcal{J}_h$ is well-defined.

4.2 Well-definiteness of $\mathcal{J}_h$

To establish the well-definiteness of $\mathcal{J}_h$ we proceed analogously to the continuous case and first establish the stability properties of the forms involved. We start by noticing that the estimates (3.4), (3.6) and (3.7) are also valid in our discrete setting with the same constants. In turn, according to the definitions in (4.3) and (4.4), and proceeding similarly to (3.3) and (3.5), it is easy to see that

$$|A^h_{\mathbf{w},\mathbf{c}}((\mathbf{u}, \boldsymbol{\varepsilon}), (\mathbf{v}, \boldsymbol{\phi}))| \le \widehat{C}_{A_{\mathbf{w},\mathbf{c}}} \|(\mathbf{u}, \boldsymbol{\varepsilon})\| \|(\mathbf{v}, \boldsymbol{\phi})\|, \quad \forall\, (\mathbf{u}, \boldsymbol{\varepsilon}),\, (\mathbf{v}, \boldsymbol{\phi}) \in \mathbf{H}_{h,0} \times \mathbf{C}_h,$$

and

$$|\mathcal{A}^h_{\mathbf{w},\mathbf{c}}(\boldsymbol{\sigma}, \boldsymbol{\tau})| \le \widehat{C}_{\mathcal{A}_{\mathbf{w},\mathbf{c}}} \|\boldsymbol{\sigma}\|_{\mathcal{H}} \|\boldsymbol{\tau}\|_{\mathcal{H}}, \quad \forall\, \boldsymbol{\sigma},\, \boldsymbol{\tau} \in \mathcal{H}_h, \tag{4.6}$$

where $\widehat{C}_{A_{\mathbf{w},\mathbf{c}}}$ and $\widehat{C}_{\mathcal{A}_{\mathbf{w},\mathbf{c}}}$ are given by

$$\widehat{C}_{A_{\mathbf{w},\mathbf{c}}} := \widehat{C} \left(\nu + S_c \nu_m^{-1} + \|\mathbf{w}\|_{1,\Omega} + S_c \nu_m^{-1} \|\mathbf{c}\|^2_{3,\mathrm{div};\Omega} + S_c \nu_m^{-1} \|\mathbf{c}\|_{3,\mathrm{div};\Omega} \right), \tag{4.7}$$

and

$$\widehat{C}_{\mathcal{A}_{\mathbf{w},\mathbf{c}}} := \widehat{C}_{A_{\mathbf{w},\mathbf{c}}} + 2 S_c, \tag{4.8}$$

with $\widehat{C} > 0$, independent of the physical parameters.

Now, we let $\mathcal{K}_h$ be the discrete kernel of the bilinear form $\mathcal{B}$, that is:

$$\mathcal{K}_h := \{\boldsymbol{\tau}_h \in \mathcal{H}_h \,:\, \mathcal{B}(\boldsymbol{\tau}_h, \mathbf{q}_h) = 0 \quad \forall\, \mathbf{q}_h \in \mathcal{Q}_h\}.$$

Thanks to the definition of $\mathcal{B}$ (cf. (2.11)) and the choice of the discrete spaces (cf. (4.1)), it is clear that $\mathcal{K}_h$ can be decomposed as

$$\mathcal{K}_h = \mathcal{N}_h \times \mathcal{M}_h,$$

with

$$\begin{aligned} \mathcal{N}_h := & \quad \{(\mathbf{v}_h, \boldsymbol{\phi}_h) \in \mathbf{H}_{h,0} \times \boldsymbol{\Phi}_h \,:\, (\operatorname{div} \mathbf{v}_h, q_h) \,=\, 0 \quad \forall\, q_h \in Q_{h,0}\}, \\ \mathcal{M}_h := & \quad \mathbf{C}_h^0 = \{\mathbf{d}_h \in \mathbf{C}_h \,:\, \operatorname{div} \mathbf{d}_h \,=\, 0 \quad \text{in} \quad \Omega\}. \end{aligned} \tag{4.9}$$

In what follows we establish suitable hypotheses under which, for given $\mathbf{w}_h \in \mathbf{H}_{h,0}$, the bilinear form $\mathcal{A}_{\mathbf{w}_h,\mathbf{c}_h}(\cdot,\cdot)$ induces an invertible operator in $\mathcal{K}_h$, which in a finite dimensional setting, is equivalent to prove that the following inf-sup condition holds:

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_h\in\mathcal{K}_h} \frac{\mathcal{A}^h_{\mathbf{w}_h,\mathbf{c}_h}(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathcal{H}}} \geq \widehat{\gamma}\|\boldsymbol{\zeta}_h\|_{\mathcal{H}}, \quad \forall\, \boldsymbol{\zeta}_h \in \mathcal{K}_h, \tag{4.10}$$

with $\widehat{\gamma}$ a positive constant independent of h (to be specified below). To do that, and according to the saddle-point structure of $\mathcal{A}_{\mathbf{w}_h,\mathbf{c}_h}(\cdot,\cdot)$ in the sequel we prove that $A^h_{\mathbf{w}_h,\mathbf{c}_h}$ and B and satisfy the discrete versions of the estimates (3.20) and (3.17), respectively. To that end, we need to define the discrete version of $\mathcal{K}_0$ (cf. (3.18)), that is

$$\begin{aligned} \mathcal{K}_{h,0} := & \quad \{(\mathbf{v}_h, \boldsymbol{\phi}_h) \in \mathcal{N}_h \,:\, B((\mathbf{v}_h, \boldsymbol{\phi}_h), \mathbf{d}_h) = 0 \quad \forall\, \mathbf{d} \in \mathcal{M}_h\} \\ = & \quad \{(\mathbf{v}_h, \boldsymbol{\phi}_h) \in \mathbf{H}_{h,0} \times \boldsymbol{\Phi}_h \,:\, (\operatorname{div} \mathbf{v}_h, q_h) \,=\, 0 \quad \forall\, q_h \in Q_{h,0} \\ & \qquad \text{and} \quad (\mathbf{d}_h, \operatorname{curl} \boldsymbol{\phi}_h)_\Omega = 0 \quad \forall\, \mathbf{d}_h \in \mathcal{M}_h\}\,. \end{aligned}$$

Notice that for a given $(\mathbf{v}_h, \boldsymbol{\phi}_h) \in \mathcal{K}_{h,0}$, it follows that $\operatorname{curl} \boldsymbol{\phi}_h \in \mathcal{M}_h$ and $(\mathbf{d}_h, \operatorname{curl} \boldsymbol{\phi}_h)_\Omega = 0$, $\forall\, \mathbf{d}_h \in \mathcal{M}_h$, which implies that particularly $\boldsymbol{\phi}_h$ satisfies $\operatorname{curl} \boldsymbol{\phi}_h = 0$ in Ω. According to this, $\mathcal{K}_{h,0}$ can be characterized as follows

$$\mathcal{K}_{h,0} = \{(\mathbf{v}_h, \boldsymbol{\phi}_h) \in \mathbf{H}_{h,0} \times \boldsymbol{\Phi}_h \,:\, (\operatorname{div} \mathbf{v}_h, q_h) \,=\, 0 \quad \forall\, q_h \in Q_{h,0} \quad \text{and} \quad \operatorname{curl} \boldsymbol{\phi}_h = 0 \quad \text{in} \quad \Omega\}\,.$$

The following result establishes suitable conditions under which, for given $\mathbf{w}_h$ and $\mathbf{c}_h$, $A^h_{\mathbf{w}_h,\mathbf{c}_h}$ is elliptic on $\mathcal{K}_{h,0}$. The proof, which is analogous to the proof of Lemma 3.6, is omitted.

Lemma 4.2 *Let $(\mathbf{w}_h, \mathbf{c}_h) \in \mathbf{H}_{h,0} \times \mathbf{C}_h$, and assume that $\mathbf{c}_h$ satisfies (3.19). Then, there holds*

$$A^h_{\mathbf{w}_h,\mathbf{c}_h}((\mathbf{v}_h, \boldsymbol{\phi}_h), (\mathbf{v}_h, \boldsymbol{\phi}_h)) \geq \alpha_1 \left(\|\mathbf{v}_h\|^2_{1,\Omega} \,+\, \|\boldsymbol{\phi}_h\|^2_{\operatorname{curl}_{\frac{3}{2}};\Omega} \right) \quad \forall\, (\mathbf{v}_h, \phi_h) \in \mathcal{K}_{h,0},$$

with $\alpha_1 > 0$ given by (3.22).

Now we turn to prove the counterpart of estimate (3.17). To that end, we let $\mathcal{E}_h$ be the set of faces of $\mathcal{T}_h$, whose corresponding diameters are denoted h_e, and define

$$\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h : e \subseteq \Omega\} \quad \text{and} \quad \mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h : e \subseteq \Gamma\}.$$

We also let $[\![\cdot]\!]$ be the usual jump operator across internal faces defined for piecewise continuous functions v, by

$$[\![v]\!] \,=\, (v\big|_{T_+})\big|_e - (v\big|_{T_-})\big|_e \quad \text{with} \quad e \,=\, \partial T_+ \cap \partial T_-,$$

where T_+ and T_- are the elements of $\mathcal{T}_h$ having e as a common face. Then, we introduce the well-known Crouzeix–Raviart space:

$$\begin{aligned} CR_h := & \quad \Big\{ v_h : \Omega \to \mathbb{R} \,:\, v_h|_T \in P_1(T), \forall\, T \in \mathcal{T}_h, \quad \int_e [\![v_h]\!] = 0, \quad \forall e \in \mathcal{E}_h(\Omega) \\ & \qquad \text{and} \quad \int_e v_h = 0, \quad \forall e \in \mathcal{E}_h(\Gamma) \Big\}\,. \end{aligned}$$

We recall from [30, Theorem 4.9] that the following orthogonal decomposition holds:

$$\mathbf{S}_h := [S_h]^3 \; = \; \mathrm{curl}\,(\mathbf{\Phi}_h) \oplus \nabla CR_h, \tag{4.11}$$

where

$$\nabla CR_h \;:=\; \{\mathbf{s}_h \in \mathbf{S}_h \,:\, \exists\, v_h \in CR_h \text{ such that } \mathbf{s}_h|_T = \nabla(v_h|_T), \quad \forall\, T \in \mathcal{T}_h\}\,.$$

Notice also that this decomposition is stable, in particular, in $\mathbf{L}^{\frac{3}{2}}(\Omega)$, in the sense that there exists $C > 0$, such that for any $\mathbf{s}_h \in \mathbf{S}_h$, $\boldsymbol{\varphi}_h \in \mathbf{\Phi}_h$ and $\chi_h \in CR_h$ satisfying $\mathbf{s}_h = \mathrm{curl}\,\boldsymbol{\varphi}_h + \nabla\chi_h$, there holds

$$\|\mathrm{curl}\,\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} + \|\nabla\chi_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \le C\|\mathbf{s}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}.$$

Now, given $T \in \mathcal{T}_h$, $s > 1/2$ and $p > 2$, we let $\mathbf{N}(T) := \{\mathbf{w} \in \mathbf{H}^s(T) \,:\, \mathrm{curl}\,(\mathbf{w}) \in \mathbf{L}^p(T)\}$ and denote by $\mathbf{I}_{N,T} : \mathbf{N}(T) \to \mathbf{N}_0(T)$ and $\mathbf{I}_{R,T} : \mathbf{H}^1(T) \to \mathbf{RT}_0(T)$ the local Nédélec and Raviart-Thomas interpolation operators, respectively.

Let us recall from [26, Theorem 5.4.1], that for all $T \in \mathcal{T}_h$ and $0 < \delta \le 1/2$, $\mathbf{I}_{N,T}$ satisfies the local approximation property

$$\|\mathbf{w} - \mathbf{I}_{N,T}(\mathbf{w})\|_{0,T} \le C\left(h_T^{1/2+\delta}\|\mathbf{w}\|_{\mathbf{H}^{1/2+\delta}(T)} + h_T\|\mathrm{curl}\,\mathbf{w}\|_{0,T}\right), \tag{4.12}$$

for all $\mathbf{w} \in \mathbf{H}^{1/2+\delta}(T)$, such that $\mathrm{curl}\,\mathbf{w} \in \mathbf{RT}_0(T)$. In addition, both $\mathbf{I}_{N,T}$ and $\mathbf{I}_{R,T}$ satisfy the following relations (see [13, Lemma 1.44] and [5, Section 4.2.1]):

$$\mathbf{I}_{R,T}(\mathrm{curl}\,\mathbf{w}) = \mathrm{curl}\,(\mathbf{I}_{N,T}(\mathbf{w})), \tag{4.13}$$

for all $\mathbf{w} \in \mathbf{N}(T)$, such that $\mathrm{curl}\,\mathbf{w} \in \mathbf{H}^1(T)$, and

$$\mathrm{div}\,(\mathbf{I}_{R,T}(\mathbf{d})) = \mathcal{P}_T(\mathrm{div}\,\mathbf{d}), \quad \forall\, \mathbf{d} \in \mathbf{H}^1(T), \tag{4.14}$$

where $\mathcal{P}_T : L^2(T) \to P_0(T)$ is the L^2–projection.

On the other hand, the Raviart–Thomas operator satisfies the following local estimates (see for instance [5, Lemma 4.1])

$$|\mathbf{d} - \mathbf{I}_{R,T}(\mathbf{d})|_{\mathbf{W}^{m,r}(T)} \le c_1 h_T^{1-m}|\mathbf{d}|_{\mathbf{W}^{1,r}(T)}, \quad \mathbf{d} \in \mathbf{W}^{1,r}(T),\; r > 1,\, m \in \{0,1\}, \tag{4.15}$$

and

$$|\mathrm{div}\,\mathbf{d} - \mathrm{div}\,(\mathbf{I}_{R,T}(\mathbf{d}))|_{\mathbf{W}^{m,r}(T)} \le c_2 h_T^{1-m}|\mathbf{d}|_{\mathbf{W}^{1,r}(T)}, \quad \mathbf{d} \in \mathbf{W}^{1,r}(T),\; r > 1,\, m \in \{0,1\}.$$

In particular, employing (4.15), it is possible to obtain the following estimate:

Lemma 4.3 *There exists $C > 0$, independent of h, such that*

$$\|\mathbf{d} - \mathbf{I}_{R,T}(\mathbf{d})\|_{\mathbf{L}^3(T)} \le C h_T^{1/2}|\mathbf{d}|_{1,T}, \quad \mathbf{d} \in \mathbf{H}^1(T). \tag{4.16}$$

Proof. Similarly to the proof of [9, Lemma 5.4], given $T \in \mathcal{T}_h$, we denote by B_T the square matrix of the local affine transformation $J_T : \widehat{T} \to T$ and recall that $|det(B_T)| = O(h_T^3)$ and $\|B_T\| \le ch_T$. Then, denoting by $\widehat{v} := v \circ J_T$ for any sufficiently smooth function v, and applying the Denny–Lions Lemma [13, Lemma B.67], the scaling estimates given in [13, Lemma 1.101], the Sobolev embedding (2.3) with $q=3$, and estimate (4.15), we deduce that for all $\mathbf{d}\in\mathbf{H}^1(\Omega)$,

$$\begin{aligned}\|\mathbf{d}-\mathbf{I}_{R,T}(\mathbf{d})\|_{\mathbf{L}^3(T)} &\le c|det(B_T)|^{1/3}\|\widehat{\mathbf{d}}-\mathbf{I}_{R,\widehat{T}}(\widehat{\mathbf{d}})\|_{1,\widehat{T}},\\ &\le c|det(B_T)|^{1/3}\left(\|\widehat{\mathbf{d}}-\mathbf{I}_{R,\widehat{T}}(\widehat{\mathbf{d}})\|_{0,\widehat{T}}+|\widehat{\mathbf{d}}-\mathbf{I}_{R,\widehat{T}}(\widehat{\mathbf{d}})|_{1,\widehat{T}}\right),\\ &\le c|det(B_T)|^{-1/6}\left(\|\mathbf{d}-\mathbf{I}_{R,T}(\mathbf{d})\|_{0,T}+\|B_T\||\mathbf{d}-\mathbf{I}_{R,T}(\mathbf{d})|_{1,T}\right),\\ &\le c h_T^{1/2}|\mathbf{d}|_{1,T}.\end{aligned}$$

□

In the sequel we will also employ the global Nédélec and Raviart-Thomas interpolation operators, denoted respectively by $\mathbf{I}_N$ and $\mathbf{I}_R$, which satisfy $\mathbf{I}_N|_T=\mathbf{I}_{N,T}$ and $\mathbf{I}_R|_T=\mathbf{I}_{R,T}$, for all $T\in\mathcal{T}_h$.

Finally, we recall that for all piece-wise polynomial functions v, the following local inverse inequality holds

$$\|v\|_{0,T}\le c h_T^{-1/2}\|v\|_{\mathrm{L}^{\frac{3}{2}}(T)},\quad \forall T\in\mathcal{T}_h,$$

and assuming further that $\mathcal{T}_h$ is a quasi-uniform mesh, the latter implies the following global inverse inequality

$$\|v\|_{0,\Omega}\le c h^{-1/2}\|v\|_{\mathrm{L}^{\frac{3}{2}}(\Omega)}. \tag{4.17}$$

Now, we are in position of establishing the discrete inf-sup condition of B.

Lemma 4.4 *Let $\mathcal{T}_h$ be a regular famililly of quasi-uniform meshes. Then, there exists $\widehat{\beta}_1>0$, independent of h, such that*

$$\sup_{\mathbf{0}\neq(\mathbf{v}_h,\boldsymbol{\phi}_h)\in\mathcal{N}_h}\frac{B((\mathbf{v}_h,\boldsymbol{\phi}_h),\mathbf{d}_h)}{\|\mathbf{v}_h\|_{1,\Omega}+\|\boldsymbol{\phi}_h\|_{\mathrm{curl}_{\frac{3}{2}};\Omega}}\ge\widehat{\beta}_1\|\mathbf{d}_h\|_{\mathbf{L}^3(\Omega)},\quad\forall\,\mathbf{d}_h\in\mathcal{M}_h.$$

Proof. Given $\mathbf{d}_h\in\mathcal{M}_h$, we first observe that, since $\operatorname{div}\mathbf{d}_h=0$ in Ω, according to [16, Theorem 3.3], $\mathbf{d}_h$ is a piece-wise constant vector field, that is, $\mathbf{d}_h\in\mathbf{S}_h$ (cf. (4.11)). Then, we let $\mathbf{f}(\mathbf{d}_h):=\mathbf{d}_h|\mathbf{d}_h|\in\mathbf{S}_h$ and owing to the decomposition (4.11), let $\boldsymbol{\varphi}_h\in\boldsymbol{\Phi}_h$ and $\chi_h\in CR_h$, be such that

$$\mathbf{f}(\mathbf{d}_h)=\operatorname{curl}\boldsymbol{\varphi}_h+\nabla\chi_h, \tag{4.18}$$

and

$$\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}+\|\nabla\chi_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\le C_1\|\mathbf{f}(\mathbf{d}_h)\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}=C_1\|\mathbf{d}_h\|^2_{\mathbf{L}^3(\Omega)}. \tag{4.19}$$

In turn, noticing that $\operatorname{curl}\boldsymbol{\varphi}_h\in\mathbf{C}_h^0\subseteq\mathbf{H}(\operatorname{div}^0;\Omega)$ (cf. (4.9)), we apply Lemma 3.3 and let $\boldsymbol{\psi}\in\mathbf{H}^1(\Omega)$ be such that

$$\operatorname{curl}\boldsymbol{\psi}=\operatorname{curl}\boldsymbol{\varphi}_h\quad\text{in}\quad\Omega\,,\quad\operatorname{div}\boldsymbol{\psi}=0\quad\text{in}\quad\Omega\quad\text{and}\quad\|\boldsymbol{\psi}\|_{1,\Omega}\le C_2\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{0,\Omega}, \tag{4.20}$$

and since $\operatorname{curl}\boldsymbol{\varphi}_h\in\mathbf{L}^{\frac{3}{2}}(\Omega)$, from Remark 3.4 we additionally deduce that $\boldsymbol{\psi}$ also belongs to $\mathbf{W}^{1,\frac{3}{2}}(\Omega)$ and satisfies

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,\frac{3}{2}}(\Omega)}\le C_3\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}. \tag{4.21}$$

Hence, since $\operatorname{curl}\boldsymbol{\varphi}_h \in \mathbf{C}_h^0$, from (4.12), (4.17) and (4.20), we deduce that

$$\begin{aligned}\|\boldsymbol{\psi}-\mathbf{I}_N(\boldsymbol{\psi})\|_{0,\Omega}^2 &= \sum_{T\in\mathcal{T}_h}\|\boldsymbol{\psi}-\mathbf{I}_{N,T}(\boldsymbol{\psi})\|_{0,T}^2 \le C_4\sum_{T\in\mathcal{T}_h}\left(h_T^2\|\boldsymbol{\psi}\|_{1,T}^2 + h_T^2\|\operatorname{curl}\boldsymbol{\psi}\|_{0,T}^2\right)\\ &\le C_4 h^2\left(\|\boldsymbol{\psi}\|_{1,\Omega}^2+\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{0,\Omega}^2\right)\le C_5h^2\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{0,\Omega}^2 \le C_6 h\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}^2,\end{aligned}$$

which implies that

$$\|\boldsymbol{\psi}-\mathbf{I}_N(\boldsymbol{\psi})\|_{0,\Omega} \le C_7 h^{1/2}\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}. \tag{4.22}$$

In this way, we define $\widetilde{\boldsymbol{\varphi}}_h := \mathbf{I}_N(\boldsymbol{\psi})$ and utilize (4.13) to obtain

$$\operatorname{curl}\widetilde{\boldsymbol{\varphi}}_h = \operatorname{curl}\left(\mathbf{I}_N(\boldsymbol{\psi})\right) = \mathbf{I}_R\left(\operatorname{curl}\boldsymbol{\psi}\right) = \mathbf{I}_R\left(\operatorname{curl}\boldsymbol{\varphi}_h\right) = \operatorname{curl}\boldsymbol{\varphi}_h. \tag{4.23}$$

Notice that estimates (4.21) and (4.22), the triangle inequality and the Sobolev embedding $\mathbf{W}^{1,\frac{3}{2}}(\Omega)$ into $\mathbf{L}^2(\Omega)$, imply that $\widetilde{\boldsymbol{\varphi}}_h$ satisfies

$$\begin{aligned}\|\widetilde{\boldsymbol{\varphi}}_h\|_{0,\Omega} \le\ & \|\boldsymbol{\psi}-\mathbf{I}_N(\boldsymbol{\psi})\|_{0,\Omega}+\|\boldsymbol{\psi}\|_{0,\Omega} \le C_7h^{1/2}\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} + C_8\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,\frac{3}{2}}(\Omega)}\\ \le\ & C_7h^{1/2}\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} + C_8C_3\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \le C_9\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)},\end{aligned}$$

which together with (4.19) and (4.23), implies

$$\begin{aligned}\|\widetilde{\boldsymbol{\varphi}}_h\|_{\operatorname{curl}_{\frac{3}{2}},\Omega} =\ & \left(\|\widetilde{\boldsymbol{\varphi}}_h\|_{0,\Omega}^2+\|\operatorname{curl}\widetilde{\boldsymbol{\varphi}}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}^2\right)^{1/2} \le (C_9+1)\|\operatorname{curl}\boldsymbol{\varphi}_h\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\\ \le\ & (C_9+1)C_1\|\mathbf{d}_h\|_{\mathbf{L}^3(\Omega)}^2.\end{aligned} \tag{4.24}$$

Moreover, making use of the decomposition (4.18), integrating by parts, and using the fact that $\displaystyle\int_e \xi = 0\ \forall e\in\mathcal{E}_h(\Gamma)$, $\operatorname{div}\mathbf{d}_h = 0$, in Ω and $\mathbf{d}_h\cdot\mathbf{n}|_e \in P_0(e)\ \forall e\in\mathcal{E}_h(\Gamma)$, from (4.23) we easily obtain

$$(\mathbf{d}_h,\operatorname{curl}\widetilde{\boldsymbol{\varphi}}_h)_\Omega = (\mathbf{d}_h,\operatorname{curl}\boldsymbol{\varphi}_h)_\Omega = (\mathbf{d}_h,\mathbf{f}(\mathbf{d}_h)-\nabla\chi_h)_\Omega = \|\mathbf{d}_h\|_{\mathbf{L}^3(\Omega)}^3. \tag{4.25}$$

According to the above, from (4.24) and (4.25), we finally obtain

$$\begin{aligned}\sup_{\mathbf{0}\neq(\mathbf{v}_h,\boldsymbol{\phi}_h)\in\mathcal{N}_h}\frac{B((\mathbf{v}_h,\boldsymbol{\phi}_h),\mathbf{d}_h)}{\|\mathbf{v}_h\|_{1,\Omega}+\|\boldsymbol{\phi}_h\|_{\operatorname{curl}_{\frac{3}{2}};\Omega}} &\ge \frac{B((\mathbf{0},\widetilde{\boldsymbol{\varphi}}_h),\mathbf{d}_h)}{\|\widetilde{\boldsymbol{\varphi}}_h\|_{\operatorname{curl}_{\frac{3}{2}};\Omega}}\\ = \frac{S_c(\mathbf{d}_h,\operatorname{curl}\widetilde{\boldsymbol{\varphi}}_h)_\Omega}{\|\widetilde{\boldsymbol{\varphi}}_h\|_{\operatorname{curl}_{\frac{3}{2}};\Omega}} \ge \widehat{\beta}_1\frac{\|\mathbf{d}_h\|_{\mathbf{L}^3(\Omega)}^3}{\|\mathbf{d}_h\|_{\mathbf{L}^3(\Omega)}^2} &= \widehat{\beta}_1\|\mathbf{d}_h\|_{\mathbf{L}^3(\Omega)}.\end{aligned}$$

with $\widehat{\beta}_1 := S_c/((C_9+1)C_1)$. □

Given the above discussion, we can now confirm that the inf-sup condition (4.10) is satisfied, ensuring the invertibility of $\mathcal{A}_{\mathbf{w}_h,\mathbf{c}_h}(\cdot,\cdot)$ on $\mathcal{K}_h$. The proof of this result is provided in the following Lemma, which follows a similar argument to that of Lemma 3.6, and hence is omitted.

Lemma 4.5 *Let $\mathcal{T}_h$ be a regular famililiy of quasi-uniform meshes and let $(\mathbf{w}_h, \mathbf{c}_h) \in \mathbf{H}_{h,0} \times \mathbf{C}_h$, with $\mathbf{c}_h$ satisfying* (3.19)*. Then, estimate* (4.10) *holds with $\widehat{\gamma}$ given by*

$$\widehat{\gamma}(\mathbf{w}_h) := (\widehat{\gamma}_1(\mathbf{w}_h) + \widehat{\gamma}_2(\mathbf{w}_h))^{-1},$$

where $\widehat{\gamma}_1(\mathbf{w}_h)$ and $\widehat{\gamma}_2(\mathbf{w}_h)$ are defined by

$$\widehat{\gamma}_1(\mathbf{w}_h) := \frac{1}{\alpha_1 \widehat{\beta}_1} \left(\alpha_1 + \widehat{\beta}_1 + \widehat{C}_{A_{\mathbf{w}_h}} \right) \quad \text{and} \quad \widehat{\gamma}_2(\mathbf{w}_h) := \frac{1}{\alpha_1 \widehat{\beta}_1^2} \left(\alpha_1 \widehat{\beta}_1 + \widehat{C}_{A_{\mathbf{w}_h}} (\alpha_1 + \widehat{\beta}_1 + \widehat{C}_{A_{\mathbf{w}_h}}) \right).$$

Above, $\widehat{C}_{A_{\mathbf{w}_h}}$ is a positive constant that bounds (4.7) *as follows*

$$\widehat{C}_{A_{\mathbf{w}_h,\mathbf{c}_h}} \le \widehat{C}_{A_{\mathbf{w}_h}} := \widehat{C} \left(\kappa(\nu, \nu_m, S_c) + \|\mathbf{w}_h\|_{1,\Omega} \right),$$

with κ being the positive constant defined in (3.32) *and $\widehat{C}$ a positive constant independent of h and the physical parameters.*

Remark 4.6 *As in the continuous case, given $\mathbf{w}_h \in \mathbf{H}_{h,0}$, $\widehat{\gamma}_1(\mathbf{w}_h)$ and $\widehat{\gamma}_2(\mathbf{w}_h)$ can be bounded in terms of $\|\mathbf{w}_h\|_{1,\Omega}$ as follows*

$$\widehat{\gamma}_1(\mathbf{w}_h) \le \widehat{c}_1 \left(S_c^{-1} + \alpha_1^{-1} + S_c^{-1} \alpha_1^{-1} \kappa(\nu, \nu_m, S_c) + \|\mathbf{w}_h\|_{1,\Omega} \right)$$

and

$$\widehat{\gamma}_2(\mathbf{w}_h) \le \widehat{c}_2 S_c^{-1} \left(1 + \kappa(\nu, \nu_m, S_c) + \alpha_1^{-1} S_c^{-1} \kappa^2(\nu, \nu_m, S_c) \right.$$
$$\left. + (1 + \kappa(\nu, \nu_m, S_c)) \|\mathbf{w}_h\|_{1,\Omega} + \alpha_1^{-1} S_c^{-1} \|\mathbf{w}_h\|_{1,\Omega}^2 \right),$$

where $\widehat{c}_1, \widehat{c}_2 > 0$ are positive constants independent of the physical parameters. Then $\widehat{\gamma}^{-1}(\mathbf{w}_h) = \widehat{\gamma}_1(\mathbf{w}_h) + \widehat{\gamma}_2(\mathbf{w}_h)$, can be bounded as follows

$$\widehat{\gamma}^{-1}(\mathbf{w}_h) = \widehat{\gamma}_1(\mathbf{w}_h) + \widehat{\gamma}_2(\mathbf{w}_h) \le \widehat{\delta}_1 + \widehat{\delta}_2 \|\mathbf{w}_h\|_{1,\Omega} + \widehat{\delta}_3 \|\mathbf{w}_h\|_{1,\Omega}^2, \tag{4.26}$$

where $\widehat{\delta}_1$, $\widehat{\delta}_2$ and $\widehat{\delta}_3$ are positive constants that depend on ν, ν_m and S_c.

Finally, we establish the discrete version of Lemma 3.1.

Lemma 4.7 *There exists $\widehat{\beta} > 0$, independent of h, such that*

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \mathcal{H}_h} \frac{\mathcal{B}(\boldsymbol{\tau}_h, \mathbf{q}_h)}{\|\boldsymbol{\tau}_h\|_{\mathcal{H}}} \ge \widehat{\beta} \|\mathbf{q}_h\|_{\mathcal{Q}}, \quad \forall\, \mathbf{q}_h \in \mathcal{Q}_h. \tag{4.27}$$

Proof. Given $\mathbf{q}_h = (q_h, s_h) \in \mathcal{Q}_h$, we first recall that the pair $(\mathbf{H}_{h,0}, Q_{h,0})$ is inf-sup stable, thus

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{H}_{h,0}} \frac{(q_h, \operatorname{div} \mathbf{v}_h)_\Omega}{\|\mathbf{v}_h\|_{1,\Omega}} \ge \widehat{c}_1 \|q_h\|_{0,\Omega}. \tag{4.28}$$

On the other hand, similarly to the proof of Lemma 3.1, given $\mathcal{O} \subseteq \mathbb{R}^3$ an open ball satisfying $\Omega \subset \mathcal{O}$, we let $\varphi \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ be the unique weak solution of the boundary value problem

$$-\Delta \varphi = K(s_h) \quad \text{in} \quad \mathcal{O}, \qquad \varphi = 0 \quad \text{on} \quad \partial \mathcal{O}, \quad \text{with} \quad K(s_h) = \left\{ \begin{array}{ll} -s_h & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \mathcal{O} \backslash \overline{\Omega} \end{array} \right.,$$

which satisfies

$$\|\varphi\|_{2,\mathcal{O}} \le c\|K(s_h)\|_{0,\mathcal{O}} = c\|s_h\|_{0,\Omega},$$

with $c>0$, independent of h. Then, we let $\widehat{\mathbf{d}} = \nabla\varphi|_\Omega \in \mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$, which clearly satisfies

$$\operatorname{div}\widehat{\mathbf{d}} = s_h \quad \text{in} \quad \Omega \quad \text{and} \quad \|\widehat{\mathbf{d}}\|_{\mathbf{L}^3(\Omega)} \le C_{Sob}(3)\|\widehat{\mathbf{d}}\|_{1,\Omega} \le c\|\varphi\|_{2,\mathcal{O}} \le \hat{c}\|s_h\|_{0,\Omega}.$$

Then, we let $\widehat{\mathbf{d}}_h = \mathbf{I}_R(\widehat{\mathbf{d}}) \in \mathbf{C}_h$ and observe from the latter, estimate (4.16) and the subadditivity inequality $\left(\sum_i a_i\right)^{2/3} \le \sum_i a_i^{2/3}$, that there holds

$$\begin{aligned}
\|\widehat{\mathbf{d}}_h\|_{\mathbf{L}^3(\Omega)} \le\ & \|\widehat{\mathbf{d}}_h - \widehat{\mathbf{d}}\|_{\mathbf{L}^3(\Omega)} + \|\widehat{\mathbf{d}}\|_{\mathbf{L}^3(\Omega)} \le \left\{\sum_{T\in\mathcal{T}_h} \|\widehat{\mathbf{d}}_h - \widehat{\mathbf{d}}\|^3_{\mathbf{L}^3(T)}\right\}^{1/3} + \hat{c}\|s_h\|_{0,\Omega} \\
\le\ & c\left\{\sum_{T\in\mathcal{T}_h} h_T^{\frac{3}{2}}|\widehat{\mathbf{d}}|^3_{1,T}\right\}^{1/3} + \hat{c}\|s_h\|_{0,\Omega} \le c\left\{\sum_{T\in\mathcal{T}_h} h_T|\widehat{\mathbf{d}}|^2_{1,T}\right\}^{1/2} + \hat{c}\|s_h\|_{0,\Omega} \\
\le\ & ch^{1/2}|\widehat{\mathbf{d}}|_{1,\Omega} + \hat{c}\|s_h\|_{0,\Omega} \le (\widetilde{c}h^{1/2} + \hat{c})\|s_h\|_{0,\Omega},
\end{aligned}$$

which together with the fact that $\operatorname{div}\widehat{\mathbf{d}} = s_h$ in Ω, implies

$$\|\widehat{\mathbf{d}}_h\|_{3,\operatorname{div};\Omega} \le \hat{C}\|s_h\|_{0,\Omega}. \tag{4.29}$$

In turn, using (4.14) it is clear that

$$\operatorname{div}\widehat{\mathbf{d}}_h = \operatorname{div}\widehat{\mathbf{d}} = s_h \quad \text{in} \quad \Omega. \tag{4.30}$$

Hence, combining (4.29) and (4.30) we readily obtain

$$\sup_{\mathbf{0}\neq\mathbf{d}\in\mathbf{C}_h} \frac{(s,\operatorname{div}\mathbf{d})_\Omega}{\|\mathbf{d}\|_{3,\operatorname{div};\Omega}} \ge \frac{(s,\operatorname{div}\widehat{\mathbf{d}}_h)_\Omega}{\|\widehat{\mathbf{d}}_h\|_{3,\operatorname{div};\Omega}} \ge \hat{C}^{-1}\|s_h\|_{0,\Omega}. \tag{4.31}$$

In this way, from (4.28) and (4.31), we readily obtain (4.27), which concludes the proof. □

Similarly to the continuous case, Lemmas 4.5 and 4.7 guarantee the well-definedness of $\mathcal{J}_h$. This conclusion is summarized in the following theorem, the proof of which is omitted due to its similarity to that of Theorem 3.8.

Theorem 4.8 *Let $\mathcal{T}_h$ be a regular famililly of quasi-uniform meshes and let $(\mathbf{w}_h,\mathbf{c}_h) \in \mathbf{H}_{h,0} \times \mathbf{C}_h$, with $\mathbf{c}_h$ satisfying (3.19). Then, there exists a unique $(\mathbf{u}_h,\mathbf{b}_h) \in \mathbf{H}_{h,0}\times\mathbf{C}_h$, such that $\mathcal{J}(\mathbf{w}_h,\mathbf{c}_h) = (\mathbf{u}_h,\mathbf{b}_h)$. In addition, the following estimate holds:*

$$\|\mathcal{J}(\mathbf{w}_h,\mathbf{c}_h)\| \le \left(\widehat{\delta}_1 + \widehat{\delta}_2\|\mathbf{w}_h\|_{1,\Omega} + \widehat{\delta}_3\|\mathbf{w}_h\|^2_{1,\Omega}\right)\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\right),$$

where $\widehat{\delta}_1$, $\widehat{\delta}_2$ and $\widehat{\delta}_3$ are the positive constants satisfying (4.26).

4.3 Well-posedness of the Galerkin scheme

Theorem 4.9 *Let $\mathcal{T}_h$ be a regular famililiy of quasi-uniform meshes. Given* $t_0 \leq \left(\dfrac{\nu_m \nu}{C_1(\Omega) S_c}\right)^{1/2}$, *with* $C_1(\Omega)$ *being the positive defined in* (3.21)*, let* $t \in (0, t_0)$*, and assume that* $\mathbf{f}$ *and* $\mathbf{g}$ *satisfy*

$$\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \leq \frac{t}{\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2}, \tag{4.32}$$

where $\widehat{\delta}_1$*,* $\widehat{\delta}_2$ *and* $\widehat{\delta}_3$ *are the positive constants satisfying* (4.26)*. Assume further that* t *satisfies*

$$\widehat{C}_2(\Omega) t\,(\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2)(1 + S_c \nu_m^{-1}(t+1)) < 1, \tag{4.33}$$

with $\widehat{C}_2(\Omega) > 0$*, independent of the physical parameters. Then, there exists a unique* $(\boldsymbol{\sigma}_h, \mathbf{p}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$ *solution to* (4.2)*. In addition,*

$$\|\boldsymbol{\sigma}_h\|_{\mathcal{H}} \leq \left(\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2\right)\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\right), \tag{4.34}$$

$$\|\mathbf{p}_h\|_{\mathcal{Q}} \leq \widehat{C}\left(1 + (\kappa(\nu, \nu_m, S_c) + S_c + t)\left(\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2\right)\right)\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\right), \tag{4.35}$$

with $\widehat{C} > 0$*, independet of the physical parameters.*

Proof. Analogously to the proof of Theorem 3.9, we let $t \in (0, t_0)$, define the convex and bounded set

$$\mathbf{K}_h := \left\{(\mathbf{v}_h, \mathbf{b}_h) \in \mathbf{H}_{h,0} \times \mathbf{C}_h \,:\, \|(\mathbf{v}_h, \mathbf{b}_h)\| \leq \left(\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2\right)\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)}\right)\right\},$$

let $(\mathbf{w}_1, \mathbf{c}_1)$, $(\mathbf{w}_2, \mathbf{c}_2) \in \mathbf{K}_h$, and observe from assumption (4.32) that

$$\|(\mathbf{w}_1, \mathbf{c}_1)\| \leq t \quad \text{and} \quad \|(\mathbf{w}_2, \mathbf{c}_2)\| \leq t,$$

which implies that $\mathbf{c}_1$ and $\mathbf{c}_2$ satisfy (3.19). Then we let $(\mathbf{u}_1, \mathbf{b}_1), (\mathbf{u}_2, \mathbf{b}_2) \in \mathbf{H}_{h,0} \times \mathbf{C}_h$, be such that

$$(\mathbf{u}_1, \mathbf{b}_1) = \mathcal{J}_h(\mathbf{w}_1, \mathbf{c}_1) \quad \text{and} \quad (\mathbf{u}_2, \mathbf{b}_2) = \mathcal{J}_h(\mathbf{w}_2, \mathbf{c}_2),$$

and proceed as in the proof of Theorem 3.9, to deduce that there exists $\widehat{C}_2(\Omega) > 0$, independent of the physical parameters, such that

$$\|\mathcal{J}_h(\mathbf{w}_1, \mathbf{c}_1) - \mathcal{J}_h(\mathbf{w}_2, \mathbf{c}_2)\| \leq \widehat{C}_2(\Omega) t(\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2)(1 + S_c \nu_m^{-1}(t+1))\|(\mathbf{w}_1 - \mathbf{w}_2, \mathbf{c}_1 - \mathbf{c}_2)\|.$$

The latter and (4.33) imply that there exists a unique $(\mathbf{u}_h, \mathbf{b}_h) \in \mathbf{K}_h$, such that $(\mathbf{u}_h, \mathbf{b}_h) = \mathcal{J}_h(\mathbf{u}_h, \mathbf{b}_h)$, which in turn implies that there exist uniques $\boldsymbol{\varepsilon}_h \in \boldsymbol{\Phi}_h$ and $\mathbf{p}_h := (p_h, r_h) \in \mathcal{Q}_h$, such that $(\boldsymbol{\sigma}_h, \mathbf{p}_h) = ((\mathbf{u}_h, \boldsymbol{\varepsilon}_h, \mathbf{b}_h), \mathbf{p}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$, is the unique solution of (4.2). In addition, proceeding analogously to the proof of Theorem 3.9, we can also deduce that $(\boldsymbol{\sigma}_h, \mathbf{p}_h)$ satisfies (4.34)–(4.35), which concludes the proof. □

5 Cea's estimate and theoretical rate of convergence

In this section we study the convergence of the Galerkin scheme (4.2). More precisely, we first deduce that the error satisfies a Cea's-type estimate and later on, under an extra regularity assumption of the exact solution, and employing the approximation properties of the discrete spaces introduced in (4.1), we derive the theoretical rate of convergence. To do that, and for the sake of simplicity, we define the errors:

$$\mathbf{e}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \quad \mathbf{e}_{\mathbf{p}} := \mathbf{p} - \mathbf{p}_h$$

and for any $(\widehat{\boldsymbol{\tau}}_h, \widehat{\mathbf{q}}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$, we write

$$\mathbf{e}_{\boldsymbol{\sigma}} := \boldsymbol{\xi}_{\boldsymbol{\sigma}} + \boldsymbol{\chi}_{\boldsymbol{\sigma}}, \quad \mathbf{e}_{\mathbf{p}} := \boldsymbol{\xi}_{\mathbf{p}} + \boldsymbol{\chi}_{\mathbf{p}}$$

where

$$\boldsymbol{\xi}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \widehat{\boldsymbol{\tau}}_h, \quad \boldsymbol{\chi}_{\boldsymbol{\sigma}} := \widehat{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h, \quad \boldsymbol{\xi}_{\mathbf{p}} := \mathbf{p} - \widehat{\mathbf{q}}_h, \quad \boldsymbol{\chi}_{\mathbf{p}} := \widehat{\mathbf{q}}_h - \mathbf{p}_h. \tag{5.1}$$

The following result establishes the aforementioned Cea's estimate.

Theorem 5.1 *Let $\mathcal{T}_h$ be a regular famlily of quasi-uniform meshes. Given $t_0 \leq \left(\dfrac{\nu_m \nu}{C_1(\Omega) S_c}\right)^{1/2}$, with $C_1(\Omega)$ being the positive defined in* (3.21)*, let $t \in (0, t_0)$, and assume that $\mathbf{f}$ and $\mathbf{g}$ satisfy* (3.37) *and* (4.32)*. Assume further that t satisfies* (3.38) *and* (4.33)*, and the estimate*

$$\Lambda t \left(1 + 2 S_c \nu_m^{-1} t \, + \, 2 S_c \nu_m^{-1}\right) (\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2) \leq \frac{1}{2}, \tag{5.2}$$

where Λ is the parameter–free positive constant satisfying (5.7) *below. Let $(\boldsymbol{\sigma}, \mathbf{p}) \in \mathcal{H} \times \mathcal{Q}$ and $(\boldsymbol{\sigma}_h, \mathbf{p}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$ be the unique solutions of problems* (2.10) *and* (4.2)*, respectively. Then, there exists a positive constant C_{cea}, independent of h, such that*

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} + \|\mathbf{e}_{\mathbf{p}}\|_{\mathcal{Q}} \leq C_{cea} \left\{ \inf_{\boldsymbol{\tau}_h \in \mathcal{H}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathcal{H}} + \inf_{\mathbf{q}_h \in \mathcal{Q}_h} \|\mathbf{p} - \mathbf{q}_h\|_{\mathcal{Q}} \right\}.$$

Proof. Let $(\boldsymbol{\sigma}, \mathbf{p}) = ((\mathbf{u}, \boldsymbol{\varepsilon}, \mathbf{b}), \mathbf{p}) \in \mathcal{H} \times \mathcal{Q}$ and $(\boldsymbol{\sigma}_h, \mathbf{p}_h) = ((\mathbf{u}_h, \boldsymbol{\varepsilon}_h, \mathbf{b}_h), \mathbf{p}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$ be the unique solutions of problems (2.10) and (4.2), and observe that from (3.37), (3.39), (4.32) and (4.34), there hold

$$\begin{aligned} &\|\mathbf{u}\|_{1,\Omega} \leq t, \quad \|\mathbf{b}\|_{3,\mathrm{div};\Omega} \leq t, \quad \|\boldsymbol{\varepsilon}\|_{\mathrm{curl}_{\frac{3}{2}};\Omega} \leq t, \\ &\|\mathbf{u}_h\|_{1,\Omega} \leq t, \quad \|\mathbf{b}_h\|_{3,\mathrm{div};\Omega} \leq t, \quad \|\boldsymbol{\varepsilon}_h\|_{\mathrm{curl}_{\frac{3}{2}};\Omega} \leq t. \end{aligned} \tag{5.3}$$

On the other hand, recalling that the exact velocity $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ satisfies $\mathrm{div}\, \mathbf{u} = 0$ in Ω, we observe that $\mathcal{A}^h_{\mathbf{u},\mathbf{b}}$ (cf. (4.3)), satisfies

$$\mathcal{A}_{\mathbf{u},\mathbf{b}}(\boldsymbol{\sigma}, \boldsymbol{\tau}_h) = \mathcal{A}^h_{\mathbf{u},\mathbf{b}}(\boldsymbol{\sigma}, \boldsymbol{\tau}_h), \qquad \forall\, \boldsymbol{\tau}_h \in \mathcal{H}_h.$$

Then, subtracting equations (2.10) and (4.2), the former with $\boldsymbol{\tau}_h \in \mathcal{H}_h$, adding and subtracting suitable terms and using the decompositions (5.1), we deduce that $(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, \boldsymbol{\chi}_{\mathbf{p}}) \in \mathcal{H}_h \times \mathcal{Q}_h$ satisfies

$$\begin{aligned} \mathcal{A}^h_{\mathbf{u}_h,\mathbf{b}_h}(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) + \mathcal{B}(\boldsymbol{\tau}_h, \boldsymbol{\chi}_{\mathbf{p}}) &= \mathcal{L}_1(\boldsymbol{\tau}_h), \quad \forall\, \boldsymbol{\tau}_h \in \mathcal{H}_h, \\ \mathcal{B}(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, \mathbf{q}_h) &= \mathcal{L}_2(\mathbf{q}_h), \quad \forall\, \mathbf{q}_h \in \mathcal{Q}_h, \end{aligned}$$

with

$$\begin{aligned} \mathcal{L}_1(\boldsymbol{\tau}_h) &:= \mathcal{A}^h_{\mathbf{u}_h,\mathbf{b}_h}(\boldsymbol{\sigma},\boldsymbol{\tau}_h) - \mathcal{A}^h_{\mathbf{u},\mathbf{b}}(\boldsymbol{\sigma},\boldsymbol{\tau}_h) - \mathcal{A}^h_{\mathbf{u}_h,\mathbf{b}_h}(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\tau}_h) - \mathcal{B}(\boldsymbol{\tau}_h,\boldsymbol{\xi}_{\mathbf{p}}), \\ \mathcal{L}_2(\mathbf{q}_h) &:= -\mathcal{B}(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\mathbf{q}_h). \end{aligned} \tag{5.4}$$

Then, recalling that $\mathcal{A}_{\mathbf{u}_h,\mathbf{b}_h}$ and $\mathcal{B}$ satisfy the hypotheses of [13, Theorem 2.34], namely estimates (4.10) and (4.27), we observe that (4.8), (4.26) and (5.3), imply

$$\widehat{\gamma}^{-1}(\mathbf{u}_h) \le \widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2,$$

and

$$\widehat{C}_{\mathcal{A}_{\mathbf{u}_h,\mathbf{b}_h}} \le \mathcal{C}_1 := \widehat{C}\left(\nu + S_c + S_c\nu_m^{-1} + t + S_c\nu_m^{-1}t^2 + S_c\nu_m^{-1}t\right), \tag{5.5}$$

and employ [13, eq. (2.30)], to obtain

$$\begin{aligned} \|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} &\le (\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2)\|\mathcal{L}_1\|_{\mathcal{H}_h'} + \widehat{\beta}^{-1}\left(1 + \mathcal{C}_1\left(\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2\right)\right)\|\mathcal{L}_2\|_{\mathcal{Q}_h'}, \\ \|\boldsymbol{\chi}_{\mathbf{p}}\|_{\mathcal{Q}} &\le \widehat{\beta}^{-1}\left(1 + \mathcal{C}_1\left(\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2\right)\right)\|\mathcal{L}_1\|_{\mathcal{H}_h'} + \widehat{\beta}^{-2}\Upsilon\left(1 + \mathcal{C}_1\left(\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2\right)\right)\|\mathcal{L}_2\|_{\mathcal{Q}_h'}, \end{aligned} \tag{5.6}$$

where $\Upsilon > 0$ is independent of h and the physical parameters.

Now we turn to bound $\|\mathcal{L}_1\|_{\mathcal{H}_h'}$ and $\|\mathcal{L}_2\|_{\mathcal{Q}_h'}$. We begin by proceeding similarly as for estimate (3.44), to deduce that

$$\begin{aligned} &|\mathcal{A}^h_{\mathbf{u}_h,\mathbf{b}_h}(\boldsymbol{\sigma},\boldsymbol{\tau}_h) - \mathcal{A}^h_{\mathbf{u},\mathbf{b}}(\boldsymbol{\sigma},\boldsymbol{\tau}_h)| \\ &\le c_1\left(\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega} + S_c\nu_m^{-1}\|\mathbf{u}\|_{1,\Omega}\|\mathbf{b}\|_{3,\mathrm{div};\Omega}\|\mathbf{b}-\mathbf{b}_h\|_{3,\mathrm{div};\Omega}\right. \\ &\left.+ S_c\nu_m^{-1}\|\mathbf{u}\|_{1,\Omega}\|\mathbf{b}-\mathbf{b}_h\|_{3,\mathrm{div};\Omega}\|\mathbf{b}_h\|_{3,\mathrm{div};\Omega} + S_c\nu_m^{-1}\|\boldsymbol{\varepsilon}\|_{\mathrm{curl}_{\frac{3}{2}};\Omega}\|\mathbf{b}-\mathbf{b}_h\|_{3,\mathrm{div};\Omega}\right)\|\mathbf{v}_h\|_{1,\Omega} \\ &+ c_2 S_c\nu_m^{-1}\|\mathbf{b}-\mathbf{b}_h\|_{3,\mathrm{div};\Omega}\|\mathbf{u}\|_{1,\Omega}\|\boldsymbol{\phi}_h\|_{\mathrm{curl}_{\frac{3}{2}};\Omega}, \\ &\le c_1\left(t\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega} + 2S_c\nu_m^{-1}t^2\|\mathbf{b}-\mathbf{b}_h\|_{3,\mathrm{div};\Omega} + S_c\nu_m^{-1}t\|\mathbf{b}-\mathbf{b}_h\|_{3,\mathrm{div};\Omega}\right)\|\mathbf{v}_h\|_{1,\Omega} \\ &+ c_2 S_c\nu_m^{-1}t\|\mathbf{b}-\mathbf{b}_h\|_{3,\mathrm{div};\Omega}\|\boldsymbol{\phi}_h\|_{\mathrm{curl}_{\frac{3}{2}};\Omega}, \end{aligned}$$

which implies that

$$|\mathcal{A}^h_{\mathbf{u}_h,\mathbf{b}_h}(\boldsymbol{\sigma},\boldsymbol{\tau}_h) - \mathcal{A}^h_{\mathbf{u},\mathbf{b}}(\boldsymbol{\sigma},\boldsymbol{\tau}_h)| \le \Lambda\left(t + 2S_c\nu_m^{-1}t^2 + 2S_c\nu_m^{-1}t\right)\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathcal{H}}\|\boldsymbol{\tau}_h\|_{\mathcal{H}}, \tag{5.7}$$

for all $\boldsymbol{\tau}_h = (\mathbf{v}_h,\boldsymbol{\phi}_h,\mathbf{d}_h) \in \mathcal{H}_h$, with $\Lambda > 0$ independent of h and the physical parameters. In turn, using (5.5), from (4.6), we deduce that

$$|\mathcal{A}_{\mathbf{u}_h,\mathbf{b}_h}(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\tau}_h)| \le \mathcal{C}_1\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}}\|\boldsymbol{\tau}_h\|_{\mathcal{H}}, \tag{5.8}$$

whereas from (3.6) we get

$$|\mathcal{B}(\boldsymbol{\tau}_h,\boldsymbol{\xi}_{\mathbf{p}})| \le \|\boldsymbol{\tau}_h\|_{\mathcal{H}}\|\boldsymbol{\xi}_{\mathbf{p}}\|_{\mathcal{Q}}. \tag{5.9}$$

In this way, from (5.4), (5.7) and (5.8) and (5.9), we obtain

$$\begin{aligned} |\mathcal{L}_1(\boldsymbol{\tau}_h)| &\le \Lambda t\left(1 + 2S_c\nu_m^{-1}t + 2S_c\nu_m^{-1}\right)\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}}\|\boldsymbol{\tau}_h\|_{\mathcal{H}} + \mathcal{C}_2(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} + \|\boldsymbol{\xi}_{\mathbf{p}}\|_{\mathcal{Q}})\|\boldsymbol{\tau}_h\|_{\mathcal{H}}, \\ |\mathcal{L}_2(\mathbf{q}_h)| &= |\mathcal{B}(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\mathbf{q}_h)| \le \|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}}\|\mathbf{q}_h\|_{\mathcal{Q}} \le \|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}}\|\mathbf{q}_h\|_{\mathcal{Q}}, \end{aligned} \tag{5.10}$$

with $\mathcal{C}_2 > 0$ a positive constant that depends on ν, ν_m, S_c and t.

Having derived the corresponding estimates for $\mathcal{L}_1$ and $\mathcal{L}_2$, now we proceed to bound $\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}}$ and $\|\boldsymbol{\chi}_{\mathbf{p}}\|_{\mathcal{Q}}$. We start by noticing that from the first estimate in (5.6) and (5.10), we easily deduce that

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} \leq \Lambda t\left(1 + 2S_c\nu_m^{-1}t \,+\, 2S_c\nu_m^{-1}\right)(\widehat{\delta}_1 + \widehat{\delta}_2 t + \widehat{\delta}_3 t^2)\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} \,+\, \mathcal{C}_3(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} + \|\boldsymbol{\xi}_{\mathbf{p}}\|_{\mathcal{Q}}),$$

which together with assumption (5.2) implies

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} \leq 2\mathcal{C}_3(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} + \|\boldsymbol{\xi}_{\mathbf{p}}\|_{\mathcal{Q}}), \tag{5.11}$$

with $\mathcal{C}_3 > 0$ independent of h. Finally, from (5.10), (5.11) and the second estimate in (5.6), we obtain

$$\|\boldsymbol{\chi}_{\mathbf{p}}\|_{\mathcal{Q}} \leq \mathcal{C}_4(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} + \|\boldsymbol{\xi}_{\mathbf{p}}\|_{\mathcal{Q}}). \tag{5.12}$$

We conclude the proof by observing that the desired result follows from (5.11), (5.12), the triangle inequality and the fact that $(\widehat{\boldsymbol{\tau}}_h, \widehat{\mathbf{q}}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$ is arbitrary. □

We end this section by providing the theoretical rate of convergence for the numerical method studied in this work. We begin by recalling the approximation properties of the numerical spaces introduced in Section 4.1:

$$\inf_{\mathbf{v}_h \in \mathbf{H}_{h,0}} \|\mathbf{v} - \mathbf{v}_h\|_{1,\Omega} \leq Ch\|\mathbf{v}\|_{2,\Omega}, \quad \forall\, \mathbf{v} \in \mathbf{H}^2(\Omega), \tag{5.13}$$

$$\inf_{\boldsymbol{\phi}_h \in \times\boldsymbol{\Phi}_h} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{\mathrm{curl}_{\frac{3}{2}};\Omega} \leq Ch\left(\|\boldsymbol{\phi}\|_{2,\Omega} + \|\mathrm{curl}\,\boldsymbol{\phi}\|_{\mathbf{W}^{1,\frac{3}{2}}(\Omega)}\right), \quad \forall\, \boldsymbol{\phi} \in \mathbf{H}^2(\Omega), \tag{5.14}$$

$$\text{such that} \quad \mathrm{curl}\,\boldsymbol{\phi} \in \mathbf{W}^{1,\frac{3}{2}}(\Omega),$$

$$\inf_{\mathbf{d}_h \in \mathbf{C}_h} \|\mathbf{d} - \mathbf{d}_h\|_{3,\mathrm{div};\Omega} \leq Ch\left(\|\mathbf{d}\|_{\mathbf{W}^{1,3}(\Omega)} + \|\mathrm{div}\,\mathbf{d}\|_{1,\Omega}\right), \quad \forall\, \mathbf{d} \in \mathbf{W}^{1,3}(\Omega), \tag{5.15}$$

$$\text{such that } \mathrm{div}\,\mathbf{d} \in \mathrm{H}^1(\Omega),$$

$$\inf_{q_h \in Q_{h,0}} \|q - q_h\|_{0,\Omega} \leq Ch\|q\|_{1,\Omega}, \quad q \in H^1(\Omega), \tag{5.16}$$

$$\inf_{s_h \in S_h} \|s - s_h\|_{0,\Omega} \leq Ch\|s\|_{1,\Omega}, \quad \forall\, s \in H^1(\Omega). \tag{5.17}$$

For (5.13), (5.16) and (5.17) we refer the reader to [13, Proposition 1.134, Section 1.6.3]. In turn, (5.14) and (5.15) can be easily deduced from [14, Section 16.2] and [5, Lemma 4.1], respectively.

The following result establishes the theoretical rate of convergence associated to the Galerkin scheme (4.2).

Theorem 5.2 *Assume that the hypotheses of Theorem 5.1 hold and let* $(\boldsymbol{\sigma}, \mathbf{p}) \in \mathcal{H} \times \mathcal{Q}$ *and* $(\boldsymbol{\sigma}_h, \mathbf{p}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$ *be the unique solutions of problems* (2.10) *and* (4.2), *respectively. Assume* *further that* $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $\boldsymbol{\varepsilon} \in \mathbf{H}^2(\Omega)$, *with* $\operatorname{curl} \boldsymbol{\varepsilon} \in \mathbf{W}^{1,3/2}(\Omega)$, $\mathbf{b} \in \mathbf{W}^{1,3}(\Omega)$, *with* $\operatorname{div} \mathbf{b} \in \mathrm{H}^1(\Omega)$ *and* p, $r \in H^1(\Omega)$. *Then there exists* $C > 0$, *independent of* h, *such that*

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathcal{H}} + \|\mathbf{e}_{\mathbf{p}}\|_{\mathcal{Q}} \leq Ch\Big(\|\mathbf{u}\|_{2,\Omega} + \|\boldsymbol{\varepsilon}\|_{2,\Omega} + \|\operatorname{curl} \boldsymbol{\varepsilon}\|_{\mathbf{W}^{1,3/2}(\Omega)} + \|\mathbf{b}\|_{\mathbf{W}^{1,3}(\Omega)} + \|\operatorname{div} \mathbf{b}\|_{1,\Omega} + \|p\|_{1,\Omega} + \|r\|_{1,\Omega}\Big).$$

Proof. The result is a straightforward application of Theorem 5.1 and estimates (5.13)–(5.17). □

6 Numerical results

In this section, we present computed errors and orders of convergence for a three-dimensional MHD problem (2.4) with a smooth solution. Our goal is to confirm the convergence rates in Theorem 5.2. Our implementation is based on the *FreeFem++* finite element library (see [21]), in conjunction with the direct linear solver UMFPACK (see [12]).

In our test, the computational domain is taken as $\Omega = (0,1)^3$, and we consider a sequence of uniformly refined tetrahedral meshes $\{\mathcal{T}_h\}_{h>0}$ of mesh size reported in Table 6.1. We take $\nu = \nu_m = S_c = 1$, and prescribe boundary data and additional right-hand sides so that the test solution is given by the smooth functions:

$$\mathbf{u}(x,y,z) = \mathbf{b}(x,y,z) := \operatorname{curl}\left(\sin^2(\pi x)\sin^2(\pi y)\sin^2(\pi z)(1,1,1)^t\right),$$

$$p(x,y,z) = yz\sin(\pi x) \quad \text{and} \quad r(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z).$$

On each mesh, we simply take $(\mathbf{w}_h, \mathbf{c}_h) = (\mathbf{0}, \mathbf{0})$ as initial guess and solve iteratively the linearized problem (4.5) until the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, i.e.,

$$\frac{|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m|}{|\mathbf{coeff}^{m+1}|} \leq tol,$$

where *tol*= 1e-6 and $|\cdot|$ is the standard euclidean norm $\mathbb{R}^N$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathcal{H}_h$ and $\mathcal{Q}_h$.

We now introduce some additional notations. The individual errors are denoted as:

$$\mathsf{e}_{\mathbf{u}} := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}, \quad \mathsf{e}_{\boldsymbol{\varepsilon}} := \|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h\|_{\operatorname{curl}_{\frac{3}{2}};\Omega}, \quad \mathsf{e}_{\mathbf{b}} := \|\mathbf{b} - \mathbf{b}_h\|_{3,\operatorname{div};\Omega},$$
$$\mathsf{e}_p := \|p - p_h\|_{0,\Omega}, \quad \mathsf{e}_r := \|r - r_h\|_{0,\Omega}.$$

and we let $R(\mathbf{u})$, $R(\boldsymbol{\varepsilon})$, $R(\mathbf{b})$, $R(p)$ and $R(r)$ be the experimental rates of convergence given by

$$R(\mathbf{u}) := \frac{\log(\mathsf{e}_{\mathbf{u}}/\mathsf{e}'_{\mathbf{u}})}{\log(h/h')}, \quad R(\boldsymbol{\varepsilon}) := \frac{\log(\mathsf{e}_{\boldsymbol{\varepsilon}}/\mathsf{e}'_{\boldsymbol{\varepsilon}})}{\log(h/h')}, \quad R(\mathbf{b}) := \frac{\log(\mathsf{e}_{\mathbf{b}}/\mathsf{e}'_{\mathbf{b}})}{\log(h/h')}$$
$$R(p) := \frac{\log(\mathsf{e}_p/\mathsf{e}'_p)}{\log(h/h')}, \quad R(r) := \frac{\log(\mathsf{e}_r/\mathsf{e}'_r)}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with errors e and e'.

The computed errors and convergence rates for all the unknowns are listed in Table 6.1. From the table, it can be seen that the $\mathbf{H}^1$-norm errors for $\mathbf{u}$, the $\mathbf{H}^3(\mathrm{div})$-norm errors for $\mathbf{b}$, and the $\mathbf{H}(\mathrm{curl}_{\frac{3}{2}})$-norm errors for $\boldsymbol{\varepsilon}$ converge at an order of $O(h)$, in agreement with Theorem 5.2. A similar behavior is observed for the Lagrange multiplier r, where for coarser meshes the order of convergence is higher and quickly approaches $O(h)$. In Table 6.1 we also observe that for the pressure, optimal rates of order $O(h^{\frac{3}{2}})$ are observed (although this is not corroborated by our theoretical results). In [8], the same phenomenon has been observed for the Mini-element approximation of the Stokes problem. In addition, the property that the approximate magnetic fields are exactly divergence-free is verified by evaluating $\|\mathrm{div}\,\mathbf{b}_h\|_{l^\infty}$ for each case. Finally, for the sake of completeness, Table 6.2 reports the number of degrees of freedom used on each mesh and the number of iterations required for the fixed-point scheme to converge.

Fluid variables

h	$\mathsf{e}(\mathbf{u})$	$R(\mathbf{u})$	$\mathsf{e}(p)$	$R(p)$
0,1101	8,4220	–	11,5698	–
0,0550	4,1758	1,0121	3,8657	1,5816
0,0367	2,7678	1,0143	2,0534	1,5603
0,0275	2,0690	1,0115	1,3000	1,5890
0,0220	1,6518	1,0093	0,9148	1,5746
0,0183	1,3745	1,0078	0,6881	1,5619

Magnetic variables

$\mathsf{e}(\boldsymbol{\varepsilon})$	$r(\boldsymbol{\varepsilon})$	$\mathsf{e}(\mathbf{b})$	$R(\mathbf{b})$	$\mathsf{e}(r)$	$R(r)$	$\|\mathrm{div}\,\mathbf{b}\|_{l^\infty}$
69,1574	–	0,8011	–	0,4706	–	3,1974e-14
35,9918	0,9422	0,4058	0,9813	0,0953	2,3038	6,3949e-14
24,1753	0,9815	0,2715	0,9914	0,0430	1,9622	1,0658e-13
18,1795	0,9908	0,2039	0,9954	0,0267	1,6514	1,7053e-13
14,5617	0,9944	0,1632	0,9972	0,0194	1,4511	2,1316e-13
12,1428	0,9963	0,1360	0,9981	0,0152	1,3250	2,5580e-13

Table 6.1: Mesh size, errors, rates of convergence and l^∞–norm of $\mathrm{div}\,\mathbf{b}$ for the mixed approximation of the MHD problem.

N	6.629	49.854	164.929	387.104	751.629	1.293.754
# Iterations	6	8	8	7	7	8

Table 6.2: Degrees of freedom and corresponding number of iterations for the fixed-point method

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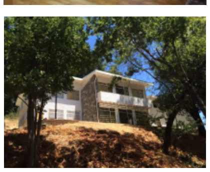

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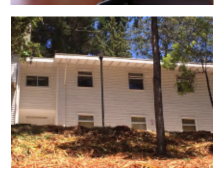
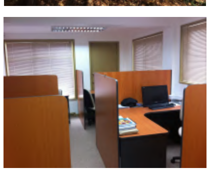

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