UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en
Ingeniería Matemática (CI^{2}MA)

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PREPRINT 2024-22

SERIE DE PRE-PUBLICACIONES

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Abstract

We perform intensive computations of Generalised Langton's Ants, discovering rules with a big number of highways. We depict the structure of some of them, formally proving that the number of highways which are possible for a given rule does not need to be bounded, moreover it can be infinite. Our simulations show that for some rules different highways appear with very unequal frecuencies, in some cases in a ratio of 2/1 000 000 000, suggesting that those highways that appear as the only possible asymptotic behaviours of some rules, might be accompanied by several very infrequent highways which are very hard to find.

Keywords: Langton's ant, Turing machines, emergent behaviour

1 Introduction

Introduced in the 80's independently by different researchers [1, 2, 3], the automaton mostly known as *Langton's ant* remains intriguing. It is cited in vulgarisation media as a paradigm of 'emergent phenomena', 'unpredictability', 'abnormal diffusion' and as evidence of the impossibility of a 'theory of everything'.

Its *emergent phenomenon* consists in a pattern that makes the ant to move in such a way that, after 104 steps, the ant is shifted by (-2,2), and the same pattern appears again in the configuration, also shifted. Given this, the ant movements will repeat again and again, producing a periodic behaviour known as the *highway* (see Figure 2(a)). In the terminology of [4], this behaviour is due to the presence of a periodic point in the *Moving Tape Model*, with the particularity that the background is homogeneous in the propagating direction.

The intriguing fact is that this pattern spontaneously seems to appear on every simulation over initially finite configurations. The transient phase varies mostly from 1000 to 100,000 iterations for initial configurations with only one black cell in a radius of 3 cells around the ant position; and, until now, no simulation suggests the conjecture might be false.

Several generalisations have been proposed: with more movements [5], with more states [6], on other grids [7], and on other dimensions [8]. The generalization that seems to better preserve the Langton's ant properties is the one we call *generalised ants* [6]. In this class, emergent behaviours are also observed, with different periods or different shapes. In addition, there are rules where no highway or any similar asymptotic behaviour arrives for some initial finite configurations.

Motivated by the fact that computational simulations can show only a very restricted part of the whole (and infinite) space of possibilities, and that, in highly unpredictable systems like this, strange phenomena may appear only on extremely big initial configurations, we started a series of high performance simulations, looking for new behaviours that could evidence this limitation of the computational tools.

Section 2 introduces the model and makes a journey over the main theoretical results about generalised ants since their discovery; it also formally defines the concept of "*highway*". Section 3 summarises the results given by our simulations. In Section 4, we establish our first theoretical result: *there is an infinite family with a growing set of highways*; while our second result is demonstrated in Section 5: *there is a generalised ant with an infinite set of highways.* We leave some reflections and proposals in Section 6.

2 Generalised ants

Langton's ant is an agent, provided with an orientation in the plane $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$ that moves and turns either to the right or to the left depending on the colour of its underlying cell, which it flips after leaving the cell. Cells can have only one of two colours, *white* or *black* (0 or 1). Over a white cell, the ant turns to the left, and, over a black cell, the ant turns to the right. Generalised ants have more colours. Their movement rule is given by the meaning of each colour, and this one changes following a simple increasing cycle.

Definition 1. *[6] A* generalised ant *with rule word $w \in \{L, R\}^+$ is an automaton that moves over the grid $\mathbb{Z}^2$, with internal state set $Q = \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$, representing the four unitary vectors of $\mathbb{Z}^2$, for example $\leftarrow = (-1, 0)$, and alphabet $A = \{0, 1, ..., |w| - 1\} = \mathbb{Z}_{|w|}$. A configuration is an element from $A^{\mathbb{Z}^2} \times \mathbb{Z}^2 \times Q$, that represents:*

- *an assignment of symbols from A to each cell in $\mathbb{Z}^2$ called* picture*,*
- *a marked cell representing the* position *of the ant, and*
- *the current* state *of the ant.*

For a given configuration $(C, (i, j), d) \in A^{\mathbb{Z}^2} \times \mathbb{Z}^2 \times Q$, the global transition function T_w is defined by $T_w(C, (i, j), d) = (C', (i', j'), d')$, where:

- $C'(k, l) = \begin{cases} C(k, l) + 1 \mod |w| & \text{if } (k, l) = (i, j) \\ C(k, l) & \text{otherwise} \end{cases}$,
- *d' is a rotation of d by $90°$ clockwise if $w_{C(i,j)} = R$, or counterclockwise if $w_{C(i,j)} = L$,*

- $(i', j') = (i, j) + d'$.

We call *trace* of ant w from configuration $(C, (i, j), d)$ the sequence of symbols that the ant encounters when evolving from configuration $(C, (i, j), d)$. A trace contains all the necessary information to recover the initial picture, up to rotations, at least on the set of visited cells [9].

We assume that the rule word has at least one L and one R. Otherwise, we call it *trivial*, because in that case its behaviour will simply consist in constantly turning in a fixed direction.

An ant simulator is provided at `lutfalla.fr/ant/`. Most of the results we present in this paper are illustrated there.

Very few theoretical results exist about this system. The first remark that should be done is that the dynamics is reversible, that is, the transition function T_w is bijective, thus no information is lost through time. After this, the most important result about ants establishes that: *independently from the initial configuration, the ant orbit will always be unbounded.* This result was formally proved by Bunimovitch and Troubetzkoy in [5] for the rule LR, but it is easy to see that it is also true for the whole family of non-trivial ants. The proof is very simple but clever, and it is based on a very strong property of this system: if the ant starts with an horizontal direction over a cell (i, j), then it will always be with horizontal direction over every cell (i', j') such that $i + j = i' + j' \mod 2$, and with vertical direction otherwise (Figure 1). This is because generalised ants always alternate between vertical and horizontal directions, and also the parity of the cell where they land is always alternating between odd and even, which couples these two situations.

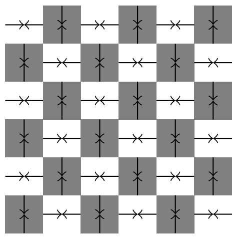
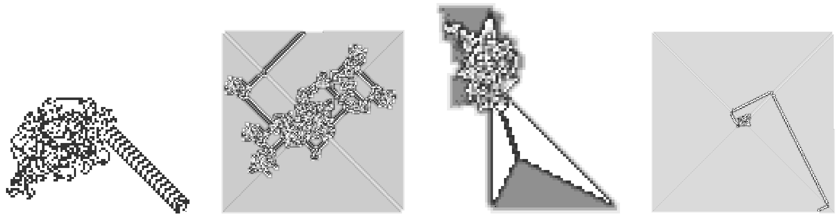


Figure 1: Checkerboard distribution of ant entrance to cells.

The most important consequence of this fact is that each cell has only two fixed entering sides: the ant can only enter a horizontal cell from east or west, and it can only enter a vertical cell from north or south. Then, if an ant trajectory is finite, the set of visited cells should contain corners, that is, cells with only two adjacent cells inside the set, and these forming an angle of $\pi/2$. Independently on the orientation of this cell, horizontal or vertical, it has only one entering cell inside the set and also only one output, which is either to the left or to the right of the first, This imply that the cell can be only visited finitely many times, since the rule word contains at least one L and one R. But this is impossible, because the system is reversible, thus any orbit on a finite set is periodic, and the corner must be visited infinitely many times.

As a corollary, if in a given trajectory we consider the set of cells which are visited infinitely many times, we know that this set (if not empty) cannot have any corner.

Up to our knowledge, the class of generalised ants was introduced for the first time in [6], where different behaviours were presented. Since then, other researchers started making simulations, finding quite surprising behaviours [10]. Figure 2 shows simulations of three nice rules taken from `mathtician.weebly.com/langtons-ant.html`.



(a) (b) (c) (d)

Figure 2: Zoology of observed emergent behaviours of generalised ants: (a) rule *LR*, the original rule; (b) rule *LRRRRRLLR* builds a textured square over which highways are repetitively born and crash; (c) rule *LLRRRLRRRLLL* builds a growing triangle that travels on the plane; (d) rule *LRRRRLLLRRRL* builds a square with a logarithmic spiral inside.

Another beautiful result is proved in [11]: *All non-trivial rules whose code word w belongs to* $\{LL, RR\}^+$ *will produce an evolution composed by an infinite sequence of closed trajectories that always comes back to the initial cell if the initial configuration satisfy the appropriate conditions* [11]. This result has two consequences: on the initially white configuration the ant evolution produces patterns with bilateral symmetry, as butterflies (Figure 3); and also, over this set of initial configurations, no *highway* will ever appear.

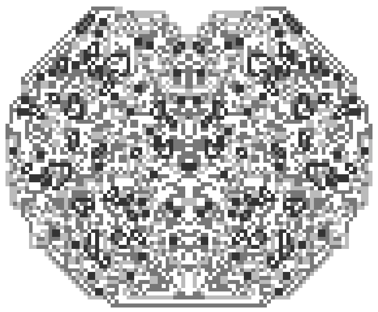


Figure 3: Configuration at iteration 817888 of rule LLRR starting with the initial configuration where all the cells are white (0).

The last known result about this class shows that predicting the trajectory of a non-trivial generalised ant is P-complete and some predictions are undecidable. More precisely, any non-trivial ant rule can compute logical circuits if the appropriate initial configuration is given. Moreover, there exists a periodical initial configuration such that the dynamics of the ant can simulate a universal Turing machine over any finite input, if the appropriate finite perturbing pattern is applied [12]. This result somehow explains why the ant analysis is so hard.

In this paper we are interested in highways, that is, emergent patterns that make the ant to follow a periodical movement that will propagate over an homogeneous background. In order to formalise these phenomena, we need to introduce some notations and concepts.

We say that a configuration $(C,(i,j),d) \in A^{\mathbb{Z}^2} \times \mathbb{Z}^2 \times Q$ is *finite* if the set of cells with a symbol different than 0 is finite: $|\{(a,b) \;:\; C(a,b) \neq 0\}| < \infty$, in this case we also say that the picture C is finite. The picture full of 0s is called *white*.

A *pattern* is an assignment P of symbols to a finite set of cells S, that is, a function $P : S \to A$ with $S \subseteq \mathbb{Z}^2$. The set S is the *support of* P, and it is denoted by $\mathrm{supp}(P)$. A restriction of a picture C to a finite set S is *the pattern of* C *on* S.

The transition function T_w of a generalised ant can be extended to patterns if also a position and direction inside the pattern support is given, that is: T_w may be applied to a tuple $(P,(i,j),d)$ if and only if $(i,j) \in \mathrm{supp}(P)$.

Finally, patterns, pictures and configurations can be shifted by a vector $(a,b) \in \mathbb{Z}^2$. The shift function $\sigma^{(a,b)}$ is defined on pictures, configurations, sets, and patterns by:

- $\sigma^{(a,b)}(C)_{(i,j)} = C_{(i-a,j-b)}$,
- $\sigma^{(a,b)}(C,(i,j),d) = (\sigma^{(a,b)}(C),(i+a,i+j),d)$,
- $\sigma^{(a,b)}(S) = S + \{(a,b)\}$,
- $\sigma^{(a,b)}(P) : \sigma^{(a,b)}(S) \to A$, $\sigma^{(a,b)}(P)_{(i,j)} = P_{(i-a,j-b)}$.

Definition 2. *A* highway *of period N and drift (a,b) of an ant rule w is represented by a pattern P, an ant position $(i,j) \in \mathrm{supp}(P)$ and an ant state d such that the transition function T_w can be applied N times over it, after which the new pattern $T_w^N(P,(i,j),d)$ has the ant at position $(i,j)+(a,b)$ in state d and the pattern P appears inside $T_w^N(P,(i,j),d)$ but shifted by (a,b), more precisely, $\forall (x,y) \in \mathrm{supp}(P)$:*

$$P(x,y) = \begin{cases} T_w^N(P,(i,j),d)_{(x+a,y+b)} & \textit{if } (x+a,y+b) \in \mathrm{supp}(P) \\ 0 & \textit{otherwise} \end{cases}$$

This definition ensures that if the pattern appears in a configuration where all the cells in direction (a,b) from $\mathrm{supp}(P)$ are in state 0, then the same pattern will appear every N steps and both the ant trace and movements will be N periodic. Behind, the cells in $\mathrm{supp}(P) \setminus \mathrm{supp}(\sigma^{(a,b)}(P))$ will not be visited any more; they will register the *prints* left by the ant on the highway.

Of course, any of the patterns appearing during the evolution of a highway can be used to represent the same highway; we can understand a highway as a global phenomenon, or as a cyclic sequence of patterns. Also, any rotation of a highway is the same highway, but with a rotated drift.

A highway can be characterized in terms of the trace of the ant. In fact, if the trace of an ant that started on a finite configuration is eventually periodic, it is clear that the ant has entered a highway, because every periodic trace comes from a semiinfinite periodic pattern [9], and the only semiinfinite periodic pattern contained in an finite configuration is the white one. We use this characterisation to automatically detect highways in computer simulations.

Example 1. *The ant LR has a highway of period 104 and drift* $(-2, 2)$*. Figure 4 shows one of the patterns P that represent it. The set* $\mathrm{supp}(P) \cap \mathrm{supp}(\sigma^{(2,-2)}(P))$ *is marked in gray. This smaller pattern is the one that we find shifted after 104 steps. The cells whose symbol must be 0 are drawn in light gray. If the background in direction* $(-2, 2)$ *is full of 0s, the ant finds the whole pattern after 104 iterations, shifted by* $(-2, 2)$ *and the history repeats.*

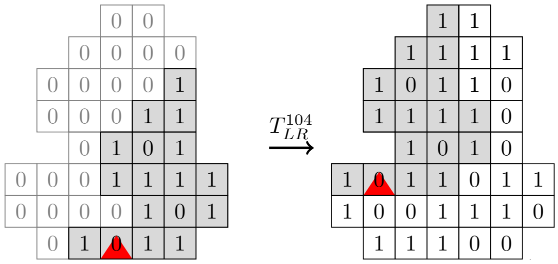


Figure 4: After 104 iterations the ant has moved by the vector $(-2, 2)$, and the configuration which produced that movement is also shifted by $(-2, 2)$. Cells left behind register the print of the highway.

3 Simulations and classification

Our goal is to tackle the highway conjecture. In order to better understand the classical Langton's ant, we decided to study the highways of generalised ants. The first step in this study was a large number of simulations of generalised ants over random perturbations of the 0-uniform configuration. The protocol we used was as follows.

For each computation, an initial pattern P_0 of size 11×11 is taken uniformly at random. We then compute the trace t of the ant for 10^5 steps starting from the white C_0 perturbated at the origin with the pattern P_0 and the ant at the centre of P_0. The trace t is then analysed to determine if a highway was reached. For each ant rule, we repeated the procedure at least 10^8 times.

Table 1 shows the statistics of these simulations for some generalised ants which have remarkable properties. For these statistics we only take in to account the length or period of the highways and not their exact configuration, pattern or trajectory. This means that multiple highways with different trace, drift or trajectory are counted as the same if they have the same period[1].

Note that each highway that was found is an actual highway of the corresponding ant. However, we make no claim that we found all the possible highways for any of these ants, nor that the proportions we observed are close to the actual probability of reaching the corresponding highway from an initial perturbation of size $n \times n$ taken uniformly at random. We also tried the same process with another shape of the inital random pattern (for example, cross shaped support instead of square support). In our experiments, this did not change significantly the behaviours or frequencies.

From these computations, we draw a few observations.

[1] Traces of every highway found is avilable here `https://www.ing-mat.udec.c/GenAnt/`

Table 1: Highway behaviours from simulations.
The table is read as follows: for ant $LLRL$, in our computations from initial random configurations, we reached a highway behaviour in 24.20% of the computations. Amongst those highway behaviour there is a dominant highway of length 384 which was reached in 99.9997% of cases and other rare highways of length 308 (freq. $3.2 \cdot 10^{-4}$%), 380 (freq. $4.0 \cdot 10^{-7}$%), 388 (freq. $4.0 \cdot 10^{-6}$%) and 928 (freq. $2.0 \cdot 10^{-7}$%). This is the only rule with such unequal frequencies, that is why we use exponential notation only for it.

Ant word	**Highways**	**Period and frequency of the highways**			
LR	100.00%	104	(100.00%)		
LLR	100.00%	18	(100.00%)		
$LLLR$	100.00%	52	(99.88%)	156	(0.12%)
$LLLLR$	100.00%	34	(30.10%)	68	(69.90%)
L^5R	100.00%	84	(100.00%)		
L^6R	100.00%	50	(18.63%)	100	(81.37%)
L^7R	100.00%	116	(100.00%)		
L^8R	100.00%	66	(13.40%)	132	(86.60%)
L^9R	100.00%	148	(100.00%)		
$L^{10}R$	100.00%	82	(10.45%)	164	(89.55%)
$L^{11}R$	100.00%	180	(100.00%)		
$L^{12}R$	100.00%	98	(8.58%)	196	(91.42%)
$LLRL$	24.20%	308	($3.2 \cdot 10^{-4}$%)	380	($4.0 \cdot 10^{-7}$%)
		384	(99.9997%)	388	($4.0 \cdot 10^{-6}$%)
		928	($2.0 \cdot 10^{-7}$%)		
$LLRLRL$	1.52%	220	(29.81%)	244	(10.58%)
		268	(5.25%)	284	(0.68%)
		292	(2.46%)	300	(9.44%)
		308	(0.87%)	316	(1.50%)
		324	(8.00%)	332	(0.55%)
		340	(0.98%)	348	(6.65%)
		356	(0.89%)	364	(0.98%)
		372	(4.14%)	380	(0.81%)
		388	(4.19%)	396	(2.93%)
		404	(0.32%)	412	(0.97%)
		420	(2.17%)	428	(0.32%)
		436	(0.81%)	444	(1.48%)
		452	(0.40%)	460	(0.56%)
		468	(0.71%)	1196	(0.64%)
		1268	(0.32%)	1292	(0.11%)
		1340	(0.35%)	1388	(0.03%)
		1412	(0.10%)		

The L^+R ants behave nicely. These ants are the simplest generalised ants and they seem to behave very nicely, though the original LR ant stands out in the behaviour we observe. Our first observation is that a highway behaviour is reached in every computation we ran. We then observe two clear subfamilies :

- the $L^{2k}R$ ants seem to each have a highway of period $16k + 2$ and, for $k \geq 2$, a second highway of period $32k + 4$;

- the $L^{2k+1}R$ ants seem to each have only a highway of period $32k + 20$, but the rule L^3R has two highways, one of them quite infrequent, 3 times longer and with a much bigger support.

Additionally, we observe that for ant $L^{2k}R$, as k grows, the short (called *fundamental*) highway gets rare and the long (called *harmonic*) highway gets more and more dominant. This is partly explained by the fact that the harmonic highway is actually $k - 1$ distinct highways with the same period as detailed in Section 4.

The $LLRL$ ant has an overwhelmingly dominant highway, and very rare highways. From most initial configurations, no highway behaviour seems to be reached quickly (that is, before $\sim 10^5$ steps). Amongst the configuration where a highway is reached there is a dominant highway of length 384 that is reached in 99.9997% of cases and a few very rare other highways of length 308, 380, 388 and 928 which all combined are reached in about 0.0003% of cases. In particular, from the initial 0-uniform grid, the ant reaches the dominant 384 period highway after 256 100 steps. These highways are illustrated on our online simulator[2].

The $LLRLRL$ ant has infinitely many highways. Among the few initial configurations that reached a highway, it is quite remarkable that no highway is too infrequent, as was the case for rule LLRL. More remarkably still, we observe arithmetic progressions in the period of the highways, correlated with some tipicity in their frequencies. Particularly, we find a family[3] of highways of length $220 + 24n$, though we also find lengths of the form $300 + 24n$ and $308+24n$. When looking at the highways themselves, we see that the period does not determine the highway. We find different traces with the same period. Note that, from the initial 0-uniform grid, and even after 10^{10} steps, the $LLRLRL$ ant does not present any highway behaviour (nor any emergent behaviour).

This ant, though remarkable, does not seem to be the only ant with infinitely many highways. Our simulations suggest that the ants with rule words $LLRLRLL$, $LLRLRRLL$, $LLRLRRLL$, $LLRLRRLR$ and $LLRRRLL$ all have infinitely many highways.

4 Multiple highways for simple family of ants

The simplest generalised ants are the L^+R ants, that is, ants whose rule word has n L states and then a R state, the classical Langton's ant being the special case for $n = 1$.

This family of generalised ants already contains many asymptotic highway behaviours, and in this section we are interested in the subfamily $(LL)^+R$, or $L^{2k}R$ ants.

These ants and the highways we present in this section are illustrated on `lutfalla.fr/ant/highway.html?antword=L2KR&k=4&i=2`. Values for `k` (number of (LL)s in the ant word) and `i` (variant of the highway) can be modified in the interface and url.

[2] `lutfalla.fr/ant/highway.html?antword=LLRL`.
[3] `lutfalla.fr/ant/highway.html?antword=LLRLRL`.

Lemma 1 ($L^{2k}R$ cycles)**.** *Let $a, b, c, d \in \mathbb{N}$ such that $2k \geq a \geq b, c, d \geq 0$. Let $b' = b + (2k - a)$, $c' = c + (2k - a)$, $d' = d + (2k - a)$. Let P and P' be the patterns of Figure 5.*

Then $T^{4(2k-a)}_{L^{2k}R}(P) = P'$.

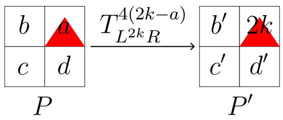


Figure 5: An elementary cycle of the $L^{2k}R$ ant.

Proof. We first remark that the lemma is trivially true when $a = 2k$, because in that case $P = P'$ and $4(2k - a) = 0$, we can thus assume $a < 2k$. As we have $a, b, c, d < 2k$, the ant $L^{2k}R$ turns left and stays in the domain $\mathrm{supp}(P)$ until it reaches a state $2k$. As $a \geq b, c, d$, the first state to become $2k$ and the first $2k$ state to be reached will be in the lower left position (where the state a and the ant initially were). Thus, the ant turns left and does $2k - a$ complete left 4-cycles and reaches configuration P'. □

Denote $\widehat{a.b.c.d} = a.b.c.d.a{+}1.\cdots.2k{-}1.b{+}2k{-}a{-}1.c{+}2k{-}a{-}1.d{+}2k{-}a{-}1$ the trace of the $L^{2k}R$ ant from P to P', .

Lemma 2 ($L^{2k}R$ almost highways)**.** *Let $0 < i < 2k$.*

Let P_i and P'_{2k-i} be the patterns of Figure 6.

We have $T^{24k-8i+2}_{L^{2k}R}(P_i) = P'_{2k-i}$.

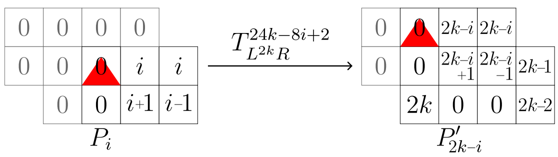


Figure 6: The almost highway of the $L^{2k}R$ ant.

Proof. We decompose the trace of the $L^{2k}R$ ant starting from P_i in three elementary cycles of lengths $8k$, $8k - 4i$, $8k - 4(i + 1)$ and in six individual moves as depicted in Figure 7.

This means that the trace from P_i to P'_{2k-i} is $t_{k,i}$, where

$$t_{k,i} := \widehat{0.0.0.0}.2k.\widehat{i.0.0.0}.2k.\widehat{i{+}1.i{-}1.i.0}.2k.2k.2k{-}i.2k.$$

□

Additionally, we remark that for $i \neq j$ the trace $t_{k,i} \cdot t_{k,i}$ does not contain the factor subword $j + 1.j - 1.j.0$ contained in $t_{k,j}$ so none of these almost-highways is a sub-dynamic of another.

Now we explain how the "almost highways" can be used to construct complete highways.

Theorem 1 (Fundamental and harmonic highway for $L^{2k}R$)**.** *For any integer $k > 1$, the $L^{2k}R$ ant admits at least two different highways :*

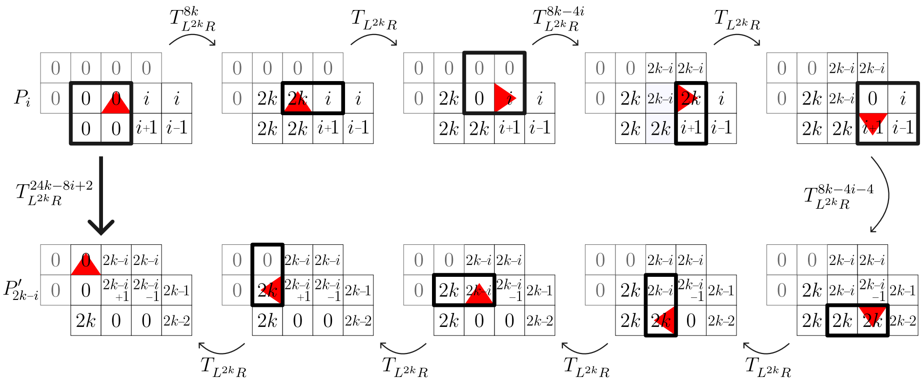


Figure 7: Decomposition of the almost highway for the $L^{2k}R$ ant.
At each step, the domain of the next move or elementary cycle is emphasised with bold boundary.

- *a fundamental highway of period* $16k + 2$ *and drift* $(\pm 1, \pm 1)$,
- *a harmonic highway of period* $32k + 4$ *and drift* $(\pm 2, \pm 2)$.

Proof. Let $k > 1$ and consider the and $L^{2k}R$.

For $n \geq 0$, let $C^n_{k,i}$ be the $\mathbb{Z}^2$ picture defined as

$$C^n_{k,i}(x,y) = \begin{cases} i & \text{if} \quad y = n \wedge x \in \{1-n, 2-n\} \\ i+1 & \text{if} \quad (x,y) = (1-n, n-1) \\ i-1 & \text{if} \quad (x,y) = (2-n, n-1) \\ 2k & \text{if} \quad y = -x-2 \wedge -n \leq x \leq -1 \\ 2k-1 & \text{if} \quad y = -x+2 \wedge 2 \leq x \leq n+1 \\ 2k-2 & \text{if} \quad y = -x+1 \wedge 2 \leq x \leq n+1 \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

In particular, $C^0_{k,k}$ is the 0-uniform picture with the pattern P_k of Lemma 2 at the origin, $C^1_{k,k}$ is the same with pattern P'_k, and in general $C^n_{k,i}$ has pattern P_i at position $(-n, n)$. We remark that P'_k also has pattern P_k but at position $(-1, 1)$.

We first show the fundamental highway. Applying Lemma 2 with $i = k$ we obtain that for any n,

$$T^{16k+2}_{L^{2k}R}(C^n_{k,k}, (-n,n), \uparrow) = (C^{n+1}_{k,k}, (-n-1, n+1), \uparrow)$$

and the trace of this transition is $t_{k,k}$. In other words, $(C^0_{k,k}, (0,0), \uparrow)$ starts a highway of period $16k+2$, drift $(-1, 1)$ and trace $t_{k,k}$.

Now we prove the harmonic highway. Remark that, by applying Lemma 2 with $i = 1$ and $i = 2k-1$ we have for any n both

$$T^{24k-6}_{L^{2k}R}(C^n_{k,1}, (-n,n), \uparrow) = (C^{n+1}_{k,2k-1}, (-n-1, n+1), \uparrow)$$

with trace $t_{k,1}$ and

$$T^{8k+10}_{L^{2k}R}(C^{n+1}_{k,2k-1}, (-n-1, n+1), \uparrow) = (C^{n+2}_{k,1}, (-n-2, n+2), \uparrow),$$

with trace $t_{k,2k-1}$.

Combining the two we obtain

$$T^{32k+4}_{L^{2k}R}(C^n_{k,1}, (-n, n), \uparrow) = (C^{n+2}_{k,1}, (-n-2, n+2), \uparrow).$$

In other words, $(C^0_{k,1}, (0, 0), \uparrow)$ starts a highway of period $32k + 4$, drift $(-2, 2)$ and trace $t_{k,1} \cdot t_{k,2k-1}$.

The periodic repetition $(t_{k,1} \cdot t_{k,2k-1})^\omega$ has $32k + 4$ as a smallest period because the factor subword 2.0.1.0 only appears once in $t_{k,1}$ and does not appear in $t_{k,2k-1}$; so $32k + 4$ is indeed the highway period of this dynamics. □

But what is done for $i = 1$ can be done for every $0 < i < k$, thus we have a family of harmonic highways.

Theorem 2 (Variants of the harmonic highway for $L^{2k}R$)**.** *For any integer $k > 1$, there are at least $k - 1$ distinct harmonic highways of the $L^{2k}R$ ant.*

Proof. Let $k > 1$ and define the $\mathbb{Z}^2$ pictures $C^n_{k,i}$ as in the proof of Theorem 1. For any $0 < i < k$, we have for any n, we apply Lemma 2 for i and $2k - i$ we obtain:

$$T^{24k-8i+2}_{L^{2k}R}(C^n_{k,i}, (-n, n), \uparrow) = (C^{n+1}_{k,2k-i}, (-n-1, n+1), \uparrow)$$

with trace $t_{k,i}$ and

$$T^{8k+8i+2}_{L^{2k}R}(C^{n+1}_{k,2k-i}, (-n-1, n+1), \uparrow) = (C^{n+2}_{k,i}, (-n-2, n+2), \uparrow),$$

with trace $t_{k,2k-i}$. Combining the two we obtain:

$$T^{32k+4}_{L^{2k}R}(C^n_{k,i}, (-n, n), \uparrow) = (C^{n+2}_{k,i}, (-n-2, n+2), \uparrow)$$

with trace $t_{k,i} \cdot t_{k,2k-i}$. In other words, $(C^0_{k,i}, (0, 0), \uparrow)$ starts a highway of period $32k + 4$, drift $(-2, 2)$ and trace $t_{k,i} \cdot t_{k,2k-i}$.

These highways are indeed distinct, as mentioned above, since the trace factor subword $i + 1.i - 1.i.0$ does not appear in $t_{k,j} \cdot t_{k,2k-j}$ for $i \neq j$. □

Remark 1. *It is quite remarkable that, for a given $k > 1$, all of these k highways (the fundamental and the $k - 1$ variants of the harmonic) leave the same print behind consisting of a diagonal of $2k$, a diagonal of $2k - 1$ and a diagonal of $2k - 2$. See Figure 8.*

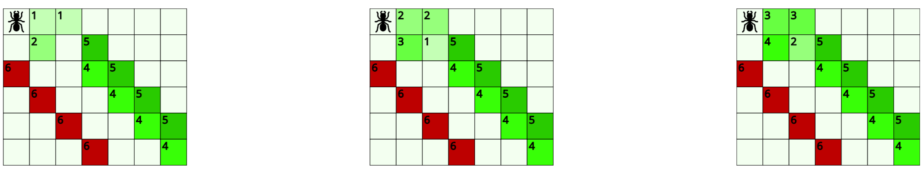


Figure 8: The three highways for the L^6R ant, the rightmost is the fundamental highway.

Remark 2 (LLR)**.** *The LLR ant does not have these harmonic highways of length 36 as there is no i such that $0 < i < 1$. However the fundamental highway of length 18 described above exists.*

Remark 3 ($L^{2k+1}R$ ants)**.** *Analogously, it can be proved that for any* $k > 1$*, the ant with rule word* $L^{2k+1}R$ *has* k *distinct highways of same length* $32k + 20$ *and drift* $(\pm 2, \pm 2)$[4]*.*

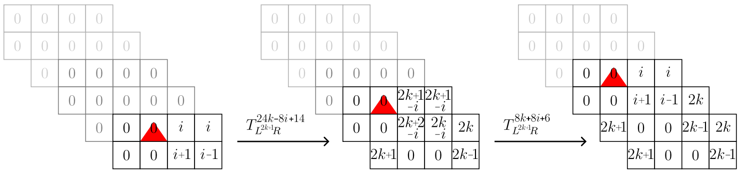


Figure 9: The configuration that starts a variant-i highway of period $32k + 20$ for the ant $L^{2k+1}R$.

5 Infinitely many highways for a single ant

Theorem 3. *The ant LLRLRL has infinitely many highways.*
In particular for any n*, LLRLRL admits a highway of period* $220 + 24n$*.*

Proof. To prove this result we introduce three widgets, two of them in two versions: a main widget M_1 and M_2 and a bounce back widget B_1 and B_2; also a link widget in four versions: L_1, L_2, L_3, and L_4.

We will prove that, for any $n \in \mathbb{N}$, the pattern $c_n = M_1 \cdot L_1^n \cdot B_1$ starts a highway of period $220 + 24n$, and drift $(-2, -2)$. Figure 14 shows a picture of this highway where the widgets can be identified, together with the print left after several periods. The proof is performed by describing the different stages of the ant movement, which can be verified by the reader by hand or by means of any ant simulator, for example `lutfalla.fr/ant/highway.html?antword=LLRLRL`.

Start. The ant start over $c_n = M_1 \cdot L_1^n \cdot B_1$ at the red arrow, as Figure 10 shows. The ant will spend 84 steps inside M_1, to finally exit and enter the first of the series of n L_1s at the position and direction marked by the yellow triangle. It will take 12 steps to transverse each L_1 to finally enter pattern B_1 at the position and direction marked by the cyan arrow, where it will spend only 4 steps. At this point, M_1 has been transformed into M_2 concatenated with a half L_2 and each of the L_1s has been transformed into an L_2, except for the last one, which slightly differs from the others and will be called L_2'.

Rebound 1. Now the ant faces L_2' and a series of $n-1$ L_2s, it takes 4 steps to transverse each of them (see Figure 11). After the ant's visit, this series of n widgets is modified. The column of the first L_2, that was adjacent to B_1 in the previous stage, will fuse with B_1 to become B_2. Starting from the other column of this widget, a new series of n link widgets is formed, which we call L_3.

New start. Now the ant is on $M_2 \cdot L_3^n \cdot B_2$ at the position marked by the red arrow (see Figure 12). The ant will spend 100 steps inside M_2, to finally exit it to enter the series of n L_3s at the position and direction marked by the yellow triangle. It will take only 4 steps to transverse each L_3 to finally enter pattern B_2 at the position and direction marked by the cyan arrow, where it will spend 30 steps. At this point, M_2 has been transformed into M_1 concatenated with a

[4]In `lutfalla.fr/ant/highway.html?antword=L2K1R&k=3&i=2` their dynamics can be seen for any `k`, and its initial pattern is shown in Figure 9

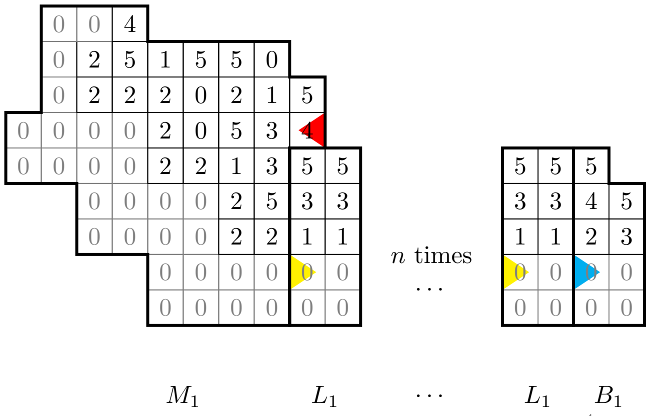


Figure 10: Patterns M_1, B_1, and L_1. The red triangle marks the starting ant's position and direction. The other 3 triangles mark the entering positions of the ant to the respective widget.

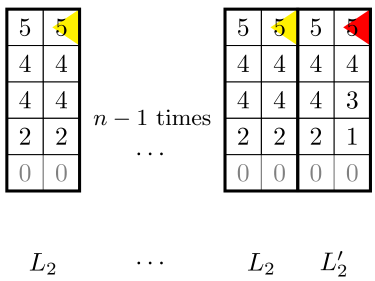


Figure 11: Pattern L_2. The ant will transverse each block in 4 steps, transforming them into a series of L_3s.

half of L_4, and each of the L_3s has been transformed into an L_4, except for the last one, which slightly differs from the others and will be called L'_4.

Rebound 2. Now the ant faces L'_4 and a series of $n-1$ L_4s. Again, it takes 4 steps to transverse each of them (see Figure 13). After the ant's visit, this series of n widgets is modified. The column of the first L_4, that was adjacent to B_2 in the previous stage, together with two cells of B_2 become B_1. Starting from the other column of this widget, the series of n L_1 appears.

After these four stages, c_n appears again, but shifted by (-2,-2), covering the gray 0s, and if enough 0s are placed in the background, a new cycle of $220+24n$ steps starts.

□

6 Discussion

The highway of period 18 and drift $(-1,1)$ of the rule LLR is the shortest known within the class of generalised ants. If any tool exists for proving the highway

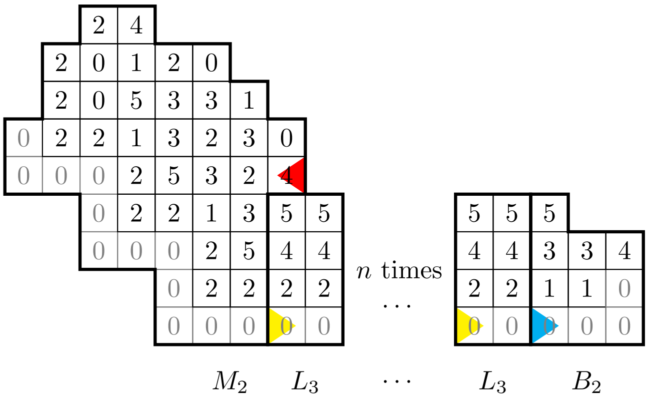


Figure 12: Patterns M_2, B_2, and L_3. The red triangle marks the starting ant's position and direction of this stage. The other 3 triangles mark the entering positions of the ant to the respective widget.

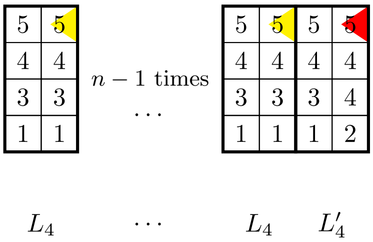


Figure 13: Pattern L_4. The ant will transverse each block in 4 steps, transforming it into L_1.

conjecture, it should work with this highway. Its structure can be visualised in Figure 15. At each period, the ant visits 5 new cells (in gray). The smallest pattern producing the highway is the one on the top of Figure 15. It is composed by only 3 cells, and we will call it *the seed*.

Since the drift of this highway always belong to $(\pm 1, \pm 1)$, and the seed is like a stairway step in its orthogonal direction, it is pertinent to study patterns around the ant with the shape of an Aztec diamond, as in Figure 16. Thanks to the property of *vertical* and *horizontal* cells, recalled in Section 2, the boundary of the Aztec diamond is such that the ant can only exit it in a position and orientation that leaves a set like seed support behind (maybe rotated), and if it exits the seed, the highway will start.

There are 27 different patterns with the support of the seed, but only 6 of them allow the ant to transverse the pattern and exit the Aztec diamond. The seed is one of these 6. The ant always exits the diamond. At each exit, it has 1 chance over 6 to start the highway. Nevertheless, the configurations on the boundary of the Aztec diamond are not random. They are completely determined by the inner configuration, and probability arguments like this one cannot be applied.

Ants with rule word $L^{2k+1}R$ also have highways engendered by small seeds

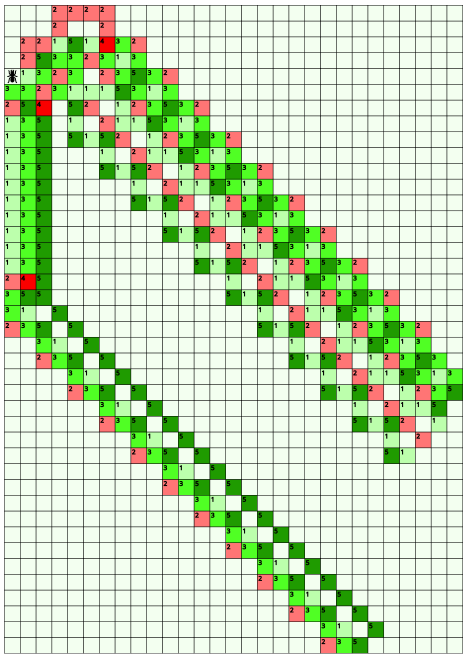


Figure 14: The configuration produced by a highway of period $220+5\cdot 24$ for the $LLRLRL$ ant after $11\cdot(220+5\cdot 24)$ iterations. We can identify the L widgets acting as a corridor between the two “walls” M, on top, and B, in the opposite extreme of the series of Ls. The two diagonals on the picture are the prints left by widgets M and B.
Note that left-turning (resp. right-turning) states are emphasized in green (resp. red).

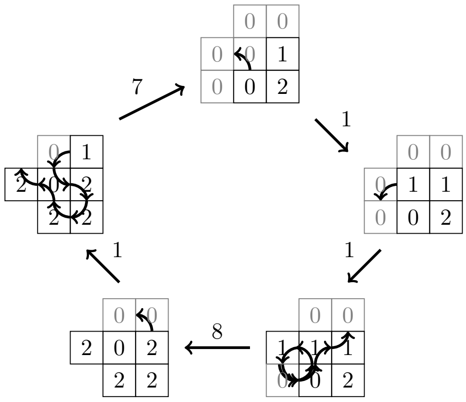


Figure 15: The basic cycle that is repeated during the only known highway of rule LLR. Draws contain only the state of the cells that will be visited during the highway. At each step, five new cells are visited, in gray.

like the one of ant LLR, but the rule $LLLR$ has two highways, one of them quite infrequent, 3 times longer and with a much bigger support. One can ask whether other $L^{2k+1}R$ ants have another lager and very infrequent highway as L^3R does.

The case of $LLRL$ ant is very suggesting. There are ant rules with a highway which exists but is extremely rare, this conforts us in the idea that it just might be possible that the LR ant has other asymptotic behaviours but they are so extremely rare that were never encountered in computations.

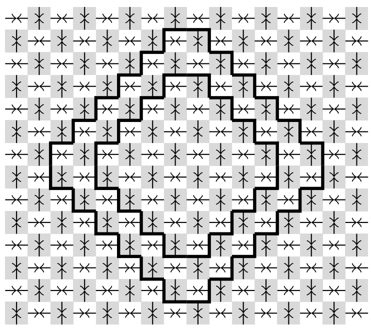


Figure 16: The Aztec diamond with vertical and horizontal cells.

Acknowledgement. A. Gajardo acknowledges the *Institut de Mathématiques de Marseille* for partially funding her research from February to July 2024.

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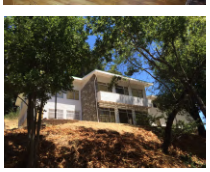
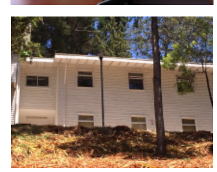

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