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A twofold perturbed saddle point-based fully mixed finite element method for the coupled Brinkman–Forchheimer/Darcy problem*

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Abstract

We introduce and analyze a new mixed finite element method for the stationary model arising from the coupling of the Brinkman–Forchheimer and Darcy equations. While the original unknowns are given by the velocities and pressures of the more and less permeable porous media, our approach is based on the introduction of the Brinkman–Forchheimer pseudostress as a further variable, which allows us to eliminate the respective pressure. Needless to say, the latter can be recovered later on by a postprocessing formula that depends only on the former. Next, aiming to perform a proper treatment of the transmission conditions, the traces on the interface, of both the Brinkman–Forchheimer velocity and the Darcy pressure, are also incorporated as auxiliary unknowns. Thus, the resulting fully-mixed variational formulation can be seen as a nonlinear perturbation of, in turn, a twofold perturbed saddle point operator equation. Additionally, the diagonal feature of some of the bilinear forms involved, facilitates the proof of their corresponding inf-sup conditions. Then, the fixed-point strategy arising from a linearization of the Forchheimer term, along with suitable abstract results exploiting the aforementioned structure, and the classical Banach theorem, are employed to prove the well-posedness of the continuous and discrete schemes. In particular, Raviart–Thomas and piecewise polynomial subspaces of the lowest degree for the domain unknowns, as well as continuous piecewise linear polynomials for the interface ones, constitute a feasible choice. Optimal error estimates and associated rates of convergence are established. Finally, several numerical results illustrating the good performance of the method and confirming the theoretical findings, are reported.

Key words: Brinkman–Forchheimer problem, Darcy problem, pseudostress-velocity formulation, mixed finite element methods, *a priori* error analysis

Mathematics subject classifications (2010): 65N30, 65N12, 65N15, 74F10, 76D05, 76S05

1 Introduction

The phenomenon of filtration of an incompressible fluid through a non-deformable saturated porous medium with heterogeneous permeability has a wide range of applications, including processes in

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chemical, environmental, and petroleum engineering. For instance, in air filtration systems with multiple layers, where one layer is more permeable than another, the differences in permeability significantly influence the flow through each section. Similarly, in groundwater remediation and oil and gas extraction, the flow can be fast near injection or production wells, especially if the aquifer or reservoir is highly porous. Accurate modeling and simulation of such flows are crucial in these fields to optimize processes, ensure safety, and minimize environmental impact. Mathematical models have been developed to capture different aspects of these flows. In particular, when two distinct regions are present in the porous medium, Darcy's law [18] is applicable in areas of low permeability and Reynolds number, effectively describing fluid motion in these less permeable regions. However, in regions where permeability is higher and flow rates rise, Darcy's law becomes inadequate, and the nonlinear Brinkman–Forchheimer model (see, e.g., [20], [12], [11]) is employed to account for the effects of viscous forces and increased flow rates. Consequently, the combination of these models, along with mass conservation and momentum continuity at the interface between the two regions, leads to the coupled Brinkman–Forchheimer/Darcy problem.

Regarding the literature, and to the best of the authors' knowledge, we begin by mentioning [7] as the first work to propose and analyze the coupled Brinkman–Forchheimer/Darcy model. Specifically, a standard mixed formulation was considered in the Brinkman–Forchheimer region, while a dual-mixed formulation was used in the Darcy region, with the continuity of normal velocities enforced through the introduction of a suitable Lagrange multiplier. For the discretization, Bernardi–Raugel and Raviart–Thomas elements were used for the velocities, piecewise constant elements for the pressures, and continuous piecewise linear elements for the Lagrange multiplier. Similar models have been explored in [36], where the coupling of the Brinkman–Forchheimer, Darcy, and heat equations was proposed to study the continuous dependence of the solution on variations in the heat source and the Forchheimer coefficient.

On the other hand, several papers have been devoted to the design and analysis of numerical schemes for simulating related coupled problems, such as the (Navier–)Stokes/Darcy(–Forchheimer) models (see, e.g., [3], [27], [29], [30], [1], [13], [17], [10], [8], and references therein). In particular, in [29], a fully-mixed finite element method was proposed and analyzed for the Stokes–Darcy coupled problem, where the Fredholm and Babuška–Brezzi theories were employed to derive sufficient conditions for the unique solvability of the resulting continuous and discrete formulations. In [30], an extension of [29] to the coupling of Stokes and nonlinear Darcy models was developed. Both *a priori* and *a posteriori* error analyses were carried out in this work. Subsequently, a fully-mixed finite element method was developed and analyzed for the coupling of the Stokes and Darcy–Forchheimer problems in [1]. This new approach yields non-Hilbertian normed spaces and a twofold saddle point structure for the corresponding operator equation, whose continuous and discrete solvabilities are analyzed using a suitable abstract theory developed for this purpose. We also refer to [17] for the analysis of a conforming mixed finite element method for the Navier–Stokes/Darcy problem. Given that this coupled system is nonlinear (due to the convective term in the free fluid region), the analysis of the continuous problem starts with the linearization of the Oseen problem in the free fluid domain. This simplified model is then studied using the classical Babuška–Brezzi theory, similarly to how it was done for the Stokes–Darcy coupling in [27]. Meanwhile, the coupling of a 2D reservoir model with a 1.5D vertical wellbore model was investigated in [3] using the compressible Navier–Stokes equations coupled with the Darcy–Forchheimer model. In [13], a penalization approach was introduced and analyzed for the Navier–Stokes/Darcy–Forchheimer model in both 2D and 3D domains, motivated by the study of internal ventilation in motorcycle helmets. In this work, the authors considered the velocity and pressure throughout the entire domain as the main unknowns of the system, employing a Galerkin approximation with piecewise quadratic elements for the velocity and linear elements for the pressure.

More recently, in [8], a primal-mixed formulation of the Navier–Stokes/Darcy–Forchheimer system was analyzed using a fixed-point argument and classical results on nonlinear monotone operators.

The goal of the present paper is to develop and analyze a new mixed variational formulation for the model introduced in [7]. Unlike [7] and similarly to [29], [1], this approach considers dual-mixed formulations in both domains. Following the strategy in [29], we introduce the pseudostress tensor as an auxiliary variable and eliminate the Brinkman–Forchheimer pressure unknown using the incompressibility condition. The transmission conditions, which involve mass conservation and momentum continuity, are imposed weakly, leading to the inclusion of additional Lagrange multipliers: the traces of the Brinkman–Forchheimer velocity and the Darcy pressure on the interface. The resulting variational system is formulated within a Banach space framework due to the presence of the Forchheimer nonlinear term and exhibits of both the continuous and discrete formulations using a fixed-point argument, abstract results from [28] and [15], the Banach–Nečas–Babuška theorem, small data assumptions, and the Banach fixed-point theorem. Since the formulation shares a similar structure with those analyzed in [29], our analysis extends or leverages the corresponding results available there, including the continuous and discrete inf-sup conditions. Additionally, by applying an ad hoc Strang-type lemma for Banach spaces, which is a slight variant of its Hilbert space counterpart developed in [22], we derive the corresponding *a priori* error estimates. Finally, using Raviart–Thomas and piecewise polynomial subspaces of the lowest degree for the domain unknowns, along with continuous piecewise linear polynomials for the interface unknowns, we prove that the method converges with optimal rates.

This work is organized as follows. The remainder of this section describes the standard notation and functional spaces used throughout the paper. In Section 2, we introduce the model problem, followed by the derivation of the fully-mixed variational formulation within a Banach space framework and the establishment of the well-posedness of the continuous scheme in Section 3. The corresponding Galerkin system is introduced and analyzed in Section 4, where the discrete analogue of the theory used in the continuous case is applied to prove the existence and uniqueness of the solution. In Section 5, we derive the *a priori* error estimate and establish the corresponding rates of convergence. Finally, the performance of the method is studied in Section 6 with several numerical examples in 2D, including cases with and without manufactured solutions, verifying the aforementioned rates of convergence, as well as illustrating its flexibility to handle spatially varying parameters in complex geometries.

Preliminary notations

Given an arbitrary domain $\mathcal{O} \subset \mathrm{R}^n$, $n \in \{2,3\}$, with polyhedral boundary $\partial\mathcal{O}$, we adopt the standard notation for Lebesgue spaces $\mathrm{L}^t(\mathcal{O})$ and Sobolev spaces $\mathrm{W}^{s,t}(\mathcal{O})$, with $s \in \mathrm{R}$ and $t > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,t;\mathcal{O}}$ and $\|\cdot\|_{s,t;\mathcal{O}}$, respectively. Note that actually $\mathrm{W}^{0,t}(\mathcal{O}) = \mathrm{L}^t(\mathcal{O})$. In turn, when $t = 2$, we simply write $\mathrm{H}^s(\mathcal{O})$ instead of $\mathrm{W}^{s,2}(\mathcal{O})$, and denote the corresponding norm by $\|\cdot\|_{s,\mathcal{O}}$. In particular, when $s = 1$ we let $\mathrm{H}^{1/2}(\partial\mathcal{O})$ be the space of traces of functions of $\mathrm{H}^s(\mathcal{O}) = \mathrm{H}^1(\mathcal{O})$, and $\mathrm{H}^{-1/2}(\partial\mathcal{O})$ stands for its dual. In addition, given any generic scalar functional space S, we let $\mathbf{S}$ and $\mathbb{S}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Also, $|\cdot|$ denotes the Euclidean norm in both R^n and $\mathrm{R}^{n\times n}$, and as usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$. Furthermore, for any vector field $\mathbf{v} = (v_i)_{i=1,n}$, we set the gradient and divergence operators as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n} \qquad \text{and} \qquad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}\,,$$

whereas for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the deviatoric tensor, and the tensor inner product, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\,\zeta_{ij}\,.$$

On the other hand, for each $t \in [1, +\infty)$ we introduce the Banach spaces

$$\begin{aligned} \mathbf{H}(\mathrm{div}_t;\mathcal{O}) &:= \Big\{ \boldsymbol{\eta} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}(\boldsymbol{\eta}) \in \mathrm{L}^t(\mathcal{O}) \Big\}, \quad \text{and} \\ \mathbb{H}(\mathbf{div}_t;\mathcal{O}) &:= \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\mathcal{O}) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\mathcal{O}) \Big\}, \end{aligned} \tag{1.1}$$

equipped with the natural norms

$$\begin{aligned} \|\boldsymbol{\eta}\|_{\mathrm{div}_t;\mathcal{O}} &:= \|\boldsymbol{\eta}\|_{0,\mathcal{O}} + \|\mathrm{div}(\boldsymbol{\eta})\|_{0,t;\mathcal{O}} \quad \forall\, \boldsymbol{\eta} \in \mathbf{H}(\mathrm{div}_t;\mathcal{O})\,, \quad \text{and} \\ \|\boldsymbol{\tau}\|_{\mathbf{div}_t;\mathcal{O}} &:= \|\boldsymbol{\tau}\|_{0,\mathcal{O}} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\mathcal{O}} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\mathcal{O})\,. \end{aligned}$$

We notice that when $t = 2$, we just write $\mathbf{H}(\mathrm{div};\mathcal{O})$, $\|\cdot\|_{\mathrm{div};\mathcal{O}}$, $\mathbb{H}(\mathbf{div};\mathcal{O})$, and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$ instead of $\mathbf{H}(\mathrm{div}_2;\mathcal{O})$, $\|\cdot\|_{\mathrm{div}_2;\mathcal{O}}$, $\mathbb{H}(\mathbf{div}_2;\mathcal{O})$, and $\|\cdot\|_{\mathbf{div}_2;\mathcal{O}}$, respectively. Additionally, we recall that, proceeding as in [24, eq. (1.43), Section 1.3.4] (see also [14, Section 3.1]), one can prove that for $t \in \left\{ \begin{array}{l} (1,+\infty] \text{ in } \mathrm{R}^2\,, \\ [\frac{6}{5},+\infty] \text{ in } \mathrm{R}^3\,, \end{array} \right.$ there holds

$$\langle \boldsymbol{\eta}\cdot\boldsymbol{\nu}, v \rangle = \int_{\mathcal{O}} \Big\{ \boldsymbol{\eta}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\eta}) \Big\} \quad \forall\, (\boldsymbol{\eta}, v) \in \mathbf{H}(\mathrm{div}_t;\mathcal{O}) \times \mathrm{H}^1(\mathcal{O})\,, \tag{1.2}$$

and

$$\langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{v} \rangle = \int_{\mathcal{O}} \Big\{ \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \Big\} \quad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\mathcal{O}) \times \mathbf{H}^1(\mathcal{O})\,, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ denotes in (1.2) (resp. (1.3)) the duality pairing between $\mathrm{H}^{1/2}(\partial\mathcal{O})$ (resp. $\mathbf{H}^{1/2}(\partial\mathcal{O})$) and $\mathrm{H}^{-1/2}(\partial\mathcal{O})$ (resp. $\mathbf{H}^{-1/2}(\partial\mathcal{O})$).

2 The model problem

In order to describe the geometry of the coupled Brinkman–Forchheimer/Darcy model, we let Ω_{B} and Ω_{D} be bounded and simply connected open polyhedral domains in R^n, $n \in \{2,3\}$, such that $\partial\Omega_{\mathrm{B}} \cap \partial\Omega_{\mathrm{D}} = \Sigma \neq \emptyset$ and $\Omega_{\mathrm{B}} \cap \Omega_{\mathrm{D}} = \emptyset$. Then, we let $\Gamma_{\mathrm{B}} := \partial\Omega_{\mathrm{B}} \setminus \overline{\Sigma}$, $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_{\mathrm{B}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ and Ω_{B} (and hence inward to Ω_{D} when seen on Σ). A sketch of a 2D geometry is displayed in Figure 2.1. The mathematical model is defined by two separate groups of equations and by a set of coupling terms. In the more permeable porous medium domain Ω_{B}, the governing equations are those of the Brinkman–Forchheimer problem, which are written in the following pseudostress-velocity-pressure formulation:

$$\begin{aligned} \boldsymbol{\sigma}_{\mathrm{B}} = \mu\,\nabla\mathbf{u}_{\mathrm{B}} - p_{\mathrm{B}}\,\mathbb{I} \quad \text{in} \quad \Omega_{\mathrm{B}}, \quad \mathrm{div}(\mathbf{u}_{\mathrm{B}}) = 0 \quad \text{in} \quad \Omega_{\mathrm{B}}, \\ \mathbf{K}_{\mathrm{B}}^{-1}\mathbf{u}_{\mathrm{B}} + \mathtt{F}\,|\mathbf{u}_{\mathrm{B}}|^{\rho-2}\mathbf{u}_{\mathrm{B}} - \mathbf{div}(\boldsymbol{\sigma}_{\mathrm{B}}) = \mathbf{f}_{\mathrm{B}} \quad \text{in} \quad \Omega_{\mathrm{B}}, \quad \mathbf{u}_{\mathrm{B}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{B}}, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\sigma}_{\mathrm{B}}$ is the pseudostress tensor, $\mathbf{u}_{\mathrm{B}}$ is the fluid velocity, p_{B} is the pressure, μ is the kinematic viscosity of the fluid, $\mathbf{K}_{\mathrm{B}}$ is an invertible symmetric tensor in Ω_{B}, equal to the symmetric permeability tensor scaled by the kinematic viscosity, $\mathtt{F} > 0$ is the Forchheimer coefficient, ρ is a number in $[3, 4]$, and $\mathbf{f}_{\mathrm{B}}$ is a given external force. In turn, in the less permeable porous medium domain Ω_{D}, we consider the Darcy equations to approximate the velocity $\mathbf{u}_{\mathrm{D}}$ and the pressure p_{D}, which read

$$\mathbf{K}_{\mathrm{D}}^{-1}\mathbf{u}_{\mathrm{D}} + \nabla p_{\mathrm{D}} = \mathbf{f}_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \mathrm{div}(\mathbf{u}_{\mathrm{D}}) = g_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\,, \tag{2.2}$$

where $\mathbf{K}_{\mathrm{D}}$ is an invertible symmetric tensor in Ω_{D}, equal to the permeability tensor scaled by the kinematic viscosity, and $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$ and $g_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$ are sources terms. Finally, to couple the Brinkman–Forchheimer and the Darcy models, we proceed as in [7] (see similar approaches in [21, 19, 36]), and consider transmission conditions that impose, respectively, the mass conservation and continuity of momentum across the interface Σ:

$$\mathbf{u}_{\mathrm{B}} \cdot \mathbf{n} \,=\, \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} \quad \text{and} \quad \boldsymbol{\sigma}_{\mathrm{B}}\mathbf{n} = -p_{\mathrm{D}}\mathbf{n} \quad \text{on} \quad \Sigma\,. \tag{2.3}$$

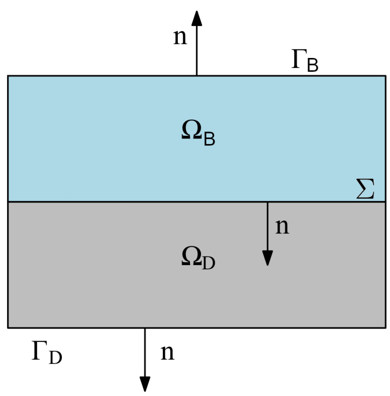

Figure 2.1: Sketch of a 2D geometry of the coupled Brinkman–Forchheimer/Darcy model

Throughout the paper we assume that for each $\star \in \{\mathrm{B}, \mathrm{D}\}$, $\mathbf{K}_\star$, $\mathbf{K}_\star^{-1} \in \mathbb{L}^\infty(\Omega_\star)$ and there exists a constant $C_{\mathbf{K}_\star} > 0$ such that

$$\mathbf{w} \cdot \mathbf{K}_\star^{-1}(\mathbf{x})\mathbf{w} \geq C_{\mathbf{K}_\star} |\mathbf{w}|^2\,, \tag{2.4}$$

for almost all $\mathbf{x} \in \Omega_\star$, and for all $\mathbf{w} \in \mathrm{R}^n$. In addition, according to the incompressibility of the fluid, the boundary conditions on $\mathbf{u}_{\mathrm{B}}$ and $\mathbf{u}_{\mathrm{D}}$, and the principle of mass conservation (cf. first equation in (2.3)), the datum g_{D} must satisfy

$$\int_{\Omega_{\mathrm{D}}} g_{\mathrm{D}} \,=\, 0\,. \tag{2.5}$$

3 The continuous formulation

In this section we proceed analogously to [29] (see also [30, 10]) and derive a fully-mixed formulation of the coupled problem given by (2.1), (2.2), and (2.3).

3.1 Preliminaries

We first observe, owing to the fact that $\mathrm{tr}(\nabla\mathbf{u}_{\mathrm{B}}) = \mathrm{div}(\mathbf{u}_{\mathrm{B}}) = 0$, that the first two equations in (2.1) are equivalent to

$$\boldsymbol{\sigma}_{\mathrm{B}} = \mu\nabla\mathbf{u}_{\mathrm{B}} - p_{\mathrm{B}}\mathbb{I}\,, \quad p_{\mathrm{B}} = -\frac{1}{n}\mathrm{tr}(\boldsymbol{\sigma}_{\mathrm{B}}) \quad \text{in} \quad \Omega_{\mathrm{B}}, \tag{3.1}$$

and hence, eliminating the pressure p_{B} (which anyway can be approximated later on by the post-processed formula suggested by the second equation of (3.1)), the Brinkman–Forchheimer problem (2.1) can be rewritten as

$$\frac{1}{\mu}\boldsymbol{\sigma}_{\mathrm{B}}^{\mathrm{d}} = \nabla\mathbf{u}_{\mathrm{B}} \quad \text{in} \quad \Omega_{\mathrm{B}}, \quad \mathbf{K}_{\mathrm{B}}^{-1}\mathbf{u}_{\mathrm{B}} + \mathtt{F}\,|\mathbf{u}_{\mathrm{B}}|^{\rho-2}\mathbf{u}_{\mathrm{B}} - \mathbf{div}(\boldsymbol{\sigma}_{\mathrm{B}}) = \mathbf{f}_{\mathrm{B}} \quad \text{in} \quad \Omega_{\mathrm{B}}, \quad \mathbf{u}_{\mathrm{B}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{B}}\,. \tag{3.2}$$

Hence, gathering (3.2), (2.2), and (2.3), the coupled Brinkman–Forchheimer/Darcy model, without the pressure p_{B}, can be summarized as follows

$$\begin{aligned} \frac{1}{\mu}\boldsymbol{\sigma}_{\mathrm{B}}^{\mathrm{d}} &= \nabla\mathbf{u}_{\mathrm{B}} && \text{in} \quad \Omega_{\mathrm{B}}\,, \\ \mathbf{K}_{\mathrm{B}}^{-1}\mathbf{u}_{\mathrm{B}} + \mathtt{F}\,|\mathbf{u}_{\mathrm{B}}|^{\rho-2}\mathbf{u}_{\mathrm{B}} - \mathbf{div}(\boldsymbol{\sigma}_{\mathrm{B}}) &= \mathbf{f}_{\mathrm{B}} && \text{in} \quad \Omega_{\mathrm{B}}\,, \\ \mathbf{K}_{\mathrm{D}}^{-1}\mathbf{u}_{\mathrm{D}} + \nabla p_{\mathrm{D}} &= \mathbf{f}_{\mathrm{D}} && \text{in} \quad \Omega_{\mathrm{D}}\,, \\ \operatorname{div}(\mathbf{u}_{\mathrm{D}}) &= g_{\mathrm{D}} && \text{in} \quad \Omega_{\mathrm{D}}\,, \\ \mathbf{u}_{\mathrm{B}}\cdot\mathbf{n} = \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \quad \text{and} \quad \boldsymbol{\sigma}_{\mathrm{B}}\mathbf{n} &= -p_{\mathrm{D}}\mathbf{n} && \text{on} \quad \Sigma\,, \\ \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\,, \quad \mathbf{u}_{\mathrm{B}} &= \mathbf{0} && \text{on} \quad \Gamma_{\mathrm{B}}\,. \end{aligned} \tag{3.3}$$

We now provide further notations and definitions. Firstly, for each $\star \in \{\mathrm{B}, \mathrm{D}\}$ we set

$$(p,q)_\star := \int_{\Omega_\star} p\,q, \quad (\mathbf{u},\mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u}\cdot\mathbf{v} \quad \text{and} \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}. \tag{3.4}$$

Next, denoting by $\mathrm{E}_{0,\star} : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{L}^2(\partial\Omega_\star)$ the extension operator given by

$$\mathrm{E}_{0,\star}(\psi) := \begin{cases} \psi & \text{on} \quad \Sigma \\ 0 & \text{on} \quad \Gamma_\star \end{cases} \qquad \forall\, \psi \in \mathrm{H}^{1/2}(\Sigma)\,,$$

and proceeding as in [29] (see also [30, 10]), we define the space of traces

$$\mathrm{H}_{00}^{1/2}(\Sigma) := \Big\{ \psi \in \mathrm{H}^{1/2}(\Sigma) : \quad \mathrm{E}_{0,\star}(\psi) \in \mathrm{H}^{1/2}(\partial\Omega_\star) \Big\}\,, \tag{3.5}$$

which is endowed with the norm

$$\|\psi\|_{1/2,00;\Sigma} := \|\mathrm{E}_{0,\star}(\psi)\|_{1/2,\partial\Omega_\star}\,. \tag{3.6}$$

Note that (3.5) actually says that $\mathrm{H}_{00}^{1/2}(\Sigma)$ can be defined in two different, but equivalent, ways, namely by performing the extension by 0 to either Γ_{B} or Γ_{D}. Then, we let $\mathbf{H}_{00}^{1/2}(\Sigma) := [\mathrm{H}_{00}^{1/2}(\Sigma)]^n$, denote the dual spaces of $\mathrm{H}_{00}^{1/2}(\Sigma)$ and $\mathbf{H}_{00}^{1/2}(\Sigma)$ by $\mathrm{H}_{00}^{-1/2}(\Sigma)$ and $\mathbf{H}_{00}^{-1/2}(\Sigma)$, respectively, and let $\langle\cdot,\cdot\rangle_\Sigma$ be the duality pairing for both cases. Since it is clear that $\mathrm{H}_{00}^{1/2}(\Sigma) \subseteq \mathrm{H}^{1/2}(\Sigma)$, there holds $\mathrm{H}^{-1/2}(\Sigma) \subseteq \mathrm{H}_{00}^{-1/2}(\Sigma)$, and analogously $\mathbf{H}^{-1/2}(\Sigma) \subseteq \mathbf{H}_{00}^{-1/2}(\Sigma)$. In addition, letting for each $\star \in \{\mathrm{B}, \mathrm{D}\}$, $\langle\cdot,\cdot\rangle_{\partial\Omega_\star}$ be the duality pairing between $\mathrm{H}^{-1/2}(\partial\Omega_\star)$ and $\mathrm{H}^{1/2}(\partial\Omega_\star)$, and given any $\eta \in \mathrm{H}^{-1/2}(\partial\Omega_\star)$, its restriction to Σ, denoted $\eta|_\Sigma$, is defined as

$$\langle \eta|_\Sigma, \psi\rangle_\Sigma := \langle \eta, \mathrm{E}_{0,\star}(\psi)\rangle_{\partial\Omega_\star} \qquad \forall\, \psi \in \mathrm{H}_{00}^{1/2}(\Sigma)\,. \tag{3.7}$$

Then, letting $\|\cdot\|_{-1/2,00;\Sigma}$ be the norm of both $\mathrm{H}_{00}^{-1/2}(\Sigma)$ and $\mathbf{H}_{00}^{-1/2}(\Sigma)$, it is easily seen from (3.6) and (3.7) that $\eta|_\Sigma \in \mathrm{H}_{00}^{-1/2}(\Sigma)$, and that

$$\|\eta|_\Sigma\|_{-1/2,00;\Sigma} \,\leq\, \|\eta\|_{-1/2,\partial\Omega_\star}\,.$$

Moreover, it can be proved (see, e.g. [23, Section 2]) that in the particular case in which $\eta|_{\Gamma_\star}$ is the null functional of $\mathrm{H}_{00}^{-1/2}(\Gamma_\star)$, there actually holds $\eta|_\Sigma \in \mathrm{H}^{-1/2}(\Sigma)$.

Certainly, the above also holds for the corresponding vector versions of the spaces involved.

3.2 The Banach spaces-based fully-mixed variational formulation

We now proceed with the derivation of our Banach spaces-based fully-mixed variational formulation for the coupled Brinkman–Forchheimer/Darcy problem. To this end, we test the first and second equations of (3.3) against functions $\boldsymbol{\tau}_{\mathrm{B}}$ and $\mathbf{v}_{\mathrm{B}}$ associated with the unknowns $\boldsymbol{\sigma}_{\mathrm{B}}$ and $\mathbf{u}_{\mathrm{B}}$, respectively, whence, using the identity $\boldsymbol{\sigma}_{\mathrm{B}}^{\mathrm{d}} : \boldsymbol{\tau}_{\mathrm{B}} = \boldsymbol{\sigma}_{\mathrm{B}}^{\mathrm{d}} : \boldsymbol{\tau}_{\mathrm{B}}^{\mathrm{d}}$ and the notations from (3.4), we formally get

$$\frac{1}{\mu}(\boldsymbol{\sigma}_{\mathrm{B}}^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{B}}^{\mathrm{d}})_{\mathrm{B}} - (\nabla\mathbf{u}_{\mathrm{B}},\boldsymbol{\tau}_{\mathrm{B}})_{\mathrm{B}} \,=\, 0\,, \tag{3.8}$$

$$(\mathbf{v}_{\mathrm{B}},\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{B}}))_{\mathrm{B}} - (\mathbf{K}_{\mathrm{B}}^{-1}\mathbf{u}_{\mathrm{B}},\mathbf{v}_{\mathrm{B}})_{\mathrm{B}} - \mathtt{F}(|\mathbf{u}_{\mathrm{B}}|^{\rho-2}\mathbf{u}_{\mathrm{B}},\mathbf{v}_{\mathrm{B}})_{\mathrm{B}} \,=\, -(\mathbf{f}_{\mathrm{B}},\mathbf{v}_{\mathrm{B}})_{\mathrm{B}}\,. \tag{3.9}$$

Notice that the first term of (3.8) is well-defined for $\boldsymbol{\sigma}_{\mathrm{B}},\boldsymbol{\tau}_{\mathrm{B}} \in \mathbb{L}^2(\Omega_{\mathrm{B}})$. In turn, applying the Hölder inequality twice, we find that the Forchheimer term, given by the third expression in (3.9), can be bounded as

$$\left|(|\mathbf{w}_{\mathrm{B}}|^{\rho-2}\mathbf{u}_{\mathrm{B}},\mathbf{v}_{\mathrm{B}})_{\mathrm{B}}\right| \,\leq\, \|\mathbf{w}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}^{\rho-2}\,\|\mathbf{u}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,\|\mathbf{v}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,, \tag{3.10}$$

which shows that it is well-defined for all $\mathbf{w}_{\mathrm{B}}$, $\mathbf{u}_{\mathrm{B}}$, $\mathbf{v}_{\mathrm{B}} \in \mathbf{L}^\rho(\Omega)$. We stress here that the above bounding is more general than the one employed for the related model studied in [6], which, involving the usual convective term from the Navier–Stokes equations, is forced to require $\mathbf{u}_{\mathrm{B}}$, $\mathbf{v}_{\mathrm{B}} \in \mathbf{L}^4(\Omega)$, and hence $\mathbf{w}_{\mathrm{B}} \in \mathbf{L}^{2(\rho-2)}(\Omega_{\mathrm{B}})$. In this way, using that $2(\rho-2)\leq 4$, $\|\mathbf{w}_{\mathrm{B}}\|_{0,2(\rho-2);\Omega_{\mathrm{B}}}$ is bounded in [6] by $C\,\|\mathbf{w}_{\mathrm{B}}\|_{0,4;\Omega_{\mathrm{B}}}$, where C is the norm of the continuous injection from $\mathbf{L}^4(\Omega_{\mathrm{B}})$ into $\mathbf{L}^{2(\rho-2)}(\Omega_{\mathrm{B}})$. Not having that convective term in the present case, the estimate (3.10) does not need to restrict to $\rho = 4$, and it is actually valid not only for $\rho \in [3,4]$, but also for an even larger range of this exponent.

Furthermore, since $\mathbf{K}_{\mathrm{B}}^{-1} \in \mathbb{L}^\infty(\Omega_{\mathrm{B}})$ and $\mathbf{L}^\rho(\Omega_{\mathrm{B}})$ is certainly contained in $\mathbf{L}^2(\Omega_{\mathrm{B}})$, the second term in (3.9) does also make sense. Next, knowing the space in which $\mathbf{v}_{\mathrm{B}}$ is taken, we deduce that the source term of (3.9) is well-defined if $\mathbf{f}_{\mathrm{B}}$ belongs to $\mathbf{L}^\varrho(\Omega_{\mathrm{B}})$, with ϱ the conjugate of ρ, that is $\varrho \in [4/3,3/2]$ and $1/\rho + 1/\varrho = 1$, which is assumed from now on, whereas the first term of (3.9) makes sense if $\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{B}})$ lies in $\mathbf{L}^\varrho(\Omega_{\mathrm{B}})$ as well, and thus initially we look for $\boldsymbol{\sigma}_{\mathrm{B}}$ in the Banach space $\mathbb{H}(\mathbf{div}_\varrho;\Omega_{\mathrm{B}})$ (cf. (1.1)). Moreover, choosing also $\mathbb{H}(\mathbf{div}_\varrho;\Omega_{\mathrm{B}})$ as the space to which the test functions $\boldsymbol{\tau}_{\mathrm{B}}$ belong, and assuming originally that $\mathbf{u}_{\mathrm{B}} \in \mathbf{H}^1(\Omega_{\mathrm{B}})$, we can integrate by parts the second term in (3.8), so that, using the Dirichlet boundary condition $\mathbf{u}_{\mathrm{B}} = \mathbf{0}$ on Γ_{B}, and defining the auxiliary unknown

$$\boldsymbol{\varphi} \,:=\, -\mathbf{u}_{\mathrm{B}}|_\Sigma \,\in\, \mathbf{H}_{00}^{1/2}(\Sigma)\,,$$

that equation becomes

$$\frac{1}{\mu}(\boldsymbol{\sigma}_{\mathrm{B}}^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{B}}^{\mathrm{d}})_{\mathrm{B}} + \langle\boldsymbol{\tau}_{\mathrm{B}}\mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma + (\mathbf{u}_{\mathrm{B}},\mathbf{div}(\boldsymbol{\tau}_{\mathrm{B}}))_{\mathrm{B}} \,=\, 0 \quad \forall\,\boldsymbol{\tau}_{\mathrm{B}} \in \mathbb{H}(\mathbf{div}_\varrho;\Omega_{\mathrm{B}})\,, \tag{3.11}$$

whereas, according to the previous discussion, (3.9) is tested against $\mathbf{v}_{\mathrm{B}} \in \mathbf{L}^\rho(\Omega_{\mathrm{B}})$.

In turn, as suggested by the boundary condition on $\mathbf{u}_{\mathrm{D}}$, we introduce the space

$$\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \,:=\, \Big\{ \mathbf{v}_{\mathrm{D}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}) : \quad \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} \Big\}.$$

Thus, similarly to the procedure employed in [29] and [7], we test the third and fourth equations of (3.3) against $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$ and $q_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$, respectively, and then impose weakly the transmission conditions on Σ (cf. fifth equation of (3.3)). In this way, introducing the additional unknown

$$\lambda \,:=\, p_{\mathrm{D}}|_{\Sigma} \,\in\, \mathrm{H}^{1/2}(\Sigma),$$

we arrive at

$$\begin{aligned}
(\mathbf{K}_{\mathrm{D}}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} - \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\lambda\rangle_{\Sigma} - (p_{\mathrm{D}},\mathrm{div}(\mathbf{v}_{\mathrm{D}}))_{\mathrm{D}} &= (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} && \forall\, \mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}),\\
(q_{\mathrm{D}},\mathrm{div}(\mathbf{u}_{\mathrm{D}}))_{\mathrm{D}} &= (g_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}} && \forall\, q_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}}),\\
-\langle \boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_{\Sigma} - \langle \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\Sigma} &= 0 && \forall\, \xi \in \mathrm{H}^{1/2}(\Sigma),\\
\langle \boldsymbol{\sigma}_{\mathrm{B}}\mathbf{n},\boldsymbol{\psi}\rangle_{\Sigma} + \langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_{\Sigma} &= 0 && \forall\, \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma).
\end{aligned} \tag{3.12}$$

We remark here that, being $\boldsymbol{\varphi}\cdot\mathbf{n} = 0$ on Γ_{B} and $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0$ on Γ_{D}, it follows that both $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\Sigma}$ and $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n}|_{\Sigma}$ belong to $\mathrm{H}^{-1/2}(\Sigma)$, which explains the fact that the third equation of (3.12) is tested against $\xi \in \mathrm{H}^{1/2}(\Sigma)$. In turn, since $\boldsymbol{\sigma}_{\mathrm{B}}\mathbf{n} \in \mathbf{H}^{-1/2}(\partial\Omega_{\mathrm{B}})$ and $\lambda\mathbf{n} \in \mathbf{L}^2(\partial\Omega_{\mathrm{D}}) \subseteq \mathbf{H}^{-1/2}(\partial\Omega_{\mathrm{D}})$, it is clear that both $\boldsymbol{\sigma}_{\mathrm{B}}\mathbf{n}|_{\Sigma}$ and $\lambda\mathbf{n}|_{\Sigma}$ belong to $\mathbf{H}^{-1/2}_{00}(\Sigma)$, which confirms the validity of the fourth equation of (3.12).

Now, let us observe that if $(\boldsymbol{\sigma}_{\mathrm{B}},\mathbf{u}_{\mathrm{B}},\boldsymbol{\varphi},\mathbf{u}_{\mathrm{D}},p_{\mathrm{D}},\lambda)$ is a solution of (3.9), (3.11), and (3.12), then for all $c \in \mathrm{R}$, $(\boldsymbol{\sigma}_{\mathrm{B}} - c\,\mathbb{I},\mathbf{u}_{\mathrm{B}},\boldsymbol{\varphi},\mathbf{u}_{\mathrm{D}},p_{\mathrm{D}}+c,\lambda+c)$ is also a solution. Then, we avoid the non-uniqueness of solution by requiring from now on that $p_{\mathrm{D}} \in \mathrm{L}^2_0(\Omega_{\mathrm{D}})$, where

$$\mathrm{L}^2_0(\Omega_{\mathrm{D}}) \,:=\, \Big\{ q_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}}) : \quad (q_{\mathrm{D}},1)_{\mathrm{D}} = 0 \Big\}.$$

On the other hand, for convenience of the subsequent analysis, we consider the decomposition (see, for instance, [5], [24])

$$\mathbb{H}(\mathbf{div}_{\varrho};\Omega_{\mathrm{B}}) \,=\, \mathbb{H}_0(\mathbf{div}_{\varrho};\Omega_{\mathrm{B}}) \oplus \mathrm{R}\,\mathbb{I}, \tag{3.13}$$

where

$$\mathbb{H}_0(\mathbf{div}_{\varrho};\Omega_{\mathrm{B}}) \,:=\, \Big\{ \boldsymbol{\tau}_{\mathrm{B}} \in \mathbb{H}(\mathbf{div}_{\varrho};\Omega_{\mathrm{B}}) : \quad (\mathrm{tr}(\boldsymbol{\tau}_{\mathrm{B}}),1)_{\mathrm{B}} = 0 \Big\},$$

and redefine the pseudostress tensor as $\boldsymbol{\sigma}_{\mathrm{B}} := \boldsymbol{\sigma}_{\mathrm{B}} + \ell\,\mathbb{I}$, with the new unknowns $\boldsymbol{\sigma}_{\mathrm{B}} \in \mathbb{H}_0(\mathbf{div}_{\varrho};\Omega_{\mathrm{B}})$ and $\ell \in \mathrm{R}$. In this way, (3.11) and the fourth equation of (3.12) are rewritten, equivalently, as

$$\begin{aligned}
\frac{1}{\mu}(\boldsymbol{\sigma}_{\mathrm{B}}^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{B}}^{\mathrm{d}})_{\mathrm{B}} + \langle \boldsymbol{\tau}_{\mathrm{B}}\mathbf{n},\boldsymbol{\varphi}\rangle_{\Sigma} + (\mathbf{u}_{\mathrm{B}},\mathbf{div}(\boldsymbol{\tau}_{\mathrm{B}}))_{\mathrm{B}} &= 0 \quad \forall\, \boldsymbol{\tau}_{\mathrm{B}} \in \mathbb{H}_0(\mathbf{div}_{\varrho};\Omega_{\mathrm{B}}),\\
\jmath\,\langle \boldsymbol{\varphi}\cdot\mathbf{n},1\rangle_{\Sigma} &= 0 \quad \forall\, \jmath \in \mathrm{R},\\
\langle \boldsymbol{\sigma}_{\mathrm{B}}\mathbf{n},\boldsymbol{\psi}\rangle_{\Sigma} + \langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_{\Sigma} + \ell\,\langle \boldsymbol{\psi}\cdot\mathbf{n},1\rangle_{\Sigma} &= 0 \quad \forall\, \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma),
\end{aligned} \tag{3.14}$$

so that the whole variational formulation reduces to (3.9), the first three rows of (3.12), and (3.14). Note here that, due to (2.5) and the transmission and boundary conditions satisfied by $\mathbf{u}_{\mathrm{D}}$ and $\mathbf{u}_{\mathrm{B}}$, the second row of (3.12) is equivalently tested against $q_{\mathrm{D}} \in \mathrm{L}^2_0(\Omega_{\mathrm{D}})$.

Now, it is clear that there are many different ways of ordering the aforementioned equations, but for the sake of the subsequent analysis, we proceed closely to [28] (see also [30] and [10] for similar works), and adopt one leading to a nonlinear perturbation of a twofold perturbed saddle point problem in a Banach spaces framework, namely: Find $\big(\boldsymbol{\sigma}_{\mathrm{B}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda, \mathbf{u}_{\mathrm{B}}, p_{\mathrm{D}}, \ell\big) \in \mathbb{H}_0(\mathbf{div}_\varrho;\Omega_{\mathrm{B}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbf{H}^{1/2}_{00}(\Sigma) \times \mathrm{H}^{1/2}(\Sigma) \times \mathbf{L}^\rho(\Omega_{\mathrm{B}}) \times \mathrm{L}^2_0(\Omega_{\mathrm{D}}) \times \mathrm{R}$, such that

$$
\begin{array}{ccccccccc}
\frac{1}{\mu}(\boldsymbol{\sigma}_{\mathrm{B}},\boldsymbol{\tau}_{\mathrm{B}})_{\mathrm{B}} & & +\langle \boldsymbol{\tau}_{\mathrm{B}}\,\mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma & & +(\mathbf{u}_{\mathrm{B}},\mathbf{div}(\boldsymbol{\tau}_{\mathrm{B}}))_{\mathrm{B}} & & & = & 0\\
 & (\mathbf{K}_{\mathrm{D}}^{-1}\,\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} & & -\,\langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\lambda\rangle_\Sigma & & -(p_{\mathrm{D}},\mathrm{div}(\mathbf{v}_{\mathrm{D}}))_{\mathrm{D}} & & = & (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}}\\
-\,\langle \boldsymbol{\sigma}_{\mathrm{B}}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma & & & -\,\langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_\Sigma & & & -\,\ell\,\langle \boldsymbol{\psi}\cdot\mathbf{n},1\rangle_\Sigma & = & 0\\
 & \langle \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_\Sigma & +\,\langle \boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_\Sigma & & & & & = & 0\\
(\mathbf{v}_{\mathrm{B}},\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{B}}))_{\mathrm{B}} & & & & -(\mathbf{K}_{\mathrm{B}}^{-1}\mathbf{u}_{\mathrm{B}},\mathbf{v}_{\mathrm{B}})_{\mathrm{B}} & -\mathtt{F}(|\mathbf{u}_{\mathrm{B}}|^{\rho-2}\mathbf{u}_{\mathrm{B}},\mathbf{v}_{\mathrm{B}})_{\mathrm{B}} & & = & -(\mathbf{f}_{\mathrm{B}},\mathbf{v}_{\mathrm{B}})_{\mathrm{B}}\\
 & -(q_{\mathrm{D}},\mathrm{div}(\mathbf{u}_{\mathrm{D}}))_{\mathrm{D}} & & & & & & = & -(g_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}}\\
 & & -\,\jmath\,\langle \boldsymbol{\varphi}\cdot\mathbf{n},1\rangle_\Sigma & & & & & = & 0
\end{array}
\tag{3.15}
$$

for all $\big(\boldsymbol{\tau}_{\mathrm{B}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi, \mathbf{v}_{\mathrm{B}}, q_{\mathrm{D}}, \jmath\big) \in \mathbb{H}_0(\mathbf{div}_\varrho;\Omega_{\mathrm{B}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbf{H}^{1/2}_{00}(\Sigma) \times \mathrm{H}^{1/2}(\Sigma) \times \mathbf{L}^\rho(\Omega_{\mathrm{B}}) \times \mathrm{L}^2_0(\Omega_{\mathrm{D}}) \times \mathrm{R}$. According to (3.15), we introduce the spaces

$$
\begin{gathered}
\mathbf{H}_1 := \mathbb{H}_0(\mathbf{div}_\varrho;\Omega_{\mathrm{B}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\,, \quad \mathbf{H}_2 := \mathbf{H}^{1/2}_{00}(\Sigma) \times \mathrm{H}^{1/2}(\Sigma)\,,\\
\mathbf{H} := \mathbf{H}_1 \times \mathbf{H}_2\,, \quad \text{and} \quad \mathbf{Q} := \mathbf{L}^\rho(\Omega_{\mathrm{B}}) \times \mathrm{L}^2_0(\Omega_{\mathrm{D}}) \times \mathrm{R}\,,
\end{gathered}
$$

and set the following notations for the unknowns and corresponding test functions

$$
\begin{array}{lll}
\vec{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_{\mathrm{B}},\mathbf{u}_{\mathrm{D}}) \in \mathbf{H}_1\,, & \vec{\boldsymbol{\varphi}} := (\boldsymbol{\varphi},\lambda) \in \mathbf{H}_2\,, & \vec{\mathbf{u}} := (\mathbf{u}_{\mathrm{B}}, p_{\mathrm{D}}, \ell) \in \mathbf{Q}\,,\\
\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{B}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{H}_1\,, & \vec{\boldsymbol{\psi}} := (\boldsymbol{\psi},\xi) \in \mathbf{H}_2\,, & \vec{\mathbf{v}} := (\mathbf{v}_{\mathrm{B}}, q_{\mathrm{D}}, \jmath) \in \mathbf{Q}\,,\\
\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}_{\mathrm{B}},\mathbf{z}_{\mathrm{D}}) \in \mathbf{H}_1\,, & \vec{\boldsymbol{\phi}} := (\boldsymbol{\phi},\vartheta) \in \mathbf{H}_2\,, & \vec{\mathbf{z}} := (\mathbf{z}_{\mathrm{B}}, r_{\mathrm{D}}, \kappa) \in \mathbf{Q}\,,
\end{array}
\tag{3.16}
$$

so that $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}$, and $\mathbf{Q}$ are endowed with the norms

$$
\begin{array}{rcll}
\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}_1} & := & \|\boldsymbol{\tau}_{\mathrm{B}}\|_{\mathbf{div}_\varrho;\Omega_{\mathrm{B}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}} & \forall\,\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{B}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{H}_1\,,\\
\|\vec{\boldsymbol{\psi}}\|_{\mathbf{H}_2} & := & \|\boldsymbol{\psi}\|_{1/2,00;\Sigma} + \|\xi\|_{1/2,\Sigma} & \forall\,\vec{\boldsymbol{\psi}} := (\boldsymbol{\psi},\xi) \in \mathbf{H}_2\,,\\
\|(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\|_{\mathbf{H}} & := & \|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}_1} + \|\vec{\boldsymbol{\psi}}\|_{\mathbf{H}_2} & \forall\,(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}) \in \mathbf{H}\,,\\
\|\vec{\mathbf{v}}\|_{\mathbf{Q}} & := & \|\mathbf{v}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}} + \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + |\jmath| & \forall\,\vec{\mathbf{v}} := (\mathbf{v}_{\mathrm{B}}, q_{\mathrm{D}}, \jmath) \in \mathbf{Q}\,.
\end{array}
$$

Hence, the mixed formulation (3.15) can be rewritten as: Find $\big((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}\big) \in \mathbf{H}\times\mathbf{Q}$ such that

$$
\begin{array}{lcl}
\mathbf{A}\big((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\big) \;+\; \mathbf{B}\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{u}}\big) & = & \mathbf{F}\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\big)\,,\\
\mathbf{B}\big((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{v}}\big) \;-\; \mathbf{C}_{\mathbf{u}_{\mathrm{B}}}\big(\vec{\mathbf{u}},\vec{\mathbf{v}}\big) & = & \mathbf{G}\big(\vec{\mathbf{v}}\big)\,,
\end{array}
\tag{3.17}
$$

for all $\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}}\big) \in \mathbf{H}\times\mathbf{Q}$, where the bilinear forms $\mathbf{A} : \mathbf{H}\times\mathbf{H} \to \mathrm{R}$, $\mathbf{B} : \mathbf{H}\times\mathbf{Q} \to \mathrm{R}$, and $\mathbf{C}_{\mathbf{w}_{\mathrm{B}}} : \mathbf{Q}\times\mathbf{Q} \to \mathrm{R}$, for each $\mathbf{w}_{\mathrm{B}} \in \mathbf{L}^\rho(\Omega_{\mathrm{B}})$, and the linear functionals $\mathbf{F} : \mathbf{H} \to \mathrm{R}$ and $\mathbf{G} : \mathbf{Q} \to \mathrm{R}$, are defined in what follows. Indeed, there holds

$$
\mathbf{A}\big((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}),(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\big) := \mathbf{a}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) + \mathbf{b}_1(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\phi}}) + \mathbf{b}_2(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\psi}}) - \mathbf{c}(\vec{\boldsymbol{\phi}},\vec{\boldsymbol{\psi}})\,,
\tag{3.18}
$$

with

$$
\begin{array}{rcll}
\mathbf{a}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) & := & \dfrac{1}{\mu}(\boldsymbol{\zeta}_{\mathrm{B}},\boldsymbol{\tau}_{\mathrm{B}})_{\mathrm{B}} + (\mathbf{K}_{\mathrm{D}}^{-1}\mathbf{z}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} & \forall\,\vec{\boldsymbol{\zeta}},\,\vec{\boldsymbol{\tau}} \in \mathbf{H}_1\,,\\
\mathbf{b}_1(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}) & := & \langle \boldsymbol{\tau}_{\mathrm{B}}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma - \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_\Sigma & \forall\,(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}) \in \mathbf{H} := \mathbf{H}_1\times\mathbf{H}_2\,,\\
\mathbf{b}_2(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}) & := & -\,\langle \boldsymbol{\tau}_{\mathrm{B}}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_\Sigma & \forall\,(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}) \in \mathbf{H} := \mathbf{H}_1\times\mathbf{H}_2\,,\\
\mathbf{c}(\vec{\boldsymbol{\phi}},\vec{\boldsymbol{\psi}}) & := & \langle \boldsymbol{\psi}\cdot\mathbf{n},\vartheta\rangle_\Sigma - \langle \boldsymbol{\phi}\cdot\mathbf{n},\xi\rangle_\Sigma & \forall\,(\vec{\boldsymbol{\phi}},\vec{\boldsymbol{\psi}}) \in \mathbf{H}_2\times\mathbf{H}_2\,,
\end{array}
\tag{3.19}
$$

whereas

$$\mathbf{B}\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}}\big) \,:=\, (\mathbf{v}_{\rm B},\mathbf{div}(\boldsymbol{\tau}_{\rm B}))_{\rm B} - \,(q_{\rm D},\mathrm{div}(\mathbf{v}_{\rm D}))_{\rm D} \,-\, \jmath\langle \boldsymbol{\psi}\cdot\mathbf{n},1\rangle_\Sigma \quad \forall\,\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}}\big)\in\mathbf{H}\times\mathbf{Q}\,, \tag{3.20}$$

and

$$\mathbf{C}_{\mathbf{w}_{\rm B}}(\vec{\mathbf{z}},\vec{\mathbf{v}}) \,:=\, (\mathbf{K}_{\rm B}^{-1}\mathbf{z}_{\rm B},\mathbf{v}_{\rm B})_{\rm B} \,+\, \mathtt{F}(|\mathbf{w}_{\rm B}|^{\rho-2}\mathbf{z}_{\rm B},\mathbf{v}_{\rm B})_{\rm B} \quad \forall\,(\vec{\mathbf{z}},\vec{\mathbf{v}})\in\mathbf{Q}\times\mathbf{Q}\,. \tag{3.21}$$

In turn,

$$\mathbf{F}\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\big) \,:=\, (\mathbf{f}_{\rm D},\mathbf{v}_{\rm D})_{\rm D} \qquad \forall\,(\vec{\boldsymbol{\tau}},\xi)\in\mathbf{Q}\,,$$

and

$$\mathbf{G}(\vec{\mathbf{v}}) \,:=\, -\,(\mathbf{f}_{\rm B},\mathbf{v}_{\rm B})_{\rm B} \,-\, (g_{\rm D},q_{\rm D})_{\rm D} \qquad \forall\,\vec{\mathbf{v}}\in\mathbf{Q}\,.$$

Equivalently, letting $\mathcal{A}_{\mathbf{w}_{\rm B}} : \big(\mathbf{H}\times\mathbf{Q}\big)\times\big(\mathbf{H}\times\mathbf{Q}\big)\to \mathrm{R}$ be the bilinear form defined by

$$\begin{aligned} \mathcal{A}_{\mathbf{w}_{\rm B}}\big(((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}),\vec{\mathbf{z}}),((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\big) \,&:=\, \mathbf{A}\big((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}),(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\big) \,+\, \mathbf{B}\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{z}}\big) \\ &+\quad \mathbf{B}\big((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}),\vec{\mathbf{v}}\big) \,-\, \mathbf{C}_{\mathbf{w}_{\rm B}}(\vec{\mathbf{z}},\vec{\mathbf{v}})\,, \end{aligned} \tag{3.22}$$

we deduce that (3.17) can be stated, equivalently, as: Find $\big((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}\big)\in\mathbf{H}\times\mathbf{Q}$, such that

$$\mathcal{A}_{\mathbf{u}_{\rm B}}\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}),((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\big) \,=\, \mathcal{F}\big(((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\big) \qquad \forall\,\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}}\big)\in\mathbf{H}\times\mathbf{Q}\,, \tag{3.23}$$

where $\mathcal{F} : \mathbf{H}\times\mathbf{Q}\to\mathrm{R}$ is defined by the addition of $\mathbf{F}$ and $\mathbf{G}$, that is

$$\mathcal{F}\big(((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\big) := (\mathbf{f}_{\rm D},\mathbf{v}_{\rm D})_{\rm D} - (\mathbf{f}_{\rm B},\mathbf{v}_{\rm B})_{\rm B} - (g_{\rm D},q_{\rm D})_{\rm D} \quad \forall\,\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}}\big)\in\mathbf{H}\times\mathbf{Q}\,.$$

It is readily seen, particularly according to (3.18) and (3.19), that a matrix representation of the bilinear form $\mathcal{A}_{\mathbf{w}_{\rm B}}$ is given by

$$\mathcal{A}_{\mathbf{w}_{\rm B}} \,:=\, \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & -\mathbf{C}_{\mathbf{w}_{\rm B}} \end{pmatrix} \,=\, \left(\begin{array}{c|c} \begin{array}{c|c} \mathbf{a} & \mathbf{b}_1 \\ \hline \mathbf{b}_2 & -\mathbf{c} \end{array} & \mathbf{B} \\ \hline \mathbf{B} & -\mathbf{C}_{\mathbf{w}_{\rm B}} \end{array}\right)\,, \tag{3.24}$$

from which its twofold perturbed saddle point structure is evident. On the other hand, for further use throughout the rest of the paper, we remark that $\mathbf{b}_2 \,=\, -\,\mathbf{b}_1$ and $\mathbf{c}(\vec{\boldsymbol{\psi}},\vec{\boldsymbol{\psi}}) \,=\, 0$ for all $\vec{\boldsymbol{\psi}}\in\mathbf{H}_2$, which, along with (2.4), yields

$$\begin{aligned} \mathbf{A}\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\big) \,&=\, \mathbf{a}(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\tau}}) \,=\, \frac{1}{\mu}\|\boldsymbol{\tau}_{\rm B}\|^2_{0,\Omega_{\rm B}} + (\mathbf{K}_{\rm D}^{-1}\mathbf{v}_{\rm D},\mathbf{v}_{\rm D})_{\rm D} \\ &\geq\, \frac{1}{\mu}\|\boldsymbol{\tau}_{\rm B}\|^2_{0;\Omega_{\rm B}} \,+\, C_{\mathbf{K}_{\rm D}}\|\mathbf{v}_{\rm D}\|^2_{0;\Omega_{\rm D}} \,\geq\, 0 \qquad \forall\,(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\in\mathbf{H}\,. \end{aligned} \tag{3.25}$$

In addition, besides being clearly symmetric, we notice that $\mathbf{C}_{\mathbf{w}_{\rm B}}$ is positive semi-definite as well since, according to (3.21), and employing again (2.4), it follows that

$$\begin{aligned} \mathbf{C}_{\mathbf{w}_{\rm B}}(\vec{\mathbf{v}},\vec{\mathbf{v}}) \,&:=\, (\mathbf{K}_{\rm B}^{-1}\mathbf{v}_{\rm B},\mathbf{v}_{\rm B})_{\rm B} \,+\, \mathtt{F}(|\mathbf{w}_{\rm B}|^{\rho-2}\mathbf{v}_{\rm B},\mathbf{v}_{\rm B})_{\rm B} \\ &\geq\, C_{\mathbf{K}_{\rm B}}\|\mathbf{v}_{\rm B}\|^2_{0;\Omega_{\rm B}} \,+\, \mathtt{F}\,(|\mathbf{w}_{\rm B}|^{\rho-2},|\mathbf{v}_{\rm B}|^2)_{\rm B} \,\geq\, 0 \quad \forall\,\vec{\mathbf{v}}\,\in\,\mathbf{Q}\,. \end{aligned} \tag{3.26}$$

Furthermore, we notice that $\mathbf{A} \,:=\, \left(\begin{array}{c|c} \mathbf{a} & \mathbf{b}_1 \\ \hline \mathbf{b}_2 & -\mathbf{c} \end{array}\right)$ is invertible in a determined space, say the kernel of $\mathbf{B}$, and hence satisfy global inf-sup conditions there, if and only if

$$\widetilde{\mathbf{A}} \,:=\, \left(\begin{array}{c|c} \mathbf{a} & \mathbf{b} \\ \hline \mathbf{b} & -\widetilde{\mathbf{c}} \end{array}\right) \tag{3.27}$$

is invertible, where $\mathbf{b} = \mathbf{b}_1$ and $\widetilde{\mathbf{c}} = -\mathbf{c}$. Note that $\widetilde{\mathbf{A}}$ arises from $\mathbf{A}$ after multiplying by -1 the second row of the later, and that obviously the resulting $\widetilde{\mathbf{c}}$ also satisfies the aforementioned property of $\mathbf{c}$, that is

$$\widetilde{\mathbf{c}}(\vec{\boldsymbol{\psi}}, \vec{\boldsymbol{\psi}}) \,=\, 0 \qquad \forall\, \vec{\boldsymbol{\psi}} \in \mathbf{H}_2\,. \tag{3.28}$$

Finally, it is interesting to observe that the bilinear forms $\mathbf{b}_1$ (and hence $\mathbf{b}_2$) as well as $\mathbf{B}$ have diagonal structures, whence proving the corresponding inf-sup conditions reduces, basically, to showing this property for each one of their diagonal components.

We end this section with the stability properties of the bilinear forms and functionals involved in (3.17). In fact, direct applications of the Cauchy-Schwarz and Hölder inequalities, along with the boundedness of the normal trace on $\mathbb{H}(\mathbf{div}_\varrho;\Omega_{\mathrm{B}})$ and $\mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})$, yield the existence of positive constants, denoted and given as:

$$\begin{gathered}
\|\mathbf{a}\| \,:=\, \max\big\{\mu^{-1},\, \|\mathbf{K}_{\mathrm{D}}^{-1}\|_{\infty;\Omega_{\mathrm{D}}}\big\}\,, \quad \|\mathbf{b}_1\| \,=\, \|\mathbf{b}_2\| \,:=\, \max\big\{1, \|\mathbf{i}_\rho\|\big\}\,,\\
\|\mathbf{c}\| \,:=\, 2\,, \quad \|\mathbf{A}\| \,:=\, \|\mathbf{a}\| \,+\, 2\,\|\mathbf{b}_1\| \,+\, \|\mathbf{c}\|\,, \quad \|\mathbf{B}\| \,:=\, 3\,,\\
\text{and} \quad \|\mathcal{F}\| \,:=\, \|\mathbf{f}_{\mathrm{D}}\|_{0;\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}} + \|g_{\mathrm{D}}\|_{0;\Omega_{\mathrm{D}}}\,,
\end{gathered} \tag{3.29}$$

where $\|\mathbf{i}_\rho\|$ is the norm of the continuous injection $\mathbf{i}_\rho$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^\rho(\Omega)$, such that there hold

$$\begin{array}{ll}
|\mathbf{a}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}})| \,\leq\, \|\mathbf{a}\|\,\|\vec{\boldsymbol{\zeta}}\|_{\mathbf{H}_1}\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}_1} & \forall\, \vec{\boldsymbol{\zeta}},\, \vec{\boldsymbol{\tau}} \in \mathbf{H}_1\,,\\
|\mathbf{b}_i(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}})| \,\leq\, \|\mathbf{b}_i\|\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}_1}\,\|\vec{\boldsymbol{\psi}}\|_{\mathbf{H}_2} & \forall\, (\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}) \in \mathbf{H}\,,\\
|\mathbf{c}(\vec{\boldsymbol{\phi}}, \vec{\boldsymbol{\psi}})| \,\leq\, \|\mathbf{c}\|\,\|\vec{\boldsymbol{\phi}}\|_{\mathbf{H}_2}\,\|\vec{\boldsymbol{\psi}}\|_{\mathbf{H}_2} & \forall\, \vec{\boldsymbol{\phi}},\, \vec{\boldsymbol{\psi}} \in \mathbf{H}_2\,,\\
|\mathbf{A}\big((\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\phi}}), (\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}})\big)| \,\leq\, \|\mathbf{A}\|\,\|(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\phi}})\|_{\mathbf{H}}\,\|(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}})\|_{\mathbf{H}} & \forall\, (\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\phi}}),\, (\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}) \in \mathbf{H}\,,\\
|\mathbf{B}((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}})| \,\leq\, \|\mathbf{B}\|\,\|(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}})\|_{\mathbf{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} & \forall\, ((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q}\,, \quad \text{and}\\
|\mathcal{F}\big(((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}})\big)| \,\leq\, \|\mathcal{F}\|\,\|\big((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}}\big)\|_{\mathbf{H}\times\mathbf{Q}} & \forall\, \big((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}}\big) \in \mathbf{H}\times\mathbf{Q}\,.
\end{array} \tag{3.30}$$

In turn, employing (3.10), we readily find that for each $\mathbf{w}_{\mathrm{B}} \in \mathbf{L}^\rho(\Omega_{\mathrm{B}})$ there holds (cf. (3.16))

$$\begin{aligned}
|\mathtt{F}(|\mathbf{w}_{\mathrm{B}}|^{\rho-2}\mathbf{z}_{\mathrm{B}}, \mathbf{v}_{\mathrm{B}})_{\mathrm{B}}| \,&\leq\, \mathtt{F}\|\mathbf{w}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}^{\rho-2}\,\|\mathbf{z}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,\|\mathbf{v}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\\
&\leq\, \mathtt{F}\|\mathbf{w}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}^{\rho-2}\,\|\vec{\mathbf{z}}\|_{\mathbf{Q}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} \qquad \forall\, \vec{\mathbf{z}}\,, \vec{\mathbf{v}} \,\in\, \mathbf{Q}\,,
\end{aligned}$$

and thus, in virtue of the definition of $\mathbf{C}_{\mathbf{w}_{\mathrm{B}}}$ (cf. (3.21)), and using again Hölder's inequality, we get

$$|\mathbf{C}_{\mathbf{w}_{\mathrm{B}}}(\vec{\mathbf{z}}, \vec{\mathbf{v}})| \,\leq\, \Big\{\|\mathbf{C}\| + \mathtt{F}\|\mathbf{w}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}^{\rho-2}\Big\}\,\|\vec{\mathbf{z}}\|_{\mathbf{Q}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} \quad \forall\, \vec{\mathbf{z}}\,, \vec{\mathbf{v}} \,\in\, \mathbf{Q}\,, \tag{3.31}$$

with

$$\|\mathbf{C}\| \,:=\, |\Omega|^{(\rho-2)/\rho}\,\|\mathbf{K}_{\mathrm{B}}^{-1}\|_{\infty;\Omega_{\mathrm{B}}}\,. \tag{3.32}$$

3.3 Some abstract results on perturbed saddle point problems

In this section we collect two abstract theorems in Banach spaces that are employed later on to analyze the solvability of (3.23) (equivalently (3.17)). The first one, taken from [28, Theorem 3.2] and stated next, constitutes a slight improvement of the original result provided in [15, Theorem 3.4].

Theorem 3.1 *Let H and Q be reflexive Banach spaces, and let $a : H\times H \to \mathrm{R}$, $b : H\times Q \to \mathrm{R}$, and $c : Q\times Q \to \mathrm{R}$ be given bounded bilinear forms. In addition, let $\mathcal{B} : H \to Q'$ be the bounded linear operator induced by b, and let $\mathcal{K} = N(\mathcal{B})$ be the respective null space. Assume that:*

i) *a and c are positive semi-definite, that is*

$$a(\tau,\tau) \,\geq\, 0 \quad \forall\, \tau \,\in\, H \quad \text{and} \quad c(v,v) \,\geq\, 0 \quad \forall\, v \,\in\, Q\,,$$

and that c is symmetric.

ii) *there exists a constant $\alpha > 0$ such that*

$$\sup_{\substack{\tau\in\mathcal{K}\\ \tau\neq 0}} \frac{a(\zeta,\tau)}{\|\tau\|_H} \,\geq\, \alpha\, \|\zeta\|_H \quad \forall\, \zeta\in\mathcal{K}\,, \quad \text{and}$$

$$\sup_{\substack{\zeta\in\mathcal{K}\\ \zeta\neq 0}} \frac{a(\zeta,\tau)}{\|\zeta\|_H} \,\geq\, \alpha\, \|\tau\|_H \quad \forall\, \tau\in\mathcal{K}\,,$$

iii) *and there exists a constant $\beta > 0$ such that*

$$\sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{b(\tau,v)}{\|\tau\|_H} \,\geq\, \beta\, \|v\|_Q \quad \forall v \,\in\, \mathrm{Q}\,.$$

Then, for each pair $(F,G)\in H'\times Q'$ there exists a unique $(\sigma,u)\in H\times Q$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) \;&=\; F(\tau) \qquad \forall\, \tau \,\in\, H\,, \\ b(\sigma,v) - c(u,v) \;&=\; G(v) \qquad \forall v \,\in\, Q\,. \end{aligned} \tag{3.33}$$

Moreover, there exists a constant $C > 0$, depending only on $\|a\|$, $\|c\|$, α, and β, such that

$$\|(\sigma,u)\|_{H\times Q} \,\leq\, C\left\{\|F\|_{H'} \,+\, \|G\|_{Q'}\right\}. \tag{3.34}$$

We remark here that (3.34) is equivalent to a global inf-sup condition for the bilinear form A that arises by summing up the equations in (3.33), namely

$$\sup_{\substack{(\tau,v)\in H\times Q\\ (\tau,v)\neq 0}} \frac{A((\zeta,w),(\tau,v))}{\|(\tau,v)\|_{H\times Q}} \,\geq\, C\,\|(\zeta,w)\|_{H\times Q} \qquad \forall\,(\zeta,w)\,\in\, H\times Q\,,$$

where

$$A((\zeta,w),(\tau,v)) \,:=\, a(\zeta,\tau) + b(\tau,w) + b(\zeta,v) - c(w,v)\,.$$

Now, we present a variation of Theorem 3.1 in which the symmetry of the perturbation c is dropped but the bilinear form a is required to be elliptic in the whole space.

Theorem 3.2 *Let* H *and* Q *be reflexive Banach spaces, and let $a : \mathrm{H}\times\mathrm{H}\to\mathrm{R}$, $b : \mathrm{H}\times\mathrm{Q}\to\mathrm{R}$, and $c : \mathrm{Q}\times\mathrm{Q}\to\mathrm{R}$ be bounded bilinear forms with boundedness constants denoted $\|a\|$, $\|b\|$, and $\|c\|$, respectively. Assume that:*

i) *c is positive semidefinite, that is $c(v,v) \,\geq\, 0$ for all $v\in\mathrm{Q}$.*

ii) *a is* H*-elliptic, that is there exists a positive constant $\alpha > 0$ such that*

$$a(\tau,\tau) \,\geq\, \alpha\, \|\tau\|^2_{\mathrm{H}} \quad \forall\, \tau \,\in\, \mathrm{H}\,, \quad \text{and}$$

iii) *b verifies the inf-sup condition, that is there exists a positive constant β such that*

$$\sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{b(\tau,v)}{\|\tau\|_H} \,\geq\, \beta\,\|v\|_Q \quad \forall v \,\in\, \mathrm{Q}\,,$$

Then, for each pair $(F,G)\in \mathrm{H}'\times \mathrm{Q}'$ there exists a unique $(\sigma,u)\in \mathrm{H}\times \mathrm{Q}$ such that

$$\begin{aligned} a(\sigma,\tau)+b(\tau,u) &= F(\tau) \qquad \forall\,\tau\,\in\,\mathrm{H}\,,\\ b(\sigma,v)-c(u,v) &= G(v) \qquad \forall\,v\,\in\,\mathrm{Q}\,. \end{aligned} \tag{3.35}$$

Moreover, there exists a positive constant C, depending only on $\|a\|$, $\|b\|$, α, and β, such that

$$\|\sigma\|_{\mathrm{H}}+\|u\|_{\mathrm{Q}} \,\leq\, C\,\Big\{\|F\|_{\mathrm{H}'}\,+\,\|G\|_{\mathrm{Q}'}\Big\}\,. \tag{3.36}$$

Proof. The proof proceeds as a natural simplification of the corresponding analysis developed in [1, Section 3] for a nonlinear version of (3.35). We begin by establishing existence of solution, for which we first observe, thanks to ii) and the Banach-Nečas-Babuška theorem (cf. [22, Theorem 2.6]), that there exists a unique $\sigma_0\in\mathrm{H}$ such that

$$a(\sigma_0,\tau)\,=\,F(\tau) \qquad \forall\,\tau\in\mathrm{H}\,, \tag{3.37}$$

and that for each $w\in\mathrm{Q}$ there exists a unique $\sigma_w\in\mathrm{H}$ such that

$$a(\sigma_w,\tau)\,=\,-\,b(\tau,w) \qquad \forall\,\tau\in\mathrm{H}\,. \tag{3.38}$$

The corresponding a priori estimates are given, respectively, by

$$\|\sigma_0\|_{\mathrm{H}}\,\leq\,\frac{1}{\alpha}\,\|F\|_{H'} \quad\text{and}\quad \|\sigma_w\|_{\mathrm{H}}\,\leq\,\frac{\|b\|}{\alpha}\,\|w\|_{\mathrm{Q}} \qquad \forall\,w\in\mathrm{Q}\,. \tag{3.39}$$

Next, employing iii) and (3.38) we get for each $w\in\mathrm{Q}$

$$\beta\,\|w\|_{\mathrm{Q}}\,\leq\,\sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}}\frac{b(\tau,w)}{\|\tau\|_{\mathrm{H}}}\,=\,\sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}}\frac{a(\sigma_w,\tau)}{\|\tau\|_{\mathrm{H}}}\,,$$

from which it readily follows

$$\frac{\beta}{\|a\|}\,\|w\|_{\mathrm{Q}}\,\leq\,\|\sigma_w\|_{\mathrm{H}} \qquad \forall\,w\in\mathrm{Q}\,. \tag{3.40}$$

Now, noting from (3.38) that σ_w depends linearly on w, we can introduce the bilinear form

$$\Theta(w,v)\,:=\,c(w,v)-b(\sigma_w,v) \qquad \forall\,w,\,v\in\mathrm{Q}\,,$$

which is clearly bounded due to the same property of c and b, and the second estimate in (3.39). In addition, according to (3.38), i), ii), and (3.40), we deduce that for each $v\in\mathrm{Q}$ there holds

$$\Theta(v,v)\,=\,c(v,v)-b(\sigma_v,v)\,=\,c(v,v)+a(\sigma_v,\sigma_v)\,\geq\,\alpha\,\|\sigma_v\|^2_{\mathrm{H}}\,\geq\,\frac{\alpha\,\beta^2}{\|a\|^2}\,\|v\|^2_{\mathrm{Q}}\,,$$

which shows that Θ is Q-elliptic. Thus, applying again the Banach-Nečas-Babuška theorem, we conclude that there exists a unique $u\in\mathrm{Q}$ such that

$$\Theta(u,v)\,=\,b(\sigma_0,v)-G(v) \qquad \forall\,v\in\mathrm{Q}\,,$$

that is

$$c(u,v) - b(\sigma_u, v) \;=\; b(\sigma_0, v) - G(v) \qquad \forall\, v \in \mathrm{Q}\,,$$

which can be rearranged as

$$b(\sigma_0 + \sigma_u, v) - c(u,v) \;=\; G(v) \qquad \forall\, v \in Q\,. \tag{3.41}$$

Now, letting $\sigma \,:=\, \sigma_0 + \sigma_u \in \mathrm{H}$, it follows from (3.37) and (3.38) that

$$a(\sigma,\tau) \;=\; a(\sigma_0,\tau) + a(\sigma_u,\tau) \;=\; F(\tau) - b(\tau,u)\,,$$

that is

$$a(\sigma,\tau) + b(\tau,u) \;=\; F(\tau) \qquad \forall\, \tau \in \mathrm{H}\,,$$

which, along with (3.41), shows that $(\sigma,u) \in \mathrm{H}\times\mathrm{Q}$ is solution of (3.35). In turn, the a priori estimate for u reads

$$\|u\|_{\mathrm{Q}} \;\leq\; \frac{\|a\|^2}{\alpha\,\beta^2}\left\{\|b\|\,\|\sigma_0\|_{\mathrm{H}} + \|G\|_{\mathrm{Q}'}\right\},$$

which, using the first inequality in (3.39), becomes

$$\|u\|_{\mathrm{Q}} \;\leq\; \frac{\|a\|^2\,\|b\|}{\alpha^2\,\beta^2}\,\|F\|_{\mathrm{H}'} + \frac{\|a\|^2}{\alpha\,\beta^2}\,\|G\|_{\mathrm{Q}'}\,, \tag{3.42}$$

whereas, employing both estimates in (3.39), and (3.42), we find that

$$\|\sigma\|_{\mathrm{H}} \;\leq\; \frac{1}{\alpha}\left(1 + \frac{\|a\|^2\,\|b\|^2}{\alpha^2\,\beta^2}\right)\|F\|_{\mathrm{H}'} \;+\; \frac{\|a\|^2\,\|b\|}{\alpha^2\,\beta^2}\,\|G\|_{\mathrm{Q}'}\,. \tag{3.43}$$

Having proved the existence of a solution (σ,u) of (3.35) satisfying (3.42) and (3.43), it only remains to show the uniqueness, for which we let $(\widetilde{\sigma},\widetilde{u}) \in \mathrm{H}\times\mathrm{Q}$ be such that

$$\begin{aligned} a(\widetilde{\sigma},\tau) + b(\tau,\widetilde{u}) \;&=\; 0 \qquad \forall\, \tau \,\in\, \mathrm{H}\,,\\ b(\widetilde{\sigma},v) - c(\widetilde{u},v) \;&=\; 0 \qquad \forall\, v \,\in\, \mathrm{Q}\,. \end{aligned} \tag{3.44}$$

Then, taking $\tau = \widetilde{\sigma}$ and $v = \widetilde{u}$ in (3.44), and then subtracting the resulting equations and using ii), we get

$$0 = a(\widetilde{\sigma},\widetilde{\sigma}) + c(\widetilde{u},\widetilde{u}) \;\geq\; \alpha\,\|\widetilde{\sigma}\|_{\mathrm{H}}^2\,,$$

from which $\widetilde{\sigma} = 0$. In addition, it is clear from the first row of (3.44) and (3.38) that $\widetilde{\sigma}_{\widetilde{u}} \,=\, \widetilde{\sigma}$, which, invoking (3.40), yields $\widetilde{u} = 0$, thus confirming the uniqueness of solution for (3.35). Finally, (3.42) and (3.43) imply (3.36) and complete the proof. □

3.4 Solvability analysis

In this section we adopt a fixed-point strategy (see, e.g. [6], [28] and some references therein) to address the solvability of the variational formulation (3.23) (equivalently, that of (3.17)). To this end, we introduce the operator $\mathbf{T} : \mathbf{L}^\rho(\Omega_{\mathrm{B}}) \to \mathbf{L}^\rho(\Omega_{\mathrm{B}})$ defined by

$$\mathbf{T}(\mathbf{w}_{\mathrm{B}}) \;:=\; \mathbf{u}_{\mathrm{B}} \qquad \forall\, \mathbf{w}_{\mathrm{B}} \in \mathbf{L}^\rho(\Omega_{\mathrm{B}})\,, \tag{3.45}$$

where $\big((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}\big) \in \mathbf{H}\times\mathbf{Q}$, with $\vec{\mathbf{u}} := (\mathbf{u}_{\mathrm{B}}, p_{\mathrm{D}}, \ell) \in \mathbf{Q}$, is the unique solution (to be confirmed later) of the linear problem arising from (3.23) when $\mathcal{A}_{\mathbf{u}_{\mathrm{B}}}$ is replaced by $\mathcal{A}_{\mathbf{w}_{\mathrm{B}}}$, that is

$$\mathcal{A}_{\mathbf{w}_{\mathrm{B}}}\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}),((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\big) \;=\; \mathcal{F}\big(((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\big) \qquad \forall\,\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}}\big) \in \mathbf{H}\times\mathbf{Q}\,. \tag{3.46}$$

It follows that (3.23) can be rewritten as the fixed-point equation: Find $\mathbf{u}_{\mathrm{B}} \in \mathbf{L}^{\rho}(\Omega_{\mathrm{B}})$ such that

$$\mathbf{T}(\mathbf{u}_{\mathrm{B}}) \,=\, \mathbf{u}_{\mathrm{B}}\,. \tag{3.47}$$

Now, as suggested by the matrix representation of $\mathcal{A}_{\mathbf{w}_{\mathrm{B}}}$ (cf. (3.24)), we plan to apply Theorem 3.1 to prove the well-posedness of (3.46), thus confirming that $\mathbf{T}$ is well-defined. To this end, we first recall that the stability properties of all the forms involved in (3.46) were established in (3.30) and (3.31). Next, and due to the diagonal structure of $\mathbf{B}$, we realize that its kernel $\mathbf{V}$ reduces to $\mathbf{V} := \mathbf{V}_1 \times \mathbf{V}_2$, where

$$\mathbf{V}_1 \,:=\, \Big\{ \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{B}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{H}_1 : \quad \mathbf{div}(\boldsymbol{\tau}_{\mathrm{B}}) = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{B}} \quad \text{and} \quad \mathrm{div}(\mathbf{v}_{\mathrm{D}}) \in \mathrm{P}_0(\Omega_{\mathrm{D}}) \Big\}\,, \quad \text{and} \tag{3.48}$$

$$\mathbf{V}_2 \,:=\, \Big\{ \vec{\boldsymbol{\psi}} := (\boldsymbol{\psi}, \xi) \in \mathbf{H}_2 : \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_{\Sigma} = 0 \Big\}\,. \tag{3.49}$$

Hereafter, we refer to the null space of the bounded linear operator induced by a bilinear form as the kernel of the latter. Then, in order to prove the invertibility of $\mathbf{A} \,=\, \begin{pmatrix} \mathbf{a} & \mathbf{b}_1 \\ \mathbf{b}_2 & -\mathbf{c} \end{pmatrix}$ in $\mathbf{V}$, which, as said in Section 3.2, is equivalent to that of $\widetilde{\mathbf{A}} \,=\, \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & -\widetilde{\mathbf{c}} \end{pmatrix}$, we proceed in what follows to show that $\widetilde{\mathbf{A}}$ satisfy the hypotheses of Theorem 3.2. We begin with the $\mathbf{V}_1$-ellipticity of a.

Lemma 3.3 *There exists a positive constant $\alpha_{\mathbf{a}}$, depending only on μ and $C_{\mathbf{K}_{\mathrm{D}}}$ (cf. (2.4)), such that*

$$\mathbf{a}(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \,\geq\, \alpha_{\mathbf{a}}\, \|\vec{\boldsymbol{\tau}}\|^2_{\mathbf{H}_1} \qquad \forall\, \vec{\boldsymbol{\tau}} \in \mathbf{V}_1\,.$$

Proof. Given $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{B}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{V}_1$, and thanks to (2.4) and the divergence free property of $\boldsymbol{\tau}_{\mathrm{B}}$, we obtain

$$\mathbf{a}(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \,\geq\, \frac{1}{\mu}\|\boldsymbol{\tau}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, C_{\mathbf{K}_{\mathrm{D}}}\|\mathbf{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}} \,=\, \frac{1}{\mu}\|\boldsymbol{\tau}_{\mathrm{B}}\|^2_{\mathbf{div}_{\varrho};\Omega_{\mathrm{B}}} \,+\, C_{\mathbf{K}_{\mathrm{D}}}\|\mathbf{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}}\,. \tag{3.50}$$

In turn, since $\mathrm{div}(\mathbf{v}_{\mathrm{D}}) \in \mathrm{P}_0(\Omega_{\mathrm{D}})$, it follows from [29, Lemma 3.2] that there exists a positive constant c such that

$$\|\mathbf{v}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \,\geq\, c\,\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}\,, \tag{3.51}$$

which, along with (3.50), conclude the proof. $\square$

Next, the required inf-sup condition for $\mathbf{b} = \mathbf{b}_1$ is stated as follows.

Lemma 3.4 *There exists a positive constant β, depending only on Ω_{B} and Ω_{D}, such that*

$$\sup_{\substack{\vec{\boldsymbol{\tau}} \in \mathbf{V}_1 \\ \vec{\boldsymbol{\tau}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}})}{\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}_1}} \,\geq\, \beta \|\vec{\boldsymbol{\psi}}\|_{\mathbf{H}_2} \qquad \forall\, \vec{\boldsymbol{\psi}} \in \mathbf{V}_2\,. \tag{3.52}$$

Proof. As remarked in Section 3.2 (see also the matrix structure in (3.15)), we now take advantage of the diagonal structure of $\mathbf{b}$ to facilitate the derivation of (3.52). Indeed, given $\vec{\boldsymbol{\psi}} = (\boldsymbol{\psi}, \xi) \in \mathbf{V}_2$, and bearing in mind (3.48), it is easily seen that

$$\mathcal{R}_1(\boldsymbol{\psi}) \,+\, \mathcal{R}_2(\xi) \,\geq\, \sup_{\substack{\vec{\boldsymbol{\tau}} \in \mathbf{V}_1 \\ \vec{\boldsymbol{\tau}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}})}{\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}_1}} \,\geq\, \frac{1}{2}\Big(\mathcal{R}_1(\boldsymbol{\psi}) \,+\, \mathcal{R}_2(\xi)\Big)\,, \tag{3.53}$$

where

$$\mathcal{R}_1(\boldsymbol{\psi}) := \sup_{\substack{\boldsymbol{\tau}_{\mathrm{B}}\in\mathbb{H}_0(\mathbf{div}_\varrho;\Omega_{\mathrm{B}})\backslash\{0\}\\ \mathbf{div}(\boldsymbol{\tau}_{\mathrm{B}})=\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_{\mathrm{B}}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{B}}\|_{\mathbf{div}_\varrho;\Omega_{\mathrm{B}}}}\,, \quad \text{and} \quad \mathcal{R}_2(\xi) := \sup_{\substack{\mathbf{v}_{\mathrm{D}}\in\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\backslash\{0\}\\ \mathrm{div}(\mathbf{v}_{\mathrm{D}})\in\mathrm{P}_0(\Omega_{\mathrm{D}})}} \frac{\langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}}\,, \tag{3.54}$$

and hence, in order to prove (3.52), it suffices to suitably bound from below the above suprema. We begin with $\mathcal{R}_1(\boldsymbol{\psi})$ by reasoning as in the proof of [30, Lemma 3.3] (see, also [4, Theorem 2.1]), that is, by taking $\boldsymbol{\eta}\in\mathbf{H}_{00}^{-1/2}(\Sigma)$ and defining $\widetilde{\boldsymbol{\tau}}_{\mathrm{B}} := \nabla\mathbf{z}_{\mathrm{B}} - d_{\mathrm{B}}\,\mathbb{I}$, where $\mathbf{z}_{\mathrm{B}}\in\mathbf{H}^1(\Omega_{\mathrm{B}})$ is the unique solution of

$$-\Delta\mathbf{z}_{\mathrm{B}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{B}}\,, \quad \mathbf{z}_{\mathrm{B}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{B}}\,, \quad \nabla\mathbf{z}_{\mathrm{B}}\,\mathbf{n} = \boldsymbol{\eta} \quad \text{on} \quad \Sigma\,, \tag{3.55}$$

and $d_{\mathrm{B}}\in\mathrm{R}$ is chosen such that $\int_{\Omega_{\mathrm{B}}}\mathrm{tr}(\widetilde{\boldsymbol{\tau}}_{\mathrm{B}}) = 0$, that is $d_{\mathrm{B}} := \frac{1}{n|\Omega_{\mathrm{B}}|}\int_{\Omega_{\mathrm{B}}}\mathrm{tr}(\nabla\mathbf{z}_{\mathrm{B}})$. It follows that $\mathbf{div}(\widetilde{\boldsymbol{\tau}}_{\mathrm{B}}) = \mathbf{0}$ in Ω_{B}, $\widetilde{\boldsymbol{\tau}}_{\mathrm{B}}\mathbf{n} = \boldsymbol{\eta} - d_{\mathrm{B}}\,\mathbf{n}$ on Σ, and, thanks to the a priori estimate for the solution of (3.55), there exists a constant $C_{\mathrm{B}}>0$, depending only on Ω_{B}, such that

$$\|\widetilde{\boldsymbol{\tau}}_{\mathrm{B}}\|_{\mathbf{div}_\varrho;\Omega_{\mathrm{B}}} = \|\widetilde{\boldsymbol{\tau}}_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}} \le C_{\mathrm{B}}\,\|\boldsymbol{\eta}\|_{-1/2,00;\Sigma}\,.$$

Thus, recalling from (3.49) that $\langle\boldsymbol{\psi}\cdot\mathbf{n},1\rangle_\Sigma = 0$, we deduce that

$$\mathcal{R}_1(\boldsymbol{\psi}) \ge \frac{\langle\widetilde{\boldsymbol{\tau}}_{\mathrm{B}}\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma}{\|\widetilde{\boldsymbol{\tau}}_{\mathrm{B}}\|_{\mathbf{div}_\varrho;\Omega_{\mathrm{B}}}} = \frac{\langle\boldsymbol{\eta},\boldsymbol{\psi}\rangle_\Sigma}{\|\widetilde{\boldsymbol{\tau}}_{\mathrm{B}}\|_{\mathbf{div}_\varrho;\Omega_{\mathrm{B}}}} \ge \frac{1}{C_{\mathrm{B}}}\,\frac{\langle\boldsymbol{\eta},\boldsymbol{\psi}\rangle_\Sigma}{\|\boldsymbol{\eta}\|_{-1/2,00;\Sigma}}\,,$$

from which, taking supremum over $\boldsymbol{\eta}\in\mathbf{H}_{00}^{-1/2}(\Sigma)$, $\boldsymbol{\eta}\neq\mathbf{0}$, we get

$$\mathcal{R}_1(\boldsymbol{\psi}) \ge \beta_1\,\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}\,. \tag{3.56}$$

with $\beta_1 := C_{\mathrm{B}}^{-1}$. We proceed similarly with $\mathcal{R}_2(\xi)$. In fact, given $\eta\in\mathrm{H}^{-1/2}(\Sigma)$, we extend it by zero to Γ_{D} by defining $\widetilde{\eta}\in\mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{D}})$ as

$$\langle\widetilde{\eta},\phi\rangle_{\partial\Omega_{\mathrm{D}}} := \langle\eta,\phi|_\Sigma\rangle_\Sigma \qquad \forall\,\phi\in\mathrm{H}^{1/2}(\partial\Omega_{\mathrm{D}})\,. \tag{3.57}$$

In fact, by exchanging the roles of Σ and Γ_{D} in (3.7), which means extending by 0 from Γ_{D} to Σ, it is easily seen, according to (3.57), that $\widetilde{\eta}|_{\Gamma_{\mathrm{D}}}$ becomes the null functional of $\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})$, and hence, as stated at the end of Section 3.1, $\widetilde{\eta}|_\Sigma$ can be identified with a functional in $\mathrm{H}^{-1/2}(\Sigma)$, namely

$$\langle\widetilde{\eta},\psi\rangle_\Sigma := \langle\widetilde{\eta},\mathrm{E}_{\mathrm{D}}(\psi)\rangle_{\partial\Omega_{\mathrm{D}}} \qquad \forall\,\psi\in\mathrm{H}^{1/2}(\Sigma)\,, \tag{3.58}$$

where $\mathrm{E}_{\mathrm{D}}:\mathrm{H}^{1/2}(\Sigma)\to\mathrm{H}^{1/2}(\partial\Omega_{\mathrm{D}})$ is any bounded linear extension operator. In this way, it is clear from (3.58) and (3.57) that

$$\langle\widetilde{\eta},\psi\rangle_\Sigma = \langle\eta,\psi\rangle_\Sigma \qquad \forall\,\psi\in\mathrm{H}^{1/2}(\Sigma)\,. \tag{3.59}$$

In addition, it is not difficult to show (see, e.g. [23, Section 2]) that there exists a constant $\widetilde{c}_{\mathrm{D}}>0$, depending only on Ω_{D}, such that

$$\|\widetilde{\eta}\|_{-1/2,\partial\Omega_{\mathrm{D}}} \le \widetilde{c}_{\mathrm{D}}\,\|\eta\|_{-1/2,\Sigma}\,. \tag{3.60}$$

Having established the above, we now set $\widetilde{\mathbf{v}}_{\mathrm{D}} := \nabla z_{\mathrm{D}}$, where $z_{\mathrm{D}}\in\mathrm{H}^1(\Omega_{\mathrm{D}})$ is the unique solution of

$$-\Delta z_{\mathrm{D}} = -\frac{1}{|\Omega_{\mathrm{D}}|}\,\langle\widetilde{\eta},1\rangle_{\partial\Omega_{\mathrm{D}}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \quad \nabla z_{\mathrm{D}}\cdot\mathbf{n} = \widetilde{\eta} \quad \text{on} \quad \partial\Omega_{\mathrm{D}}\,, \quad \int_{\Omega_{\mathrm{D}}} z_{\mathrm{D}} = 0\,. \tag{3.61}$$

Note that the right hand sides of the first and second equalities in (3.61) satisfy the compatibility condition required by this Neumann boundary value problem. It follows that $\mathrm{div}(\widetilde{\mathbf{v}}_{\mathrm{D}})\in\mathrm{P}_0(\Omega_{\mathrm{D}})$, and

$\widetilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n} = \widetilde{\eta}$ on $\partial\Omega_{\mathrm{D}}$, so that, in particular, $\widetilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n}|_{\Gamma_{\mathrm{D}}} = \widetilde{\eta}|_{\Gamma_{\mathrm{D}}} = 0$. In addition, the a priori estimate for the solution of (3.61) ensures the existence of a constant $\widetilde{C}_{\mathrm{D}} > 0$, depending only on Ω_{D}, such that $\|z_{\mathrm{D}}\|_{1,\Omega} \le \widetilde{C}_{\mathrm{D}} \|\widetilde{\eta}\|_{-1/2,\partial\Omega_{\mathrm{D}}}$, and thus, invoking (3.51) and (3.60), we find that

$$\|\widetilde{\mathbf{v}}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}} \le c^{-1} \|\widetilde{\mathbf{v}}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} = c^{-1} |z_{\mathrm{D}}|_{1,\Omega} \le c^{-1} \widetilde{C}_{\mathrm{D}} \|\widetilde{\eta}\|_{-1/2,\partial\Omega_{\mathrm{D}}} \le C_{\mathrm{D}} \|\eta\|_{-1/2,\Sigma}, \tag{3.62}$$

with $C_{\mathrm{D}} := c^{-1} \widetilde{C}_{\mathrm{D}} \widetilde{c}_{\mathrm{D}}$. Consequently, employing (3.59) and (3.62), we deduce that

$$\mathcal{R}_2(\xi) \ge \frac{\langle \widetilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n}, \xi \rangle_\Sigma}{\|\widetilde{\mathbf{v}}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} = \frac{\langle \widetilde{\eta}, \xi \rangle_\Sigma}{\|\widetilde{\mathbf{v}}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} = \frac{\langle \eta, \xi \rangle_\Sigma}{\|\widetilde{\mathbf{v}}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \ge \frac{1}{C_{\mathrm{D}}} \frac{\langle \eta, \xi \rangle_\Sigma}{\|\eta\|_{-1/2,\Sigma}},$$

from which, taking supremum over $\eta \in \mathrm{H}^{1/2}(\Sigma)$, $\eta \neq 0$, we obtain

$$\mathcal{R}_2(\xi) \ge \beta_2 \|\xi\|_{1/2,\Sigma}. \tag{3.63}$$

with $\beta_2 := C_{\mathrm{D}}^{-1}$. Finally, (3.53), (3.56), and (3.63) lead to (3.52) with $\beta := \frac{1}{2} \min\{\beta_1, \beta_2\}$. □

Bearing in mind (3.28), along with Lemmas 3.3 and 3.4, we conclude that $\widetilde{\mathbf{A}}$ (cf. (3.27)) satisfies the hypotheses of Theorem 3.2, whence this matrix operator, and thus $\mathbf{A}$ as well, is invertible in $\mathbf{V}$. Moreover, it is readily seen that the same holds by exchanging the roles of $\mathbf{b}_1$ and $\mathbf{b}_2$ in $\mathbf{A}$, so that we can finally establish the following result.

Lemma 3.5 *There exists a positive constant $\alpha_{\mathbf{A}}$, depending only on $\|\mathbf{a}\|$, $\|\mathbf{b}\| = \|\mathbf{b}_1\| = \|\mathbf{b}_2\|$, $\alpha_{\mathbf{a}}$, and β, such that*

$$\sup_{\substack{(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\in\mathbf{V}\\(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\neq\mathbf{0}}} \frac{\mathbf{A}((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}),(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}))}{\|(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\|_{\mathbf{H}}} \ge \alpha_{\mathbf{A}} \|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}})\|_{\mathbf{H}} \qquad \forall (\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}) \in \mathbf{V},$$

and

$$\sup_{\substack{(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}})\in\mathbf{V}\\(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}})\neq\mathbf{0}}} \frac{\mathbf{A}((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}),(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}))}{\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}})\|_{\mathbf{H}}} \ge \alpha_{\mathbf{A}} \|(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\|_{\mathbf{H}} \qquad \forall (\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}) \in \mathbf{V}.$$

We continue the analysis by proving the continuous inf-sup condition for $\mathbf{B}$.

Lemma 3.6 *There exists a positive constant $\boldsymbol{\beta}$ such that*

$$\sup_{\substack{(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\in\mathbf{H}\\(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\neq\mathbf{0}}} \frac{\mathbf{B}((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})}{\|(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\|_{\mathbf{H}}} \ge \boldsymbol{\beta} \|\vec{\mathbf{v}}\|_{\mathbf{Q}} \quad \forall \vec{\mathbf{v}} \in \mathbf{Q}. \tag{3.64}$$

Proof. We begin by noticing that, given $\vec{\mathbf{v}} := (\mathbf{v}_{\mathrm{B}}, q_{\mathrm{D}}, \jmath) \in \mathbf{Q}$, the diagonal structure of $\mathbf{B}$ (cf. (3.20)) allows to show that

$$\mathcal{S}_1(\mathbf{v}_{\mathrm{B}}) + \mathcal{S}_2(q_{\mathrm{D}}) + \mathcal{S}_3(\jmath) \ge \sup_{\substack{(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\in\mathbf{H}\\(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\neq\mathbf{0}}} \frac{\mathbf{B}((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})}{\|(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}})\|_{\mathbf{H}}} \ge \frac{1}{3}\Big(\mathcal{S}_1(\mathbf{v}_{\mathrm{B}}) + \mathcal{S}_2(q_{\mathrm{D}}) + \mathcal{S}_3(\jmath)\Big), \tag{3.65}$$

where

$$\mathcal{S}_1(\mathbf{v}_{\mathrm{B}}) = \sup_{\substack{\boldsymbol{\tau}_{\mathrm{B}}\in\mathbb{H}_0(\mathbf{div}_\varrho;\Omega_{\mathrm{B}})\\\boldsymbol{\tau}_{\mathrm{B}}\neq\mathbf{0}}} \frac{(\mathbf{v}_{\mathrm{B}}, \mathbf{div}(\boldsymbol{\tau}_{\mathrm{B}}))_{\mathrm{B}}}{\|\boldsymbol{\tau}_{\mathrm{B}}\|_{\mathbf{div}_\varrho;\Omega_{\mathrm{B}}}}, \quad \mathcal{S}_2(q_{\mathrm{D}}) := \sup_{\substack{\mathbf{v}_{\mathrm{D}}\in\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\\\mathbf{v}_{\mathrm{D}}\neq\mathbf{0}}} \frac{(q_{\mathrm{D}}, \mathrm{div}(\mathbf{v}_{\mathrm{D}}))_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}}, \tag{3.66}$$

and

$$\mathcal{S}_3(\jmath) \,:=\, \sup_{\substack{\boldsymbol{\psi}\in\mathbf{H}^{1/2}_{00}(\Sigma)\\ \boldsymbol{\psi}\neq\mathbf{0}}} \frac{\jmath\,\langle\boldsymbol{\psi}\cdot\mathbf{n},1\rangle_\Sigma}{\|\boldsymbol{\psi}\|_{-1/2,00;\Sigma}}\,, \tag{3.67}$$

so that, similarly to the proof of Lemma 3.4, the rest of the proof reduces to bounding from below the above suprema. Indeed, we begin with $\mathcal{S}_1$ by letting, as in [16, Section 4.2.1], $\mathbf{v}_\varrho := |\mathbf{v}_{\rm B}|^{\rho-2}\mathbf{v}_{\rm B}$, which is easily seen to belong to $\mathbf{L}^\varrho(\Omega_{\rm B})$ and satisfy

$$\int_{\Omega_{\rm B}} \mathbf{v}_{\rm B}\cdot\mathbf{v}_\varrho \,=\, \|\mathbf{v}_{\rm B}\|_{0,\rho;\Omega_{\rm B}}\,\|\mathbf{v}_\varrho\|_{0,\varrho;\Omega_{\rm B}}\,. \tag{3.68}$$

Then, we let $\mathbf{w}_{\rm B}$ be the unique element in $\mathbf{H}^1_0(\Omega_{\rm B})$ such that

$$\int_{\Omega_{\rm B}} \nabla\mathbf{w}_{\rm B} : \nabla\mathbf{z} \,=\, -\int_\Omega \mathbf{v}_\varrho\cdot\mathbf{z} \qquad \forall\,\mathbf{z}\in\mathbf{H}^1_0(\Omega_{\rm B})\,,$$

which is guaranteed by the Lax-Milgram lemma, and notice, thanks to the corresponding a priori estimate, that $\|\mathbf{w}_{\rm B}\|_{1,\Omega} \,\le\, \frac{\|\mathbf{i}_\rho\|}{c_P}\,\|\mathbf{v}_\varrho\|_{0,\varrho;\Omega_{\rm B}}$. Hereafter, $\mathbf{i}_\rho$ stands for the continuous injection from $\mathbf{H}^1(\Omega_{\rm B})$ into $\mathbf{L}^\rho(\Omega_{\rm B})$, and c_P is the positive constant establishing $c_P\,\|\cdot\|_{1,\Omega_{\rm B}} \,\le\, |\cdot|_{1,\Omega_{\rm B}}$ in $\mathbf{H}^1_0(\Omega_{\rm B})$. Next, defining $\boldsymbol{\zeta} := \nabla\mathbf{w}_{\rm B}$, we readily see that $\mathbf{div}(\boldsymbol{\zeta}) \,=\, \mathbf{v}_\varrho$ in $\Omega_{\rm B}$, so that

$$\boldsymbol{\zeta}\in\mathbb{H}(\mathbf{div}_\varrho;\Omega_{\rm B}) \quad\text{and}\quad \|\boldsymbol{\zeta}\|_{\mathbf{div}_\varrho;\Omega_{\rm B}} \,\le\, \left(1+\frac{\|\mathbf{i}_\rho\|}{c_P}\right)\|\mathbf{v}_\varrho\|_{0,\varrho;\Omega_{\rm B}}\,. \tag{3.69}$$

Thus, letting $\boldsymbol{\zeta}_0$ be the $\mathbb{H}_0(\mathbf{div}_\varrho;\Omega_{\rm B})$-component of $\boldsymbol{\zeta}$, we observe that $\mathbf{div}(\boldsymbol{\zeta}_0) = \mathbf{v}_\varrho$, whence bounding $\mathcal{S}_1(\mathbf{v}_{\rm B})$ by below with $\boldsymbol{\tau}_{\rm B} \,=\, \boldsymbol{\zeta}_0$, noting that $\|\boldsymbol{\zeta}_0\|_{\mathbf{div}_\varrho;\Omega_{\rm B}} \,\le\, \|\boldsymbol{\zeta}\|_{\mathbf{div}_\varrho;\Omega_{\rm B}}$, and employing (3.68) and (3.69), we deduce that

$$\mathcal{S}_1(\mathbf{v}_{\rm B}) \,\ge\, \frac{(\mathbf{v}_{\rm B},\mathbf{div}(\boldsymbol{\zeta}_0))_{\rm B}}{\|\boldsymbol{\zeta}_0\|_{\mathbf{div}_\varrho;\Omega_{\rm B}}} \,=\, \frac{\displaystyle\int_{\Omega_{\rm B}}\mathbf{v}_{\rm B}\cdot\mathbf{v}_\varrho}{\|\boldsymbol{\zeta}_0\|_{\mathbf{div}_\varrho;\Omega_{\rm B}}} \,\ge\, \boldsymbol{\beta}_1\,\|\mathbf{v}_{\rm B}\|_{0,\rho;\Omega_{\rm B}}\,, \tag{3.70}$$

with $\boldsymbol{\beta}_1 \,:=\, \left(1+\frac{\|\mathbf{i}_\rho\|}{c_P}\right)^{-1}$. In turn, regarding $\mathcal{S}_2(q_{\rm D})$, we let z be the unique element in $\widetilde{\mathrm{H}}^1(\Omega_{\rm D}) := \left\{v\in\mathrm{H}^1(\Omega_{\rm D}) : \quad \int_{\Omega_{\rm D}} v = 0\right\}$, whose existence follows from the Lax-Milgram lemma as well, such that

$$\int_{\Omega_{\rm D}} \nabla z\cdot\nabla v \,=\, -\int_{\Omega_{\rm D}} q_{\rm D}\,v \qquad \forall\,v\in\widetilde{\mathrm{H}}^1(\Omega_{\rm D})\,, \tag{3.71}$$

and define $\mathbf{w}_{\rm D} \,:=\, \nabla z$. The fact that $q_{\rm D}\,\in\,\mathrm{L}^2_0(\Omega_{\rm D})$ implies that (3.71) is equivalent to requiring it for all $v\,\in\,\mathrm{H}^1(\Omega_{\rm D})$, from which it is easy to see that $\mathrm{div}(\mathbf{w}_{\rm D}) \,=\, q_{\rm D}$ in $\Omega_{\rm D}$ and $\mathbf{w}_{\rm D}\cdot\mathbf{n} \,=\, 0$ on $\partial\Omega_{\rm D}$. It follows that $\mathbf{w}_{\rm D}\,\in\,\mathbf{H}_{\Gamma_{\rm D}}(\mathrm{div};\Omega_{\rm D})$, and that there exists a positive constant $C_{\rm D}$ such that $\|\mathbf{w}_{\rm D}\|_{\mathrm{div};\Omega_{\rm D}} \,\le\, C_{\rm D}\,\|q_{\rm D}\|_{0,\Omega_{\rm D}}$. In this way, bounding $\mathcal{S}_2(q_{\rm D})$ by below with $\mathbf{v}_{\rm D} = \mathbf{w}_{\rm D}$, we find that

$$\mathcal{S}_2(q_{\rm D}) \,\ge\, \frac{(\mathrm{div}(\mathbf{w}_{\rm D}),q_{\rm D})_{\rm D}}{\|\mathbf{w}_{\rm D}\|_{\mathrm{div};\Omega_{\rm D}}} \,\ge\, \boldsymbol{\beta}_2\,\|q_{\rm D}\|_{0,\Omega_{\rm D}}\,, \tag{3.72}$$

with $\boldsymbol{\beta}_2 \,:=\, C_{\rm D}^{-1}$. In turn, following the remark right after the proof of [30, Lemma 3.2], which is actually taken from the last part of the proof of [29, Lemma 3.6], we can construct $\boldsymbol{\psi}_0\in\mathbf{H}^{1/2}_{00}(\Sigma)$ such that $\langle\boldsymbol{\psi}_0\cdot\mathbf{n},1\rangle_\Sigma \neq 0$. Thus, we readily find that

$$\mathcal{S}_3(\jmath) \,\ge\, \boldsymbol{\beta}_3\,|\jmath|\,, \tag{3.73}$$

with $\boldsymbol{\beta}_3 := \frac{|\langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1\rangle_\Sigma|}{\|\boldsymbol{\psi}_0\|_{1/2,00,\Sigma}}$. Actually, it is easy to see that the existence of such $\boldsymbol{\psi}_0$ is equivalent to proving (3.73). Finally, employing (3.70), (3.72), and (3.73) back into (3.65), we reach (3.64) with

$$\boldsymbol{\beta} := \frac{1}{3} \min\left\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3\right\}.$$

□

We are now in position to prove the well-posedness of (3.46), equivalently that $\mathbf{T}$ is well-defined.

Lemma 3.7 *Given $r > 0$, we let $\mathbf{w}_\mathrm{B} \in \mathbf{L}^\rho(\Omega_\mathrm{B})$ be such that $\|\mathbf{w}_\mathrm{B}\|_{0,\rho;\Omega_\mathrm{B}} \le r$. Then, there exists a unique solution $\big((\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\varphi}}), \vec{\mathbf{u}}\big) \in \mathbf{H} \times \mathbf{Q}$ of (3.46), with $\vec{\mathbf{u}} := (\mathbf{u}_\mathrm{B}, p_\mathrm{D}, \ell) \in \mathbf{Q}$, and hence one can define $\mathbf{T}(\mathbf{w}_\mathrm{B}) := \mathbf{u}_\mathrm{B}$. In addition, there exists a positive constant $C_\mathbf{T}$, depending on $\|\mathbf{A}\|$ (cf. (3.29) - (3.30)), $\|\mathbf{C}\|$ (cf. (3.32)), $\mathtt{F}$, r, ρ, $\alpha_\mathbf{A}$, and $\boldsymbol{\beta}$, such that*

$$\|\mathbf{T}(\mathbf{w}_\mathrm{B})\|_{0,\rho;\Omega_\mathrm{B}} = \|\mathbf{u}_\mathrm{B}\|_{0,\rho;\Omega_\mathrm{B}} \le \big\|\big((\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\varphi}}), \vec{\mathbf{u}}\big)\big\|_{\mathbf{H}\times\mathbf{Q}} \le C_\mathbf{T}\left\{\|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}} + \|\mathbf{f}_\mathrm{B}\|_{0,\varrho;\Omega_\mathrm{B}} + \|g_\mathrm{D}\|_{0,\Omega_\mathrm{D}}\right\}. \quad (3.74)$$

Proof. We begin by remarking that the bilinear forms $\mathbf{A}$ (cf. (3.18)) and $\mathbf{C}_{\mathbf{w}_\mathrm{B}}$ (cf. (3.21)) satisfy the hypothesis i) of Theorem 3.1. In particular, the semi-positiveness of them was established by (3.25) and (3.26). In addition, Lemmas 3.5 and 3.6 provide the respective assumptions ii) and iii). In this way, bearing in mind the structure described by (3.24), and applying the aforementioned abstract result, we conclude the unique solvability of (3.46), which, according to the estimate for $\|\mathcal{F}\|$ provided by (3.29), satisfies (3.74) with a positive constant $C_\mathbf{T}$, depending on $\|\mathbf{A}\|$, $\|\mathbf{C}_{\mathbf{w}_\mathrm{B}}\|$, $\alpha_\mathbf{A}$, and $\boldsymbol{\beta}$. Finally, it is clear from (3.31) that we can take $\|\mathbf{C}_{\mathbf{w}_\mathrm{B}}\| = \|\mathbf{C}\| + \mathtt{F}\, r^{\rho-2}$, which completes the proof. □

Having established the above lemma, and realizing that an analogue result is attained if we consider the transpose of $\mathcal{A}_{\mathbf{w}_\mathrm{B}}$, which simply reduces to exchange the bilinear forms $\mathbf{b}_1$ and $\mathbf{b}_2$ in (3.24), we conclude that global inf-sup conditions are satisfied by $\mathcal{A}_{\mathbf{w}_\mathrm{B}}$ with respect to both components. More precisely, there exists a positive constant $\alpha_\mathcal{A}$, which depends only on $C_\mathbf{T}$, and hence on $\|\mathbf{A}\|$, $\|\mathbf{C}\|$, $\mathtt{F}$, r, ρ, $\alpha_\mathbf{A}$, and $\boldsymbol{\beta}$, such that for each $\mathbf{w}_\mathrm{B} \in \mathbf{L}^\rho(\Omega_\mathrm{B})$ with $\|\mathbf{w}_\mathrm{B}\|_{0,\rho;\Omega_\mathrm{B}} \le r$, there holds

$$\sup_{\substack{((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\\ ((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathcal{A}_{\mathbf{w}_\mathrm{B}}\big(((\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\phi}}), \vec{\mathbf{z}}), ((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}})\big)}{\|((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}} \ge \alpha_\mathcal{A}\, \|((\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\phi}}), \vec{\mathbf{z}})\|_{\mathbf{H}\times\mathbf{Q}} \qquad \forall\, ((\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\phi}}), \vec{\mathbf{z}}) \in \mathbf{H}\times\mathbf{Q}, \quad (3.75)$$

and

$$\sup_{\substack{((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}),\vec{\mathbf{z}})\in\mathbf{H}\times\mathbf{Q}\\ ((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}),\vec{\mathbf{z}})\neq\mathbf{0}}} \frac{\mathcal{A}_{\mathbf{w}_\mathrm{B}}\big(((\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\phi}}), \vec{\mathbf{z}}), ((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}})\big)}{\|((\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\phi}}), \vec{\mathbf{z}})\|_{\mathbf{H}\times\mathbf{Q}}} \ge \alpha_\mathcal{A}\, \|((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}} \qquad \forall\, ((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q}. \quad (3.76)$$

In what follows, we apply the well-known Banach fixed-point theorem to prove the unique solvability of (3.47). To this end, given $r > 0$, we first introduce the closed ball in $\mathbf{L}^\rho(\Omega_\mathrm{B})$ centered at the origin with radius r, namely

$$\mathbf{W}_r := \left\{\mathbf{w}_\mathrm{B} \in \mathbf{L}^\rho(\Omega_\mathrm{B}) : \quad \|\mathbf{w}_\mathrm{B}\|_{0,\rho;\Omega_\mathrm{B}} \le r\right\}, \quad (3.77)$$

and notice that, under the assumption

$$C_\mathbf{T}\left\{\|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}} + \|\mathbf{f}_\mathrm{B}\|_{0,\varrho;\Omega_\mathrm{B}} + \|g_\mathrm{D}\|_{0,\Omega_\mathrm{D}}\right\} \le r, \quad (3.78)$$

the a priori estimate (3.74) guarantees that $\mathbf{T}$ maps $\mathbf{W}_r$ into itself.

Our next goal is to prove the Lipschitz continuity of the operator $\mathbf{T}$ (cf. (3.45)), for which we need the slight generalization of [6, Lemma 4.4] given by the following result.

Lemma 3.8 *For each $\rho \in [3,4]$ there exists a positive constant $C(\rho)$, depending only on ρ, such that*

$$\begin{aligned}
&\left|\big((|\mathbf{w}_{\mathrm{B}}|^{\rho-2} - |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|^{\rho-2})\mathbf{z}_{\mathrm{B}}, \mathbf{v}_{\mathrm{B}}\big)_{\mathrm{B}}\right| \\
&\quad \le\ C(\rho)\left\{\|\mathbf{w}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}} + \|\underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\right\}^{\rho-3} \|\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\, \|\mathbf{z}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\, \|\mathbf{v}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}
\end{aligned} \tag{3.79}$$

for each $\mathbf{w}_{\mathrm{B}}$, $\underset{\sim}{\mathbf{w}}_{\mathrm{B}}$, $\mathbf{z}_{\mathrm{B}}$, $\mathbf{v}_{\mathrm{B}} \in \mathbf{L}^\rho(\Omega_{\mathrm{B}})$.

Proof. We begin by recalling from the first half of the proof of [6, Lemma 4.4], which, in turn, makes use of the key estimate provided by [32, Lemma 5.3], that there holds (cf. first inequality right after [6, eq. (4.36)])

$$\left|\big((|\mathbf{w}_{\mathrm{B}}|^{\rho-2} - |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|^{\rho-2})\mathbf{z}_{\mathrm{B}}, \mathbf{v}_{\mathrm{B}}\big)_{\mathrm{B}}\right| \ \le\ C(\rho)\int_{\Omega_{\mathrm{B}}} \big(|\mathbf{w}_{\mathrm{B}}| + |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\big)^{\rho-3} |\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\, |\mathbf{z}_{\mathrm{B}}\cdot\mathbf{v}_{\mathrm{B}}|\,. \tag{3.80}$$

Next, applying Hölder's inequality with conjugate indexes $t = \frac{\rho}{\rho-2}$ and $t' = \frac{\rho}{2}$ to the right hand side of (3.80), and then Cauchy-Schwarz's inequality to the resulting second factor, we obtain

$$\begin{aligned}
&\int_{\Omega_{\mathrm{B}}} \big(|\mathbf{w}_{\mathrm{B}}| + |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\big)^{\rho-3} |\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\, |\mathbf{z}_{\mathrm{B}}\cdot\mathbf{v}_{\mathrm{B}}| \\
&\qquad \le\ \big\|\big(|\mathbf{w}_{\mathrm{B}}| + |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\big)^{\rho-3} |\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\,\big\|_{0,t;\Omega_{\mathrm{B}}}\, \|\mathbf{z}_{\mathrm{B}}\cdot\mathbf{v}_{\mathrm{B}}\|_{0,t';\Omega_{\mathrm{B}}} \\
&\qquad \le\ \big\|\big(|\mathbf{w}_{\mathrm{B}}| + |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\big)^{\rho-3} |\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\,\big\|_{0,t;\Omega_{\mathrm{B}}}\, \|\mathbf{z}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\, \|\mathbf{v}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,,
\end{aligned} \tag{3.81}$$

which, along with (3.80), easily yields (3.79) for $\rho = 3$. In turn, when $\rho \in (3,4]$, the first factor above is bounded by employing Hölder's inequality again, but now with conjugate indexes $r = \frac{\rho-2}{\rho-3}$ and $r' = \rho - 2$. In this way, noting in this case that $tr = \frac{\rho}{\rho-3}$ and $tr' = \rho$, and using the triangle inequality in the last step, we are led to

$$\begin{aligned}
\big\|\big(|\mathbf{w}_{\mathrm{B}}| + |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\big)^{\rho-3} |\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\,\big\|_{0,t;\Omega_{\mathrm{B}}} \ &\le\ \big\|\big(|\mathbf{w}_{\mathrm{B}}| + |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\big)^{\rho-3}\big\|_{0,\frac{\rho}{\rho-3};\Omega_{\mathrm{B}}}\, \|\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}} \\
&=\ \big\|\,|\mathbf{w}_{\mathrm{B}}| + |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|\,\big\|_{0,\rho;\Omega_{\mathrm{B}}}^{\rho-3}\, \|\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}} \\
&\le\ \big(\|\mathbf{w}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}} + \|\underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\big)^{\rho-3}\, \|\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,,
\end{aligned}$$

which, jointly with (3.81) and (3.80), imply (3.79) and complete the proof. $\square$

We are now in position to establish the announced result on $\mathbf{T}$.

Lemma 3.9 *There exists a positive constant $L_{\mathbf{T}}$, depending only on $\alpha_{\mathcal{A}}$, $\mathtt{F}$, ρ, r, and $C_{\mathbf{T}}$, such that*

$$\|\mathbf{T}(\mathbf{w}_{\mathrm{B}}) - \mathbf{T}(\underset{\sim}{\mathbf{w}}_{\mathrm{B}})\|_{0,\rho;\Omega_{\mathrm{B}}} \ \le\ L_{\mathbf{T}}\left\{\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}} + \|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\right\} \|\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,, \tag{3.82}$$

for all $\mathbf{w}_{\mathrm{B}}$, $\underset{\sim}{\mathbf{w}}_{\mathrm{B}} \in \mathbf{W}_r$.

Proof. Given $\mathbf{w}_{\mathrm{B}}$, $\underset{\sim}{\mathbf{w}}_{\mathrm{B}} \in \mathbf{W}_r$, let $\mathbf{T}(\mathbf{w}_{\mathrm{B}}) := \mathbf{u}_{\mathrm{B}}$ and $\mathbf{T}(\underset{\sim}{\mathbf{w}}_{\mathrm{B}}) := \underset{\sim}{\mathbf{u}}_{\mathrm{B}}$, where $\big((\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\varphi}}), \vec{\mathbf{u}}\big) \in \mathbf{H}\times\mathbf{Q}$ and $\big((\underset{\sim}{\vec{\boldsymbol{\sigma}}}, \underset{\sim}{\vec{\boldsymbol{\varphi}}}), \underset{\sim}{\vec{\mathbf{u}}}\big) \in \mathbf{H}\times\mathbf{Q}$ are the corresponding unique solutions of (3.46), with $\vec{\mathbf{u}} := (\mathbf{u}_{\mathrm{B}}, p_{\mathrm{D}}, \ell)$ and $\underset{\sim}{\vec{\mathbf{u}}} := (\underset{\sim}{\mathbf{u}}_{\mathrm{B}}, \underset{\sim}{p}_{\mathrm{D}}, \underset{\sim}{\ell})$. Then, according to the definitions of the forms $\mathbf{C}_{\mathbf{w}_{\mathrm{B}}}$ and $\mathcal{A}_{\mathbf{w}_{\mathrm{B}}}$ (cf. (3.21)), (3.22)), and bearing in mind (3.46), we find

$$\begin{aligned}
\mathcal{A}_{\mathbf{w}_{\mathrm{B}}}\big(((\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\varphi}}), \vec{\mathbf{u}}) - ((\underset{\sim}{\vec{\boldsymbol{\sigma}}}, \underset{\sim}{\vec{\boldsymbol{\varphi}}}), \underset{\sim}{\vec{\mathbf{u}}}), ((\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\psi}}), \vec{\mathbf{v}})\big) \ &=\ \big(\mathbf{C}_{\mathbf{w}_{\mathrm{B}}} - \mathbf{C}_{\underset{\sim}{\mathbf{w}}_{\mathrm{B}}}\big)(\underset{\sim}{\vec{\mathbf{u}}}, \vec{\mathbf{v}}) \\
&=\ \mathtt{F}\big((|\mathbf{w}_{\mathrm{B}}|^{\rho-2} \,-\, |\underset{\sim}{\mathbf{w}}_{\mathrm{B}}|^{\rho-2})\,\underset{\sim}{\mathbf{u}}_{\mathrm{B}}, \mathbf{v}_{\mathrm{B}}\big)_{\mathrm{B}}\,,
\end{aligned}$$

for all $\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}}\big) \in \mathbf{H}\times\mathbf{Q}$, from which, invoking (3.79) and the fact that both $\|\mathbf{w}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}$ and $\|\underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}$ are bounded by r, we deduce that

$$\begin{aligned}&\mathcal{A}_{\mathbf{w}_{\mathrm{B}}}\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}) - ((\underset{\sim}{\vec{\boldsymbol{\sigma}}},\underset{\sim}{\vec{\boldsymbol{\varphi}}}),\underset{\sim}{\vec{\mathbf{u}}}),((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\big)\\ &\qquad\le \mathtt{F}\,C(\rho)\,(2\,r)^{\rho-3}\,\|\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,\|\underset{\sim}{\mathbf{u}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,\|\mathbf{v}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,.\end{aligned} \tag{3.83}$$

Hence, applying the inf-sup condition (3.75) to $\big((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\phi}}),\vec{\mathbf{z}}\big) = \big((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}\big) - \big((\underset{\sim}{\vec{\boldsymbol{\sigma}}},\underset{\sim}{\vec{\boldsymbol{\varphi}}}),\underset{\sim}{\vec{\mathbf{u}}}\big)$, and then using (3.83), we readily get

$$\begin{aligned}\|\mathbf{T}(\mathbf{w}_{\mathrm{B}}) - \mathbf{T}(\underset{\sim}{\mathbf{w}}_{\mathrm{B}})\|_{0,\rho;\Omega_{\mathrm{B}}} &= \|\mathbf{u}_{\mathrm{B}} - \underset{\sim}{\mathbf{u}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}} \le \big\|\big((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}\big) - \big((\underset{\sim}{\vec{\boldsymbol{\sigma}}},\underset{\sim}{\vec{\boldsymbol{\varphi}}}),\underset{\sim}{\vec{\mathbf{u}}}\big)\big\|_{\mathbf{H}\times\mathbf{Q}}\\ &\le \alpha_{\mathcal{A}}^{-1} \sup_{\substack{((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\\ ((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathcal{A}_{\mathbf{w}_{\mathrm{B}}}\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}) - \big((\underset{\sim}{\vec{\boldsymbol{\sigma}}},\underset{\sim}{\vec{\boldsymbol{\varphi}}}),\underset{\sim}{\vec{\mathbf{u}}}\big),\big((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}}\big)\big)}{\|((\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\psi}}),\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}}\\ &\le \alpha_{\mathcal{A}}^{-1}\,\mathtt{F}\,C(\rho)\,(2\,r)^{\rho-3}\,\|\underset{\sim}{\mathbf{u}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,\|\mathbf{w}_{\mathrm{B}} - \underset{\sim}{\mathbf{w}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}\,.\end{aligned} \tag{3.84}$$

Finally, bounding in (3.84) $\|\underset{\sim}{\mathbf{u}}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}} = \|\mathbf{T}(\underset{\sim}{\mathbf{w}}_{\mathrm{B}})\|_{0,\rho;\Omega_{\mathrm{B}}}$ by (3.74) instead of directly by r, we obtain (3.82) with the constant

$$L_{\mathbf{T}} := \alpha_{\mathcal{A}}^{-1}\,\mathtt{F}\,C(\rho)\,(2\,r)^{\rho-3}\,C_{\mathbf{T}}\,,$$

thus concluding the proof. □

The main result concerning the solvability of the fixed-point equation (3.47), equivalently, that of (3.23) (or (3.17)), is stated as follows.

Theorem 3.10 *Assume that the data satisfy* (3.78) *and*

$$L_{\mathbf{T}}\Big\{\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}} + \|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\} < 1\,. \tag{3.85}$$

Then, the operator $\mathbf{T}$ *has a unique fixed-point* $\mathbf{u}_{\mathrm{B}} \in \mathbf{W}_r$. *Equivalently,* (3.23) *has a unique solution* $\big((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}\big) := \big((\boldsymbol{\sigma}_{\mathrm{B}},\mathbf{u}_{\mathrm{D}},\boldsymbol{\varphi},\lambda),(\mathbf{u}_{\mathrm{B}},p_{\mathrm{D}},\ell)\big) \in \mathbf{H}\times\mathbf{Q}$ *with* $\mathbf{u}_{\mathrm{B}} \in \mathbf{W}_r$. *Moreover, there holds*

$$\|((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}})\|_{\mathbf{H}\times\mathbf{Q}} \le C_{\mathbf{T}}\Big\{\|\mathbf{f}_{\mathrm{D}}\|_{0;\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}} + \|g_{\mathrm{D}}\|_{0;\Omega_{\mathrm{D}}}\Big\}\,. \tag{3.86}$$

Proof. It is clear from Lemma 3.9 and the assumptions (3.78) and (3.85) that $\mathbf{T}$ is a contraction that maps $\mathbf{W}_r$ into itself. Hence, a straightforward application of the classical Banach fixed-point theorem implies the existence of a unique fixed point $\mathbf{u}_{\mathrm{B}} \in \mathbf{W}_r$ of $\mathbf{T}$, and therefore the solvability of (3.23). Finally, the *a priori* estimate (3.86) follows from (3.74). □

4 The Galerkin scheme

Here, we introduce a generic Galerkin scheme for the problem (3.23) (equivalently (3.17)), and, under suitable sufficient conditions on the finite element subspaces involved, establish its well-posedness and derive the associated Céa estimate. In particular, the respective solvability analysis is carried out by means of a discrete version of the fixed-point strategy from Section 3.4, which, in turn, employs the discrete versions of Theorems 3.1 (cf. [28, Theorem 4.1]) and 3.2 to analyze the corresponding Galerkin scheme of (3.46).

4.1 The discrete problem

Let us consider arbitrary finite element subspaces

$$
\begin{gathered}
\widetilde{\mathbb{H}}_h(\Omega_{\mathrm{B}}) \subseteq \mathbb{H}(\mathbf{div}_\varrho;\Omega_{\mathrm{B}}), \quad \widetilde{\mathbf{H}}_h(\Omega_{\mathrm{D}}) \subseteq \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}), \quad \Lambda_h^{\mathrm{B}}(\Sigma) \subseteq \mathrm{H}_{00}^{1/2}(\Sigma), \\
\Lambda_h^{\mathrm{D}}(\Sigma) \subseteq \mathrm{H}^{1/2}(\Sigma), \quad \mathbf{L}_h(\Omega_{\mathrm{B}}) \subseteq \mathbf{L}^\rho(\Omega_{\mathrm{B}}), \quad \text{and} \quad \widetilde{\mathrm{L}}_h(\Omega_{\mathrm{D}}) \subseteq \mathrm{L}^2(\Omega_{\mathrm{D}}),
\end{gathered}
\tag{4.1}
$$

and define

$$
\begin{gathered}
\mathbb{H}_h(\Omega_{\mathrm{B}}) := \widetilde{\mathbb{H}}_h(\Omega_{\mathrm{B}}) \cap \mathbb{H}_0(\mathbf{div}_\varrho;\Omega_{\mathrm{B}}), \quad \mathbf{H}_h(\Omega_{\mathrm{D}}) := \widetilde{\mathbf{H}}_h(\Omega_{\mathrm{D}}) \cap \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}), \\
\boldsymbol{\Lambda}_h^{\mathrm{B}}(\Sigma) := [\Lambda_h^{\mathrm{B}}(\Sigma)]^n, \quad \text{and} \quad \mathrm{L}_h(\Omega_{\mathrm{D}}) := \widetilde{\mathrm{L}}_h(\Omega_{\mathrm{D}}) \cap \mathrm{L}_0^2(\Omega_{\mathrm{D}}).
\end{gathered}
$$

Then, we introduce the global finite element spaces

$$
\begin{gathered}
\mathbf{H}_{h,1} := \mathbb{H}_h(\Omega_{\mathrm{B}}) \times \mathbf{H}_h(\Omega_{\mathrm{D}}), \quad \mathbf{H}_{h,2} := \boldsymbol{\Lambda}_h^{\mathrm{B}}(\Sigma) \times \Lambda_h^{\mathrm{D}}(\Sigma), \\
\mathbf{H}_h := \mathbf{H}_{h,1} \times \mathbf{H}_{h,2}, \quad \text{and} \quad \mathbf{Q}_h := \mathbf{L}_h(\Omega_{\mathrm{B}}) \times \mathrm{L}_h(\Omega_{\mathrm{D}}) \times \mathrm{R},
\end{gathered}
\tag{4.2}
$$

and set the unknowns and test functions as

$$
\begin{array}{lll}
\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{\mathrm{B},h}, \mathbf{u}_{\mathrm{D},h}) \in \mathbf{H}_{h,1}, & \vec{\boldsymbol{\varphi}}_h := (\boldsymbol{\varphi}_h, \lambda_h) \in \mathbf{H}_{h,2}, & \vec{\mathbf{u}}_h := (\mathbf{u}_{\mathrm{B},h}, p_{\mathrm{D},h}, \ell_h) \in \mathbf{Q}_h, \\
\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{\mathrm{B},h}, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{H}_{h,1}, & \vec{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h, \xi_h) \in \mathbf{H}_{h,2}, & \vec{\mathbf{v}}_h := (\mathbf{v}_{\mathrm{B},h}, q_{\mathrm{D},h}, \jmath_h) \in \mathbf{Q}_h, \\
\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_{\mathrm{B},h}, \mathbf{z}_{\mathrm{D},h}) \in \mathbf{H}_{h,1}, & \vec{\boldsymbol{\phi}}_h := (\boldsymbol{\phi}_h, \vartheta_h) \in \mathbf{H}_{h,2}, & \vec{\mathbf{z}}_h := (\mathbf{z}_{\mathrm{B},h}, r_{\mathrm{D},h}, \kappa_h) \in \mathbf{Q}_h.
\end{array}
$$

Hence, the Galerkin scheme of (3.23) reads: Find $((\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\varphi}}_h), \vec{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that:

$$
\mathcal{A}_{\mathbf{u}_{\mathrm{B},h}}\big(((\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\varphi}}_h), \vec{\mathbf{u}}_h), ((\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h), \vec{\mathbf{v}}_h)\big) = \mathcal{F}(((\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h), \vec{\mathbf{v}}_h)) \quad \forall\, ((\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h), \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h, \tag{4.3}
$$

where $\mathcal{A}_{\mathbf{w}_{\mathrm{B},h}}$ is defined as in (3.22) with $\mathbf{w}_{\mathrm{B},h}$ instead of $\mathbf{w}_{\mathrm{B}}$.

Note that throughout this section, h stands just for the index of each subspace. Later one, it will be utilized to refer also to the sizes of triangulations of Ω_{B} and Ω_{D}.

In order to analyze the solvability of (4.3), and analogously to the continuous formulation, we realize that this problem can be rewritten equivalently as the fixed-point equation: Find $\mathbf{u}_{\mathrm{B},h} \in \mathbf{L}_h(\Omega_{\mathrm{B}})$ such that

$$
\mathbf{T}_h(\mathbf{u}_{\mathrm{B},h}) = \mathbf{u}_{\mathrm{B},h}, \tag{4.4}
$$

where $\mathbf{T}_h : \mathbf{L}_h(\Omega_{\mathrm{B}}) \to \mathbf{L}_h(\Omega_{\mathrm{B}})$ is the discrete version of $\mathbf{T}$ (cf. (3.45)), that is, given $\mathbf{w}_{\mathrm{B},h} \in \mathbf{L}_h(\Omega_{\mathrm{B}})$, $\mathbf{T}_h(\mathbf{w}_{\mathrm{B},h}) := \mathbf{u}_{\mathrm{B},h}$, where $\big((\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\varphi}}_h), \vec{\mathbf{u}}_h\big) \in \mathbf{H}_h \times \mathbf{Q}_h$, with $\vec{\mathbf{u}}_h := (\mathbf{u}_{\mathrm{B},h}, p_{\mathrm{D},h}, \ell_h) \in \mathbf{Q}_h$, is the unique solution (to be confirmed below) of the linearized version of (4.3), namely

$$
\mathcal{A}_{\mathbf{w}_{\mathrm{B},h}}\big(((\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\varphi}}_h), \vec{\mathbf{u}}_h), ((\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h), \vec{\mathbf{v}}_h)\big) = \mathcal{F}(((\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h), \vec{\mathbf{v}}_h)) \quad \forall\, ((\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h), \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h. \tag{4.5}
$$

4.2 Solvability analysis

In this section we address the solvability of (4.3), equivalently of (4.4), for which we previously need to focus on that of (4.5). For this purpose, and as the respective discussion progresses, we introduce suitable hypotheses on the finite element subspaces (4.2), which facilitate the corresponding analysis.

We begin by noticing, similarly as done in Section 3.4 for the continuous case, that the kernel $\mathbf{V}_h$ of $\mathbf{B}|_{\mathbf{H}_h\times\mathbf{Q}_h}$ reduces to $\mathbf{V}_h := \mathbf{V}_{h,1}\times\mathbf{V}_{h,2}$, where

$$\begin{aligned}\mathbf{V}_{h,1} \,:=\, \Big\{\vec{\boldsymbol{\tau}}_h \,=\, \big(\boldsymbol{\tau}_{\mathrm{B},h},\mathbf{v}_{\mathrm{D},h}\big)\in\mathbf{H}_{h,1}: \quad &(\mathbf{v}_{\mathrm{B},h},\mathbf{div}(\boldsymbol{\tau}_{\mathrm{B},h}))_{\mathrm{B}}=0 \quad \forall\,\mathbf{v}_{\mathrm{B},h}\in\mathbf{L}_h(\Omega_{\mathrm{B}})\\ \text{and}\quad &(q_{\mathrm{D},h},\mathrm{div}(\mathbf{v}_{\mathrm{D},h}))_{\mathrm{D}}=0 \quad \forall\,q_{\mathrm{D},h}\in\mathrm{L}_h(\Omega_{\mathrm{D}})\Big\}\,,\end{aligned} \tag{4.6}$$

and

$$\mathbf{V}_{h,2} \,:=\, \Big\{\vec{\boldsymbol{\psi}}_h\in\mathbf{H}_{h,2}: \quad \langle\boldsymbol{\psi}_h\cdot\mathbf{n},1\rangle_\Sigma \,=\, 0\Big\}\,.$$

Next, we introduce the first hypotheses on the finite element subspaces, namely

(**H.1**) $\widetilde{\mathbb{H}}_h(\Omega_{\mathrm{B}})$ contains the multiplies of the identity tensor $\mathbb{I}$,

(**H.2**) $\mathrm{P}_0(\Omega_{\mathrm{D}})\,\subseteq\,\widetilde{\mathrm{L}}_h(\Omega_{\mathrm{D}})$,

(**H.3**) $\mathbf{div}(\widetilde{\mathbb{H}}_h(\Omega_{\mathrm{B}}))\,\subseteq\,\mathbf{L}_h(\Omega_{\mathrm{B}})$, and

(**H.4**) $\mathrm{div}(\widetilde{\mathbf{H}}_h(\Omega_{\mathrm{D}}))\,\subseteq\,\widetilde{\mathrm{L}}_h(\Omega_{\mathrm{D}})$.

Note that, as a consequence of (**H.1**) and the decomposition (3.13), the subspace $\mathbb{H}_h(\Omega_{\mathrm{B}})$ (cf. (4.1)) can be redefined as

$$\mathbb{H}_h(\Omega_{\mathrm{B}}) \,:=\, \Big\{\boldsymbol{\tau}_{\mathrm{B},h}-\Big(\frac{1}{n|\Omega_{\mathrm{B}}|}\int_{\Omega_{\mathrm{B}}}\mathrm{tr}(\boldsymbol{\tau}_{\mathrm{B},h})\Big)\mathbb{I}: \quad \boldsymbol{\tau}_{\mathrm{B},h}\in\widetilde{\mathbb{H}}_h(\Omega_{\mathrm{B}})\Big\}\,,$$

while it readily follows from (**H.2**) that there holds the decomposition

$$\widetilde{\mathrm{L}}_h(\Omega_{\mathrm{D}}) \,=\, \mathrm{L}_h(\Omega_{\mathrm{D}})\,\oplus\,\mathrm{P}_0(\Omega_{\mathrm{D}})\,.$$

In addition, thanks to (**H.3**) and (**H.4**), it follows from (4.6) that

$$\begin{aligned}\mathbf{V}_{h,1} \,=\, \Big\{\vec{\boldsymbol{\tau}}_h = \big(\boldsymbol{\tau}_{\mathrm{B},h},\mathbf{v}_{\mathrm{D},h}\big)\in\mathbf{H}_{h,1}: \quad &\mathbf{div}(\boldsymbol{\tau}_{\mathrm{B},h}) \,=\, \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{B}}\,,\\ \text{and}\quad &\mathrm{div}(\mathbf{v}_{\mathrm{D},h})\,\in\,\mathrm{P}_0(\Omega_{\mathrm{D}}) \quad\text{in}\quad \Omega_{\mathrm{D}}\Big\}\,,\end{aligned}$$

so that $\mathbf{V}_{h,1}\,\subseteq\,\mathbf{V}_1$ (cf. (3.48)), and hence Lemma 3.3 is also valid in the discrete setting, which means that, denoting $\alpha_{\mathbf{a},\mathrm{d}} := \alpha_{\mathbf{a}}$, there holds

$$\mathbf{a}(\vec{\boldsymbol{\tau}}_h,\vec{\boldsymbol{\tau}}_h)\,\geq\,\alpha_{\mathbf{a},\mathrm{d}}\,\|\vec{\boldsymbol{\tau}}_h\|^2_{\mathbf{H}_1} \qquad \forall\,\vec{\boldsymbol{\tau}}_h\in\mathbf{V}_{h,1}\,.$$

Now, in order to apply Theorem 3.2 to $\mathbf{A}|_{\mathbf{V}_h\times\mathbf{V}_h}$, we add the remaining assumption iii) of that result, which is the discrete counterpart of Lemma 3.4, as the following hypothesis:

(**H.5**) there exists a positive constant β_{d}, independent of h, such that

$$\sup_{\substack{\vec{\boldsymbol{\tau}}_h\in\mathbf{V}_{h,1}\\ \vec{\boldsymbol{\tau}}_h\neq\mathbf{0}}}\frac{\mathbf{b}(\vec{\boldsymbol{\tau}}_h,\vec{\boldsymbol{\psi}}_h)}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}_1}}\,\geq\,\beta_{\mathrm{d}}\,\|\vec{\boldsymbol{\psi}}_h\|_{\mathbf{H}_2} \quad \forall\,\vec{\boldsymbol{\psi}}_h\,\in\,\mathbf{V}_{h,2}\,.$$

Analogously as remarked in the proof of Lemma 3.4, and due again to the diagonal structure of $\mathbf{b}$, we find it important to remark here that (**H.5**) is equivalent to the existence of positive constants

$\beta_{i,\mathrm{d}}$, independent of h, such that the discrete counterparts of $\mathcal{R}_i$ (cf. (3.54)), $i \in \{1,2\}$, satisfy the corresponding discrete inf-sup conditions, that is for each $\vec{\boldsymbol{\psi}}_h = (\boldsymbol{\psi}_h, \xi_h) \in \mathbf{V}_{h,2}$ there hold

$$\mathcal{R}_{1,h}(\boldsymbol{\psi}_h) := \sup_{\substack{\boldsymbol{\tau}_{\mathrm{B},h} \in \mathbb{H}_h(\Omega_{\mathrm{B}}) \setminus \{0\} \\ \mathbf{div}(\boldsymbol{\tau}_{\mathrm{B},h}) = \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_{\mathrm{B},h}\, \mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{B},h}\|_{\mathbf{div}_\varrho;\Omega_{\mathrm{B}}}} \geq \beta_{1,\mathrm{d}}\, \|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma} \tag{4.7}$$

and

$$\mathcal{R}_{2,h}(\xi_h) := \sup_{\substack{\mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_h(\Omega_{\mathrm{D}}) \setminus \{0\} \\ \mathrm{div}(\mathbf{v}_{\mathrm{D},h}) \in \mathrm{P}_0(\Omega_{\mathrm{D}})}} \frac{\langle \mathbf{v}_{\mathrm{D},h} \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \geq \beta_{2,\mathrm{d}}\, \|\xi_h\|_{1/2,\Sigma}\,. \tag{4.8}$$

Next, noting that certainly there holds (cf. (3.19)) $\mathbf{c}(\vec{\boldsymbol{\psi}}_h, \vec{\boldsymbol{\psi}}_h) = 0$ for all $\vec{\boldsymbol{\psi}}_h \in \mathbf{V}_{h,2}$, we deduce, as a straightforward application of Theorem 3.2, that $\mathbf{A}|_{\mathbf{V}_h \times \mathbf{V}_h}$ satisfies the discrete counterpart of Lemma 3.5, that is, there exists a positive constant $\alpha_{\mathbf{A},\mathrm{d}}$, depending only on $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, $\alpha_{\mathbf{a},\mathrm{d}}$, and β_{d}, and hence independent of h, such that

$$\sup_{\substack{(\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h) \in \mathbf{V}_h \\ (\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h) \neq \mathbf{0}}} \frac{\mathbf{A}((\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\phi}}_h), (\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h))}{\|(\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h)\|_{\mathbf{H}}} \geq \alpha_{\mathbf{A},\mathrm{d}}\, \|(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\phi}}_h)\|_{\mathbf{H}} \qquad \forall\, (\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\phi}}_h) \in \mathbf{V}_h\,. \tag{4.9}$$

The inf-sup condition with respect to the second component of $\mathbf{A}$, being equivalent to (4.9), and with the same constant $\alpha_{\mathbf{A},\mathrm{d}}$, is omitted.

Finally, and aiming to apply the discrete version of Theorem 3.1 (cf. [28, Theorem 4.1]) to conclude the well-posedness of (4.5), equivalently that $\mathbf{T}_h$ (cf. (4.4)) is well-defined, we assume the remaining assumption as the following hypothesis:

(**H.6**) there exists a positive constant $\boldsymbol{\beta}_{\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{(\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h) \in \mathbf{H}_h \\ (\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h) \neq \mathbf{0}}} \frac{\mathbf{B}((\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h), \vec{\mathbf{v}}_h)}{\|(\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h)\|_{\mathbf{H}}} \geq \boldsymbol{\beta}_{\mathrm{d}}\, \|\vec{\mathbf{v}}_h\|_{\mathbf{Q}} \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{Q}_h\,.$$

Similarly as observed for (**H.5**), and due again to the diagonal structure of $\mathbf{B}$ exploited in the proof of Lemma 3.6, we stress here that (**H.6**) is equivalent to the existence of positive constants $\boldsymbol{\beta}_{i,\mathrm{d}}$, independent of h, such that the discrete counterparts of $\mathcal{S}_i$ (cf. (3.66), (3.67)), $i \in \{1,2,3\}$, satisfy the corresponding discrete inf-sup conditions, that is for each $\vec{\mathbf{v}}_h := (\mathbf{v}_{\mathrm{B},h}, q_{\mathrm{D},h}, \jmath_h) \in \mathbf{Q}_h$ there hold

$$\mathcal{S}_{1,h}(\mathbf{v}_{\mathrm{B},h}) := \sup_{\substack{\boldsymbol{\tau}_{\mathrm{B},h} \in \mathbb{H}_h(\Omega_{\mathrm{B}}) \\ \boldsymbol{\tau}_{\mathrm{B},h} \neq \mathbf{0}}} \frac{(\mathbf{v}_{\mathrm{B},h}, \mathbf{div}(\boldsymbol{\tau}_{\mathrm{B},h}))_{\mathrm{B}}}{\|\boldsymbol{\tau}_{\mathrm{B},h}\|_{\mathbf{div}_\varrho;\Omega_{\mathrm{B}}}} \geq \boldsymbol{\beta}_{1,\mathrm{d}}\, \|\mathbf{v}_{\mathrm{B},h}\|_{0,\rho;\Omega_{\mathrm{B}}}\,, \tag{4.10}$$

$$\mathcal{S}_{2,h}(q_{\mathrm{D},h}) := \sup_{\substack{\mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_h(\Omega_{\mathrm{D}}) \\ \mathbf{v}_{\mathrm{D},h} \neq \mathbf{0}}} \frac{(q_{\mathrm{D},h}, \mathrm{div}(\mathbf{v}_{\mathrm{D},h}))_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \geq \boldsymbol{\beta}_{2,\mathrm{d}}\, \|q_{\mathrm{D},h}\|_{0;\Omega_{\mathrm{D}}}\,, \tag{4.11}$$

and

$$\mathcal{S}_{3,h}(\jmath_h) := \sup_{\substack{\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^{\mathrm{B}}(\Sigma) \\ \boldsymbol{\psi}_h \neq \mathbf{0}}} \frac{\jmath_h\, \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1 \rangle_\Sigma}{\|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma}} \geq \boldsymbol{\beta}_{3,\mathrm{d}}\, |\jmath_h|\,. \tag{4.12}$$

Hence, proceeding as in the proof of Lemma 3.6, we readily find that $\boldsymbol{\beta}_{\mathrm{d}} = \dfrac{1}{3}\min\Big\{\boldsymbol{\beta}_{1,\mathrm{d}}, \boldsymbol{\beta}_{2,\mathrm{d}}, \boldsymbol{\beta}_{3,\mathrm{d}}\Big\}$.

In addition to the above discussion, we observe here, thanks to (3.25) and (3.26), that $\mathbf{A}|_{\mathbf{H}_h\times\mathbf{H}_h}$ and $\mathbf{C}_{\mathbf{w}_{\mathrm{B},h}}|_{\mathbf{Q}_h\times\mathbf{Q}_h}$ are certainly positive semi-definite, besides the obvious fact that $\mathbf{C}_{\mathbf{w}_{\mathrm{B},h}}|_{\mathbf{Q}_h\times\mathbf{Q}_h}$ is symmetric as well. Hence, as a straightforward application of [28, Theorem 4.1], and making use again of the estimate for $\|\mathcal{F}\|$ provided in (3.29), we are led to the discrete counterpart of Lemma 3.7, which is stated as follows.

Lemma 4.1 *Given $r > 0$, we let $\mathbf{w}_{\mathrm{B},h} \in \mathbf{L}_h(\Omega_{\mathrm{B}})$ be such that $\|\mathbf{w}_{\mathrm{B},h}\|_{0,\rho;\Omega_{\mathrm{B}}} \le r$. Then, there exists a unique solution $\big((\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\varphi}}_h), \vec{\mathbf{u}}_h\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ of (4.5), with $\vec{\mathbf{u}}_h := (\mathbf{u}_{\mathrm{B},h}, p_{\mathrm{D},h}, \ell_h) \in \mathbf{Q}_h$, and hence one can define $\mathbf{T}_h(\mathbf{w}_{\mathrm{B},h}) := \mathbf{u}_{\mathrm{B},h}$. In addition, there exists a positive constant $C_{\mathbf{T},\mathrm{d}}$, depending only on $\|\mathbf{A}\|$ (cf. (3.29) - (3.30)), $\|\mathbf{C}\|$ (cf. (3.32)), $\mathtt{F}$, r, ρ, $\alpha_{\mathbf{A},\mathrm{d}}$, and $\boldsymbol{\beta}_{\mathrm{d}}$, such that*

$$
\begin{aligned}
\|\mathbf{T}_h(\mathbf{w}_{\mathrm{B},h})\|_{0,\rho;\Omega_{\mathrm{B}}} &= \|\mathbf{u}_{\mathrm{B},h}\|_{0,\rho;\Omega_{\mathrm{B}}} \le \big\|\big((\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\varphi}}_h), \vec{\mathbf{u}}_h\big)\big\|_{\mathbf{H}\times\mathbf{Q}} \\
&\le C_{\mathbf{T},\mathrm{d}}\Big\{\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}} + \|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\}.
\end{aligned} \tag{4.13}
$$

As a consequence of Lemma 4.1, we conclude the discrete versions of (3.75) and (3.76), which means that there exists a positive constant $\alpha_{\mathcal{A},\mathrm{d}}$, depending only on $\|\mathbf{A}\|$, $\|\mathbf{C}\|$, $\mathtt{F}$, r, ρ, $\alpha_{\mathbf{A},\mathrm{d}}$, and $\boldsymbol{\beta}_{\mathrm{d}}$, and hence independent of h, such that for each $\mathbf{w}_{\mathrm{B},h} \in \mathbf{L}_h(\Omega_{\mathrm{B}})$ with $\|\mathbf{w}_{\mathrm{B},h}\|_{0,\rho;\Omega_{\mathrm{B}}} \le r$, there holds

$$
\sup_{\substack{((\vec{\boldsymbol{\tau}}_h,\vec{\boldsymbol{\psi}}_h),\vec{\mathbf{v}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h \\ ((\vec{\boldsymbol{\tau}}_h,\vec{\boldsymbol{\psi}}_h),\vec{\mathbf{v}}_h)\neq\mathbf{0}}} \frac{\mathcal{A}_{\mathbf{w}_{\mathrm{B},h}}\big(((\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\phi}}_h), \vec{\mathbf{z}}_h), ((\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h), \vec{\mathbf{v}}_h)\big)}{\|\big((\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\psi}}_h), \vec{\mathbf{v}}_h\big)\|_{\mathbf{H}\times\mathbf{Q}}} \ge \alpha_{\mathcal{A},\mathrm{d}}\,\|\big((\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\phi}}_h), \vec{\mathbf{z}}_h\big)\|_{\mathbf{H}\times\mathbf{Q}} \tag{4.14}
$$

for all $\big((\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\phi}}_h), \vec{\mathbf{z}}_h\big) \in \mathbf{H}_h \times \mathbf{Q}_h$. Similarly as for $\mathbf{A}|_{\mathbf{V}_h\times\mathbf{V}_h}$ (cf. (4.9)), the inf-sup condition with respect to the second component of $\mathcal{A}_{\mathbf{w}_{\mathrm{B},h}}$, being equivalent to (4.14), and with the same constant $\alpha_{\mathcal{A},\mathrm{d}}$, is omitted.

We now aim to apply the Banach fixed-point theorem to establish the unique solvability of (4.4). Indeed, given the same $r > 0$ as before, we first introduce the discrete ball

$$
\mathbf{W}_{r,h} := \Big\{\mathbf{w}_{\mathrm{B},h} \in \mathbf{L}_h(\Omega_{\mathrm{B}}) : \quad \|\mathbf{w}_{\mathrm{B},h}\|_{0,\rho;\Omega_{\mathrm{B}}} \le r\Big\}, \tag{4.15}
$$

and observe from (4.13) that, under the assumption

$$
C_{\mathbf{T},\mathrm{d}}\Big\{\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}} + \|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\} \le r, \tag{4.16}
$$

there holds $\mathbf{T}_h(\mathbf{W}_{r,h}) \subseteq \mathbf{W}_{r,h}$.

Furthermore, employing now the discrete global inf-sup condition (4.14) along with the property provided by Lemma 3.8, and following analogue arguments to those utilized in the proof of Lemma 3.9, we are able to prove the discrete counterpart of this latter result. More precisely, the Lipschitz-continuity of $\mathbf{T}_h$ is stated as follows.

Lemma 4.2 *There exists a positive constant $L_{\mathbf{T},\mathrm{d}}$, depending only on $\alpha_{\mathcal{A},\mathrm{d}}$, $\mathtt{F}$, ρ, r, and $C_{\mathbf{T},\mathrm{d}}$, such that*

$$
\begin{aligned}
&\|\mathbf{T}_h(\mathbf{w}_{\mathrm{B},h}) - \mathbf{T}_h(\underset{\sim}{\mathbf{w}}_{\mathrm{B},h})\|_{0,\rho;\Omega_{\mathrm{B}}} \\
&\qquad \le L_{\mathbf{T},\mathrm{d}}\Big\{\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}} + \|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\}\|\mathbf{w}_{\mathrm{B},h} - \underset{\sim}{\mathbf{w}}_{\mathrm{B},h}\|_{0,\rho;\Omega_{\mathrm{B}}},
\end{aligned}
$$

for all $\mathbf{w}_{\mathrm{B},h}$, $\underset{\sim}{\mathbf{w}}_{\mathrm{B},h} \in \mathbf{W}_{r,h}$.

We are now in position to state the main result of this section.

Theorem 4.3 *Assume that the data satisfy* (4.16) *and*

$$L_{\mathbf{T},\mathrm{d}}\left\{\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}+\|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}}+\|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\right\} < 1\,.$$

Then, the operator $\mathbf{T}_h$ *has a unique fixed-point* $\mathbf{u}_{\mathrm{B},h}\in\mathbf{W}_{r,h}$. *Equivalently,* (4.3) *has a unique solution* $\big((\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\varphi}}_h),\vec{\mathbf{u}}_h\big) := \big((\boldsymbol{\sigma}_{\mathrm{B},h},\mathbf{u}_{\mathrm{D},h},\boldsymbol{\varphi}_h,\lambda_h),(\mathbf{u}_{\mathrm{B},h},p_{\mathrm{D},h},\ell_h)\big)\in\mathbf{H}_h\times\mathbf{Q}_h$ *with* $\mathbf{u}_{\mathrm{B},h}\in\mathbf{W}_{r,h}$. *Moreover, there holds*

$$\big\|\big((\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\varphi}}_h),\vec{\mathbf{u}}_h\big)\big\|_{\mathbf{H}\times\mathbf{Q}}\leq C_{\mathbf{T},\mathrm{d}}\left\{\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}+\|\mathbf{f}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}+\|g_{\mathrm{D}}\|_{0;\Omega_{\mathrm{D}}}\right\}.$$

Proof. It proceeds analogously to the proof of Theorem 3.10. □

4.3 A priori error analysis

In this section, we derive the a priori error estimate for the Galerkin scheme (4.3) with arbitrary finite element subspaces satisfying the hypotheses (**H.1**)-(**H.6**) from Section 4.2. In other words, our main goal is to establish the Céa estimate for the global error

$$\|((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}})-((\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\varphi}}_h),\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\,,$$

where $((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}})\in\mathbf{H}\times\mathbf{Q}$ and $((\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\varphi}}_h),\vec{\mathbf{u}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h$ are the unique solutions of (3.23) and (4.3), respectively, with $\mathbf{u}_{\mathrm{B}}\in\mathbf{W}_r$ (cf. (3.77)) and $\mathbf{u}_{\mathrm{B},h}\in\mathbf{W}_{r,h}$ (cf. (4.15)). Hereafter, given a subspace X_h of a generic Banach space $(\mathrm{X},\|\cdot\|_{\mathrm{X}})$, we set as usual $\operatorname{dist}(x,\mathrm{X}_h):=\inf\limits_{x_h\in\mathrm{X}_h}\|x-x_h\|_{\mathrm{X}}$ for all $x\in\mathrm{X}$.

We begin by recalling from Sections 3.4 and 4.2 that, given $r>0$, and thanks to the global inf-sup conditions provided by (3.75), (3.76), and (4.14), the bilinear forms $\mathcal{A}_{\mathbf{u}_{\mathrm{B}}}$ and $\mathcal{A}_{\mathbf{u}_{\mathrm{B},h}}$, with $\mathbf{u}_{\mathrm{B}}\in\mathbf{W}_r$ and $\mathbf{u}_{\mathrm{B},h}\in\mathbf{W}_{r,h}$, satisfy the hypotheses of the Banach–Nečas–Babuška theorem (cf. [22, Theorem 2.6]) on $\mathbf{H}\times\mathbf{Q}$ and $\mathbf{H}_h\times\mathbf{Q}_h$, respectively. Thus, applying a slight variant of the first Strang Lemma (cf. [22, Lemma 2.27]) to the context given by (3.23) and (4.3), we deduce the existence of a positive constant $C_{\mathcal{A}}$, depending only on $\|\mathbf{A}\|$, $\|\mathbf{B}\|$, $\|\mathbf{C}\|$, $\mathtt{F}$, r, ρ, and $\alpha_{\mathcal{A},\mathrm{d}}$, and hence independent of h, such that

$$\begin{aligned}\|((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}})-((\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\varphi}}_h),\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\leq C_{\mathcal{A}}\Big\{&\operatorname{dist}\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}),\mathbf{H}_h\times\mathbf{Q}_h\big)\\&+\|\big(\mathcal{A}_{\mathbf{u}_{\mathrm{B}}}-\mathcal{A}_{\mathbf{u}_{\mathrm{B},h}}\big)\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}),\cdot\big)\|_{(\mathbf{H}_h\times\mathbf{Q}_h)'}\Big\},\end{aligned}\tag{4.17}$$

where the consistency term from (4.17) is defined as

$$\begin{aligned}&\|\big(\mathcal{A}_{\mathbf{u}_{\mathrm{B}}}-\mathcal{A}_{\mathbf{u}_{\mathrm{B},h}}\big)\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}),\cdot\big)\|_{(\mathbf{H}_h\times\mathbf{Q}_h)'}\\&\quad:=\sup_{\substack{((\vec{\boldsymbol{\tau}}_h,\vec{\boldsymbol{\psi}}_h),\vec{\mathbf{v}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h\\((\vec{\boldsymbol{\tau}}_h,\vec{\boldsymbol{\psi}}_h),\vec{\mathbf{v}}_h)\neq\mathbf{0}}}\frac{\big(\mathcal{A}_{\mathbf{u}_{\mathrm{B}}}-\mathcal{A}_{\mathbf{u}_{\mathrm{B},h}}\big)\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}),((\vec{\boldsymbol{\tau}}_h,\vec{\boldsymbol{\psi}}_h),\vec{\mathbf{v}}_h)\big)}{\|((\vec{\boldsymbol{\tau}}_h,\vec{\boldsymbol{\psi}}_h),\vec{\mathbf{v}}_h)\|_{\mathbf{H}\times\mathbf{Q}}}\,.\end{aligned}\tag{4.18}$$

We point out here that the aforementioned variant (see, e.g. [9, Lemma 5.1]) is motivated in this case by the fact that $\mathcal{A}_{\mathbf{u}_{\mathrm{B},h}}$ can be evaluated in the exact solution $((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}})$ as well. Hence, after subtracting and adding the latter in the first component of both $\mathcal{A}_{\mathbf{u}_{\mathrm{B}}}$ and $\mathcal{A}_{\mathbf{u}_{\mathrm{B},h}}$, the respective consistency term from [22, Lemma 2.27] becomes separated from the infimum defining the distance to the subspaces involved, thus yielding the resulting simplified estimate (4.17). Then, bearing in mind the definitions of $\mathcal{A}_{\mathbf{w}_{\mathrm{B}}}$ (cf. (3.22)) and $\mathbf{C}_{\mathbf{w}_{\mathrm{B}}}$ (cf. (3.21)), and employing Lemma 3.8, and the fact that both $\|\mathbf{u}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}$ and $\|\mathbf{u}_{\mathrm{B},h}\|_{0,\rho;\Omega_{\mathrm{B}}}$ are bounded by r, as well as the upper bound for $\|\mathbf{u}_{\mathrm{B}}\|_{0,\rho;\Omega_{\mathrm{B}}}$ given by Theorem 3.10, it readily follows from (4.18) that

$$\begin{aligned}\|\big(\mathcal{A}_{\mathbf{u}_{\mathrm{B}}}-\mathcal{A}_{\mathbf{u}_{\mathrm{B},h}}\big)\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}),\cdot\big)\|_{(\mathbf{H}_h\times\mathbf{Q}_h)'} \,&\le\, \|\big(\mathbf{C}_{\mathbf{u}_{\mathrm{B}}}-\mathbf{C}_{\mathbf{u}_{\mathrm{B},h}}\big)(\vec{\mathbf{u}},\cdot)\|_{\mathbf{Q}_h'}\\ &\le\, L_{\mathcal{A}}\Big\{\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}+\|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}}+\|g_{\mathrm{D}}\|_{0;\Omega_{\mathrm{D}}}\Big\}\|\mathbf{u}_{\mathrm{B}}-\mathbf{u}_{\mathrm{B},h}\|_{0,\rho;\Omega_{\mathrm{B}}}\,.\end{aligned}\tag{4.19}$$

with $L_{\mathcal{A}} := \mathtt{F}\, C(\rho)\,(2r)^{\rho-3}\,C_{\mathbf{T}}$.

We are now in position to establish the required a priori error estimate.

Theorem 4.4 *Assume that the data satisfy*

$$C_{\mathcal{A}}\,L_{\mathcal{A}}\Big\{\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}+\|\mathbf{f}_{\mathrm{B}}\|_{0,\varrho;\Omega_{\mathrm{B}}}+\|g_{\mathrm{D}}\|_{0;\Omega_{\mathrm{D}}}\Big\}\,\le\,\frac{1}{2}\,.\tag{4.20}$$

Then, there holds

$$\|((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}})\,-\,((\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\varphi}}_h),\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\,\le\,2\,C_{\mathcal{A}}\,\mathrm{dist}\big(((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}),\mathbf{H}_h\times\mathbf{Q}_h\big)\,.\tag{4.21}$$

Proof. It suffices to replace (4.19) back into (4.17) and then use the assumption (4.20). □

5 A particular choice of finite element subspaces

In this section we proceed similarly to [2] and [29] (see also [30]), and specify a concrete example of finite element subspaces satisfying the hypotheses (**H.1**) - (**H.6**). The approximation properties of them and the consequent rates of convergence of the resulting Galerkin scheme are also established.

5.1 Preliminaries

We begin by letting $\mathcal{T}_h^{\mathrm{B}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be triangulations of the domains Ω_{B} and Ω_{D}, respectively, formed by shape-regular triangles (in R^2) or tetrahedra (in R^3) of diameter h_T, which are assumed to match in Σ. In particular, we may think of Σ as a polygonal curve in R^2 (resp. a polyhedral region in R^3). In this way, being $\mathcal{T}_h^{\mathrm{B}}\cup\mathcal{T}_h^{\mathrm{D}}$ a triangulation of $\Omega_{\mathrm{B}}\cup\Sigma\cup\Omega_{\mathrm{D}}$, we denote by Σ_h the partition of Σ inherited either from $\mathcal{T}_h^{\mathrm{B}}$ or $\mathcal{T}_h^{\mathrm{D}}$. Also, we define $h_* := \max\{h_T : T\in\mathcal{T}_h^*\}$ $(*\in\{\mathrm{B},\mathrm{D}\})$ and $h := \max\{h_{\mathrm{B}},h_{\mathrm{D}}\}$. In addition, for each $T\in\mathcal{T}_h^{\mathrm{B}}\cup\mathcal{T}_h^{\mathrm{D}}$ we let $\mathrm{P}_0(T)$ be the space of polynomials on T of degree $=0$, and, according to the notation introduced in Section 1, we put $\mathbf{P}_0(T) := [\mathrm{P}_0(T)]^n$. Then, we set the vector and tensor local Raviart-Thomas spaces of order 0 as

$$\mathbf{RT}_0(T) := \mathbf{P}_0(T)\oplus\mathrm{P}_0(T)\mathbf{x}\,,\quad\text{and}\quad \mathbb{RT}_0(T) := \Big\{\boldsymbol{\tau}\in\mathbb{L}^2(T):\ \boldsymbol{\tau}_i\in\mathbf{RT}_0(T)\ \forall\, i\in\{1,\ldots,n\}\Big\},$$

where $\mathbf{x} := (x_1,...,x_n)^{\mathtt{t}}$ is a generic vector of R^n, and $\boldsymbol{\tau}_i$ stands for the i-th row of the tensor $\boldsymbol{\tau}$. Next, we introduce the discrete domain subspaces in (4.1):

$$\begin{aligned}\widetilde{\mathbb{H}}_h(\Omega_{\mathrm{B}}) \,&:=\, \Big\{\boldsymbol{\tau}_{\mathrm{B},h}\in\mathbb{H}(\mathbf{div}_\varrho;\Omega_{\mathrm{B}}):\quad \boldsymbol{\tau}_{\mathrm{B},h}|_T\in\mathbb{RT}_0(T)\quad\forall\,T\in\mathcal{T}_h^{\mathrm{B}}\Big\},\\ \widetilde{\mathbf{H}}_h(\Omega_{\mathrm{D}}) \,&:=\, \Big\{\mathbf{v}_{\mathrm{D},h}\in\mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}):\quad \mathbf{v}_{\mathrm{D},h}|_T\in\mathbf{RT}_0(T)\quad\forall\,T\in\mathcal{T}_h^{\mathrm{D}}\Big\},\\ \mathbf{L}_h(\Omega_{\mathrm{B}}) \,&:=\, \Big\{\mathbf{v}_{\mathrm{B},h}\in\mathbf{L}^\rho(\Omega_{\mathrm{B}}):\quad \mathbf{v}_{\mathrm{B},h}|_T\in\mathbf{P}_0(T)\quad\forall\,T\in\mathcal{T}_h^{\mathrm{B}}\Big\},\quad\text{and}\\ \widetilde{\mathrm{L}}_h(\Omega_{\mathrm{D}}) \,&:=\, \Big\{q_{\mathrm{D},h}\in\mathrm{L}^2(\Omega_{\mathrm{D}}):\quad q_{\mathrm{D},h}|_T\in\mathrm{P}_0(T)\quad\forall\,T\in\mathcal{T}_h^{\mathrm{D}}\Big\},\end{aligned}\tag{5.1}$$

whereas, denoting by $\partial\Sigma$ the extreme points of Σ in 2D, or the polygonal boundary of Σ in 3D, the discrete interface subspaces in (4.1) are initially defined as:

$$\begin{aligned} \Lambda_h^{\mathrm{B}}(\Sigma) &:= \Big\{ \psi_h : \Sigma \to \mathrm{R} \text{ continuous} : \quad \psi_h|_e \in \mathrm{P}_1(e) \quad \forall \text{ edge/face } e \in \Sigma_h\,, \quad \psi_h|_{\partial\Sigma} = 0 \Big\} \\ \text{and} \quad \Lambda_h^{\mathrm{D}}(\Sigma) &:= \Big\{ \xi_h : \Sigma \to \mathrm{R} \text{ continuous} : \quad \xi_h|_e \in \mathrm{P}_1(e) \quad \forall \text{ edge/face } e \in \Sigma_h \Big\}\,. \end{aligned} \tag{5.2}$$

5.2 Verification of the assumptions

We first notice that, under the choice of finite element subspaces defined by (5.1), (**H.1**) up to (**H.4**) are clearly satisfied.

Now, we jump to (**H.6**) and stress first that (4.10) follows from a simple extension of the vector version of it provided by [26, Lemma 4.8] for any $\rho > 2$. Alternatively, the proof of (4.10) proceeds almost verbatim to that for the particular case $\rho = 4$ given by [14, Lemma 5.5]. In turn, while most of the main aspects regarding the proof of (4.11) are available in the literature, for sake of completeness we provide next a full proof of it. To this end, we resort to some properties of the Raviart-Thomas interpolation operator $\Pi_h^{\mathrm{D}} : \mathbf{H}^1(\Omega_{\mathrm{D}}) \to \widetilde{\mathbf{H}}_h(\Omega_{\mathrm{D}})$, which are collected, for instance, in [2, Section 4.2.2, items (a), (b), (c), and (d)] (see also [29, Section 5.2, items **a**), **b**), **c**), and **d**)]).

Lemma 5.1 *There exists a positive constant $\boldsymbol{\beta}_{2,\mathrm{d}}$, independent of h, such that*

$$\mathcal{S}_{2,h}(q_{\mathrm{D},h}) \,\geq\, \boldsymbol{\beta}_{2,\mathrm{d}}\, \|q_{\mathrm{D},h}\|_{0;\Omega_{\mathrm{D}}} \qquad \forall\, q_{\mathrm{D},h} \in \mathrm{L}_h(\Omega_{\mathrm{D}})\,.$$

Proof. We proceed analogously to the proof of (3.72), the continuous version of (4.11). Indeed, given $q_{\mathrm{D},h} \in \mathrm{L}_h(\Omega_{\mathrm{D}})$, we now let z be the unique element in $\widetilde{\mathrm{H}}^1(\Omega_{\mathrm{D}})$, such that

$$\int_{\Omega_{\mathrm{D}}} \nabla z \cdot \nabla v \,=\, -\int_{\Omega_{\mathrm{D}}} q_{\mathrm{D},h}\, v \qquad \forall\, v \in \widetilde{\mathrm{H}}^1(\Omega_{\mathrm{D}})\,, \tag{5.3}$$

for which there exists a constant $\widetilde{c}_{\mathrm{D},\mathtt{d}} > 0$, depending only on Ω_{D}, such that $\|z\|_{1,\Omega_{\mathrm{D}}} \leq \widetilde{c}_{\mathrm{D},\mathtt{d}}\, \|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}$. Being (5.3) a particular case of (3.71) with $q_{\mathrm{D},h}$ instead of q_{D}, it is clear that $\mathrm{div}(\nabla z) = q_{\mathrm{D},h}$ in Ω_{D} and $\nabla z \cdot \mathbf{n} = 0$ on $\partial\Omega_{\mathrm{D}}$. In addition, the corresponding elliptic regularity result (cf. [33], [34]) establishes the existence of $\delta > 0$ and another constant $c_{\mathrm{D},\mathtt{d}} > 0$, such that, actually, $z \in \mathrm{H}^{1+\delta}(\Omega_{\mathrm{D}})$ and $\|z\|_{1+\delta,\Omega_{\mathrm{D}}} \leq c_{\mathrm{D},\mathtt{d}}\, \|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}$, from which it follows that $\nabla z \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$ and

$$\|\nabla z\|_{\delta,\Omega_{\mathrm{D}}} \,\leq\, \|z\|_{1+\delta,\Omega_{\mathrm{D}}} \,\leq\, c_{\mathrm{D},\mathtt{d}}\, \|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,. \tag{5.4}$$

Then, bearing in mind [2, Section 4.2.2, items (a), (b), and (c)] we can define $\mathbf{w}_{\mathrm{D},h} := \Pi_h^{\mathrm{D}}(\nabla z) \in \widetilde{\mathbf{H}}_h(\Omega_{\mathrm{D}})$, which satisfies $\mathrm{div}(\mathbf{w}_{\mathrm{D},h}) = q_{\mathrm{D},h}$ in Ω_{D} and $\mathbf{w}_{\mathrm{D},h} \cdot \mathbf{n} = 0$ on $\partial\Omega_{\mathrm{D}}$, so that, in particular $\mathbf{w}_{\mathrm{D},h} \cdot \mathbf{n} = 0$ on Γ_{D}, and hence $\mathbf{w}_{\mathrm{D},h} \in \mathbf{H}_h(\Omega_{\mathrm{D}})$. Additionally, using the a priori estimate for $\|z\|_{1,\Omega_{\mathrm{D}}}$, we readily obtain

$$\begin{aligned} \|\mathbf{w}_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \,&=\, \|\Pi_h^{\mathrm{D}}(\nabla z)\|_{0,\Omega_{\mathrm{D}}} \,\leq\, \|\nabla z - \Pi_h^{\mathrm{D}}(\nabla z)\|_{0,\Omega_{\mathrm{D}}} \,+\, \|\nabla z\|_{0,\Omega_{\mathrm{D}}} \\ &\leq\, \|\nabla z - \Pi_h^{\mathrm{D}}(\nabla z)\|_{0,\Omega_{\mathrm{D}}} + \widetilde{c}_{\mathrm{D},\mathtt{d}}\, \|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,. \end{aligned} \tag{5.5}$$

In turn, employing the interpolation error estimate from [2, Section 4.2.2, item (d)], and invoking (5.4), we find that

$$\begin{aligned} \|\nabla z - \Pi_h^{\mathrm{D}}(\nabla z)\|_{0,\Omega_{\mathrm{D}}}^2 \,&=\, \sum_{T \in \mathcal{T}_h^{\mathrm{D}}} \|\nabla z - \Pi_h^{\mathrm{D}}(\nabla z)\|_{0,T}^2 \,\leq\, C \sum_{T \in \mathcal{T}_h^{\mathrm{D}}} h_T^{2\delta} \Big\{ \|\nabla z\|_{\delta,T}^2 \,+\, \|\mathrm{div}(\nabla z)\|_{0,T}^2 \Big\} \\ &\leq\, C\, h_{\mathrm{D}}^{2\delta} \sum_{T \in \mathcal{T}_h^{\mathrm{D}}} \Big\{ \|\nabla z\|_{\delta,T}^2 \,+\, \|q_{\mathrm{D},h}\|_{0,T}^2 \Big\} \,\leq\, C\, h_{\mathrm{D}}^{2\delta} \Big\{ \|\nabla z\|_{\delta,\Omega_{\mathrm{D}}}^2 \,+\, \|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}^2 \Big\} \\ &\leq\, C\, h_{\mathrm{D}}^{2\delta} \Big(c_{\mathrm{D},\mathtt{d}}^2 + 1 \Big)\, \|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}^2\,, \end{aligned}$$

which, along with (5.5) and the identity satisfied by $\mathrm{div}(\mathbf{w}_{\mathrm{D},h})$, yields

$$\|\mathbf{w}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}} \,\le\, C_{\mathrm{D},\mathtt{d}}\,\|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,, \tag{5.6}$$

with a positive constant $C_{\mathrm{D},\mathtt{d}}$, depending only on $\widetilde{c}_{\mathrm{D},\mathtt{d}}$, C, $|\Omega_{\mathrm{D}}|$, δ, and $c_{\mathrm{D},\mathtt{d}}$. In this way, from the definition of $\mathcal{S}_{2,h}(q_{\mathrm{D},h})$ (cf. (4.11)) and (5.6), we conclude that

$$\mathcal{S}_{2,h}(q_{\mathrm{D},h}) \,\ge\, \frac{(q_{\mathrm{D},h},\mathrm{div}(\mathbf{w}_{\mathrm{D},h}))_{\mathrm{D}}}{\|\mathbf{w}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \,=\, \frac{\|q_{\mathrm{D},h}\|^2_{0,\Omega_{\mathrm{D}}}}{\|\mathbf{w}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \,\ge\, \boldsymbol{\beta}_{2,\mathrm{d}}\,\|q_{\mathrm{D},h}\|_{0;\Omega_{\mathrm{D}}}\,,$$

with $\boldsymbol{\beta}_{2,\mathrm{d}} := C^{-1}_{\mathrm{D},\mathtt{d}}$, thus ending the proof. □

On the other hand, similarly as for the proof of (3.73), if we assume that there exists $\boldsymbol{\psi}_{0,\mathrm{d}} \in \mathbf{H}^{1/2}_{00}(\Sigma)$ such that $\boldsymbol{\psi}_{0,\mathrm{d}} \in \boldsymbol{\Lambda}^{\mathrm{B}}_h(\Sigma)$ for all $h>0$, and $\langle \boldsymbol{\psi}_{0,\mathrm{d}}\cdot\mathbf{n},1\rangle_\Sigma \neq 0$, then it is easy to show that there holds (4.12) with $\boldsymbol{\beta}_{3,\mathrm{d}} := \frac{\langle \boldsymbol{\psi}_{0,\mathrm{d}}\cdot\mathbf{n},1\rangle_\Sigma}{\|\boldsymbol{\psi}_{0,\mathrm{d}}\|_{1/2,00;\Sigma}}$. In this regard, and as noticed at the beginning of [29, Section 5.3], the existence of such $\boldsymbol{\psi}_{0,\mathrm{d}}$ is guaranteed, in particular, if the sequence of subspaces $\big\{\boldsymbol{\Lambda}^{\mathrm{B}}_h(\Sigma)\big\}_{h>0}$ is nested. In this case, and as already mentioned in the proof of (3.73), $\boldsymbol{\psi}_{0,\mathrm{d}}$ can be constructed, for instance, as indicated in the last part of the proof of [29, Lemma 3.6].

In what follows we focus on the verification of (**H.5**), which reduces to proving (4.7) and (4.8). To this end, and proceeding as in [24, Section 4.4] (which collects the results from [29, Section 5]), and [2, Section 4.2], we assume from now that $\mathcal{T}^{\mathrm{B}}_h$ and $\mathcal{T}^{\mathrm{D}}_h$ are quasi-uniform around Σ, which means that there exists a Lipschitz-continuous open neighborhood Ω_Σ of Σ, such that the elements of $\mathcal{T}^{\mathrm{B}}_h$ and $\mathcal{T}^{\mathrm{D}}_h$ intersecting that region are roughly of the same size. More precisely, defining

$$\mathcal{T}_{h,\Sigma} \,:=\, \Big\{ T\in \mathcal{T}^{\mathrm{B}}_h \cup \mathcal{T}^{\mathrm{D}}_h : \quad T\cap\Omega_\Sigma \neq \emptyset \Big\}\,,$$

there exists a positive constant c, independent of h, such that

$$\max_{T\in\mathcal{T}_{h,\Sigma}} h_T \,\le\, c \min_{T\in\mathcal{T}_{h,\Sigma}} h_T\,.$$

Then, defining the subspaces of $\mathrm{H}^{-1/2}(\Sigma)$ and $\mathbf{H}^{-1/2}(\Sigma)$ given, respectively, by

$$\Phi_h(\Sigma) \,:=\, \Big\{ \phi_h\in\mathrm{L}^2(\Sigma) : \quad \phi_h|_e \in \mathrm{P}_0(e) \quad \forall\ \text{edge/face } e\in\Sigma_h \Big\} \quad \text{and} \quad \boldsymbol{\Phi}_h(\Sigma) \,:=\, [\Phi_h(\Sigma)]^n\,,$$

one can show (cf. [24, Theorem 4.1] and [2, Lemma 4.4] for the 2D and 3D cases, respectively) that there exist $\mathcal{E}^{\mathrm{D}}_h \in \mathcal{L}\big(\Phi_h(\Sigma),\mathbf{H}_h(\Omega_{\mathrm{D}})\big)$ and $\boldsymbol{\mathcal{E}}^{\mathrm{B}}_h \in \mathcal{L}\big(\boldsymbol{\Phi}_h(\Sigma),\mathbb{H}_h(\Omega_{\mathrm{B}})\big)$, with norms $\|\mathcal{E}^{\mathrm{D}}_h\|$ and $\|\boldsymbol{\mathcal{E}}^{\mathrm{B}}_h\|$ independent of h, such that

$$\mathrm{div}\big(\mathcal{E}^{\mathrm{D}}_h(\phi_h)\big)\in\mathrm{P}_0(\Omega_{\mathrm{D}}) \quad\text{and}\quad \mathcal{E}^{\mathrm{D}}_h(\phi_h)\cdot\mathbf{n} \,=\, \phi_h \quad\text{on}\quad \Sigma \qquad \forall\,\phi_h\in\Phi_h(\Sigma)\,, \tag{5.7}$$

$$\mathbf{div}\big(\boldsymbol{\mathcal{E}}^{\mathrm{B}}_h(\boldsymbol{\phi}_h)\big) = \mathbf{0} \quad\text{in}\quad \Omega_{\mathrm{B}} \quad\text{and}\quad \boldsymbol{\mathcal{E}}^{\mathrm{B}}_h(\boldsymbol{\phi}_h)\,\mathbf{n} \,=\, \boldsymbol{\phi}_h \quad\text{on}\quad \Sigma \qquad \forall\,\boldsymbol{\phi}_h\in\boldsymbol{\Phi}_h(\Sigma)\,. \tag{5.8}$$

In this way, having these so-called discrete lifting operators $\mathcal{E}^{\mathrm{D}}_h$ and $\boldsymbol{\mathcal{E}}^{\mathrm{B}}_h$ satisfying (5.7) and (5.8), it is not difficult to prove (cf. [24, Lemma 4.9] or [29, Lemma 4.2], and [2, proof of Lemma 4.6] for the 2D and 3D cases, respectively) that (4.7) and (4.8) are equivalent to the existence of positive constants $\gamma_{1,\mathrm{d}}$ and $\gamma_{2,\mathrm{d}}$, respectively, such that

$$\sup_{\boldsymbol{\phi}_h\in\boldsymbol{\Phi}_h(\Sigma)\backslash\{0\}} \frac{\langle\boldsymbol{\phi}_h,\boldsymbol{\psi}_h\rangle_\Sigma}{\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}} \,\ge\, \gamma_{1,\mathrm{d}}\,\|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma} \quad \forall\,\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}^{\mathrm{B}}_h(\Sigma) \quad\text{such that}\quad \langle\boldsymbol{\psi}_h\cdot\mathbf{n},1\rangle_\Sigma = 0\,, \tag{5.9}$$

and

$$\sup_{\phi_h \in \Phi_h(\Sigma)\setminus\{0\}} \frac{\langle \phi_h, \xi_h \rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \,\geq\, \gamma_{2,\mathrm{d}}\,\|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \in \Lambda_h^{\mathrm{D}}(\Sigma)\,. \tag{5.10}$$

For the 2D case there are several ways of yielding the verification of (5.9) and (5.10), which usually involve suitable modifications of the original mesh Σ_h when defining $\Lambda_h^{\mathrm{B}}(\Sigma)$ and $\Lambda_h^{\mathrm{D}}(\Sigma)$ (cf. (5.2)). In particular, three options are described in [29, Section 5.3] (see also [24, Section 4.4] for two of them), so that, being the third one the easiest to implement, here we stay with it. Its definition is based on the assumption that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ that arises by joining pairs of adjacent edges of Σ_h, denote the resulting edges still by e, and define $h_\Sigma := \max\{h_e : \quad e \in \Sigma_{2h}\}$. If the number of edges of Σ_h were odd, we first reduce it to the even case by joining any pair of two adjacent elements, construct Σ_{2h} from this reduced partition, and define h_Σ as indicated above. In this way, redefining $\Lambda_h^{\mathrm{B}}(\Sigma)$ and $\Lambda_h^{\mathrm{D}}(\Sigma)$ in (5.2) with Σ_{2h} instead of Σ_h, the proofs of (5.9) and (5.10) follow directly from [29, Lemma 5.2] (see also [24, Lemma 4.12]).

For the 3D case, and up to the authors' knowledge, there is no approach similar to the above one available in the literature. Instead of it, we introduce now a partition $\Sigma_{\widetilde{h}}$ of Σ, which is independent of Σ_h, and which is formed by triangles $\widetilde{e}$ of diameter $h_{\widetilde{e}}$, so that we set $\widetilde{h} := \max\big\{h_{\widetilde{e}} : \quad \widetilde{e} \in \Sigma_{\widetilde{h}}\big\}$. Then, denoting $h_\Sigma := \max\big\{h_e : \quad e \in \Sigma_h\big\}$, and redefining $\Lambda_h^{\mathrm{B}}(\Sigma)$ and $\Lambda_h^{\mathrm{D}}(\Sigma)$ in (5.2) with $\Sigma_{\widetilde{h}}$ instead of Σ_h, it is possible to prove that, under a suitable relationship between $\widetilde{h}$ and h_Σ, the required inequalities hold. More precisely, it is shown in [2, Lemma 4.5] (see also [25, Lemma 7.5]) that there exists a positive constant C_0 such that whenever $h_\Sigma \,\leq\, C_0\,\widetilde{h}$, (5.9) and (5.10) are satisfied.

According to the different 2D and 3D notations for the meshsize in the interface, we now unify them by defining $\widetilde{h}_\Sigma := \begin{cases} h_\Sigma & \text{in 2D} \\ \widetilde{h} & \text{in 3D} \end{cases}$.

5.3 Rates of convergence

The approximation properties of the finite element subspaces involved, which are named after the unknowns to which they are applied on, are collected next (cf. [22], [24], [31]):

$\big(\mathbf{AP}_h^{\boldsymbol{\sigma}_{\mathrm{B}}}\big)$ there exists a positive constant C, independent of h, such that for each $s \in (0,1]$, and for each $\boldsymbol{\tau}_{\mathrm{B}} \in \mathbb{H}^s(\Omega_{\mathrm{B}}) \cap \mathbb{H}_0(\mathbf{div}_\varrho;\Omega_{\mathrm{B}})$ with $\mathbf{div}(\boldsymbol{\tau}_{\mathrm{B}}) \in \mathbf{W}^{s,\varrho}(\Omega_{\mathrm{B}})$, there holds

$$\mathrm{dist}\,\big(\boldsymbol{\tau}_{\mathrm{B}}, \mathbb{H}_h(\Omega_{\mathrm{B}})\big) \,\leq\, C\,h^s \Big\{\|\boldsymbol{\tau}_{\mathrm{B}}\|_{s,\Omega_{\mathrm{B}}} \,+\, \|\mathbf{div}(\boldsymbol{\tau}_{\mathrm{B}})\|_{s,\varrho;\Omega_{\mathrm{B}}}\Big\}\,,$$

$\big(\mathbf{AP}_h^{\mathbf{u}_{\mathrm{D}}}\big)$ there exists a positive constant C, independent of h, such that for each $s \in (0,1]$, and for each $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}^s(\Omega_{\mathrm{D}}) \cap \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$ with $\mathrm{div}(\mathbf{v}_{\mathrm{D}}) \in \mathrm{H}^s(\Omega_{\mathrm{D}})$, there holds

$$\mathrm{dist}\,\big(\mathbf{v}_{\mathrm{D}}, \mathbf{H}_h(\Omega_{\mathrm{D}})\big) \,\leq\, C\,h^s \Big\{\|\mathbf{v}_{\mathrm{D}}\|_{s;\Omega_{\mathrm{D}}} \,+\, \|\mathrm{div}(\mathbf{v}_{\mathrm{D}})\|_{s;\Omega_{\mathrm{D}}}\Big\}\,,$$

$\big(\mathbf{AP}_h^{\boldsymbol{\varphi}}\big)$ there exists a positive constant C, independent of h and $\widetilde{h}_\Sigma$, such that for each $s \in [0,1]$, and for each $\boldsymbol{\psi} \in \mathbf{H}^{1/2+s}(\Sigma) \cap \mathbf{H}_{00}^{1/2}(\Sigma)$, there holds

$$\mathrm{dist}\,\big(\boldsymbol{\psi}, \boldsymbol{\Lambda}_h^{\mathrm{B}}\big) \,\leq\, C\,\widetilde{h}_\Sigma^s\,\|\boldsymbol{\psi}\|_{1/2+s;\Sigma}\,,$$

$\left(\mathbf{AP}_h^{\lambda}\right)$ there exists a positive constant C, independent of h and $\widetilde{h}_\Sigma$, such that for each $s \in [0,1]$, and for each $\xi \in \mathrm{H}^{1/2+s}(\Sigma)$, there holds

$$\operatorname{dist}\left(\xi, \Lambda_h^{\mathrm{D}}\right) \,\le\, C\, \widetilde{h}_\Sigma^s \, \|\xi\|_{1/2+s;\Sigma}\,,$$

$\left(\mathbf{AP}_h^{\mathbf{u}_{\mathrm{B}}}\right)$ there exists a positive constant C, independent of h, such that for each $s \in [0,1]$, and for each $\mathbf{v}_{\mathrm{B}} \in \mathbf{W}^{s,\rho}(\Omega_{\mathrm{B}})$, there holds

$$\operatorname{dist}\left(\mathbf{v}_{\mathrm{B}}, \mathbf{L}_h(\Omega_{\mathrm{B}})\right) \,\le\, C\, h^s \, \|\mathbf{v}_{\mathrm{B}}\|_{s,\rho;\Omega_{\mathrm{B}}}\,,$$

$\left(\mathbf{AP}_h^{p_{\mathrm{D}}}\right)$ there exists a positive constant C, independent of h, such that for each $s \in [0,1]$, and for each $q_{\mathrm{D}} \in \mathrm{H}^s(\Omega_{\mathrm{D}}) \cap \mathrm{L}_0^2(\Omega_{\mathrm{D}})$, there holds

$$\operatorname{dist}\left(q_{\mathrm{D}}, \mathrm{L}_h(\Omega_{\mathrm{D}})\right) \,\le\, C\, h^s \, \|q_{\mathrm{D}}\|_{s;\Omega_{\mathrm{D}}}\,.$$

Hence, we are now in position to provide the rates of convergence of the Galerkin scheme (4.3) with the finite element subspaces defined throughout this section.

Theorem 5.2 *In addition to the hypotheses of the Theorems* 3.10*,* 4.3 *and* 4.4*, assume that there exists* $s \in (0,1]$ *such that* $\boldsymbol{\sigma}_{\mathrm{B}} \in \mathbb{H}^s(\Omega_{\mathrm{B}}) \cap \mathbb{H}_0(\mathbf{div}_\varrho;\Omega_{\mathrm{B}})$*,* $\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{B}}) \in \mathbf{W}^{s,\varrho}(\Omega_{\mathrm{B}})$*,* $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^s(\Omega_{\mathrm{D}}) \cap \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$*,* $\mathrm{div}(\mathbf{u}_{\mathrm{D}}) \in \mathrm{H}^s(\Omega_{\mathrm{D}})$*,* $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+s}(\Sigma) \cap \mathbf{H}_{00}^{1/2}(\Sigma)$*,* $\lambda \in \mathrm{H}^{1/2+s}(\Sigma)$*,* $\mathbf{u}_{\mathrm{B}} \in \mathbf{W}^{s,\rho}(\Omega_{\mathrm{B}})$*, and* $p_{\mathrm{D}} \in \mathrm{H}^s(\Omega_{\mathrm{D}}) \cap \mathrm{L}_0^2(\Omega_{\mathrm{D}})$*. Then, there exists a positive constant* C*, independent of* h *and* $\widetilde{h}_\Sigma$*, such that*

$$\begin{aligned}\|((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}}),\vec{\mathbf{u}}) \,-\, ((\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\varphi}})_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\le\, C\,\Big\{ h^s \Big(&\|\boldsymbol{\sigma}_{\mathrm{B}}\|_{s,\Omega_{\mathrm{B}}} \,+\, \|\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{B}})\|_{s,\varrho;\Omega_{\mathrm{B}}} \,+\, \|\mathbf{u}_{\mathrm{D}}\|_{s;\Omega_{\mathrm{D}}} \\ &+\, \|\mathrm{div}(\mathbf{u}_{\mathrm{D}})\|_{s;\Omega_{\mathrm{D}}} \,+\, \|\mathbf{u}_{\mathrm{B}}\|_{s,\rho;\Omega_{\mathrm{B}}} \,+\, \|p_{\mathrm{D}}\|_{s;\Omega_{\mathrm{D}}}\Big) \,+\, \widetilde{h}_\Sigma^s \Big(\|\boldsymbol{\varphi}\|_{1/2+s;\Sigma} \,+\, \|\lambda\|_{1/2+s;\Sigma}\Big)\Big\}\,.\end{aligned}$$

Proof. It follows straightforwardly from the Céa estimate (4.21) and the approximation properties $\left(\mathbf{AP}_h^{\boldsymbol{\sigma}_{\mathrm{B}}}\right)$, $\left(\mathbf{AP}_h^{\mathbf{u}_{\mathrm{D}}}\right)$, $\left(\mathbf{AP}_h^{\boldsymbol{\varphi}}\right)$, $\left(\mathbf{AP}_h^{\lambda}\right)$, $\left(\mathbf{AP}_h^{\mathbf{u}_{\mathrm{B}}}\right)$, and $\left(\mathbf{AP}_h^{p_{\mathrm{D}}}\right)$. □

6 Numerical results

In this section we present three examples illustrating the performance of the mixed finite element scheme (4.3) on a set of quasi-uniform triangulations of the respective domains, and considering the finite element subspaces defined by (5.1)–(5.2) (cf. Section 5). The implementation of the numerical method is based on a `FreeFem++` code [35]. A Newton–Raphson algorithm with a fixed tolerance $\mathsf{tol} = 1\mathrm{E}-6$ is used for the resolution of the nonlinear problem (4.3). As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\ell^2}}{\|\mathbf{coeff}^{m+1}\|_{\ell^2}} \,\le\, \mathsf{tol}\,,$$

where $\|\cdot\|_{\ell^2}$ is the standard ℓ^2-norm in $\mathrm{R}^{\mathtt{DoF}}$, with `DoF` denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_{h,1}, \mathbf{H}_{h,2}$, and $\mathbf{Q}_h$ (cf. (4.2) and (5.1)–(5.2)).

We now introduce some additional notation. The individual errors are denoted by

$$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{B}}) := \|\boldsymbol{\sigma}_{\mathrm{B}} - \boldsymbol{\sigma}_{\mathrm{B},h}\|_{\mathbf{div}_{\varrho},\Omega_{\mathrm{B}}}\,, \quad \mathsf{e}(\mathbf{u}_{\mathrm{B}}) := \|\mathbf{u}_{\mathrm{B}} - \mathbf{u}_{\mathrm{B},h}\|_{0,\rho;\Omega_{\mathrm{B}}}\,, \quad \mathsf{e}(p_{\mathrm{B}}) := \|p_{\mathrm{B}} - p_{\mathrm{B},h}\|_{0,\Omega_{\mathrm{B}}}\,,$$

$$\mathsf{e}(\mathbf{u}_{\mathrm{D}}) := \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}\,, \quad \mathsf{e}(p_{\mathrm{D}}) := \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,,$$

$$\mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}\,, \quad \mathsf{e}(\lambda) := \|\lambda - \lambda_h\|_{1/2,\Sigma}\,,$$

with $\rho \in [3,4]$ and $\varrho \in [4/3, 3/2]$ satisfying $1/\rho + 1/\varrho = 1$, to be specified in the examples below. In turn, $p_{\mathrm{B},h}$ stands for the post-processed Brinkman–Forchheimer pressure suggested by the second formula in (3.1) and the decomposition (3.13), that is

$$p_{\mathrm{B},h} \,=\, -\frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma}_{\mathrm{B},h}) - \ell_h \quad \text{in} \quad \Omega_{\mathrm{B}}\,.$$

Notice that, for ease of computation, the interface norm $\|\lambda-\lambda_h\|_{1/2,\Sigma}$ will be replaced by $\|\lambda-\lambda_h\|_{(0,1),\Sigma}$ with

$$\|\xi\|_{(0,1),\Sigma} \,:=\, \|\xi\|_{0,\Sigma}^{1/2}\,\|\xi\|_{1,\Sigma}^{1/2} \quad \forall\,\xi \in \mathrm{H}^1(\Sigma)\,,$$

owing to the fact that $\mathrm{H}^{1/2}(\Sigma)$ is the interpolation space with index $1/2$ between $\mathrm{H}^1(\Sigma)$ and $\mathrm{L}^2(\Sigma)$. Similarly, the interface norm $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}$ will be replaced by $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{(0,1),\Sigma}$. Furthermore, the respective experimental rates of convergence are computed as

$$\mathsf{r}(\diamond) \,:=\, \frac{\log(\mathsf{e}(\diamond)/\widehat{\mathsf{e}}(\diamond))}{\log(h/\widehat{h})} \quad \text{for each } \diamond \in \Big\{\boldsymbol{\sigma}_{\mathrm{B}}, \mathbf{u}_{\mathrm{B}}, p_{\mathrm{B}}, \mathbf{u}_{\mathrm{D}}, p_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda\Big\}\,,$$

where h and $\widehat{h}$ denote two consecutive mesh sizes, taken accordingly from $h \in \{h_{\mathrm{B}}, h_{\mathrm{D}}, h_{\Sigma}\}$, with their respective errors e and $\widehat{\mathsf{e}}$.

The examples considered in this section are described below. In all cases, we use $\mathbf{u}_{\mathrm{B},h}^0 = (0, 1\mathrm{E}-6)^{\mathrm{t}}$ as the initial guess. Additionally, the conditions $(\mathrm{tr}(\boldsymbol{\sigma}_{\mathrm{B},h}), 1)_{\mathrm{B}} = 0$ and $(p_{\mathrm{D},h}, 1)_{\mathrm{D}} = 0$ are imposed using a penalization strategy.

Example 1: 2D convex domain with varying $\mu, \mathtt{F}$, and $\mathbf{K}_{\mathrm{D}}$ parameters

In the first example, inspired by [7, Example 1 in Section 5], we validate the rates of convergence in a two-dimensional convex domain and also study the performance of the numerical method with respect to the number of Newton iterations when different values of the parameters μ, $\mathtt{F}$, and $\mathbf{K}_{\mathrm{D}}$ are considered. More precisely, we consider a semi-disk-shaped porous domain coupled with a porous unit square, i.e.,

$$\Omega_{\mathrm{B}} := \Big\{(x_1, x_2) : \quad x_1^2 + (x_2 - 0.5)^2 < 0.5^2,\ x_2 > 0.5\Big\} \quad \text{and} \quad \Omega_{\mathrm{D}} := (-0.5, 0.5)^2\,,$$

with interface $\Sigma := (-0.5, 0.5) \times \{0.5\}$. We consider the model parameters $\rho = 3$, $\varrho = 3/2$, $\mu = 1$, $\mathtt{F} = 10$, $\mathbf{K}_{\mathrm{B}} = \mathbb{I}$, and $\mathbf{K}_{\mathrm{D}} = 10^{-1}\,\mathbb{I}$. The data $\mathbf{f}_{\mathrm{B}}$, $\mathbf{f}_{\mathrm{D}}$, and g_{D} are chosen such that the exact solution in the tombstone-shaped porous domain $\Omega = \Omega_{\mathrm{B}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ is given by the smooth functions

$$\mathbf{u}_{\mathrm{B}} := \left(\begin{array}{c} \cos(\pi x_1)\sin(\pi x_2) \\ -\sin(\pi x_1)\cos(\pi x_2) \end{array}\right), \quad \mathbf{u}_{\mathrm{D}} := \left(\begin{array}{c} \cos(\pi x_1)\exp(x_2) \\ \exp(x_1)\cos(\pi x_2) \end{array}\right),$$

$$p_\star := \sin(\pi x_1)\sin(\pi x_2) \quad \text{in} \quad \Omega_\star, \qquad \text{with } \star \in \{\mathrm{B}, \mathrm{D}\}.$$

Note that this solution satisfies mass conservation on the interface, i.e., $\mathbf{u}_{\mathrm{B}} \cdot \mathbf{n} = \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n}$ on Σ. However, the continuity of momentum (cf. the second transmission condition in (2.3)) is not met. Additionally,

DoF	$h_{\rm B}$	$\mathsf{e}(\boldsymbol{\sigma}_{\rm B})$	$r(\boldsymbol{\sigma}_{\rm B})$	$\mathsf{e}(\mathbf{u}_{\rm B})$	$r(\mathbf{u}_{\rm B})$	$\mathsf{e}(p_{\rm B})$	$r(p_{\rm B})$	h_Σ	$\mathsf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$
197	0.330	1.8E-00	–	1.5E-01	–	1.8E-01	–	1/2	2.5E-01	–
733	0.191	9.5E-01	1.162	7.9E-02	1.188	9.3E-02	1.209	1/4	9.3E-02	1.415
2736	0.091	4.6E-01	0.976	3.8E-02	0.989	3.9E-02	1.161	1/8	3.3E-02	1.506
10718	0.049	2.3E-01	1.112	1.9E-02	1.112	1.9E-02	1.145	1/16	1.2E-02	1.512
42915	0.024	1.1E-01	1.013	9.4E-03	1.009	1.0E-02	0.937	1/32	4.1E-03	1.503
170305	0.013	5.8E-02	1.156	4.7E-03	1.159	4.9E-03	1.219	1/64	1.4E-03	1.506

$h_{\rm D}$	$\mathsf{e}(\mathbf{u}_{\rm D})$	$r(\mathbf{u}_{\rm D})$	$\mathsf{e}(p_{\rm D})$	$r(p_{\rm D})$	h_Σ	$\mathsf{e}(\lambda)$	$r(\lambda)$	iter
0.373	7.3E-01	–	1.3E-01	–	1/2	4.9E-01	–	4
0.190	3.2E-01	1.217	7.5E-02	0.842	1/4	2.0E-01	1.275	4
0.095	1.6E-01	0.963	3.0E-02	1.330	1/8	5.1E-02	1.969	4
0.054	8.4E-02	1.168	1.5E-02	1.196	1/16	1.7E-02	1.568	4
0.025	4.2E-02	0.908	7.5E-03	0.911	1/32	6.2E-03	1.478	4
0.014	2.1E-02	1.290	3.7E-03	1.291	1/64	2.2E-03	1.528	4

Table 6.1: [EXAMPLE 1] Degrees of freedom, mesh sizes, errors, convergence history, and Newton iteration count for the approximation of the coupled Brinkman–Forchheimer/Darcy problem with $\rho = 3$, $\mu = 1$, $\mathbf{K}_{\rm B} = \mathbb{I}$, $\mathbf{K}_{\rm D} = 10^{-1}\mathbb{I}$, and $\mathtt{F} = 10$.

the Dirichlet boundary condition for the Brinkman–Forchheimer velocity on $\Gamma_{\rm B}$ and the Neumann boundary condition for the Darcy velocity on $\Gamma_{\rm D}$ are both non-homogeneous, leading to extra contributions on the right-hand side of the resulting system. The results reported in Table 6.1 are consistent with the theoretical optimal convergence rate of $\mathcal{O}(h)$, as stated in Theorem 5.2. The domain configuration and some components of the numerical solution are shown in Figure 6.1, computed using the fully-mixed approximation with a mesh size of $h = 0.014$ and $53,511$ triangular elements (corresponding to $170,305$ DoF). We observe that the continuity of the normal component of the velocities on Σ is maintained, as the second components of $\mathbf{u}_{\rm B}$ and $\mathbf{u}_{\rm D}$ match on Σ, as expected. It can also be noted that the pressure remains continuous throughout the domain and retains its sinusoidal pattern.

Table 6.2 presents the number of Newton iterations as a function of the parameters μ, $\mathtt{F}$, and $\mathbf{K}_{\rm D} = \kappa_{\rm D}\,\mathbb{I}$, with $\mathbf{K}_{\rm B} = \mathbb{I}$ and different mesh sizes h. It can be observed that Newton's method remains robust with respect to both h and $\mathbf{K}_{\rm D}$. However, the number of iterations increases for smaller values of μ and larger values of $\mathtt{F}$, respectively. This dependence aligns with the theoretical rate of convergence of the mixed approach (4.3) (cf. Theorem 5.2). In particular, the behavior of the iterative method with varying Forchheimer numbers $\mathtt{F} \in \{1, 10, 10^2, 10^3, 10^4\}$ is justified by the greater influence of the nonlinear term $\mathtt{F}|\mathbf{u}_{\rm B}|\mathbf{u}_{\rm B}$ in the Brinkman–Forchheimer model.

Example 2: Accuracy assessment in a 2D non-convex domain

In the second example, we test the fully-mixed scheme (4.3) in a 2D non-convex domain. Specifically, we consider the 2D helmet-shaped domain defined by $\Omega = \Omega_{\rm B} \cup \Sigma \cup \Omega_{\rm D}$, where

$$\Omega_{\rm B} := (-1,1)\times(0,1.25)\backslash(-0.75,0.75)\times(0.25,1.25)\,,\quad \Omega_{\rm D} := (-1,1)\times(-0.5,0)\,,$$

and $\Sigma := (-1,1)\times\{0\}$ (see the first plot of Figure 6.2 below). We use the model parameters $\rho = 7/2$, $\varrho = 7/5$, $\mu = 10^{-1}$, $\mathtt{F} = 10$, $\mathbf{K}_{\rm B} = 10^{-1}\,\mathbb{I}$, and $\mathbf{K}_{\rm D} = 10^{-2}\,\mathbb{I}$. The data $\mathbf{f}_{\rm B}, \mathbf{f}_{\rm D}$, and $g_{\rm D}$ are adjusted so

μ	F	κ_D	$h = 0.373$	$h = 0.191$	$h = 0.095$	$h = 0.054$	$h = 0.025$	$h = 0.014$
1	10	10^{-1}	4	4	4	4	4	4
1	10	10^{-2}	4	4	4	4	4	4
1	10	10^{-3}	4	4	4	4	4	4
1	10	10^{-4}	4	4	4	4	4	4
10^{-1}	10	10^{-1}	6	6	6	6	6	6
10^{-2}	10	10^{-1}	8	7	7	7	7	7
10^{-3}	10	10^{-1}	8	9	9	9	9	9
10^{-4}	10	10^{-1}	9	9	9	10	10	10
1	1	10^{-1}	4	4	4	4	4	4
1	10^2	10^{-1}	6	6	6	6	6	6
1	10^3	10^{-1}	9	10	9	9	9	9
1	10^4	10^{-1}	13	13	13	13	13	13

Table 6.2: [EXAMPLE 1] Number of Newton iterations for different values of μ, $\mathtt{F}$, and $\mathbf{K}_{\mathrm{D}} = \kappa_{\mathrm{D}}\,\mathbb{I}$.

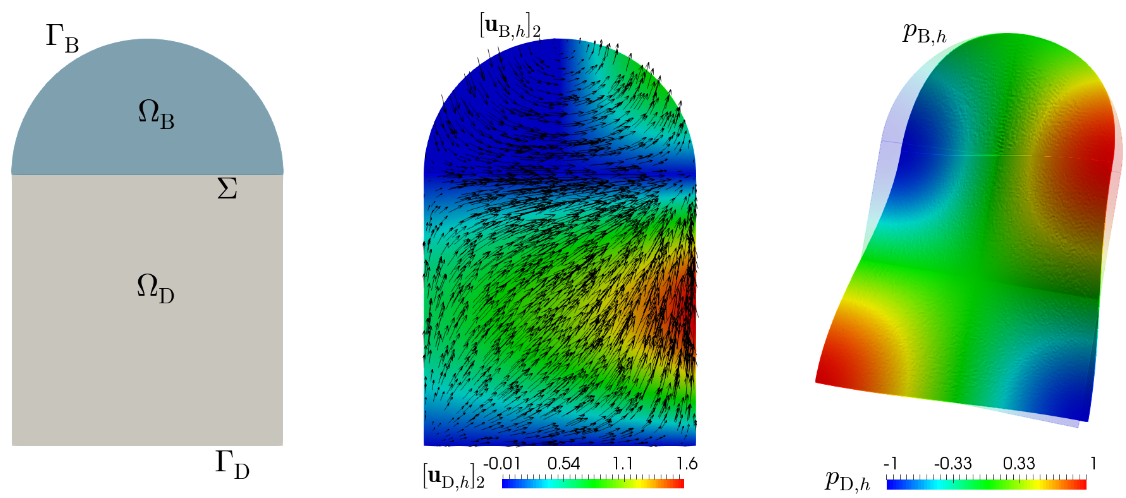

Figure 6.1: [EXAMPLE 1] Domain configuration, computed velocity field and magnitude of its second component, and pressure field in the whole domain.

that the exact solution in the 2D helmet-shaped domain Ω is given by the smooth functions

$$\mathbf{u}_{\mathrm{B}} = \begin{pmatrix} -\sin(\pi x_1)\cos(\pi x_2) \\ \cos(\pi x_1)\sin(\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{B}}, \quad \mathbf{u}_{\mathrm{D}} = \begin{pmatrix} \sin(2\pi x_1)\exp(x_2) \\ \exp(x_1)\sin(2\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{D}},$$
$$p_\star = \sin(\pi x_1)\exp(x_2) \quad \text{in} \quad \Omega_\star, \qquad \text{with } \star \in \{\mathrm{B}, \mathrm{D}\}\,.$$

The model problem is then complemented with the appropriate boundary conditions. Some components of the numerical solution are displayed in Figure 6.2, which were obtained using the mixed approximation (4.3) with a mesh size of $h = 0.007$ and 284,356 triangular elements (representing a total of 1,072,673 DoF).

The convergence history for a series of quasi-uniform mesh refinements using the particular discrete spaces (5.1)–(5.2) is presented in Table 6.3. Once again, the mixed finite element method exhibits optimal convergence with an order of $\mathcal{O}(h)$, as established by Theorem 5.2.

DoF	h_{B}	$\mathtt{e}(\boldsymbol{\sigma}_{\mathrm{B}})$	$r(\boldsymbol{\sigma}_{\mathrm{B}})$	$\mathtt{e}(\mathbf{u}_{\mathrm{B}})$	$r(\mathbf{u}_{\mathrm{B}})$	$\mathtt{e}(p_{\mathrm{B}})$	$r(p_{\mathrm{B}})$	h_Σ	$\mathtt{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$
1137	0.188	1.3E-00	–	1.2E-01	–	6.6E-01	–	1/4	2.4E-01	–
4578	0.100	4.4E-01	1.717	5.7E-02	1.251	2.1E-01	1.796	1/8	9.6E-02	1.299
17075	0.050	1.4E-01	1.616	2.7E-02	1.062	5.8E-02	1.895	1/16	3.4E-02	1.518
68304	0.026	5.6E-02	1.421	1.3E-02	1.069	1.8E-02	1.742	1/32	1.2E-02	1.490
267557	0.014	2.6E-02	1.325	6.7E-03	1.190	7.6E-03	1.487	1/64	4.2E-03	1.515
1072673	0.007	1.2E-02	0.974	3.3E-03	0.932	3.5E-03	1.047	1/128	1.5E-03	1.518

h_{D}	$\mathtt{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathtt{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	h_Σ	$\mathtt{e}(\lambda)$	$r(\lambda)$	iter
0.200	1.3E-00	–	2.7E-01	–	1/4	1.1E-00	–	5
0.095	6.2E-01	0.983	6.1E-02	2.009	1/8	4.1E-01	1.351	5
0.049	3.2E-01	1.036	1.9E-02	1.805	1/16	1.6E-01	1.411	5
0.026	1.6E-01	1.082	7.3E-03	1.483	1/32	5.6E-02	1.467	5
0.013	7.9E-02	0.967	3.3E-03	1.093	1/64	1.9E-02	1.534	5
0.007	4.0E-02	1.205	1.6E-03	1.253	1/128	6.1E-03	1.670	5

Table 6.3: [Example 2] Degrees of freedom, mesh sizes, errors, convergence history, and Newton iteration count for the approximation of the coupled Brinkman–Forchheimer/Darcy problem with $\rho = 7/2$, $\mu = 10^{-1}$, $\mathbf{K}_{\mathrm{B}} = 10^{-1}\mathbb{I}$, $\mathbf{K}_{\mathrm{D}} = 10^{-2}\mathbb{I}$, and $\mathtt{F} = 10$.

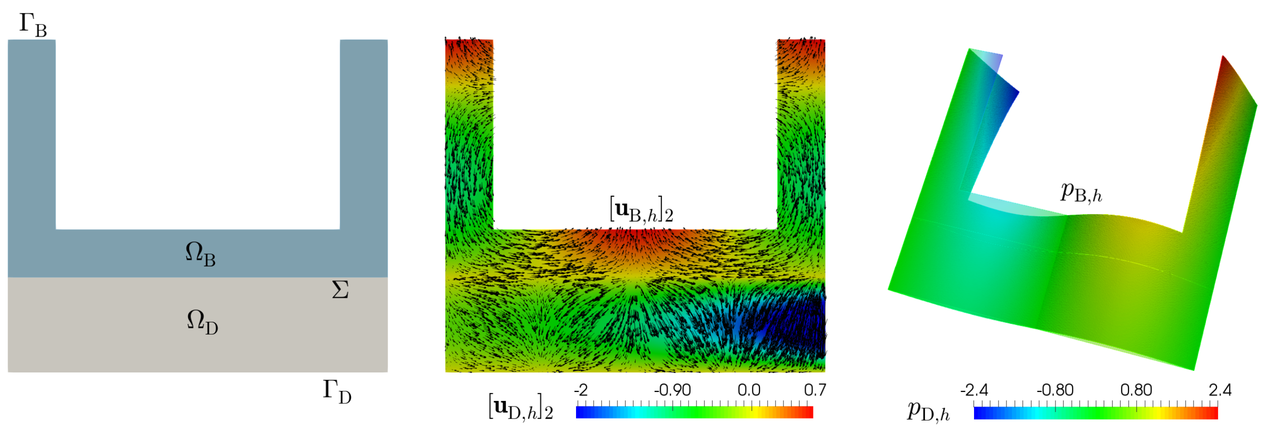

Figure 6.2: [Example 2] Domain configuration, computed velocity field and magnitude of its second component, and pressure field in the whole domain.

Example 3: Flow through a heterogeneous porous media

In the final example, we examine the behavior of the numerical method for different values of $\mathtt{F}$ with $\rho = 4$, in order to model the higher-order inertial correction $\mathtt{F}\,|\mathbf{u}_{\mathrm{B}}|^2\mathbf{u}_{\mathrm{B}}$ discussed in [7, Example 2 in Section 5]. We consider the rectangular domain $\Omega = \Omega_{\mathrm{B}} \cup \Sigma \cup \Omega_{\mathrm{D}}$, where

$$\Omega_{\mathrm{B}} := (0,2)\times(0,1), \quad \Sigma := (0,2)\times\{0\}, \quad \text{and} \quad \Omega_{\mathrm{D}} := (0,2)\times(-1,0)\,,$$

with boundaries $\Gamma_{\mathrm{B}} = \Gamma_{\mathrm{B,left}} \cup \Gamma_{\mathrm{B,top}} \cup \Gamma_{\mathrm{B,right}}$ and $\Gamma_{\mathrm{D}} = \Gamma_{\mathrm{D,left}} \cup \Gamma_{\mathrm{D,bottom}} \cup \Gamma_{\mathrm{D,right}}$, respectively. The problem parameters are $\mu = 1$, $\mathbf{K}_{\mathrm{B}} = 10^{-1}\,\mathbb{I}$ and $\mathbf{K}_{\mathrm{D}} = 10^{-3}\,\mathbb{I}$. The right-hand side data $\mathbf{f}_{\mathrm{B}}, \mathbf{f}_{\mathrm{D}}$, and g_{D} are chosen as zero, and the boundary conditions are

$$\mathbf{u}_{\mathrm{B}} = (-10\,x_2\,(x_2-1), 0)^{\mathrm{t}} \quad \text{on} \quad \Gamma_{\mathrm{B,left}}\,, \quad \mathbf{u}_{\mathrm{B}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{B,top}}\,, \quad \boldsymbol{\sigma}_{\mathrm{B}}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{B,right}}\,,$$

$$p_{\mathrm{D}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D,bottom}}\,, \quad \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D,left}} \cup \Gamma_{\mathrm{D,right}}\,.$$

In Figure 6.3, we plot the magnitude of the second component of the velocity across the entire domain for $\mathtt{F} \in \{0, 10^1, 10^2, 10^4\}$, computed using the mixed approximation (4.3) with a mesh size of $h = 0.027$ and $37,238$ triangular elements (corresponding to $141,032$ DoF). The number of Newton iterations for the different values of $\mathtt{F}$ is $\{1, 5, 6, 9\}$, respectively, indicating an increase as $\mathtt{F}$ becomes larger, which is consistent with the observations from Example 1. Note that when $\mathtt{F} = 0$, the problem becomes linear, requiring only one Newton iteration. As expected, we observe that most of the flow moves from left to right within the more permeable Brinkman–Forchheimer domain, while part of it is diverted into the less permeable Darcy medium due to the zero pressure condition at the bottom of the domain. For all considered values of $\mathtt{F}$, the continuity of the normal velocity across the interface is preserved, illustrating mass conservation on Σ. Finally, we observe that as $\mathtt{F}$ increases, the magnitude of the vertical component of the velocity decreases at the interface. This behavior illustrates the role of the inertial term $\mathtt{F}\,|\mathbf{u}_{\mathrm{B}}|^2\mathbf{u}_{\mathrm{B}}$ in correcting the potential overestimation of fluid flow between the more and less permeable porous media when using the Brinkman/Darcy model (i.e., when $\mathtt{F} = 0$).

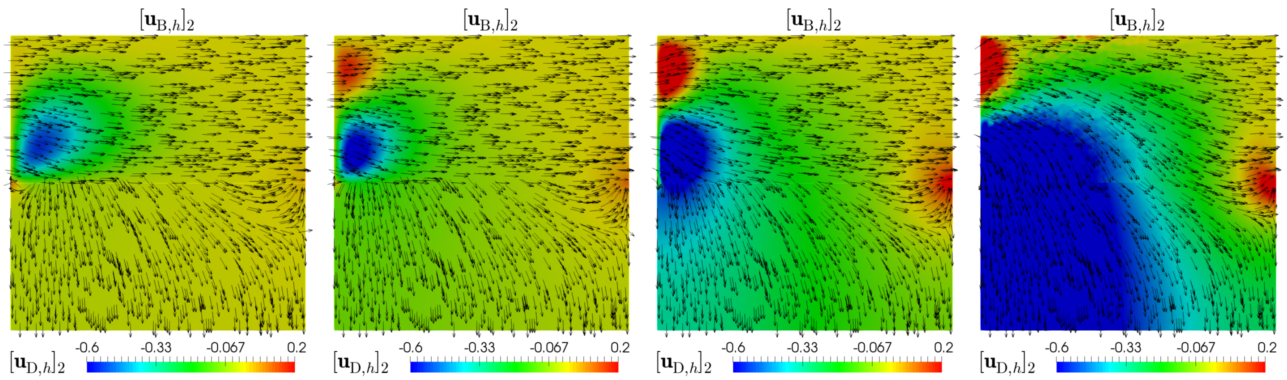

Figure 6.3: [Example 3] From left to right: magnitude of the second component of the velocity in the whole domain for $\mathtt{F} \in \{0, 10^1, 10^2, 10^4\}$.

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