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THEORETICAL AND NUMERICAL RESULTS FOR THE EXPONENTIAL STABILITY OF THE ROTATING DISK-BEAM SYSTEM WITH A BOUNDARY INFINITE MEMORY OF TYPE ANGULAR VELOCITY

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Abstract. This paper is concerned with the investigation of the effect of a boundary infinite memory term on the stability of the rotating disk-beam system. Assuming the infinite memory is of angular velocity type, the minimal state approach is employed to handle the memory term. Under specific conditions on the memory kernel function, we demonstrate that the problem is well-posed and its solutions are exponentially stable. Notably, the beam's vibrations are suppressed, and the disk achieves a desired angular velocity provided the latter remains bounded. Using the Finite Volumes Method, a comprehensive numerical study validates the theoretical stability results.

Keywords: Rotating disk-beam; infinite memory; minimal state; exponential stability, Finite Volumes Method.

1. INTRODUCTION AND PRELIMINARIES

The stability and stabilization of systems consisting of coupled elastic and rigid parts has stimulated a substantial amount of endeavor of scholars from different disciplines. Among the numerous examples of coupled elastic-rigid systems, the rotating disk-beam system remains at the forefront of the study of such systems. Indeed, the disk-beam system has been introduced in [4] to model the dynamics of a large-scale flexible-rigid space structure. Roughly speaking, it consists of two main parts: the first one is a flexible beam of length ℓ, representing a flexible robot arm, while the second one is a rigid body such as a disk. The beam is supposed to be clamped at the center of the disk and free at the other end. In turn, the disk rotates around its axis, while it is assumed that the beam's motion remains in a plane perpendicular to the disk (see Figure 1).

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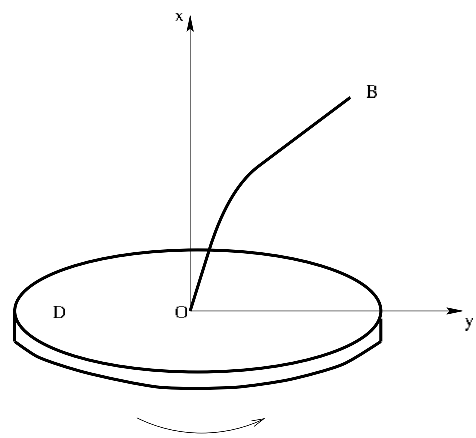

FIGURE 1. The disk-beam system

Under the above assumptions, the dynamics of the coupled beam-disk system is modeled by the following hybrid system (integro-differential equation) [4] :

$$\begin{cases} \rho(x)y_{tt}(x,t) + \big(EI(x)y_{xx}\big)_{xx}(x,t) = \rho(x)\omega^2(t)y(x,t) + U_{\text{int}}(x,t), & x \in (0,\ell),\ t>0, \\ y(0,t) = y_x(0,t) = 0, & t>0, \\ \big(EI(x)y_{xx}\big)_x(\ell,t) = U_{\text{f}}(t), & t>0, \\ (EIy_{xx})(\ell,t) = \mathcal{U}_{\text{m}}(t), & t>0, \\ \dfrac{d}{dt}\left\{\omega(t)\left[I_d + \displaystyle\int_0^\ell y^2(x,t)\,dx\right]\right\} = T(t), & t>0, \end{cases} \tag{1.1}$$

where $D = (0,\ell)\times(0,\infty)$, y represents the beam's displacement, ω is the angular velocity of the disk whose moment of inertia is I_d. In turn, EI and ρ are respectively the flexural rigidity and the mass per unit length of the beam, assumed to depend on the space variable x. Finally, U_{int} is the interior control, U_{f} is the force control, U_{m} is the moment control. These controls are to be applied on the beam. Furthermore, $T(t)$ is the torque control exerted on the disk.

The stabilization problem of numerous variants of (1.1) have been actively treated in a huge number of articles and hence it is difficult to mention all of them and even harder to discuss their contents in the present paper. Notwithstanding that, we shall name the most significant ones. To the authors' best knowledge, the first result goes back to Baillieul and Levil [4], where it is shown that a sole structural damping control

$$U_{\text{int}}(x,t) = \alpha y_{xxxxt}(x,t),\ \alpha>0,$$

leads the system to have a finite number of rotating equilibrium states. In turn, with a weaker viscous damping control

$$U_{\text{int}} = \alpha y_t(x,t),\ \alpha>0,$$

the system possesses a flat linear inertial manifold [6]. These outcomes motivated the authors in [43] to propose a feedback law that combines a viscous damping and torque control

$$\begin{cases} U_{\mathrm{int}}(x,t) = \alpha y_t(x,t),\ \alpha > 0, \\ T(t) = -\gamma(\omega(t) - \widehat{\omega}),\ \ \gamma > 0,\ \widehat{\omega} \in \mathbb{R}, \end{cases}$$

where $\widehat{\omega}$ is the desired angular velocity. The closed-loop system is exponentially stabilized, provided that the desired angular velocity $\widehat{\omega}$ is bounded [43]. The first attempt to stabilize the system (1.1) via boundary controls is due to Morgül [37, 38]. Nevertheless, the system is supposed to have a constant angular velocity in addition to constant flexural rigidity and the mass per unit length of the beam, that is,

$$\begin{cases} \rho y_{tt}(x,t) + EIy_{xxxx}(x,t) = \rho\widehat{\omega}^2 y(x,t), & (x,t) \in (0,\ell) \times (0,\infty), \\ y(0,t) = y_x(0,t) = 0, & t > 0, \\ EIy_{xx}(\ell,t) = U_{\mathrm{m}}(t) := -k_1 y_{xt}(\ell,t), & t > 0, \\ EIy_{xxx}(\ell,t) = U_{\mathrm{f}}(t) := k_2 y_t(\ell,t), & t > 0, \\ \omega(t) = \widehat{\omega}\ (\text{constant}), & t > 0, \end{cases}$$

where k_1 and k_2 are positive constants. In this event, the system is uniformly exponential stabilized [37, 38]. These findings are improved in [35, 13, 14] to the original system (1.1) and when only one static or dynamic boundary control (either force or moment) is applied, in addition to a control torque $T(t)$. Note also that similar results are obtained in [3, 10], where an extra mass is attached to the beam. In turn, nonlinear controls are also proposed in several papers so that the system (1.1) is stabilized. For instance, only a nonlinear feedback torque control is designed in [25] so that the system is asymptotically stable, whereas a torque control and a class of nonlinear controls are suggested in [2, 11] to guarantee the exponential stability of the system (1.1). It is noteworthy that non-classical feedback control laws are also designed in [9, 17, 30, 42] to ensure the exponential stabilization of (1.1).

Thereafter, a growing attention is paid to the stabilization of (1.1) when a time-delay occurs in the interior or boundary control. Basically, it is shown that the stability of the system (1.1) is robust regardless of the occurrence of an interior or boundary delay (constant or time-dependent) [12, 15, 16, 18, 20, 21, 23]. Of course, the delay term "is small" as it is known in literature. To be more precise, let us mention the first result related to this point [15], where the feedback law consists of the torque control applied on the disk, that is, $T(t) = \omega(t) - \widehat{\omega}$ and a delayed boundary control

$$U_{\mathrm{f}}(t) := \sigma y_t(\ell,t) + \beta y_t(\ell,t-\tau),\ \sigma \in \mathbb{R}, \quad t > 0.$$

It is then proved that (1.1) is exponentially stabilized if $|\sigma| < \beta$, that is, the delayed term $y_t(\ell, t-\tau)$ is "smaller" than the compensating one $y_t(\ell,t)$.

More recently, many authors consider the occurrence of an infinite memory term in the problem (1.1). Indeed, motivated by the fact that materials with memory feature are abundant in practice and specially for mechanical and thermodynamical systems, they are modeled by equations with memory. Roughly speaking, the past values of the state variable are inevitable and hence one has to take into account their influence if the model. In our case, we may say that the memory feature takes place at the right end of the beam in the sense that the stress depends on the history of small deformations [23]. Additionally, the presence of a memory term could be a raison of loosing the stability of systems [39].

Going back to our discussion, the first paper that treated the stabilization of (1.1) with memory is [19]. There, a boundary infinite memory term of velocity type is proposed and in order to contrast the eventual negative effect of the memory term, the author suggested a boundary velocity control. This permitted to establish the exponential stability of the system [19] via the energy method. The same approach is used when an interior infinite memory term of velocity type occurs in [22], where a boundary velocity control is invoked in order to compensate such memory term. In these special cases, the proof of the stability result is not arduous since the energy method works fine. A more challenging case has been examined in [31], where a moment control with infinite memory is considered. Since the energy method fails, the authors had to adopt the so-called the resolvent method to obtain the stability result of the system. In the same spirit, the stabilization problem of the rotating body-beam system is studied in [24] when either an interior or a boundary infinite memory term occurs and the compensating term is not of the same type as the memory term.

In the present paper, we shall investigate the case when an angular velocity memory type arises in the boundary, while the compensating distributed term is of the same type. In other words, we suggest the following feedback law:

$$\begin{cases} U_{\text{int}} = -\alpha(x) y_{tx}(x,t), & (x,t) \in D, \\ \mathcal{U}_{\text{m}}(t) = -\beta \displaystyle\int_0^\infty \xi(s) y_{tx}(\ell, t-s)\, ds, & t > 0, \end{cases} \tag{1.2}$$

and hence the closed-loop system (1.1)-(1.2) is

$$\begin{cases} \rho(x) y_{tt}(x,t) + (EI(x) y_{xx})_{xx}(x,t) = \rho(x)\omega^2(t) y(x,t) - \alpha(x) y_{tx}(x,t), & (x,t) \in D, \\ y(0,t) = y_x(0,t) = (EI(x) y_{xx})_x(\ell,t) = 0, & t > 0, \\ (EI y_{xx})(\ell,t) = -\beta \displaystyle\int_0^\infty \xi(s) y_{tx}(\ell, t-s)\, ds, & t > 0, \\ \dfrac{d}{dt}\left\{\omega(t)\left[I_d + \displaystyle\int_0^\ell \rho(x) y^2(x,t)\, dx\right]\right\} = -\gamma(\omega(t) - \widehat{\omega}), & t > 0, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & x \in (0,\ell), \\ y_x(\ell, -\tau) = \varphi_0(\tau), & (x,\tau) \in D, \\ \omega(0) = \omega_0 \in \mathbb{R}, \end{cases} \tag{1.3}$$

where α is the spatial distribution of the interior control $y_{tx}(x,t)$, while β and γ are positive constants. Moreover, ξ represents the kernel of the memory term and $\widehat{\omega}$ is the desired angular velocity. Finally, y_0, y_1, φ_0 and ω_0 are known initial data.

The following hypotheses are assumed to be fulfilled throughout this work:

A.I: The memory kernel satisfies:
- $\xi(s) \in L^1(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$ such that $\int_0^\infty \xi(s)\,ds > 0$ and $\xi(s) \geq 0$, for any $s \in \mathbb{R}^+$;
- $\xi^{'}(s) < 0$, $\xi^{''}(s) \geq -C\xi^{'}(s)$, for any $s \in \mathbb{R}^+$, and for some positive constant C.

A.II: The flexural rigidity $EI(x)$ and the mass per unit length of the beam $\rho(x)$ are in $C^4[0,\ell]$ and there exist two positive constants ρ_0 and EI_0 such that

$$0 < \rho_0 \leq \rho(x), \ \ 0 < EI_0 \leq EI(x), \quad \forall x \in [0,\ell].$$

A.III: The angular velocity $\widehat{\omega}$ is bounded in the following sense:

$$|\widehat{\omega}| < \frac{2}{\ell^2}\sqrt{\frac{3EI_0}{\|\rho\|_\infty}}.$$

A.IV: $\alpha(x)$ is a positive and non-increasing function of $W^{1,\infty}(0,\ell)$.

The principal contribution of the paper is to prove, theoretically and numerically, that the system (1.3) is exponentially stable. This shows, contrary to [24], that there is no need to invoke a compensating term different from the memory term. As a matter of fact, the authors in [24] had to consider this difference in order to get a dissipative system.

Subsequently, let

$$H_\star^k = \Big\{ f \in H^k(0,\ell); f(0) = f_x(0) = 0 \Big\}, \quad \text{for } k = 2, 3, \cdots.$$

Thanks to the hypothesis **A.I**, we have

$$\xi(\infty) := \lim_{s\to\infty} \xi(s) = 0. \tag{1.4}$$

In order to study the above system, let us consider the minimal state variable defined by (see [1, 29, 27])

$$\eta(t,s) = \int_0^\infty \xi^{'}(s+\tau)(y_x(\ell,t) - y_x(\ell,t-\tau))d\tau, \tag{1.5}$$

which satisfies the following equation

$$\eta(.,\infty) = \lim_{s\to\infty} \eta(.,s) = 0, \quad \eta_t(t,s) = \eta_s(t,s) - \xi(s) y_{tx}(\ell,t). \tag{1.6}$$

Next, let

$$\mathcal{K} = \left\{ \eta(s);\, \beta \int_0^\infty \frac{-\eta_s^2(s)}{\xi'(s)}\,ds < \infty \right\},$$

endowed with the following inner product:

$$\langle \eta, \tilde{\eta} \rangle_{\mathcal{K}} = \int_0^\infty \frac{-\beta}{\xi'(s)} \eta_s(s) \tilde{\eta}_s(s)\,ds.$$

Then, consider the state space $\mathcal{X} = H^2_\star \times L^2(0,1) \times \mathcal{K}$ equipped with the following real inner product:

$$\begin{aligned}\langle (y,z,\eta),(\tilde{y},\tilde{z},\tilde{\eta})\rangle_{\mathcal{X}} &= \int_0^\ell \left(EI(x)y_{xx}\tilde{y}_{xx} - \widehat{\omega}^2\rho(x)y\tilde{y} + \rho(x)z\tilde{z}\right)dx \\ &\quad - \int_0^\infty \frac{\beta}{\xi'(s)}\eta_s(s)\tilde{\eta}_s(s)\,ds. \end{aligned} \tag{1.7}$$

In light of the Assumptions **A.I-A.III**, one can readily verify that the space $\mathcal{X}$ is a Hilbert space. This implies that the space

$$\mathcal{Y} = \mathcal{X} \times \mathbb{R},$$

equipped with inner product

$$\langle (y,z,\eta,\omega),(\tilde{y},\tilde{z},\tilde{\eta},\tilde{\omega})\rangle_{\mathcal{Y}} = \langle (y,z,\eta),(\tilde{y},\tilde{z},\tilde{\eta})\rangle_{\mathcal{X}} + \omega\tilde{\omega}, \tag{1.8}$$

is also a Hilbert space. Whereupon, the system (1.3) has the following form:

$$\frac{d\psi}{dt} = \left[\begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{B}\right]\psi, \tag{1.9}$$

where $\psi = (y,z,\eta,\omega)$, $\psi(0) = \psi_0 = (y_0, y_1, \eta_0, \omega_0) \in \mathcal{Y}$. Moreover, the operator $\mathcal{A}$ is defined by:

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \Big\{ &(y,z,\nu) \in H^4_\star \times H^2_\star \times \mathcal{K};\ \eta_s - \xi(s)z_x(\ell) \in \mathcal{K}, \\ &\qquad (EI(x)y_{xx})_x(\ell) = 0,\ \ (EI(x)y_{xx})(\ell) = \beta\eta(0)\Big\} \\ \mathcal{A}(y,z,\eta) = \Big(&z, -\tfrac{1}{\rho(x)}\left(EI(x)y_{xx}\right)_{xx} + \widehat{\omega}^2 y - \frac{\alpha(x)}{\rho(x)}z_x, \eta_s - \xi(s)z_x(\ell)\Big),\ \forall\,\phi \in \mathcal{D}(\mathcal{A}), \end{aligned} \tag{1.10}$$

and $\mathcal{B}$ is the nonlinear operator defined as

$$\mathcal{B}\psi = \left(0, (\omega^2 - \widehat{\omega}^2)y, 0, \frac{-\gamma\,(\omega - \widehat{\omega}) - 2\,\omega < \rho(x)y, z >_{L^2(0,\ell)}}{I_d + \|\sqrt{\rho(x)}\,y\|^2_{L^2(0,\ell)}}\right),\ \ \forall\,\psi \in \mathcal{Y}. \tag{1.11}$$

It is noteworthy that the initial condition η_0 depends on y_0 and φ_0 given in (1.3). Indeed, in light of (1.5) and (1.6) along with (1.3), we have

$$\begin{aligned}\eta_0(s) &= \int_0^\infty \xi^{'}(s+\tau)(y_x(\ell,0) - y_x(\ell,-\tau))d\tau \\ &= -y_{0x}(\ell)\xi(s) - \int_0^\infty \xi^{'}(s+\tau)\varphi_0(\tau)\,d\tau.\end{aligned}$$

We conclude this section by discussing the advantage of using the minimal state approach. As the reader can notice, the nonlocal character of (1.1)-(1.2) does not appear in (1.9) as it is hidden in the minimal state variable η. This represents a major advantage of this approach. Furthermore, compared to the classical past history methodology [26], the past history variable defined by $v(t,s) = y(t) - y(t-s)$, where t, $s \geq 0$ does not discern between solutions stemmed from two different initial past histories [40, 29].

Now, let us provide an overview of the article. In section 2, the linear operator $\mathcal{A}$ is shown to generate a C_0 semigroup of contractions $e^{t\mathcal{A}}$ on $\mathcal{X}$ and then conclude that the system (1.9) is well-posed in $\mathcal{Y}$. Section 3 is devoted to establish the uniform stability of the semigroup $e^{t\mathcal{A}}$ by means of the resolvent method.

Thereafter, the exponential stability of the global system (1.9) is deduced. Furthermore, a short discussion about the constant case of α is provided. In Section 4, a Finite volumes method is used in order to conduct a complete numerical study of the problem and provide numerous examples that validate the theoretical outcomes. Finally, we end the paper with concluding remarks.

2. WELL-POSEDNESS OF THE PROBLEM

In this section, we will prove the global existence and the uniqueness of the solution to the problem (1.9) (see also (1.3)). We will first prove the well-posedness of its main linear part which is identified by the linear operator $\mathcal{A}$. This part of the proof is based on the semigroup approach and specifically Lumer-Phillips' Theorem. Then, we have to consider the following linear subsystem

$$\begin{cases} \dot{\phi}(t) = \mathcal{A}\phi, \\ \phi(0) = (y_0, y_1, \eta_0). \end{cases} \tag{2.1}$$

Theorem 1. *Assume that the Assumptions **A.I-A.IV** are satisfied. Then, the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ on $\mathcal{X}$. Moreover, if $\phi(0) \in \mathcal{D}(\mathcal{A})$ the problem (2.1) admits a unique strong solution $\phi \in C((0,\infty);\mathcal{D}(\mathcal{A})) \cap C^1((0,\infty);\mathcal{X})$. However, if $\phi(0) \in \mathcal{X}$, then the solution ϕ is called mild solution and belongs to $C((0,\infty);\mathcal{X})$.*

In order to show the above theorem, we shall first establish the following result:

Lemma 1. *Under the Assumptions **A.I-A.IV**, we have:*

(1) *The operators $\mathcal{A}$ and its adjoint $\mathcal{A}^*$ are both dissipative in $\mathcal{X}$.*
(2) *For all positive real number γ, the operator $\gamma I - \mathcal{A}$ is surjective in $\mathcal{X}$.*

Proof. Let $\phi = (y, z, \eta) \in \mathcal{D}(\mathcal{A})$. Then, in light of (1.7) and (1.10), a simple integration by parts gives

$$\begin{aligned} \langle \mathcal{A}\phi, \phi\rangle_{\mathcal{X}} &= \int_0^\ell \{EI(x)y_{xx}z_{xx} - (EI(x)y_{xx})_{xx}z - \alpha(x)z_xz\}\, dx - \beta\int_0^\infty \frac{1}{\xi'(s)}(\eta_s - \xi(s)z_x(\ell))_s\eta_s\, dx \\ &= \beta\eta(0)z_x(\ell) - \int_0^\ell \alpha(x)z_xz\, dx - \beta\int_0^\infty \frac{1}{\xi'(s)}(\eta_s - \xi(s)z_x(\ell))_s\eta_s\, dx \\ &= -\frac{1}{2}\alpha(\ell)z^2(\ell) + \frac{1}{2}\int_0^\ell \alpha'(x)z^2\, dx + \beta\frac{\eta_s^2(0)}{2\xi'(0)} - \beta\int_0^\infty \frac{\xi''(s)}{2(\xi'(s))^2}\eta_s^2\, ds. \end{aligned} \tag{2.2}$$

Amalgamating (2.2) with the hypotheses **A.I-A.IV** on the functions ξ and α, it follows that $\mathcal{A}$ is dissipative in $\mathcal{X}$.

Now, we turn to the adjoint of the operator $\mathcal{A}$. First, we need to identify the operator $\mathcal{A}^*$. To do so, consider $\phi = (y, z, \eta) \in \mathcal{D}(\mathcal{A})$ and $\hat{\phi} = (\hat{y}, \hat{z}, \hat{\eta}) \in \mathcal{X}$. Using (1.7) and (1.10) once more, after straightforward computation based on some integrations by parts, we obtain

$$
\begin{aligned}
\langle \mathcal{A}\phi, \hat{\phi}\rangle_{\mathcal{X}} &= \int_0^\ell \left\{ EI(x) y_{xx}\hat{z}_{xx} - \hat{\omega}^2\rho(x) z\hat{y} - (EI(x)\hat{y}_{xx})_{xx} z + \hat{\omega}\rho(x) y\hat{z} - \alpha(x) z_x \hat{z}\right\}\, dx \\
&\quad -\beta \int_0^\infty \frac{1}{\xi'(s)} (\eta_s - \xi(s) z_x(\ell))_s \hat{\eta}_s \, dx \\
&= -z(\ell)\left[(EI(x)\hat{y}_{xx})_x(\ell) + \alpha(\ell)\hat{z}(\ell)\right] + z_x(\ell)[(EI(x)\hat{y}_{xx})(\ell) - \beta\hat{\eta}(0)] \\
&\quad + \int_0^\ell \rho(x) z \left\{ \frac{1}{\rho(x)} (EI(x)\hat{y}_{xx})_{xx} - \hat{\omega}^2 \hat{y} + \frac{\alpha(x)}{\rho(x)}\hat{z}_x + \frac{\alpha'(x)}{\rho(x)}\hat{z}\right\} dx \\
&\quad + \int_0^\ell \left[EI(x) y_{xx}(-\hat{z}_{xx}) - \hat{\omega}^2 \rho(x) y(-\hat{z})\right] dx + \beta \frac{\eta_s(0)\hat{\eta}(0)}{\xi'(0)} \\
&\quad -\beta \int_0^\infty \frac{\xi''(s)}{(\xi'(s))^2} \eta_s \hat{\eta}_s \, ds - \beta \int_0^\infty \frac{1}{\xi'(s)} \eta_s \left(-\hat{\eta}_s + \xi(s)\hat{z}_x(\ell)\right)_s \, ds.
\end{aligned} \tag{2.3}
$$

Taking the Heaviside function H, let us define a functional $\mathcal{Z}$ by the following identity

$$
\left(\mathcal{Z}(\eta(s))\right)_s = -\xi'(s)\left(\frac{H(s)}{\xi'(s)}\right)' \eta_s(s), \quad \text{for all } \eta \in \mathcal{K}, \tag{2.4}
$$

which, together with (2.3), implies that

$$
\begin{aligned}
\langle \mathcal{A}\phi, \hat{\phi}\rangle_{\mathcal{X}} &= -z(\ell)\left[(EI(x)\hat{y}_{xx})_x(\ell) + \alpha(\ell)\hat{z}(\ell)\right] + z_x(\ell)[(EI(x)\hat{y}_{xx})(\ell) - \beta\hat{\eta}(0)] \\
&\quad + \int_0^\ell \rho(x) z \left\{ \frac{1}{\rho(x)} (EI(x)\hat{y}_{xx})_{xx} - \hat{\omega}^2 \hat{y} + \frac{\alpha(x)}{\rho(x)}\hat{z}_x + \frac{\alpha'(x)}{\rho(x)}\hat{z}\right\} dx \\
&\quad + \int_0^\ell \left[EI(x) y_{xx}(-\hat{z}_{xx}) - \hat{\omega}^2 \rho(x) y(-\hat{z})\right] dx \\
&\quad -\beta \int_0^\infty \frac{1}{\xi'(s)} \left(\mathcal{Z}(\hat{\eta}) - \hat{\eta}_s + \xi(s)\hat{z}_x(\ell)\right)_s \hat{\eta}_s \, ds.
\end{aligned} \tag{2.5}
$$

Finally, letting $(EI(x)\hat{y}_{xx})(\ell) = \beta\hat{\eta}(0)$ and $(EI(x)\hat{y}_{xx})_x(\ell) = -\alpha(\ell)\hat{z}(\ell)$ and keeping in mind the identity $\langle \mathcal{A}\phi, \hat{\phi}\rangle_{\mathcal{X}} = \langle \phi, \mathcal{A}^*\hat{\phi}\rangle_{\mathcal{X}}$, we it follows from (2.5) that the desired adjoint operator is

$$
D(\mathcal{A}^*) = \left\{ \begin{array}{l} \phi = (y, z, \eta) \in H^4_\star \times H^2_\star \times \mathcal{K};\ \ \mathcal{Z}(\eta) - \eta_s + \xi(s) z_x(\ell) \in \mathcal{K}, \\ (EI(x) y_{xx})(\ell) = \beta\eta(0) \text{ and } (EI(x) y_{xx})_x(\ell) + \alpha(\ell) z(\ell) = 0 \end{array} \right\} \tag{2.6}
$$

and

$$
\mathcal{A}^*(\phi) = \left(-z, \frac{1}{\rho(x)}\left[(EI(x) y_{xx})_{xx} + \alpha(x) z_x + \alpha'(x) z\right] - \hat{\omega}^2 y, \mathcal{Z}(\eta) - \eta_s + \xi(s) z_x(\ell)\right), \quad \forall \phi \in D(\mathcal{A}^*). \tag{2.7}
$$

Next, performing a number of integrations by parts on $(0, \infty)$, we find

$$
-\beta \int_0^\infty \frac{1}{\xi'(s)} \left(\mathcal{Z}(\eta) - \eta_s + \xi(s) z_x(\ell)\right)_s \eta_s \, ds = \beta z_x(\ell)\eta(0) + \beta \frac{\eta_s(0)^2}{2\xi'(0)} - \beta \int_0^\infty \frac{\xi''(s)}{2(\xi'(s))^2} \eta_s^2 \, ds, \tag{2.8}
$$

and hence (2.6)-(2.8) yield

$$
\langle \mathcal{A}^*\phi, \phi\rangle_{\mathcal{X}} = -\frac{1}{2}\alpha(\ell) z^2(\ell) + \frac{1}{2}\int_0^\ell \alpha'(x) z^2 \, dx + \beta \frac{\eta_s(0)^2}{2\xi'(0)} - \beta \int_0^\infty \frac{\xi''(s)}{2(\xi'(s))^2} \eta_s^2 \, ds.
$$

Thus, thanks to **A.I-A.IV**, the adjoint $\mathcal{A}^*$ of $\mathcal{A}$ is also dissipative in $\mathcal{X}$.

To show the second point of Lemma 1, let us consider $F = (f, g, h)$ as an element of $\mathcal{X}$ and we look for an element $\phi = (y, z, \eta) \in \mathcal{D}(\mathcal{A})$ such that

$$
(\gamma I - \mathcal{A})\phi = F, \quad \text{for } \gamma > 0.
$$

This can be rewritten as the problem

$$\begin{cases} \gamma y - z = f, \\ \gamma z + \frac{1}{\rho(x)}[(EI(x)y_{xx} + \alpha(x)z_x] - \hat{\omega}^2 y = g, \\ \gamma\eta - \eta_s + \xi(s)z_x(\ell) = h. \end{cases} \tag{2.9}$$

Equivalently, it amounts to solving the following system

$$\begin{cases} z = \gamma y - f, \\ (EI(x)y_{xx} + \rho(x)(\gamma^2 - \hat{\omega}^2)y + \gamma\alpha(x)y_x = \rho(x)(g + \gamma f) + \alpha(x)f_x, \\ \eta_x - \gamma\eta = \xi(s)[\gamma y_x(\ell) - f(\ell)] - h, \end{cases} \tag{2.10}$$

with

$$\begin{cases} y(0) = y_x(0) = (EI(x)y_{xx})_x(\ell) = 0, \\ (EI(x)y_{xx}(\ell) = \beta\eta(0). \end{cases} \tag{2.11}$$

Clearly, the third equation in (2.10) leads to the following identity

$$\eta(s) = \int_s^\infty e^{\gamma(s-\tau)}(h(\tau) + \xi(\tau)[f(\ell) - \gamma y_x(\ell)])\, d\tau. \tag{2.12}$$

Subsequently, substituting (2.12) in the second equation in (2.10), multiplying the result equation by $\psi \in H^2_\star$ and integrating it over $(0, \ell)$, we immediately obtain the following variational problem

$$\mathcal{G}(y, \psi) = \mathcal{L}(\psi),$$

where $\mathcal{G}$ is the real bilinear form in $H^2_\star \times H^2_\star$ given by

$$\mathcal{G}(y, \psi) = \int_0^\ell \left\{EI(x)y_{xx}\psi_{xx} + \rho(x)(\gamma^2 - \hat{\omega}^2)y\psi + \gamma\alpha(x)y_x\psi\right\}\, dx + \gamma\left(\int_0^\infty e^{-\gamma\tau}\xi(\tau)d\tau\right) y_x(l)\psi_x(l),$$

and $\mathcal{L}$ is a real linear form defined in $H^2_\star$ by

$$\mathcal{L}(\psi) = \beta\psi_x(\ell)\int_0^\infty e^{-\gamma\tau}[\xi(\tau)f(\ell) + h]\, d\tau + \int_0^\ell [\rho(x)(g + \gamma f) + \alpha(x)f_x]\, dx.$$

Now, it is an immediate task to check that $\mathcal{G}$ is a continuous coercive bilinear form, whereas $\mathcal{L}$ is a continuous linear form as long as the assumptions of the lemma. Applying Lax-Milgram Theorem (see for instance [7]), one can deduce the existence of a unique solution $y \in H^4_\star$ of (2.10) as long as $\gamma > 0$. Thus, the operator $\gamma I - \mathcal{A}$ is surjective in $\mathcal{X}$. □

Proof. of Theorem 1: Using Lemma 1 and thanks to Lumer-Phillips Theorem (see Theorem 4.3 or Corollary 4.4 in [41]), we can deduce the first result of Theorem 1. The second one is an immediate consequence of semigroups theory [7, 41]. □

The task ahead is to study the existence and uniqueness of solutions of the nonlinear problem (1.3) which is equivalent to the system (1.9) governed by the operators $\mathcal{A}$ and $\mathcal{B}$ defined respectively by (1.10) and (1.11). We have the following well-posedness result:

Proposition 1. *Suppose that the Assumptions **A.I-A.IV** are fulfilled. If $\psi(0) \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}$, the problem (1.9) posses a unique classical global bounded solution $\psi(t) \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}$. Furthermore, if $\psi(0) \in \mathcal{Y}$, the problem (1.9) posses a unique classical global bounded solution $\psi(t) \in \mathcal{Y}$.*

Proof. Taking in mind that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions and the operator $\mathcal{B}$ is differentiable, Theorem 1.5 of chapter 6 in [41] guarantees that if $\psi(0) \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}$, the nonlinear problem (1.9) admits a unique classical local (continuously differentiable) solution $\psi(t) \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}$ on some interval $[0, T]$, for some positive real T. Moreover, If $\psi(0) \in \mathcal{Y}$ there is a unique local mild solution $\psi(t) \in C([0, T]; \mathcal{Y})$ given by the variation of parameters formula [41]. Thereby, we are going to use a Lyapunov function to show that, this solution can be extended to the interval $(0, \infty)$. Thenceforth, let us consider:

(2.13)
$$\begin{aligned}\mathcal{P}(t) \;=\;& \frac{1}{2}\int_0^\ell \left\{EI(x)y_{xx}^2 + \rho(x)y_t^2 - \hat{\omega}^2\rho(x)y^2\right\}\,dx + \frac{I_d}{2}(\omega(t)-\hat{\omega})^2 + \frac{1}{2}(\omega(t)-\hat{\omega})^2\int_0^\ell \rho(x)y^2\,dx\\ &-\frac{1}{2}\int_0^\infty \frac{\beta}{\xi'(s)}\eta_s^2\,ds.\end{aligned}$$

Owing to the hypotheses **A.I-A.II**, this functional is positive for all $t \geq 0$ and, furthermore, differentiating it along regular solutions in $\mathcal{D}(\mathcal{A}) \times \mathbb{R}$, we find

$$\mathcal{P}'(t) = -\gamma(\omega(t)-\hat{\omega})^2 + \frac{1}{2}\int_0^\ell \alpha'(x)y_t^2\,dx + \beta\frac{\eta_s(0)^2}{2\xi'(0)} - \beta\int_0^\infty \frac{\xi''(s)}{2(\xi'(s))^2}\eta_s^2\,ds$$

which implies that for all $t \geq 0$ that $\mathcal{P}'(t) \leq 0$ by virtue of **A.I-A.IV**. Therefore, $\mathcal{P}$ is a Lyapunov function, and, hence the solution $\psi(t)$ previously obtained is global. This achieves the proof of the Proposition 1. □

Remark 1. *It is worth mentioning that all the previous results remain valid without using the condition that appears in **A.I**: $\xi''(s) \geq -C\xi'(s)$, for any $s \in \mathbb{R}^+$, and for some positive constant C. The latter will be invoked for the stability result.*

3. EXPONENTIAL STABILITY

In this section, we are going to state the second result of the paper, that is, the exponential stability of the system (1.9). For that, we suppose that α, in addition of **A.IV**, is decreasing and so

$$\alpha'(x) < 0, \quad \forall x \in (0, \ell). \tag{3.1}$$

The second main result is given by the following theorem:

Theorem 2. *Suppose that the assumption **A.I-A.IV** hold. Then, for any initial data $(y_0, y_1, \eta_0, \omega_0)$ in $\mathcal{D}(\mathcal{A}) \times \mathbb{R}$, the solution (y, y_t, η, ω) of the system (1.9), or equivalently of (1.3), tends exponentially in $\mathcal{Y}$ to $(0, 0, 0, \widehat{\omega})$.*

The proof of Theorem 2 is based on the study of the linear part of the system (1.9) governed by the operator $\mathcal{A}$ by showing that the semigroup $e^{t\mathcal{A}}$ is exponentially stable on $\mathcal{X}$. To prove such result, we apply the following frequency domain theorem (see [32] for more details):

Theorem 3. *There exist two positive constant A and σ such that a C_0-semigroup of contractions $S(t)$ on a Hilbert space $\mathcal{H}$ satisfies the estimate*

$$||S(t)||_{\mathcal{L}(\mathcal{H})} \leq Ae^{-\sigma t}, \;\; \textit{for all}\, t > 0,$$

if and only if

$$\rho(\mathcal{A}) \supset \{i\gamma;\, \gamma \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{3.2}$$

and

$$\limsup_{|\gamma|\to\infty} \|(i\gamma I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{3.3}$$

where $\mathcal{A}$ is the generator of $S(t)$ and $\rho(\mathcal{A})$ is the resolvent set of the operator $\mathcal{A}$.

Now, we are ready to state our result which gives the fact that the semigroup operator $e^{t\mathcal{A}}$ is exponentially stable in $\mathcal{X}$.

Theorem 4. *Assume that **A.I-A.IV** are satisfied. Then, there exists $C > 0$ and $\delta > 0$ such that for all $t > 0$, we have*

$$\left\|e^{t\mathcal{A}}\right\|_{\mathcal{L}(\mathcal{X})} \leq Ce^{-\delta t}.$$

Proof. The proof is divided into three steps:

First step. First of all, we have to prove that the resolvent set $\rho(\mathcal{A})$ contains the zero complex number. To do so, let us consider $F = (f_1, f_2, f_3)$ an element of $\mathcal{X}$ and solve, for $\phi = (y, z, \eta) \in \mathcal{D}(\mathcal{A})$, the equation

$$\mathcal{A}_2\phi = F. \tag{3.4}$$

Equivalently, we have to consider

$$\begin{gathered} z = f_1, \\ -\frac{1}{\rho(x)}\big(EI(x)y_{xx}\big)_{xx} + \widehat{\omega}^2 y - \frac{\alpha(x)}{\rho(x)} z_x = f_2, \\ \eta_s - \xi(s) z_x(\ell) = f_3. \end{gathered}$$

Solving the last equation and using the first one, we get

$$\eta(s) = -\int_s^\infty [\xi(s) f_{1,x}(\ell) + f_3(s)]\, ds,$$

which gives in particular

$$\eta(0) = -\int_0^\infty [\xi(s) f_{1,x}(\ell) + h(s)]\, ds. \tag{3.5}$$

Taking in mind this expression, the equations above and the definition of $\mathcal{D}(\mathcal{A})$, it suffices to study the problem

$$\begin{cases} \big(EI(x)y_{xx}\big)_{xx} - \rho(x)\widehat{\omega}^2 y = -\alpha(x) f_{1,x} + f_2, \\ y(0) = y_x(0) = \big(EI(x)y_{xx}\big)_x(\ell) = 0, \\ \big(EI(x)y_{xx}\big)(\ell) = -\beta \displaystyle\int_0^\infty \left(\xi(s) f_{1,x}(\ell) + f_3(s)\right) ds. \end{cases} \tag{3.6}$$

Thereafter, the classic elliptic theory implies that the problem (3.6) and hence (3.4) admits a unique solution. Finally, we can deduce that $0 \in \rho(\mathcal{A})$.

Second step. The objective of this step is to prove the condition (3.2) for the operator $\mathcal{A}$. We shall use a contradiction argument together with a result from [36]. To begin, taking the result of the first step in mind and suppose that the condition (3.2) is not satisfied which translates that there, at least, exists $\varpi \in \mathbb{R}$ with $\left\|\mathcal{A}^{-1}\right\| \leq |\varpi|^{-1} < \infty$ such that

$$\{i\gamma, \;\; |\gamma| < |\varpi|\} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup\left\{(i\gamma - \mathcal{A})^{-1}; \;\; |\gamma| < |\varpi|\right\} = \infty.$$

Thus, thanks to Banach-Steinhaus Theorem, we can deduce that there exist a sequence of vectors $\phi_n = (y_n, z_n, \eta_n) \in \mathcal{D}(\mathcal{A})$ with $\|\phi_n\|_{\mathcal{H}} = 1$ and a sequence of real numbers $\gamma_n \to \varpi$ such that

$$(i\gamma_n - \mathcal{A})\phi_n = (f_n, g_n, h_n) \to 0 \text{ in } \mathcal{X}. \tag{3.7}$$

This later equation is equivalent to

$$i\gamma_n y_n - z_n = f_n \to 0, \text{ in } H^2_\star, \tag{3.8}$$

$$i\gamma_n z_n + \frac{1}{\rho(x)}\big(EI(x)y_{n,xx}\big)_{xx} - \widehat{\omega}^2 y_n + \frac{\alpha(x)}{\rho(x)} z_{n,x} = g_n \to 0, \text{ in } L^2(0,\ell), \tag{3.9}$$

$$i\gamma_n \eta_n - \eta_{n,s} + \xi(s) z_{n,x}(\ell) = h_n \to 0, \text{ in } \mathcal{K}. \tag{3.10}$$

These equations are coupled with the following boundary conditions

$$y_n(0) = y_{n,x}(0) = \big(EI(x)y_{n,xx}\big)_x(\ell) = 0, \tag{3.11}$$

$$\big(EI(x)y_{n,xx}\big)(\ell) = \beta\eta_n(0). \tag{3.12}$$

Taking the inner product of (3.7) with ϕ_n and using the fact that $\mathcal{A}$ is dissipative, we find

$$z_n(\ell) \to 0 \text{ in } \mathbb{C}, \tag{3.13}$$

$$\frac{\eta^2_{n,s}(0)}{\xi'(0)} \to 0 \text{ in } \mathbb{C}, \tag{3.14}$$

$$\int_0^\ell \alpha'(x)|z_n|^2 \, dx \to 0 \text{ in } \mathbb{R}, \tag{3.15}$$

and

$$\int_0^\ell \frac{\xi''(s)}{(\xi'(s))^2} |\eta_{n,s}|^2 \, ds \to 0 \text{ in } \mathbb{R}. \tag{3.16}$$

According to the assumption $\xi''(s) \geq -C\xi'(s)$, $\forall s \in \mathbb{R}$, in **A.I**, we get

$$0 \leq -C \int_0^\infty \frac{1}{\xi'(s)} |\eta_{n,s}|^2(s) \, ds \leq \int_0^\infty \frac{\xi''(s)}{(\xi'(s))^2} |\eta_{n,s}|^2(s) \, ds \to 0 \text{ in } \mathbb{C}, \tag{3.17}$$

which implies that

$$\eta_n \to 0 \text{ in } \mathcal{K}. \tag{3.18}$$

It follows from Cauchy-Schwartz's inequality that

$$|\eta_n|^2(0) \leq \xi(0) \, \|\eta_n\|^2_{\mathcal{S}} \, ,$$

and, hence,

$$\eta_n(0) \to 0 \text{ in } \mathbb{C}. \tag{3.19}$$

Then, (3.19) together with (3.12), yields that

$$\big(EI(x) y_{n,xx}\big)(\ell) \to 0 \text{ in } \mathbb{C}. \tag{3.20}$$

On the other hand, we express the component z_n in terms of y_n by the equation (3.8) and substitute it into (3.9), to find

$$\big(EI(x) y_{n,xx}\big)_{xx} - \rho(x)\big(\widehat{\omega}^2 + \gamma_n^2\big) y_n + i\gamma_n \alpha(x) y_{n,x} = \rho(x) g_n + i\gamma_n \rho(x) f_n + \alpha(x) f_{n,x}, \tag{3.21}$$

which clearly gives

$$\int_0^\ell \left\{ \big(EI(x) y_{n,xx}\big)_{xx} - \rho(x)\big(\widehat{\omega}^2 + \gamma_n^2\big) y_n + i\gamma_n \alpha(x) y_{n,x} \right\} \overline{y_n} \, dx = \int_0^\ell \left(\rho(x) g_n + i\gamma_n \rho(x) f_n + \alpha(x) f_{n,x}\right) \overline{y_n} \, dx. \tag{3.22}$$

It is clear that the right-hand side of (3.22) converges to 0 thanks to the boundedness of the component y_n and the fact that f_n and g_n converge to 0 in $H^2_\star$ and $L^2(0, \ell)$ respectively.

As for the last integral in left-hand side, since (3.8), (3.13) and (3.15), a straightforward calculation yields

$$\begin{aligned} \gamma_n \int_0^\ell \alpha(x) y_{n,x} \overline{y_n} \, dx &= -\frac{\alpha(\ell)}{2\gamma_n} |z_n(\ell) + f_n(\ell)|^2 - \frac{1}{2\gamma_n} \int_0^\ell \alpha'(x) |z_n + f_n|^2 \, dx \\ &= o(1). \end{aligned} \tag{3.23}$$

For the first one, we have

$$\int_0^\ell \big(EI(x) y_{n,xx}\big)_{xx} \overline{y_n} \, dx = -\big(EI(x) y_{n,xx}\big)(\ell) \overline{y_{n,x}}(\ell) + \int_0^\ell EI(x) |y_{n,xx}|^2 \, dx. \tag{3.24}$$

In addition, as $y_{n,x}(\ell) = \int_0^\ell y_{n,xx} \, dx$, one can use Cauchy-Schwartz inequality and the boundedness of y_n in $H^2_\star$ to deduce that

$$y_{n,x}(\ell) \text{ is bounded.} \tag{3.25}$$

Thus, combining (3.20), (3.24) and (3.25), we obtain

$$\int_0^\ell \big(EI(x)y_{n,xx}\big)_{xx}\overline{y_n}\,dx - \int_0^\ell EI(x)|y_{n,xx}|^2\,dx = o(1). \tag{3.26}$$

Therefore, we conclude from (3.22) along with (3.23),(3.26) that

$$\|z_n\|^2_{L^2(0,\ell)} - \|y_n\|^2_{H^2_\star} = o(1). \tag{3.27}$$

Finally, using the assumption **A.IV** and (3.1) in (3.15), it follows that $\|z_n\|_{L^2(0,\ell)} = o(1)$, and thus (3.27) yields

$$\|y_n\|_{H^2_\star} = o(1).$$

This contradicts the fact that $\|\phi_n\|_{\mathcal{H}} = o(1)$ and the desired condition (3.2) is shown.

Third step. This last step is devoted to the proof of the second requirement (3.3). We will argue by contradiction. Suppose that it is not true, that is, there exists a sequences $\phi_n = (y_n, z_n, \eta_n) \in \mathcal{D}(\mathcal{A})$ and $\gamma_n \in \mathbb{R}$ with $\|\phi_n\|_{\mathcal{H}} = 1$ and $\gamma_n \to \infty$ such that

$$\lim_{n\to+\infty} \|(i\gamma_n - \mathcal{A})\phi_n\| = 0.$$

This implies that even when $\gamma_n \to \infty$, (3.8)-(3.12) hold, as well as (3.13)-(3.19).

Due to trace theorem applied on f_n and using (3.8) and (3.13), we find

$$\gamma_n y_n(\ell) \to 0 \text{ in } \mathbb{C}, \tag{3.28}$$

and substituting (3.19) in (3.12), we obtain

$$(EI(x)y_{n,xx})(\ell) \to 0 \text{ in } \mathbb{C}. \tag{3.29}$$

To do so, we express z_n as a function from (3.8) and substitute it into (3.9), to get

$$\big(EI(x)y_{n,xx}\big)_{xx} - \rho(x)\big(\widehat{\omega}^2 + \gamma_n^2\big)y_n + i\gamma_n\alpha(x)y_{n,x} = \rho(x)g_n + i\gamma_n\rho(x)f_n + \alpha(x)f_{n,x}. \tag{3.30}$$

Similarly to [36] (see also [14, 24]), consider the smooth function

$$q(x) = e^{\mu x} - 1,$$

with $\mu = \max\left\{\dfrac{\|\rho'\|_\infty}{\rho_0}, \dfrac{\|EI'\|_\infty}{EI_0}\right\}$. Thereafter, taking the inner product of (3.30) with $q(x)y_{n,x}$, we obtain

$$EI(\ell)q''(\ell)|y_{n,x}(\ell)|^2 + \int_0^\ell \left\{3EI(x)q'(x) - EI'(x)q(x)\right\}|y_{n,xx}|^2\,dx + \int_0^\ell (\rho(x)q(x))'\,(\widehat{\omega}^2 + \gamma_n^2)|y_n|^2\,dx = o(1). \tag{3.31}$$

A careful look at the function $q(x)$ defined above leads to claim that $3EI(x)q'(x) - EI'(x)q(x)$ and $(\rho(x)q(x))'$ are strictly positive and hence (3.31) yields

$$y_{n,x}(\ell) \to 0,$$

and

$$\|\gamma_n y_n\| \to 0, \quad \|y_n\|_{H_c^2} \to 0.$$

This, together with (3.8) and (3.18), gives the desired contradiction $\|\phi_n\| = o(1)$ which completes the proof of the Theorem 4. $\square$

Proof of Theorem 2. First, given a regular initial data $(\phi_0, \omega_0) \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}$, we decompose the corresponding strong solution $\Phi = (y, z, \eta, \omega) \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}$ into two components as follows:

$$\Phi(t) = (\phi(t), \omega(t)),$$

where $\phi(t) = (y, z, \eta)$ is the unique solution of the linear subsystem

$$\phi_t(t) = [\mathcal{A} + (\omega^2(t) - \hat{\omega}^2)\mathcal{R}]\phi(t), \tag{3.32}$$

with $\mathcal{R}$ is the compact operator defined by $\mathcal{R}(u, v, \chi) = (0, u, 0)$, for all $(u, v, \chi) \in \mathcal{X}$. Furthermore, $\omega(t)$ is the solution to the ordinary differential equation

$$\frac{d\omega(t)}{dt} = \frac{-\gamma(\omega(t) - \hat{\omega}) - 2\omega(t) \displaystyle\int_0^1 y(x)z(x)\,dx}{I_d + \displaystyle\int_0^1 y^2(x)\,dx}. \tag{3.33}$$

Recall that the solution of (3.32) is given by

$$\phi(t) = e^{(t-T)\mathcal{A}_2}\phi(T) + \int_T^t e^{(t-s)\mathcal{A}_2}(\omega^2(t) - \hat{\omega}^2)\mathcal{R}\phi(s)\,ds, \ \forall t \geq T, \tag{3.34}$$

for any $T > 0$. Then, it suffices to argue as in [35] (see also [9] and [43]). In fact, thanks to Theorem 1 and (2.13), one can use Barbalat's Lemma [33] to obtain

$$\lim_{t \to \infty} (\omega(t) - \hat{\omega}) = 0.$$

In turn, we know, by virtue of Theorem 4, that $\left\|e^{t\mathcal{A}}\right\| \leq Ce^{-\delta t}$, for all $t \geq 0$. This, together with Gronwall's inequality, leads to the exponential stability in $\mathcal{X}$ of the component $\phi(t)$. Finally, as in [43] (see also [9]), the second component $\omega(t)$ is exponentially converges to the desired angular velocity $\hat{\omega}$ of the disk.

Remark 2. *It is a simple task to verify that the well-posedness result shown in Section 2 is true even if $\alpha(x)$ is a positive constant function (denoted by α) and hence **A.IV** is fulfilled. Nevertheless, in this case, the stability result of Section 3 requires a more careful investigation as the condition* (3.1) *is not satisfied. This will be treated in the next paragraph when the physical parameters EI and ρ are constant and taken to be 1. This means that the **A.II** is obviously satisfied, while **A.III** takes the form $|\widehat{\omega}| < \dfrac{2\sqrt{3}}{\ell^2}$.*

Theorem 5. *Suppose that the assumption **A.I** is fulfilled. Moreover, we assume that $\alpha(x) = \alpha$ a positive constant and the desired angular velocity is small such as $|\widehat{\omega}| < \dfrac{2\sqrt{3}}{\ell^2}$. Then, for any initial data*

$(y_0, y_1, \eta_0, \omega_0)$ *in* $\mathcal{D}(\mathcal{A}) \times \mathbb{R}$, *the solution* (y, y_t, η, ω) *of the closed-loop system (1.9) exponentially goes to the equilibrium state* $(0, 0, 0, \widehat{\omega})$ *in* $\mathcal{Y}$.

Proof. Running on exactly the same lines as in the proof of Theorem 2, it suffices to show the exponential stability of the linear subsystem governed by the operator $\mathcal{A}$. To fulfill this task, one has merely to follow the proof of Theorem 4. Indeed, the first step is entirely the same. For the second and third steps, we just need to remove (3.15) and ignore any term containing $\alpha'(x)$. Otherwise, all the results up to (3.27) remain true.

Now, in order to conclude that $\|z_n\|_{L^2(0,\ell)} = o(1)$ and $\|y_n\|_{H^2_\star} = o(1)$, let us take the inner product in $L^2(0, \ell)$ of (3.21) with $xy_{n,xxx}$. On one hand, integrating by parts yields

$$\begin{aligned} &\frac{\ell}{2}(\widehat{\omega}^2 + \gamma_n^2)|y_{n,x}(1)|^2 - \frac{3}{2}(\widehat{\omega}^2 + \gamma_n^2) \int_0^1 |y_{n,x}|^2 dx - \frac{1}{2} \int_0^1 |y_{n,xxx}|^2 dx \\ &-(\widehat{\omega} + \gamma_n^2)\text{Re } (y_n(\ell)\bar{y}_{n,x}(\ell)) + \alpha\ell\gamma_n \text{Re } (i\,(y_{n,x}(\ell)\bar{y}_{n,xx}(\ell))) = o(1). \end{aligned} \tag{3.35}$$

On the other hand, we infer from (3.8) and (3.13) together with (3.20) and (3.25) that (3.35) gives

$$\|y_{n,xxx}\|_{L^2(0,1)} \to 0,\ \|y_{n,x}\|_{L^2(0,1)} = o(1),$$

which, by interpolation, leads to $y_n \to 0$ in $H^2_c(0, 1)$. This, together with (3.27), implies that $\|z_n\|_{L^2(0,\ell)} = o(1)$ and the contradiction is reached.

Lastly, the reader can notice that (3.3) is also fulfilled if α is constant. □

4. NUMERICAL SCHEME

In this section, we will numerically replicate our stability results using a finite volumes method for the space variable, and a Newmark method for the time variable.

4.1. **Foundations of the scheme.** Follow the ideas contained in [28], let $L \in \mathbb{N}$, and let $\mathcal{T}$ an admissible mesh of the interval $(0, \ell)$ given by the family $\{K_i\}_{i=1,\ldots,L} : K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, and a family $\{x_i\}_{i=0,\ldots,L+1}$ such that

$$x_0 = x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{3}{2}} < \cdots < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < \cdots < x_L < x_{L+\frac{1}{2}} = x_{L+1} = \ell.$$

We will set $|K_i| = \Delta x$ and $x_i = \frac{1}{2}\left(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}\right)$, $\forall i = 1, 2, \ldots, L$; this is, our mesh will be uniform and discretized using L *control volumes* or *cells* with center x_i. In order to deduce our numerical scheme, we will integrate equation (1.3) over each cell K_i to obtain

$$\int_{K_i} \rho(x)y_{tt}(x,t)dx + \int_{K_i} (EI(x)y_{xx})_{xx}(x,t)dx - \int_{K_i} \rho(x)\omega^2(t)y(x,t)dx + \int_{K_i} \alpha(x)y_{tx}(x,t)dx = 0. \tag{4.1}$$

Observe that

$$\int_{K_i}(EI(x)y_{xx})_{xx}(x,t)dx = (EI(x)y_{xx})_x(x,t)\Big|_{x=x_{i+\frac{1}{2}}} - (EI(x)y_{xx})_x(x,t)\Big|_{x=x_{i-\frac{1}{2}}}.$$

While the last term in (4.1) can be rewritten as

$$\int_{K_i}\alpha(x)y_{tx}(x,t)dx = \alpha(x_{i+\frac{1}{2}})y_t(x_{i+\frac{1}{2}},t) - \alpha(x_{i-\frac{1}{2}})y_t(x_{i-\frac{1}{2}},t) - \int_{K_i}\alpha'(x)y_t(x,t)dx.$$

With this, equation (4.1) turns into

$$\begin{aligned}\int_{K_i}\rho(x)y_{tt}(x,t)dx + (EI(x)y_{xx})_x(x,t)\Big|_{x=x_{i+\frac{1}{2}}} - (EI(x)y_{xx})_x(x,t)\Big|_{x=x_{i-\frac{1}{2}}}\\ -\int_{K_i}\rho(x)\omega^2(t)y(x,t)dx + \alpha(x_{i+\frac{1}{2}})y_t(x_{i+\frac{1}{2}},t) - \alpha(x_{i-\frac{1}{2}})y_t(x_{i-\frac{1}{2}},t) - \int_{K_i}\alpha'(x)y_t(x,t)dx = 0,\end{aligned}\tag{4.2}$$

which will be the expression to be studied under the finite volumes approach. For a given function $f(x,t)$, we will write $f_i(t) = f(x_i,t)$, $\dfrac{\partial f}{\partial t}(x_i,t) = f_{t,i}(t)$, and so on for other derivatives. In (4.2), each term will be approximated as follows:

$$\begin{aligned}\int_{K_i}\rho(x)y_{tt}(x,t)dx &\approx \rho_i y_{tt,i}(t)\Delta x;\\ (EI(x)y_{xx})_x(x,t)\Big|_{x=x_{i+\frac{1}{2}}} &\approx \frac{EI_{i+1}y_{xx,i+1}(t) - EI_i y_{xx,i}(t)}{\Delta x};\\ \int_{K_i}\rho(x)\omega^2(t)y(x,t)dx &\approx \rho_i\omega^2(t)y_i(t)\Delta x;\\ \alpha(x_{i+\frac{1}{2}})y_t(x_{i+\frac{1}{2}},t) &\approx \alpha_i y_{t,i}(t);\\ \int_{K_i}\alpha'(x)y_t(x,t)dx &\approx \alpha_{x,i}y_{t,i}(t)\Delta x.\end{aligned}$$

We thus obtain the following semi discrete approximation of (1.3):

$$\begin{aligned}\rho_i y_{tt,i}(t)\Delta x + \frac{EI_{i+1}y_{xx,i+1}(t) - EI_i y_{xx,i}(t)}{\Delta x} - \frac{EI_i y_{xx,i}(t) - EI_{i-1}y_{xx,i-1}(t)}{\Delta x}\\ -\rho_i\omega^2(t)y_i(t)\Delta x + \alpha_i y_{t,i}(t) - \alpha_{i-1}y_{t,i-1}(t) - \alpha_{x,i}y_{t,i}(t)\Delta x = 0.\end{aligned}\tag{4.3}$$

Observe that the derivative of $\alpha(x)$ can be obtained exactly since it is datum, while $y_{xx}(x,t)$ will be approximated using a centered finite difference

$$y_{xx,i}(t) \approx \frac{y_{i+1}(t) - 2y_i(t) + y_{i-1}(t)}{\Delta x^2}.$$

Therefore, (4.3) turns into

$$\begin{aligned}\rho_i y_{tt,i}(t)\Delta x + \frac{EI_{i+1}\left(y_{i+2}(t) - 2y_{i+1}(t) + y_i(t)\right) - EI_i\left(y_{i+1}(t) - 2y_i(t) + y_{i-1}(t)\right)}{\Delta x^3}\\ -\frac{EI_i\left(y_{i+1}(t) - 2y_i(t) + y_{i-1}(t)\right) - EI_{i-1}\left(y_i(t) - 2y_{i-1}(t) + y_{i-2}(t)\right)}{\Delta x^3}\\ -\rho_i\omega^2(t)y_i(t)\Delta x + \alpha_i y_{t,i}(t) - \alpha_{i-1}y_{t,i-1}(t) - \alpha_{x,i}y_{t,i}(t)\Delta x = 0,\end{aligned}$$

after re-ordering terms, we get

$$\begin{aligned}(4.4)\quad &\rho_i y_{tt,i}(t)\Delta x^4 + \left(\alpha_i \Delta x^3 - \alpha_{x,i}\Delta x^4\right) y_{t,i}(t) - \alpha_{i-1} y_{t,i-1}(t)\Delta x^3 + EI_{i+1} y_{i+2}(t) - 2\left(EI_{i+1} + EI_i\right) y_{i+1}(t) \\ &+ \left(EI_{i+1} + 4EI_i + EI_{i-1} - \rho_i \omega^2(t)\Delta x^4\right) y_i(t) - 2\left(EI_i + EI_{i-1}\right) y_{i-1}(t) + EI_{i-1} y_{i-2}(t) = 0,\ i = 1, 2, \ldots, L-2.\end{aligned}$$

As this is a second order ODE for $y_i(t)$, we will find its solutions using the Newmark algorithm described in [34]. Let us write $\boldsymbol{y}(t) \in \mathbb{R}^L : \boldsymbol{y}(t) = (y_1(t)\, y_2(t)\, \ldots\, y_L(t))^T$. In order to implement the Newmark algorithm, we need to re-write (4.4) as follows:

$$\boldsymbol{M}\boldsymbol{y}''(t) + \boldsymbol{C}\boldsymbol{y}'(t) + \boldsymbol{K}\boldsymbol{y}(t) = \boldsymbol{f}(t) \tag{4.5}$$

where $\boldsymbol{f}(t) \in \mathbb{R}^L$ and $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{K} \in \mathbb{R}^{L\times L}$ will be defined in the following subsection, as we first need to consider the boundary conditions associated to $y(x,t)$ in Problem (1.3). Given $T > 0$, the Newmark algorithm will allow us to compute an approximation of $\boldsymbol{y}(t)$ at $t = t_n \in [0, T]$, where for $\Delta t < 1$, $t_n = n\Delta t$, $n = 0, 1, \ldots, N$ and $N = \lceil \frac{T}{\Delta t} \rceil$. We will denote by $\boldsymbol{y}^{(n)}$ the approximation of $\boldsymbol{y}(t_n)$.

4.2. **Boundary conditions at $x = \ell$.** Regarding the boundary conditions at $x = \ell$, we will have $(EIy_{xx})_x(x_{L+\frac{1}{2}}, t) = 0$. Therefore, in the cell L we will have

$$\int_{K_L} (EI(x)y_{xx})_{xx}(x,t)dx = -(EI(x)y_{xx})_x(x,t)\Big|_{x=x_{L-\frac{1}{2}}} \approx -\frac{EI_L y_{xx,L}(t) - EI_{L-1} y_{xx,L-1}(t)}{\Delta x}.$$

Observe the appearance of the term $EI_L y_{xx,L}(t)$, which contains the memory effect. Writing $\mathcal{F}(t) := -\beta \displaystyle\int_0^\infty \xi(s) y_{tx}(\ell, t-s)ds$, that is, $EI_L y_{xx,L}(t) = \mathcal{F}(t)$, one can write

$$\int_{K_L} (EI(x)y_{xx})_{xx}(x,t)dx \approx -\frac{\mathcal{F}(t)}{\Delta x} + \frac{EI_{L-1}(y_{L(t)} - 2y_{L-1(t)} + y_{L-2(t)})}{\Delta x^3}. \tag{4.6}$$

On the other hand, at the cell $L-1$ we have

$$\begin{aligned}\int_{K_{L-1}} (EI(x)y_{xx})_{xx}(x,t)dx &= (EI(x)y_{xx})_x(x,t)\Big|_{x=x_{L-\frac{1}{2}}} - (EI(x)y_{xx})_x(x,t)\Big|_{x=x_{L-\frac{3}{2}}} \\ &\approx +\frac{EI_L\left(y_{L+1}(t) - 2y_L(t) + y_{L-1}(t)\right) - EI_{L-1}\left(y_L(t) - 2y_{L-1}(t) + y_{L-2}(t)\right)}{\Delta x^3} \\ &\quad - \frac{EI_{L-1}\left(y_L(t) - 2y_{L-1}(t) + y_{L-2}(t)\right) - EI_{L-2}\left(y_{L-1}(t) - 2y_{L-2}(t) + y_{L-3}(t)\right)}{\Delta x^3};\end{aligned}$$

Note that the term $y_{L+1}(t)$ appears. Because $y_{L+1}(t) = y_{L+\frac{1}{2}}(t)$ and due to the memory term, we have

$$\mathcal{F}(t) = EI_L y_{xx,L}(t) \approx EI_L \frac{y_{L+1}(t) - 2y_L(t) + y_{L-1}(t)}{\Delta x^2},$$

this motivates the following

$$y_{L+1}(t) = \Delta x^2 \frac{\mathcal{F}(t)}{EI_L} + 2y_L(t) - y_{L-1}(t).$$

The first term at the right hand side explains the consideration of a *source term* $\boldsymbol{f}(t)$. Therefore, matrices $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{K}$ (now $\boldsymbol{K}(t)$) and the vector $\boldsymbol{f}(t)$ in (4.5) will have the following form

$$\boldsymbol{M} = \Delta x^4 \begin{pmatrix} \rho_1 & & & & \\ & \rho_2 & & & \\ & & \ddots & & \\ & & & \rho_{L-1} & \\ & & & & \rho_L \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} \gamma_1 & & & & \\ \delta_1 & \gamma_2 & & & \\ & \ddots & \ddots & & \\ & & \delta_{L-2} & \gamma_{L-1} & \\ & & & \delta_{L-1} & \gamma_L \end{pmatrix} \tag{4.7}$$

$$\boldsymbol{K}(t) = \begin{pmatrix} a_1(t) & b_1 & c_1 & & & & \\ b_1 & a_2(t) & b_2 & c_2 & & & \\ c_1 & b_2 & a_3(t) & b_3 & c_3 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & c_{L-4} & b_{L-3} & a_{L-2}(t) & b_{L-2} & c_{L-2} \\ & & & c_{L-3} & b_{L-2} & \tilde{a}(t) & \tilde{c} \\ & & & & c_{L-2} & \tilde{c} & \tilde{b}(t) \end{pmatrix}, \quad \boldsymbol{f}(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -\mathcal{F}(t)\Delta x^2 \\ \mathcal{F}(t)\Delta x^2 \end{pmatrix}. \tag{4.8}$$

where in $\boldsymbol{C}$, $\gamma_i = \alpha_i \Delta x^3 - \alpha_{x,i}\Delta x^4$ and $\delta_i = -\alpha_i \Delta x^3$, $i = 1, 2, \dots, L$; and in $\boldsymbol{K}(t)$, $a_i(t) = EI_{i+1} + 4EI_i + EI_{i-1} - \rho_i \omega^2(t)\Delta x^4$, $b_i = -2(EI_i + EI_{i+1})$, $c_i = EI_{i+1}$ for $i = 1, 2, \dots, L-2$, while $\tilde{a}(t) = EI_{L-2} + 4EI_{L-1} - \rho_i \omega^2(t)\Delta x^4$, $\tilde{b}(t) = EI_{L-1} - \rho_i \omega^2(t)\Delta x^4$, $\tilde{c} = -2EI_{L-1}$. Thereby, the problem (4.5) turns into

$$\boldsymbol{M}\boldsymbol{y}''(t) + \boldsymbol{C}\boldsymbol{y}'(t) + \boldsymbol{K}(t)\boldsymbol{y}(t) = \boldsymbol{f}(t).$$

Regarding the $\omega(t)$ function, writing $\omega^{(n)}$ the approximation of $\omega(t_n)$, we will compute it using an implicit scheme given by

$$\omega^{(n+1)} = \frac{I^{(n)} - \frac{\Delta t \gamma}{2}\omega^{(n)} + \Delta t \gamma \hat{\omega}}{I^{(n+1)} + \frac{\Delta t \gamma}{2}},$$

where $I^{(n)} := I_d + \Delta x \sum_{j=1}^{L} \rho(x_j)(y_j^n)^2$ is the approximation of $I_d + \int_0^\ell \rho(x) y^2(x, t_n)dx$, and $\omega^{(0)} = \omega_0$.

4.3. **Treatment of the $\boldsymbol{K}(t)$ matrix and boundary conditions at $x = 0$.** The Newmark scheme needs the $\boldsymbol{K}(t)$ matrix to be symmetric and positive definite. To our inconvenience, this is not the case as it is computationally found that a negative $\lambda \approx 0$ is an eigenvalue of $\boldsymbol{K}(t)$. Instead of considering this matrix, we use the projection of $\boldsymbol{K}(t)$ over the vector space orthogonal to the eigenvector $\boldsymbol{v}$ associated to this eigenvalue. This is done by using $(\boldsymbol{I} - \frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}})\boldsymbol{K}(t)$ instead of $\boldsymbol{K}(t)$. Once this is set, we consider the last boundary condition left to apply at $x = 0$. It is clear that $y_{\frac{1}{2}}(t) = 0$ and $y_{x,\frac{1}{2}}(t) = 0$, $t > 0$. Due to this last condition, we will write $y_1(t) = y_0(t) = 0$, and thus after the projection we discard the first row and column of all matrices involved in our calculations. This procedure is similar to other cases found in the literature (see, for example, [8]).

4.4. **Computation of the memory term.** The Newmark algorithm requires to compute $\boldsymbol{f}(t)$ at $t = t_{n+1}$. Because the memory terms are inside that source term and those depend on the same numerical solution $\boldsymbol{y}^{(n+1)}$ we need to compute, a priori it is not possible to implement this algorithm. To overcome this difficulty, we will treat (4.5) as a fixed point problem. Therefore, the problem to solve will be given by

$$\boldsymbol{M}\boldsymbol{y}''(t) + \boldsymbol{C}\boldsymbol{y}'(t) + \boldsymbol{K}\boldsymbol{y}(t) = \hat{\boldsymbol{f}} \tag{4.9}$$

where

$$\hat{\boldsymbol{f}} = \begin{pmatrix} 0 & 0 & \dots & 0 & -\hat{\mathcal{F}}^{(n+1,p)}\Delta x^2 & \hat{\mathcal{F}}^{(n+1,p)}\Delta x^2 \end{pmatrix}^T \in \mathbb{R}^{J-1}, \qquad \hat{\mathcal{F}}^{(n+1,p)} = EI_L y_{xx,L}^{(n+1,p)};$$

and $\{\boldsymbol{y}^{(n+1,p)}\}_{p\in\mathbb{N}}$ is the sequence of numerical solutions obtained by the Newmark algorithm for approximating the solution at $t = t_{n+1}$. Once a stopping criteria is fulfilled, we do $\boldsymbol{y}^{(n+1)} = \boldsymbol{y}^{(n+1,p)}$ and move to the next timestep.

To compute $\hat{\mathcal{F}}^{(n+1,p)}$, we will borrow the idea from [5]. Invoking (1.5), we obtain

$$\beta\eta(t,0) = -\beta\int_0^\infty \xi(\tau)y_{xt}(\ell, t-\tau)d\tau = EI(\ell)y_{xx}(\ell,t).$$

Since η satisfies a transport equation, we solve it numerically through the following scheme

$$\frac{\eta_j^{n+1} - \eta_j^n}{\Delta t} - \frac{\eta_{j+1}^{n+\frac{1}{2}} - \eta_j^{n+\frac{1}{2}}}{\Delta s} = -\xi(s_j)\frac{y_{xt}(\ell)^{(n+1,p)} + y_{xt}(\ell)^n}{2\Delta t},$$

in which $\eta_j^n = \eta(s_j, t_n)$, $\eta_j^{n+\frac{1}{2}} = \dfrac{\eta_j^{n+1} + \eta_j^n}{2}$, and $s_i = j\Delta s$, $j = 0, 1, 2, \dots, K$. We assume $s \in [0, s_f]$, where s_f sufficiently big and $\Delta s < 1 : K\Delta s = s_f$. Subsequently, we set $\hat{\mathcal{F}}^{(n+1,p)} = \beta\eta_0^{n+1}$.

4.5. **First computational example.** Let us consider $t \in [0, 20]$, $x \in [0, 1]$, $s \in [0, 50]$; $\Delta t = 0.025$, $\Delta x = 0.05$, $\Delta s = 0.025$; $\beta = 0.1$, $\gamma = 1$, $\omega_0 = 5$, $\hat{\omega} = 1$, $I_d = 1$; $y_0(x) = x^4 - 4x^3 + 6x^2$, $y_1(x) = -y_0(x)$, $\alpha(x) = \frac{19}{25}$, $EI(x) = 1 + x$, $\varphi_0(\tau) = -2t^2 + 4t + 4$, $\rho(x) = 1 + x$, and $\xi(s) = e^{-s}$. Figure 2 illustrates our results from different angles, while Figure 3 shows the numerical behavior of $\omega(t)$ and $||y||_{\mathcal{Y}}$ with time. Results are in line with Theorem 5.

4.6. **Second computational example.** Let us consider $t \in [0, 50]$, $x \in [0, 1]$, $s \in [0, 50]$; $\Delta t = 0.025$, $\Delta x = 0.05$, $\Delta s = 0.025$; $\beta = 3$, $\gamma = 1$, $\omega_0 = 5$, $\hat{\omega} = \frac{3}{2}$, $I_d = 5$; $y_0(x) = ((1-x)\sin(\pi x))^2$, $y_1(x) = 0$, $\alpha(x) = e^{-x}$, $EI(x) = 1 + x$, $\varphi_0(\tau) = -2t^2 + 4t + 4$, $\rho(x) = 3 + \sin(x)$, and $\xi(s) = e^{-s}$. Figure 4 illustrates our results from different angles, while Figure 5 shows the numerical behavior of $\omega(t)$ and $||y||_{\mathcal{Y}}$ with time. Results are in line with Theorem 2. Some figures consider $t \in [0, 15]$ in order to improve its visibility.

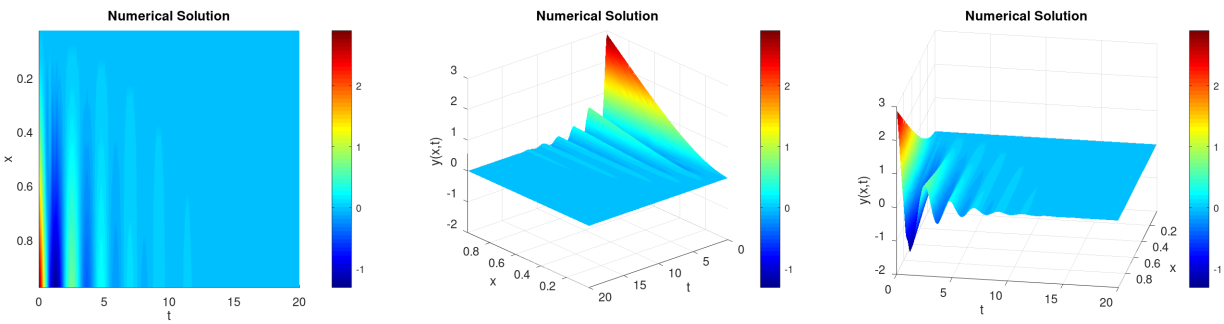

FIGURE 2. First experiment. Numerical solution of $y(x,t)$ from different angles.

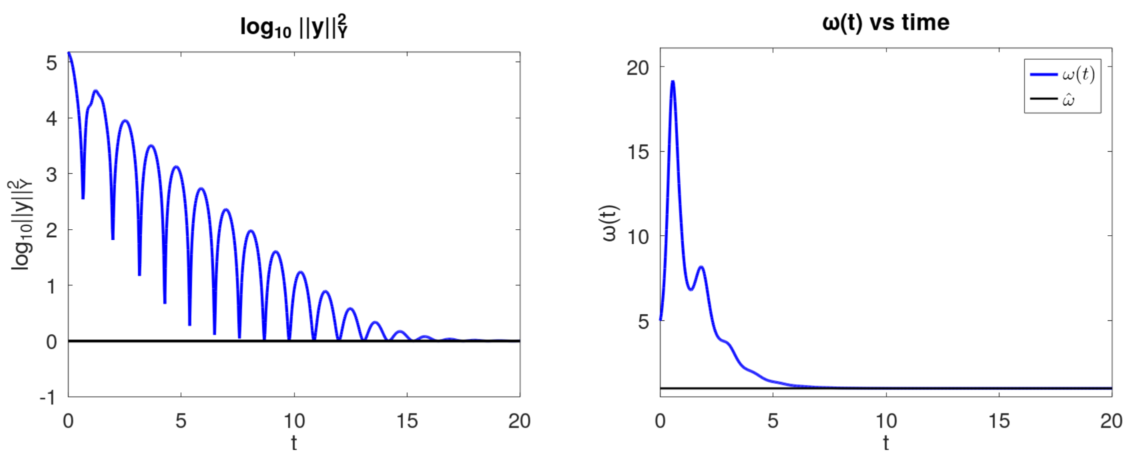

FIGURE 3. First experiment. Left: time volution of the numerical solution of $y(x,t)$ in $\mathcal{Y}$ norm. Right: time evolution of the numerical solution of $\omega(t)$.

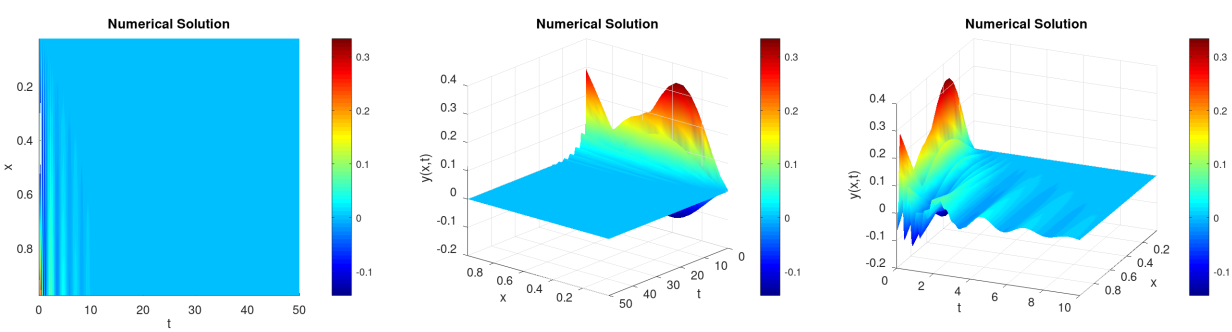

FIGURE 4. Second experiment. Numerical solution of $y(x,t)$ from different angles. Right figure considers $t \in [0,10]$.

5. CONCLUSION

In this article, we managed to show the stability result of the rotating disk-beam system under the effect of a boundary infinite memory of type angular velocity. The minimal state approach is adopted so that the infinite memory term is handled. Our findings are obtained under standard assumptions on the memory kernel, the angular velocity of the disk and the parameters of the beam but also a pretty restrictive condition on the spatial distribution $\alpha(x)$ of the compensation term $y_{tx}(x,t)$. More precisely, the hypothesis

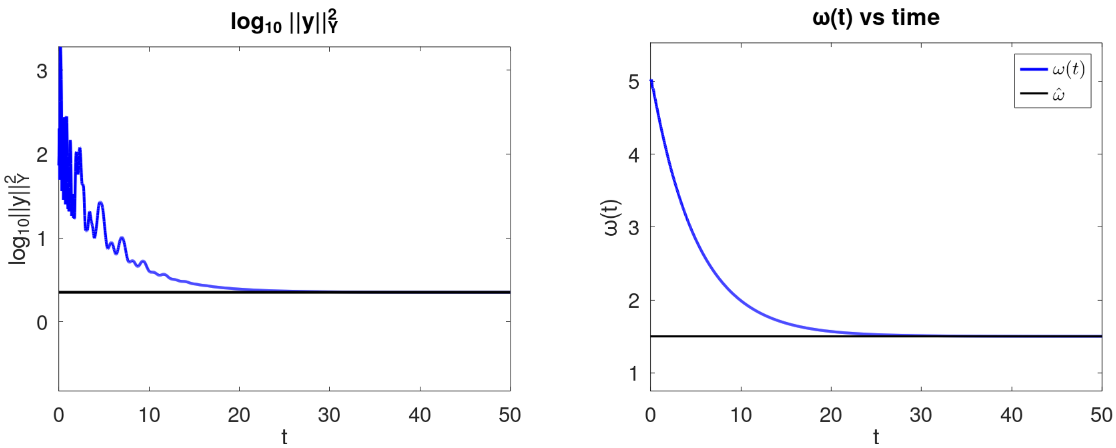

FIGURE 5. Second experiment. Left: time volution of the numerical solution of $y(x,t)$ in $\mathcal{Y}$ norm for $t \in [0,15]$. Right: time evolution of the numerical solution of $\omega(t)$.

$\alpha(x)$ is non-increasing is needed to have a dissipative system. Moreover, the stability outcome of the system is established under a stronger requirement, that is, $\alpha(x)$ is decreasing. Lastly, the case α constant is also briefly discussed. These stability outcomes are validated through a numerical analysis by using a Finite volumes method. Several examples are provided concerning the solution and the corresponding energy norm.

In a future work, we aspire to try to relax the conditions on $\alpha(x)$. Moreover, it would be desirable to consider a localized compensation term by assuming that $\alpha(x)$ acts only on a subdomain of the spatial interval $[0,\ell]$.

Statements and Declarations

The authors have no competing interests to declare that are relevant to the content of this article.

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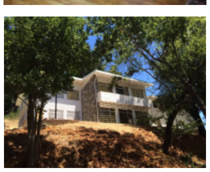

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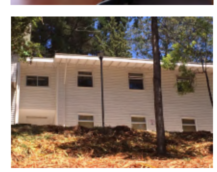
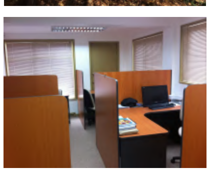

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