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New fully mixed finite element methods for the coupled convective Brinkman–Forchheimer and nonlinear transport equations*

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Abstract

We introduce and analyze new Banach spaces-based fully-mixed finite element methods for the convective Brinkman–Forchheimer equations coupled with a nonlinear transport phenomenon. Our approach is based on the incorporation of the fluid velocity gradient, the incomplete nonlinear fluid pseudostress, the concentration gradient, and the total (diffusive plus advective) flux for the concentration, as auxiliary variables, which, along with the velocity and concentration themselves, constitute the set of unknowns of the model. The resulting mixed variational formulation can be written as two coupled nonlinear saddle point systems, which are then reformulated as an equivalent fixed-point equation defined in terms of the operators solving the corresponding decoupled problems. An analogue approach is utilized for the associated Galerkin scheme. In this way, the Babuška–Brezzi theory, some abstract results on monotone operators, and the classical Banach fixed-point theorem are employed to establish the well-posedness of both the continuous and discrete schemes. In particular, for each integer $k \geq 0$, vector and tensor Raviart–Thomas subspaces of order k for the pseudostress and the total flux, respectively, as well as piecewise polynomial subspaces of degree $\leq k$ for the velocity, the concentration, and their respective gradients, yield stable Galerkin schemes. Optimal a priori error estimates along with the corresponding rates of convergence are also established. Finally, several numerical experiments confirming the latter and illustrating the good performance of the method in 2D and 3D, are reported.

Key words: convective Brinkman–Forchheimer equations, nonlinear transport, pseudostress-velocity formulation, fixed point theory, mixed finite elements

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

The transport of species density in a saturated porous medium fluid, which involves three main fields: the velocity of the flow, pressure, and local solids concentration, has a wide range of applications in

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chemical, environmental, and petroleum engineering. Examples include chemical distillation processes, sedimentation-consolidation processes, solid-liquid separation, aluminum production, and natural and thermal convection, among others. Accurate modeling and simulation of such flows are essential to optimize processes, ensure safety, and minimize environmental impact. Over the years, mathematical models have been developed to capture various aspects of these flows, with much of the research focusing on coupling the Stokes (or Brinkman) model with transport equations. However, these equations may be inadequate for modeling fluid flow through porous media with high Reynolds numbers or highly porous materials. To address these limitations, the convective Brinkman–Forchheimer (CBF) equations have been proposed (see, e.g., [18], [32], [30], [13], and [14]). The CBF equations extend both the Stokes and Brinkman models by incorporating additional terms to account for the physical phenomena described above. Accordingly, the present work focuses on the coupled flow and transport problem governed by a nonlinear convection-diffusion equation interacting with the CBF equations.

Concerning the literature devoted to studying the coupling of the Stokes and transport equations, we begin by mentioning [1], where an augmented mixed formulation for the fluid equations and the standard primal scheme for the transport equation were proposed and analyzed. Subsequently, in [2], the approach from [1] was extended to the case of a strongly coupled flow and transport system. This system was modeled using the Brinkman problem with variable viscosity, expressed in terms of Cauchy pseudo-stresses and the bulk velocity of the mixture, coupled with a nonlinear advection-diffusion equation describing the transport of the solids volume fraction. Furthermore, the existence of solutions to a related model for chemically reacting non-Newtonian fluids was established in [8]. Regarding the analysis developed in [2], an augmented mixed approach was employed for the Brinkman problem, while the usual primal weak formulation was applied to the transport equation to derive the variational formulation of the coupled problem. Similarly to [1], the continuous and discrete solvability analyses were carried out by combining fixed-point arguments, elliptic regularity estimates, sufficiently small data assumptions, and classical results on Hilbert space frameworks [7, 19, 23]. More recently, in [3], a model describing the flow-transport interaction in a porous-fluidic domain was analyzed using the techniques developed in [1] and [2]. In this case, the medium consists of a highly permeable material, where the flow of an incompressible viscous fluid is governed by the Brinkman equations formulated in terms of vorticity, velocity, and pressure, and a porous medium, where Darcy's law describes fluid motion in terms of filtration velocity and pressure. Additionally, an augmented fully-mixed variational formulation for the model introduced in [1] was proposed and analyzed in [25]. In this work, the authors employed a dual-mixed method and an augmentation procedure for both the Stokes and transport equations. We conclude by mentioning [5] and [6], where a mixed-primal formulation and a fully-mixed formulation for the coupled problem analyzed in [1] and [25], respectively, both posed within Banach space frameworks, were proposed and analyzed.

Regarding the analysis of the CBF equations in the literature, we first refer to [18], where the authors studied the continuous dependence of solutions to the CBF equations, expressed in velocity-pressure formulation, on the Forchheimer coefficient in the H^1 norm. Subsequently, [32] proposed and developed an approximation of solutions for the incompressible CBF equations using the artificial compressibility method. Furthermore, [30] analyzed the well-posedness of the velocity-pressure variational formulation for the two-dimensional stationary CBF equations. This study also included error estimates for a mixed finite element approximation and proposed a one-step Newton iteration algorithm initialized with a fixed-point iteration. More recently, [13] examined an augmented mixed pseudostress-velocity formulation. In this case, the well-posedness of the problem was achieved through a combination of the Lax–Milgram theorem, Schauder and Banach fixed-point theorems, and a fixed-point strategy. In [14], a mixed formulation in Banach spaces was proposed and analyzed for the CBF problem. Unlike the approach in [13], this formulation did not require an augmentation procedure for either the formulation itself or the solvability analysis. Instead, the non-augmented scheme was equivalently reformulated as a fixed-point equation, enabling the use of results from [21] on perturbed saddle-point problems in Banach spaces. These results, together with the Banach–Nečas–Babuška and Banach theorems, were applied to establish the well-posedness of both the continuous and discrete systems.

The purpose of this work is to develop and analyze a new fully mixed formulation for the coupling of the CBF and nonlinear transport equations, as well as to study a suitable numerical discretization. Motivated by [20], [15], [17], and [4], we introduce additional unknowns, including the fluid velocity gradient, the incomplete nonlinear fluid pseudostress, the concentration gradient, and the total (diffusive plus advective) flux for the concentration, alongside the fluid velocity and concentration, while eliminating the pressure using the incompressibility condition. The resulting mixed variational formulation consists of two coupled nonlinear saddle-point systems, which are reformulated as an equivalent fixed-point equation defined by the operators solving the corresponding decoupled problems. Following a methodology similar to [20] and [4], we establish the existence and uniqueness of solutions for both the continuous and discrete formulations by combining a fixed-point argument, the Babuška–Brezzi theory, abstract results on monotone operators, sufficiently small data assumptions, and the classical Banach fixed-point theorem. Additionally, we derive the corresponding *a priori* error estimates using ad-hoc Strang-type lemmas in Banach spaces. Finally, employing Raviart–Thomas spaces of order $k \geq 0$ for the pseudostress and total flux, along with discontinuous piecewise polynomials of degree $\leq k$ for the velocity, concentration, and their respective gradients, we prove that the method converges with optimal rates.

This work is organized as follows. The remainder of this section describes standard notation and functional spaces to be employed throughout the paper. In Section 2 we introduce the model problem. Next, in Section 3 we derive the mixed variational formulation in Banach spaces and establish the well-posedness of this continuous scheme. The corresponding Galerkin system is introduced and analyzed in Section 4, where the discrete counterpart of the theory applied in the continuous case is used to prove the existence and uniqueness of the solution, as well as the *a priori* error estimates for general discrete spaces. Convergence rates for specific finite element subspaces are derived in Section 5. Finally, the performance of the method is illustrated in Section 6 through numerical examples in both 2D and 3D, including cases with and without manufactured solutions, validating the accuracy and flexibility of our Banach spaces-based mixed finite element method.

Preliminary notations

Let $\Omega \subset \mathrm{R}^n, n \in \{2,3\}$, be a bounded domain with polyhedral boundary Γ, and let $\boldsymbol{\nu}$ be the outward unit normal vector on Γ. In what follows, standard notation is adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. On the other hand, given any generic scalar functional space S, we let $\mathbf{S}$ and $\mathbb{S}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Also, $|\cdot|$ denotes the Euclidean norm in both R^n and $\mathrm{R}^{n\times n}$, and as usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$. In addition, for normed vector spaces H and Q, with norms $\|\cdot\|_H$ and $\|\cdot\|_Q$ respectively, we endow the product space $H \times Q$ with the natural norm

$$\|(u,v)\|_{H\times Q} := \|u\|_H + \|v\|_Q \qquad \forall\,(u,v) \in H \times Q\,.$$

Also, given any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}\,,$$

whereas for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the deviatoric tensor, and the tensor inner product, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}, \quad \text{and} \quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\,\zeta_{ij}\,.$$

Next, for each $t \in [1,+\infty)$ we introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \Big\{\boldsymbol{\tau}\in\mathbf{L}^2(\Omega): \quad \mathrm{div}(\boldsymbol{\tau})\in\mathrm{L}^t(\Omega)\Big\} \quad \text{and}$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau}\in\mathbb{L}^2(\Omega): \quad \mathbf{div}(\boldsymbol{\tau})\in\mathbf{L}^t(\Omega)\Big\}\,,$$

which are equipped, respectively, with the natural norms

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \quad \forall\,\boldsymbol{\tau}\in\mathbf{H}(\mathrm{div}_t;\Omega) \quad \text{and} \tag{1.1}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_t;\Omega)\,.$$

Furthermore, we consider the canonical injections $i_{p,q} : \mathrm{L}^p(\Omega)\to\mathrm{L}^q(\Omega)$ for all $p,q\in[1,+\infty)$, $p\geq q$, and $i_{\mathrm{H},p} : \mathrm{H}^1(\Omega)\to\mathrm{L}^p(\Omega)$ for all $p\in(1,+\infty)$, which are continuous with norms depending on the domain. In particular, we have

$$\|i_{p,q}\| \leq |\Omega|^{(p-q)/(pq)}\,. \tag{1.2}$$

In turn, we let $\boldsymbol{i}_{p,q}$ and $\boldsymbol{i}_{\mathbf{H},p}$ be the corresponding vector counterparts of $i_{p,q}$ and $i_{\mathrm{H},p}$, respectively. Note that the norm of $\boldsymbol{i}_{p,q}$ also achieves the bound (1.2). Additionally, we recall that, proceeding as in [23, eq. (1.43), Section 1.3.4] (see also [9, Section 4.1] and [20, Section 3.1]), one can prove that for $t \in \begin{cases} (1,+\infty] \text{ in } \mathrm{R}^2\,, \\ [\frac{6}{5},+\infty] \text{ in } \mathrm{R}^3\,, \end{cases}$ there holds

$$\langle\boldsymbol{\xi}\cdot\boldsymbol{\nu},\varphi\rangle = \int_\Omega \Big\{\boldsymbol{\xi}\cdot\nabla\varphi + \varphi\,\mathrm{div}(\boldsymbol{\xi})\Big\} \quad \forall\,(\boldsymbol{\xi},\varphi)\in\mathbf{H}(\mathrm{div}_t;\Omega)\times\mathrm{H}^1(\Omega) \quad \text{and} \tag{1.3}$$

$$\langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{v}\rangle = \int_\Omega \Big\{\boldsymbol{\tau}:\nabla\mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\Big\} \quad \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathbb{H}(\mathbf{div}_t;\Omega)\times\mathbf{H}^1(\Omega)\,, \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ in (1.3) and (1.4) denotes the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, and between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$, respectively.

2 The model problem

We consider a porous medium occupying the region Ω, and assume that a viscous fluid governed by the convective Brinkman–Forchheimer equations flows through it, so that the sought variables are its pressure p and velocity $\mathbf{u}$. In addition, we let ϕ be the concentration of a chemical component transported by the fluid, which is advected and diffused in Ω according to the corresponding physical principle. Alternatively, ϕ could represent the temperature of the fluid, among several other possibilities. In this way, the coupled model of interest is given by the following system of partial differential equations:

$$
\begin{aligned}
-\mu\,\Delta\mathbf{u} + \big(\nabla\mathbf{u}\big)\mathbf{u} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} + \nabla p &= \phi\,\mathbf{f} && \text{in} \quad \Omega\,,\\
\mathrm{div}(\mathbf{u}) &= 0 && \text{in} \quad \Omega\,,\\
\mathrm{div}\big(\kappa(|\nabla\phi|)\,\nabla\phi - \phi\,\mathbf{u} - f(\phi)\,\mathbf{g}\big) &= g && \text{in} \quad \Omega\,,\\
\mathbf{u} = \mathbf{u}_{\mathrm{D}} \quad \text{and} \quad \phi &= \phi_{\mathrm{D}} && \text{on} \quad \Gamma\,,
\end{aligned} \tag{2.1}
$$

where μ is the constant viscosity of the fluid, $\mathtt{D}, \mathtt{F} > 0$ are the Darcy and Forchheimer coefficients, respectively, ρ is a given number in $[3,4]$, $\kappa : \mathrm{R}^+ \to \mathrm{R}^+$ is a nonlinear diffusivity function, f is a nonlinear flux acting in the direction of $\mathbf{g}$, which, in turn, is a constant vector pointing in the direction of gravity, $\mathbf{f}$ and g are given source functions, and $\mathbf{u}_{\mathrm{D}}$ and ϕ_{D} are Dirichlet data for $\mathbf{u}$ and ϕ, respectively. Regarding κ, we assume that there exist constants κ_1, $\kappa_2 > 0$ such that

$$
\kappa_1 \,\le\, \kappa(t) \,\le\, \kappa_2 \quad \text{and} \quad \kappa_1 \,\le\, \kappa(t) + t\,\kappa'(t) \,\le\, \kappa_2 \qquad \forall\, t \in \mathrm{R}^+\,. \tag{2.2}
$$

In addition, f is required to be bounded and Lipschitz-continuous, which means that there exist constants f_1, f_2, $\mathrm{L}_f > 0$ such that

$$
f_1 \,\le\, f(t) \,\le\, f_2 \quad \text{and} \quad |f(t) - f(s)| \,\le\, \mathrm{L}_f\,|s-t| \qquad \forall\, s,\, t \in \mathrm{R}^+\,. \tag{2.3}
$$

Now, due to the incompressibility of the fluid (cf. second row of (2.1)), $\mathbf{u}_{\mathrm{D}}$ must formally satisfy the compatibility condition

$$
\int_\Gamma \mathbf{u}_{\mathrm{D}}\cdot\boldsymbol{\nu} \,=\, 0\,. \tag{2.4}
$$

On the other hand, for the uniqueness of p we look for this unknown in the space

$$
\mathrm{L}^2_0(\Omega) \,:=\, \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\}\,.
$$

Next, in order to derive a fully-mixed formulation for (2.1), in which the Dirichlet boundary condition becomes natural, we first define as auxiliary unknowns the velocity gradient

$$
\boldsymbol{\chi} \,:=\, \nabla\mathbf{u}\,,
$$

and the incomplete nonlinear fluid pseudostress

$$
\boldsymbol{\sigma} \,:=\, \mu\,\boldsymbol{\chi} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u}) - p\,\mathbb{I}\,, \tag{2.5}
$$

so that the first row of (2.1) becomes

$$
-\,\mathbf{div}(\boldsymbol{\sigma}) + \frac{1}{2}\boldsymbol{\chi}\,\mathbf{u} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} \,=\, \phi\,\mathbf{f}\,.
$$

Thus, taking matrix trace along with the fact that $\mathrm{tr}(\boldsymbol{\chi}) = \mathrm{tr}(\nabla\mathbf{u}) = \mathrm{div}(\mathbf{u}) = 0$, and applying the deviatoric operator, we deduce from (2.5) that

$$
p \,=\, -\frac{1}{n}\,\mathrm{tr}\Big(\boldsymbol{\sigma} + \frac{1}{2}(\mathbf{u}\otimes\mathbf{u})\Big) \quad \text{and} \quad \boldsymbol{\sigma}^{\mathrm{d}} \,:=\, \mu\,\boldsymbol{\chi} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}}\,, \tag{2.6}
$$

which are equivalent to the pair of equations formed by the incompressibility condition and (2.5). In turn, regarding the concentration equation, we introduce the further unknowns given by the concentration gradient

$$\mathbf{t} := \nabla\phi$$

and its total flux

$$\boldsymbol{\eta} := \kappa(|\mathbf{t}|)\,\mathbf{t} - \phi\,\mathbf{u} - f(\phi)\,\mathbf{g}\,,$$

so that the third row of (2.1) is rewritten as

$$\operatorname{div}(\boldsymbol{\eta}) = g\,.$$

Hence, eliminating the unknown p, and computing it afterwards according to the identity provided by (2.6), the original system (2.1) can be stated, equivalently, as: Find $\boldsymbol{\chi}$, $\mathbf{u}$, $\boldsymbol{\sigma}$, $\mathbf{t}$, ϕ, and $\boldsymbol{\eta}$ in suitable spaces to be indicated below, such that

$$\begin{aligned}
\nabla\mathbf{u} &= \boldsymbol{\chi} && \text{in } \Omega\,,\\
\mu\,\boldsymbol{\chi} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} &= \boldsymbol{\sigma}^{\mathrm{d}} && \text{in } \Omega\,,\\
-\,\mathbf{div}(\boldsymbol{\sigma}) + \frac{1}{2}\boldsymbol{\chi}\,\mathbf{u} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} &= \phi\,\mathbf{f} && \text{in } \Omega\,,\\
\nabla\phi &= \mathbf{t} && \text{in } \Omega\,,\\
\kappa(|\mathbf{t}|)\,\mathbf{t} - \phi\,\mathbf{u} - f(\phi)\,\mathbf{g} &= \boldsymbol{\eta} && \text{in } \Omega\,,\\
\operatorname{div}(\boldsymbol{\eta}) &= g && \text{in } \Omega\,,\\
\mathbf{u} = \mathbf{u}_{\mathrm{D}} \quad \text{and} \quad \phi &= \phi_{\mathrm{D}} && \text{on } \Gamma\,,\\
\int_\Omega \operatorname{tr}\Big(\boldsymbol{\sigma} + \frac{1}{2}(\mathbf{u}\otimes\mathbf{u})\Big) &= 0\,.
\end{aligned} \tag{2.7}$$

3 The continuous formulation

3.1 The variational formulation

In this section, we deduce the fully mixed formulation of our coupled model, for which we begin with the convective Brinkman–Forchheimer equations. Indeed, testing the second row of (2.7) with a tensor field $\boldsymbol{\vartheta} \in \mathbb{L}^2(\Omega)$ we formally get

$$\mu \int_\Omega \boldsymbol{\chi} : \boldsymbol{\vartheta} - \frac{1}{2}\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \boldsymbol{\vartheta} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\vartheta} = 0\,, \tag{3.1}$$

whose first and third terms are well-defined if $\boldsymbol{\chi}$ and $\boldsymbol{\sigma}$ belong to $\mathbb{L}^2(\Omega)$ as well. In turn, straightforward applications of the Cauchy–Schwarz inequality yield

$$\left|\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \boldsymbol{\vartheta}\right| \le \|(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}}\|_{0,\Omega}\,\|\boldsymbol{\vartheta}\|_{0,\Omega} \le \|\mathbf{u}\otimes\mathbf{u}\|_{0,\Omega}\,\|\boldsymbol{\vartheta}\|_{0,\Omega} \le n^{1/2}\,\|\mathbf{u}\|_{0,4;\Omega}^2\,\|\boldsymbol{\vartheta}\|_{0,\Omega}\,,$$

from which we deduce that the second term of (3.1) is well-defined if $\mathbf{u} \in \mathbf{L}^4(\Omega)$. In addition, looking at (3.1) with the particular choice $\boldsymbol{\vartheta} := \zeta\,\mathbb{I}$ and $\zeta \in C_0^\infty(\Omega)$, we readily find that $\operatorname{tr}(\boldsymbol{\chi}) = 0$, whence $\boldsymbol{\chi}$ must be sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, where

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) := \Big\{\boldsymbol{\vartheta} \in \mathbb{L}^2(\Omega) : \quad \operatorname{tr}(\boldsymbol{\vartheta}) = 0\Big\}\,.$$

Alternatively, the above can also be derived by applying the matrix trace to the second row of (2.7). Moreover, from the decomposition $\mathbb{L}^2(\Omega) = \mathbb{L}^2_{\mathrm{tr}}(\Omega) \oplus \mathbb{L}^2_{\mathrm{tr}}(\Omega)^\perp$, where $\mathbb{L}^2_{\mathrm{tr}}(\Omega)^\perp := \big\{\zeta\,\mathbb{I} : \quad \zeta \in \mathrm{L}^2(\Omega)\big\}$, we readily deduce that testing (3.1) against $\mathbb{L}^2(\Omega)$ is equivalent to doing it against $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, which, noting that $\int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \boldsymbol{\vartheta} = \int_\Omega \boldsymbol{\tau} : \boldsymbol{\vartheta}$ for all $(\boldsymbol{\tau}, \boldsymbol{\vartheta}) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, becomes

$$\mu \int_\Omega \boldsymbol{\chi} : \boldsymbol{\vartheta} - \frac{1}{2}\int_\Omega (\mathbf{u}\otimes\mathbf{u}) : \boldsymbol{\vartheta} - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\vartheta} \,=\, 0 \qquad \forall\, \boldsymbol{\vartheta} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,. \tag{3.2}$$

Next, we formally test the third equation of (2.7) with a vector field $\mathbf{v}$, which gives

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) + \frac{1}{2}\int_\Omega \boldsymbol{\chi}\,\mathbf{u}\cdot\mathbf{v} + \mathtt{D}\int_\Omega \mathbf{u}\cdot\mathbf{v} + \mathtt{F}\int_\Omega |\mathbf{u}|^{\rho-2}\mathbf{u}\cdot\mathbf{v} \,=\, \int_\Omega \phi\,\mathbf{f}\cdot\mathbf{v}\,. \tag{3.3}$$

Then, knowing where to look for $\mathbf{u}$, we notice that the third term on the left-hand side of (3.3) is well-defined if $\mathbf{v}$ belongs to $\mathbf{L}^4(\Omega)$ as well, whence for the first one to also make sense we need that $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$, thus requiring finally that $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$. Additionally, since $2(\rho-2) \le 4$, the space $\mathbf{L}^4(\Omega)$ is continuously embedded into $\mathbf{L}^{2(\rho-2)}(\Omega)$, so that employing Cauchy–Schwarz's inequality and the estimate (1.2), we easily find that

$$\int_\Omega |\mathbf{z}|^{\rho-2}\mathbf{u}\cdot\mathbf{v} \,\le\, |\Omega|^{(4-\rho)/4}\,\|\mathbf{z}\|^{\rho-2}_{0,4;\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega} \quad \forall\,\mathbf{z},\,\mathbf{u},\,\mathbf{v} \in \mathbf{L}^4(\Omega)\,,$$

which says that the fourth term on the left-hand side of (3.3) is well-defined for $\mathbf{u}$ and $\mathbf{v}$ in the space indicated. Similarly, but employing only Cauchy–Schwarz's inequality again, we obtain

$$\left|\int_\Omega \boldsymbol{\chi}\,\mathbf{u}\cdot\mathbf{v}\right| \,\le\, \|\boldsymbol{\chi}\|_{0,\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,,$$

which, given that $\boldsymbol{\chi} \in \mathbf{L}^2(\Omega)$, guarantees that the second term on the left-hand side of (3.3) is also well-defined. The right-hand side of (3.3) will be addressed when deriving the variational formulation for the transport equations. In this way, adding (3.2) and (3.3), and reordering some of the terms, we arrive at

$$\begin{aligned} &\mu\int_\Omega \boldsymbol{\chi} : \boldsymbol{\vartheta} \,+\, \frac{1}{2}\int_\Omega \big(\boldsymbol{\chi}\,\mathbf{u}\cdot\mathbf{v} - (\mathbf{u}\otimes\mathbf{u}) : \boldsymbol{\vartheta}\big) \,+\, \mathtt{D}\int_\Omega \mathbf{u}\cdot\mathbf{v} \,+\, \mathtt{F}\int_\Omega |\mathbf{u}|^{\rho-2}\mathbf{u}\cdot\mathbf{v} \\ &-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\vartheta} \,-\, \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \,=\, \int_\Omega \phi\,\mathbf{f}\cdot\mathbf{v} \qquad \forall\,(\boldsymbol{\vartheta},\mathbf{v}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{L}^4(\Omega)\,. \end{aligned} \tag{3.4}$$

Furthermore, the first equation of (2.7) and the fact that $\boldsymbol{\chi} \in \mathbb{L}^2(\Omega)$ imply that $\mathbf{u} \in \mathbf{H}^1(\Omega)$, so that applying the integration by parts formula (1.4) to $(\boldsymbol{\tau},\mathbf{u}) \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)\times\mathbf{H}^1(\Omega)$, and assuming that the Dirichlet datum $\mathbf{u}_{\mathrm{D}}$ belongs to $\mathbf{H}^{1/2}(\Gamma)$, we get

$$\int_\Omega \boldsymbol{\chi} : \boldsymbol{\tau} \,+\, \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,=\, \langle \boldsymbol{\tau\nu}, \mathbf{u}_{\mathrm{D}}\rangle \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)\,, \tag{3.5}$$

from which we see that it suffices to look for $\mathbf{u}$ in $\mathbf{L}^4(\Omega)$, thus reconfirming the previous choice for it. In addition, according to the decomposition $\mathbb{H}(\mathbf{div}_t;\Omega) = \mathbb{H}_0(\mathbf{div}_t;\Omega) \oplus \mathrm{R}\,\mathbb{I}$, which is valid for each $t \in (1,+\infty)$, where

$$\mathbb{H}_0(\mathbf{div}_t;\Omega) \,:=\, \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\right\},$$

$\boldsymbol{\sigma}$ can be decomposed uniquely as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + d\,\mathbb{I}$, with $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $d \in \mathrm{R}$, where, invoking the last equation of (2.7), there holds

$$d \,:=\, \frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, -\frac{1}{2\,n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\,.$$

Thus, having expressed d in terms of $\mathbf{u}$, and realizing that $\boldsymbol{\sigma}$ can be replaced by $\boldsymbol{\sigma}_0$ in (3.4) without altering the meaning of that equation, in what follows we redenote the remaining unknown $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Moreover, thanks to the compatibility condition given by (2.4), (3.5) is trivially satisfied for $\boldsymbol{\tau} = \mathbb{I}$, and hence imposing this equation with $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it with $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$.

We continue the derivation of the fully-mixed formulation by dealing now with the equations forming the nonlinear transport part. In fact, testing the fifth row of (2.7) against $\mathbf{s} \in \mathbf{L}^2(\Omega)$, we obtain

$$\int_\Omega \kappa(|\mathbf{t}|)\,\mathbf{t}\cdot\mathbf{s} \,-\, \int_\Omega \phi\,\mathbf{u}\cdot\mathbf{s} \,-\, \int_\Omega f(\phi)\,\mathbf{g}\cdot\mathbf{s} \,=\, \int_\Omega \boldsymbol{\eta}\cdot\mathbf{s}\,, \tag{3.6}$$

from which, bearing in mind the boundedness of κ and f (cf. (2.2), (2.3)), we deduce that all the terms, except the second one on the left-hand side, are well-defined if $\mathbf{t}$, $\mathbf{g}$, and $\boldsymbol{\eta}$ belong to $\mathbf{L}^2(\Omega)$ as well. However, we remark in advance that in Section 3.4 we will actually require the datum $\mathbf{g}$ to be in $\mathbf{L}^4(\Omega)$ (see (3.53)), so that for simplicity we adopt this latter assumption from now on. In turn, regarding the aforementioned second term, and recalling from the previous analysis that $\mathbf{u} \in \mathbf{L}^4(\Omega)$, straightforward applications of the Cauchy–Schwarz's inequality yield

$$\left|\int_\Omega \phi\,\mathbf{u}\cdot\mathbf{s}\right| \,\le\, \|\phi\|_{0,4;\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,,$$

thus showing that the above expression makes sense if the unknown ϕ is sought in $\mathrm{L}^4(\Omega)$. Having observed this, and as previously announced, we now notice that the right-hand side of (3.3) is well defined if we ask for $\mathbf{f}$ to be in $\mathbf{L}^2(\Omega)$. Next, in order to use $\mathrm{L}^4(\Omega)$ as both the unknown and test spaces associated with ϕ, we assume the datum g to belong to $\mathrm{L}^{4/3}(\Omega)$, which yields $\boldsymbol{\eta}$ to be sought in $\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, and hence the sixth equation of (2.7) is tested as

$$\int_\Omega \varphi\,\mathrm{div}(\boldsymbol{\eta}) \,=\, \int_\Omega g\,\varphi \qquad \forall\,\varphi \in \mathrm{L}^4(\Omega)\,. \tag{3.7}$$

Thus, suitably gathering (3.6) and (3.7), we arrive at

$$\begin{gathered}\int_\Omega \kappa(|\mathbf{t}|)\,\mathbf{t}\cdot\mathbf{s} \,-\, \int_\Omega \phi\,\mathbf{u}\cdot\mathbf{s} \,-\, \int_\Omega \boldsymbol{\eta}\cdot\mathbf{s} \,-\, \int_\Omega \varphi\,\mathrm{div}(\boldsymbol{\eta}) \,=\, -\int_\Omega g\,\varphi + \int_\Omega f(\phi)\,\mathbf{g}\cdot\mathbf{s} \\ \forall\,(\mathbf{s},\varphi) \,\in\, \mathbf{L}^2(\Omega)\times\mathrm{L}^4(\Omega)\,.\end{gathered} \tag{3.8}$$

On the other hand, thanks to the fourth equation of (2.7) and the fact that $\mathbf{t} \in \mathbf{L}^2(\Omega)$, we see that $\phi \in \mathrm{H}^1(\Omega)$, so that applying the integration by parts formula (1.3) to $(\boldsymbol{\xi},\phi) \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\times\mathrm{H}^1(\Omega)$, and assuming that $\phi_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$, we get

$$\int_\Omega \mathbf{t}\cdot\boldsymbol{\xi} \,+\, \int_\Omega \phi\,\mathrm{div}(\boldsymbol{\xi}) \,=\, \langle\boldsymbol{\xi}\cdot\boldsymbol{\nu},\phi_{\mathrm{D}}\rangle \qquad \forall\,\boldsymbol{\xi} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,, \tag{3.9}$$

from which we reconfirm $\mathrm{L}^4(\Omega)$ as the space where to seek ϕ.

In this way, defining the spaces

$$\begin{gathered}\mathbf{H}_1 \,:=\, \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,,\quad \mathbf{H}_2 \,:=\, \mathbf{L}^4(\Omega)\,,\quad \mathbf{Q} \,:=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\ \mathrm{X}_1 \,:=\, \mathbf{L}^2(\Omega)\,,\quad \mathrm{X}_2 \,:=\mathrm{L}^4(\Omega)\,,\quad \mathrm{Y} \,:=\, \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,,\end{gathered}$$

introducing the notations

$$\vec{\chi} := (\boldsymbol{\chi}, \mathbf{u}), \quad \vec{\boldsymbol{\vartheta}} := (\boldsymbol{\vartheta}, \mathbf{v}), \quad \vec{\boldsymbol{\varrho}} := (\boldsymbol{\varrho}, \mathbf{w}) \in \mathbf{H} := \mathbf{H}_1 \times \mathbf{H}_2,$$

$$\vec{\mathbf{t}} := (\mathbf{t}, \phi), \quad \vec{\mathbf{s}} := (\mathbf{s}, \varphi), \quad \vec{\mathbf{r}} := (\mathbf{r}, \psi) \in \mathrm{X} := \mathrm{X}_1 \times \mathrm{X}_2,$$

and denoting by $[\cdot,\cdot]$ the duality pairing between X' and X, we conclude that the fully mixed variational formulation of the coupled problem (2.7), which consists of (3.4), (3.5), (3.8), and (3.9), can be stated as: Find $(\vec{\chi}, \boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$ and $(\vec{\mathbf{t}}, \boldsymbol{\eta}) \in \mathrm{X} \times \mathrm{Y}$ such that

$$\begin{aligned}
\mathbf{A}_{\mathbf{u}}(\vec{\chi}, \vec{\boldsymbol{\vartheta}}) + \mathbf{B}(\vec{\boldsymbol{\vartheta}}, \boldsymbol{\sigma}) &= \mathbf{F}_\phi(\vec{\boldsymbol{\vartheta}}) & \forall\, \vec{\boldsymbol{\vartheta}} \in \mathbf{H}, \\
\mathbf{B}(\vec{\chi}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \mathbf{Q}, \\
[\mathbf{a}_{\mathbf{u}}(\vec{\mathbf{t}}), \vec{\mathbf{s}}] + \mathbf{b}(\vec{\mathbf{s}}, \boldsymbol{\eta}) &= \mathrm{F}_\phi(\vec{\mathbf{s}}) & \forall\, \vec{\mathbf{s}} \in \mathrm{X}, \\
\mathbf{b}(\vec{\mathbf{t}}, \boldsymbol{\xi}) &= \mathrm{G}(\boldsymbol{\xi}) & \forall\, \boldsymbol{\xi} \in \mathrm{Y},
\end{aligned} \tag{3.10}$$

where the bilinear forms $\mathbf{A}_{\mathbf{z}} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$, for each $\mathbf{z} \in \mathbf{H}_2$, and $\mathbf{B} : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$, and the linear functionals $\mathbf{F}_\psi : \mathbf{H} \to \mathrm{R}$, for each $\psi \in \mathrm{X}_2$, and $\mathbf{G} : \mathbf{Q} \to \mathrm{R}$, are defined as

$$\mathbf{A}_{\mathbf{z}}(\vec{\boldsymbol{\varrho}}, \vec{\boldsymbol{\vartheta}}) := \mu \int_\Omega \boldsymbol{\varrho} : \boldsymbol{\vartheta} + \frac{1}{2} \int_\Omega \Big(\boldsymbol{\varrho}\,\mathbf{z} \cdot \mathbf{v} - (\mathbf{w} \otimes \mathbf{z}) : \boldsymbol{\vartheta} \Big) + \mathtt{D} \int_\Omega \mathbf{w} \cdot \mathbf{v} + \mathtt{F} \int_\Omega |\mathbf{z}|^{\rho - 2} \mathbf{w} \cdot \mathbf{v}, \tag{3.11}$$

$$\mathbf{B}(\vec{\boldsymbol{\vartheta}}, \boldsymbol{\tau}) := -\int_\Omega \boldsymbol{\tau} : \boldsymbol{\vartheta} - \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}),$$

$$\mathbf{F}_\psi(\vec{\boldsymbol{\vartheta}}) := -\int_\Omega \psi\, \mathbf{f} \cdot \mathbf{v}, \quad \text{and} \quad \mathbf{G}(\boldsymbol{\tau}) := -\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_{\mathrm{D}} \rangle,$$

for all $\vec{\boldsymbol{\varrho}}$, $\vec{\boldsymbol{\vartheta}} \in \mathbf{H}$, for all $\boldsymbol{\tau} \in \mathbf{Q}$, whereas the nonlinear operator $\mathbf{a}_{\mathbf{z}} : \mathrm{X} \to \mathrm{X}'$, for each $\mathbf{z} \in \mathbf{H}_2$, the bilinear form $\mathbf{b} : \mathrm{X} \times \mathrm{Y} \to \mathrm{R}$, and the functionals $\mathrm{F}_\psi : \mathrm{X} \to \mathrm{R}$, for each $\psi \in \mathrm{X}_2$, and $\mathrm{G} : \mathrm{Y} \to \mathrm{R}$, are given by

$$[\mathbf{a}_{\mathbf{z}}(\vec{\mathbf{r}}), \vec{\mathbf{s}}] := \int_\Omega \kappa(|\mathbf{r}|)\mathbf{r} \cdot \mathbf{s} - \int_\Omega \psi\, \mathbf{z} \cdot \mathbf{s}, \qquad \mathbf{b}(\vec{\mathbf{s}}, \boldsymbol{\xi}) := -\int_\Omega \boldsymbol{\xi} \cdot \mathbf{s} - \int_\Omega \varphi\, \mathrm{div}(\boldsymbol{\xi}), \tag{3.12}$$

$$\mathrm{F}_\psi(\vec{\mathbf{s}}) := -\int_\Omega g\, \varphi + \int_\Omega f(\psi)\mathbf{g} \cdot \mathbf{s}, \quad \text{and} \quad \mathrm{G}(\boldsymbol{\xi}) := -\langle \boldsymbol{\xi} \cdot \boldsymbol{\nu}, \phi_{\mathrm{D}} \rangle, \tag{3.13}$$

for all $\vec{\mathbf{r}}$, $\vec{\mathbf{s}} \in \mathrm{X}$, for all $\boldsymbol{\xi} \in \mathrm{Y}$.

3.2 A fixed-point strategy

We now employ a fixed-point approach to reformulate (3.10). Indeed, we first let $\mathbf{S} : \mathbf{H}_2 \times \mathrm{X}_2 \to \mathbf{H}_2$ be the operator defined by

$$\mathbf{S}(\mathbf{z}, \psi) := \underset{\sim}{\mathbf{u}} \qquad \forall (\mathbf{z}, \psi) \in \mathbf{H}_2 \times \mathrm{X}_2,$$

where $(\underset{\sim}{\vec{\chi}}, \underset{\sim}{\boldsymbol{\sigma}}) := \big((\underset{\sim}{\boldsymbol{\chi}}, \underset{\sim}{\mathbf{u}}), \underset{\sim}{\boldsymbol{\sigma}}\big) \in (\mathbf{H}_1 \times \mathbf{H}_2) \times \mathbf{Q}$ is the unique solution of the problem arising from the first two rows of (3.10) after replacing $\mathbf{A}_{\mathbf{u}}$ and $\mathbf{F}_\phi$ by $\mathbf{A}_{\mathbf{z}}$ and $\mathbf{F}_\psi$, respectively, that is

$$\begin{aligned}
\mathbf{A}_{\mathbf{z}}\big(\underset{\sim}{\vec{\chi}}, \vec{\boldsymbol{\vartheta}}\big) + \mathbf{B}(\vec{\boldsymbol{\vartheta}}, \underset{\sim}{\boldsymbol{\sigma}}) &= \mathbf{F}_\psi(\vec{\boldsymbol{\vartheta}}) & \forall\, \vec{\boldsymbol{\vartheta}} \in \mathbf{H}, \\
\mathbf{B}\big(\underset{\sim}{\vec{\chi}}, \boldsymbol{\tau}\big) &= \mathbf{G}(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \mathbf{Q}.
\end{aligned} \tag{3.14}$$

Similarly, we let $\mathbf{T} : \mathbf{H}_2 \times \mathrm{X}_2 \to \mathrm{X}_2$ be the operator defined by

$$\mathbf{T}(\mathbf{z}, \psi) := \underset{\sim}{\phi} \qquad \forall (\mathbf{z}, \psi) \in \mathbf{H}_2 \times \mathrm{X}_2,$$

where $(\vec{\underset{\sim}{\mathrm{t}}}, \underset{\sim}{\boldsymbol{\eta}}) := ((\underset{\sim}{\mathrm{t}}, \underset{\sim}{\phi}), \underset{\sim}{\boldsymbol{\eta}}) \in (\mathrm{X}_1 \times \mathrm{X}_2) \times \mathrm{Y}$ is the unique solution of the problem arising from the last two rows of (3.10) after replacing $\mathbf{a_u}$ and F_ϕ by $\mathbf{a_z}$ and F_ψ, respectively, that is

$$\begin{aligned} \big[\mathbf{a_z}(\vec{\underset{\sim}{\mathrm{t}}}), \vec{\mathrm{s}}\big] + \mathbf{b}(\vec{\mathrm{s}}, \underset{\sim}{\boldsymbol{\eta}}) &= \mathrm{F}_\psi(\vec{\mathrm{s}}) \qquad \forall\, \vec{\mathrm{s}} \in \mathrm{X}\,, \\ \mathbf{b}(\vec{\underset{\sim}{\mathrm{t}}}, \boldsymbol{\xi}) &= \mathrm{G}(\boldsymbol{\xi}) \qquad \forall\, \boldsymbol{\xi} \in \mathrm{Y}\,. \end{aligned} \tag{3.15}$$

Finally, we define the operator $\Theta : \mathbf{H}_2 \times \mathrm{X}_2 \to \mathbf{H}_2 \times \mathrm{X}_2$ by

$$\Theta(\mathbf{z}, \psi) := \big(\mathbf{S}(\mathbf{z}, \psi), \mathbf{T}(\mathbf{z}, \psi)\big) \qquad \forall\, (\mathbf{z}, \psi) \in \mathbf{H}_2 \times \mathrm{X}_2, \tag{3.16}$$

and see that solving (3.10) is equivalent to finding a fixed point of Θ, that is $(\mathbf{u}, \phi) \in \mathbf{H}_2 \times \mathrm{X}_2$ such that

$$\Theta(\mathbf{u}, \phi) = (\mathbf{u}, \phi). \tag{3.17}$$

We stress that, as an alternative to the definition adopted for the operator Θ in (3.16), and similarly to [6], we can consider either $\Theta(\mathbf{z}, \psi) := \mathbf{T}(\mathbf{S}(\mathbf{z}, \psi), \psi)$ or $\Theta(\mathbf{z}, \psi) := \mathbf{S}(\mathbf{z}, \mathbf{T}(\mathbf{z}, \psi))$. Both definitions lead to a well-defined fixed-point approach, and the analysis developed in the next section remains valid with slight modifications to Lemma 3.10 and Theorem 3.13.

In the next section, we prove that $\mathbf{S}$ and $\mathbf{T}$ are well defined (equivalently, that (3.14) and (3.15) are uniquely solvable), and, as a consequence, that Θ is well defined. Finally, we prove that, under suitable assumptions on the data, Θ has a unique fixed-point.

3.3 Well-posedness of the uncoupled problems

In what follows we address the solvability analysis of the uncoupled problems (3.14) and (3.15). We begin with the stability properties of the previously defined bilinear forms and functionals. In fact, applying the Cauchy–Schwarz and Hölder inequalities, and employing the boundedness of the normal trace operators $\boldsymbol{\gamma_\nu} : \mathbb{H}(\mathbf{div}_{4/3}; \Omega) \to \mathbf{H}^{-1/2}(\Gamma)$ and $\gamma_{\boldsymbol{\nu}} : \mathbf{H}(\mathrm{div}_{4/3}; \Omega) \to \mathrm{H}^{-1/2}(\Gamma)$ along with corresponding duality parings, we deduce that, for each $\mathbf{z} \in \mathbf{H}_2$, $\psi \in \mathrm{X}_2$, $\vec{\boldsymbol{\vartheta}}, \vec{\boldsymbol{\varrho}} \in \mathbf{H}$, $\boldsymbol{\tau} \in \mathbf{Q}$, $\vec{\mathrm{s}} \in \mathrm{X}$, and $\boldsymbol{\xi} \in \mathrm{Y}$, there hold:

$$\begin{gathered} |\mathbf{A_z}(\vec{\boldsymbol{\varrho}}, \vec{\boldsymbol{\vartheta}})| \le \|\mathbf{A_z}\| \, \|\vec{\boldsymbol{\varrho}}\|_{\mathbf{H}} \, \|\vec{\boldsymbol{\vartheta}}\|_{\mathbf{H}}\,, \qquad |\mathbf{B}(\vec{\boldsymbol{\vartheta}}, \boldsymbol{\tau})| \le \|\mathbf{B}\| \, \|\vec{\boldsymbol{\vartheta}}\|_{\mathbf{H}} \, \|\boldsymbol{\tau}\|_{\mathbf{Q}}\,, \\ |\mathbf{F}_\psi(\vec{\boldsymbol{\vartheta}})| \le \|\mathbf{F}_\psi\| \, \|\vec{\boldsymbol{\vartheta}}\|_{\mathbf{H}}\,, \qquad |\mathbf{G}(\boldsymbol{\tau})| \le \|\mathbf{G}\| \, \|\boldsymbol{\tau}\|_{\mathbf{Q}}\,, \\ |\mathbf{b}(\vec{\mathrm{s}}, \boldsymbol{\xi})| \le \|\mathbf{b}\| \, \|\vec{\mathrm{s}}\|_{\mathrm{X}} \, \|\boldsymbol{\xi}\|_{\mathrm{Y}}\,, \qquad |\mathrm{F}_\psi(\vec{\mathrm{s}})| \le \|\mathrm{F}_\psi\| \, \|\vec{\mathrm{s}}\|_{\mathrm{X}}\,, \\ \text{and} \qquad |\mathrm{G}(\boldsymbol{\xi})| \le \|\mathrm{G}\| \, \|\boldsymbol{\xi}\|_{\mathrm{Y}}\,, \end{gathered} \tag{3.18}$$

where

$$\begin{gathered} \|\mathbf{A_z}\| := \mu + \|\mathbf{z}\|_{0,4;\Omega} + \mathtt{D}\, |\Omega|^{1/2} + \mathtt{F}\, |\Omega|^{(4-\rho)/4} \, \|\mathbf{z}\|_{0,4;\Omega}^{\rho-2}\,, \qquad \|\mathbf{B}\| := 1, \\ \|\mathbf{F}_\psi\| := \|\psi\|_{0,4;\Omega} \, \|\mathbf{f}\|_{0,\Omega}\,, \qquad \|\mathbf{G}\| := \|\boldsymbol{\gamma_\nu}\| \, \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\,, \\ \|\mathbf{b}\| := 1\,, \qquad \|\mathrm{F}_\psi\| := \|g\|_{0,4/3;\Omega} + f_2\, |\Omega|^{1/4} \, \|\mathbf{g}\|_{0,4;\Omega}\,, \\ \text{and} \qquad \|\mathrm{G}\| := \|\gamma_{\boldsymbol{\nu}}\| \, \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\,. \end{gathered} \tag{3.19}$$

For the well-posedness of (3.14) we need to recall next the Babuška–Brezzi theorem in Banach spaces, whose proof can be found, among several other places, in [22, Theorem 2.34].

Theorem 3.1 *Let H and Q be reflexive Banach spaces, and let $a : H \times H \to \mathrm{R}$ and $b : H \times Q \to \mathrm{R}$ be bounded bilinear forms with induced operators $A \in \mathcal{L}(H, H')$ and $B \in \mathcal{L}(H, Q')$, respectively. In addition, let V be the null space of B, and assume that*

i) *there exists $\alpha > 0$ such that*

$$\sup_{\substack{v \in V \\ v \neq 0}} \frac{a(u,v)}{\|v\|_H} \geq \alpha \|u\|_H \qquad \forall u \in V\,,$$

ii) *there holds*

$$\sup_{u \in V} a(u,v) > 0 \qquad \forall\, v \in V, \quad v \neq 0\,,$$

iii) *there exists β such that*

$$\sup_{\substack{v \in H \\ v \neq 0}} \frac{b(v,\tau)}{\|v\|_H} \geq \beta \|\tau\|_Q \qquad \forall \tau \in Q.$$

Then, there exists a unique $(u, \sigma) \in H \times Q$ such that

$$\begin{aligned} a(u,v) + b(v,\sigma) &= F(v) \quad \forall\, v \in H, \\ b(v,\tau) &= G(\tau) \quad \forall\, \tau \in Q\,, \end{aligned}$$

and the following a priori estimates hold:

$$\|u\| \leq \frac{1}{\alpha}\|F\| + \frac{1}{\beta}\left(1 + \frac{\|A\|}{\alpha}\right)\|G\|\,, \quad \textit{and} \tag{3.20}$$

$$\|\sigma\| \leq \frac{1}{\beta}\left(1 + \frac{\|A\|}{\alpha}\right)\|F\| + \frac{\|A\|}{\beta^2}\left(1 + \frac{\|A\|}{\alpha}\right)\|G\|\,. \tag{3.21}$$

We remark here that if a is V-elliptic, that is, if there exists a positive constant α such that there holds $a(v,v) \geq \alpha\,\|v\|^2 \quad \forall\, v \in V$, then the first two hypotheses of Theorem 3.1 are satisfied straightforwardly with the same constant α. In particular, regarding (3.14), we notice that the null space of the linear and bounded operator induced by $\mathbf{B}$ is given by

$$\mathcal{V} := \left\{ (\boldsymbol{\vartheta}, \mathbf{v}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{L}^4(\Omega) : \quad \int_\Omega \boldsymbol{\tau} : \boldsymbol{\vartheta} + \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) = 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \right\}\,,$$

from which we easily deduce that

$$\mathcal{V} = \left\{ (\boldsymbol{\vartheta}, \mathbf{v}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{L}^4(\Omega) : \quad \boldsymbol{\vartheta} = \nabla \mathbf{v} \quad \text{and} \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega) \right\}. \tag{3.22}$$

In the next lemma we prove precisely that the family $\left\{\mathbf{A}_{\mathbf{z}}\right\}_{\mathbf{z} \in \mathbf{H}_2}$ is uniformly $\mathcal{V}$-elliptic.

Lemma 3.2 *There exists a positive constant α, depending only on μ, $\mathtt{D}$ and $|\Omega|$, such that for each $\mathbf{z} \in \mathbf{H}_2$ there holds*

$$\mathbf{A}_{\mathbf{z}}(\vec{\boldsymbol{\vartheta}}, \vec{\boldsymbol{\vartheta}}) \geq \alpha\,\|\vec{\boldsymbol{\vartheta}}\|^2_{\mathbf{H}} \qquad \forall\, \vec{\boldsymbol{\vartheta}} \in \mathcal{V}\,. \tag{3.23}$$

Proof. Given $\mathbf{z} \in \mathbf{H}_2$ and $\vec{\boldsymbol{\vartheta}} := (\boldsymbol{\vartheta}, \mathbf{v}) \in \mathcal{V}$, it follows from (3.11) that

$$\mathbf{A}_{\mathbf{z}}(\vec{\boldsymbol{\vartheta}}, \vec{\boldsymbol{\vartheta}}) \,=\, \mu\,\|\boldsymbol{\vartheta}\|_{0,\Omega}^2 + \frac{1}{2}\int_\Omega \Big(\boldsymbol{\vartheta}\,\mathbf{z}\cdot\mathbf{v} - (\mathbf{z}\otimes\mathbf{v}) : \boldsymbol{\vartheta}\Big) \,+\, \mathtt{D}\,\|\mathbf{v}\|_{0,\Omega}^2 \,+\, \mathtt{F}\int_\Omega |\mathbf{z}|^{\rho-2}\,|\mathbf{v}|^2\,,$$

from which, noting that $\boldsymbol{\vartheta}\,\mathbf{z}\cdot\mathbf{v} - (\mathbf{z}\otimes\mathbf{v}) : \boldsymbol{\vartheta} \,=\, 0$, the last term in positive, and using that $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and $\boldsymbol{\vartheta} = \nabla\mathbf{v}$, we get

$$\begin{aligned}
\mathbf{A}_{\mathbf{z}}(\vec{\boldsymbol{\vartheta}}, \vec{\boldsymbol{\vartheta}}) \,&=\, \mu\,\|\boldsymbol{\vartheta}\|_{0,\Omega}^2 \,+\, \mathtt{D}\,\|\mathbf{v}\|_{0,\Omega}^2 \,+\, \mathtt{F}\int_\Omega |\mathbf{z}|^{\rho-2}\,|\mathbf{v}|^2 \,\ge\, \mu\,\|\boldsymbol{\vartheta}\|_{0,\Omega}^2 \,+\, \mathtt{D}\,\|\mathbf{v}\|_{0,\Omega}^2 \\
&=\, \frac{\mu}{2}\,\|\boldsymbol{\vartheta}\|_{0,\Omega}^2 \,+\, \frac{\mu}{2}\,\|\nabla\mathbf{v}\|_{0,\Omega}^2 \,+\, \mathtt{D}\|\mathbf{v}\|_{0,\Omega}^2 \,\ge\, \min\Big\{\frac{\mu}{2}, \mathtt{D}\Big\}\,\Big(\|\boldsymbol{\vartheta}\|_{0,\Omega}^2 \,+\, \|\mathbf{v}\|_{1,\Omega}^2\Big)\,.
\end{aligned}$$

Then, invoking the continuous injection $\boldsymbol{i}_{\mathbf{H},4} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, whose norm depends on $|\Omega|$ (cf. (1.2)), we arrive at the required inequality (3.23) with $\alpha \,:=\, \dfrac{1}{2}\,\min\Big\{\dfrac{\mu}{2}, \mathtt{D}\Big\}\,\min\big\{1, \|\boldsymbol{i}_{\mathbf{H},4}\|^{-2}\big\}$. □

Next, we recall that a slight modification of the proof of [23, Lemma 2.3] allows to show that for each $t \in \begin{cases} (1,+\infty) & \text{if } n = 2 \\ [6/5,+\infty) & \text{if } n = 3 \end{cases}$, there exists a positive constant C_t, depending only on Ω, such that

$$C_t\,\|\boldsymbol{\tau}\|_{0,\Omega}^2 \,\le\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega}^2 \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_t;\Omega)\,. \tag{3.24}$$

Then, the inf-sup condition for $\mathbf{B}$, originally proved in [20] (cf. [20, Lemma 3.3, eq. (3.44)]), is established as follows (see also [11, eq. (3.16)]).

Lemma 3.3 *There exists a positive constant β, depending only on $C_{4/3}$ (cf. (3.24)), such that*

$$\sup_{\substack{\vec{\boldsymbol{\vartheta}}\in\mathbf{H}\\ \vec{\boldsymbol{\vartheta}}\neq\mathbf{0}}} \frac{\mathbf{B}(\vec{\boldsymbol{\vartheta}}, \boldsymbol{\tau})}{\|\vec{\boldsymbol{\vartheta}}\|_{\mathbf{H}}} \,\ge\, \beta\,\|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\,\boldsymbol{\tau} \in \mathbf{Q}\,.$$

We are now in position to show that the operator $\mathbf{S}$ is well defined.

Lemma 3.4 *For each $(\mathbf{z},\psi) \in \mathbf{H}_2 \times \mathrm{X}_2$ there exists a unique $(\vec{\underset{\sim}{\boldsymbol{\chi}}}, \underset{\sim}{\boldsymbol{\sigma}}) \,:=\, \big((\underset{\sim}{\boldsymbol{\chi}}, \underset{\sim}{\mathbf{u}}), \underset{\sim}{\boldsymbol{\sigma}}\big) \in (\mathbf{H}_1 \times \mathbf{H}_2) \times \mathbf{Q}$ solution of (3.14), and hence one can define $\mathbf{S}(\mathbf{z},\psi) \,:=\, \underset{\sim}{\mathbf{u}} \in \mathbf{H}_2$. Moreover, there exists a positive constant C_{BF}, depending only on α, β, $\|\mathbf{A}_{\mathbf{z}}\|$, and $\|\boldsymbol{\gamma}_{\boldsymbol{\nu}}\|$, such that*

$$\|\mathbf{S}(\mathbf{z},\psi)\| \,\le\, C_{\mathrm{BF}}\,\Big\{\|\psi\|_{0,4;\Omega}\,\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,. \tag{3.25}$$

Proof. Since $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbf{L}^4(\Omega)$, and $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ are clearly reflexive Banach spaces, the well-posedness of (3.14) follows from Lemmas 3.2 and 3.3, and a straightforward application of Theorem 3.1. In particular, the a priori estimate (3.20) yields

$$\|\mathbf{S}(\mathbf{z},\psi)\| \,=\, \|\underset{\sim}{\mathbf{u}}\|_{0,4;\Omega} \,\le\, \|\vec{\underset{\sim}{\boldsymbol{\chi}}}\|_{\mathbf{H}} \,\le\, \frac{1}{\alpha}\,\|\mathbf{F}_\psi\| \,+\, \frac{1}{\beta}\,\Big(1 + \frac{\|\mathbf{A}_{\mathbf{z}}\|}{\alpha}\Big)\,\|\mathbf{G}\|\,,$$

which, along with the bounds for $\|\mathbf{F}_\psi\|$ and $\|\mathbf{G}\|$ (cf. (3.19)), implies (3.25) and ends the proof. □

We remark here that for each $\mathbf{z} \in \mathbf{L}^4(\Omega)$ satisfying $\|\mathbf{z}\|_{0,4;\Omega} \,\le\, \delta$, with $\delta > 0$ given, and thanks to the corresponding bound in (3.19), $\|\mathbf{A}_{\mathbf{z}}\|$ can be bounded by a constant depending only on δ, μ, D, F, $|\Omega|$ and ρ. Hence, according to the dependence specified in Lemmas 3.2 and 3.3 regarding the constants α and β, we rephrase (3.25) by stating the existence of a positive constant C_{BF}, depending only on α, β, δ, μ, D, F, $|\Omega|$, ρ, $C_{4/3}$, and $\|\boldsymbol{\gamma}_{\boldsymbol{\nu}}\|$, such that

$$\|\mathbf{S}(\mathbf{z},\psi)\| \,=\, \|\underset{\sim}{\mathbf{u}}\|_{0,4;\Omega} \,\le\, \|\vec{\underset{\sim}{\boldsymbol{\chi}}}\|_{\mathbf{H}} \,\le\, C_{\mathrm{BF}}\left\{\|\psi\|_{0,4;\Omega}\,\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right\} \tag{3.26}$$

for all $(\mathbf{z},\psi)\in\mathbf{H}_2\times\mathrm{X}_2$ such that $\|\mathbf{z}\|_{0,4;\Omega}\,\le\,\delta$. Moreover, according to the a priori estimate (3.21), the second component of the solution of (3.14) is bounded as

$$\|\underset{\sim}{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \,\le\, \widehat{C}_{\mathrm{BF}}\left\{\|\psi\|_{0,4;\Omega}\,\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right\}, \tag{3.27}$$

where $\widehat{C}_{\mathrm{BF}}$ is a positive constant having a similar dependence to that of C_{BF}.

We now address the well-posedness of the uncoupled transport problem (3.15). Due to the nonlinear character of the operator $\mathbf{a}_{\mathbf{z}}$, instead of the Babuška–Brezzi theory in Banach spaces (cf. Theorem 3.1), we now need to consider the abstract result given by the following theorem.

Theorem 3.5 *Let* X *and* Y *be separable and reflexive Banach spaces, with* X *being uniformly convex, and denote by* $[\,\cdot,\cdot\,]$ *the duality paring between* X' *and* X*. In addition, let* $a:\mathrm{X}\to\mathrm{X}'$ *be a nonlinear operator and* $b\in\mathcal{L}(\mathrm{X},\mathrm{Y}')$*. In turn, let* V *be the null space of* b*, and assume that*

i) *a is Lipschitz continuous, that is there exists a positive constant L such that*

$$\|a(v)-a(w)\|_{\mathrm{X}'} \,\le\, L\,\|v-w\|_{\mathrm{X}} \qquad \forall\, v,\, w\in\mathrm{X}\,,$$

ii) *the family of operators $a(\cdot+z):\mathrm{V}\to\mathrm{V}'$, with $z\in\mathrm{X}$, is uniformly strongly monotone, that is there exists a positive constant $\widetilde{\alpha}$ such that*

$$[a(v+z)-a(w+z), v-w] \,\ge\, \widetilde{\alpha}\,\|v-w\|_{\mathrm{X}}^2 \qquad \forall\, z\in\mathrm{X}, \quad \forall\, v,\, w\in\mathrm{V}\,,$$

iii) *there exists a positive constant $\widetilde{\beta}$ such that*

$$\sup_{\substack{v\in\mathrm{X}\\ v\neq 0}} \frac{[b(v),\xi]}{\|v\|_{\mathrm{X}}} \,\ge\, \widetilde{\beta}\,\|\xi\|_{\mathrm{Y}} \qquad \forall\,\xi\in\mathrm{Y}\,.$$

Then, for each $(\mathrm{F},\mathrm{G})\in\mathrm{X}'\times\mathrm{Y}'$ there exists a unique pair $(u,\eta)\in\mathrm{X}\times\mathrm{Y}$ such that

$$\begin{aligned} [a(u),v]+b(v,\eta) \;&=\; \mathrm{F}(v) \qquad \forall\, v\in\mathrm{X}\,,\\ b(u,\xi) \;&=\; \mathrm{G}(\xi) \qquad \forall\,\xi\in\mathrm{Y}\,. \end{aligned}$$

Moreover, there hold

$$\|u\|_{\mathrm{X}} \,\le\, \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}\|_{\mathrm{X}'} \,+\, \frac{1}{\widetilde{\beta}}\left(1+\frac{L}{\widetilde{\alpha}}\right)\|\mathrm{G}\|_{\mathrm{Y}'} \,+\, \frac{1}{\widetilde{\alpha}}\,\|a(0)\|_{\mathrm{X}'}\,, \quad \textit{and} \tag{3.28}$$

$$\|\eta\|_{\mathrm{Y}} \,\le\, \frac{1}{\widetilde{\beta}}\left(1+\frac{L}{\widetilde{\alpha}}\right)\|\mathrm{F}\|_{\mathrm{X}'} \,+\, \frac{L}{\widetilde{\beta}^2}\left(1+\frac{L}{\widetilde{\alpha}}\right)\|\mathrm{G}\|_{\mathrm{Y}'} \,+\, \frac{1}{\widetilde{\beta}}\left(1+\frac{L}{\widetilde{\alpha}}\right)\|a(0)\|_{\mathrm{X}'}\,. \tag{3.29}$$

Proof. It reduces to the particular case arising from [12, Theorem 3.1] when the corresponding index there is taken as $p = 2$. Further details are omitted. □

In what follows we apply Theorem 3.5 to the context given by (3.15). We begin by noticing that the uniform convexity and separability of the L^t spaces, with $t \in (1, +\infty)$, imply the same properties on the spaces involved, thus guaranteeing that the corresponding assumptions required by Theorem 3.5, are satisfied. Next, regarding the nonlinear part of $\mathbf{a_z}$ (cf. first term of the respective definition in (3.12)), and bearing in mind the hypotheses on κ (cf. (2.2)), we recall from the proof of [26, Theorem 3.8] that, denoting $L_\kappa := \max\big\{\kappa_2, 2\kappa_2 - \kappa_1\big\} = 2\kappa_2 - \kappa_1$, there hold

$$\int_\Omega \Big(\kappa(|\mathbf{r}|)\,\mathbf{r} - \kappa(|\mathbf{s}|)\,\mathbf{s}\Big)\cdot\mathbf{t} \,\le\, L_\kappa\,\|\mathbf{r}-\mathbf{s}\|_{X_1}\,\|\mathbf{t}\|_{X_1} \quad \forall\,\mathbf{r},\,\mathbf{s},\,\mathbf{t}\in X_1\,, \tag{3.30}$$

and

$$\int_\Omega \Big(\kappa(|\mathbf{r}|)\,\mathbf{r} \,-\, \kappa(|\mathbf{s}|)\,\mathbf{s}\Big)\cdot(\mathbf{r}-\mathbf{s}) \,\ge\, \kappa_1\,\|\mathbf{r}-\mathbf{s}\|^2_{X_1} \quad \forall\,\mathbf{r},\,\mathbf{s}\in X_1\,. \tag{3.31}$$

On the other hand, we now let $\mathbf{V}$ be the null space of the operator induced by $\mathbf{b}$, which, similarly as for the derivation of (3.22), becomes

$$\mathbf{V} \,:=\, \Big\{\vec{\mathbf{s}} := (\mathbf{s},\varphi)\in\mathbf{L}^2(\Omega)\times L^4(\Omega):\quad \varphi\in H^1_0(\Omega)\quad\text{and}\quad \mathbf{s} = \nabla\varphi\Big\}\,. \tag{3.32}$$

Also, we recall the well-known Poincaré inequality (see, for instance, [29, Theorem 5.11.2]), which guarantees the existence of a positive constant c_p, depending only on Ω, such that

$$\|\varphi\|_{1,\Omega} \,\le\, c_p\,|\varphi|_{1,\Omega} \qquad \forall\,\varphi\in H^1_0(\Omega)\,. \tag{3.33}$$

We are now ready to establish the Lipschitz-continuity and strong monotonicity properties of $\mathbf{a_z}$, as required by hypotheses i) and ii) of Theorem 3.5. Indeed, we have the following results.

Lemma 3.6 *For each $\mathbf{z}\in\mathbf{H}_2$ there exists a positive constant $L_{\mathbf{z}}$, depending only on κ_1, κ_2, and $\|\mathbf{z}\|_{0,4;\Omega}$, such that*

$$\|\mathbf{a_z}(\vec{\mathbf{r}}) - \mathbf{a_z}(\vec{\mathbf{s}})\|_{X'} \,\le\, L_{\mathbf{z}}\,\|\vec{\mathbf{r}}-\vec{\mathbf{s}}\|_X \quad \forall\,\vec{\mathbf{r}},\,\vec{\mathbf{s}}\in X\,. \tag{3.34}$$

Proof. Given $\mathbf{z}\in\mathbf{H}_2$, and $\vec{\mathbf{r}} := (\mathbf{r},\psi)$, $\vec{\mathbf{s}} := (\mathbf{s},\varphi)$, $\vec{\mathbf{t}} := (\mathbf{t},\phi)\in X$, we invoke the definition of $\mathbf{a_z}$ (cf. (3.12)) to obtain

$$[\,\mathbf{a_z}(\vec{\mathbf{r}}) - \mathbf{a_z}(\vec{\mathbf{s}}),\vec{\mathbf{t}}\,] \,=\, \int_\Omega \Big(\kappa(|\mathbf{r}|)\,\mathbf{r} - \kappa(|\mathbf{s}|)\,\mathbf{s}\Big)\cdot\mathbf{t} \,-\, \int_\Omega (\psi-\varphi)\,\mathbf{z}\cdot\mathbf{t}\,. \tag{3.35}$$

Then, using (3.30) and applying Cauchy–Schwarz's inequality twice, it follows from (3.35) that

$$\begin{aligned}
[\,\mathbf{a_z}(\vec{\mathbf{r}}) - \mathbf{a_z}(\vec{\mathbf{s}}),\vec{\mathbf{t}}\,] \,&\le\, L_\kappa\,\|\mathbf{r}-\mathbf{s}\|_{X_1}\,\|\mathbf{t}\|_{X_1} \,+\, \|\mathbf{z}\|_{\mathbf{H}_2}\,\|\psi-\varphi\|_{X_2}\,\|\mathbf{t}\|_{X_1}\\
&\le\, \Big\{L_\kappa\,\|\mathbf{r}-\mathbf{s}\|_{X_1} \,+\, \|\mathbf{z}\|_{\mathbf{H}_2}\,\|\psi-\varphi\|_{X_2}\Big\}\,\|\vec{\mathbf{t}}\|_X\,,
\end{aligned}$$

which readily yields (3.34) with $L_{\mathbf{z}} := \max\big\{L_\kappa, \|\mathbf{z}\|_{\mathbf{H}_2}\big\}$. □

Lemma 3.7 *There exists a positive constant $\widetilde{\alpha}$, depending only on κ_1, c_p, and $\|i_{H,4}\|$ (cf. (1.2)), such that, for each $\mathbf{z}\in\mathbf{H}_2$ satisfying $\|\mathbf{z}\|_{\mathbf{H}_2} \le \frac{\kappa_1}{2\,\|i_{H,4}\|\,c_p}$, there holds*

$$[\,\mathbf{a_z}(\vec{\mathbf{r}}+\vec{\mathbf{k}}) - \mathbf{a_z}(\vec{\mathbf{s}}+\vec{\mathbf{k}}),\vec{\mathbf{r}}-\vec{\mathbf{s}}\,] \,\ge\, \widetilde{\alpha}\,\|\vec{\mathbf{r}}-\vec{\mathbf{s}}\|^2_X \qquad \forall\,\vec{\mathbf{k}}\in X\,,\quad \forall\,\vec{\mathbf{r}},\,\vec{\mathbf{s}}\in\mathbf{V}\,. \tag{3.36}$$

Proof. Given $\mathbf{z} \in \mathbf{H}_2$, $\vec{\mathbf{k}} := (\mathbf{k}, \theta) \in \mathrm{X}$, and $\vec{\mathbf{r}} := (\mathbf{r}, \psi)$, $\vec{\mathbf{s}} := (\mathbf{s}, \varphi) \in \mathbf{V}$, we get from the definition of $\mathbf{a}_{\mathbf{z}}$ (cf. (3.12)) that

$$[\,\mathbf{a}_{\mathbf{z}}(\vec{\mathbf{r}} + \vec{\mathbf{k}}) - \mathbf{a}_{\mathbf{z}}(\vec{\mathbf{s}} + \vec{\mathbf{k}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}\,] \,=\, [\,\mathbf{a}_{\mathbf{z}}(\vec{\mathbf{r}} + \vec{\mathbf{k}}) - \mathbf{a}_{\mathbf{z}}(\vec{\mathbf{s}} + \vec{\mathbf{k}}), (\vec{\mathbf{r}} + \vec{\mathbf{k}}) - (\vec{\mathbf{s}} + \vec{\mathbf{k}})\,]$$

$$= \int_\Omega \Big(\kappa(|\mathbf{r}+\mathbf{k}|)\,(\mathbf{r}+\mathbf{k}) - \kappa(|\mathbf{s}+\mathbf{k}|)\,(\mathbf{s}+\mathbf{k})\Big) \cdot \big((\mathbf{r}+\mathbf{k}) - (\mathbf{s}+\mathbf{k})\big) \,-\, \int_\Omega (\psi - \varphi)\,\mathbf{z} \cdot (\mathbf{r} - \mathbf{s})\,,$$

which, employing (3.31) and Cauchy–Schwarz's inequality, yields

$$[\,\mathbf{a}_{\mathbf{z}}(\vec{\mathbf{r}} + \vec{\mathbf{k}}) - \mathbf{a}_{\mathbf{z}}(\vec{\mathbf{s}} + \vec{\mathbf{k}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}\,] \,\geq\, \kappa_1\, \|\mathbf{r} - \mathbf{s}\|^2_{0,\Omega} \,-\, \|\mathbf{z}\|_{0,4;\Omega}\, \|\psi - \varphi\|_{0,4;\Omega}\, \|\mathbf{r} - \mathbf{s}\|_{0,\Omega}\,.$$

Then, using from the characterization of $\mathbf{V}$ (cf. (3.32)) that $\mathbf{r} = \nabla\psi$ and $\mathbf{s} = \nabla\varphi$, with ψ, $\varphi \in \mathrm{H}^1_0(\Omega)$, and applying the continuous injection $i_{\mathrm{H},4} : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$ and (3.33), we deduce that

$$\|\psi - \varphi\|_{0,4;\Omega}\, \|\mathbf{r} - \mathbf{s}\|_{0,\Omega} \,=\, \|\psi - \varphi\|_{0,4;\Omega}\, |\psi - \varphi|_{1,\Omega} \,\leq\, \|i_{\mathrm{H},4}\|\, c_p\, |\psi - \varphi|^2_{1,\Omega} \,=\, \|i_{\mathrm{H},4}\|\, c_p\, \|\mathbf{r} - \mathbf{s}\|^2_{0,\Omega}\,,$$

which replaced back in the foregoing inequality implies

$$[\,\mathbf{a}_{\mathbf{z}}(\vec{\mathbf{r}} + \vec{\mathbf{k}}) - \mathbf{a}_{\mathbf{z}}(\vec{\mathbf{s}} + \vec{\mathbf{k}}), \vec{\mathbf{r}} - \vec{\mathbf{s}}\,] \,\geq\, \Big\{\kappa_1 \,-\, \|\mathbf{z}\|_{0,4;\Omega}\, \|i_{\mathrm{H},4}\|\, c_p\Big\}\, \|\mathbf{r} - \mathbf{s}\|^2_{0,\Omega} \,\geq\, \frac{\kappa_1}{2}\, \|\mathbf{r} - \mathbf{s}\|^2_{0,\Omega}$$

for each $\mathbf{z} \in \mathbf{H}_2$ such that $\|\mathbf{z}\|_{\mathbf{H}_2} \,\leq\, \frac{\kappa_1}{2\,\|i_{\mathrm{H},4}\|\, c_p}$. Finally, splitting $\|\mathbf{r} - \mathbf{s}\|^2_{0,\Omega}$ as $\frac{1}{2}\,\|\mathbf{r} - \mathbf{s}\|^2_{0,\Omega} + \frac{1}{2}\,|\psi - \varphi|^2_{1,\Omega}$, we readily arrive at (3.36) with $\widetilde{\alpha} \,:=\, \frac{\kappa_1}{4}\, \min\big\{1, c_p^{-2}\, \|i_{\mathrm{H},4}\|^{-2}\big\}$. □

The inf-sup condition for $\mathbf{b}$, originally stated in [20, Lemma 3.3, eq. (3.45)], is established next.

Lemma 3.8 *There exists a positive constant* $\widetilde{\beta}$ *such that*

$$\sup_{\substack{\vec{\mathbf{s}} \in \mathrm{X} \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{s}},\, \boldsymbol{\xi})}{\|\vec{\mathbf{s}}\|_{\mathrm{X}}} \,\geq\, \widetilde{\beta}\, \|\boldsymbol{\xi}\|_{\mathrm{Y}} \qquad \forall\, \boldsymbol{\xi} \in \mathrm{Y}\,. \tag{3.37}$$

Proof. While the main arguments of this proof are given in the aforementioned reference, for sake of completeness we provide all details in what follows. Indeed, given $\boldsymbol{\xi} \in \mathrm{Y} \,:=\, \mathbf{H}(\mathrm{div}_{4/3}; \Omega)$, we first let $\vec{\mathbf{r}} = (\mathbf{r}, \psi) := (-\boldsymbol{\xi}, 0) \in \mathrm{X} := \mathbf{L}^2(\Omega) \times \mathrm{L}^4(\Omega)$ and easily obtain

$$\sup_{\substack{\vec{\mathbf{s}} \in \mathrm{X} \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{s}},\, \boldsymbol{\xi})}{\|\vec{\mathbf{s}}\|_{\mathrm{X}}} \,\geq\, \frac{\mathbf{b}(\vec{\mathbf{r}},\, \boldsymbol{\xi})}{\|\vec{\mathbf{r}}\|_{\mathrm{X}}} \,=\, \|\boldsymbol{\xi}\|_{0,\Omega}\,,$$

whereas, letting $\vec{\mathbf{r}} = (\mathbf{r}, \psi) := \big(\mathbf{0}, -\big(\mathrm{div}(\boldsymbol{\xi})\big)^{1/3}\big) \in \mathrm{X} := \mathbf{L}^2(\Omega) \times \mathrm{L}^4(\Omega)$, we are led to

$$\sup_{\substack{\vec{\mathbf{s}} \in \mathrm{X} \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{s}},\, \boldsymbol{\xi})}{\|\vec{\mathbf{s}}\|_{\mathrm{X}}} \,\geq\, \frac{\mathbf{b}(\vec{\mathbf{r}},\, \boldsymbol{\xi})}{\|\vec{\mathbf{r}}\|_{\mathrm{X}}} \,=\, \frac{\|\mathrm{div}(\boldsymbol{\xi})\|^{4/3}_{0,4/3;\Omega}}{\|\mathrm{div}(\boldsymbol{\xi})\|^{1/3}_{0,4/3;\Omega}} \,=\, \|\mathrm{div}(\boldsymbol{\xi})\|_{0,4/3;\Omega}\,.$$

In this way, the above inequalities along with (1.1) yield (3.37) with $\widetilde{\beta} = 1/2$. □

The well-posedness of problem (3.15) can be stated now.

Lemma 3.9 *For each* $(\mathbf{z}, \psi) \in \mathbf{H}_2 \times \mathrm{X}_2$ *such that* $\|\mathbf{z}\|_{\mathbf{H}_2} \,\leq\, \frac{\kappa_1}{2\,\|i_{\mathrm{H},4}\|\, c_p}$*, there exists a unique solution* $(\vec{\underset{\sim}{\mathbf{t}}}, \underset{\sim}{\boldsymbol{\eta}}) \,:=\, ((\underset{\sim}{\mathbf{t}}, \underset{\sim}{\phi}), \underset{\sim}{\boldsymbol{\eta}}) \in (\mathrm{X}_1 \times \mathrm{X}_2) \times \mathrm{Y}$ *of* (3.15)*, and hence one can define* $\mathbf{T}(\mathbf{z}, \psi) \,:=\, \underset{\sim}{\phi} \in \mathrm{X}_2$*. Moreover, there exists a positive constant* C_{NT}*, depending only on* κ_1*,* κ_2*,* $\|i_{\mathrm{H},4}\|$*,* c_p*,* f_2*,* $|\Omega|$*, and* $\|\gamma_{\boldsymbol{\nu}}\|$*, such that*

$$\|\mathbf{T}(\mathbf{z}, \psi)\| \,=\, \|\underset{\sim}{\phi}\|_{0,4;\Omega} \,\leq\, \|\vec{\underset{\sim}{\mathbf{t}}}\|_{X} \,\leq\, C_{\mathrm{NT}} \Big\{\|g\|_{0,4/3;\,\Omega} \,+\, \|\mathbf{g}\|_{0,4;\,\Omega} \,+\, \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,. \tag{3.38}$$

Proof. Being the spaces $X = X_1 \times X_2 := \mathbf{L}^2(\Omega) \times L^4(\Omega)$ and $Y := \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ clearly reflexive, and X uniformly convex, and bearing in mind Lemmas 3.6 and 3.7, the well-posedness of (3.15) follows from a direct application of Theorem 3.5. In particular, noting from (3.12) that $\mathbf{a_z}(\vec{\mathbf{0}}) = \mathbf{0}$, the a priori estimate (3.28) yields

$$\|\mathbf{T}(\mathbf{z},\psi)\| \,=\, \|\underset{\sim}{\phi}\|_{0,4;\Omega} \,\leq\, \|\vec{\underset{\sim}{\mathbf{t}}}\|_X \,\leq\, \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}_\psi\|_{\mathrm{X}'} \,+\, \frac{1}{\widetilde{\beta}}\left(1+\frac{L_{\mathbf{z}}}{\widetilde{\alpha}}\right)\|\mathrm{G}\|_{\mathrm{Y}'}\,,$$

which, according to the upper bounds of $\|\mathrm{F}_\psi\|_{\mathrm{X}'}$ and $\|\mathrm{G}\|_{\mathrm{Y}'}$ (cf. (3.19)), and the fact that (see the end of the proof of Lemma 3.6)

$$L_{\mathbf{z}} \,:=\, \max\big\{L_\kappa, \|\mathbf{z}\|_{\mathbf{H}_2}\big\} \,\leq\, \max\left\{L_\kappa, \frac{\kappa_1}{2\,\|i_{\mathrm{H},4}\|\,c_p}\right\}, \quad \text{with} \quad L_\kappa := 2\kappa_2 - \kappa_1\,,$$

implies (3.38) and completes the proof. □

Analogously to (3.27), and employing now the a priori estimate (3.29), the second component of the solution of (3.15) is bounded as

$$\|\underset{\sim}{\boldsymbol{\eta}}\|_{\mathrm{Y}} \,\leq\, \widehat{C}_{\mathrm{NT}}\left\{\|g\|_{0,4/3;\,\Omega} \,+\, \|\mathbf{g}\|_{0,4;\,\Omega} \,+\, \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\right\}, \tag{3.39}$$

where, as for C_{NT}, $\widehat{C}_{\mathrm{NT}}$ is a positive constant depending on κ_1, κ_2, $\|i_{\mathrm{H},4}\|$, c_p, f_2, $|\Omega|$, and $\|\gamma_{\boldsymbol{\nu}}\|$.

3.4 Solvability analysis of the fixed point equation

In this section we apply the classical Banach theorem (see, e.g. [19, Lemma 3.7-1]) to prove that, under suitable assumptions on the data, the operator Θ (cf. (3.16)) has a unique fixed point. To this end, and coherently with (3.26) and the assumption on $\mathbf{z}$ in Lemma 3.9, we first let $\delta := \frac{\kappa_1}{2\,\|i_{\mathrm{H},4}\|\,c_p}$, consider an arbitrary positive constant r, and introduce the closed subset of $\mathbf{H}_2 \times \mathrm{X}_2$, and hence complete metric space,

$$\mathrm{W}(r) \,:=\, \Big\{(\mathbf{z},\psi) \in \mathbf{H}_2 \times \mathrm{X}_2 : \quad \|\mathbf{z}\|_{0,4;\Omega} \,\leq\, \delta\,, \quad \|\psi\|_{0,4;\Omega} \,\leq\, r\Big\}. \tag{3.40}$$

Then, it is easily seen from Lemmas 3.4 and 3.9 that the restriction of Θ to $\mathrm{W}(r)$ is well-defined. In turn, the next lemma provides necessary conditions guaranteeing that Θ maps $\mathrm{W}(r)$ into itself.

Lemma 3.10 *Assume that there hold*

$$C_{\mathrm{BF}}\left\{r\,\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right\} \,\leq\, \delta\,, \quad \textit{and} \tag{3.41}$$

$$C_{\mathrm{NT}}\left\{\|g\|_{0,4/3;\,\Omega} \,+\, \|\mathbf{g}\|_{0,4;\Omega} \,+\, \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\right\} \,\leq\, r\,. \tag{3.42}$$

Then, $\Theta\big(\mathrm{W}(r)\big) \,\subseteq\, \mathrm{W}(r)$.

Proof. It follows straightforwardly from the definition of Θ (cf. (3.16)) along with the estimates (3.26) and (3.38), and the assumptions (3.41) and (3.42). Further details are omitted. □

In order to prove that Θ is a contraction, we need to show first the Lipschitz continuity properties of $\mathbf{S}$ and $\mathbf{T}$. In fact, we have the results provided by the following lemmas.

Lemma 3.11 *There exists a positive constant $L_{\mathbf{S}}$, depending only on α, n, $\mathtt{F}$, $|\Omega|$, ρ, δ, r, and C_{BF}, such that*

$$\|\mathbf{S}(\mathbf{z},\psi) - \mathbf{S}(\mathbf{w},\varphi)\|_{\mathbf{H}_2} \,\le\, L_{\mathbf{S}} \Big\{ \|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \Big\} \, \|(\mathbf{z},\psi) - (\mathbf{w},\varphi)\|_{\mathbf{H}_2\times \mathrm{X}_2} \tag{3.43}$$

for all $(\mathbf{z},\psi)$, $(\mathbf{w},\varphi) \in \mathrm{W}(r)$.

Proof. Given $(\mathbf{z},\psi)$ and $(\mathbf{w},\varphi)$ in $\mathrm{W}(r)$, we let

$$(\vec{\underset{\sim}{\chi}},\underset{\sim}{\boldsymbol{\sigma}}) \,:=\, \big((\underset{\sim}{\chi},\underset{\sim}{\mathbf{u}}),\underset{\sim}{\boldsymbol{\sigma}}\big) \in (\mathbf{H}_1 \times \mathbf{H}_2) \times \mathbf{Q} \quad \text{and} \quad (\underline{\vec{\chi}},\underline{\boldsymbol{\sigma}}) \,:=\, \big((\underline{\chi},\underline{\mathbf{u}}),\underline{\boldsymbol{\sigma}}\big) \in (\mathbf{H}_1 \times \mathbf{H}_2) \times \mathbf{Q}$$

be the respective solutions of (3.14), so that

$$\mathbf{S}(\mathbf{z},\psi) \,:=\, \underset{\sim}{\mathbf{u}} \quad \text{and} \quad \mathbf{S}(\mathbf{w},\varphi) \,:=\, \underline{\mathbf{u}}\,.$$

Then, subtracting the corresponding equations defining (3.14), we obtain

$$\begin{array}{lcll} \mathbf{A}_{\mathbf{z}}(\vec{\underset{\sim}{\chi}},\vec{\boldsymbol{\vartheta}}) - \mathbf{A}_{\mathbf{w}}(\underline{\vec{\chi}},\vec{\boldsymbol{\vartheta}}) \,+\, \mathbf{B}(\vec{\boldsymbol{\vartheta}},\underset{\sim}{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}) & = & \mathbf{F}_{\psi}(\vec{\boldsymbol{\vartheta}}) - \mathbf{F}_{\varphi}(\vec{\boldsymbol{\vartheta}}) & \forall\, \vec{\boldsymbol{\vartheta}} \in \mathbf{H}\,, \\ \mathbf{B}(\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}},\boldsymbol{\tau}) & = & 0 & \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,. \end{array} \tag{3.44}$$

It is clear from the second equation of (3.44) that $\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}} \in \mathcal{V}$ (cf. (3.22)), so that taking $\vec{\boldsymbol{\vartheta}} = \vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}}$ in the first one, and invoking the definitions of $\mathbf{F}_{\psi}$ and $\mathbf{F}_{\varphi}$ (cf. (3.13)), we deduce that

$$\mathbf{A}_{\mathbf{z}}(\vec{\underset{\sim}{\chi}},\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}}) - \mathbf{A}_{\mathbf{w}}(\underline{\vec{\chi}},\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}}) \,=\, - \int_{\Omega} (\psi - \varphi)\,\mathbf{f}\cdot(\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}})\,. \tag{3.45}$$

In turn, applying the $\mathcal{V}$-ellipticity for $\mathbf{A}_{\mathbf{z}}$ (cf. Lemma 3.2), employing the identity (3.45), and bearing in mind the definitions of $\mathbf{A}_{\mathbf{w}}$ and $\mathbf{A}_{\mathbf{z}}$ (cf. (3.11)), we find that

$$\begin{aligned} \alpha\,\|\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}}\|^2_{\mathbf{H}} \,&\le\, \mathbf{A}_{\mathbf{z}}\big(\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}},\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}}\big) \,=\, \mathbf{A}_{\mathbf{z}}\big(\vec{\underset{\sim}{\chi}},\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}}\big) \,-\, \mathbf{A}_{\mathbf{z}}\big(\underline{\vec{\chi}},\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}}\big) \\ &=\, \mathbf{A}_{\mathbf{w}}(\underline{\vec{\chi}},\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}}) \,-\, \mathbf{A}_{\mathbf{z}}\big(\underline{\vec{\chi}},\vec{\underset{\sim}{\chi}} - \underline{\vec{\chi}}\big) \,-\, \int_{\Omega} (\psi - \varphi)\,\mathbf{f}\cdot(\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}}) \\ &=\, \frac{1}{2}\int_{\Omega} \underline{\chi}\,(\mathbf{w} - \mathbf{z})\cdot(\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}}) \,-\, \frac{1}{2}\int_{\Omega} \underline{\mathbf{u}}\otimes(\mathbf{w} - \mathbf{z}) : (\underset{\sim}{\chi} - \underline{\chi}) \\ &\quad +\, \mathtt{F}\int_{\Omega}\big(|\mathbf{w}|^{\rho-2} - |\mathbf{z}|^{\rho-2}\big)\,\underline{\mathbf{u}}\cdot(\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}}) \,+\, \int_{\Omega} (\varphi - \psi)\,\mathbf{f}\cdot(\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}})\,. \end{aligned} \tag{3.46}$$

Then, simple applications of the Cauchy–Schwarz inequality yield

$$\left|\int_{\Omega} \underline{\chi}\,(\mathbf{w} - \mathbf{z})\cdot(\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}})\right| \,\le\, \|\underline{\chi}\|_{0,\Omega}\,\|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}\,\|\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}}\|_{0,4;\Omega}\,, \tag{3.47}$$

$$\left|\int_{\Omega} \underline{\mathbf{u}}\otimes(\mathbf{w} - \mathbf{z}) : (\underset{\sim}{\chi} - \underline{\chi})\right| \,\le\, \sqrt{n}\,\|\underline{\mathbf{u}}\|_{0,4;\Omega}\,\|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}\,\|\underset{\sim}{\chi} - \underline{\chi}\|_{0,\Omega}\,, \quad \text{and}$$

$$\left|\int_{\Omega} (\varphi - \psi)\,\mathbf{f}\cdot(\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}})\right| \,\le\, \|\mathbf{f}\|_{0,\Omega}\,\|\varphi - \psi\|_{0,4;\Omega}\,\|\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}}\|_{0,4;\Omega}\,.$$

On the other hand, proceeding exactly as for the proof of [10, Lemma 4.4, eq. (4.33)], which, in turn, makes use of the key estimate provided by [27, Lemma 5.3], we derive the existence of a positive constant $L_{\mathtt{F}}$, depending only on $\mathtt{F}$, $|\Omega|$, and ρ, such that

$$\begin{aligned} &\left|\mathtt{F}\int_{\Omega}\big(|\mathbf{w}|^{\rho-2} - |\mathbf{z}|^{\rho-2}\big)\,\underline{\mathbf{u}}\cdot(\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}})\right| \\ &\qquad \le\, L_{\mathtt{F}}\Big\{\|\mathbf{w}\|_{0,4;\Omega} + \|\mathbf{z}\|_{0,4;\Omega}\Big\}^{\rho-3}\,\|\underline{\mathbf{u}}\|_{0,4;\Omega}\,\|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}\,\|\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}}\|_{0,4;\Omega}\,. \end{aligned} \tag{3.48}$$

In this way, replacing (3.47) up to (3.48) back into (3.46), using that $\|\mathbf{w}\|_{0,4;\Omega}$, $\|\mathbf{z}\|_{0,4;\Omega} \,\le\, \delta$ (cf. (3.40)), and performing some algebraic manipulations, in particular bounding both $\|\underset{\sim}{\boldsymbol{\chi}} - \underline{\boldsymbol{\chi}}\|_{0,\Omega}$ and $\|\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}}\|_{0,4;\Omega}$ by $\|\underset{\sim}{\vec{\boldsymbol{\chi}}} - \underline{\vec{\boldsymbol{\chi}}}\|_{\mathbf{H}}$, and then simplifying the latter, we arrive at

$$\|\underset{\sim}{\vec{\boldsymbol{\chi}}} - \underline{\vec{\boldsymbol{\chi}}}\|_{\mathbf{H}} \,\le\, \alpha^{-1} \,\max\left\{\sqrt{n}, 2\,L_{\mathbf{F}}\,(2\delta)^{\rho-3}\right\} \left\{\|\mathbf{f}\|_{0,\Omega} \,+\, \|\underline{\vec{\boldsymbol{\chi}}}\|_{\mathbf{H}}\right\} \|(\mathbf{z},\psi) - (\mathbf{w},\varphi)\|_{\mathbf{H}_2\times \mathrm{X}_2}\,. \tag{3.49}$$

Finally, recalling that $\|\mathbf{S}(\mathbf{z},\psi) - \mathbf{S}(\mathbf{w},\varphi)\|_{\mathbf{H}_2} \,=\, \|\underset{\sim}{\mathbf{u}} - \underline{\mathbf{u}}\|_{0,4;\Omega} \,\le\, \|\underset{\sim}{\vec{\boldsymbol{\chi}}} - \underline{\vec{\boldsymbol{\chi}}}\|_{\mathbf{H}}$, and using from (3.26) and (3.40) that

$$\|\underline{\vec{\boldsymbol{\chi}}}\|_{\mathbf{H}} \,\le\, C_{\mathrm{BF}}\left\{r\,\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\right\},$$

we conclude from (3.49) the required inequality (3.43) with a positive constant $L_{\mathbf{S}}$ depending only on α, n, $L_{\mathbf{F}}$, δ, ρ, r, and C_{BF}, and hence as originally indicated. □

Lemma 3.12 *There exists a positive constant $L_{\mathbf{T}}$, depending only on $\widetilde{\alpha}$, C_{NT}, and L_f, such that*

$$\|\mathbf{T}(\mathbf{z},\psi) - \mathbf{T}(\mathbf{w},\varphi)\|_{\mathrm{X}_2} \,\le\, L_{\mathbf{T}}\left\{\|g\|_{0,4/3;\Omega} \,+\, \|\mathbf{g}\|_{0,4;\Omega} \,+\, \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\right\} \|(\mathbf{z},\psi) - (\mathbf{w},\varphi)\|_{\mathbf{H}_2\times \mathrm{X}_2} \tag{3.50}$$

for all $(\mathbf{z},\psi)$, $(\mathbf{w},\varphi) \in \mathrm{W}(r)$.

Proof. It proceeds analogously to that of Lemma 3.11. In fact, given $(\mathbf{z},\psi)$ and $(\mathbf{w},\varphi)$ in $\mathrm{W}(r)$, we first let

$$(\underset{\sim}{\vec{\mathbf{t}}}, \underset{\sim}{\boldsymbol{\eta}}) \,:=\, ((\underset{\sim}{\mathbf{t}}, \underset{\sim}{\phi}), \underset{\sim}{\boldsymbol{\eta}}) \in (\mathrm{X}_1\times\mathrm{X}_2)\times \mathrm{Y} \quad \text{and} \quad (\underline{\vec{\mathbf{t}}}, \underline{\boldsymbol{\eta}}) \,:=\, ((\underline{\mathbf{t}}, \underline{\phi}), \underline{\boldsymbol{\eta}}) \in (\mathrm{X}_1\times\mathrm{X}_2)\times \mathrm{Y}$$

be the respective solutions of (3.15), so that

$$\mathbf{T}(\mathbf{z},\psi) \,:=\, \underset{\sim}{\phi} \quad \text{and} \quad \mathbf{T}(\mathbf{w},\varphi) \,:=\, \underline{\phi}\,.$$

Thus, similarly to the derivation of (3.44), the subtraction of the equations defining (3.15) leads to

$$\begin{aligned} \big[\mathbf{a}_{\mathbf{z}}(\underset{\sim}{\vec{\mathbf{t}}}), \vec{\mathbf{s}}\big] \,-\, \big[\mathbf{a}_{\mathbf{w}}(\underline{\vec{\mathbf{t}}}), \vec{\mathbf{s}}\big] \,+\, \mathbf{b}(\vec{\mathbf{s}}, \underset{\sim}{\boldsymbol{\eta}} - \underline{\boldsymbol{\eta}}) \quad &= \quad \mathrm{F}_\psi(\vec{\mathbf{s}}) - \mathrm{F}_\varphi(\vec{\mathbf{s}}) \qquad \forall\, \vec{\mathbf{s}} \in \mathrm{X}\,, \\ \mathbf{b}\big(\underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}, \boldsymbol{\xi}\big) \quad &= \quad 0 \qquad \forall\, \boldsymbol{\xi} \in \mathrm{Y}\,, \end{aligned} \tag{3.51}$$

so that, taking in the first row of (3.51) $\vec{\mathbf{s}} = \underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}$, which clearly belongs to $\mathbf{V}$ (cf. (3.32)), and bearing in mind the definitions of F_ψ and F_φ (cf. (3.13)), we find that

$$\big[\mathbf{a}_{\mathbf{z}}(\underset{\sim}{\vec{\mathbf{t}}}), \underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}\big] \,=\, \big[\mathbf{a}_{\mathbf{w}}(\underline{\vec{\mathbf{t}}}), \underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}\big] \,+\, \int_\Omega \big(f(\psi) - f(\varphi)\big)\,\mathbf{g}\cdot(\underset{\sim}{\mathbf{t}} - \underline{\mathbf{t}})\,. \tag{3.52}$$

Next, applying the strong monotonicity of $\mathbf{a}_{\mathbf{z}}$ (cf. (3.36)) with $\vec{\mathbf{k}} = \underline{\vec{\mathbf{t}}} \in \mathrm{X}$, $\vec{\mathbf{r}} = \underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}} \in \mathbf{V}$, and $\vec{\mathbf{s}} = \vec{\mathbf{0}} \in \mathbf{V}$, employing (3.52), and invoking the definitions of $\mathbf{a}_{\mathbf{w}}$ and $\mathbf{a}_{\mathbf{z}}$ (cf. (3.12)), we obtain

$$\begin{aligned} \widetilde{\alpha}\,\|\underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}\|_{\mathrm{X}}^2 \,\le\, \big[\mathbf{a}_{\mathbf{z}}(\underset{\sim}{\vec{\mathbf{t}}}) - \mathbf{a}_{\mathbf{z}}(\underline{\vec{\mathbf{t}}}), \underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}\big] &= \big[\mathbf{a}_{\mathbf{w}}(\underline{\vec{\mathbf{t}}}) - \mathbf{a}_{\mathbf{z}}(\underline{\vec{\mathbf{t}}}), \underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}\big] + \int_\Omega \big(f(\psi) - f(\varphi)\big)\,\mathbf{g}\cdot(\underset{\sim}{\mathbf{t}} - \underline{\mathbf{t}}) \\ &= \int_\Omega \underline{\phi}\,(\mathbf{z} - \mathbf{w})\cdot(\underset{\sim}{\mathbf{t}} - \underline{\mathbf{t}}) \,+\, \int_\Omega \big(f(\psi) - f(\varphi)\big)\,\mathbf{g}\cdot(\underset{\sim}{\mathbf{t}} - \underline{\mathbf{t}})\,, \end{aligned}$$

from which, employing Cauchy–Schwarz's inequality, using the Lipschitz-continuity of f (cf. (2.3)), bounding $\|\underset{\sim}{\mathbf{t}} - \underline{\mathbf{t}}\|_{0,\Omega}$ by $\|\underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}\|_{\mathrm{X}}$, and then simplifying the latter, we arrive at

$$\|\underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}\|_{\mathrm{X}} \,\le\, \widetilde{\alpha}^{-1}\left\{\|\underline{\phi}\|_{0,4;\Omega} \,+\, \mathrm{L}_f\,\|\mathbf{g}\|_{0,4;\Omega}\right\} \|(\mathbf{z},\psi) - (\mathbf{w},\varphi)\|_{\mathbf{H}_2\times\mathrm{X}_2}\,. \tag{3.53}$$

In this way, noting that $\|\mathbf{T}(\mathbf{z},\psi) - \mathbf{T}(\mathbf{w},\varphi)\|_{X_2} = \|\underset{\sim}{\phi} - \underline{\phi}\|_{0,4;\Omega} \le \|\underset{\sim}{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}\|_{X}$, and using the a priori estimate (3.38) for $\|\underline{\phi}\|_{0,4;\Omega}$, we deduce from (3.53) the required inequality (3.50). $\square$

As a straightforward consequence of Lemmas 3.11 and 3.12 we conclude the Lipschitz-continuity of the operator Θ. Indeed, given $(\mathbf{z},\psi)$, $(\mathbf{w},\varphi) \in \mathrm{W}(r)$, we readily obtain from (3.16), (3.43), and (3.50) that

$$\|\Theta(\mathbf{z},\psi) - \Theta(\mathbf{w},\varphi)\|_{\mathbf{H}_2\times \mathrm{X}_2} \le L_\Theta\, \mathcal{C}(\mathtt{data})\, \|(\mathbf{z},\psi) - (\mathbf{w},\varphi)\|_{\mathbf{H}_2\times \mathrm{X}_2}\,,$$

where $L_\Theta := \max\big\{L_{\mathbf{S}}, L_{\mathbf{T}}\big\}$ and

$$\mathcal{C}(\mathtt{data}) := \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|g\|_{0,4/3;\,\Omega} + \|\mathbf{g}\|_{0,4;\,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\,. \tag{3.54}$$

We are now in position to establish the main result of this section.

Theorem 3.13 *Given $r > 0$, assume that, in addition to the hypotheses of Lemma 3.10, the data satisfy*

$$L_\Theta\, \mathcal{C}(\mathtt{data}) < 1\,. \tag{3.55}$$

Then, there exists a unique $(\mathbf{u},\phi) \in \mathrm{W}(r)$ (cf. (3.40)) fixed point of Θ (cf. (3.17)). Equivalently, (3.10) has a unique solution $(\vec{\boldsymbol{\chi}},\boldsymbol{\sigma}) := (\underset{\sim}{\vec{\boldsymbol{\chi}}},\underset{\sim}{\boldsymbol{\sigma}}) \in \mathbf{H}\times\mathbf{Q}$ and $(\vec{\mathbf{t}},\boldsymbol{\eta}) := (\underset{\sim}{\vec{\mathbf{t}}},\underset{\sim}{\boldsymbol{\eta}}) \in \mathrm{X}\times\mathrm{Y}$, with $(\mathbf{u},\phi) \in \mathrm{W}(r)$, where $(\underset{\sim}{\vec{\boldsymbol{\chi}}},\underset{\sim}{\boldsymbol{\sigma}})$ and $(\underset{\sim}{\vec{\mathbf{t}}},\underset{\sim}{\boldsymbol{\eta}})$ are the unique solution of (3.14)–(3.15) with $(\mathbf{z},\psi) = (\mathbf{u},\phi)$. Moreover, there hold

$$\begin{aligned}
\|\vec{\boldsymbol{\chi}}\|_{\mathbf{H}} &\le C_{\mathrm{BF}}\Big\{r\,\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,,\\
\|\boldsymbol{\sigma}\|_{\mathbf{Q}} &\le \widehat{C}_{\mathrm{BF}}\Big\{r\,\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,,\\
\|\vec{\mathbf{t}}\|_{\mathrm{X}} &\le C_{\mathrm{NT}}\Big\{\|g\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{0,4;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,,\\
\|\boldsymbol{\eta}\|_{\mathrm{Y}} &\le \widehat{C}_{\mathrm{NT}}\Big\{\|g\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{0,4;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,.
\end{aligned} \tag{3.56}$$

Proof. Since (3.55) guarantees that Θ is a contraction, the unique solvability of (3.17), and hence of (3.10), with $(\mathbf{u},\phi) \in \mathrm{W}(r)$, follows from Lemma 3.10 and a direct application of the Banach fixed-point theorem. In turn, (3.26), (3.27), (3.38), and (3.39) yield the a priori estimates provided by (3.56). $\square$

4 The Galerkin scheme

In this section, we introduce and analyze a Galerkin scheme associated with the problem (3.10). In particular, the solvability analysis is carried out using a discrete adaptation of the fixed-point strategy from Section 3.2. After decoupling the problems, we provide conditions on the finite element subspaces that ensure their well-posedness. Then, under certain conditions on the data, it is proven that the discrete fixed-point equation has a unique solution, thereby ensuring the stability of the Galerkin scheme. Finally, we deduce a priori error estimates for each uncoupled problem, which lead to a global Céa estimate.

4.1 The discrete problem

We first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, and set $h := \max\{h_K :\ K \in \mathcal{T}_h\}$. We let $\mathbf{H}_h^{\boldsymbol{\chi}}$, $\mathbf{H}_h^{\mathbf{u}}$, $\widetilde{\mathbf{Q}}_h$,

$X_h^{\mathbf{t}}$, X_h^{ϕ} and Y_h be arbitrary finite element subspaces of $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbf{L}^4(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, $\mathbf{L}^2(\Omega)$, $\mathrm{L}^4(\Omega)$ and $\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, respectively, each endowed with the corresponding subspace topology. Specific choices of them, satisfying suitable hypotheses to be introduced along the discussion, will be described later. Next, we also define the product spaces

$$\mathbf{Q}_h := \widetilde{\mathbf{Q}}_h \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \qquad \mathbf{H}_h := \mathbf{H}_h^{\boldsymbol{\chi}} \times \mathbf{H}_h^{\mathbf{u}}, \qquad \mathrm{X}_h := \mathrm{X}_h^{\mathbf{t}} \times \mathrm{X}_h^{\phi},$$

and set the notations

$$\vec{\boldsymbol{\chi}}_h := (\boldsymbol{\chi}_h, \mathbf{u}_h), \quad \vec{\boldsymbol{\vartheta}}_h := (\boldsymbol{\vartheta}_h, \mathbf{v}_h), \quad \vec{\boldsymbol{\varrho}}_h := (\boldsymbol{\varrho}_h, \mathbf{w}_h) \in \mathbf{H}_h,$$

$$\vec{\mathbf{t}}_h := (\mathbf{t}_h, \phi_h), \quad \vec{\mathbf{s}}_h := (\mathbf{s}_h, \varphi_h), \quad \vec{\mathbf{r}}_h := (\mathbf{r}_h, \psi_h) \in \mathrm{X}_h.$$

Thus, the Galerkin scheme associated with (3.10) reads: Find $(\vec{\boldsymbol{\chi}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ and $(\vec{\mathbf{t}}_h, \boldsymbol{\eta}_h) \in \mathrm{X}_h \times \mathrm{Y}_h$ such that

$$\begin{aligned}
\mathbf{A}_{\mathbf{u}_h}(\vec{\boldsymbol{\chi}}_h, \vec{\boldsymbol{\vartheta}}_h) + \mathbf{B}(\vec{\boldsymbol{\vartheta}}_h, \boldsymbol{\sigma}_h) &= \mathbf{F}_{\phi_h}(\vec{\boldsymbol{\vartheta}}_h) && \forall\, \vec{\boldsymbol{\vartheta}}_h \in \mathbf{H}_h\,,\\
\mathbf{B}(\vec{\boldsymbol{\chi}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h\,,\\
[\mathbf{a}_{\mathbf{u}_h}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}_h] + \mathbf{b}(\vec{\mathbf{s}}_h, \boldsymbol{\eta}_h) &= \mathrm{F}_{\phi_h}(\vec{\mathbf{s}}_h) && \forall\, \vec{\mathbf{s}}_h \in \mathrm{X}_h\,,\\
\mathbf{b}(\vec{\mathbf{t}}_h, \boldsymbol{\xi}_h) &= \mathrm{G}(\boldsymbol{\xi}_h) && \forall\, \boldsymbol{\xi}_h \in \mathrm{Y}_h\,.
\end{aligned} \tag{4.1}$$

4.2 Discrete fixed point strategy

In order to address the solvability of (4.1), we adopt the discrete analogue of the fixed point strategy employed in section (3.2). To do so, we first introduce $\mathbf{S}_h : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi} \to \mathbf{H}_h^{\mathbf{u}}$ as the operator defined by

$$\mathbf{S}_h(\mathbf{z}_h, \psi_h) := \underset{\sim}{\mathbf{u}}_h \qquad \forall\,(\mathbf{z}_h, \psi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi}\,,$$

where $(\underset{\sim}{\vec{\boldsymbol{\chi}}}_h, \underset{\sim}{\boldsymbol{\sigma}}_h) := ((\underset{\sim}{\boldsymbol{\chi}}_h, \underset{\sim}{\mathbf{u}}_h), \underset{\sim}{\boldsymbol{\sigma}}_h) \in (\mathbf{H}_h^{\boldsymbol{\chi}} \times \mathbf{H}_h^{\mathbf{u}}) \times \mathbf{Q}_h$ is the unique solution of the problem arising from the first two rows of (4.1) after replacing $\mathbf{A}_{\mathbf{u}_h}$ and $\mathbf{F}_{\phi_h}$ by $\mathbf{A}_{\mathbf{z}_h}$ and $\mathbf{F}_{\psi_h}$, respectively, which serves as the discrete counterpart to (3.14)

$$\begin{aligned}
\mathbf{A}_{\mathbf{z}_h}(\underset{\sim}{\vec{\boldsymbol{\chi}}}_h, \vec{\boldsymbol{\vartheta}}_h) + \mathbf{B}(\vec{\boldsymbol{\vartheta}}_h, \underset{\sim}{\boldsymbol{\sigma}}_h) &= \mathbf{F}_{\psi_h}(\vec{\boldsymbol{\vartheta}}_h) && \forall\, \vec{\boldsymbol{\vartheta}}_h \in \mathbf{H}_h\,,\\
\mathbf{B}(\underset{\sim}{\vec{\boldsymbol{\chi}}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h\,.
\end{aligned} \tag{4.2}$$

Similarly, let $\mathbf{T}_h : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi} \to \mathrm{X}_h^{\phi}$ be the operator defined by

$$\mathbf{T}_h(\mathbf{z}_h, \psi_h) := \underset{\sim}{\phi}_h \qquad \forall(\mathbf{z}_h, \psi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi},$$

where $(\underset{\sim}{\vec{\mathbf{t}}}_h, \underset{\sim}{\boldsymbol{\eta}}_h) := ((\underset{\sim}{\mathbf{t}}_h, \underset{\sim}{\phi}_h), \underset{\sim}{\boldsymbol{\eta}}_h) \in (\mathrm{X}_h^{\mathbf{t}} \times \mathrm{X}_h^{\phi}) \times \mathrm{Y}_h$ is the unique solution of the discrete counterpart of the uncoupled nonlinear transport problem (3.15)

$$\begin{aligned}
[\mathbf{a}_{\mathbf{z}_h}(\underset{\sim}{\vec{\mathbf{t}}}_h), \vec{\mathbf{s}}_h] + \mathbf{b}(\vec{\mathbf{s}}_h, \underset{\sim}{\boldsymbol{\eta}}_h) &= \mathrm{F}_{\psi_h}(\vec{\mathbf{s}}_h) && \forall\, \vec{\mathbf{s}}_h \in \mathrm{X}_h\,,\\
\mathbf{b}(\underset{\sim}{\vec{\mathbf{t}}}_h, \boldsymbol{\xi}_h) &= \mathrm{G}(\boldsymbol{\xi}_h) && \forall\, \boldsymbol{\xi}_h \in \mathrm{Y}_h\,,
\end{aligned} \tag{4.3}$$

with the given pair $(\mathbf{z}_h, \psi_h)$. In addition, we define the operator $\Theta_h : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi} \to \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi}$ by

$$\Theta_h(\mathbf{z}_h, \psi_h) := \big(\mathbf{S}_h(\mathbf{z}_h, \psi_h), \mathbf{T}_h(\mathbf{z}_h, \psi_h)\big) \quad \forall\,(\mathbf{z}_h, \psi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi}\,, \tag{4.4}$$

and observe that solving (4.1) is equivalent to finding a fixed point of the operator Θ_h, i.e., seeking $(\mathbf{u}_h, \phi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi}$ such that

$$\Theta_h(\mathbf{u}_h, \phi_h) = (\mathbf{u}_h, \phi_h). \tag{4.5}$$

In the following section, we prove that $\mathbf{S}_h$ and $\mathbf{T}_h$ are well-defined (equivalently, that the problems (4.2) and (4.3) have unique solutions), and, consequently, that Θ_h is well defined. Finally, we will show that, under appropriate assumptions, there exists a unique fixed point of Θ_h.

4.3 Well-posedness of the discrete uncoupled problems

In order to carry out the solvability analysis of the uncoupled problems, we shall introduce hypotheses concerning the arbitrary finite element spaces. We begin by introducing the auxiliary spaces defined by

$$\mathbf{Q}_{0,h} := \Big\{ \boldsymbol{\tau}_h \in \mathbf{Q}_h : \ \mathbf{B}\big((0,\mathbf{v}_h),\boldsymbol{\tau}_h\big) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}} \Big\} \quad \text{and}$$

$$\mathbf{Q}_{0,h}^{\mathrm{d}} := \Big\{ \boldsymbol{\tau}_h^{\mathrm{d}} : \quad \boldsymbol{\tau}_h \in \mathbf{Q}_{0,h} \Big\}.$$

In addition, from now on, we assume that $\mathbf{H}_h^{\boldsymbol{\chi}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\widetilde{\mathbf{Q}}_h$ satisfy:

(H.1) $\widetilde{\mathbf{Q}}_h$ contains the multiples of the identity tensor $\mathbb{I}$,

(H.2) $\mathbf{div}(\mathbf{Q}_h) \subset \mathbf{H}_h^{\mathbf{u}}$,

(H.3) $\mathbf{Q}_{0,h}^{\mathrm{d}} \subset \mathbf{H}_h^{\boldsymbol{\chi}}$, and

(H.4) there exists a positive constant $\beta_{\mathrm{d},1}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{Q}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \ \geq\ \beta_{\mathrm{d},1}\, \|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}. \tag{4.6}$$

According to the decomposition $\mathbb{H}(\mathbf{div}_{4/3};\Omega) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\,\mathbb{I}$ and **(H.1)**, one can deduce that

$$\mathbf{Q}_h \ = \ \left\{ \boldsymbol{\tau}_h - \left(\frac{1}{n\,|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) \right) \mathbb{I} : \quad \boldsymbol{\tau}_h \in \widetilde{\mathbf{Q}}_h \right\}.$$

In turn, the remaining hypotheses enable us to deduce the discrete analogues of Lemmas 3.2 and 3.3. We first observe that, thanks to hypothesis **(H.2)**, the space $\mathbf{Q}_{0,h}$ can be reduced to

$$\mathbf{Q}_{0,h} = \Big\{ \boldsymbol{\tau}_h \in \mathbf{Q}_h : \ \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \Big\}. \tag{4.7}$$

Now, given $\boldsymbol{\tau}_h \in \mathbf{Q}_{0,h}$, **(H.3)** ensures that $\boldsymbol{\tau}_h^{\mathrm{d}} \in \mathbf{H}_h^{\boldsymbol{\chi}}$. Thus, using (3.24) with $t = 4/3$ and the fact that $\mathbf{div}(\boldsymbol{\tau}_h) = 0$, we obtain

$$\begin{aligned} \sup_{\substack{\boldsymbol{\vartheta}_h \in \mathbf{H}_h^{\boldsymbol{\chi}} \\ \boldsymbol{\vartheta}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\vartheta}_h}{\|\boldsymbol{\vartheta}_h\|_{0,\Omega}} \ &\geq\ \frac{\displaystyle\int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\tau}_h^{\mathrm{d}}}{\|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega}} = \|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega} \\ &\geq\ C_{4/3}^{1/2}\, \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}\,. \end{aligned} \tag{4.8}$$

Certainly, the inf-sup condition (4.6), and the above inequality (4.8), can be rewritten in terms of the bilinear form $\mathbf{B}$. In fact, denoting $\beta_{\mathrm{d},2} := C_{4/3}^{1/2}$, we have

$$\begin{aligned} \sup_{\substack{\boldsymbol{\tau}_h\in\mathbf{Q}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\mathbf{B}\big((0,\mathbf{v}_h),\boldsymbol{\tau}_h\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \,\ge\, \beta_{\mathrm{d},1}\,\|\mathbf{v}_h\|_{0,4;\Omega} \quad \text{and} \\ \sup_{\substack{\boldsymbol{\vartheta}_h\in\mathbf{H}_h^{\boldsymbol{\chi}}\\ \boldsymbol{\vartheta}_h\neq\mathbf{0}}} \frac{\mathbf{B}\big((\boldsymbol{\vartheta}_h,0),\boldsymbol{\tau}_h\big)}{\|\boldsymbol{\vartheta}_h\|_{0,\Omega}} \,\ge\, \beta_{\mathrm{d},2}\,\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}\,, \end{aligned} \tag{4.9}$$

for all $\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}$ and $\boldsymbol{\tau}_h \in \mathbf{Q}_{0,h}$, which provide the necessary conditions for establishing both the discrete analogue of Lemma 3.3 and an intermediate result that is required in the proof of the discrete analogue of Lemma 3.2. More precisely, recalling that the discrete kernel of $\mathbf{B}$ is defined as

$$\mathcal{V}_h := \Big\{\vec{\boldsymbol{\vartheta}}_h \in \mathbf{H}_h :\ \mathbf{B}(\vec{\boldsymbol{\vartheta}}_h,\boldsymbol{\tau}_h) = 0 \quad \forall\,\boldsymbol{\tau}_h\in\mathbf{Q}_h\Big\}\,,$$

we can state the following results.

Lemma 4.1 *There exist positive constants β_{d} and c_{d}, independent of h, such that*

$$\sup_{\substack{\vec{\boldsymbol{\vartheta}}_h\in\mathbf{H}_h\\ \vec{\boldsymbol{\vartheta}}_h\neq\mathbf{0}}} \frac{\mathbf{B}(\vec{\boldsymbol{\vartheta}}_h,\boldsymbol{\tau}_h)}{\|\vec{\boldsymbol{\vartheta}}_h\|_{\mathbf{H}}} \,\ge\, \beta_{\mathrm{d}}\,\|\boldsymbol{\tau}_h\|_{\mathrm{div}_{4/3};\Omega} \quad \forall\,\boldsymbol{\tau}_h\in\mathbf{Q}_h \quad \textit{and} \tag{4.10}$$

$$\|\boldsymbol{\vartheta}_h\|_{0,\Omega} \,\ge\, c_{\mathrm{d}}\,\|\mathbf{v}_h\|_{0,4;\Omega} \quad \forall\,\vec{\boldsymbol{\vartheta}}_h := (\boldsymbol{\vartheta}_h,\mathbf{v}_h)\in\mathcal{V}_h\,. \tag{4.11}$$

Proof. Thanks to the abstract result given by [20, Lemma 5.1] with local notation $X = \mathbf{H}_h^{\mathbf{u}}$, $Y = Y_1 = \mathbf{H}_h^{\boldsymbol{\chi}}$, $Y_2 = \{0\}$, $Z = \mathbf{Q}_h$, and $b\big((\mathbf{v}_h,\boldsymbol{\vartheta}_h),\boldsymbol{\tau}_h\big) := \mathbf{B}\big((\boldsymbol{\vartheta}_h,\mathbf{v}_h),\boldsymbol{\tau}_h\big)$, for all $\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}$, $\boldsymbol{\vartheta}_h\in\mathbf{H}_h^{\boldsymbol{\chi}}$ and $\boldsymbol{\tau}_h\in\mathbf{Q}_h$, we deduce that (4.9) is equivalent to (4.10) along with (4.11). □

Lemma 4.2 *For all $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$, the bilinear form $\mathbf{A}_{\mathbf{z}_h}$ is $\mathcal{V}_h$-elliptic, that is, there exists a positive constant α_{d}, independent of h, such that*

$$\big|\mathbf{A}_{\mathbf{z}_h}(\vec{\boldsymbol{\vartheta}}_h,\vec{\boldsymbol{\vartheta}}_h)\big| \,\ge\, \alpha_{\mathrm{d}}\,\|\vec{\boldsymbol{\vartheta}}_h\|^2_{\mathbf{H}} \qquad \forall\,\vec{\boldsymbol{\vartheta}}_h\in\mathcal{V}_h\,. \tag{4.12}$$

Proof. Given $\vec{\boldsymbol{\vartheta}}_h := (\boldsymbol{\vartheta}_h,\mathbf{v}_h)\in\mathcal{V}_h$, we easily deduce, as in the continuous case (see Lemma 3.2), that

$$\mathbf{A}_{\mathbf{z}_h}(\vec{\boldsymbol{\vartheta}}_h,\vec{\boldsymbol{\vartheta}}_h) \,=\, \mu\,\|\boldsymbol{\vartheta}_h\|^2_{0,\Omega} + \mathtt{D}\,\|\mathbf{v}_h\|^2_{0,\Omega} + \mathtt{F}\int_\Omega |\mathbf{z}_h|^{\rho-2}\,|\mathbf{v}_h|^2\,.$$

Then, employing (4.11), we obtain

$$\mathbf{A}_{\mathbf{z}_h}(\vec{\boldsymbol{\vartheta}}_h,\vec{\boldsymbol{\vartheta}}_h) \,\ge\, \mu\,\|\boldsymbol{\vartheta}_h\|^2_{0,\Omega} \,\ge\, \frac{\mu}{2}\,\|\boldsymbol{\vartheta}_h\|^2_{0,\Omega} + \frac{\mu\,c_{\mathrm{d}}^2}{2}\,\|\mathbf{v}_h\|^2_{0,4;\Omega}\,,$$

which easily yields (4.12) with $\alpha_{\mathrm{d}} := \frac{\mu}{2}\,\min\{1,c_{\mathrm{d}}^2\}$. □

Lemma 4.3 *For each $(\mathbf{z}_h,\psi_h)\in\mathbf{H}_h^{\mathbf{u}}\times\mathrm{X}_h^{\phi}$ there exists a unique solution $(\vec{\underset{\sim}{\boldsymbol{\chi}}}_h,\underset{\sim}{\boldsymbol{\sigma}}_h) := ((\underset{\sim}{\boldsymbol{\chi}}_h,\underset{\sim}{\mathbf{u}}_h),\underset{\sim}{\boldsymbol{\sigma}}_h)\in(\mathbf{H}_h^{\boldsymbol{\chi}}\times\mathbf{H}_h^{\mathbf{u}})\times\mathbf{Q}_h$ of (4.2), and hence one can define $\mathbf{S}_h(\mathbf{z}_h,\psi_h) := \underset{\sim}{\mathbf{u}}_h\in\mathbf{H}_h^{\mathbf{u}}$. Moreover, there exists a positive constant $C_{\mathrm{BF,d}}$, depending only on α_{d}, β_{d}, and $\|\mathbf{A}_{\mathbf{z}_h}\|$, such that*

$$\|\mathbf{S}_h(\mathbf{z},\psi)\| \,\le\, C_{\mathrm{BF,d}}\Big\{\|\psi_h\|_{0,4;\Omega}\,\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,. \tag{4.13}$$

Proof. Thanks to (4.10) and Lemma 4.2, Theorem 3.1 ensures that (4.2) is well-posed. Moreover, the a priori estimate (3.20) yields

$$\|\mathbf{S}_h(\mathbf{z}_h,\psi_h)\| = \|\underset{\sim}{\mathbf{u}}_h\| \;\leq\; \|\vec{\underset{\sim}{\chi}}_h\| \;\leq\; \frac{1}{\alpha_{\mathrm{d}}}\|\mathbf{F}_{\psi_h}\| + \frac{1}{\beta_{\mathrm{d}}}\left(1+\frac{\|\mathbf{A}_{\mathbf{z}_h}\|}{\alpha_{\mathrm{d}}}\right)\|\mathbf{G}\|.$$

Then, employing the stability properties to bound $\|\mathbf{F}_{\psi_h}\|$ and $\|\mathbf{G}\|$, we arrive at (4.13). □

At this point, similarly to the continuous case, we remark that, given $\delta > 0$, for each $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ satisfying $\|\mathbf{z}_h\|_{0,4;\Omega} \leq \delta$, and according to (3.19), $\|\mathbf{A}_{\mathbf{z}_h}\|$ can be bounded by a constant depending only on δ,μ,ρ and $|\Omega|$. Therefore, we can restate the a priori estimate (4.13) as the existence of a positive constant $C_{\mathrm{BF,d}}$, depending only on α_{d}, $\beta_{\mathrm{d}}, \delta, \mu, \rho$ and $|\Omega|$, such that

$$\|\mathbf{S}_h(\mathbf{z}_h,\psi_h)\| \;=\; \|\underset{\sim}{\mathbf{u}}_h\| \;\leq\; \|\vec{\underset{\sim}{\chi}}_h\|_{\mathbf{H}} \;\leq\; C_{\mathrm{BF,d}}\Big\{\|\psi_h\|_{0,4;\Omega}\,\|\mathbf{f}\|_{0,\Omega} \;+\; \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}, \tag{4.14}$$

for all $(\mathbf{z}_h,\psi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi}$ satisfying $\|\mathbf{z}_h\|_{0,4;\Omega} \leq \delta$. Moreover, according to the a priori estimate (3.21), the second component of the solution of (4.2) is bounded as

$$\|\underset{\sim}{\boldsymbol{\sigma}}_h\| \;\leq\; \widehat{C}_{\mathrm{BF,d}}\Big\{\|\psi_h\|_{0,4;\Omega}\,\|\mathbf{f}\|_{0,\Omega} \;+\; \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}, \tag{4.15}$$

where $\widehat{C}_{\mathrm{BF,d}}$ is a positive constant having a similar dependence to that of $C_{\mathrm{BF,d}}$.

Our next goal is to show that the uncoupled problem (4.3) is well-posed. To this end, we will provide sufficient assumptions on the finite element spaces to prove the discrete versions of Lemmas 3.6, 3.7 and 3.8. Finally, we apply Theorem 3.5 to conclude the desired result. Similarly to the discrete analysis of the convective Brinkman–Forchheimer equations, we begin by introducing the auxiliary space

$$\mathrm{Y}_{0,h} \;:=\; \Big\{\boldsymbol{\xi}_h \in \mathrm{Y}_h : \quad \mathbf{b}\big((0,\varphi_h),\boldsymbol{\xi}_h\big) = 0 \quad \forall\,\varphi_h \in \mathrm{X}_h^{\phi}\Big\}.$$

Furthermore, in what follows, we assume that $\mathrm{X}_h^{\mathbf{t}}$, X_h^{ϕ}, and Y_h are such that:

(H.5) $\mathrm{div}(\mathrm{Y}_h) \subset \mathrm{X}_h^{\phi}$,

(H.6) $\mathrm{Y}_{0,h} \subset \mathrm{X}_h^{\mathbf{t}}$, and

(H.7) there exists a positive constant $\widetilde{\beta}_{\mathrm{d},1}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\xi}_h\in\mathrm{Y}_h\\ \boldsymbol{\xi}_h\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \varphi_h\,\mathrm{div}(\boldsymbol{\xi}_h)}{\|\boldsymbol{\xi}_h\|_{\mathrm{div}_{4/3};\Omega}} \;\geq\; \widetilde{\beta}_{\mathrm{d},1}\,\|\varphi_h\|_{0,4;\Omega} \qquad \forall\,\varphi_h \in \mathrm{X}_h^{\phi}. \tag{4.16}$$

We notice that these hypotheses are similar to **(H.2)**, **(H.3)**, and **(H.4)**. Indeed, the analysis of this uncoupled problem follows a similar approach to the previous one. In particular, we note that hypothesis **(H.5)** provides the convenient characterization

$$\mathrm{Y}_{0,h} = \Big\{\boldsymbol{\xi}_h \in \mathrm{Y}_h :\ \mathrm{div}(\boldsymbol{\xi}_h) = 0 \quad \text{in} \quad \Omega\Big\}.$$

Consequently, according to **(H.6)**, for each $\boldsymbol{\xi}_h \in \mathrm{Y}_{0,h}$ we readily obtain

$$\sup_{\substack{\mathbf{s}_h\in\mathrm{X}_h^{\mathbf{t}}\\ \mathbf{s}_h\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\xi}_h\cdot\mathbf{s}_h}{\|\mathbf{s}_h\|_{0,\Omega}} \;\geq\; \|\boldsymbol{\xi}_h\|_{0,\Omega} = \|\boldsymbol{\xi}_h\|_{\mathrm{div}_{4/3};\Omega}, \tag{4.17}$$

which, along with (4.16), yields the discrete inf-sup condition for $\mathbf{b}$, as well as an inequality that will be instrumental in the discrete version of Lemma 3.7. To this end, we now let $\mathbf{V}_h$ be the discrete kernel of $\mathbf{b}$, that is

$$\mathbf{V}_h := \left\{ \vec{\mathbf{s}}_h \in \mathrm{X}_h : \quad \int_\Omega \boldsymbol{\xi}_h \cdot \mathbf{s}_h + \int_\Omega \varphi_h \, \mathrm{div}(\boldsymbol{\xi}_h) = 0 \quad \forall\, \boldsymbol{\xi}_h \in \mathrm{Y}_h \right\}.$$

Then, we have the following lemma.

Lemma 4.4 *There exist positive constants $\widetilde{\beta}_{\mathrm{d}}$ and $\widetilde{c}_{\mathrm{d}}$, independent of h, such that*

$$\sup_{\substack{\vec{\mathbf{s}}_h \in \mathrm{X}_h \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{s}}_h, \boldsymbol{\xi}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathrm{X}}} \geq \widetilde{\beta}_{\mathrm{d}} \, \|\boldsymbol{\xi}_h\|_{\mathrm{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\xi}_h \in \mathrm{Y}_h \,, \quad \text{and}$$

$$\|\mathbf{s}_h\|_{0,\Omega} \geq \widetilde{c}_{\mathrm{d}} \, \|\varphi_h\|_{0,4;\Omega} \qquad \forall\, \vec{\mathbf{s}}_h := (\mathbf{s}_h, \varphi_h) \in \mathbf{V}_h \tag{4.18}$$

Proof. From (4.16) and (4.17), and using [20, Lemma 5.1] in a similar way as in the proof of Lemma 4.1, we obtain the result. Further details are omitted. □

Next, we present results regarding the Lipschitz continuity and strong monotonicity properties of $\mathbf{a}_{\mathbf{z}_h}|_{\mathrm{X}_h}$, which constitute the discrete analogues of Lemmas 3.6 and 3.7, respectively.

Lemma 4.5 *For each $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$, there exists a positive constant $L_{\mathbf{z}_h}$, depending on κ_1, κ_2, and $\|\mathbf{z}_h\|_{0,4;\Omega}$, such that*

$$\|\mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{r}}_h) - \mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{s}}_h)\|_{\mathrm{X}_h'} \leq L_{\mathbf{z}_h} \|\vec{\mathbf{r}}_h - \vec{\mathbf{s}}_h\|_{\mathrm{X}} \qquad \forall\, \vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h \in \mathrm{X}_h \,. \tag{4.19}$$

Proof. Noticing that

$$\|\mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{r}}_h) - \mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{s}}_h)\|_{\mathrm{X}_h'} \leq \|\mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{r}}_h) - \mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{s}}_h)\|_{\mathrm{X}'} \,,$$

and then applying Lemma 3.6, we conclude (4.19) with $L_{\mathbf{z}_h} := \max\big\{ L_\kappa, \|\mathbf{z}_h\|_{\mathbf{H}_2} \big\}$. □

Lemma 4.6 *There exists a positive constant $\widetilde{\alpha}_{\mathrm{d}}$, depending only on κ_1 and $\widetilde{c}_{\mathrm{d}}$, such that, for each $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ satisfying $\|\mathbf{z}_h\|_{0,4;\,\Omega} \leq \frac{\kappa_1}{2} \min\{1, \widetilde{c}_{\mathrm{d}}^2\}$, there holds*

$$[\, \mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{r}}_h + \vec{\mathbf{k}}_h) - \mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{s}}_h + \vec{\mathbf{k}}_h), \vec{\mathbf{r}}_h - \vec{\mathbf{s}}_h \,] \geq \widetilde{\alpha}_{\mathrm{d}} \, \|\vec{\mathbf{r}}_h - \vec{\mathbf{s}}_h\|_{\mathrm{X}}^2 \qquad \forall\, \vec{\mathbf{k}}_h \in \mathrm{X}_h \,, \quad \forall\, \vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h \in \mathbf{V}_h \,. \tag{4.20}$$

Proof. Given $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$, $\vec{\mathbf{k}}_h \in \mathrm{X}_h$, and $\vec{\mathbf{r}}_h := (\mathbf{r}_h, \psi_h)$, $\vec{\mathbf{s}}_h := (\mathbf{s}_h, \varphi_h) \in \mathbf{V}_h$, we proceed as in the beginning of the proof of Lemma 3.7 to obtain

$$[\, \mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{r}}_h + \vec{\mathbf{k}}_h) - \mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{s}}_h + \vec{\mathbf{k}}_h), \vec{\mathbf{r}}_h - \vec{\mathbf{s}}_h \,] \geq \|\mathbf{r}_h - \mathbf{s}_h\|_{0,\Omega}^2 - \|\mathbf{z}_h\|_{0,4;\Omega} \, \|\psi_h - \varphi_h\|_{0,4;\Omega} \, \|\mathbf{r}_h - \mathbf{s}_h\|_{0,\Omega} \,.$$

Next, splitting $\|\mathbf{r}_h - \mathbf{s}_h\|_{0,\Omega}^2$ in two halves, applying (4.18) to $\vec{\mathbf{r}}_h - \vec{\mathbf{s}}_h \in \mathbf{V}_h$, and employing Young's inequality, we find that

$$\begin{aligned} &[\, \mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{r}}_h + \vec{\mathbf{k}}_h) \,-\, \mathbf{a}_{\mathbf{z}_h}(\vec{\mathbf{s}}_h + \vec{\mathbf{k}}_h), \vec{\mathbf{r}}_h - \vec{\mathbf{s}}_h \,] \\ &\quad \geq \frac{1}{2} \, (\kappa_1 - \|\mathbf{z}_h\|_{0,4;\Omega}) \, \|\mathbf{r}_h - \mathbf{s}_h\|_{0,\Omega}^2 + \frac{1}{2} \, \big(\kappa_1 \, \widetilde{c}_{\mathrm{d}}^2 - \|\mathbf{z}_h\|_{0,4;\Omega}\big) \, \|\psi_h - \varphi_h\|_{0,4;\Omega}^2 \,, \end{aligned}$$

from which, assuming additionally that $\|\mathbf{z}_h\|_{0,4;\,\Omega} \leq \frac{\kappa_1}{2} \min\{1, \widetilde{c}_{\mathrm{d}}^2\}$, we deduce (4.20) with the constant $\widetilde{\alpha}_{\mathrm{d}} := \frac{\kappa_1}{4} \min\big\{1, \widetilde{c}_{\mathrm{d}}^2\big\}$. □

As a straight consequence of the previous lemmas, we are ready now to establish the well-posedness of (4.3).

Lemma 4.7 *For each* $(\mathbf{z}_h, \psi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi}$ *such that* $\|\mathbf{z}_h\|_{0,4;\Omega} \le \frac{\kappa_1}{2}\min\{1, \widetilde{c}_{\mathrm{d}}^2\}$, *there exists a unique solution* $(\vec{\underset{\sim}{\mathbf{t}}}_h, \underset{\sim}{\boldsymbol{\eta}}_h) := ((\underset{\sim}{\mathbf{t}}_h, \underset{\sim}{\phi}_h), \underset{\sim}{\boldsymbol{\eta}}_h) \in (\mathrm{X}_h^{\mathbf{t}} \times \mathrm{X}_h^{\phi}) \times \mathrm{Y}_h$ *of* (4.3), *and hence one can define* $\mathbf{T}_h(\mathbf{z}_h, \psi_h) := \underset{\sim}{\phi}_h \in \mathrm{X}_h^{\phi}$. *Moreover, there exists a positive constant* $C_{\mathrm{NT,d}}$ *depending only on* $\widetilde{\beta}_{\mathrm{d}}$, κ_1, κ_2, $\widetilde{c}_{\mathrm{d}}$, f_2, $|\Omega|$ *and* $\|\gamma_{\boldsymbol{\nu}}\|$ *such that*

$$\|\mathbf{T}_h(\mathbf{z}_h, \psi_h)\| \le C_{\mathrm{NT,d}}\Big\{\|g\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{0,4;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}. \tag{4.21}$$

Proof. In virtue of Lemmas 4.4, 4.5, and 4.6, the application of Theorem 3.5 to this context yields the desired result. □

In a similar way to that of estimate (4.15), employing (3.29), we are able to bound the second component of the solution of (4.3) as

$$\|\underset{\sim}{\boldsymbol{\eta}}_h\|_{\mathrm{Y}_h} \le \widehat{C}_{\mathrm{NT,d}}\Big\{\|g\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{0,4;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}, \tag{4.22}$$

where $\widehat{C}_{\mathrm{NT,d}}$ is a constant with similar dependence to that of $C_{\mathrm{NT,d}}$.

4.4 Solvability analysis of the discrete fixed point equation

In the same fashion as in Section 3.4, we now aim to apply the Banach theorem to prove that, under suitable assumptions on the data, the operator Θ_h has a fixed point. To this end, following the continuous analysis, and according to estimate (4.14) and the assumption of Lemma 4.7, we set $\delta_{\mathrm{d}} := \frac{\kappa_1}{2}\min\{1, \widetilde{c}_{\mathrm{d}}^2\}$, consider a positive real number r, and define

$$\mathrm{W}_h(r) := \Big\{(\mathbf{z}_h, \psi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi} : \quad \|\mathbf{z}_h\|_{0,4;\Omega} \le \delta_{\mathrm{d}}, \quad \|\psi_h\|_{0,4;\Omega} \le r\Big\}, \tag{4.23}$$

which is a closed subset of $\mathbf{H}_h^{\mathbf{u}} \times \mathrm{X}_h^{\phi}$ and hence a complete metric space. It is clear from lemmas 4.3 and 4.7 that the restriction of Θ_h to $\mathrm{W}_h(r)$ is well-defined. The following result provides sufficient conditions to ensure that Θ_h maps $\mathrm{W}_h(r)$ into itself. We stress here that, as stated in Section 4.3, we are certainly assuming that the finite element subspaces satisfy **(H.1)**–**(H.7)**.

Lemma 4.8 *Assume that there hold*

$$C_{\mathrm{BF,d}}\Big\{r\,\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\} \le \delta_{\mathrm{d}}, \quad \textit{and} \tag{4.24}$$

$$C_{\mathrm{NT,d}}\Big\{\|g\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{0,4;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\} \le r. \tag{4.25}$$

Then, $\Theta_h(\mathrm{W}_h(r)) \subseteq \mathrm{W}_h(r)$.

Proof. It follows from the definition of Θ_h (see (4.4)), together with the a priori estimates (4.14) and (4.21), and the assumptions (4.24) and (4.25). □

In turn, the following two lemmas establish the Lipschitz continuity of the operators $\mathbf{S}_h$ and $\mathbf{T}_h$. Since their proofs are completely analogous to those of Lemmas 3.11 and 3.12, respectively, we omit further details.

Lemma 4.9 *There exists a positive constant* $L_{\mathbf{S},\mathrm{d}}$ *independent of* h, *but depending on* α_{d}, n, $\mathtt{F}$, $|\Omega|$, ρ, δ_{d}, r, *and* $C_{\mathrm{BF,d}}$, *such that*

$$\|\mathbf{S}_h(\mathbf{z}_h, \psi_h) - \mathbf{S}_h(\mathbf{w}_h, \varphi_h)\|_{\mathbf{H}_2} \le L_{\mathbf{S},\mathrm{d}}\Big(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big)\,\|(\mathbf{z}_h, \psi_h) - (\mathbf{w}_h, \varphi_h)\|_{\mathbf{H}_2 \times \mathrm{X}_2},$$

for all $(\mathbf{z}_h, \psi_h), (\mathbf{w}_h, \varphi_h) \in \mathrm{W}_h(r)$.

Lemma 4.10 *There exists a positive constant $L_{\mathbf{T},\mathrm{d}}$ independent of h, but depending on $\widetilde{\alpha}_{\mathrm{d}}$, $C_{\mathrm{NT,d}}$ and L_f, such that*

$$\begin{aligned}&\|\mathbf{T}_h(\mathbf{z}_h,\psi_h)-\mathbf{T}_h(\mathbf{w}_h,\varphi_h)\|_{\mathrm{X}_2}\\&\quad\le\ L_{\mathbf{T},\mathrm{d}}\Big(\|g\|_{0,4/3;\Omega}+\|\mathbf{g}\|_{0,4;\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big)\|(\mathbf{z}_h,\psi_h)-(\mathbf{w}_h,\varphi_h)\|_{\mathbf{H}_2\times\mathrm{X}_2}\,,\end{aligned}$$

for all $(\mathbf{z}_h,\psi_h)$, $(\mathbf{w}_h,\varphi_h)\in\mathrm{W}_h(r)$.

Note that, as in the continuous part, the previous two lemmas have as a direct consequence, the Lipschitz continuity of the (discrete) global fixed point operator Θ_h. In fact, given $(\mathbf{z}_h,\psi_h)$, $(\mathbf{w}_h,\varphi_h)\in\mathrm{W}_h(r)$, according to the preceding two lemmas, we are able to write

$$\|\Theta_h(\mathbf{z}_h,\psi_h)-\Theta_h(\mathbf{w}_h,\varphi_h)\|_{\mathbf{H}_2\times\mathrm{X}_2}\ \le\ L_{\Theta,\mathrm{d}}\mathcal{C}(\mathtt{data})\,\|(\mathbf{z}_h,\psi_h)-(\mathbf{w}_h,\varphi_h)\|_{\mathbf{H}_2\times\mathrm{X}_2},$$

where $L_{\Theta,\mathrm{d}}:=\max\{L_{\mathbf{S},\mathrm{d}},L_{\mathbf{T},\mathrm{d}}\}$ and $\mathcal{C}(\mathtt{data})$ is already defined in (3.54).

Theorem 4.11 *Given $r>0$, assume that, in addition to the hypotheses of Lemma 4.8, the data satisfy*

$$L_{\Theta,\mathrm{d}}\,\mathcal{C}(\mathtt{data})\ <\ 1. \tag{4.26}$$

Then, there exists a unique $(\mathbf{u}_h,\phi_h)\in\mathrm{W}_h(r)$ (cf. (4.23)) fixed point of Θ_h (cf. (4.5)). Equivalently, (4.1) has a unique solution $(\vec{\boldsymbol{\chi}}_h,\boldsymbol{\sigma}_h):=(\underset{\sim}{\boldsymbol{\chi}}_h,\underset{\sim}{\boldsymbol{\sigma}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h$ and $(\vec{\mathbf{t}}_h,\boldsymbol{\eta}_h):=(\underset{\sim}{\mathbf{t}}_h,\underset{\sim}{\boldsymbol{\eta}}_h)\in\mathrm{X}_h\times\mathrm{Y}_h$, with $(\mathbf{u}_h,\phi_h)\in\mathrm{W}_h(r)$, where $(\underset{\sim}{\boldsymbol{\chi}}_h,\underset{\sim}{\boldsymbol{\sigma}}_h)$ and $(\underset{\sim}{\mathbf{t}}_h,\underset{\sim}{\boldsymbol{\eta}}_h)$ are the unique solution of (4.2)–(4.3) with $(\mathbf{z}_h,\psi_h)=(\mathbf{u}_h,\phi_h)$. Moreover, there hold

$$\begin{aligned}\|\vec{\boldsymbol{\chi}}_h\|_{\mathbf{H}_h}\ &\le\ C_{\mathrm{BF,d}}\Big\{\|\mathbf{f}\|_{0,\Omega}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\},\\ \|\boldsymbol{\sigma}_h\|_{\mathbf{Q}_h}\ &\le\ \widehat{C}_{\mathrm{BF,d}}\Big\{\|\mathbf{f}\|_{0,\Omega}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\},\\ \|\vec{\mathbf{t}}_h\|_{\mathrm{X}_h}\ &\le\ C_{\mathrm{NT,d}}\Big\{\|g\|_{0,4/3;\Omega}+\|\mathbf{g}\|_{0,4;\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\},\\ \|\boldsymbol{\eta}_h\|_{\mathrm{Y}_h}\ &\le\ \widehat{C}_{\mathrm{NT,d}}\Big\{\|g\|_{0,4/3;\Omega}+\|\mathbf{g}\|_{0,4;\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}.\end{aligned}\tag{4.27}$$

Proof. Recalling that solving the Galerkin scheme is equivalent to finding a fixed point of Θ_h and noting that (4.26) ensures that Θ_h is a contraction, according to the Banach fixed-point theorem, we have existence and uniqueness of solution for the problem (4.1). In addition, (4.14), (4.15), (4.21) and (4.22) yield the a priori estimates (4.27). □

4.5 A priori error analysis

In this section, our main goal is to deduce optimal a priori error estimates. To this end, given $r>0$ we let $(\vec{\boldsymbol{\chi}},\boldsymbol{\sigma},\vec{\mathbf{t}},\boldsymbol{\eta})\in\mathbf{H}\times\mathbf{Q}\times\mathrm{X}\times\mathrm{Y}$ and $(\vec{\boldsymbol{\chi}}_h,\boldsymbol{\sigma}_h,\vec{\mathbf{t}}_h,\boldsymbol{\eta}_h)\in\mathbf{H}_h\times\mathbf{Q}_h\times\mathrm{X}_h\times\mathrm{Y}_h$ be the solutions of the coupled problem (3.10) and its Galerkin scheme (4.1), respectively, with $(\mathbf{u},\phi)\in\mathrm{W}(r)$ and $(\mathbf{u}_h,\phi_h)\in\mathrm{W}_h(r)$. We stress once more that, in order for this to make sense, the data must satisfy the assumptions studied in the previous sections, and the finite element spaces must fulfill the conditions **(H.1)** through **(H.7)**. Given a subspace V_h of a generic Banach space $(V,\|\cdot\|_V)$, it is customary to set the distance of $v\in V$ to V_h as

$$\mathrm{dist}(v,V_h):=\inf_{v_h\in V_h}\|v-v_h\|_V.$$

In order to derive an a priori estimate for the global error

$$\|(\vec{\boldsymbol{\chi}},\boldsymbol{\sigma}) - (\vec{\boldsymbol{\chi}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\vec{\mathbf{t}},\boldsymbol{\eta}) - (\vec{\mathbf{t}}_h,\boldsymbol{\eta}_h)\|_{\mathrm{X}\times\mathrm{Y}}\,, \tag{4.28}$$

we establish error estimates for each problem separately, that is, we derive optimal error estimates for $\|(\vec{\boldsymbol{\chi}},\boldsymbol{\sigma}) - (\vec{\boldsymbol{\chi}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}}$ and $\|(\vec{\mathbf{t}},\boldsymbol{\eta}) - (\vec{\mathbf{t}}_h,\boldsymbol{\eta}_h)\|_{\mathrm{X}\times\mathrm{Y}}$. The following lemma addresses the estimate corresponding to the convective Brinkman–Forchheimer problem.

Lemma 4.12 *There exists a positive constant $\mathcal{C}_{\mathrm{ST},1}$, independent of h, such that*

$$\begin{aligned}\|(\vec{\boldsymbol{\chi}},\boldsymbol{\sigma}) - (\vec{\boldsymbol{\chi}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\le\, \mathcal{C}_{\mathrm{ST},1}\Big\{&\mathrm{dist}((\vec{\boldsymbol{\chi}},\boldsymbol{\sigma}),\mathbf{H}_h\times\mathbf{Q}_h)\\ &+ \big(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)\,\|(\mathbf{u},\phi) - (\mathbf{u}_h,\phi_h)\|_{\mathbf{H}_2\times\mathrm{X}_2}\Big\}\,.\end{aligned} \tag{4.29}$$

Proof. First, we recall that Lemmas 3.2 and 3.3 provide sufficient conditions to establish the well-posedness of (3.14), as summarized in Lemma 3.4, by applying the Babuška–Brezzi theory (cf. Theorem 3.1). On the other hand, the discrete counterpart, provided by Lemmas 4.1 and 4.2, leads to Lemma 4.3. These results enable the application of [24, Theorem 2.2] in our context, yielding a Strang-type error estimate for this uncoupled problem. Specifically, by setting $X_1 = X_2 = \mathbf{H}$, $M_1 = M_2 = \mathbf{Q}$, $F = \mathbf{F}_\phi$, $G = \mathbf{G}$, $a = \mathbf{A}_{\mathbf{u}}$, $b_1 = b_2 = \mathbf{B}$, $\{X_{1,h}\}_{h>0} = \{X_{2,h}\}_{h>0} = \{\mathbf{H}_h\}_{h>0}$, $\{M_{1,h}\}_{h>0} = \{M_{2,h}\}_{h>0} = \{\mathbf{Q}_h\}_{h>0}$, $a_h = \mathbf{A}_{\mathbf{u}_h}|_{\mathbf{H}_h\times\mathbf{H}_h}$, $b_{1,h} = b_{2,h} = \mathbf{B}|_{\mathbf{H}_h\times\mathbf{Q}_h}$, $F_h = \mathbf{F}_{\phi_h}|_{\mathbf{H}_h}$, and $G_h = \mathbf{G}|_{\mathbf{Q}_h}$, we deduce the existence of a positive constant Λ_{ST} depending only on α, β, α_{d}, β_{d}, δ, δ_{d}, μ, $\mathtt{D}$, $\mathtt{F}$, $|\Omega|$, and ρ, such that

$$\begin{aligned}&\|(\vec{\boldsymbol{\chi}},\boldsymbol{\sigma}) - (\vec{\boldsymbol{\chi}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}}\\ &\quad\le\, \Lambda_{\mathrm{ST}}\Big\{\mathrm{dist}(\vec{\boldsymbol{\chi}},\mathbf{H}_h) + \mathrm{dist}(\boldsymbol{\sigma},\mathbf{Q}_h) + \|(\mathbf{A}_{\mathbf{u}} - \mathbf{A}_{\mathbf{u}_h})(\vec{\boldsymbol{\chi}},\cdot)\|_{\mathcal{V}_h'} + \|\mathbf{F}_\phi - \mathbf{F}_{\phi_h}\|_{\mathcal{V}_h'}\Big\}\,.\end{aligned} \tag{4.30}$$

Next, we proceed to bound the last two terms on the right-hand side of (4.30). Specifically, for the last term, we apply the Cauchy–Schwarz inequality twice to obtain

$$|\mathbf{F}_\phi(\vec{\boldsymbol{\vartheta}}_h) - \mathbf{F}_{\phi_h}(\vec{\boldsymbol{\vartheta}}_h)| \,\le\, \|\mathbf{f}\|_{0,\Omega}\,\|\phi - \phi_h\|_{0,4;\Omega}\,\|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\,\vec{\boldsymbol{\vartheta}}_h \in \mathbf{H}_h\,. \tag{4.31}$$

In turn, given $\vec{\boldsymbol{\varrho}}_h, \vec{\boldsymbol{\vartheta}}_h \in \mathbf{H}_h$, the triangle inequality allows us to bound the third term as

$$\Big|\mathbf{A}_{\mathbf{u}_h}(\vec{\boldsymbol{\varrho}}_h,\vec{\boldsymbol{\vartheta}}_h) - \mathbf{A}_{\mathbf{u}}(\vec{\boldsymbol{\varrho}}_h,\vec{\boldsymbol{\vartheta}}_h)\Big| \,\le\, \Big|\mathbf{A}_{\mathbf{u}}(\vec{\boldsymbol{\chi}} - \vec{\boldsymbol{\varrho}}_h,\vec{\boldsymbol{\vartheta}}_h)\Big| + \Big|\mathbf{A}_{\mathbf{u}_h}(\vec{\boldsymbol{\varrho}}_h - \vec{\boldsymbol{\chi}},\vec{\boldsymbol{\vartheta}}_h)\Big| + \Big|(\mathbf{A}_{\mathbf{u}_h} - \mathbf{A}_{\mathbf{u}})(\vec{\boldsymbol{\chi}},\vec{\boldsymbol{\vartheta}}_h)\Big|, \tag{4.32}$$

where we observe that the first two terms can be bounded using the stability properties of $\mathbf{A}_{\mathbf{u}}$ and $\mathbf{A}_{\mathbf{u}_h}$ (cf. (3.18)), namely

$$\big|\mathbf{A}_{\mathbf{u}}(\vec{\boldsymbol{\chi}} - \vec{\boldsymbol{\varrho}}_h,\vec{\boldsymbol{\vartheta}}_h)\big| + \big|\mathbf{A}_{\mathbf{u}_h}(\vec{\boldsymbol{\varrho}}_h - \vec{\boldsymbol{\chi}},\vec{\boldsymbol{\vartheta}}_h)\big| \,\le\, \big(\|\mathbf{A}_{\mathbf{u}}\| + \|\mathbf{A}_{\mathbf{u}_h}\|\big)\,\|\vec{\boldsymbol{\chi}} - \vec{\boldsymbol{\varrho}}_h\|_{\mathbf{H}}\,\|\vec{\boldsymbol{\vartheta}}_h\|_{\mathbf{H}}\,. \tag{4.33}$$

On the other hand, the third term on the right-hand side of (4.32) can be estimated using similar arguments as in (3.46), yielding

$$\begin{aligned}&|(\mathbf{A}_{\mathbf{u}} - \mathbf{A}_{\mathbf{u}_h})(\vec{\boldsymbol{\chi}},\vec{\boldsymbol{\vartheta}}_h)|\\ &\quad\le\, \left|\frac{1}{2}\int_\Omega \boldsymbol{\chi}(\mathbf{u}_h - \mathbf{u})\cdot\mathbf{v}_h - \frac{1}{2}\int_\Omega (\mathbf{u}\otimes(\mathbf{u}_h - \mathbf{u})) : \boldsymbol{\vartheta}_h + \mathtt{F}\int_\Omega (|\mathbf{u}_h|^{\rho-2} - |\mathbf{u}|^{\rho-2})\mathbf{u}\cdot\mathbf{v}_h\right|\\ &\quad\le\, \Big(1 + \mathtt{F}\,\beta\,|\Omega|^{(4-\rho)/4}\,(\delta_{\mathrm{d}} + \delta)^{\rho-3}\Big)\,\|\vec{\boldsymbol{\chi}}\|_{\mathbf{H}}\,\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,\|\vec{\boldsymbol{\vartheta}}_h\|_{\mathbf{H}}.\end{aligned} \tag{4.34}$$

Thus, by substituting (4.33) and (4.34) back into (4.32), and using the fact that $\|\vec{\boldsymbol{\chi}}\|_{\mathbf{H}}$ is bounded as shown in (3.26) in conjunction with the estimate $\|\phi_h\|_{0,4;\Omega} \le r$, we deduce the existence of a positive constant $\widetilde{\Lambda}$, independent of h, such that

$$\left|\mathbf{A}_{\mathbf{u}_h}(\vec{\boldsymbol{\varrho}}_h,\vec{\boldsymbol{\vartheta}}_h) - \mathbf{A}_{\mathbf{u}}(\vec{\boldsymbol{\varrho}}_h,\vec{\boldsymbol{\vartheta}}_h)\right| \le \widetilde{\Lambda}\left\{\|\vec{\boldsymbol{\chi}} - \vec{\boldsymbol{\varrho}}_h\|_{\mathbf{H}} + \Big(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\Big)\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\right\}\|\vec{\boldsymbol{\vartheta}}_h\|_{\mathbf{H}}, \quad (4.35)$$

for all $\vec{\boldsymbol{\varrho}}_h, \vec{\boldsymbol{\vartheta}}_h \in \mathbf{H}_h$. Finally, by combining (4.30), (4.31), and (4.35), and performing simple algebraic manipulations, we obtain (4.29), thus completing the proof. □

Lemma 4.13 *There exists a positive constant $\mathcal{C}_{\mathrm{ST},2}$, independent of h, such that*

$$\begin{aligned}\|(\vec{\mathbf{t}},\boldsymbol{\eta}) - (\vec{\mathbf{t}}_h,\boldsymbol{\eta}_h)\|_{\mathrm{X}\times\mathrm{Y}} \le \mathcal{C}_{\mathrm{ST},2}\Big\{&\mathrm{dist}\big((\vec{\mathbf{t}},\boldsymbol{\eta}),\mathrm{X}_h\times\mathrm{Y}_h\big)\\ &+\Big(\|g\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{0,4;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big)\|(\mathbf{u},\phi)-(\mathbf{u}_h,\phi_h)\|_{\mathbf{H}_2\times\mathrm{X}_2}\Big\}.\end{aligned} \quad (4.36)$$

Proof. We first apply [6, Lemma 5.1] with $H = \mathrm{X}$, $Q = \mathrm{Y}$, $a = \mathbf{a}_{\mathbf{u}}$, $b = \mathbf{b}$, $\{H_h\}_{h>0} = \{\mathrm{X}_h\}_{h>0}$, $\{Q_h\}_{h>0} = \{\mathrm{Y}_h\}_{h>0}$, $a_h = \mathbf{a}_{\mathbf{u}_h}|_{\mathrm{X}_h\times\mathrm{X}_h}$, $F = \mathrm{F}_\phi$, $G = \mathrm{G}$, $F_h = \mathrm{F}_{\phi_h}|_{\mathrm{X}_h}$, $G_h = \mathrm{G}|_{\mathrm{Y}_h}$, to get

$$\begin{aligned}\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathrm{X}} + \|\boldsymbol{\eta}-\boldsymbol{\eta}_h\|_{\mathrm{Y}} \le\ & C_{\mathrm{S},1}\,\mathrm{dist}(\vec{\mathbf{t}},\mathrm{X}_h) + C_{\mathrm{S},2}\,\mathrm{dist}(\boldsymbol{\eta},\mathrm{Y}_h)\\ &+ C_{\mathrm{S},3}\Big(\|\mathrm{F}_\phi - \mathrm{F}_{\phi_h}\| + \|\mathbf{a}_{\mathbf{u}}(\vec{\mathbf{t}}) - \mathbf{a}_{\mathbf{u}_h}(\vec{\mathbf{t}})\|\Big),\end{aligned} \quad (4.37)$$

where $C_{\mathrm{S},1}$, $C_{\mathrm{S},2}$ and $C_{\mathrm{S},3}$ are positive constants depending only on $L_{\mathbf{u}}$, $\widetilde{\alpha}_{\mathbf{u}_h,\mathrm{d}}$, $\widetilde{\beta}_{\mathrm{d}}$ and $L_{\mathbf{u}_h}$ (cf. Lemmas 3.6, 4.6, 4.4 and 4.5).

Next, we bound the last two terms on the right-hand side of (4.37). In fact, let us consider $\vec{\mathbf{s}}_h \in \mathrm{X}_h$, and by applying Cauchy–Schwarz's inequality twice and the Lipschitz continuity of f, we deduce that

$$\left|\mathrm{F}_\phi(\vec{\mathbf{s}}_h) - \mathrm{F}_{\phi_h}(\vec{\mathbf{s}}_h)\right| = \left|\int_\Omega \Big(f(\phi) - f(\phi_h)\Big)\,\mathbf{g}\cdot\mathbf{s}_h\right| \le \mathrm{L}_f\,\|\phi-\phi_h\|_{0,4;\Omega}\,\|\mathbf{g}\|_{0,4;\Omega}\,\|\mathbf{s}_h\|_{0,\Omega},$$

which directly implies that

$$\|\mathrm{F}_\phi - \mathrm{F}_{\phi_h}\| \le \mathrm{L}_f\,\|\mathbf{g}\|_{0,4;\Omega}\,\|\phi-\phi_h\|_{0,4;\Omega}. \quad (4.38)$$

On the other hand, by using again Cauchy–Schwarz's inequality twice, we obtain

$$\big|[\mathbf{a}_{\mathbf{u}}(\vec{\mathbf{t}}) - \mathbf{a}_{\mathbf{u}_h}(\vec{\mathbf{t}}),\vec{\mathbf{s}}_h]\big| = \left|\int_\Omega \phi\,(\mathbf{u}_h - \mathbf{u})\cdot\mathbf{s}_h\right| \le \|\phi\|_{0,4;\Omega}\|\mathbf{u}_h-\mathbf{u}\|_{0,4;\Omega}\|\mathbf{s}_h\|_{0,\Omega},$$

which, along with the a priori bound for ϕ (cf. (3.56)), implies that

$$\|\mathbf{a}_{\mathbf{u}}(\vec{\mathbf{t}}) - \mathbf{a}_{\mathbf{u}_h}(\vec{\mathbf{t}})\| \le C_{NT}\left\{\|g\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{0,4;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\right\}\|\mathbf{u}_h - \mathbf{u}\|_{0,4;\Omega}. \quad (4.39)$$

Finally, replacing back (4.38) and (4.39) into (4.37) and performing simple algebraic manipulations, we arrive at (4.36). □

The required Céa estimate will now follow from Lemmas 4.12 and 4.13. In fact, from (4.29) and (4.36) we easily obtain

$$\begin{aligned}\|(\vec{\boldsymbol{\chi}},\boldsymbol{\sigma}) - (\vec{\boldsymbol{\chi}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\vec{\mathbf{t}},\boldsymbol{\eta}) - (\vec{\mathbf{t}}_h,\boldsymbol{\eta}_h)\|_{\mathrm{X}\times\mathrm{Y}} \le \mathcal{C}_{\mathrm{ST}}\Big\{&\mathrm{dist}\big((\vec{\boldsymbol{\chi}},\boldsymbol{\sigma}),\mathbf{H}_h\times\mathbf{Q}_h\big)\\ + \mathrm{dist}\big((\vec{\mathbf{t}},\boldsymbol{\eta}),\mathrm{X}_h\times\mathrm{Y}_h\big)\Big\} + \mathcal{C}_{\mathrm{ST}}\,\mathcal{C}(\mathtt{data})\,\|(\mathbf{u},\phi)-(\mathbf{u}_h,\phi_h)\|_{\mathbf{H}_2\times\mathrm{X}_2},\end{aligned} \quad (4.40)$$

where $\mathcal{C}_{\mathrm{ST}} := \max\big\{\mathcal{C}_{\mathrm{ST},1}, \mathcal{C}_{\mathrm{ST},2}\big\}$ and $\mathcal{C}(\mathtt{data})$ is defined in (3.54). Thus, by imposing that the term multiplying $\|(\mathbf{u},\phi)-(\mathbf{u}_h,\phi_h)\|_{\mathbf{H}_2\times \mathrm{X}_2}$ is sufficiently small, say $\leq 1/2$, the aforementioned Céa estimate for our Galerkin scheme (4.1) can be deduced. More precisely, we have proved the following result.

Theorem 4.14 *Assume that the data* $\mathbf{f}, g, \mathbf{g}, \mathbf{u}_{\mathrm{D}}$ *and* ϕ_{D} *(cf.* (3.54)*) are sufficiently small so that*

$$\mathcal{C}_{\mathrm{ST}}\,\mathcal{C}(\mathtt{data}) \,\leq\, \frac{1}{2}\,.$$

Then there holds

$$\begin{aligned} &\|(\vec{\boldsymbol{\chi}},\boldsymbol{\sigma})-(\vec{\boldsymbol{\chi}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \|(\vec{\mathbf{t}},\boldsymbol{\eta})-(\vec{\mathbf{t}}_h,\boldsymbol{\eta}_h)\|_{\mathrm{X}\times\mathrm{Y}} \\ &\quad\leq\, 2\,\mathcal{C}_{\mathrm{ST}}\Big\{\mathrm{dist}\big((\vec{\boldsymbol{\chi}},\boldsymbol{\sigma}),\mathbf{H}_h\times\mathbf{Q}_h\big) + \mathrm{dist}\big((\vec{\mathbf{t}},\boldsymbol{\eta}),\mathrm{X}_h\times\mathrm{Y}_h\big)\Big\}\,. \end{aligned} \tag{4.41}$$

5 Specific finite element subspaces

In this section, we provide specific examples for the choice of the spaces $\mathbf{H}_h^{\boldsymbol{\chi}}$, $\mathbf{H}_h^{\mathbf{u}}$, $\widetilde{\mathbf{Q}}_h$, $\mathrm{X}_h^{\mathbf{t}}$, X_h^{ϕ}, and Y_h that satisfy the hypotheses **(H.1)** through **(H.7)** introduced in Section 4.3. In order to do this, we introduce some preliminary notations. Given an integer $k \geq 0$ and a subset S of R^n, we denote by $\mathrm{P}_k(S)$ the space of polynomials of total degree at most k defined on S, and $\mathbf{P}_k(S)$ its vector counterpart. For each integer $k \geq 0$ and $K \in \mathcal{T}_h$, we define the local Raviart–Thomas space of order k as $\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \widetilde{\mathrm{P}}_k(K)\,\mathbf{x}$, where $\mathbf{x} := (x_1,\dots,x_n)^{\mathtt{t}}$ is a generic vector of R^n and $\widetilde{\mathrm{P}}_k(K)$ is the space of polynomials of total degree equal to k defined on K. Lastly, we define the tensorial counterpart of the Raviart–Thomas space of order k by $\mathbb{RT}_k(K) := \big\{(\tau_{ij}) \,:\, (\tau_{i1},\dots,\tau_{in})^{\mathtt{t}} \in \mathbf{RT}_k(K),\ \forall i \in \{1,\dots,n\}\big\}$.

Next, we introduce the following finite element subspaces:

$$\begin{aligned} \mathbf{H}_h^{\boldsymbol{\chi}} &:= \Big\{\boldsymbol{\chi}_h \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \boldsymbol{\chi}_h\big|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}, \\ \mathbf{H}_h^{\mathbf{u}} &:= \Big\{\mathbf{u}_h \in \mathbf{L}^4(\Omega) : \quad \mathbf{u}_h\big|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}, \\ \widetilde{\mathbf{Q}}_h &:= \Big\{\boldsymbol{\sigma}_h \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \boldsymbol{\sigma}_h\big|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}, \\ \mathrm{X}_h^{\mathbf{t}} &:= \Big\{\mathbf{t}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{t}_h\big|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}, \\ \mathrm{X}_h^{\phi} &:= \Big\{\phi_h \in \mathrm{L}^4(\Omega) : \quad \phi_h\big|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}, \\ \mathrm{Y}_h &:= \Big\{\boldsymbol{\eta}_h \in \mathbf{H}(\mathrm{div}_{4/3};\Omega) : \quad \boldsymbol{\eta}_h\big|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}. \end{aligned} \tag{5.1}$$

It is clear from its definition that $\widetilde{\mathbf{Q}}_h$ satisfies **(H.1)**. Moreover, the definition of the Raviart–Thomas space guarantee that $\mathbf{div}(\mathbf{Q}_h)$ is contained into $\mathbf{H}_h^{\mathbf{u}}$, i.e., **(H.2)** holds. This allows us to obtain the characterization (4.7), which, according to the properties of Raviart–Thomas spaces, implies that $\mathbf{Q}_{0,h}$ is contained in the space of piecewise polynomials of total degree at most k, and therefore **(H.3)** is also verified. The inf-sup condition of hypothesis **(H.4)**, with this space setting, was proved in [6, Lemma 5.5].

On the other hand, **(H.5)** and **(H.6)** hold as well. In fact, these follow from the same arguments used for **(H.2)** and **(H.3)**. We also refer to [6, Lemma 5.5], for a modification for the vectorial case instead of the tensorial one, to verify that hypothesis **(H.7)** holds.

The previous discussion shows that this discrete space setting provides an appropiate family of finite elements, in the sense that the results of the preceding sections hold. Now, our interest lies in the deduction of the rates of convergence. To this end, approximation properties of the finite element subspaces $\mathbf{H}_h^{\boldsymbol{\chi}}$, $\mathbf{H}_h^{\mathbf{u}}$, $\mathbf{Q}_h$, $\mathrm{X}_h^{\mathbf{t}}$, X_h^{ϕ} and Y_h are presented below, which follow from well-known interpolation estimates of Sobolev spaces and the approximation properties of the orthogonal projectors and the interpolation operators involved in their definitions (see, for instance, [7], [9], [20], [22], [23]).

$(\mathbf{AP}_h^{\boldsymbol{\chi}})$ there exists $C > 0$, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\boldsymbol{\vartheta} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$ there holds

$$\mathrm{dist}(\boldsymbol{\vartheta}, \mathbf{H}_h^{\boldsymbol{\chi}}) \,\leq\, C\, h^\ell \, \|\boldsymbol{\vartheta}\|_{\ell,\Omega} \,,$$

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists $C > 0$, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{\ell,4}(\Omega)$ there holds

$$\mathrm{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) \,\leq\, C\, h^\ell \, \|\mathbf{v}\|_{\ell,4;\Omega} \,,$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists $C > 0$, independent of h, such that for each $\ell \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{\ell,4/3}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau}, \mathbf{Q}_h) \,\leq\, C\, h^\ell \, \left\{ \|\boldsymbol{\tau}\|_{\ell,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\ell,4/3;\Omega} \right\} \,,$$

$(\mathbf{AP}_h^{\mathbf{t}})$ there exists $C > 0$, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\mathbf{s} \in \mathbf{H}^\ell(\Omega)$ there holds

$$\mathrm{dist}(\mathbf{s}, \mathrm{X}_h^{\mathbf{t}}) \,\leq\, C\, h^\ell \, \|\mathbf{s}\|_{\ell,\Omega} \,,$$

$(\mathbf{AP}_h^{\phi})$ there exists $C > 0$, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\varphi \in \mathrm{W}^{\ell,4}(\Omega)$ there holds

$$\mathrm{dist}(\varphi, \mathrm{X}_h^{\phi}) \,\leq\, C\, h^\ell \, \|\varphi\|_{\ell,4;\Omega} \,,$$

$(\mathbf{AP}_h^{\boldsymbol{\eta}})$ there exists $C > 0$, independent of h, such that for each $\ell \in (0, k+1]$, and for each $\boldsymbol{\xi} \in \mathbf{H}^\ell(\Omega) \cap \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ with $\mathrm{div}(\boldsymbol{\xi}) \in \mathrm{W}^{\ell,4/3}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\xi}, \mathrm{Y}_h) \,\leq\, C\, h^\ell \, \left\{ \|\boldsymbol{\xi}\|_{\ell,\Omega} + \|\mathrm{div}(\boldsymbol{\xi})\|_{\ell,4/3;\Omega} \right\} \,.$$

We conclude this section with the rates of convergence of the Galerkin scheme (4.1).

Theorem 5.1 *In addition to the hypotheses of Theorems 3.13, 4.11 and 4.14, assume that there exists $\ell \in (0, k+1]$ such that $\boldsymbol{\chi} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{\ell,4}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{\ell,4/3}(\Omega)$, $\mathbf{t} \in \mathbf{H}^\ell(\Omega)$, $\phi \in \mathrm{W}^{\ell,4}(\Omega)$, $\boldsymbol{\eta} \in \mathbf{H}^\ell(\Omega) \cap \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ and $\mathrm{div}(\boldsymbol{\eta}) \in \mathrm{W}^{\ell,4/3}(\Omega)$. Then, there exists a positive constant C, independent of h, such that*

$$\begin{aligned}
&\|(\vec{\boldsymbol{\chi}}, \boldsymbol{\sigma}) - (\vec{\boldsymbol{\chi}}_h, \boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\vec{\mathbf{t}}, \boldsymbol{\eta}) - (\vec{\mathbf{t}}_h, \boldsymbol{\eta}_h)\|_{\mathrm{X}\times\mathrm{Y}} \,\leq\, C\, h^\ell \Big\{ \|\boldsymbol{\chi}\|_{\ell,\Omega} + \|\mathbf{u}\|_{\ell,4;\Omega} \\
&\quad + \|\boldsymbol{\sigma}\|_{\ell,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\ell,4/3;\Omega} + \|\mathbf{t}\|_{\ell,\Omega} + \|\phi\|_{\ell,4;\Omega} + \|\boldsymbol{\eta}\|_{\ell,\Omega} + \|\mathrm{div}(\boldsymbol{\eta})\|_{\ell,4/3;\Omega} \Big\} \,.
\end{aligned}$$

Proof. It follows straightforwardly from (4.41) (cf. Theorem 4.14) and the approximation properties $(\mathbf{AP}_h^{\boldsymbol{\chi}})$, $(\mathbf{AP}_h^{\mathbf{u}})$, $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, $(\mathbf{AP}_h^{\mathbf{t}})$, $(\mathbf{AP}_h^{\phi})$, and $(\mathbf{AP}_h^{\boldsymbol{\eta}})$. □

6 Numerical results

In this section we present three examples illustrating the performance of the fully mixed finite element method (4.1) on a set of quasi-uniform triangulations of the respective domains, and considering the finite element subspaces defined by (5.1) (cf. Section 5). In what follows, we refer to the corresponding sets of finite element subspaces generated by $k = 0$ and $k = 1$, as simply $\mathbb{P}_0 - \mathbf{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{RT}_0$ and $\mathbb{P}_1 - \mathbf{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1 - \mathrm{P}_1 - \mathbf{RT}_1$, respectively. The implementation of the numerical method is based on a `FreeFEM` code [28]. A Newton–Raphson algorithm with a fixed tolerance $\mathsf{tol} = 1\mathrm{E}-06$ is used for the resolution of the nonlinear problem (4.1). As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\mathtt{DoF}}}{\|\mathbf{coeff}^{m+1}\|_{\mathtt{DoF}}} \leq \mathsf{tol}\,,$$

where $\|\cdot\|_{\mathtt{DoF}}$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DoF}}$ with `DoF` denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h^{\boldsymbol{\chi}}, \mathbf{H}_h^{\mathbf{u}}, \mathbf{Q}_h, \mathrm{X}_h^{\mathbf{t}}, \mathrm{X}_h^{\phi}$ and Y_h.

We now introduce some additional notation. The individual errors are denoted by

$$\mathrm{e}(\boldsymbol{\chi}) := \|\boldsymbol{\chi} - \boldsymbol{\chi}_h\|_{0,\Omega}\,, \quad \mathrm{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,, \quad \mathrm{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}\,, \quad \mathrm{e}(p) := \|p - p_h\|_{0,\Omega}\,,$$

$$\mathrm{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}\,, \quad \mathrm{e}(\phi) := \|\phi - \phi_h\|_{0,4;\Omega}\,, \quad \text{and} \quad \mathrm{e}(\boldsymbol{\eta}) := \|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{\mathrm{div}_{4/3};\Omega}\,,$$

where p_h stands for the post-processed pressure suggested by the first formula of (2.6), that is

$$p_h \;=\; -\frac{1}{n}\,\mathrm{tr}\big(\boldsymbol{\sigma}_h + \frac{1}{2}(\mathbf{u}_h \otimes \mathbf{u}_h)\big) - d_h\,, \quad \text{with } d_h \;:=\; -\frac{1}{2\,n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}_h \otimes \mathbf{u}_h)\,. \tag{6.1}$$

As usual, for each $\diamond \in \big\{\boldsymbol{\chi}, \mathbf{u}, \boldsymbol{\sigma}, p, \mathbf{t}, \phi, \boldsymbol{\eta}\big\}$ we let $\mathsf{r}(\diamond)$ be the experimental rate of convergence given by

$$\mathsf{r}(\diamond) \;:=\; \frac{\log(\mathrm{e}(\diamond)/\widehat{\mathrm{e}}(\diamond))}{\log(h/\widehat{h})}\,,$$

where h and $\widehat{h}$ denote two consecutive meshsizes with errors e and $\widehat{\mathrm{e}}$, respectively.

The examples to be considered in this section are described next. In all of them, for the sake of simplicity, we take $\mathbf{g} = (0,-1)$ when $n = 2$ and $\mathbf{g} = (0,0,-1)$ when $n = 3$, and similarly to [1, 5, 6], we choose the nonlinear functions κ and f, respectively, as:

$$\kappa(|\mathbf{t}|) = m_1 + m_2\,(1 + |\mathbf{t}|^2)^{m_3/2-1} \quad \text{and} \quad f(\phi) = c\,\phi\,(1 - c\,\phi)^2\,,$$

where $c = m_1 = m_2 = 1/2$ and $m_3 = 3/2$. In addition, the mean value of $\mathrm{tr}(\boldsymbol{\sigma}_h)$ over Ω is fixed via a Lagrange multiplier strategy (adding one row and one column to the matrix system that solves (4.2) for $\vec{\boldsymbol{\chi}}_h = (\boldsymbol{\chi}_h, \mathbf{u}_h)$ and $\boldsymbol{\sigma}_h$).

Example 1: 2D smooth exact solution with varying D, F, and ρ parameters

In this test we corroborate the rates of convergence in a two-dimensional domain and also study the performance of the numerical method with respect to the number of Newton iterations required to achieve a tolerance of $\mathsf{tol} = 1\mathrm{E}-6$ for different values of the parameters D, F, and ρ. The domain is the square $\Omega = (0,1)^2$. We set $\mu = 1$ and adjust the data $\mathbf{f}$ and g in (2.7) so that the exact solution is given by

$$\mathbf{u}(x_1,x_2) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \quad p(x_1,x_2) = \cos(\pi\, x_1)\, \sin\left(\frac{\pi}{2}\, x_2\right),$$

$$\text{and} \quad \phi(x_1,x_2) = 15 - 15\, \exp(-x_1(x_1-1)x_2(x_2-1)) .$$

Notice that ϕ vanishes at Γ and $\mathbf{u}_{\mathrm{D}}$ is imposed accordingly to the exact solution. Tables 6.1–6.2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations when $\mathtt{D} = 1, \mathtt{F} = 10$, and $\rho = 3$. Notice that we are able not only to approximate the original unknowns but also the pressure field through the formula (6.1). The results confirm that the optimal rates of convergence $\mathcal{O}(h^{k+1})$, provided by Theorem 5.1 are attained for $k = 0, 1$. The Newton method exhibits a behavior independent of the meshsize, converging in six iterations in all cases. In Figure 6.1 we display some solutions obtained with the fully mixed $\mathbb{P}_1 - \mathbf{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1 - \mathrm{P}_1 - \mathbf{RT}_1$ approximation with meshsize $h = 0.0135$ and $41,146$ triangle elements (actually representing $1,606,278\,\mathtt{DoF}$). On the other hand, in Table 6.3 we report the number of Newton iterations as a function of the parameters $\mathtt{D}$, $\mathtt{F}$, and ρ, considering polynomial degree $k = 0$ and different meshsizes h. We observe that Newton's method is robust with respect to both h and ρ, although the number of iterations increases slightly for larger values of $\mathtt{D}$ and $\mathtt{F}$. This behavior is consistent with the growing influence of the nonlinear term $\mathtt{F}\,|\mathbf{u}|\mathbf{u}$.

Example 2: Convergence against smooth exact solutions in a 3D domain

In the second example, we consider the cube domain $\Omega = (0,1)^3$ and the parameters $\mu = 1, \mathtt{D} = 1, \mathtt{F} = 10$ and $\rho = 3.5$. The manufactured solution is given by

$$\mathbf{u}(x_1,x_2,x_3) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3), \end{pmatrix}, \quad p = \cos(\pi\, x_1)\, \exp(x_2 + x_3),$$

$$\text{and} \quad \phi(x_1,x_2,x_3) = 15 - 15\, \exp(-x_1(x_1-1)x_2(x_2-1)x_3(x_3-1)) .$$

Similarly to the first example, the data $\mathbf{f}, g, \mathbf{u}_{\mathrm{D}}$, and ϕ_{D} are computed from (2.7) using the above solution. The convergence history for a set of quasi-uniform mesh refinements using $k = 0$ is shown in Table 6.4. Again, the fully mixed finite element method converges optimally with order $\mathcal{O}(h)$, as it was proved by Theorem 5.1. In addition, some components of the numerical solution are displayed in Figure 6.2, which were built using the fully mixed $\mathbb{P}_0 - \mathbf{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{RT}_0$ approximation with meshsize $h = 0.0866$ and $48,000$ tetrahedral elements (actually representing $1,113,600\,\mathtt{DoF}$).

Example 3: Fluid flow through a rectangular domain with circular obstacles

Inspired by [31, Chapter 1], we finally focus on a flow through a rectangular porous medium with circular obstacles and a non-manufactured solution. More precisely, we consider the domain $\Omega = (0,2)\times(0,0.25)\setminus\Omega_c$, where $\Omega_c := \bigcup_{i=1}^{3}\Omega_c^{\mathrm{up},i} \cup \bigcup_{j=1}^{2}\Omega_c^{\mathrm{down},j}$,

$$\Omega_c^{\mathrm{up},i} = \Big\{(x_1,x_2) : \quad (x_1 - 0.8\,i + 0.6)^2 + (x_2 - 0.15)^2 < 0.05^2\Big\}, \quad i = \{1,2,3\},$$

and

$$\Omega_c^{\mathrm{down},j} = \Big\{(x_1,x_2) : \quad (x_1 - 0.8\,j + 0.2)^2 + (x_2 - 0.1)^2 < 0.05^2\Big\}, \quad j = \{1,2\},$$

with boundary $\Gamma = \partial\Omega$, where the input and output parts are defined as $\Gamma_{\rm in} = \{0\} \times (0, 0.25)$ and $\Gamma_{\rm out} = \{2\} \times (0, 0.25)$, respectively. We consider the parameters $\mu = \exp(-x_1\, x_2)$, $\mathtt{D} = 1$, $\mathtt{F} = 10$, and $\rho = 4$, while the data is chosen as $\mathbf{f} = (0, -9.81)$ and $g = 0$. The boundary conditions are given by

$$\mathbf{u} = (-10\, x_2(x_2 - 0.25), 0)^{\rm t} \quad \text{on} \quad \Gamma_{\rm in} \cup \Gamma_{\rm out}\,, \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma \setminus \big(\Gamma_{\rm in} \cup \Gamma_{\rm out}\big)\,,$$

$$\phi = 1 \quad \text{on} \quad \Gamma_{\rm in}\,, \quad \phi = 5 \quad \text{on} \quad \Gamma_{\rm out}\,, \quad \boldsymbol{\eta} \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma \setminus \big(\Gamma_{\rm in} \cup \Gamma_{\rm out}\big)\,,$$

which drive the flow through a parabolic fluid velocity from left to right of the rectangular domain Ω. We remark that the analysis presented in the previous sections can be extended, with minor modifications, to the case of mixed boundary conditions considered in this example (see, e.g., [11, Section 2.4] and [16] for details). Additionally, the analysis can be readily adapted to scenarios where the parameter μ is spatially varying, provided it is bounded above and below by positive constants.

In Figure 6.3, we show the computed magnitudes of the velocity and concentration gradients, along with the pressure and concentration fields. These results were obtained using the fully mixed $\mathbb{P}_0 - \mathbf{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{RT}_0$ scheme on a mesh with $h = 0.0126$ and $18,916$ triangle elements (corresponding to $237,674$ DoF). As expected, the velocity moves from left to right. In addition, due to the gravitational force imposed in $\mathbf{f}$ and the fact that the flow cannot enter or exit through the top, bottom, and circular boundaries, a sinusoidal behavior is observed. This behavior is consistent with the pressure distribution, which decreases from left to right. In turn, the concentration is smaller on the left side of the domain and increases towards the right, which is consistent with the behavior observed in the magnitude of the concentration gradients. In particular, a higher concentration is observed at the right-bottom corner of the domain.

DoF	h	iter	e($\boldsymbol{\chi}$)	r($\boldsymbol{\chi}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)
1948	0.1964	6	4.42E-01	–	1.09E-01	–	1.96E+00	–
4372	0.1267	6	2.93E-01	0.9361	7.18E-02	0.9443	1.28E+00	0.9814
12020	0.0776	6	1.80E-01	0.9958	4.20E-02	1.0933	7.57E-01	1.0655
39291	0.0443	6	9.87E-02	1.0682	2.35E-02	1.0336	4.17E-01	1.0610
137358	0.0244	6	5.28E-02	1.0516	1.25E-02	1.0668	2.22E-01	1.0621
515117	0.0135	6	2.72E-02	1.1203	6.50E-03	1.1005	1.15E-01	1.1097

e(p)	r(p)	e($\mathbf{t}$)	r($\mathbf{t}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\eta}$)	r($\boldsymbol{\eta}$)
1.48E-01	–	3.57E-01	–	7.74E-01	–	7.11E-01	–
9.18E-02	1.0978	2.37E-01	0.9393	5.08E-01	0.9587	4.78E-01	0.9093
5.60E-02	1.0040	1.38E-01	1.0941	3.24E-01	0.9191	2.81E-01	1.0785
3.00E-02	1.1148	7.67E-02	1.0521	1.69E-01	1.1549	1.54E-01	1.0679
1.60E-02	1.0591	4.14E-02	1.0353	9.22E-02	1.0229	8.36E-02	1.0322
8.35E-03	1.0953	2.13E-02	1.1241	4.79E-02	1.1059	4.32E-02	1.1157

Table 6.1: [Example 1] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the fully mixed $\mathbb{P}_0 - \mathbf{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{RT}_0$ approximation.

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DoF	h	iter	e($\boldsymbol{\chi}$)	r($\boldsymbol{\chi}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(p)	r(p)	e($\mathbf{t}$)	r($\mathbf{t}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\eta}$)	r($\boldsymbol{\eta}$)
6024	0.1964	6	3.01E-02	–	6.69E-03	–	4.33E-01	–	9.86E-03	–	2.11E-02	–	6.20E-03	–	9.60E-02	–
13560	0.1267	6	1.27E-02	1.9634	2.87E-03	1.9340	2.03E-01	1.7349	4.13E-03	1.9856	9.75E-03	1.7590	2.63E-03	1.9606	4.47E-02	1.7455
37368	0.0776	6	4.35E-03	2.1888	1.11E-03	1.9269	7.82E-02	1.9374	1.38E-03	2.2289	3.58E-03	2.0427	8.75E-04	2.2372	1.42E-02	2.3379
122346	0.0443	6	1.34E-03	2.1008	3.28E-04	2.1803	2.54E-02	2.0082	4.10E-04	2.1663	1.06E-03	2.1695	2.57E-04	2.1856	3.98E-03	2.2658
428100	0.0244	6	3.76E-04	2.1332	9.41E-05	2.0971	8.16E-03	1.9070	1.13E-04	2.1628	3.10E-04	2.0653	7.25E-05	2.1267	1.12E-03	2.1339
1606278	0.0135	6	1.02E-04	2.2049	2.52E-05	2.2281	2.53E-03	1.9779	3.03E-05	2.2285	8.29E-05	2.2266	1.97E-05	2.1976	2.97E-04	2.2346

Table 6.2: [Example 1] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the fully mixed $\mathbb{P}_1 - \mathbf{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1 - \mathrm{P}_1 - \mathbf{RT}_1$ approximation.

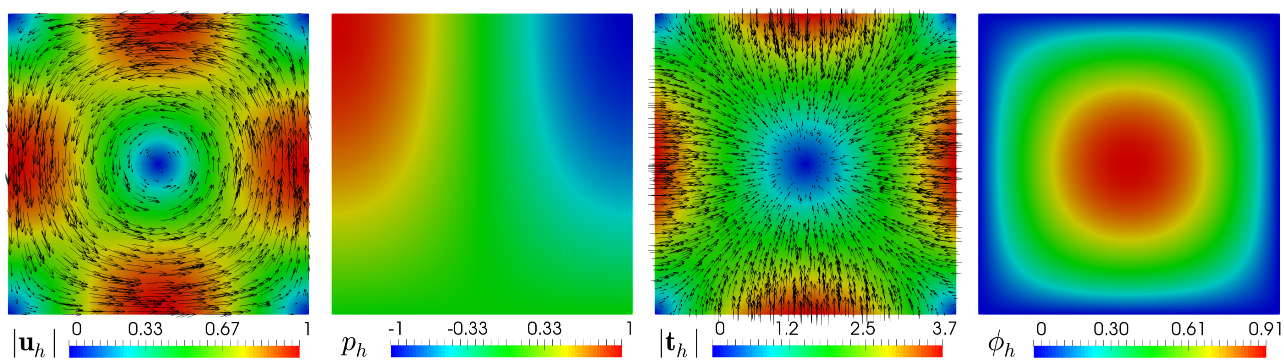

Figure 6.1: [Example 1] From left to right: computed magnitude of the velocity, pressure field, magnitude of the concentration gradient, and concentration field.

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D	F	ρ	$h = 0.1964$	$h = 0.1267$	$h = 0.0776$	$h = 0.0443$	$h = 0.0244$	$h = 0.0135$
1	1	3	6	6	6	6	6	6
10	10	3	6	6	6	6	6	6
10^2	10	3	6	6	6	6	6	6
10^3	10	3	7	7	7	7	7	7
1	10	3	6	6	6	6	6	6
1	10^2	3	6	6	6	6	6	6
1	10^3	3	7	6	7	7	7	7
1	10^4	3	7	7	7	7	7	7
1	10	3.3	6	6	6	6	6	6
1	10	3.5	6	6	6	6	6	6
1	10	3.8	6	6	6	6	6	6
1	10	4.0	6	6	6	6	6	6

Table 6.3: [Example 1] Performance of the iterative method (number of Newton iterations) upon variations of the parameters D, F, and ρ with polynomial degree $k = 0$ and $\mu = 1$.

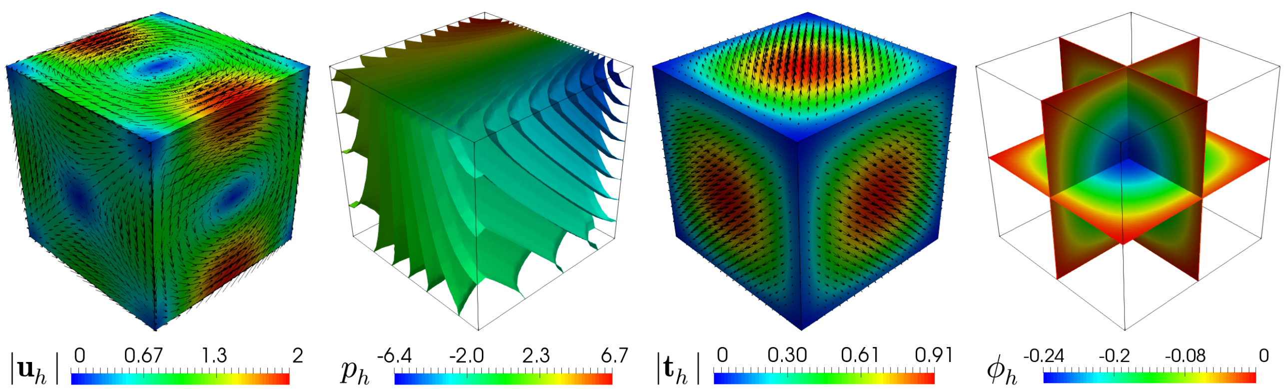

Figure 6.2: [Example 2] From left to right: computed magnitude of the velocity, pressure field, magnitude of the concentration gradient, and concentration field.

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DoF	h	iter	e($\boldsymbol{\chi}$)	r($\boldsymbol{\chi}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)
1200	0.8666	6	2.63E+00	–	5.78E-01	–	1.62E+01	–
9216	0.4330	7	1.44E+00	0.8704	3.02E-01	0.9362	8.72E+00	0.8926
72192	0.2165	7	7.42E-01	0.9549	1.55E-01	0.9618	4.38E+00	0.9929
571392	0.1083	7	3.75E-01	0.9854	7.80E-02	0.9913	2.15E+00	1.0243
1113600	0.0866	7	3.00E-01	0.9938	6.24E-02	0.9970	1.71E+00	1.0276

e(p)	r(p)	e($\mathbf{t}$)	r($\mathbf{t}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\eta}$)	r($\boldsymbol{\eta}$)
1.43E+00	–	2.78E-01	–	6.25E-02	–	1.06E+00	–
7.28E-01	0.9760	1.57E-01	0.8281	3.42E-02	0.8713	5.87E-01	0.8535
3.19E-01	1.1891	8.11E-02	0.9516	1.79E-02	0.9363	3.01E-01	0.9652
1.36E-01	1.2362	4.09E-02	0.9873	9.03E-03	0.9840	1.51E-01	0.9917
1.04E-01	1.1858	3.27E-02	0.9952	7.24E-03	0.9941	1.21E-01	0.9969

Table 6.4: [Example 2] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the fully mixed $\mathbb{P}_0 - \mathbf{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{RT}_0$ approximation.

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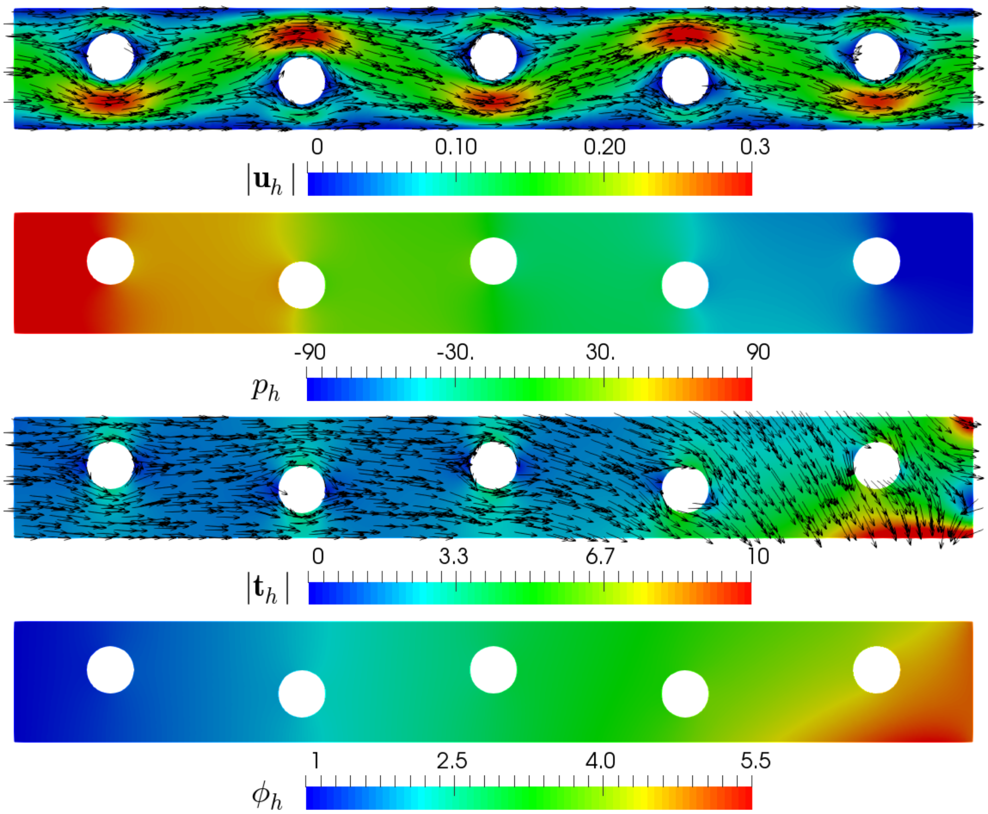

Figure 6.3: [Example 3] From top to bottom: computed magnitude of the velocity, pressure field, magnitude of the concentration gradient, and concentration field.

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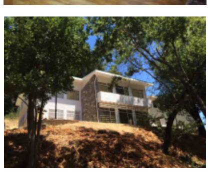

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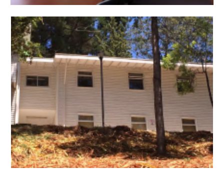

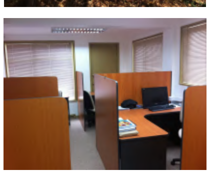

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