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Global Existence and Asymptotic Profile of an Infinite Memory Logarithmic Wave Equation with Fractional Derivative and Strong Damping

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Abstract

This paper investigates the global existence and long-term behavior of solutions to a logarithmic wave equation incorporating infinite memory, fractional derivative, and strong damping in a bounded domain. The equation features a nonlinear logarithmic source term, which is significant in various physical applications such as structural vibrations, fluid dynamics, and quantum mechanics. The presence of strong damping and fractional derivative terms is crucial in ensuring well-posedness and stabilizing the system, while the infinite memory term introduces a complex history-dependent dynamic. This manuscript is a continuation of recent work by the first two authors (Nonlinear logarithmic wave equations: Blow-up phenomena and the influence of fractional damping, infinite memory, and strong dissipation, Evol. Equ. Control Theory, 13(2024), 1423–1435). In addition, numerical simulations are presented to illustrate the asymptotic behavior of solutions.

Keywords: Fractional damping; strong damping; infinite memory; asymptotic behavior.

1. Introduction

We investigate the following problem:

$$\begin{cases} \omega_{tt} - \lambda\Delta\omega - \Delta\omega_t + \int_0^{+\infty} q(s)\Delta\omega(t-s)ds + \partial_t^{\varpi,\varsigma}\omega(t) = \omega|\omega|^{\varrho-2}\ln|\omega|, & \text{in } \Omega\times(0,\infty), \\ \omega = 0, \quad \text{on } \partial\Omega\times(0,\infty), \\ \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad \text{in } \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $q : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function such that

$$q(0) > 0, \quad q_0 = \int_0^\infty q(s)ds = 1-\lambda > 0, \; q'(t) \le -\vartheta q(t), \quad \forall\, t \ge 0,$$

for some positive constant ϑ, and ϱ satisfies

$$\varrho > 2 \text{ if } n = 1, 2 \quad \text{or} \quad 2 < \varrho < \frac{2n}{n-2} \text{ if } n \geq 3.$$

The symbol $\partial_t^{\varpi,\varsigma}$ denote the modified Caputo's fractional derivative which was first defined by Choi and Maccamy [2].

$$\partial_t^{\varpi,\varsigma}\omega(t) = \frac{1}{\Gamma(1-\varpi)} \int_0^t (t-\tau)^{-\varpi} e^{-\varsigma(t-\tau)} \omega_\tau(\tau) d\tau, \qquad 0 < \varpi < 1, \varsigma \geq 0.$$

In the modern age, numerous studies have been conducted on the nonlinear logarithmic term in (1.1), which is a theoretically rich and physically significant nonlinear component. This term plays a crucial role in the study of viscoelastic wave equations, partial differential equations, and related fields. It is particularly relevant to control theory, where the goal is to develop control laws and strategies that guide a system described by PDEs toward a desired trajectory or state.

In recent years, there has been a notable increase in interest surrounding nonlinear wave equations that incorporate logarithmic source terms. In the physical sciences, the logarithmic nonlinearity present in this wave equation model has broad implications. Such nonlinear terms arise in a wide range of theoretical and applied contexts, including nuclear physics, optics, geophysics, symmetry principles in cosmology, quantum mechanics, and fluid dynamics.

This nonlinearity captures two significant vibration events in the specific model considered here, equation (1.1): longitudinal oscillations in structural elements like bars and transverse motions in viscous strings. There are still many unanswered questions about the complex dynamics of these logarithmic wave processes, which have profound effects on both our fundamental understanding of the physical universe and our ability to describe and control complex wave-based systems.

Despite its importance, there is limited research on partial differential equations with strong damping, particularly in the context of wave equations with logarithmic nonlinearity. Strong damping ensures the existence and uniqueness of solutions, significantly enhances stability, and prevents blow-up phenomena in nonlinear logarithmic systems. It improves robustness against disturbances, efficiently dissipates energy, and simplifies numerical analysis, making it essential for both theoretical studies and practical applications. Over the past few decades, extensive research has focused on the characteristics of solutions for wave equations influenced by strong damping and source effects, investigating the existence and nonexistence of solutions, their stability, and blow-up phenomena.

The fields of biology, physics, vibrations, electronics, and other sciences extensively use partial differential equations with fractional derivatives. Recent studies have focused on the regulation of PDEs through the application of fractional derivatives (see [4, 5, 6]). In the context of linear wave equations with fractional derivatives, Matignon et al. [7] have investigated fractional damping in PDEs, demonstrating well-posedness and asymptotic stability, which was a significant contribution. Tatar and Kirane [8] studied wave equations with fractional derivatives for exponential growth. For additional literature on fractional derivatives, see [9, 10, 11].

Recently, Aslam and Hao [12] investigated the effects of strong damping and infinite memory, focusing on the blow-up phenomena of solutions under specific conditions. Their investigation into strongly damped wave equations featuring a logarithmic nonlinear source term, as well as the analysis of PDEs involving fractional derivatives, has provided significant insights into the dynamics and characteristics of these systems. This research represents a pioneering effort in exploring a nonlinear fractional wave equation within a bounded domain, characterized by fractional damping and logarithmic nonlinearity that incorporates infinite memory in $\mathbb{R}^n$. The findings have important implications across various scientific fields.

The author intends to use semigroup theory and appropriate Lyapunov functionals to address this research gap by examining the global existence and asymptotic system's behavior. This current document is outlined as follows: The presumptions and results required to establish the primary findings are given in Section 2. The global solution is demonstrated in Section 3 through the application of semi-group theory [13]. The exponential stability has been discussed in Section 4. Finally, in Section 5, we present numerical examples based on the finite-volume method for spatial discretization and the Newmark method for time integration. A carefully designed discretization is employed for the nonlinear term to ensure energy conservation in the absence of dissipation and a monotonic decrease in energy when dissipation is present.

2. PRELIMINARIES

Several key lemmas which are helpful throughout the work will be pointed out in this section.

Lemma 1. *([14]) Let ζ be defined as: $\zeta(\xi) = |\xi|^{\frac{(2\varpi-1)}{2}}$ for $\xi \in \mathbb{R}$ and $0 < \varpi < 1$. Additionally, let $b = \frac{\sin(\varpi\pi)}{\pi}$. Then, the relationship between the input U and the output O of the system*

$$\begin{cases} \partial_t\theta(\xi,t) - U(x,t)\zeta(\xi) + (\xi^2+\varsigma)\theta(\xi,t) = 0, \quad t>0, \xi\in\mathbb{R}, \varsigma\geq 0, \\ O(t) := b\int_{-\infty}^{+\infty}\theta(\xi,t)\zeta(\xi)\,d\xi \\ \theta(\xi,0)=0, \end{cases} \tag{2.1}$$

is given by

$$O := I^{1-\varpi,\varsigma}U,$$

where

$$I^{\varpi,\varsigma}u(t) := \frac{1}{\Gamma(\varpi)}\int_0^t u(\tau)(t-\tau)^{\varpi-1}e^{-\varsigma(t-\tau)}d\tau.$$

Lemma 2. *([15]) For all $y \in D_\varsigma = \{y\in\mathbb{C} : \mathrm{Re}(y)+\varsigma>0\}\bigcup\{y\in\mathbb{C} : \mathrm{Im}(y)\neq 0\}$, we have*

$$A_y := \int_{-\infty}^{+\infty}\frac{\zeta^2(\xi)}{y+\varsigma+\xi^2}d\xi = \frac{\pi}{sin(\varpi\pi)}(y+\varsigma)^{\varpi-1}.$$

Now, similarly to [3], we define a new variable:

$$o^t(x,s) = \omega(x,t) - \omega(x,t-s),$$

The variable o^t represents the relative history of ω that fulfills the following equation:

$$o^t_t(x,s) - \omega_t(x,t) + o^t_s(x,s) = 0, \quad x\in\Omega, \ \ t,s>0. \tag{2.2}$$

By using Lemma 1 and equation (2.2), the system (1.1) turns into:

$$\begin{cases} \omega_{tt} - \lambda\Delta\omega(t) - \displaystyle\int_0^{+\infty} q(s)\Delta o^t(x,s)ds - \Delta\omega_t & \\ +b\displaystyle\int_{-\infty}^{+\infty}\theta(\xi,t)\zeta(\xi)d\xi = \omega|\omega|^{\varrho-2}\ln|\omega|, & x\in\Omega, t>0, \\ \partial_t\theta(\xi,t) + (\xi^2+\varsigma)\theta(\xi,t) - \omega_t(x,t)\zeta(\xi) = 0, & \xi\in\mathbb{R}, t>0, \varsigma\geq 0, \\ o^t_t(x,s) + o^t_s(x,s) = \omega_t(x,t), & x\in\Omega, t,s>0, \\ \omega = o^t(x,s) = 0, & x\in\partial\Omega, t,s>0, \\ \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), & x\in\Omega, \\ o^t(x,0) = 0, \quad o^0(x,s) = \omega_0(x) - \omega_0(x,-s), & x\in\Omega, t,s>0, \\ \theta(\xi,0) = 0, & x\in\Omega, \xi\in\mathbb{R}. \end{cases} \tag{2.3}$$

The energy of system (2.3) is given by

$$
\begin{aligned}
E(t) := {} & \frac{1}{2}||\omega_t(t)||_2^2 + \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} |\theta(\xi,t)|^2 d\xi dx + \frac{\lambda}{2}||\nabla\omega(t)||_2^2 \\
& + \frac{1}{\varrho^2}||\omega(t)||_\varrho^\varrho - \frac{1}{\varrho}\int_\Omega \ln|\omega|\omega^\varrho dx + \frac{1}{2}\int_0^{+\infty} q(s)||\nabla o^t(s)||_2^2 ds
\end{aligned}
\tag{2.4}
$$

and satisfies

$$
\begin{aligned}
\frac{dE(t)}{dt} = {} & \frac{1}{2}\int_0^{+\infty} q'(s)||\nabla o^t(s)||_2^2 ds - \frac{1}{2}||\nabla\omega||_2^2 \\
& - b\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\varsigma)|\theta(\xi,t)|^2 d\xi dx \leq 0.
\end{aligned}
\tag{2.5}
$$

The energy space $\mathcal{H}$ is defined as follows:

$$
\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega,\mathbb{R}) \times L_q^2(\mathbb{R}_+, H_0^1(\Omega)),
$$

where

$$
L_q^2(\mathbb{R}_+, H_0^1(\Omega)) = \left\{ w : \mathbb{R}_+ \to H_0^1(\Omega) \mid \int_0^{+\infty} q(s)||\nabla w(s)||_2^2 ds < \infty \right\}.
$$

The space $L_q^2(\mathbb{R}_+, H_0^1(\Omega))$ is equipped with the inner product:

$$
\langle w_1, w_2 \rangle_{L_q^2(\mathbb{R}_+,H_0^1(\Omega))} = \int_0^{+\infty} q(s)\int_\Omega \nabla w_1(s)\nabla w_2(s) dx ds.
$$

Consequently, the inner product defined on $\mathcal{H}$ is given by:

$$
\langle U,\bar{U}\rangle_{\mathcal{H}} = \int_\Omega [\lambda\nabla\omega.\nabla\bar{\omega} + u\bar{u}]dx + b\int_\Omega\int_{-\infty}^{+\infty} \theta\bar{\theta} d\xi dx + \int_0^{+\infty} q(s)\int_\Omega \nabla o^t(s)\nabla\bar{o}^t(s) dx ds,
$$

where $U = (\omega, u, \theta, o^t)^T \in \mathcal{H}$ and $\bar{U} = (\bar{\omega}, \bar{u}, \bar{\theta}, \bar{o}^t)^T \in \mathcal{H}$. The system (2.3) can be written as:

$$
\begin{cases} U_t(t) + AU(t) = J(U(t)), \\ U(0) = U_0. \end{cases}
$$

where

$$
AU = \begin{pmatrix} -u \\ -\lambda\Delta\omega - \displaystyle\int_0^{+\infty} q(s)\Delta o^t(x,s)ds + b\int_{-\infty}^{+\infty} \theta(x,\xi,t)\zeta(\xi)d\xi - \Delta\omega_t \\ (\zeta^2+\varsigma)\theta - u(x)\zeta(\xi) \\ o_s^t(s) - u \end{pmatrix},
$$

with domain

$$
D(A) = \left\{ \begin{array}{l} U = (\omega, u, \theta, o^t)^T \in \mathcal{H}; \omega \in H^2(\Omega); u \in H_0^1(\Omega); \\ (\xi^2+\varsigma)\theta - u\zeta(\xi) \in L^2(\Omega,\mathbb{R}); \\ |\xi|\theta \in L^2(\Omega,\mathbb{R}); o_s^t \in L_q^2(\mathbb{R}_+, H_0^1(\Omega)) \end{array} \right\}
$$

and

$$
J(U) = (0, |\omega|^{\varrho-2}\omega\ln|\omega|, 0, 0)^T.
$$

Now, we present the local existence result [12] of the solution to problem (2.3).

Theorem 1. *Let $T > 0$. The system (2.3) possesses a unique solution $U = (\omega, u, \theta, o^t)^T$ that satisfies the following conditions:*

1. *If $U_0 \in \mathcal{H}$, then $U \in C([0,T);\mathcal{H})$.*
2. *If $U_0 \in D(A)$, then $U \in C^1([0,T);\mathcal{H}) \cap C([0,T);D(A))$.*

3. Global existence

In this section, we focus on proving the global existence of the solution for the given problem (2.3). To begin, we define the following functionals:

$$I(t)=\lambda\|\nabla\omega(t)\|_2^2+b\int_\Omega\int_{-\infty}^{+\infty}|\theta(\xi,t)|^2\ \mathrm{d}\xi\mathrm{dx}+\frac{1}{2}\int_0^{+\infty}q(s)\left\|\nabla o^t(t)\right\|_2^2ds-\int_\Omega\ln|\omega|\omega^\varrho dx,$$

$$J(t)=\frac{\lambda}{2}\|\nabla\omega(t)\|_2^2+\frac{1}{\varrho^2}\|\omega(t)\|_\varrho^\varrho-\frac{1}{\varrho}\int_\Omega\ln|\omega|\omega^\varrho dx+\frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\theta(\xi,t)|^2\ \mathrm{d}\xi\mathrm{dx}+\frac{1}{2}\int_0^{+\infty}q(s)\left\|\nabla o^t(t)\right\|_2^2ds.$$

We have

$$E(t)=J(t)+\frac{1}{2}\left\|\omega_t(t)\right\|_2^2.$$

In what follows, we denote by C_r^* the Sobolev embedding constant $H_0^1(\Omega)\hookrightarrow L^r(\Omega)$, i.e.

$$\|w\|_r^r\le C_r^*\|\nabla w\|_2^r,$$

for any $2<r<\frac{2n}{n-2}$.

Lemma 3. *For any $U_0\in\mathcal{H}$ satisfying*

$$\begin{cases}\chi=\frac{2C_{\varrho+l}^*}{\varrho\lambda}\left(\frac{2\varrho}{\lambda(\varrho-2)}E(0)\right)^{\frac{\varrho-2+l}{2}}<1\\ I(0)>0,\end{cases}\tag{3.1}$$

we have $I(t)>0,\forall\, t>0$.

Proof. Given the continuity of ω and the condition $I(0)>0$, it follows that there exists a $T^*<T$ such that $I(t)\ge 0,\ \forall\, t\in[0,T^*]$. Besides, we have

$$\begin{aligned}J(t)=&\lambda\left(\frac{\varrho-2}{2\varrho}\right)\|\nabla\omega(t)\|_2^2+\frac{1}{\varrho^2}\|\omega(t)\|_\varrho^\varrho+b\left(\frac{\varrho-2}{2\varrho}\right)\int_\Omega\int_{-\infty}^{+\infty}|\theta(\xi,t)|^2\,d\xi\,dx\\&+\left(\frac{\varrho-1}{2\varrho}\right)\int_0^{+\infty}q(s)\left\|\nabla o^t(t)\right\|_2^2\,ds+\frac{1}{\varrho}I(t).\end{aligned}$$

Therefore,

$$\lambda\|\nabla\omega(t))\|_2^2\le\frac{2\varrho}{\varrho-2}J(t)\le\frac{2\varrho}{\varrho-2}E(t)\le\frac{2\varrho}{\varrho-2}E(0).\tag{3.2}$$

By using the fact $\ln|\omega|<|\omega|^l$, we get

$$\int_\Omega\ln|\omega|\omega^\varrho dx\le\int_\Omega|\omega|^{\varrho+l}dx.$$

where l is chosen to be $\frac{1}{e}<l<\frac{2}{n-2}$, so that

$$\varrho+l<\frac{2n-2}{n-2}+l<\frac{2n}{n-2}.$$

Therefore, by embedding $H_0^1(\Omega)\hookrightarrow L^{\varrho+l}(\Omega)$, it holds that

$$\begin{aligned}\frac{1}{\varrho}\int_\Omega\ln|\omega|\omega^\varrho dx&\le\frac{C_{\varrho+l}^*}{\varrho}\|\nabla\omega\|_2^{\varrho+l}\le\frac{2C_{\varrho+l}^*}{\varrho\lambda}\|\nabla\omega\|_2^{\varrho+l-2}\left(\frac{\lambda}{2}\|\nabla\omega\|_2^2\right)\\&\le\frac{2C_{\varrho+l}^*}{\varrho\lambda}\left(\left(\frac{2\varrho}{\lambda(\varrho-2)}E(0)\right)\right)^{\frac{\varrho-2+l}{2}}\left(\frac{\lambda}{2}\|\nabla\omega\|_2^2\right)<\frac{\lambda}{2}\|\nabla\omega\|_2^2\end{aligned}\tag{3.3}$$

Thus, $I(t) > 0, \forall\, t \in [0, T^*]$. Repeating this process and taking into account the fact that

$$\lim_{t \to T^*} \frac{2C^*_{\varrho+l}}{\varrho\lambda} \left(\frac{2\varrho}{\lambda(\varrho-2)} E(0) \right)^{\frac{\varrho-2+l}{2}} < 1,$$

yields to $T^* = T$. Furthermore, we have

$$\frac{1}{2} \|\omega_t(t)\|_2^2 + \frac{\lambda(\varrho-2)}{2\varrho} \|\nabla\omega\|_2^2 \leq \frac{1}{2} \|\omega_t(t)\|_2^2 + J(t) = E(t) \leq E(0),$$

which implies that the solution of system (2.3) is both global in time and bounded. □

4. Exponential stability

Let us define the functional $L(t)$ by

$$L(t) = \mathrm{N}E(t) + \mathrm{N}_1\varphi(t), \tag{4.1}$$

where N and N_1 are positive constants that will be fixed later, and

$$\varphi(t) = \int_\Omega \omega_t \omega dx + \frac{b}{2} \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \varsigma\right) \mid \mathcal{M}(x,\xi) \mid^2 d\xi dx,$$

where

$$\mathcal{M}(x,\xi) = \frac{\omega_0(x)\zeta(\xi)}{(\xi^2+\varsigma)} + \int_0^t \theta(x,\xi,s)ds.$$

Lemma 4. *Let $(\omega, \omega_t, \theta, o^t)^T$ be a regular solution of the problem* (2.3)*. Then, we have*

$$\int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \varsigma\right) \theta(\xi,t)\mathcal{M}(x,\xi) d\xi dx = \int_\Omega \omega(x,t) \int_{-\infty}^{+\infty} \theta(\xi,t)\zeta(\xi) d\xi dx - \int_\Omega \int_{-\infty}^{+\infty} |\theta(\xi,t)|^2 d\xi dx. \tag{4.2}$$

Proof. Clearly, by using the second equation of (2.3) we obtain

$$\left(\xi^2 + \varsigma\right) \theta(\xi,t) = \omega_t(x,t)\zeta(\xi) - \partial_t\theta(\xi,t), \quad \forall x \in \Omega. \tag{4.3}$$

Integrating (4.3) between 0 and t yields to

$$\int_0^t \left(\xi^2 + \varsigma\right) \theta(\xi,s)ds = \omega(x,t)\zeta(\xi) - \theta(\xi,t) - \omega_0(x)\zeta(\xi), \quad \forall x \in \Omega,$$

So,

$$\left(\xi^2 + \varsigma\right) \left(\int_0^t \theta(\xi,s)ds + \frac{\omega_0(x)\zeta(\xi)}{(\xi^2+\varsigma)} \right) = \omega(x,t)\zeta(\xi) - \theta(\xi,t), \quad \forall x \in \Omega. \tag{4.4}$$

Multiplying (4.4) by θ and integrating over $\Omega \times (-\infty, +\infty)$, we obtain (4.2). □

By following the same steps as in [1, Lemma 4.4], we easily obtain:

Lemma 5. *Let $(\omega, \omega_t, \theta, o^t)$ be the solution of problem* (2.3)*. Therefore,*

$$|\varphi(t)| \leq \frac{1}{2}\|\omega_t\|_2^2 + \frac{C_2^*}{2}\left(1 + 4A_0\right)\|\nabla\omega\|_2^2 + \frac{2}{\varsigma} \int_\Omega \int_{-\infty}^{+\infty} |\theta(\xi,t)|^2 \, d\xi \, dx.$$

This last lemma and (3.3) give that $L(t) \sim E(t)$ for N large enough, i.e., there exist a positive constants c_1 and c_2 such that

$$c_1 E(t) \leq L(t) \leq c_2 E(t), \ \forall\, t \geq 0.$$

Theorem 2. *Assume that* (3.1) *holds true. Then, there exist positive constants k and K such that*

$$E(t) \leq K e^{-kt}, \ \forall\, t \geq 0.$$

Proof. We will work with regular solutions, and the decay holds true for weak solutions according to classic density arguments. We differentiate (4.1) to obtain

$$\begin{aligned}
L'(t) &= \mathrm{N}E'(t) + \mathrm{N}_1\varphi'(t)\\
&= \frac{\mathrm{N}}{2}\int_0^{+\infty} q'(s)||\nabla o^t(s)||_2^2 ds - \frac{\mathrm{N}}{2}||\nabla\omega_t||_2^2 - \mathrm{N}b\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\varsigma)|\theta(\xi,t)|^2 d\xi dx + \mathrm{N}_1\|\omega_t\|_2^2 - \mathrm{N}_1\lambda\|\nabla\omega\|_2^2\\
&\quad -\mathrm{N}_1\int_0^{+\infty} q(s)\int_\Omega \nabla\omega\nabla o^t dx d\xi - \mathrm{N}_1\int_\Omega \nabla\omega\nabla\omega_t dx - \mathrm{N}_1 b\int_\Omega \omega\int_{-\infty}^{\infty}\theta(\xi,t)\zeta(\xi)d\xi dx + \mathrm{N}_1\int_\Omega |\omega|^\varrho \ln|\omega| dx\\
&\quad +\mathrm{N}_1 b\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\varsigma\right)\theta(\xi,t)\mathcal{M}(x,\xi)d\xi dx\\
&= \frac{\mathrm{N}}{2}\int_0^{+\infty} q'(s)||\nabla o^t(s)||_2^2 ds - \frac{\mathrm{N}}{2}||\nabla\omega_t||_2^2 - \mathrm{N}b\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\varsigma)|\theta(\xi,t)|^2 d\xi dx + \mathrm{N}_1\|\omega_t\|_2^2 - \mathrm{N}_1\lambda\|\nabla\omega\|_2^2\\
&\quad -\mathrm{N}_1\int_0^{+\infty} q(s)\int_\Omega \nabla\omega\nabla o^t dx d\xi - \mathrm{N}_1\int_\Omega \nabla\omega\nabla\omega_t dx + \mathrm{N}_1\int_\Omega |\omega|^\varrho\ln|\omega| dx - \mathrm{N}_1 b\int_\Omega\int_{-\infty}^{+\infty}|\theta(\xi,t)|^2 d\xi dx
\end{aligned}\tag{4.5}$$

By combining Young's inequality and Hölder's inequality, we find that:

$$-\int_0^{+\infty} q(s)\int_\Omega \nabla\omega\nabla o^t dx d\xi \leq \frac{\lambda}{4}\|\nabla\omega\|_2^2 + \frac{1-\lambda}{\lambda}\int_0^{+\infty} q(s)\left\|\nabla o^t(x,s)\right\|_2^2 ds, \tag{4.6}$$

and

$$\int_\Omega \nabla\omega\nabla\omega_t dx \leq \frac{\lambda}{4}\|\nabla\omega\|_2^2 + \frac{1}{\lambda}\|\nabla\omega_t\|_2^2. \tag{4.7}$$

Inserting (4.6) and (4.7) in (4.5) and using the fact that $q'(s) \leq -\vartheta q(s), \|\omega_t\|_2^2 \leq C_2^*\|\nabla\omega_t\|_2^2$ (since $\omega_t \in H_0^1(\Omega)$) and (3.2), we obtain

$$\begin{aligned}
L'(t) &\leq -\mathrm{N}_1\|\omega_t\|_2^2 - \left(\frac{\mathrm{N}}{2} - \mathrm{N}_1(2C_2^* + \frac{1}{\lambda})\right)\|\nabla\omega_t\|_2^2 - \left(\frac{\mathrm{N}_1\lambda}{2} - \frac{C_\varrho^*}{\varrho^2}\left(\frac{2pE(0)}{\lambda(\varrho-2)}\right)^{\frac{\varrho-2}{2}}\right)\|\nabla\omega\|_2^2 + \mathrm{N}_1\int_\Omega|\omega|^\varrho\ln|\omega|dx\\
&\quad -\frac{1}{\varrho^2}\|\omega_t\|_\varrho^\varrho - \left(\frac{\mathrm{N}\vartheta}{2} - \frac{\mathrm{N}_1(1-\lambda)}{\lambda}\right)\int_0^{+\infty} q(s)\left\|\nabla o^t(x,s)\right\|_2^2 ds - \mathrm{N}_1 b\int_\Omega\int_{-\infty}^{+\infty}|\theta(\xi,t)|^2 d\xi dx.
\end{aligned}\tag{4.8}$$

Now, first choose N_1 such that

$$\mathrm{N}_1 > \frac{2C_\varrho^*}{\lambda\varrho^2}\left(\frac{2pE(0)}{\lambda(\varrho-2)}\right)^{\frac{\varrho-2}{2}},$$

and then N so that

$$\mathrm{N} > \max\left\{2\mathrm{N}_1(2C_2^* + \frac{1}{\lambda}), \frac{2\mathrm{N}_1(1-\lambda)}{\vartheta\lambda}\right\}.$$

Consequently, from above, we deduce that there exists a positive m such that (4.8) becomes

$$L'(t) \leq -mE(t), \quad \text{for all t} \geq 0.$$

Using Gronwall's inequality and the fact that $L(t) \sim E(t)$, we deduce that

$$E(t) \leq \frac{c_2}{c_1}E(0)e^{-\frac{m}{c_2}t}, \ t \geq 0.$$

This ends the proof of Theorem 2. □

5. Numerical Approximation

In this section, we will numerically verify the exponential decay rate of the energy obtained in the previous section for the case $n = 1$.

5.1. Finite Volume Approximation

We consider the finite volume method (FVM) for spatial discretization of the variable $\omega = \omega(x,t)$, based on a discretization of the finite difference of the flux [16]. In this sense, let there be for the one-dimensional space case ($n = 1$), a uniform discretization of the domain $\Omega = (0,l)$ in small J control volumes $K_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, with $x_{j+\frac{1}{2}} = j\delta x$, $\delta x = \dfrac{l}{J}$, $j = 1,\dots,J$. The unknown $\omega(x,t)$, are approximated by $\omega_j(t)$ in the control volume K_j. Given the uniformity of the mesh, the Laplacian operator terms are approximated as:

$$\Delta\omega = \omega_{xx} \approx \left(\mathbf{D}^2\omega\right)_j = \frac{\omega_{j-1} - 2\omega_j + \omega_{j+1}}{\delta x^2}, \qquad j = 1,\dots,J \tag{5.1}$$

with $\omega_0 = \omega_{J+1} = 0$.

5.2. Linear equations of Motion

Let the vector $\mathbf{w}(t) = [\omega_1(t),\dots,\omega_J(t)]^\top$, an approximation of $\omega(x,t)$ in $\mathbb{R}^J$. Considering (5.1), we have the following system of equations of motion

$$\mathbf{M}\ddot{\mathbf{w}}(t) + \mathbf{K}\mathbf{w}(t) + \mathbf{C}\dot{\mathbf{w}}(t) = \mathbf{J}(\mathbf{w}) \tag{5.2}$$

where $\mathbf{M} = \mathbf{I}_{J\times J}$ is the identity matrix of size J, $\mathbf{K} = -\lambda\mathbf{D}^2$ is the stiffness matrix, and

$$\mathbf{C} = \mathbf{C}_{\Delta\omega_t} + \mathbf{C}_{memo} + \mathbf{C}_{frac}$$

is the dissipation matrix given by the sum of the three matrices taking part in the approximation of the three dissipative terms of the equation (1.1):

- $\mathbf{C}_{\Delta\omega_t}$ which characterizes the Kelvin-Voight dissipative term $-\Delta\omega_t$;
- $\mathbf{C}_{memo}$ which characterizes the infinite memory dissipative term $\int_0^\infty q(s)\Delta w(t-s)\,ds$;
- $\mathbf{C}_{frac}$ which characterizes the fractional derivative dissipative term $\partial_t^{\varpi,\varsigma}\omega(t)$;

In the case of the Kelvin-Voight dissipative term, the matrix is simply given by $\mathbf{C}_{\Delta\omega_t} = -\mathbf{D}^2$. In the other two cases, since both terms are non-local in time, it is important to first specify the time discretization of our scheme.

5.3. Time discretization

In order to preserve the energy with a second-order scheme in time, we choose a β-Newmark scheme for w. The method consists of updating the displacement, velocity and acceleration vectors at the current time $t^n = n\delta t$ to the time $t^{n+1} = (n+1)\delta t$, a small time interval δt later. The Newmark algorithm [18] is based on a set of two relations expressing the forward displacement $\mathbf{w}^{n+1}$ and velocity $\dot{\mathbf{w}}^{n+1}$ in terms of their current values and the forward and current values of the acceleration:

$$\dot{\mathbf{w}}^{n+1} = \dot{\mathbf{w}}^n + (1-\gamma)\delta t\,\ddot{\mathbf{w}}^n + \gamma\delta t\,\ddot{\mathbf{w}}^{n+1}, \tag{5.3}$$

$$\mathbf{w}^{n+1} = \mathbf{w}^n + \delta t\dot{\mathbf{w}}^n + \left(\frac{1}{2} - \beta\right)\delta t^2\,\ddot{\mathbf{w}}^n + \beta\delta t^2\,\ddot{\mathbf{w}}^{n+1}, \tag{5.4}$$

where β and γ are parameters of the methods that will be fixed later. Returning now to the description of nonlocal dissipative matrices, we have the following.

5.4. Infinite memory term.

The infinite memory term $\int_0^\infty q(s)\Delta w(t-s)\,ds$ taking part in the equation (1.1), can be approximated by $\mathbf{C}_{memo}\mathbf{w}$. Before specifying the matrix associated with this decay, we must note that this type of infinite memory terms have already been treated numerically in multiple works, for example in [17] in which reasonable results are obtained, however the energy is not conserved and spurious oscillations occur in the decay. In order to avoid these unwanted oscillations, we consider another approximation in which we use the modified model (2.3), and we discretize the variable o^t introduced in (2.2). This allows us to obtain a conservative scheme whose energy will be numerically decreasing. To do this, we approximate $o^t(x,s)$ by $o_j^{m,n}$, for $j=1,\dots,J$, $n=1,\dots,N$ and $m=1,\dots,M$ in the equation (2.2), and defining

$$o_j^{m,n} := \omega_j^n - \omega_j^{n-m} \tag{5.5}$$

Replacing (5.3)-(5.4) in (5.5), we obtain

$$\begin{aligned} o_j^{m,n+1} - o_j^{m-1,n} &= \omega_j^{n+1} - \omega_j^n \\ &= \delta t\left(\frac{\beta}{\gamma}\dot{\omega}_j^{n+1} + \left(1-\frac{\beta}{\gamma}\right)\dot{\omega}_j^{n+1}\right) - \frac{2\beta-\gamma}{2\gamma}\delta t^2\ddot{\omega}_j^n \end{aligned}$$

Then $\gamma = \frac{1}{2}$, $\beta = \frac{1}{2}\gamma$ is chosen, in order to obtain the following conservative scheme

$$o_j^{m,n+1} = o_j^{m-1,n} + \delta t\dot{\omega}_j^{n+\frac{1}{2}} \tag{5.6}$$

with $\dot{\omega}_j^{n+\frac{1}{2}} = \dfrac{\dot{\omega}_j^n + \dot{\omega}_j^{n+1}}{2}$. Then from (2.3), the approximation of the infinite-memory term at time $t = t_{n+1}$ can be written as

$$\begin{aligned} -\int_0^{+\infty} q(s)\Delta o^{t_{n+1}}(x,s)ds &\approx -\delta t\sum_{m=1}^{M} q_m\mathbf{D}^2\mathbf{o}^{m,n+1} \\ &= -\delta t\sum_{m=1}^{M} q_m\mathbf{D}^2\mathbf{o}^{m-1,n} - \delta t^2\left(\sum_{m=1}^{M} q_m\right)\mathbf{D}^2\dot{\mathbf{w}}^{n+\frac{1}{2}} \end{aligned} \tag{5.7}$$

5.5. Fractional derivative term.

In order to numerically simulate the improper integral $(2.1)_2$, we consider $R>0$ sufficiently large, so that

$$\partial_t^{\varpi,\varsigma}\omega(t) \approx 2b\int_0^R \theta(\xi,t)\zeta(\xi)d\xi$$

(we note the parity of the function $\theta\zeta$ with respect to ξ from (2.1)). Let $\xi_\ell := \ell\delta\xi$ $\ell = 1,\dots,L$, $\delta\xi = L/R$. From (2.1), we define

$$\zeta_\ell = |\xi_\ell|^{(2\varpi-1)/2}, \quad \ell = 1,\dots,L, \quad 0<\varpi<1.$$

In the case of this dissipative term, we will simply be inspired by the work of [20], where considering the augmented model of [14] results in a conservative scheme and decreasing numerical energy. Thus, an approximation of the fractional derivative term, is given by

$$\partial_t^{\varpi,\varsigma}\omega(t) \approx 2b\delta\xi\sum_{\ell=1}^{L}\zeta_\ell\theta_\ell^n. \tag{5.8}$$

On the other hand, the system (2.1) can be discretized using the Crank–Nicolson method [19], in order to maintain the conservation of energy, or its nondecrease in case of dissipation. Then, we obtain the following conservative numerical scheme:

$$\theta_\ell^{n+1} = \theta_\ell^n - \delta t \left(\xi_\ell^2 + \varsigma\right) \theta_\ell^{n+\frac{1}{2}} + \delta t \zeta_\ell \dot{\mathbf{w}}^{n+\frac{1}{2}} \tag{5.9}$$

Combining then (5.7) and (5.9) with (5.3) and (5.4), and replacing these expressions in (5.2) for $t = t_{n+1}$ gives the following system of nonlinear equations describing the first part of the conservative scheme:

$$\begin{aligned}
&\left(\mathbf{M} + \gamma\delta t\, \mathbf{C} + \beta\delta t^2\, \mathbf{K}\right) \ddot{\mathbf{w}}^{n+1} - \mathbf{J}(\mathbf{w}^{n+1}) = \\
&\quad - \mathbf{C}\left(\dot{\mathbf{w}}^n + (1-\gamma)\,\delta t \ddot{\mathbf{w}}^n\right) - \mathbf{K}\left(\mathbf{w}^n + \delta t \dot{\mathbf{w}}^n + \left(\frac{1}{2} - \beta\right)\delta t^2 \ddot{\mathbf{w}}^n\right), \\
&\quad - \delta t \sum_{m=1}^{M} q_k \mathbf{D}^2 \mathbf{o}^{m-1,n} - \frac{\delta t^2}{2}\left(\sum_{m=1}^{M} q_k\right) \mathbf{D}^2 \dot{\mathbf{w}}^n - b\delta\xi \sum_{\ell=1}^{L} \widetilde{\zeta}_\ell \theta_\ell^n
\end{aligned} \tag{5.10}$$

with $\mathbf{C} = \mathbf{C}_{\Delta\omega_t} + \mathbf{C}_{memo} + \mathbf{C}_{frac}$ and

$$\begin{cases}
\mathbf{C}_{\Delta\omega_t} = -\mathbf{D}^2 \\
\mathbf{C}_{memo} = -\dfrac{\delta t}{2}\left(\displaystyle\sum_{m=1}^{M} q_k\right)\mathbf{D}^2 \\
\mathbf{C}_{frac} = \delta t b \left(\displaystyle\sum_{\ell=1}^{L} \frac{2\widetilde{\zeta}_\ell^2}{2 + \delta t\left(\xi_\ell^2 + \varsigma\right)}\right)\mathbf{I}_J, \qquad \text{with} \quad \widetilde{\zeta}_\ell = \dfrac{2 - \delta t\left(\xi_\ell^2 + \varsigma\right)}{2 + \delta t\left(\xi_\ell^2 + \varsigma\right)}\zeta_\ell
\end{cases}$$

5.6. Source term $J(\mathbf{w})$

In the first instance we propose a discretization of the nonlinear term $\mathbf{J}(\mathbf{w})$ for the scheme (5.10), in a quite natural and naive way as

$$\mathbf{J}(\mathbf{w}^{n+1})_j = \mathbf{J}(\omega_j^{n+1}) = \omega_j^{n+1}|\omega_j^{n+1}|^\varrho \ln|\omega_j^{n+1}| \tag{5.11}$$

This choice is reasonable; however, it does not preserve the system's energy when the dissipative terms in equation (1.1) and the corresponding (5.10) scheme are not taken into account. Specifically, when analyzing the (5.10) scheme combined with (5.11), a numerical dissipation of energy is observed, as shown in Figure 1, along with persistent oscillations around this decay. While dissipation is desirable, we aim for a more accurate scheme free of spurious numerical effects, enabling a clearer assessment of the performance of the approximate dissipative terms. For this reason, we subsequently propose the following conservative scheme:

$$\begin{aligned}
&\left(\mathbf{M} + \gamma\delta t\, \mathbf{C} + \beta\delta t^2\, \mathbf{K}\right) \ddot{\mathbf{w}}^{n+1} - \mathcal{J}(\mathbf{w}^n, \mathbf{w}^{n+1}) = \mathcal{L}(\mathbf{w}^n, \dot{\mathbf{w}}^n, \ddot{\mathbf{w}}^n, \mathbf{o}^{\cdot,n}, \theta^n) \\
&\quad - \mathbf{C}\left(\dot{\mathbf{w}}^n + (1-\gamma)\,\delta t \ddot{\mathbf{w}}^n\right) - \mathbf{K}\left(\mathbf{w}^n + \delta t \dot{\mathbf{w}}^n + \left(\frac{1}{2} - \beta\right)\delta t^2 \ddot{\mathbf{w}}^n\right), \\
&\quad - \delta t \sum_{m=1}^{M} q_m \mathbf{D}^2 \mathbf{o}^{m-1,n} - \frac{\delta t^2}{2}\left(\sum_{m=1}^{M} q_m\right) \mathbf{D}^2 \dot{\mathbf{w}}^n - b\delta\xi \sum_{\ell=1}^{L} \widetilde{\zeta}_\ell \theta_\ell^n
\end{aligned} \tag{5.12}$$

where

$$\mathcal{J}(\mathbf{w}^n, \mathbf{w}^{n+1})_j = \begin{cases} \dfrac{\mathbf{F}(\omega_j^{n+1}) - \mathbf{F}(\omega_j^n)}{\omega_j^{n+1} - \omega_j^n} & \text{if } \omega_j^n \neq \omega_j^{n+1} \\ 0 & \text{otherwise,} \end{cases} \qquad j = 1, \ldots, J, \tag{5.13}$$

with $\mathbf{F}(\mathbf{w})_j = \dfrac{1}{\varrho^2}|\omega_j|^\varrho(\ln|\omega_j|^\varrho - 1)$ and

$$\mathcal{L}(\mathbf{w}^n, \dot{\mathbf{w}}^n, \ddot{\mathbf{w}}^n, \mathbf{o}^{\cdot,n}, \theta^n) = -\left(\mathbf{M}\ddot{\mathbf{w}}^n + \mathbf{K}\mathbf{w}^n + \mathbf{K}\dot{\mathbf{w}}^n\right) - \delta t \sum_{m=1}^{M} q_m \mathbf{D}^2 \mathbf{o}^{m,n} - b\delta\xi \sum_{\ell=1}^{L} \zeta_\ell \theta_\ell^n \tag{5.14}$$

Remark 1. • *The function* $\mathbf{F}(\cdot)$ *is a primitive of the function* $\mathbf{J}(\cdot)$*, so the term* $\mathcal{J}$ *defined in* (5.13) *represents a finite difference of* J *and, therefore, the scheme* (5.12) *is consistent with* (2.3)*. This approach to discretizing the nonlinear term, aimed at ensuring energy conservation, is partially inspired by the work of Delfour et al. [21]."*

- *The so-called scheme without the dissipative physical terms can be rewritten and is given by*

$$\mathbf{M}\ddot{\mathbf{w}}^{n+\frac{1}{2}} + \mathbf{K}\mathbf{w}^{n+\frac{1}{2}} = \mathcal{J}(\mathbf{w}^n, \mathbf{w}^{n+1}) \tag{5.15}$$

where $\mathbf{w}^{n+\frac{1}{2}} = \dfrac{\mathbf{w}^n + \mathbf{w}^{n+1}}{2}$*. The scheme* (5.15) *represents a discretization of the equation*

$$\omega_{tt} - \Delta\omega = \omega|\omega|^{\varrho-2}\ln|\omega|$$

in which there is conservation of energy, and therefore the similar is expected with respect to the scheme (5.15)*. In fact, let the discrete energy be defined for the conservative case by*

$$\mathbf{E}^n_{\Delta,cons} = \frac{\delta x}{2}\left[(\dot{\mathbf{w}}^n)^T \mathbf{M}\dot{\mathbf{w}}^n + (\mathbf{w}^n)^T \mathbf{K}\mathbf{w}^n - \sum_{j=1}^{J} \mathbf{F}(\omega_j^n)\right] \tag{5.16}$$

Lemma 6. *Choosing* $\gamma = \dfrac{1}{2}$ *and* $\beta = \dfrac{1}{4}$*, the numerical solution of* (5.12) *with* (5.3)-(5.4) *and* (5.13) *verifies the following conservation property in time*

$$\mathbf{E}^{n+1}_{\Delta,cons} = \mathbf{E}^n_{\Delta,cons}$$

Proof. By multiplying (5.12) by $\delta t\dot{\mathbf{w}}^{n+\frac{1}{2}}$ and applying (5.3)-(5.4) with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$, we derive the desired identity. Observe that $\mathcal{J}$ is explicitly defined in (5.13), which allows us to express $\mathbf{F}(\omega_j^{n+1}) - \mathbf{F}(\omega_j^n)$ directly as the product of $\mathcal{J}(\mathbf{w}^n, \mathbf{w}^{n+1})$ and $\delta t\dot{\mathbf{w}}^{n+\frac{1}{2}}$, thereby yielding the expected result. □

In Figure 1, a comparative simulation of the conservative numerical energy (5.16) is presented for two methods: the implicit method (5.10) (assuming the three dissipative terms are null in this case) and the conservative method (5.12). The initial conditions for this example are given by

$$\omega_0(x) = \sin(10\pi x), \qquad \omega_1(x) = 0, \qquad \forall x \in (0,1).$$

The domain is defined as $\Omega = (0,1)$, with non-linearity characterized by $\varrho = 3$ and parameter $\lambda = 1$. The discretization parameters are set to $J = 250$, $T = 5$, $N = 500$, $M = 10,000$, $R = 100$, and $d\xi = R/M$.

As shown in Figure 1, the conservative method demonstrates greater accuracy in preserving energy and effectively avoids numerical dissipation when compared to the implicit method.

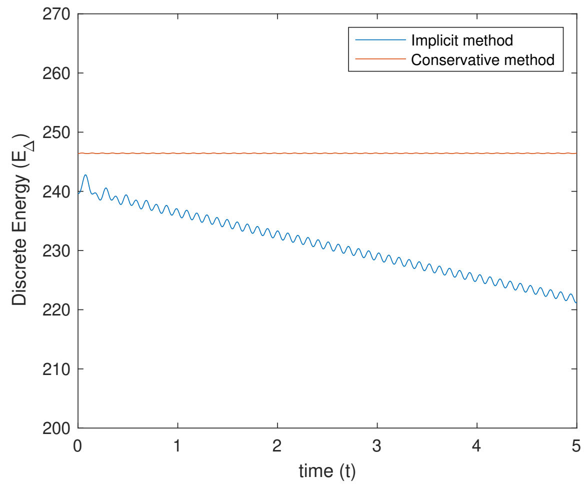

Figure 1: Comparison of Energy Computation Using Two Methods: Implicit and Conservative Approaches

In the case of the complete system (1.1) with all its dissipative terms and their numerical approximation given by (5.10), the energy to be considered is defined by the following:

$$\mathbf{E}_\Delta^n = \frac{\delta x}{2}\Bigg[(\dot{\mathbf{w}}^n)^T \mathbf{M}\dot{\mathbf{w}}^n + (\mathbf{w}^n)^T \mathbf{K}\mathbf{w}^n - \sum_{j=1}^{J} \mathbf{F}(\omega_j^n) - \delta t \sum_{m=1}^{M} q_m \left(\mathbf{o}^{m,n}\right)^T \mathbf{D}^2 \mathbf{o}^{k,n} - b\delta\xi \sum_{\ell=1}^{L}\sum_{j=1}^{J} \zeta_\ell |\theta_{\ell,j}^n|^2 \Bigg] \tag{5.17}$$

Lemma 7. *Choosing* $\gamma = \dfrac{1}{2}$ *and* $\beta = \dfrac{1}{4}$*, the numerical solution of* (5.10) *with* (5.3)-(5.4), (5.6), (5.9), (5.13) *and* (5.14) *verifies the following conservation property in time*

$$\mathbf{E}_\Delta^{n+1} - \mathbf{E}_\Delta^n = \frac{\delta x}{2}\Bigg[\left(\dot{\mathbf{w}}^{n+\frac{1}{2}}\right)^T \mathbf{D}^2 \dot{\mathbf{w}}^{n+\frac{1}{2}} - b\delta\xi \sum_{\ell=1}^{L} \left(\xi_\ell^2 + \varsigma\right) \left|\theta_\ell^{n+\frac{1}{2}}\right|^2 - \delta t \sum_{m=1}^{M} (q_{m+1} - q_m) \left(\mathbf{o}^{m,n}\right)^T \mathbf{D}^2 \mathbf{o}^{m,n} - \delta t^2 \sum_{m=1}^{M} (q_{m+1} - q_m) \left(\mathbf{o}^{m,n}\right)^T \mathbf{D}^2 \dot{\mathbf{w}}^{n+\frac{1}{2}} \Bigg] \tag{5.18}$$

Proof. The basis of the proof is similar to Lemma 6, with the additional dissipative terms. By multiplying (5.10) by $\delta t \dot{\mathbf{w}}^{n+\frac{1}{2}}$ and applying (5.3)-(5.4), (5.6), (5.9), (5.13) and (5.14), with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$, we derive the desired identity. □

Remark 2.

- *Similarly, the discrete energy* (5.18) *provides a good numerical approximation to the continuous energy* (2.4)*, and the inequality* (5.18) *in Lemma 7 corresponds to its analogous inequality in the continuous case* (2.5).
- *The first three terms on the right-hand side of* (5.18) *are clearly negative, thereby contributing to the strict decrease in energy. However, the fourth term has a changing sign and, being of order* $O(\delta t^2)$*, is expected to take small values for sufficiently small* δt*, as observed in the numerical examples presented in this study.*

5.7. Numerical examples

5.7.1. Exponential decay of the energy

Here, we simulate the discretized energy (5.17) from the numerical simulation (5.10), for different physical and discretization parameters. We first consider a simple initial condition given by

$$\omega_0(x) = \sin(p\pi x), \qquad \omega_1(x) = 0, \qquad \forall x \in (0,1).$$

The computational domain is defined as $\Omega = (0,1)$. The discretization parameters are set as follows: $J = 250$, $T = 35$, $N = 3500$, $M = 10,000$, $R = 100$, and $d\xi = R/M$. Numerical simulations are performed for $p = 4$, $\varrho = 3$, and various values of λ, as presented in Table 1 and Figure 2. The results indicate that the decay rates exhibit a significant sensitivity to the parameter λ. In contrast, additional simulations were performed for a range of values of p (from 4 to 100) and ϱ (from 3 to 20). These results revealed an almost negligible sensitivity of the decay rates to variations in these two parameters.

λ	0.1	0.3	0.5	0.7	0.9
C	3.962	11.6826	19.3526	26.7489	32.9054
k	-0.8970	-1.1053	-1.3316	-1.5781	-1.8446

Table 1: Asymptotic behavior of the energy compared with an exponential function of the form $y = Ce^{kx}$ for different values of λ

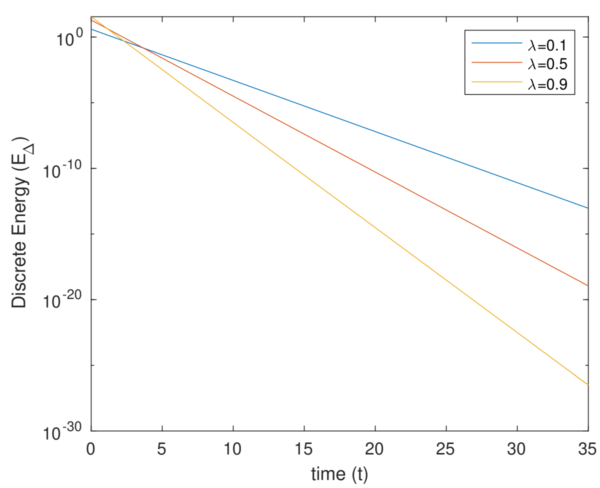

Figure 2: Exponential decay of the energy for different values of λ.

5.7.2. Blow up and exponential decay of the energy

In this numerical example, we will show the importance of hypothesis (3.1) in order to obtain the exponential decay of Theorem 2. To do this we simulate with two initial conditions:

$$\omega_0^1(x) = \sin(4\pi x), \qquad \omega_1^1(x) = 0, \tag{5.19}$$

$$\omega_0^2(x) = 20\sin(4\pi x), \qquad \omega_1^2(x) = 0, \tag{5.20}$$

$\forall x \in (0,1)$. The computational domain is defined as $\Omega = (0,1)$. The discretization parameters are set as follows: $J = 250$, $T = 3.5$, $N = 350$, $M = 10,000$, $R = 100$, and $d\xi = R/M$. Numerical simulations are performed for $p = 4$, $\varrho = 3$, and $\lambda = 0.5$.

The initial condition (5.19) is sufficiently small to satisfy hypothesis (3.1), ensuring the exponential decay observed in the left graph of Figure 3. In contrast, when the initial condition (5.20) is large enough, hypothesis (3.1) is no longer satisfied, leading to the blow-up predicted in [12] and clearly visible in the right graph of Figure 3. In particular, condition (5.20) corresponds to an amplification by a factor of 20 of the amplitude of (5.19). In a linear model, such a scaling would not alter the asymptotic behavior of the energy; however, in our model, the presence of logarithmic nonlinearity significantly affects the dynamics. Some numerical tests are carried out to illustrate these asymptotic behaviors.

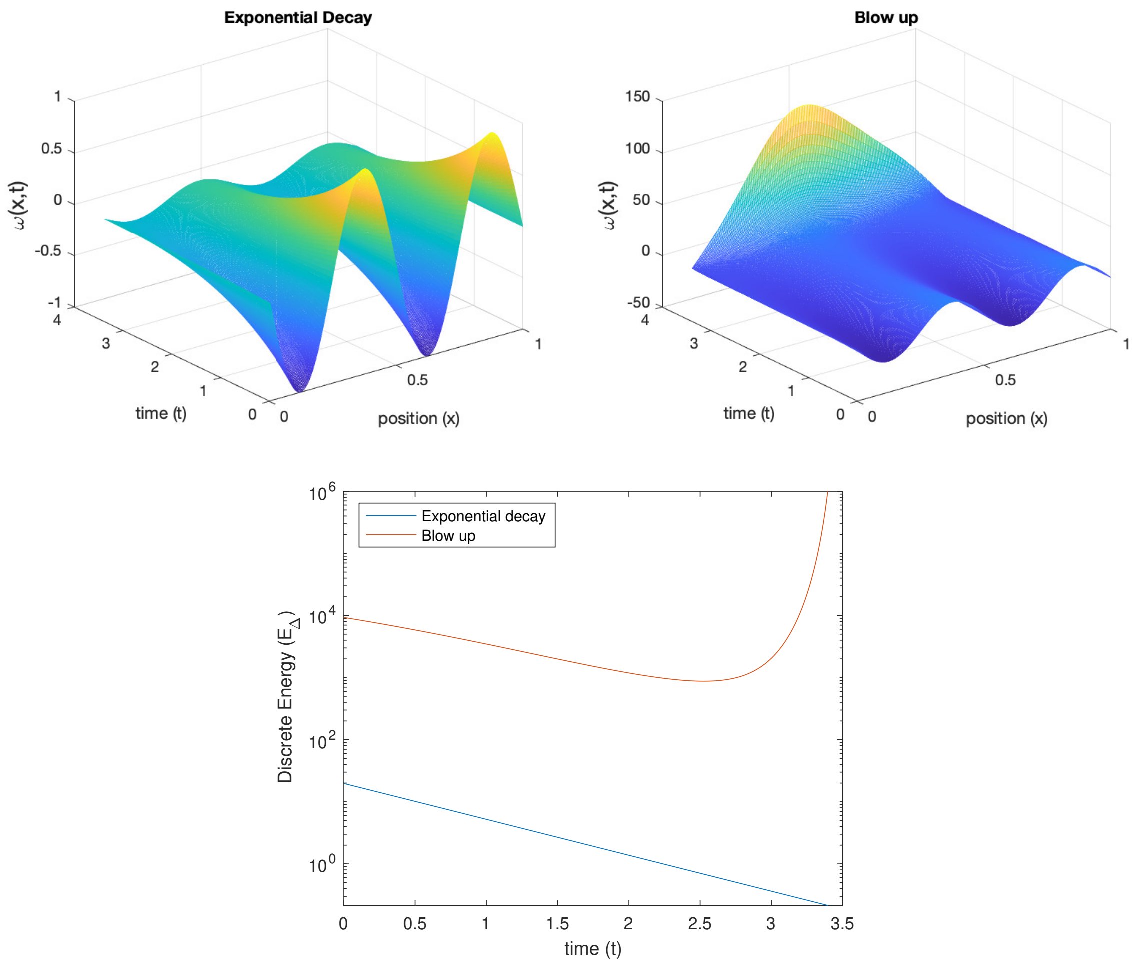

Figure 3: Comparison between two different asymptotic behaviours of the energy depending on the size of the initial condition.

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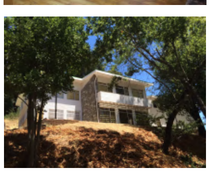

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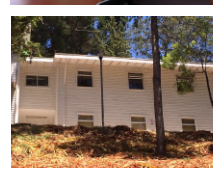

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