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Isaac Bermudez, Gabriel N. Gatica,
Juan P. Silva

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A new Banach spaces-based mixed finite element method for the coupled Navier–Stokes and Darcy equations*

Isaac Bermúdez † Gabriel N. Gatica‡ Juan P. Silva§

Abstract

In this paper we propose and analyze a new fully-mixed finite element method for the coupled model arising from the Navier-Stokes equations, with variable viscosity, in an incompressible fluid, and the Darcy equations in an adjacent porous medium, so that suitable transmission conditions are considered on the corresponding interface. The approach is based on the introduction of the further unknowns in the fluid given by the velocity gradient and the pseudostress tensor, where the latter includes the respective diffusive and convective terms. The above allows the elimination from the system of the fluid pressure, which can be calculated later on via a postprocessing formula. In addition, the traces of the fluid velocity and the Darcy pressure become the Lagrange multipliers enforcing weakly the interface conditions. In this way, the resulting variational formulation is given by a nonlinear perturbation of a threefold saddle point operator equation, where the saddle-point in the middle of them is, in turn, perturbed. A fixed-point strategy along with the generalized Babuška-Brezzi theory, a related abstract result for perturbed saddle-point problems, the Banach-Nečas-Babuška theorem, and the Banach fixed-point theorem, are employed to prove the well-posedness of the continuous and Galerkin schemes. In particular, Raviart-Thomas and piecewise polynomial subspaces of the lowest degree for the domain unknowns, as well as continuous piecewise linear polynomials for the Lagrange multipliers on the interface, constitute a feasible choice of the finite element subspaces. Optimal error estimates and associated rates of convergence are then established. Finally, several numerical results illustrating the good performance of the method in 2D and confirming the theoretical findings, are reported.

Key words: Navier–Stokes equations, Darcy equations, Banach spaces, mixed finite element methods, *a priori* error analysis

Mathematics subject classifications (2000): 35J66, 65J15, 65N12, 65N15, 65N30, 47J26, 76D07.

1 Introduction

The study of coupled fluid systems, particularly those involving free and porous media flows, governed by the Navier–Stokes and Darcy equations, respectively, and connected through a set of suitable interface conditions, has received significant attention because of their wide range of applications. In

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†CI^2MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Concepción, Chile. Present address: Universidad de Costa Rica, Alajuela, Costa Rica, email: isaac.bermudez@ucr.ac.cr.

‡CI^2MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: ggatica@ci2ma.udec.cl.

§CI^2MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: jsilva2018@udec.cl.

particular, the latter includes environmental, biological, and industrial processes, such as the interaction of surface and subsurface flows, modeling of blood flow, and others. Over the years, several papers have been devoted to numerical modeling and analysis of the Navier–Stokes/Darcy and related coupled problems (see, e.g.,[3, 10, 16, 17, 24, 26, 27, 28, 30]). In the context of the Stokes–Darcy coupled problem, the first theoretical results go back to [30] and [16]. In [16] the authors introduce an iterative subdomain method that employs the standard velocity-pressure formulation for the Stokes equation and the primal one in the Darcy domain, whereas in [30] they apply the primal method in the fluid and the dual-mixed one in the porous medium, which means that only the original velocity and pressure unknowns are considered in the Stokes domain, whereas a further unknown (velocity) is added in the Darcy region. In turn, a conforming mixed finite element discretization of the variational formulation from [30] was introduced and analyzed in [24]. In this work, the porous medium is assumed to be entirely enclosed within a fluid region, and, as in [30], the corresponding interface conditions refer to mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman (BJS) law. As a consequence, the trace of the porous medium pressure needs to be introduced as a suitable Lagrange multiplier. In addition, Bernardi–Raugel and Raviart–Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise-linear elements for the aforementioned multiplier, yield a stable Galerkin scheme. The results from [24] are then improved in [28] where a classical result on projection methods for Fredholm operators of index zero is employed to show that the use, not only of the one in [24], but of any pair of stable Stokes and Darcy elements, implies the stability of the corresponding Stokes-Darcy Galerkin scheme. Later one, a fully-mixed finite element method was proposed and analyzed in [26] for the Stokes–Darcy coupled problem, where the Babuška-Brezzi theories were used to derive sufficient conditions for the unique solvability of the resulting continuous and discrete formulations. Subsequently, in [27] the authors extend the previous results in [26] to the case of a two-dimensional nonlinear Stokes–Darcy coupled problem. Both *a priori* and *a posteriori* error analyses were developed in this work. As part of augmentation approaches, a fully-mixed finite element method for the Navier–Stokes/Darcy coupled problem with nonlinear viscosity has been introduced and analyzed in [10]. We also refer to [17] for the analysis of a conforming mixed finite element method for the Navier-–Stokes/Darcy coupled problem. In both works, and in order to stay within a Hilbertian framework, the velocity is sought in the Sobolev space of order 1, which requires to augment the variational formulation with additional Galerkin-type terms arising from the constitutive and equilibrium equations.

Although augmented methods are effective in ensuring stability, they significantly increase complexity and computational cost. This issue motivates the exploration of alternative approaches, such as those based on Banach spaces, whose main advantage is that no augmentation is required, and hence the spaces to which the unknowns belong are the natural ones arising from the application of the Cauchy–Schwarz and Hölder inequalities to the tested and eventually integrated by parts equations. A significant number of works have demonstrated the advantage of using this approach to analyze the continuous and discrete formulation of diverse problems (see, e.g, [2, 3, 9, 12, 14]). In particular a non-augmented mixed finite element method for the Navier–Stokes equations with variable viscosity was studied in [3]. More recently, a mass conservative finite element method for the Navier–Stokes/Darcy coupled system, which revisits the original primal-mixed approach from [17], was proposed in [6], whereas a conforming finite element method for a nonisothermal fluid-membrane interaction problem, modeled by the Navier-Stokes/heat system in the free-fluid region, and a Darcy-heat coupled system in the membrane, was introduced and analyzed in [7].

According to the above bibliographic discussion, the goal of this work is to extend the applicability of the Banach spaces framework by introducing a fully–mixed formulation for the coupling of fluid flow with porous media flow, without any augmentation procedure. To this end, we consider a similar approach to the one presented in [3] for the Navier-Stokes domain and adapt it to the coupled Navier-

Stokes/Darcy problem. The remainder of this paper is organized as follows. In Section 2 we introduce the governing equations and the mathematical model. Subsequently, in Section 3 we present the fully-mixed variational formulation within a Banach space framework and prove the well-posedness of the continuous problem. The corresponding Galerkin system is introduced and analyzed in Section 4, where a discrete version of the fixed-point strategy developed in Section 3 is used. In addition, we derive the associated *a priori* error estimate in the same Section. In Section 5 we specify particular choices of discrete subspaces, in 2D and 3D, that satisfy the hypotheses from Section 4 and establish the rates of convergence. Finally, in Section 6 we report on 2D numerical examples that validate the method and showcase its practical applications.

Preliminary notations

Throughout the paper, Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, whose outward normal at $\Gamma := \partial\Omega$ is denoted by $\mathbf{n}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$ and Sobolev spaces $\mathrm{W}^{l,t}(\Omega)$, with $l \geq 0$ and $t \in [1,+\infty)$, whose corresponding norms, either for the scalar or vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{l,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and if $t = 2$ we write $\mathrm{H}^l(\Omega)$ instead of $\mathrm{W}^{l,2}(\Omega)$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$, respectively. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$ will be employed for the norm of any element or operator whenever there is no confusion about the spaces to which they belong. Furthermore, as usual, $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathrm{R}^n$. Also, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product, respectively, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Additionally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} = (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{n} \boldsymbol{\tau}_{ii}, \quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}.$$

On the other hand, given $t \in (1,+\infty)$, we also introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \left\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\right\},$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\right\},$$

which are endowed with the natural norms defined, respectively, by

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega),$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega).$$

Then, proceeding as in [22, eq. (1.43), Section 1.3.4] (see also [5, Section 4.1] and [12, Section 3.1]), it is easy to show that for each $t \in \begin{cases} (1,+\infty) & \text{if} \quad n = 2 \\ [6/5,+\infty) & \text{if} \quad n = 3 \end{cases}$, there holds

$$\langle \boldsymbol{\tau}\cdot\mathbf{n}, v\rangle = \int_{\Omega} \{\boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau})\} \quad \forall\,(\boldsymbol{\tau},v) \in \mathbf{H}(\mathrm{div}_t;\Omega)\times\mathrm{H}^1(\Omega), \tag{1.1}$$

and analogously

$$\langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{v}\rangle \,=\, \int_\Omega \{\boldsymbol{\tau} : \nabla\mathbf{v} \,+\, \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\} \quad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega)\times\mathbf{H}^1(\Omega)\,, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, as well as between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. We find it important to stress here, as explained in the aforementioned references, that the second term on the right-hand side of (1.1) (resp. (1.2)) is well-defined because of the continuous embedding of $\mathrm{H}^1(\Omega)$ (resp. $\mathbf{H}^1(\Omega)$) into $\mathrm{L}^{t'}(\Omega)$ (resp. $\mathbf{L}^{t'}(\Omega)$), where t' is the conjugate of t, that is $t' \in [1,+\infty)$ such that $\frac{1}{t}+\frac{1}{t'}=1$, which holds for $t' \in \begin{cases} [1,+\infty) & \text{if} \quad n\,=\,2 \\ [1,6] & \text{if} \quad n\,=\,3 \end{cases}$.

2 The model problem

In this section we introduce the model of interest, namely the coupled Navier-Stokes and Darcy equations with variable viscosity. To this end, we first let Ω_S and Ω_D be bounded and simply connected open polyhedral domains in R^n, such that $\Omega_S\cap\Omega_D=\emptyset$ and $\partial\Omega_S\cap\partial\Omega_D=\Sigma\neq\emptyset$. The parts of the boundaries are $\Gamma_S := \partial\Omega_S\backslash\overline{\Sigma}$, $\Gamma_D := \partial\Omega_D\backslash\overline{\Sigma}$, and $\mathbf{n}$ denotes the unit normal vector on them, which is chosen pointing outward from $\Omega := \Omega_S\cup\Sigma\cup\Omega_D$ and Ω_S (and hence inward to Ω_D when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t}=\mathbf{t}_1$ when $n=2$ and by $\{\mathbf{t}_1,\mathbf{t}_2\}$ when $n=3$ (see Fig. 2.1 below for a 2D illustration of the geometry involved). The mathematical model is defined by two separate groups of equations and by a set of coupling terms. Here, Ω_S and Ω_D represent the domains in the free and porous media, respectively.

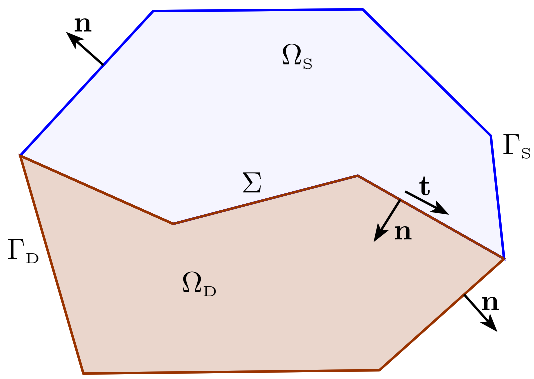

Figure 2.1: geometry of the coupled model

The governing equations in Ω_S are those of the Navier-Stokes problem with constant density ρ and variable viscosity μ, which are written in terms of the velocity $\mathbf{u}_S$ and the pressure p_S of the fluid, that is

$$\begin{gathered} -\mathbf{div}(\mu\nabla\mathbf{u}_S)+\rho(\nabla\mathbf{u}_S)\mathbf{u}_S+\nabla p_S \,=\, \mathbf{f}_S \quad \text{in} \quad \Omega_S\,, \\ \mathrm{div}(\mathbf{u}_S) \,=\, 0 \quad \text{in} \quad \Omega_S\,, \quad \mathbf{u}_S \,=\, \mathbf{g} \quad \text{on} \quad \Gamma_S\,, \end{gathered} \tag{2.1}$$

where the given data are a function $\mu:\Omega_S\to\mathrm{R}^+$ describing the viscosity, a volume force $\mathbf{f}_S$, and the boundary velocity $\mathbf{g}$. The right spaces to which $\mathbf{f}_S$ and $\mathbf{g}$ need to belong are specified later on. Furthermore, the function μ is supposed to be bounded, which means that there exist constants μ_1, $\mu_2>0$, such that

$$\mu_1 \,\le\, \mu(\mathbf{x}) \,\le\, \mu_2 \quad \forall\,\mathbf{x}\in\Omega_S. \tag{2.2}$$

Next, we introduce the pseudostress tensor unknown

$$\boldsymbol{\sigma}_S := \mu\,\nabla\mathbf{u}_S - \rho(\mathbf{u}_S\otimes\mathbf{u}_S) - p_S\mathbb{I} \quad \text{in} \quad \Omega_S\,, \tag{2.3}$$

so that, nothing that $\mathbf{div}(\mathbf{u}_S\otimes\mathbf{u}_S) = (\nabla\mathbf{u}_S)\mathbf{u}_S$, which makes use of the fact that $\mathrm{div}(\mathbf{u}_S) = 0$, we find that the first equation of (2.1) can be rewritten as

$$-\mathbf{div}(\boldsymbol{\sigma}_S) = \mathbf{f}_S \quad \text{in} \quad \Omega_S\,.$$

In turn, it is straightforward to prove, taking matrix trace and the deviatoric part of (2.3), that the latter along with the incompressibility condition are equivalent to the pair

$$\begin{aligned} \boldsymbol{\sigma}_S^{\mathtt{d}} &= \mu\nabla\mathbf{u}_S - \rho\,(\mathbf{u}_S\otimes\mathbf{u}_S)^{\mathtt{d}} \quad \text{in} \quad \Omega_S\,, \quad \text{and} \\ p_S &= -\frac{1}{n}\mathrm{tr}\big(\boldsymbol{\sigma}_S + \rho\,(\mathbf{u}_S\otimes\mathbf{u}_S)\big) \quad \text{in} \quad \Omega_S\,. \end{aligned} \tag{2.4}$$

Thus, eliminating the pressure unknown which, anyway, can be approximated later on by the post-processed formula suggested in (2.4), the Navier–Stokes problem (2.1) can be rewritten as:

$$\begin{aligned} \boldsymbol{\sigma}_S^{\mathtt{d}} &= \mu\nabla\mathbf{u}_S - \rho\,(\mathbf{u}_S\otimes\mathbf{u}_S)^{\mathtt{d}} \quad \text{in} \quad \Omega_S\,, \\ -\mathbf{div}(\boldsymbol{\sigma}_S) &= \mathbf{f}_S \quad \text{in} \quad \Omega_S\,, \quad \mathbf{u}_S = \mathbf{g} \quad \text{on} \quad \Gamma_S\,. \end{aligned} \tag{2.5}$$

Next, since we are interested in a mixed variational formulation of our problem, and in order to employ the integration by parts formula typically required by this approach, we introduce the auxiliary unknown $\mathbf{t}_S := \nabla\mathbf{u}_S$ in Ω_S. Consequently, instead of (2.5), we consider from now the set of equations with unknowns $\mathbf{t}_S$, $\mathbf{u}_S$, and $\boldsymbol{\sigma}_S$, given by

$$\begin{aligned} \mathbf{t}_S = \nabla\mathbf{u}_S \quad \text{in} \quad \Omega_S\,, \quad \boldsymbol{\sigma}_S^{\mathtt{d}} &= \mu\,\mathbf{t}_S - \rho\,(\mathbf{u}_S\otimes\mathbf{u}_S)^{\mathtt{d}} \quad \text{in} \quad \Omega_S\,, \\ -\mathbf{div}(\boldsymbol{\sigma}_S) = \mathbf{f}_S \quad \text{in} \quad \Omega_S\,, \quad \mathbf{u}_S &= \mathbf{g} \quad \text{on} \quad \Gamma_S\,. \end{aligned} \tag{2.6}$$

On the other hand, in Ω_D we consider the linearized Darcy model:

$$\begin{aligned} \mathbf{u}_D = -\mathbf{K}\nabla p_D \quad \text{in} \quad \Omega_D, \quad \mathrm{div}(\mathbf{u}_D) = f_D \quad \text{in} \quad \Omega_D\,, \\ \mathbf{u}_D\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_D\,, \end{aligned} \tag{2.7}$$

where $\mathbf{u}_D$ and p_D denote the velocity and pressure, respectively, in the porous medium, $f_D \in L^2(\Omega_D)$ is a source term and $\mathbf{K} \in [L^\infty(\Omega_D)]^{n\times n}$ is a positive definite symmetric tensor describing the permeability of Ω_D divided by a constant approximation of the viscosity, satisfying with $C_{\mathbf{K}} > 0$

$$\mathbf{w}\cdot\mathbf{K}^{-1}(\mathbf{x})\,\mathbf{w} \,\geq\, C_{\mathbf{K}}\,|\mathbf{w}|^2 \quad \forall\,(a.e.)\,\mathbf{x}\in\Omega_D, \qquad \forall\,\mathbf{w}\in\mathrm{R}^n\,.$$

Finally, following [30] and [24], the transmission conditions on Σ are given by

$$\begin{aligned} \mathbf{u}_S\cdot\mathbf{n} &= \mathbf{u}_D\cdot\mathbf{n} \quad \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}_S\mathbf{n} + \sum_{l=1}^{n-1}\omega_l^{-1}(\mathbf{u}_S\cdot\mathbf{t}_l)\mathbf{t}_l &= -p_D\mathbf{n} \quad \text{on} \quad \Sigma\,, \end{aligned} \tag{2.8}$$

where $\{\omega_1, ..., \omega_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally. The first equation in (2.8) corresponds to mass conservation on Σ, whereas the second one establishes the balance of normal forces and Beavers–Joseph–Saffman law. In addition, $\mathbf{g}$ and f_D must formally satisfy the compatibility condition

$$\int_{\Gamma_S}\mathbf{g}\cdot\mathbf{n} + \int_{\Omega_D} f_D \,=\, 0\,. \tag{2.9}$$

3 The continuous analysis

In this section we derive a Banach spaces-based fully-mixed variational formulation of the coupled problem described by (2.6), (2.7), and (2.8), and then perform its solvability analysis by means of a fixed-point strategy.

3.1 Preliminaries

Here we introduce further notations and definitions. We begin with the spaces

$$\begin{aligned}\mathbf{H}_0(\mathrm{div};\Omega_D) &:= \Big\{\mathbf{v}_D \in \mathbf{H}(\mathrm{div};\Omega_D): \quad \mathbf{v}_D\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_D\Big\},\\ \mathbb{L}^2_{\mathrm{tr}}(\Omega_S) &:= \Big\{\mathbf{r}_S \in \mathbb{L}^2(\Omega_S): \quad \mathrm{tr}(\mathbf{r}_S) = 0\Big\}.\end{aligned}$$

Furthermore, for each $* \in \{\mathrm{S,D}\}$, and given $\widetilde{\Gamma} \subset \partial\Omega_*$, we denote the space of traces

$$\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma}) := \Big\{v|_{\widetilde{\Gamma}}: \quad v \in \mathrm{H}^1(\Omega_*), \quad v = 0 \quad \text{on} \quad \partial\Omega_*\backslash\widetilde{\Gamma}\Big\}.$$

and its vector version $\mathbf{H}^{1/2}_{00}(\widetilde{\Gamma}) = \Big[\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma})\Big]^n$. Observe that, if $E_{\widetilde{\Gamma},*}: \mathrm{H}^{1/2}(\widetilde{\Gamma}) \to \mathrm{L}^2(\partial\Omega_*)$ is the extension operator defined by

$$E_{\widetilde{\Gamma},*}(\psi) := \begin{cases}\psi & \text{on} \quad \widetilde{\Gamma}\\ 0 & \text{on} \quad \partial\Omega_*\backslash\widetilde{\Gamma}\end{cases} \qquad \forall\,\psi \in \mathrm{H}^{1/2}(\widetilde{\Gamma}),$$

we have, alternatively, that

$$\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma}) = \Big\{\psi \in \mathrm{H}^{1/2}(\widetilde{\Gamma}): \quad E_{\widetilde{\Gamma},*}(\psi) \in \mathrm{H}^{1/2}(\partial\Omega_*)\Big\},$$

which is endowed with the norm $\|\psi\|_{1/2,00;\widetilde{\Gamma}} := \|E_{\widetilde{\Gamma},*}(\psi)\|_{1/2,\partial\Omega_*}$. The dual of $\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma})$ (respectively $\mathbf{H}^{1/2}_{00}(\widetilde{\Gamma})$) is denoted by $\mathrm{H}^{-1/2}_{00}(\widetilde{\Gamma})$ (respectively $\mathbf{H}^{-1/2}_{00}(\widetilde{\Gamma})$), and $\|\cdot\|_{-1/2,00;\widetilde{\Gamma}}$ is set as the corresponding norms. Next, in order to deduce the variational formulation of the Navier–Stokes problem, we first look originally for $\mathbf{u}_S \in \mathbf{H}^1(\Omega_S)$, for which we assume from now on, for simplicity, that $\mathbf{g} \in \mathbf{H}^{1/2}_{00}(\Gamma_S)$. Equivalently, letting

$$\mathbf{g}_S := E_{\Gamma_S,S}(\mathbf{g}) = \begin{cases}\mathbf{g} & \text{on} \quad \Gamma_S\\ \mathbf{0} & \text{on} \quad \Sigma\end{cases},$$

there holds $\mathbf{g}_S \in \mathbf{H}^{1/2}(\partial\Omega_S)$, and hence, using the trace operator $\gamma_0: \mathbf{H}^1(\Omega_S) \to \mathbf{H}^{1/2}(\partial\Omega_S)$ (see [22, Section 1.3.1]), we can write $\gamma_0(\mathbf{u}_S) = \mathbf{g}_S + (\gamma_0(\mathbf{u}_S) - \mathbf{g}_S)$, where

$$\gamma_0(\mathbf{u}_S) - \mathbf{g}_S = \begin{cases}\mathbf{0} & \text{on} \quad \Gamma_S\\ \gamma_0(\mathbf{u}_S) & \text{on} \quad \Sigma\end{cases} = E_{\Sigma,S}(\gamma_0(\mathbf{u}_S)|_\Sigma) \in \mathbf{H}^{1/2}(\partial\Omega_S),$$

which proves that

$$\boldsymbol{\varphi} := -\gamma_0(\mathbf{u}_S)|_\Sigma \in \mathbf{H}^{1/2}_{00}(\Sigma).$$

As a consequence, for each $\chi \in \mathbf{H}^{-1/2}(\partial\Omega_S)$ we get

$$\begin{aligned}\langle\chi,\gamma_0(\mathbf{u}_S)\rangle_{\partial\Omega_S} &= \langle\chi,\mathbf{g}_S\rangle_{\partial\Omega_S} + \langle\chi,\gamma_0(\mathbf{u}_S) - \mathbf{g}_S\rangle_{\partial\Omega_S}\\ &= \langle\chi, E_{\Gamma_S,S}(\mathbf{g})\rangle_{\partial\Omega_S} - \langle\chi, E_{\Sigma,S}(\boldsymbol{\varphi})\rangle_{\partial\Omega_S}\\ &= \langle\chi|_{\Gamma_S},\mathbf{g}\rangle_{\Gamma_S} - \langle\chi|_\Sigma,\boldsymbol{\varphi}\rangle_\Sigma,\end{aligned} \tag{3.1}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_S}$ (respectively $\langle\cdot,\cdot\rangle_\Sigma$) stands for the duality pairing between $\mathbf{H}^{-1/2}_{00}(\Gamma_S)$ (respectively $\mathbf{H}^{-1/2}_{00}(\Sigma)$) and $\mathbf{H}^{1/2}_{00}(\Gamma_S)$ (respectively $\mathbf{H}^{1/2}_{00}(\Sigma)$).

3.2 The fully-mixed formulation

Having established the above, we now multiply the first equation of (2.6) by $\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_t;\Omega_S)$, with $t \in \begin{cases} (1,+\infty) & \text{if} \quad n = 2 \\ [6/5,+\infty) & \text{if} \quad n = 3 \end{cases}$, apply the integration by parts formula (1.2), and use (3.1) with $\chi = \boldsymbol{\tau}_S\,\mathbf{n}$, to find that

$$\int_{\Omega_S} \boldsymbol{\tau}_S : \mathbf{t}_S + \int_{\Omega_S} \mathbf{u}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) \;=\; \langle \boldsymbol{\tau}_S\,\mathbf{n}, \mathbf{g}\rangle_{\Gamma_S} - \langle \boldsymbol{\tau}_S\,\mathbf{n}, \boldsymbol{\varphi}\rangle_{\Sigma} \quad \forall\, \boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_t;\Omega_S)\,. \tag{3.2}$$

It is clear from (3.2) that its first term is well-defined for $\mathbf{t}_S \in \mathbb{L}^2(\Omega_S)$, which, along with the free trace property of $\mathbf{t}_S$, suggests to look for $\mathbf{t}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$. In addition, knowing that $\mathbf{div}(\boldsymbol{\tau}_S) \in \mathbf{L}^t(\Omega_S)$, we realize from the second term and Hölder's inequality that it suffices to look for $\mathbf{u}_S \in \mathbf{L}^{t'}(\Omega_S)$, where t' is the conjugate of t. Next, it follows from the second equation of (2.6), that formally

$$\int_{\Omega_S} \mu\,\mathbf{t}_S : \mathbf{r}_S - \int_{\Omega_S} \boldsymbol{\sigma}_S^{\mathtt{d}} : \mathbf{r}_S - \rho \int_{\Omega_S} (\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathtt{d}} : \mathbf{r}_S \;=\; 0 \quad \forall\, \mathbf{r}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)\,, \tag{3.3}$$

from which we notice that the first term is well-defined, whereas the second one makes sense if $\boldsymbol{\sigma}_S$ is sought in $\mathbb{L}^2(\Omega_S)$. In turn, for the third one there holds

$$\left|\int_{\Omega_S} (\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathtt{d}} : \mathbf{r}_S\right| = \left|\int_{\Omega_S} (\mathbf{u}_S \otimes \mathbf{u}_S) : \mathbf{r}_S\right| \;\leq\; \|\mathbf{u}_S\|_{0,4;\Omega_S}\,\|\mathbf{u}_S\|_{0,4;\Omega_S}\,\|\mathbf{r}_S\|_{0,\Omega_S}\,,$$

which, necessarily yields $t' = 4$, and thus $t = 4/3$. Finally, looking for $\boldsymbol{\sigma}_S$ in the same space of its corresponding test function $\boldsymbol{\tau}_S$, that is $\boldsymbol{\sigma}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$, it follows from the third equation of (2.6) that

$$-\int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\sigma}_S) \;=\; \int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v}_S \quad \forall\, \mathbf{v}_S \in \mathbf{L}^4(\Omega_S)\,, \tag{3.4}$$

which forces $\mathbf{f}_S$ to belong to $\mathbf{L}^{4/3}(\Omega_S)$. Now for the Darcy equations given in (2.7) and the transmission conditions specified in (2.8), we proceed similarly as in [10], so that introducing the auxiliary unknown

$$\lambda := p_D|_{\Sigma} \in \mathrm{H}^{1/2}(\Sigma)\,,$$

we obtain the variational problem: Find $\mathbf{t}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, $\mathbf{u}_S \in \mathbf{L}^4(\Omega_S)$, $\boldsymbol{\sigma}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$, $\mathbf{u}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D)$, $p_D \in \mathrm{L}^2(\Omega_D)$, $\lambda \in \mathrm{H}^{1/2}(\Sigma)$ and $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, such that

$$\begin{aligned}
\int_{\Omega_S} \mathbf{t}_S : \boldsymbol{\tau}_S^{\mathtt{d}} + \int_{\Omega_S} \mathbf{u}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) + \langle \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\varphi}\rangle_{\Sigma} \quad &= \langle \boldsymbol{\tau}_S\mathbf{n}, \mathbf{g}\rangle_{\Gamma_S}\,,\\
\int_{\Omega_D} \mathbf{K}^{-1}\mathbf{u}_D \cdot \mathbf{v}_D - \int_{\Omega_D} p_D\,\mathrm{div}(\mathbf{v}_D) - \langle \mathbf{v}_D \cdot \mathbf{n}, \lambda\rangle_{\Sigma} \quad &= 0\,,\\
\int_{\Omega_S} \mu\,\mathbf{t}_S : \mathbf{r}_S - \int_{\Omega_S} \boldsymbol{\sigma}_S^{\mathtt{d}} : \mathbf{r}_S - \rho \int_{\Omega_S} (\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathtt{d}} : \mathbf{r}_S \quad &= 0\,,\\
-\int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\sigma}_S) \quad &= \int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v}_S\,,\\
\int_{\Omega_D} q_D\,\mathrm{div}(\mathbf{u}_D) \quad &= \int_{\Omega_D} f_D\,q_D\,,\\
-\langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi\rangle_{\Sigma} - \langle \mathbf{u}_D \cdot \mathbf{n}, \xi\rangle_{\Sigma} \quad &= 0\,,\\
\langle \boldsymbol{\sigma}_S\mathbf{n}, \boldsymbol{\psi}\rangle_{\Sigma} - \langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda\rangle_{\Sigma} \quad &= 0\,,
\end{aligned} \tag{3.5}$$

for all $\mathbf{r}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, $\mathbf{v}_S \in \mathbf{L}^4(\Omega_S)$, $\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$, $\mathbf{v}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D)$, $q_D \in \mathrm{L}^2(\Omega_D)$, $\xi \in \mathrm{H}^{1/2}(\Sigma)$ and $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, where:

$$\langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} = \sum_{l=1}^{n-1} w_l^{-1} \langle \boldsymbol{\varphi} \cdot \mathbf{t}_l, \boldsymbol{\psi} \cdot \mathbf{t}_l\rangle_{\Sigma}\,.$$

It is not difficult to see that the system (3.5) is not uniquely solvable since, given any solution $(\mathbf{t}_S, \mathbf{u}_S, \boldsymbol{\sigma}_S, \mathbf{u}_D, p_D, \lambda, \boldsymbol{\varphi})$ in the indicated spaces, and given any constant $c \in \mathrm{R}$, the vector defined by $(\mathbf{t}_S, \mathbf{u}_S, \boldsymbol{\sigma}_S - c\mathbb{I}, \mathbf{u}_D, p_D - c, \lambda + c, \boldsymbol{\varphi})$ also becomes a solution. In order to avoid this non-uniqueness, from now on we require the Darcy pressure p_D to be in $\mathrm{L}^2_0(\Omega_D)$, where

$$\mathrm{L}^2_0(\Omega_D) := \left\{ q_D \in \mathrm{L}^2(\Omega_D) : \quad \int_{\Omega_D} q_D = 0 \right\} .$$

On the other hand, for convenience of the subsequent analysis, we consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3}; \Omega_S) = \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S) \oplus \mathrm{R}\,\mathbb{I}\,, \tag{3.6}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega_S) : \quad \int_{\Omega_S} \mathrm{tr}(\boldsymbol{\tau}) = 0 \right\} .$$

It follows that $\boldsymbol{\sigma}_S$ can be uniquely decomposed as $\boldsymbol{\sigma}_S = \boldsymbol{\sigma}_{S,0} + l\mathbb{I}$, where

$$\boldsymbol{\sigma}_{S,0} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S) \quad \text{and} \quad l := \frac{1}{n|\Omega_S|} \int_{\Omega_S} \mathrm{tr}(\boldsymbol{\sigma}_S)\,. \tag{3.7}$$

In this regard, we notice that (3.3) and (3.4) remain unchanged if $\boldsymbol{\sigma}_S$ is replaced by $\boldsymbol{\sigma}_{S,0}$. In this way, using the compatibility condition (2.9), the first and last equations of (3.5) are rewritten equivalently as

$$\begin{aligned}
\int_{\Omega_S} \mathbf{t}_S : \boldsymbol{\tau}_S^{\mathrm{d}} + \int_{\Omega_S} \mathbf{u}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma &= \langle \boldsymbol{\tau}_S \mathbf{n}, \mathbf{g} \rangle_{\Gamma_S} \quad && \forall\, \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S)\,, \\
j \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_\Sigma &= j \langle \mathbf{g} \cdot \mathbf{n}, 1 \rangle_{\Gamma_S} && \forall\, j \in \mathrm{R}\,, \\
\langle \boldsymbol{\sigma}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma - \langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma + l \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma &= 0 && \forall\, \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)\,.
\end{aligned}$$

As a consequence of the above, we find that the resulting variational formulation reduces to: Find $\mathbf{t}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, $\mathbf{u}_D \in \mathbf{H}_0(\mathrm{div}; \Omega_D)$, $\boldsymbol{\sigma}_S \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S)$, $\lambda \in \mathrm{H}^{1/2}(\Sigma)$, $\mathbf{u}_S \in \mathbf{L}^4(\Omega_S)$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, $p_D \in \mathrm{L}^2_0(\Omega_D)$ and $l \in \mathrm{R}$, such that

$$\begin{array}{ccccccl}
\displaystyle\int_{\Omega_S} \mu \mathbf{t}_S : \mathbf{r}_S & & \displaystyle -\int_{\Omega_S} \boldsymbol{\sigma}_S^{\mathrm{d}} : \mathbf{r}_S & & & \displaystyle -\rho \int_{\Omega_S} (\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathrm{d}} : \mathbf{r}_S & = 0 \\
& \displaystyle\int_{\Omega_D} \mathbf{K}^{-1} \mathbf{u}_D \cdot \mathbf{v}_D & & -\langle \mathbf{v}_D \cdot \mathbf{n}, \lambda \rangle_\Sigma & & \displaystyle -\int_{\Omega_D} p_D \,\mathrm{div}(\mathbf{v}_D) & = 0 \\
\displaystyle\int_{\Omega_S} \boldsymbol{\tau}_S^{\mathrm{d}} : \mathbf{t}_S & & & & +\langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma & \displaystyle +\int_{\Omega_S} \mathbf{u}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) & = \langle \boldsymbol{\tau}_S \mathbf{n}, \mathbf{g} \rangle_{\Gamma_S} \\
& \langle \mathbf{u}_D \cdot \mathbf{n}, \xi \rangle_\Sigma & & & +\langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_\Sigma & & = 0 \\
& & \langle \boldsymbol{\sigma}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma & +\langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma & -\langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{\mathbf{t},\Sigma} & +l \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma & = 0 \\
& & \displaystyle\int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\sigma}_S) & & & & \displaystyle = -\int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v}_S \\
& & & & j \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_\Sigma & & = j \langle \mathbf{g} \cdot \mathbf{n}, 1 \rangle_{\Gamma_S} \\
& \displaystyle -\int_{\Omega_D} q_D \,\mathrm{div}(\mathbf{u}_D) & & & & & \displaystyle = -\int_{\Omega_D} f_D q_D
\end{array} \tag{3.8}$$

for all $\mathbf{r}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, $\mathbf{v}_D \in \mathbf{H}_0(\mathrm{div}; \Omega_D)$, $\boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S)$, $\xi \in \mathrm{H}^{1/2}(\Sigma)$, $\mathbf{v}_S \in \mathbf{L}^4(\Omega_S)$, $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, $q_D \in \mathrm{L}^2_0(\Omega_D)$ and $j \in \mathrm{R}$. Now, we group the spaces, unknowns, and test functions as follows:

$$\begin{aligned}
&\mathbf{X} := \mathbb{L}^2_{\mathrm{tr}}(\Omega_S) \times \mathbf{H}_0(\mathrm{div}; \Omega_D)\,, \qquad && \mathbf{Y} := \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S) \times \mathrm{H}^{1/2}(\Sigma) \\
&\mathbf{Z} := \mathbf{L}^4(\Omega_S) \times \mathbf{H}^{1/2}_{00}(\Sigma)\,, && \mathbb{H} := \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}\,, \\
&\mathbf{Q} := \mathrm{L}^2_0(\Omega_D) \times \mathrm{R}\,,
\end{aligned}$$

$$
\begin{aligned}
&\vec{\mathbf{t}} := (\mathbf{t}_S, \mathbf{u}_D) \in \mathbf{X}, \quad && \vec{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_S, \lambda) \in \mathbf{Y}, \quad && \vec{\mathbf{u}} := (\mathbf{u}_S, \boldsymbol{\varphi}) \in \mathbf{Z}, \quad && \vec{\boldsymbol{p}} := (p_D, l) \in \mathbf{Q}, \\
&\vec{\mathbf{r}} := (\mathbf{r}_S, \mathbf{v}_D) \in \mathbf{X}, \quad && \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \xi) \in \mathbf{Y}, \quad && \vec{\mathbf{v}} := (\mathbf{v}_S, \boldsymbol{\psi}) \in \mathbf{Z}, \quad && \vec{\boldsymbol{q}} := (q_D, j) \in \mathbf{Q}, \\
&\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}_S, \boldsymbol{z}_D) \in \mathbf{X}, \quad && \vec{\boldsymbol{\eta}} := (\boldsymbol{\eta}_S, \vartheta) \in \mathbf{Y}, \quad && \vec{\boldsymbol{z}} := (\boldsymbol{z}_S, \boldsymbol{\phi}) \in \mathbf{Z}, \quad && \vec{\mathbf{s}} := (s_D, k) \in \mathbf{Q},
\end{aligned}
$$

where $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, $\mathbb{H}$ and $\mathbf{Q}$ are respectively endowed with the norms

$$
\begin{aligned}
&\|\vec{\mathbf{r}}\|_{\mathbf{X}} := \|\mathbf{r}_S\|_{0,\Omega_S} + \|\mathbf{v}_D\|_{\mathrm{div},\Omega_D}, \qquad && \|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} := \|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S} + \|\xi\|_{1/2,\Sigma}, \\
&\|\vec{\mathbf{v}}\|_{\mathbf{Z}} := \|\mathbf{v}_S\|_{0,4;\Omega_S} + \|\boldsymbol{\psi}\|_{1/2,00;\Sigma}, \qquad && \|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}} := \|\vec{\mathbf{r}}\|_{\mathbf{X}} + \|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} + \|\vec{\mathbf{v}}\|_{\mathbf{Z}}, \\
&\|\vec{\boldsymbol{q}}\|_{\mathbf{Q}} := \|q_D\|_{0,\Omega_D} + |j|.
\end{aligned}
$$

Hence, using the same colors from (3.8), this formulation can be rewritten as: Find $((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}) \in \mathbb{H} \times \mathbf{Q}$, such that

$$
\begin{array}{llllll}
[a(\vec{\mathbf{t}}), \vec{\mathbf{r}}] & +[b_1(\vec{\mathbf{r}}), \vec{\boldsymbol{\sigma}}] & & & -\displaystyle\int_{\Omega_D} p_D \operatorname{div}(\mathbf{v}_D) \quad + b(\mathbf{u}_S; \mathbf{u}_S, \mathbf{r}_S) & = 0 \\
[b_2(\vec{\mathbf{t}}), \vec{\boldsymbol{\tau}}] & & +[\mathbf{B}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}), \vec{\mathbf{u}}] & & & = \langle \boldsymbol{\tau}_S \mathbf{n}, \mathbf{g} \rangle_{\Gamma_S} \\
 & [\mathbf{B}(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}), \vec{\mathbf{v}}] & -[\mathbf{C}(\vec{\mathbf{u}}), \vec{\mathbf{v}}] & + l \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma & & = -\displaystyle\int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v}_S \\
 & & + j \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_\Sigma & & & = j \langle \mathbf{g} \cdot \mathbf{n}, 1 \rangle_{\Gamma_S} \\
-\displaystyle\int_{\Omega_D} q_D \operatorname{div}(\mathbf{u}_D) & & & & & = -\displaystyle\int_{\Omega_D} f_D \, q_D
\end{array}
\tag{3.9}
$$

for all $((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H} \times \mathbf{Q}$, where $a : \mathbf{X} \times \mathbf{X} \to \mathrm{R}$, $b_1 : \mathbf{X} \times \mathbf{Y} \to \mathrm{R}$, $b_2 : \mathbf{X} \times \mathbf{Y} \to \mathrm{R}$, $\mathbf{B} : \mathbb{H} \to \mathrm{R}$, and $\mathbf{C} : \mathbf{Z} \times \mathbf{Z} \to \mathrm{R}$, are the bilinear forms defined by

$$
\begin{aligned}
[a(\vec{\boldsymbol{\zeta}}), \vec{\mathbf{r}}] &:= \int_{\Omega_S} \mu \, \boldsymbol{\zeta}_S : \mathbf{r}_S + \int_{\Omega_D} \mathbf{K}^{-1} \boldsymbol{z}_D \cdot \mathbf{v}_D && \forall\, \vec{\boldsymbol{\zeta}}, \vec{\mathbf{r}} \in \mathbf{X}, \\
[b_1(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}] &:= -\langle \mathbf{v}_D \cdot \mathbf{n}, \xi \rangle_\Sigma - \int_{\Omega_S} \boldsymbol{\tau}_S^{\mathrm{d}} : \mathbf{r}_S && \forall\, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X} \times \mathbf{Y}, \\
[b_2(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}] &:= -[b_1(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}] && \forall\, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X} \times \mathbf{Y}, \\
[\mathbf{B}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}] &:= \langle \boldsymbol{\psi} \cdot \mathbf{n}, \xi \rangle_\Sigma + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma + \int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) && \forall\, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H}, \\
[\mathbf{C}(\vec{\boldsymbol{z}}), \vec{\mathbf{v}}] &:= \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{\mathbf{t},\Sigma}, && \forall\, \vec{\boldsymbol{z}}, \vec{\mathbf{v}} \in \mathbf{Z},
\end{aligned}
\tag{3.10}
$$

whereas for each $\mathbf{w}_S \in \mathbf{L}^4(\Omega_S)$, $b(\mathbf{w}_S; \cdot, \cdot) : \mathbf{L}^4(\Omega_S) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega_S) \to \mathrm{R}$ is the bilinear form given by

$$
b(\mathbf{w}_S; \mathbf{v}_S, \mathbf{r}_S) := -\rho \int_{\Omega_S} (\mathbf{w}_S \otimes \mathbf{v}_S)^{\mathrm{d}} : \mathbf{r}_S . \tag{3.11}
$$

As announced in the abstract, we notice here that (3.9) can be seen as a nonlinear perturbation, given by the term $b(\mathbf{u}_S; \mathbf{u}_S, \mathbf{r}_S)$, of a threefold saddle point operator equation, whose main operator $\widetilde{\mathbf{A}}$, to be introduced below, shows a perturbed saddle-point structure (cf. [13]). Indeed, letting $\mathbf{A} : (\mathbf{X} \times \mathbf{Y}) \times (\mathbf{X} \times \mathbf{Y}) \to \mathrm{R}$ be the bilinear form that arises from the block $\begin{pmatrix} a & b_1 \\ b_2 & \end{pmatrix}$ by adding the first two equations of (3.9), that is

$$
[\mathbf{A}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})] := [a(\vec{\boldsymbol{\zeta}}), \vec{\mathbf{r}}] + [b_1(\vec{\mathbf{r}}), \vec{\boldsymbol{\eta}}] + [b_2(\vec{\boldsymbol{\zeta}}), \vec{\boldsymbol{\tau}}] \quad \forall\, (\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X} \times \mathbf{Y}, \tag{3.12}
$$

and letting $\widetilde{\mathbf{A}} : \mathbb{H} \times \mathbb{H} \to \mathrm{R}$ be the bilinear form that is derived from the block $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix}$ by adding the first three equations from (3.9), that is

$$
[\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})] := [\mathbf{A}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})] + [\mathbf{B}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}), \vec{\boldsymbol{z}}] + [\mathbf{B}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), \vec{\mathbf{v}}] - [\mathbf{C}(\vec{\boldsymbol{z}}), \vec{\mathbf{v}}] \tag{3.13}
$$

for all $(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}$, we find that (3.9) becomes: Find $((\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),\vec{\boldsymbol{p}})\in\mathbb{H}\times\mathbf{Q}$ such that

$$\begin{aligned} [\widetilde{\mathbf{A}}(\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})] \;+\; [\widetilde{\mathbf{B}}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{p}}] \;+\; b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_S) &= [\mathrm{G},(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})]\,,\\ [\widetilde{\mathbf{B}}(\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),\vec{\boldsymbol{q}}] &= [\mathrm{F},\vec{\boldsymbol{q}}]\,, \end{aligned} \tag{3.14}$$

for all $(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}$, for all $\vec{\boldsymbol{q}}\in\mathbf{Q}$, where

$$\begin{gathered} [\widetilde{\mathbf{B}}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}}] := -\int_{\Omega_D} q_D\,\mathrm{div}(\mathbf{v}_D) \;+\; j\,\langle\boldsymbol{\psi}\cdot\mathbf{n},1\rangle_\Sigma\,,\\ [\mathrm{G},(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})] := \langle\boldsymbol{\tau}_S\mathbf{n},\mathbf{g}\rangle_{\Gamma_S} - \int_{\Omega_S}\mathbf{f}_S\cdot\mathbf{v}_S \quad\text{and}\quad [\mathrm{F},\vec{\boldsymbol{q}}] := -\int_{\Omega_D} f_D q_D + j\langle\mathbf{g}\cdot\mathbf{n},1\rangle_{\Gamma_S}\,. \end{gathered} \tag{3.15}$$

Moreover, letting now $\mathbf{P}:(\mathbb{H}\times\mathbf{Q})\times(\mathbb{H}\times\mathbf{Q})\to\mathrm{R}$ be the bilinear that arises from the block $\begin{pmatrix}\widetilde{\mathbf{A}} & \widetilde{\mathbf{B}}\\ \widetilde{\mathbf{B}} & \end{pmatrix}$ by adding both equations of (3.14), that is

$$[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] := [\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})] + [\widetilde{\mathbf{B}}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\mathbf{s}}] + [\widetilde{\mathbf{B}}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\boldsymbol{q}}] \tag{3.16}$$

for all $((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}}),((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}$, we deduce that (3.14) (and hence (3.9)) can be stated, equivalently as well, as: Find $((\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),\vec{\boldsymbol{p}})\in\mathbb{H}\times\mathbf{Q}$ such that

$$[\mathbf{P}(\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}},\vec{\boldsymbol{p}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] + b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_S) = [\mathrm{H},(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] \quad \forall\,((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}\,, \tag{3.17}$$

where $\mathrm{H}\in(\mathbb{H}\times\mathbf{Q})'$ is defined by $[\mathrm{H},(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] = [\mathrm{G},(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})] + [\mathrm{F},\vec{\boldsymbol{q}}]$. Furthermore, let us introduce the operator $\mathbf{T}:\mathbf{L}^4(\Omega_S)\to\mathbf{L}^4(\Omega_S)$ defined as

$$\mathbf{T}(\mathbf{w}_S) \;:=\; \mathbf{u}_S \quad \forall\,\mathbf{w}_S\in\mathbf{L}^4(\Omega_S)\,, \tag{3.18}$$

where $\mathbf{u}_S$ is the first component of $\vec{\mathbf{u}}\in\mathbf{Z}$, which, in turn, is the third component of the unique solution $\big((\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),\vec{\boldsymbol{p}}\big)\in\mathbb{H}\times\mathbf{Q}$ (to be proved later on) of the linearized problem arising from (3.17) after replacing $b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_S)$ by $b(\mathbf{w}_S;\mathbf{u}_S,\mathbf{r}_S)$, namely:

$$[\mathbf{P}(\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}},\vec{\boldsymbol{p}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] + b(\mathbf{w}_S;\mathbf{u}_S,\mathbf{r}_S) = [\mathrm{H},(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] \quad \forall\,((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}\,. \tag{3.19}$$

Thus, we realize that solving (3.14) (or (3.17)) is equivalent to finding a fixed-point of $\mathbf{T}$, that is $\mathbf{u}_S\in\mathbf{L}^4(\Omega_S)$ such that

$$\mathbf{T}(\mathbf{u}_S) \;=\; \mathbf{u}_S. \tag{3.20}$$

3.3 Solvability analysis

In this section we analyze the solvability of (3.17) (which is equivalent to (3.9) or (3.14)), by means of the fixed-point strategy that was depicted at the end of the previous section. To this end, we first recall next some theoretical results to be applied later on.

3.3.1 Some useful abstract results

We begin with the generalized Babuška-Brezzi theory.

Theorem 3.1. *Let H_1, H_2, Q_1 and Q_2 be reflexive Banach spaces, and let $b_i : H_i\times Q_i\to R, i\in\{1,2\}$, be bounded bilinear forms with boundedness constants given by $\|a\|$ and $\|b_i\|$, $i\in\{1,2\}$, respectively. In addition, for each $i\in\{1,2\}$, let $\mathcal{K}_i$ be the kernel of the operator induced by b_i, that is*

$$\mathcal{K}_i \;:=\; \Big\{v\in H_i:\quad b_i(v,q)=0 \quad \forall\,q\in Q_i\Big\}\,,$$

and assume that

i) *there exists a positive constant α such that*

$$\sup_{\substack{v\in\mathcal{K}_1\\ v\neq 0}} \frac{a(w,v)}{\|v\|_{H_1}} \,\geq\, \alpha\,\|w\|_{H_2} \quad \forall\, w\in\mathcal{K}_2\,,$$

ii) *there holds*

$$\sup_{w\in\mathcal{K}_2} a(w,v) \,>\, 0 \quad \forall\, v\in\mathcal{K}_1,\, v\neq 0\,, \quad \textit{and}$$

iii) *for each $i\in\{1,2\}$ there exists a positive constant β_i such that*

$$\sup_{\substack{v\in H_i\\ v\neq 0}} \frac{b_i(v,q)}{\|v\|_{H_i}} \,\geq\, \beta_i\,\|q\|_{Q_i} \quad \forall\, q\in Q_i\,.$$

Then, for each $(\mathrm{F},\mathrm{G})\in H_1'\times Q_2'$ there exists a unique $(u,p)\in H_2\times Q_1$ such that

$$\begin{array}{lcll} a(u,v) + \; b_1(v,p) & = & \mathrm{F}(v) & \forall\, v\in H_1\,,\\ b_2(u,q) & = & \mathrm{G}(q) & \forall\, q\in Q_2\,, \end{array} \tag{3.21}$$

and the following a priori estimates hold

$$\|u\|_{H_2} \leq \frac{1}{\alpha}\|\mathrm{F}\|_{H_1'} + \frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right)\|\mathrm{G}\|_{Q_2'} \tag{3.22}$$

$$\|p\|_{Q_1} \leq \frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right)\|\mathrm{F}\|_{H_1'} + \frac{\|a\|}{\beta_1\beta_1}\left(1+\frac{\|a\|}{\alpha}\right)\|\mathrm{G}\|_{Q_2'}\,.$$

Moreover, i), ii) *and* iii) *are also necessary conditions for the well-posedness of* (3.21)*.*

Proof. See [4, Theorem 2.1, Corollary 2.1, Section 2.1] for the original version and its proof. For the particular case given by $H_1 = H_2$, $Q_1 = Q_2$, and $b_1 = b_2$, we also refer to [22, Theorem 2.34]. □

We remark here that the roles of $\mathcal{K}_1$ and $\mathcal{K}_2$ in the assumptions i) and ii) of Theorem 3.1 can be exchanged without altering the joint meaning of these hypotheses. In addition, it is important to stress that (3.22) is equivalent to an inf-sup condition for the bilinear form arising after adding the left-hand sides of (3.21), which means that there exists a constant $C>0$, depending only on α,β_1,β_2 and $\|a\|$, such that

$$\sup_{\substack{(v,q)\in H_1\times Q_2\\ (v,q)\neq\mathbf{0}}} \frac{a(w,v)+b_1(v,r)+b_2(w,q)}{\|(v,q)\|_{H_1\times Q_2}} \,\geq\, C\,\|(w,r)\|_{H_2\times Q_1} \quad \forall\,(w,r)\in H_2\times Q_1\,. \tag{3.23}$$

Next, we recall from [25, Theorem 3.2] (see also [13, Theorem 3.4] for the original version of it) a result providing sufficient conditions for the well-posedness of a perturbed saddle-point problem.

Theorem 3.2. *Let H and Q be reflexive Banach spaces, and let $a: H\times H\to R$, $b: H\times Q\to R$ and $c: Q\times Q\to R$ be given bounded bilinear forms. In addition, let $\mathbf{B}: H\to Q'$ be the bounded linear operator induced by b, and let $V := N(\mathbf{B})$ be the respective null space. Assume that:*

i) *a and c are positive semi-definite, that is*

$$a(\tau,\tau)\geq 0 \quad \forall\,\tau\in H \quad \textit{and} \quad c(v,v)\geq 0 \quad \forall\, v\in Q, \tag{3.24}$$

and that c is symmetric,

ii) *there exists a constant* $\alpha > 0$ *such that*

$$\sup_{\substack{\tau \in V \\ \tau \neq 0}} \frac{a(\vartheta, \tau)}{\|\tau\|_H} \geq \alpha \|\vartheta\|_H \quad \forall \vartheta \in V, \quad \textit{and} \tag{3.25}$$

$$\sup_{\substack{\vartheta \in V \\ \vartheta \neq 0}} \frac{a(\vartheta, \tau)}{\|\vartheta\|_H} \geq \alpha \|\tau\|_H \quad \forall \tau \in V, \tag{3.26}$$

iii) *and there exists a constant* $\beta > 0$ *such that*

$$\sup_{\substack{\tau \in H \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_H} \geq \beta \|v\|_Q \quad \forall v \in Q.$$

Then, for each pair $(f, g) \in H' \times Q'$ *there exists a unique* $(\sigma, u) \in H \times Q$ *such that*

$$\begin{aligned} a(\sigma, \tau) \;&+\; b(\tau, u) \;=\; f(\tau) \qquad \forall \tau \in H, \\ b(\sigma, v) \;&-\; c(u, v) \;=\; g(v) \qquad \forall v \in Q. \end{aligned} \tag{3.27}$$

Moreover, there exists a constant $\widetilde{C} > 0$*, depending only on* $\|a\|, \|c\|, \alpha,$ *and* β*, such that*

$$\|(\sigma, u)\|_{H \times Q} \leq \widetilde{C} \left\{ \|f\|_{H'} + \|g\|_{Q'} \right\}. \tag{3.28}$$

As announced before, we stress here that the foregoing theorem is referred to as a slight variant of the original version given by [13, Theorem 3.4], which requires a to be symmetric as well. Indeed, the proof reduces basically to show that there exists a positive constant $\widehat{C}$, depending on $\|a\|$, $\|c\|$, α, and β, such that the bilinear form arising from adding the left hand sides of (3.27), say $A : (H \times Q) \times (H \times Q) \to \mathrm{R}$, satisfies the inf-sup condition

$$\sup_{\substack{(\tau, v) \in H \times Q \\ (\tau, v) \neq \mathbf{0}}} \frac{A((\zeta, w), (\tau, v))}{\|(\tau, v)\|_{H \times Q}} \geq \widehat{C} \|(\zeta, w)\|_{H \times Q} \quad \forall (\zeta, w) \in H \times Q. \tag{3.29}$$

In this way, thanks to the symmetry of a and c, A is obviously symmetric, and thus (3.29) is sufficient to conclude, using the Banach–Nečas–Babuška Theorem (cf.[19, Theorem 2.6], also known as the generalized Lax–Milgram Lemma, the well-posedness of (3.27). However, if the symmetry assumption on a (and consequently on A) is dropped, as done in the present Theorem 3.2, the same conclusion is attained if additionally (3.29) is also satisfied by the bilinear form $\widetilde{A}$ that arises from A after exchanging its components. Thus, noting that the above reduces to fixing the second component of A and taking the supremum in (3.29) with respect to the first one, we realize that in order to prove this further inf-sup condition, the assumption (3.25) needs to be added, as we did in Theorem 3.2. Needless to say, and because of the same constant α in (3.24) and (3.25), the aforementioned further condition holds with the same constant $\widehat{C}$ from (3.29).

3.3.2 Well-definedness of the operator $\mathbf{T}$

We continue by establishing the well-definedness of the operator $\mathbf{T}$, equivalently, that problem (3.19) is well-posed. To this end, we first state the boundedness of all the variational forms involved by employing the Cauchy–Schwarz and Hölder inequalities, the upper bounds of μ, the continuity of the normal trace operator in $\mathbb{H}(\mathbf{div}_{4/3}; \Omega_S)$ (which follows from (1.2)), the boundedness of the injection

$\mathbf{i}_4 : \mathbf{H}^1(\Omega_S) \to \mathbf{L}^4(\Omega_S)$, the boundedness of a suitable extension operator $E_D : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{H}^{1/2}(\partial\Omega_D)$ to be defined later on in (3.37) - (3.38), and the existence of a positive constant c_s, depending only on $\partial\Omega_S$, such that $\|\boldsymbol{\psi}\|_{0,\Sigma} \le c_s\|\boldsymbol{\psi}\|_{1/2,\Sigma} \quad \forall \boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Sigma)$, which yields, in particular, $\|\boldsymbol{\psi}\|_{0,\Sigma} \le c_s\|\boldsymbol{\psi}\|_{1/2,00;\Sigma} \quad \forall \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$ (see [2, Appendix A.1]). In this way, we deduce the existence of positive constants, denoted and given as:

$$\begin{gathered}
\|a\| := \max\{\mu_2, \|\mathbf{K}^{-1}\|_\infty\}, \quad \|b_1\| = \|b_2\| := \max\left\{1, \|E_D\|\right\}, \\
\|\mathbf{A}\| = \|a\| + 2\|b_1\|, \quad \|\mathbf{B}\| = \max\{1, \|\mathbf{i}_4\|, c_s^2\}, \\
\|\mathbf{C}\| := c_s^2(n-1)\max\{\omega_1^{-1}, ..., \omega_{n-1}^{-1}\}, \quad \|\widetilde{\mathbf{A}}\| := \|\mathbf{A}\| + 2\|\mathbf{B}\| + \|\mathbf{C}\|, \\
\|\widetilde{\mathbf{B}}\| := \max\{1, c_s|\Sigma|^{1/2}\}, \quad \text{and} \quad \|\mathrm{H}\| := \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D},
\end{gathered} \tag{3.30}$$

with $\widetilde{\mathbf{g}} := \max\{1, \|\mathbf{i}_4\|, c_s|\Sigma|^{1/2}\}\mathbf{g}$, such that

$$\begin{aligned}
|[a(\vec{\boldsymbol{\zeta}}), \vec{\mathbf{r}}]| &\le \|a\|\, \|\vec{\boldsymbol{\zeta}}\|_{\mathbf{X}} \|\vec{\mathbf{r}}\|_{\mathbf{X}} && \forall\, \vec{\boldsymbol{\zeta}}, \vec{\mathbf{r}} \in \mathbf{X}, \\
|[b_i(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}]| &\le \|b_i\|\, \|\vec{\mathbf{r}}\|_{\mathbf{X}} \|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} && \forall\, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X} \times \mathbf{Y}, \\
|[\mathbf{A}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})]| &\le \|\mathbf{A}\|\, \|(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}})\|_{\mathbf{X}\times\mathbf{Y}} \|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}} && \forall\, (\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X} \times \mathbf{Y}, \\
|[\mathbf{B}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}]| &\le \|\mathbf{B}\|\, \|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}} \|\vec{\mathbf{v}}\|_{\mathbf{Z}} && \forall\, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H}, \\
|[\mathbf{C}(\vec{\mathbf{v}}), \vec{\boldsymbol{z}}]| &\le \|\mathbf{C}\|\, \|\vec{\mathbf{v}}\|_{\mathbf{X}} \|\vec{\boldsymbol{z}}\|_{\mathbf{X}} && \forall\, \boldsymbol{\psi}, \boldsymbol{\phi} \in \mathbf{H}^{1/2}_{00}(\Sigma), \\
|[\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})]| &\le \|\widetilde{\mathbf{A}}\|\, \|(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}})\|_{\mathbb{H}} \|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}} && \forall (\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H}, \\
|[\widetilde{\mathbf{B}}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}]| &\le \|\widetilde{\mathbf{B}}\|\, \|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}} \|\vec{\boldsymbol{q}}\|_{\mathbf{Q}} && \forall\, ((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H} \times \mathbf{Q}, \\
|[\mathrm{H}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})]| &\le \|\mathrm{H}\|\, \|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}} && \forall\, ((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H} \times \mathbf{Q}.
\end{aligned} \tag{3.31}$$

In turn, employing the Cauchy–Schwarz inequality twice, we find that

$$\begin{gathered}
|b(\mathbf{w}_S; \mathbf{v}_S, \mathbf{r}_S)| \le \rho\, \|\mathbf{w}_S\|_{0,4;\Omega_S}\, \|\mathbf{v}_S\|_{0,4;\Omega_S}\, \|\mathbf{r}_S\|_{0,\Omega_S} \\
\forall\, (\mathbf{w}_S, \mathbf{v}_S, \mathbf{r}_S) \in \mathbf{L}^4(\Omega_S) \times \mathbf{L}^4(\Omega_S) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega_S).
\end{gathered} \tag{3.32}$$

In what follows, and as suggested by the matrix representation $\begin{pmatrix} \widetilde{\mathbf{A}} & \widetilde{\mathbf{B}} \\ \widetilde{\mathbf{B}} & \mathbf{0} \end{pmatrix}$, we apply the symmetric case of Theorem 3.1. In particular, in order to derive the inf-sup conditions of the bilinear form $\widetilde{\mathbf{A}}$, and according to its structure given by $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix}$ (cf. (3.13)), we employ Theorem 3.2. In turn, and due to the corresponding structure $\begin{pmatrix} a & b_1 \\ b_2 & 0 \end{pmatrix}$ of $\mathbf{A}$ (cf. (3.12)), we employ Theorem 3.1 to establish the required assumptions on $\mathbf{A}$. For the above purposes, we begin by deducing from the definition (3.15) that the kernel $\widetilde{\mathrm{V}}$ of $\widetilde{\mathbf{B}}$ reduces to

$$\widetilde{\mathrm{V}} := \Big\{(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H} : \quad [\widetilde{\mathbf{B}}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}] = 0 \quad \forall\, \vec{\boldsymbol{q}} \in \mathbf{Q}\Big\} = \widetilde{\mathbf{X}} \times \mathbf{Y} \times \widetilde{\mathbf{Z}}, \tag{3.33}$$

where

$$\widetilde{\mathbf{X}} = \mathbb{L}^2_{\mathrm{tr}}(\Omega_S) \times \widetilde{\mathbf{H}}_0(\mathrm{div}; \Omega_D), \qquad \widetilde{\mathbf{Z}} = \mathbf{L}^4(\Omega_S) \times \widetilde{\mathbf{H}}^{1/2}_{00}(\Sigma), \tag{3.34}$$

with

$$\widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D) := \Big\{ \mathbf{v}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D) : \quad \mathrm{div}(\mathbf{v}_D) \in \mathrm{P}_0(\Omega_D) \Big\} ,$$

$$\widetilde{\mathbf{H}}_{00}^{1/2}(\Sigma) := \Big\{ \boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma) : \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma = 0 \Big\} .$$

Hereafter, we refer to the null space of the bounded linear operator induced by a bilinear form as the kernel of the latter. Then we let V be the kernel of $\mathbf{B}|_{\widetilde{\mathrm{V}}}$, that is

$$\mathrm{V} = \widetilde{\mathbf{X}} \times \overline{\mathbf{Y}} ,$$

where

$$\begin{aligned}\overline{\mathbf{Y}} &:= \Big\{ \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \xi) \in \mathbf{Y} : \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, \xi\rangle_\Sigma + \langle \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) = 0 \quad \forall\, \vec{\mathbf{v}} := (\mathbf{v}_S, \boldsymbol{\psi}) \in \widetilde{\mathbf{Z}} \Big\} ,\\ &= \Big\{ \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \xi) \in \mathbf{Y} : \quad \mathbf{div}(\boldsymbol{\tau}_S) = \mathbf{0}, \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, \xi\rangle_\Sigma = -\langle \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma, \quad \forall\, \boldsymbol{\psi} \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Sigma) \Big\} .\end{aligned}$$

Then for each $i \in \{1, 2\}$ we let $\mathcal{K}_i$ be the kernel of $b_i|_{\mathrm{V}}$, that is

$$\mathcal{K}_i := \Big\{ \vec{\mathbf{r}} := (\mathbf{r}_S, \mathbf{v}_D) \in \widetilde{\mathbf{X}} : \quad [b_i(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}] = 0 \quad \forall\, \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \xi) \in \overline{\mathbf{Y}} \Big\} ,$$

which, recalling from (3.10) that $b_1 = -b_2$, yields

$$\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K} \subseteq \widetilde{\mathbf{X}} .$$

At this point we recall, for later use, that there exist positive constants $c_{4/3}(\Omega_S)$ and C_{div}, such that (see, [3, Lemma 4.4] and [26, Lemma 3.2], respectively, for details)

$$c_{4/3}(\Omega)\, \|\boldsymbol{\tau}_S\|_{0,\Omega_S} \;\le\; \|\boldsymbol{\tau}_S^{\mathrm{d}}\|_{0,\Omega_S} + \|\mathbf{div}(\boldsymbol{\tau}_S)\|_{0,4/3;\Omega_S} \quad \forall\, \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) \tag{3.35}$$

and

$$\|\mathbf{v}_D\|_{0,\Omega_D}^2 \ge C_{\mathrm{div}} \|\mathbf{v}_D\|_{\mathrm{div},\Omega_D}^2 \quad \forall\, \mathbf{v}_D \in \widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D) . \tag{3.36}$$

We now follow [27] to recall some preliminary results concerning boundary conditions and extension operators. Given $\mathbf{v}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D)$, the boundary condition $\mathbf{v}_D\cdot\mathbf{n} = 0$ on Γ_D means

$$\langle \mathbf{v}_D\cdot\mathbf{n}, E_{\Gamma_D,D}(\zeta)\rangle_{\partial\Omega_D} = 0 \quad \forall\, \zeta \in \mathrm{H}_{00}^{1/2}(\Gamma_D) .$$

As a consequence, it is not difficult to show (see [21, Section 2]) that the restriction of $\mathbf{v}_D\cdot\mathbf{n}$ to Σ can be identified with an element of $\mathrm{H}^{-1/2}(\Sigma)$, namely

$$\langle \mathbf{v}_D\cdot\mathbf{n}, \xi\rangle_\Sigma := \langle \mathbf{v}_D\cdot\mathbf{n}, E_D(\xi)\rangle_{\partial\Omega_D} \quad \forall\, \xi \in \mathrm{H}^{1/2}(\Sigma) ,$$

where $E_D : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{H}^{1/2}(\partial\Omega_D)$ is any bounded extension operator. In particular, given $\xi \in \mathrm{H}^{1/2}(\Sigma)$, one could define $E_D(\xi) := z|_{\partial\Omega_D}$, where $z \in \mathrm{H}^1(\Omega_D)$ is the unique solution of the boundary value problem:

$$\Delta z = 0 \quad \text{in} \quad \Omega_D , \quad z = \xi \quad \text{on} \quad \Sigma , \quad \nabla z\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_D , \tag{3.37}$$

whose continuous dependence estimate yields $E_D \in \mathcal{L}\big(\mathrm{H}^{1/2}(\Sigma), \mathrm{H}^{1/2}(\partial\Omega_D)\big)$, and hence

$$\|E_D(\xi)\|_{1/2,\partial\Omega_D} \;\le\; \|E_D\|\, \|\xi\|_{1/2,\Sigma} . \tag{3.38}$$

In addition, one can show (see [21, Lemma 2.2]) that for all $\zeta \in \mathrm{H}^{1/2}(\partial\Omega_D)$ there exist unique elements $\zeta_\Sigma \in \mathrm{H}^{1/2}(\Sigma)$ and $\zeta_{\Gamma_D} \in \mathrm{H}^{1/2}_{00}(\Gamma_D)$ such that

$$\zeta = E_D(\zeta_\Sigma) + E_{\Gamma_D,D}(\zeta_{\Gamma_D})\,, \tag{3.39}$$

and

$$C_1 \left\{ \|\zeta_\Sigma\|_{1/2,\Sigma} + \|\zeta_{\Gamma_D}\|_{1/2,00;\Gamma_D} \right\} \,\le\, \|\zeta\|_{1/2,\partial\Omega_D} \,\le\, C_2 \left\{ \|\zeta_\Sigma\|_{1/2,\Sigma} + \|\zeta_{\Gamma_D}\|_{1/2,00;\Gamma_D} \right\},$$

with positive constants C_1 and C_2, independent of Σ.

Then, we are in position to prove the results stated by the following lemmas.

Lemma 3.3. *For each $i \in \{1,2\}$ there exists a positive constant β_i such that*

$$\sup_{\substack{\vec{\mathbf{r}}\in\widetilde{\mathbf{X}}\\ \vec{\mathbf{r}}\neq\mathbf{0}}} \frac{[b_i(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}]}{\|\vec{\mathbf{r}}\|_{\mathbf{X}}} \,\ge\, \beta_i\, \|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} \quad \forall\, \vec{\boldsymbol{\tau}} \in \overline{\mathbf{Y}}\,. \tag{3.40}$$

Proof. Since $b_1 = -b_2$, it suffices to show for one of these bilinear forms, so that we stay with b_1. Moreover, considering that $\overline{\mathbf{Y}} \subseteq \widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S) \times \mathrm{H}^{1/2}(\Sigma)$, with

$$\widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S) := \left\{ \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) : \quad \mathbf{div}(\boldsymbol{\tau}_S) = \mathbf{0} \right\}\,,$$

we need to prove that there exists a positive constant β_1 such that

$$\sup_{\substack{\vec{\mathbf{r}}\in\widetilde{\mathbf{X}}\\ \vec{\mathbf{r}}\neq\mathbf{0}}} \frac{[b_1(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}]}{\|\vec{\mathbf{r}}\|_{\mathbf{X}}} \,\ge\, \beta_1\, \|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} \quad \forall\, \vec{\boldsymbol{\tau}} \in \widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S) \times \mathrm{H}^{1/2}(\Sigma)\,. \tag{3.41}$$

In addition, due to the diagonal character of b_1 (cf. (3.10)), the proof of (3.41) reduces to establishing the following two independent inf-sup conditions

$$\sup_{\substack{\mathbf{v}_D\in\widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D)\\ \mathbf{v}_D\neq\mathbf{0}}} \frac{\langle \mathbf{v}_D\cdot\mathbf{n}, \xi\rangle_\Sigma}{\|\mathbf{v}_D\|_{\mathrm{div};\Omega_D}} \,\ge\, \beta_{1,\Sigma}\, \|\xi\|_{1/2,\Sigma} \quad \forall\, \xi \in \mathrm{H}^{1/2}(\Sigma)\,, \quad \text{and} \tag{3.42}$$

$$\sup_{\substack{\mathbf{r}_S\in\mathbb{L}^2_{\mathrm{tr}}(\Omega_S)\\ \mathbf{r}_S\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_S} \boldsymbol{\tau}_S^{\mathrm{d}} : \mathbf{r}_S}{\|\mathbf{r}_S\|_{0,\Omega_S}} \,\ge\, \beta_{1,S}\, \|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S} \quad \forall\, \boldsymbol{\tau}_S \in \widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S)\,, \tag{3.43}$$

with $\beta_{1,\Sigma}$, $\beta_{1,S} > 0$. Indeed, for (3.42) we refer to [27, Lemma 3.3]. However, for sake of completeness, most details are given in what follows. Given $\phi \in \mathrm{H}^{-1/2}(\Sigma)$, we define $\eta \in \mathrm{H}^{-1/2}(\partial\Omega_D)$ as

$$\langle \eta, \zeta\rangle_{\partial\Omega_D} := \langle \phi, \zeta_\Sigma\rangle_\Sigma \quad \forall\, \zeta \in \mathrm{H}^{1/2}(\partial\Omega_D)\,, \tag{3.44}$$

where ζ_Σ is given by the decomposition (3.39). It is not difficult to see that

$$\langle \eta, E_{\Gamma_D,D}(\rho)\rangle_{\partial\Omega_D} = 0 \quad \forall\, \rho \in \mathrm{H}^{1/2}_{00}(\Gamma_D)\,, \tag{3.45}$$

$$\langle \eta, E_D(\xi)\rangle_{\partial\Omega_D} = \langle \phi, \xi\rangle_\Sigma \quad \forall \xi \in \mathrm{H}^{1/2}(\Sigma) \tag{3.46}$$

and

$$\|\eta\|_{-1/2,\partial\Omega_D} \le C\|\phi\|_{-1/2,\Sigma}\,. \tag{3.47}$$

Hence, we now define $\mathbf{w}_D := \nabla z \in \Omega_D$, where $z \in H^1(\Omega_D)$ is the unique solution of the boundary value problem

$$\Delta z = \frac{1}{|\Omega_D|}\langle \eta, 1\rangle_{\partial\Omega_D} \quad \text{in} \quad \Omega_D, \qquad \nabla z \cdot \mathbf{n} = \eta \quad \text{on} \quad \partial\Omega_D, \qquad \int_{\partial\Omega_D} z = 0\,.$$

It follows that $\operatorname{div}(\mathbf{w}_D) = \frac{1}{|\Omega_D|}\langle \eta, 1\rangle_{\partial\Omega_D} \in \mathbb{P}_0(\Omega_D)$, $\mathbf{w}_D \cdot \mathbf{n} = \eta$ on $\partial\Omega_D$, and, using the estimate (3.47), $\|\mathbf{w}_D\|_{\operatorname{div};\Omega_D} \le C\|\eta\|_{-1/2,\partial\Omega_D} \le C\|\phi\|_{-1/2,\Sigma}$. In addition, according to (3.44), (3.45) and (3.46), we find, respectively, that

$$\langle \mathbf{w}_D \cdot \mathbf{n}, \xi\rangle_\Sigma = \langle \mathbf{w}_D \cdot \mathbf{n}, E_D(\xi)\rangle_{\partial\Omega_D} = \langle \eta, E_D(\xi)\rangle_{\partial\Omega_D} = \langle \phi, \xi\rangle_\Sigma$$

and

$$\langle \mathbf{w}_D \cdot \mathbf{n}, E_{\Gamma_D,D}(\rho)\rangle_{\partial\Omega_D} = \langle \eta, E_{\Gamma_D,D}(\rho)\rangle_{\partial\Omega_D} = 0 \quad \forall\, \rho \in \mathrm{H}^{1/2}_{00}(\Gamma_D)\,,$$

which implies that $\mathbf{w}_D \in \widetilde{\mathbf{H}}_0(\operatorname{div};\Omega_D)$. In this way, we conclude that

$$\sup_{\substack{\mathbf{v}_D \in \widetilde{\mathbf{H}}_0(\operatorname{div};\Omega_D)\\ \mathbf{v}_D \neq \mathbf{0}}} \frac{\langle \mathbf{v}_D \cdot \mathbf{n}, \xi\rangle_\Sigma}{\|\mathbf{v}_D\|_{\operatorname{div};\Omega_D}} \ge \frac{|\langle \mathbf{w}_D \cdot \mathbf{n}, \xi\rangle_\Sigma|}{\|\mathbf{w}_D\|_{\operatorname{div};\Omega_D}} \ge C\, \frac{\big|\langle \phi, \xi\rangle_\Sigma\big|}{\|\phi\|_{-1/2,\Sigma}} \quad \forall\, \phi \in \mathrm{H}^{-1/2}(\Sigma)\,,$$

and hence

$$\sup_{\substack{\mathbf{v}_D \in \widetilde{\mathbf{H}}_0(\operatorname{div};\Omega_D)\\ \mathbf{v}_D \neq \mathbf{0}}} \frac{\langle \mathbf{v}_D \cdot \mathbf{n}, \xi\rangle_\Sigma}{\|\mathbf{v}_D\|_{\operatorname{div};\Omega_D}} \ge C \sup_{\substack{\phi \in \mathrm{H}^{-1/2}(\Sigma)\\ \phi \neq \mathbf{0}}} \frac{\big|\langle \phi, \xi\rangle_\Sigma\big|}{\|\phi\|_{-1/2,\Sigma}} = C\,\|\xi\|_{1/2,\Sigma}\,,$$

which confirms (3.42). On the other hand, given $\boldsymbol{\tau}_S \in \widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S)$ such that $\boldsymbol{\tau}_S^{\mathtt{d}} \neq \mathbf{0}$, we have that $\boldsymbol{\tau}_S^{\mathtt{d}} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, so that bounding the supremum in (3.43) by below with $\mathbf{r}_S = -\boldsymbol{\tau}_S^{\mathtt{d}}$, it follows that

$$\sup_{\substack{\mathbf{r}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)\\ \mathbf{r}_S \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_S} \boldsymbol{\tau}_S^{\mathtt{d}} : \mathbf{r}_S}{\|\mathbf{r}_S\|_{0,\Omega_S}} \ge \frac{\displaystyle\int_{\Omega_S} \boldsymbol{\tau}_S^{\mathtt{d}} : \boldsymbol{\tau}_S^{\mathtt{d}}}{\|\boldsymbol{\tau}_S^{\mathtt{d}}\|_{0,\Omega_S}} = \|\boldsymbol{\tau}_S^{\mathtt{d}}\|_{0;\Omega_S}\,,$$

which, using (3.35) and the fact that $\mathbf{div}(\boldsymbol{\tau}_S) = \mathbf{0}$, implies that (3.43) is satisfied with constant $\beta_{1,S} = c_{4/3}(\Omega_S)$. On the other hand, if $\boldsymbol{\tau}_S^{\mathtt{d}} = \mathbf{0}$, it is clear from (3.35) that $\boldsymbol{\tau}_S = \mathbf{0}$, and so (3.43) is trivially satisfied. □

Lemma 3.4. *There exists a positive constant α such that*

$$[a(\vec{\mathbf{r}}), \vec{\mathbf{r}}] \;\ge\; \alpha_a\,\|\vec{\mathbf{r}}\|^2_{\mathbf{X}} \quad \forall\, \vec{\mathbf{r}} \in \widetilde{\mathbf{X}}\,.$$

Proof. Given $\vec{\mathbf{r}} := (\mathbf{r}_S, \mathbf{v}_D) \in \widetilde{\mathbf{X}}$, we use the definition of a (cf. (3.10)), (2.2), and (3.36), to obtain

$$[a(\vec{\mathbf{r}}), \vec{\mathbf{r}}] \;=\; \int_{\Omega_S} \mu\,\mathbf{r}_S : \mathbf{r}_S + \int_{\Omega_D} \mathbf{K}^{-1}\mathbf{v}_D \cdot \mathbf{v}_D \;\ge\; \mu_1\,\|\mathbf{r}_S\|^2_{0,\Omega_S} + C_{\mathbf{K}}\,\|\mathbf{v}_D\|^2_{0,\Omega_D} \;\ge\; \alpha_a\,\|\vec{\mathbf{r}}\|_{\mathbf{X}}\,,$$

with $\alpha_a := \dfrac{1}{2}\min\{\mu_1, C_{\operatorname{div}} C_{\mathbf{K}}\}$, thus confirming the required property on a. In particular, since $\mathcal{K} \subset \widetilde{\mathbf{X}}$, it is clear that a is $\mathcal{K}$-elliptic. □

As a consequence of Lemma 3.3 and Lemma 3.4, we conclude that a, b_1 and b_2 satisfy the hypotheses of Theorem 3.1, and hence, a straightforward application of this abstract result yields the existence of a positive constant $\alpha_{\mathbf{A}}$, depending on $\|a\|$, α_a and β_1, such that

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}) \in \mathrm{V}\\ (\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}) \neq \mathbf{0}}} \frac{[\mathbf{A}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})]}{\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}}} \;\ge\; \alpha_{\mathbf{A}}\,\|(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}})\|_{\mathbf{X}\times\mathbf{Y}} \quad \forall\, (\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}) \in \mathrm{V}\,. \tag{3.48}$$

Moreover, if we swap the roles of b_1 and b_2, changing the matrix from $\begin{pmatrix} a & b_1 \\ b_2 & 0 \end{pmatrix}$ to $\begin{pmatrix} a & b_2 \\ b_1 & 0 \end{pmatrix}$, we can reapply Theorem 3.1 and (3.23) to conclude that, with the same constant $\alpha_{\mathbf{A}}$ from (3.48), there holds

$$\sup_{\substack{(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}})\in \mathrm{V} \\ (\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}})\neq \mathbf{0}}} \frac{[\mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})]}{\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}})\|_{\mathbf{X}\times\mathbf{Y}}} \,\geq\, \alpha_{\mathbf{A}}\,\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}} \quad \forall\,(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\in \mathrm{V}\,.$$

Furthermore, it is evident from (3.12) and the ellipticity of a in $\widetilde{\mathbf{X}}$, that

$$[\mathbf{A}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})] \,=\, [a(\vec{\mathbf{r}}),\vec{\mathbf{r}}] \,\geq\, \alpha_a\,\|\vec{\mathbf{r}}\|_{\mathbf{X}} \quad \forall\,(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\in \widetilde{\mathbf{X}}\times\mathbf{Y}\,,$$

which proves that $\mathbf{A}$ is positive semi-definite.

Lemma 3.5. *There holds*

$$[\mathbf{C}(\vec{\mathbf{v}}),\vec{\mathbf{v}}]\geq 0 \quad \forall\,\vec{\mathbf{v}}\in\mathbf{Z}\,.$$

Proof. From the definition of the operator $\mathbf{C}$ (cf. (3.10)), it readily follows that

$$[\mathbf{C}(\vec{\mathbf{v}}),\vec{\mathbf{v}}] = \sum_{l=1}^{n-1} w_l^{-1}\|\boldsymbol{\psi}\cdot\mathbf{t}_l\|_{0,\Sigma}^2 \geq 0 \quad \vec{\mathbf{v}}\in\mathbf{Z}\,,$$

which confirms that $\mathbf{C}$ is positive semi-definite. □

In this way, we have demonstrated that $\mathbf{A}$ and $\mathbf{C}$ satisfy hypotheses i) and ii) of Theorem 3.2, and hence it only remains to show the corresponding assumption iii), which is the continuous inf-sup condition for $\mathbf{B}$ with respect to the third component $\widetilde{\mathbf{Z}}$ of the kernel $\widetilde{\mathrm{V}}$ of $\widetilde{\mathbf{B}}$ (cf. (3.33), (3.34)).

Lemma 3.6. *There exists a positive constant $\boldsymbol{\beta}_S$ such that*

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\in\widetilde{\mathbf{X}}\times\mathbf{Y} \\ (\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\neq\mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}]}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}}} \,\geq\, \boldsymbol{\beta}_S\,\|\vec{\mathbf{v}}\|_{\mathbf{Z}} \qquad \forall\,\vec{\mathbf{v}}\in\widetilde{\mathbf{Z}}\,. \tag{3.49}$$

Proof. Given $\vec{\mathbf{v}} := (\mathbf{v}_S,\boldsymbol{\psi})\in\widetilde{\mathbf{Z}} := \mathbf{L}^4(\Omega_S)\times\widetilde{\mathbf{H}}_{00}^{1/2}(\Sigma)$, we first realize, taking $\vec{\mathbf{r}} := (\mathbf{r}_S,\mathbf{v}_D) = \vec{\mathbf{0}}$ and $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S,\xi) = (\boldsymbol{\tau}_S,0)$, that

$$\begin{aligned} \sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\in\mathbb{H} \\ (\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\neq\mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}]}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\|_{\mathbb{H}}} &\geq \sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) \\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{0}},(\boldsymbol{\tau}_S,0)),\vec{\mathbf{v}}]}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}} \\ &= \sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) \\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}}\,. \end{aligned} \tag{3.50}$$

Next, setting $\boldsymbol{\tau}_S := \boldsymbol{\tau}_{S,0} + c\mathbb{I}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$ with the respective components $\boldsymbol{\tau}_{S,0}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)$ and $c\in\mathrm{R}$, we observe that

$$\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S) = \int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_{S,0})\,, \qquad \langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma = \langle\boldsymbol{\tau}_{S,0}\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma\,, \quad \text{and}$$

$$\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}^2 \,=\, \|\boldsymbol{\tau}_{S,0}\|_{\mathbf{div}_{4/3};\Omega_S}^2 \,+\, 2\,c^2\,|\Omega_S|\,.$$

Hence, noting that $\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S} \geq \|\boldsymbol{\tau}_{S,0}\|_{\mathbf{div}_{4/3};\Omega_S}$, we find that

$$\sup_{\substack{\boldsymbol{\tau}_{S,0}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_{S,0}\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_{S,0}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_{S,0})}{\|\boldsymbol{\tau}_{S,0}\|_{\mathbf{div}_{4/3};\Omega_S}} \geq \sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}},$$

which, along with (3.50), implies that in order to conclude (3.49), it suffices to show that there exists a positive constant $\boldsymbol{\beta}_S$, independent of the given $\vec{\mathbf{v}} := (\mathbf{v}_S,\boldsymbol{\psi}) \in \widetilde{\mathbf{Z}}$, such that

$$\sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}} \geq \boldsymbol{\beta}_S\left\{\|\boldsymbol{\psi}\|_{1/2,00;\Sigma} + \|\mathbf{v}_S\|_{0,4;\Omega_S}\right\}. \tag{3.51}$$

To this end, we now set $\widehat{\mathbf{v}}_S := |\mathbf{v}_S|^2\,\mathbf{v}_S$ and notice that $\|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S}^{4/3} = \|\mathbf{v}_S\|_{0,4;\Omega_S}^4$, which says that $\widehat{\mathbf{v}}_S \in \mathbf{L}^{4/3}(\Omega_S)$, and

$$\int_{\Omega_S}\mathbf{v}_S\cdot\widehat{\mathbf{v}}_S \;=\; \|\mathbf{v}_S\|_{0,4;\Omega_S}\,\|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S}\,. \tag{3.52}$$

Then, we let $\boldsymbol{z} \in \mathbf{H}^1(\Omega_S)$ be the unique solution of

$$-\Delta\boldsymbol{z} \;=\; \widehat{\mathbf{v}}_S \quad\text{in}\quad \Omega_S, \qquad \boldsymbol{z} = 0 \quad\text{on}\quad \Gamma_S\,, \qquad\text{and}\qquad \nabla\boldsymbol{z}\,\mathbf{n} = 0 \quad\text{on}\quad \Sigma\,,$$

whose variational formulation reads: Find $\boldsymbol{z} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ such that

$$\int_{\Omega_S}\nabla\boldsymbol{z}\cdot\nabla\mathbf{w} = \int_{\Omega_S}\widehat{\mathbf{v}}_S\cdot\mathbf{w} \quad \forall\,\mathbf{w}\in\mathbf{H}^1_{\Gamma_S}(\Omega_S)\,, \tag{3.53}$$

where

$$\mathbf{H}^1_{\Gamma_S}(\Omega_S) \;:=\; \Big\{\mathbf{w}\in\mathbf{H}^1(\Omega_S): \quad \mathbf{w} = \mathbf{0} \quad\text{on}\quad \Gamma_S\Big\}\,.$$

In fact, we first notice that the left-hand side of (3.53) defines an $\mathbf{H}^1_{\Gamma_S}(\Omega_S)$-elliptic bilinear form. In addition, Hölder's inequality and the continuous injection $\mathbf{i}_4$ from $\mathbf{H}^1(\Omega_S)$ into $\mathbf{L}^4(\Omega_S)$ guarantee that the right-hand side of (3.53) constitutes a functional in $\mathbf{H}^1_{\Gamma_S}(\Omega_S)'$. Consequently, a straightforward application of the classical Lax–Milgram Lemma implies the existence of a unique $\boldsymbol{z} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ solution to (3.53). Moreover, it follows from (3.53) that

$$|\boldsymbol{z}|_{1,\Omega_S} \;\leq\; c_s\,\|\mathbf{i}_4\|\,\|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S}\,, \tag{3.54}$$

where c_s is the positive constant, depending only on Ω_S, provided by the Poincaré inequality, that is such that $\|\mathbf{v}\|_{1,\Omega_S} \leq c_s\,|\mathbf{v}|_{1,\Omega_S}$ for all $\mathbf{v}\in\mathbf{H}^1_{\Gamma_S}(\Omega_S)$. Then, defining $\widetilde{\boldsymbol{\tau}}_S := -\nabla\boldsymbol{z} \in \mathbf{L}^2(\Omega_S)$, we see that $\mathbf{div}(\widetilde{\boldsymbol{\tau}}_S) = \widehat{\mathbf{v}}_S$ in Ω_S, which says that actually $\widetilde{\boldsymbol{\tau}}_S \in \mathbb{H}(\mathbf{div}_{4/3},\Omega_S)$, and that $\widetilde{\boldsymbol{\tau}}_S\mathbf{n} = \mathbf{0}$ on Σ, so that using (3.54), we get

$$\|\widetilde{\boldsymbol{\tau}}_S\|_{\mathrm{div}_{4/3};\Omega_S} \;=\; |\boldsymbol{z}|_{1,\Omega_S} + \|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S} \;\leq\; \big(1 \,+\, c_s\,\|\mathbf{i}_4\|\big)\,\|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S}\,. \tag{3.55}$$

In this way, bounding by below with $\widetilde{\boldsymbol{\tau}}_S$, and employing (3.52) and (3.55), we deduce that

$$\begin{aligned}\sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}} &\geq \frac{\displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\widetilde{\boldsymbol{\tau}}_S)}{\|\widetilde{\boldsymbol{\tau}}_S\|_{\mathbf{div}_{4/3};\Omega_S}}\\ = \frac{\displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\widehat{\mathbf{v}}_S}{\|\widetilde{\boldsymbol{\tau}}_S\|_{\mathbf{div}_{4/3};\Omega_S}} = \frac{\|\mathbf{v}_S\|_{0,4;\Omega_S}\,\|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S}}{\|\widetilde{\boldsymbol{\tau}}_S\|_{\mathbf{div}_{4/3};\Omega_S}} &\geq \boldsymbol{\beta}_{S,1}\,\|\mathbf{v}_S\|_{0,4;\Omega_S}\,,\end{aligned} \tag{3.56}$$

with $\boldsymbol{\beta}_{S,1} := \big(1 + c_s \|\mathbf{i}_4\|\big)^{-1}$. On the other hand, given $\boldsymbol{\eta} \in \mathbf{H}_{00}^{-1/2}(\Sigma)$, we let $\widehat{\boldsymbol{z}} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ be the unique solution of

$$-\Delta \widehat{\boldsymbol{z}} = \mathbf{0} \quad \text{in} \quad \Omega_S\,, \quad \widehat{\boldsymbol{z}} = \mathbf{0} \quad \text{on} \quad \Gamma_S\,, \quad \nabla \widehat{\boldsymbol{z}}\,\mathbf{n} = \boldsymbol{\eta} \quad \text{on} \quad \Sigma\,,$$

and define $\widehat{\boldsymbol{\tau}}_S := \nabla \widehat{\boldsymbol{z}}$ in Ω_S. It follows that $\mathbf{div}(\widehat{\boldsymbol{\tau}}_S) = \mathbf{0}$ in Ω_S, $\widehat{\boldsymbol{\tau}}_S\,\mathbf{n} = \boldsymbol{\eta}$ on Γ_S, and $\|\widehat{\boldsymbol{\tau}}_S\|_{\mathbf{div}_{4/3};\Omega_S} = \|\widehat{\boldsymbol{\tau}}_S\|_{0,\Omega_S} \le \widehat{C}\,\|\boldsymbol{\eta}\|_{-1/2,00;\Sigma}$, which yields

$$\sup_{\substack{\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_S \neq \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}} \ge \frac{\langle \widehat{\boldsymbol{\tau}}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma}{\|\widehat{\boldsymbol{\tau}}_S\|_{\mathrm{div}_{4/3};\Omega_S}} \ge \boldsymbol{\beta}_{S,2}\,\frac{\big|\langle \boldsymbol{\eta}, \boldsymbol{\psi}\rangle_\Sigma\big|}{\|\boldsymbol{\eta}\|_{-1/2,00;\Sigma}}\,,$$

with $\boldsymbol{\beta}_{S,2} := \widehat{C}^{-1}$. Since $\boldsymbol{\eta} \in \mathbf{H}_{00}^{-1/2}(\Sigma)$ is arbitrary, the foregoing inequality leads to

$$\sup_{\substack{\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_S \neq \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}} \ge \boldsymbol{\beta}_{S,2}\,\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}\,,$$

which, along with (3.56), shows (3.51), and hence (3.49), with $\boldsymbol{\beta}_S := \frac{1}{2}\min\big\{\boldsymbol{\beta}_{S,1}, \boldsymbol{\beta}_{S,2}\big\}$. □

Consequently, having the bilinear forms $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ satisfied the three hypotheses of Theorem 3.2, a straightforward application of this abstract result yields the existence of a positive constant $\widetilde{\alpha}$, depending on $\|\mathbf{A}\|$, $\|\mathbf{C}\|$, $\alpha_{\mathbf{A}}$, and $\boldsymbol{\beta}_S$ such that

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}) \in \widetilde{\mathrm{V}}\\ (\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}) \neq \mathbf{0}}} \frac{[\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})]}{\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}}} \ge \widetilde{\alpha}\,\|(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}})\|_{\mathbb{H}} \qquad \forall\,(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}) \in \widetilde{\mathrm{V}}\,,$$

and

$$\sup_{\substack{(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}) \in \widetilde{\mathrm{V}}\\ (\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}) \neq \mathbf{0}}} \frac{[\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})]}{\|(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}})\|_{\mathbb{H}}} \ge \widetilde{\alpha}\,\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}} \qquad \forall\,(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \widetilde{\mathrm{V}}\,,$$

which means that $\widetilde{\mathbf{A}}$ satisfies the assumptions i) and ii) of Theorem 3.1. Thus, it only remains to demonstrate the corresponding assumption iii), which is the continuous inf-sup condition for $\widetilde{\mathbf{B}}$.

Lemma 3.7. *There exists a positive constant $\widetilde{\beta}$ such that*

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}) \in \mathbb{H}\\ (\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}) \neq \mathbf{0}}} \frac{[\widetilde{\mathbf{B}}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}]}{\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}}} \ge \widetilde{\beta}\,\|\vec{\boldsymbol{q}}\|_{\mathbf{Q}} \qquad \forall\,\vec{\boldsymbol{q}} \in \mathbf{Q}\,. \tag{3.57}$$

Proof. We first observe that the diagonal character of $\widetilde{\mathbf{B}}$ (cf. (3.15)) says that proving (3.57) is equivalent to establishing the following two independent inf–sup conditions

$$\sup_{\substack{\mathbf{v}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D)\\ \mathbf{v}_D \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_D} q_D\,\mathrm{div}(\mathbf{v}_D)}{\|\mathbf{v}_D\|_{\mathrm{div},\Omega_D}} \ge \widetilde{\beta}_D\,\|q_D\|_{0,\Omega_D} \quad \forall\, q_D \in \mathrm{L}^2_0(\Omega_D)\,, \tag{3.58}$$

$$\sup_{\substack{\boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma)\\ \boldsymbol{\psi} \neq \mathbf{0}}} \frac{j\,\langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma}{\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}} \ge \widetilde{\beta}_S\,|j| \quad \forall\, j \in \mathrm{R}\,. \tag{3.59}$$

To this end, we proceed similarly to the proof of [26, Lemma 3.6]. We define $\mathbf{v}_D := \nabla z$, where $z \in \mathrm{H}^1_\Sigma(\Omega_D)$ is the unique solution of the boundary value problem:

$$\Delta z = q_D \quad \text{in} \quad \Omega_D\,, \quad z = 0 \quad \text{on} \quad \Sigma\,, \quad \nabla z \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_D\,.$$

It follows that $\mathbf{v}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D)$ and $\mathrm{div}(\mathbf{v}_D) = q_D$, which yields the surjectivity of the operator $\mathrm{div} : \mathbf{H}_0(\mathrm{div};\Omega_D) \to \mathrm{L}^2_0(\Omega_D)$, which is (3.58). On the other hand, the inf-sup condition (3.59) reduces to the surjectivity of the operator $\boldsymbol{\psi} \to \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma$ from $\mathrm{H}^{1/2}(\Sigma) \to \mathrm{R}$, which in turn is equivalent to showing the existence of $\boldsymbol{\psi}_0 \in \mathrm{H}^{1/2}(\Sigma)$ such that $\langle \boldsymbol{\psi}_0\cdot\mathbf{n}, 1\rangle_\Sigma \neq 0$. In fact, we pick one corner point of Σ and define a function v that is continuous, linear on each side of Σ, equal to one in the chosen vertex, and zero on all other ones. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors on the two sides of Σ that meet at the corner point, then $\boldsymbol{\psi}_0 := \nu(\mathbf{n}_1 + \mathbf{n}_2)$ satisfies the required property. Finally, the required inequality (3.57) is obtained with $\widetilde{\beta} := \min\left\{\widetilde{\beta}_S, \widetilde{\beta}_D\right\}$. □

Now, having the bilinear forms $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ satisfied the assumptions of Theorem 3.1, a direct application of this abstract result guarantees the global inf-sup condition for $\mathbf{P}$ (cf. (3.16)), that is the existence of a positive constant $\alpha_{\mathbf{P}}$, depending on $\widetilde{\alpha}$, $\widetilde{\beta}$, and $\|\widetilde{\mathbf{A}}\|$, such that

$$\sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}\\ ((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})]}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}} \geq \alpha_{\mathbf{P}}\,\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}})\|_{\mathbb{H}\times\mathbf{Q}} \quad \forall\,((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}}) \in \mathbb{H}\times\mathbf{Q}\,. \tag{3.60}$$

In turn, if we consider the transpose of $\mathbf{P}$, which simply reduces to exchange the bilinear forms b_1 and b_2 in (3.12), we conclude that inf-sup conditions are satisfied by $\mathbf{P}$ with respect to the other component, that is

$$\sup_{\substack{((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\in\mathbb{H}\times\mathbf{Q}\\ ((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})]}{\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\mathbf{s})\|_{\mathbb{H}\times\mathbf{Q}}} \geq \alpha_{\mathbf{P}}\,\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}} \quad \forall\,((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}}) \in \mathbb{H}\times\mathbf{Q}\,. \tag{3.61}$$

Moreover, employing (3.60) and the boundedness property of b (cf. (3.32)), it readily follows that, given $\mathbf{w}_S \in \mathbf{L}^4(\Omega_S)$, there holds

$$\sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}\\ ((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] + b(\mathbf{w}_S;\mathbf{u}_S,\mathbf{r}_S)}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}} \geq \big(\alpha_{\mathbf{P}} - \rho\,\|\mathbf{w}_S\|_{0,4;\Omega_S}\big)\,\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}})\|_{\mathbb{H}\times\mathbf{Q}}$$

for all $((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}}) \in \mathbb{H}\times\mathbf{Q}$, and hence, for each $\mathbf{w}_S \in \mathbf{L}^4(\Omega_S)$ such that $\|\mathbf{w}_S\|_{0,4;\Omega_S} \leq \dfrac{\alpha_{\mathbf{P}}}{2\rho}$, we get

$$\sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}\\ ((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] + b(\mathbf{w}_S;\mathbf{u}_S,\mathbf{r}_S)}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}} \geq \frac{\alpha_{\mathbf{P}}}{2}\,\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}})\|_{\mathbb{H}\times\mathbf{Q}} \tag{3.62}$$

for all $((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}}) \in \mathbb{H}\times\mathbf{Q}$. Similarly, but now using (3.61), and under the same assumption on $\mathbf{w}_S$, we arrive at

$$\sup_{\substack{((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\in\mathbb{H}\times\mathbf{Q}\\ ((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] + b(\mathbf{w}_S;\mathbf{u}_S,\mathbf{r}_S)}{\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}})\|_{\mathbb{H}\times\mathbf{Q}}} \geq \frac{\alpha_{\mathbf{P}}}{2}\,\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}} \tag{3.63}$$

for all $((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}}) \in \mathbb{H}\times\mathbf{Q}$.

Consequently, the well-definedness of the operator $\mathbf{T}$ can be stated as follows.

Theorem 3.8. *For each* $\mathbf{w}_S \in \mathbf{L}^4(\Omega_S)$ *such that* $\|\mathbf{w}_S\|_{0,4;\Omega_S} \le \dfrac{\alpha_{\mathbf{P}}}{2\rho}$*, there exists a unique solution* $\big((\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),\vec{\boldsymbol{p}}\big) \in \mathbb{H}\times\mathbf{Q}$ *solution to* (3.19)*, and hence we can define* $\mathbf{T}(\mathbf{w}_S) := \mathbf{u}_S \in \mathbf{L}^4(\Omega_S)$*. Moreover, there holds*

$$\begin{aligned} \|\mathbf{T}(\mathbf{w}_S)\|_{0,4;\Omega_S} = \|\mathbf{u}_S\|_{0,4;\Omega_S} &\le \|(\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}},\vec{\boldsymbol{p}})\|_{\mathbb{H}\times\mathbf{Q}} \\ &\le \frac{2}{\alpha_{\mathbf{P}}}\Big\{\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D}\Big\}. \end{aligned} \tag{3.64}$$

Proof. Given $\mathbf{w}_S$ as indicated, the existence of a unique solution to (3.19) follows from (3.62), (3.63), and a direct application of the Banach–Nečas–Babuška Theorem (see [19, Theorem 2.6]). In turn, the corresponding *a priori* estimate and the boundedness of H (cf. (3.31)) yield (3.64). □

3.3.3 Solvability analysis of the fixed-point scheme

Knowing that the operator $\mathbf{T}$ (cf. (3.18)) is well-defined, in this section we proceed to establish the existence of a unique solution of the fixed-point equation (3.20). To this end, in what follows we will first derive sufficient conditions on $\mathbf{T}$ to map a closed ball of $\mathbf{L}^4(\Omega_S)$ into itself. This will allow us to apply the Banach Theorem later on. Indeed, from now on we let

$$\mathrm{W} := \left\{\mathbf{w}_S \in \mathbf{L}^4(\Omega_S): \quad \|\mathbf{w}_S\|_{0,4;\Omega_S} \le \frac{\alpha_{\mathbf{P}}}{2\rho}\right\}.$$

Lemma 3.9. *Assume that*

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} \le \frac{\alpha_{\mathbf{P}}^2}{4\rho}. \tag{3.65}$$

Then, there holds $\mathbf{T}\big(\mathrm{W}\big) \subseteq \mathrm{W}$.

Proof. Given $\mathbf{w}_S \in \mathrm{W}$, we know from Theorem 3.8 that $\mathbf{T}(\mathbf{w}_S)$ is well-defined and that there holds

$$\|\mathbf{T}(\mathbf{w}_S)\|_{0,4;\Omega_S} \le \frac{2}{\alpha_{\mathbf{P}}}\Big\{\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega} + \|f_D\|_{0,\Omega}\Big\} \le \frac{\alpha_{\mathbf{P}}}{2\rho}, \tag{3.66}$$

which shows that $\mathbf{T}(\mathbf{w}_S) \in \mathrm{W}$. □

We continue with the following result providing the required continuity of $\mathbf{T}$.

Lemma 3.10. *There holds*

$$\|\mathbf{T}(\mathbf{w}_S) - \mathbf{T}(\underline{\mathbf{w}}_S)\|_{0,4:\Omega_S} \le \frac{4\rho}{\alpha_{\mathbf{P}}^2}\Big\{\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D}\Big\}\|\mathbf{w}_S - \underline{\mathbf{w}}_S\|_{0,4;\Omega_S} \tag{3.67}$$

for all $\mathbf{w}_S, \underline{\mathbf{w}}_S \in \mathrm{W}$.

Proof. Given $\mathbf{w}_S, \underline{\mathbf{w}}_S \in \mathbf{L}^4(\Omega_S)$, we let $\mathbf{T}(\mathbf{w}_S) := \mathbf{u}_S$ and $\mathbf{T}(\underline{\mathbf{w}}_S) := \underline{\mathbf{u}}_S$, where $((\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),\vec{\boldsymbol{p}}) \in \mathbb{H}\times\mathbf{Q}$ and $((\underline{\vec{\mathbf{t}}},\underline{\vec{\boldsymbol{\sigma}}},\underline{\vec{\mathbf{u}}}),\underline{\vec{\boldsymbol{p}}}) \in \mathbb{H}\times\mathbf{Q}$ are the corresponding unique solutions of (3.19), that is

$$[\mathbf{P}(\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}},\vec{\boldsymbol{p}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] + b(\mathbf{w}_S;\mathbf{u}_S,\mathbf{r}_S) = [\mathrm{H},(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] \quad \forall\,((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}}) \in \mathbb{H}\times\mathbf{Q} \tag{3.68}$$

and

$$[\mathbf{P}(\underline{\vec{\mathbf{t}}},\underline{\vec{\boldsymbol{\sigma}}},\underline{\vec{\mathbf{u}}},\underline{\vec{\boldsymbol{p}}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] + b(\underline{\mathbf{w}}_S;\underline{\mathbf{u}}_S,\mathbf{r}_S) = [\mathrm{H},(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] \quad \forall\,((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}}) \in \mathbb{H}\times\mathbf{Q}. \tag{3.69}$$

Then, applying the inf-sup condition (3.62) to $(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}, \mathbf{s}) = (\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\underline{\vec{\mathbf{t}}}, \underline{\vec{\boldsymbol{\sigma}}}, \underline{\vec{\mathbf{u}}}, \underline{\vec{\boldsymbol{p}}})$, we obtain

$$\begin{aligned}
&\frac{\alpha_{\mathbf{P}}}{2} \, \|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\underline{\vec{\mathbf{t}}}, \underline{\vec{\boldsymbol{\sigma}}}, \underline{\vec{\mathbf{u}}}, \underline{\vec{\boldsymbol{p}}})\|_{\mathbb{H}\times\mathbf{Q}} \\
&\quad \le \sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q} \\ ((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{[\mathbf{P}((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\underline{\vec{\mathbf{t}}}, \underline{\vec{\boldsymbol{\sigma}}}, \underline{\vec{\mathbf{u}}}, \underline{\vec{\boldsymbol{p}}})), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] + b(\mathbf{w}_S; \mathbf{u}_S - \underline{\mathbf{u}}_S, \mathbf{r}_S)}{\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}} \,,
\end{aligned}$$

from which, employing (3.68) and (3.69), we arrive at

$$\|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\underline{\vec{\mathbf{t}}}, \underline{\vec{\boldsymbol{\sigma}}}, \underline{\vec{\mathbf{u}}}, \underline{\vec{\boldsymbol{p}}})\|_{\mathbb{H}\times\mathbf{Q}} \le \frac{2}{\alpha_{\mathbf{P}}} \sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q} \\ ((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{b(\underline{\mathbf{w}}_S - \mathbf{w}_S; \underline{\mathbf{u}}_S, \mathbf{r}_S)}{\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}} \,. \tag{3.70}$$

In turn, using the boundedness of b (cf. (3.32)) and the *a priori* estimate for

$$\|\underline{\mathbf{u}}_S\|_{0,4;\Omega_S} \;=\; \|\mathbf{T}(\underline{\mathbf{w}}_S)\|_{0,4;\Omega_S}$$

given by (3.64) (cf. Theorem 3.8), it follows from (3.70) that

$$\begin{aligned}
\|\mathbf{T}(\mathbf{w}_S) - \mathbf{T}(\underline{\mathbf{w}}_S)\|_{0,4;\Omega_S} \;&=\; \|\mathbf{u}_S - \underline{\mathbf{u}}_S\|_{0,4;\Omega_S} \;\le\; \frac{2\rho}{\alpha_{\mathbf{P}}} \, \|\mathbf{w}_S - \underline{\mathbf{w}}_S\|_{0,4;\Omega_S} \, \|\underline{\mathbf{u}}_S\|_{0,4;\Omega_S} \\
&\le\; \frac{4\rho}{\alpha_{\mathbf{P}}^2} \left\{ \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \;+\; \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \;+\; \|f_D\|_{0,\Omega_D} \right\} \|\mathbf{w}_S - \underline{\mathbf{w}}_S\|_{0,4;\Omega_S} \,,
\end{aligned}$$

which confirms the announced property on $\mathbf{T}$ (cf. (3.67)). □

The main result concerning the solvability of the fixed-point equation (3.20) is stated as follows.

Theorem 3.11. *Assume that*

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \;+\; \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \;+\; \|f_D\|_{0,\Omega_D} \;<\; \frac{\alpha_{\mathbf{P}}^2}{4\rho} \,.$$

Then, the operator $\mathbf{T}$ *has a unique fixed-point* $\mathbf{u}_S \in \mathrm{W}$*. Equivalently, problem* (3.17) *has a unique solution* $((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}) \in \mathbb{H} \times \mathbf{Q}$ *with* $\mathbf{u}_S \in \mathrm{W}$*. Moreover, there holds*

$$\|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}})\|_{\mathbb{H}\times\mathbf{Q}} \;\le\; \frac{2}{\alpha_{\mathbf{P}}} \left\{ \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \;+\; \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \;+\; \|f_D\|_{0,\Omega_D} \right\}. \tag{3.71}$$

Proof. Thanks to Lemma 3.9, we have that $\mathbf{T}$ maps W into itself. Then, bearing in mind the Lipschitz-continuity of $\mathbf{T} : \mathrm{W} \to \mathrm{W}$ (cf. (3.67)) and the assumption (3.65), a straightforward application of the classical Banach theorem yields the existence of a unique fixed-point $\mathbf{u}_S \in \mathrm{W}$ of this operator, and hence a unique solution to (3.14). Finally, it is easy to see that the *a priori* estimate is provided by (3.28) (cf. Theorem 3.1), which finishes the proof. □

4 The discrete analysis

In order to approximate the solution of (3.9), we now introduce its associated Galerkin scheme, analyze its solvability by applying a discrete version of the fixed-point approach introduced for the continuous analysis, and derive the corresponding *a priori* error estimates.

4.1 The Galerkin scheme

We first consider a set of arbitrary discrete subspaces, namely

$$\begin{gathered}\mathrm{L}^2_h(\Omega_*) \subset \mathrm{L}^2(\Omega_*) \quad * \in \{S, D\}\,, \quad \mathbf{H}_h(\Omega_D) \subset \mathbf{H}(\mathrm{div};\Omega_D)\,, \quad \mathbf{H}_h(\Omega_S) \subset \mathbf{H}(\mathrm{div}_{4/3};\Omega_S)\,,\\ \mathbf{L}^4_h(\Omega_S) \subset \mathbf{L}^4(\Omega_S)\,, \qquad \Lambda^S_h(\Sigma) \subset \mathrm{H}^{1/2}_{00}(\Sigma)\,, \quad \text{and} \quad \Lambda^D_h(\Sigma) \subset \mathrm{H}^{1/2}(\Sigma)\,,\end{gathered} \tag{4.1}$$

so that, denoting by $\boldsymbol{\tau}_{S,i}$ the i-th row of a tensor $\boldsymbol{\tau}_S$, we set

$$\begin{gathered}\mathbb{L}^2_{\mathrm{tr},h}(\Omega_S) := [\mathrm{L}^2_h(\Omega_S)]^{n\times n} \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)\,, \qquad \mathbf{H}_{h,0}(\Omega_D) := \mathbf{H}_h(\Omega_D) \cap \mathbf{H}_0(\mathrm{div};\Omega_D)\,,\\ \mathbb{H}_h(\Omega_S) := \Big\{\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S) : \quad \boldsymbol{\tau}_{S,i} \in \mathbf{H}_h(\Omega_S) \quad \forall\, i\Big\}\,, \quad \boldsymbol{\Lambda}^S_h(\Sigma) := [\Lambda^S_h(\Sigma)]^n\,,\\ \mathbb{H}_{h,0}(\Omega_S) := \mathbb{H}_h(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)\,, \quad \text{and} \quad \mathrm{L}^2_{h,0}(\Omega_D) := \mathrm{L}^2_h(\Omega_D) \cap \mathrm{L}^2_0(\Omega_D)\,.\end{gathered} \tag{4.2}$$

Then, defining the global spaces, unknowns, and test functions as follows

$$\begin{gathered}\mathbf{X}_h := \mathbb{L}^2_{\mathrm{tr},h}(\Omega_S) \times \mathbf{H}_{h,0}(\Omega_D)\,, \quad \mathbf{Y}_h := \mathbb{H}_{h,0}(\Omega_S) \times \Lambda^D_h(\Sigma)\,, \quad \mathbf{Z}_h := \mathbf{L}^4_h(\Omega_S) \times \boldsymbol{\Lambda}^S_h(\Sigma)\,,\\ \mathbb{H}_h := \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h\,, \qquad \mathbf{Q}_h := \mathrm{L}^2_{h,0}(\Omega_D) \times \mathrm{R}\,,\end{gathered} \tag{4.3}$$

$$\vec{\mathbf{t}}_h := (\mathbf{t}_{S,h}, \mathbf{u}_{D,h}) \in \mathbf{X}_h\,, \quad \vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{S,h}, \lambda_h) \in \mathbf{Y}_h\,, \quad \vec{\mathbf{u}}_h := (\mathbf{u}_{S,h}, \boldsymbol{\varphi}_h) \in \mathbf{Z}_h\,, \quad \vec{\boldsymbol{p}}_h := (p_{D,h}, l_h) \in \mathbf{Q}_h\,,$$

$$\vec{\mathbf{r}}_h := (\mathbf{r}_{S,h}, \mathbf{v}_{D,h}) \in \mathbf{X}_h\,, \quad \vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{S,h}, \xi_h) \in \mathbf{Y}_h\,, \quad \vec{\mathbf{v}}_h := (\mathbf{v}_{S,h}, \boldsymbol{\psi}_h) \in \mathbf{Z}_h\,, \quad \vec{\boldsymbol{q}}_h := (q_{D,h}, j) \in \mathbf{Q}_h\,,$$

$$\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_{S,h}, \boldsymbol{z}_{D,h}) \in \mathbf{X}_h\,, \quad \vec{\boldsymbol{\eta}}_h := (\boldsymbol{\eta}_{S,h}, \vartheta_h) \in \mathbf{Y}_h\,, \quad \vec{\boldsymbol{z}}_h := (\boldsymbol{z}_{S,h}, \boldsymbol{\phi}_h) \in \mathbf{Z}_h\,, \quad \vec{\mathbf{s}}_h := (s_{D,h}, k) \in \mathbf{Q}_h\,,$$

the Galerkin scheme associated with (3.9) reads: Find $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \vec{\boldsymbol{p}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ such that

$$\begin{array}{llllll} [a(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}_h] & +[b_1(\vec{\mathbf{r}}_h), \vec{\boldsymbol{\sigma}}_h] & & -\displaystyle\int_{\Omega_D} p_{D,h}\,\mathrm{div}(\mathbf{v}_{D,h}) & -b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) & = 0\\ [b_2(\vec{\mathbf{t}}_h), \vec{\boldsymbol{\tau}}_h] & & +[\mathbf{B}(\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h), \vec{\mathbf{u}}_h] & & & = \langle \boldsymbol{\tau}_{S,h}\mathbf{n}, g\rangle_{\Gamma_S}\\ & [\mathbf{B}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h), \vec{\mathbf{v}}_h] & -[\mathbf{C}(\vec{\mathbf{v}}_h), \vec{\mathbf{u}}_h] & +l\langle \boldsymbol{\psi}_h\cdot\mathbf{n}, 1\rangle_\Sigma & & = -\displaystyle\int_{\Omega_S} \mathbf{f}_S\cdot\mathbf{v}_{S,h}\\ & & +j\langle \boldsymbol{\varphi}_h\cdot\mathbf{n}, 1\rangle_\Sigma & & & = j\langle \mathbf{g}\cdot\mathbf{n}, 1\rangle_{\Gamma_S}\\ -\displaystyle\int_{\Omega_D} q_{D,h}\,\mathrm{div}(\mathbf{u}_{D,h}) & & & & & = -\displaystyle\int_{\Omega_D} f_D\,q_{D,h} \end{array} \tag{4.4}$$

for all $((\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{q}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$. Similarly, the ones associated with (3.14) and (3.17), which are certainly equivalent to (4.4), become, respectively: Find $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \boldsymbol{p}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ such that

$$\begin{array}{rcl} [\widetilde{\mathbf{A}}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)] + [\widetilde{\mathbf{B}}(\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{p}}_h] + b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) & = & [\mathrm{G}, (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)]\\ [\widetilde{\mathbf{B}}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \vec{\boldsymbol{q}}_h] & = & [\mathrm{F}, \vec{\boldsymbol{q}}_h] \end{array}$$

for all $((\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{q}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ and: Find $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \vec{\boldsymbol{p}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ such that

$$[\mathbf{P}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{p}}_h), (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{q}}_h)] + b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) = [\mathrm{H}, (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{q}}_h)]\,, \tag{4.5}$$

for all $((\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{q}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$.

In what follows, we adopt the discrete version of the fixed-point strategy employed in Section 3 (at the end of Subsection 3.2) to study the solvability of (4.5). For this purpose, we now let $\mathbf{T}_h : \mathbf{L}^4_h(\Omega_S) \to \mathbf{L}^4_h(\Omega_S)$ be the operator defined by

$$\mathbf{T}_h(\mathbf{w}_{S,h}) := \mathbf{u}_{S,h} \quad \forall\, \mathbf{w}_{S,h} \in \mathbf{L}^4_h(\Omega_S)\,, \tag{4.6}$$

where $\mathbf{u}_{S,h}$ is the first component of $\vec{\mathbf{u}}_h \in \mathbf{Z}_h$, which in turn is the third component of the unique solution $(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h)$ (to be proved later on) of the linearized problem arising from (4.5) after replacing $b(\mathbf{u}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h})$ by $b(\mathbf{w}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h})$, namely:

$$[\mathbf{P}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{p}}_h), (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{q}}_h)] + b(\mathbf{w}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h}) = [\mathrm{H}, (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{q}}_h)]\,, \tag{4.7}$$

for all $((\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{q}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$. Thus, we realize that solving (4.5) is equivalent to finding a fixed-point of $\mathbf{T}_h$, that is $\mathbf{u}_{S,h} \in \mathbf{L}^4_h(\Omega_S)$ such that

$$\mathbf{T}_h(\mathbf{u}_{S,h}) \,=\, \mathbf{u}_{S,h}\,. \tag{4.8}$$

4.2 Solvability analysis

Similarly to Section 3.3, in what follows we address the solvability of (4.5) by means of the corresponding analysis of (4.8).

4.2.1 Preliminaries

In addition to the finite dimensional versions of the Babuška-Brezzi theory in Banach spaces (cf. Theorem 3.1) and the Banach-Nečas-Babuška theorem, here we will also need the discrete version of Theorem 3.2, which is stated next.

Theorem 4.1. *Let H and Q be reflexive Banach spaces, and let $a : H \times H \to \mathrm{R}$, $b : H \times Q \to \mathrm{R}$ and $c : Q \times Q \to \mathrm{R}$ be given bounded bilinear forms. In addition, let $\{H_h\}_{h>0}$ and $\{Q_h\}_{h>0}$ be families of finite dimensional subspaces of H and Q, respectively, and let V_h be the kernel of $b|_{H_h\times Q_h}$ that is*

$$\mathrm{V}_h \,:=\, \Big\{ \tau_h \in H_h : \quad b(\tau_h, v_h) = 0 \quad \forall\, v_h \in Q_h \Big\}\,.$$

Assume that

i) *a and c are positive semi-definite, and that c is symmetric,*

ii) *there exists a constant $\alpha_{\mathtt{d}} > 0$ such that*

$$\sup_{\substack{\tau_h \in V_h \\ \tau_h \neq \mathbf{0}}} \frac{a(\vartheta_h, \tau_h)}{\|\tau_h\|_H} \,\geq\, \alpha_{\mathtt{d}}\, \|\vartheta_h\|_H \quad \forall\, \vartheta_h \in V_h,$$

iii) *and there exists a constant $\beta_{\mathtt{d}} > 0$ such that*

$$\sup_{\substack{\tau_h \in H_h \\ \tau_h \neq \mathbf{0}}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_H} \,\geq\, \beta_{\mathtt{d}}\, \|v_h\|_Q \quad \forall\, v_h \in Q_h\,.$$

Then, for each pair $(f, g) \in H' \times Q'$ there exists a unique $(\sigma_h, u_h) \in H_h \times Q_h$ such that

$$\begin{array}{rclcll} a(\sigma_h, \tau_h) & + & b(\tau_h, u_h) & = & f(\tau_h) & \forall\, \tau_h \in H_h\,, \\ b(\sigma_h, v_h) & - & c(u_h, v_h) & = & g(v_h) & \forall\, v_h \in Q_h\,. \end{array} \tag{4.9}$$

Moreover, there exists a constant $\widetilde{C}_{\mathtt{d}} > 0$, depending only on $\|a\|, \|c\|, \alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$, such that

$$\|(\sigma_h, u_h)\|_{H\times Q} \leq \widetilde{C}_{\mathtt{d}} \left\{ \|f\|_{H'} + \|g\|_{Q'} \right\}\,.$$

We stress here that the discrete analogue of (3.26) is not required for Theorem 4.1. Indeed, since $H_h \times Q_h$ is the space to which both the unknowns and test functions of (4.9) belong, the corresponding finite dimensional version of the Banach–Nečas–Babuška Theorem (cf. [19, Theorem 2.22]) only requires the discrete analogue of (3.29), for which the already described hypotheses of Theorem 4.1 suffice.

4.2.2 Well-definedness of the operator $\mathbf{T}_h$

We begin by providing the preliminary results that are necessary to show that (4.7) is uniquely solvable. Once this is established, we address later on the well-posedness of (4.8), and consequently of (4.5). Indeed, following a similar procedure to that of Section 3.3.2, we first note that the kernel $\widetilde{\mathrm{V}}_h$ of $\widetilde{\mathbf{B}}|_{\mathbb{H}_h\times \mathrm{Q}_h}$ reduces to

$$\widetilde{\mathrm{V}}_h := \widetilde{\mathbf{X}}_h \times \mathbf{Y}_h \times \widetilde{\mathbf{Z}}_h \,,$$

where

$$\widetilde{\mathbf{X}}_h \,:=\, \mathbb{L}^2_{\mathrm{tr},h}(\Omega_S)\times \widetilde{\mathbf{H}}_{h,0}(\Omega_D) \quad\text{and}\quad \widetilde{\mathbf{Z}}_h \,:=\, \mathbf{L}^4_h(\Omega_S)\times \widetilde{\mathbf{\Lambda}}^S_h(\Sigma)\,,$$

with

$$\begin{aligned}\widetilde{\mathbf{H}}_{h,0}(\Omega_D) \,&:=\, \left\{\mathbf{v}_D\in \mathbf{H}_{h,0}(\Omega_D):\quad \int_{\Omega_D} q_D\,\mathrm{div}(\mathbf{v}_{D,h}) = 0 \quad \forall\, q_D\in \mathrm{L}^2_{h,0}(\Omega_D)\right\}\,,\quad\text{and}\\ \widetilde{\mathbf{\Lambda}}^S_h(\Sigma) \,&:=\, \left\{\boldsymbol{\psi}_h\in \mathbf{\Lambda}^S_h(\Sigma):\quad \langle \boldsymbol{\psi}_h\cdot\mathbf{n},1\rangle_\Sigma = 0\right\}.\end{aligned} \tag{4.10}$$

Then, the kernel V_h of $\mathbf{B}|_{\widetilde{\mathrm{V}}_h}$ reduces to

$$\mathrm{V}_h \,=\, \widetilde{\mathbf{X}}_h\times\overline{\mathbf{Y}}_h\,,$$

where

$$\begin{aligned}\overline{\mathbf{Y}}_h \,:=\, \Big\{\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{S,h},\xi_h)\in \mathbf{Y}_h:\quad &\int_{\Omega_S}\mathbf{v}_{S,h}\cdot\mathbf{div}(\boldsymbol{\tau}_{S,h}) \,=\, 0 \quad\text{and}\\ &\langle \boldsymbol{\psi}_h\cdot\mathbf{n},\xi_h\rangle_\Sigma \,=\, -\langle\boldsymbol{\tau}_{S,h}\mathbf{n},\boldsymbol{\psi}_h\rangle_\Sigma \quad \forall\,\vec{\mathbf{v}}_{S,h} := (\mathbf{v}_{S,h},\boldsymbol{\psi}_h)\in\mathbf{Z}_h\Big\}.\end{aligned}$$

At this point, we notice that $\overline{\mathbf{Y}}_h \subseteq \widetilde{\mathbb{H}}_{h,0}(\Omega_S)\times \Lambda^D_h(\Sigma)$, where

$$\widetilde{\mathbb{H}}_{h,0}(\Omega_S) \,:=\, \left\{\boldsymbol{\tau}_{S,h}\in\mathbb{H}_{h,0}(\Omega_S):\quad \int_{\Omega_S}\mathbf{v}_{S,h}\cdot\mathbf{div}(\boldsymbol{\tau}_{S,h}) \,=\, 0 \quad \forall\,\mathbf{v}_{S,h}\in\mathbf{L}^4_h(\Omega_S)\right\}. \tag{4.11}$$

We now proceed similarly to [10], and introduce suitable hypotheses on the spaces defined in (4.3) to ensure the well-posedness of (4.7). We begin by noticing that, in order to have meaningful spaces $\mathbb{H}_{h,0}(\Omega_S)$ and $\mathrm{L}^2_{h,0}(\Omega_D)$, we need to be able to eliminate multiples of the identity matrix and constant polynomials from $\mathbb{H}_{h,0}(\Omega_S)$ and $\mathrm{L}^2_{h,0}(\Omega_D)$, respectively. This is certainly satisfied if we assume:

(**H.0**) $\mathrm{P}_0(\Omega_D) \subseteq \mathrm{L}^2_h(\Omega_D)$ and $\mathbb{I}\in\mathbb{H}_h(\Omega_S)$.

In addition, we consider the following further hypotheses

(**H.1**) $\mathrm{div}(\mathbf{H}_h(\Omega_D)) \subseteq \mathrm{L}^2_h(\Omega_D)$,

(**H.2**) $\mathbf{div}(\mathbb{H}_h(\Omega_S) \subseteq \mathbf{L}^4_h(\Omega_S)$,

(**H.3**) $\widetilde{\mathbb{H}}^{\mathtt{d}}_{h,0} \,:=\, \left\{\boldsymbol{\tau}^{\mathtt{d}}_{S,h}:\quad \boldsymbol{\tau}_{S,h}\in\widetilde{\mathbb{H}}_{h,0}\right\} \subseteq \mathbb{L}^2_{\mathrm{tr},h}(\Omega_S)$, (**H.4**) there holds the discrete analogue of (3.42), that is there exists a positive constant $\beta_{1,\Sigma}^{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\mathbf{v}_{D,h}\in\widetilde{\mathbf{H}}_{h,0}(\Omega_D)\\ \mathbf{v}_{D,h}\neq\mathbf{0}}} \frac{\langle \mathbf{v}_{D,h}\cdot\mathbf{n},\xi_h\rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{\mathrm{div};\Omega_D}} \,\geq\, \beta_{1,\Sigma}^{\mathtt{d}}\,\|\xi_h\|_{1/2,\Sigma} \quad \forall\,\xi\in\Lambda_h^D(\Sigma)\,, \tag{4.12}$$

(**H.5**) there holds the discrete analogue of (3.51), that is there exists a positive constant $\boldsymbol{\beta}_S^{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{S,h}\in\mathbb{H}_h(\Omega_S)\\ \boldsymbol{\tau}_{S,h}\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_{S,h}\mathbf{n},\boldsymbol{\psi}_h\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_{S,h}\cdot\mathbf{div}(\boldsymbol{\tau}_{S,h})}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}_{4/3};\Omega_S}} \,\geq\, \boldsymbol{\beta}_S^{\mathtt{d}}\Big\{\|\mathbf{v}_{S,h}\|_{0,4;\Omega_S} + \|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma}\Big\}\,, \tag{4.13}$$

for all $\vec{\mathbf{v}}_{S,h} := (\mathbf{v}_{S,h},\boldsymbol{\psi}_h)\in\mathbf{L}_h^4(\Omega_S)\times\boldsymbol{\Lambda}_h^S(\Sigma)$,

(**H.6**) there hold the discrete analogue of (3.58) and a sufficient condition for the discrete analogue of (3.59), that is there exist a positive constant $\widetilde{\beta}_D^{\mathtt{d}}$, independent of h, and $\boldsymbol{\psi}_0\in\mathbf{H}_{00}^{1/2}(\Sigma)$, such that

$$\sup_{\substack{\mathbf{v}_{D,h}\in\mathbf{H}_{h,0}(\Omega_D)\\ \mathbf{v}_{D,h}\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_D} q_{D,h}\,\mathrm{div}(\mathbf{v}_{D,h})}{\|\mathbf{v}_{D,h}\|_{\mathrm{div};\Omega_D}} \,\geq\, \widetilde{\beta}_D^{\mathtt{d}}\,\|q_{D,h}\|_{0,\Omega_D} \quad \forall\, q_{D,h}\in\mathrm{L}_{h,0}^2(\Omega_D)\,, \quad \text{and} \tag{4.14}$$

$$\boldsymbol{\psi}_0\in\boldsymbol{\Lambda}_h^S(\Sigma) \quad \forall\, h, \qquad \langle\boldsymbol{\psi}_0\cdot\mathbf{n},1\rangle_\Sigma\neq 0\,. \tag{4.15}$$

We highlight here that as a consequence of (**H.0**) we can employ the discrete version of the decomposition $\mathbb{H}(\mathbf{div}_{4/3};\Omega_S) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)\oplus\mathrm{R}\,\mathbb{I}$, namely $\mathbb{H}_h(\Omega_S) = \mathbb{H}_{h,0}(\Omega_S)\oplus\mathrm{R}\,\mathbb{I}$, thanks to which $\mathbb{H}_{h,0}(\Omega_S)$ can be used as the subspace where the unknown $\boldsymbol{\sigma}_{S,h}$ is sought. However, for the computational implementation of the Galerkin scheme (4.7), which will be addressed later on in Section 6, we will utilize a real Lagrange multiplier to impose the mean value condition on the trace of the unknown tensor lying in $\mathbb{H}_{0,h}(\Omega_S)$. In turn, it follows from (**H.1**) and (4.10) that $\widetilde{\mathbf{H}}_{h,0}(\Omega_D)$ reduces to

$$\widetilde{\mathbf{H}}_{h,0}(\Omega_D) \,:=\, \Big\{\mathbf{v}_{D,h}\in\mathbf{H}_{h,0}(\Omega_D): \quad \mathrm{div}(\mathbf{v}_{D,h})\in\mathrm{P}_0(\Omega_D)\Big\}\,.$$

Similarly, thanks to (**H.2**) and (4.11), $\widetilde{\mathbb{H}}_{h,0}(\Omega_S)$ becomes

$$\widetilde{\mathbb{H}}_{h,0}(\Omega_S) \,:=\, \Big\{\boldsymbol{\tau}_{S,h}\in\mathbb{H}_{h,0}(\Omega_S): \quad \mathbf{div}(\boldsymbol{\tau}_{S,h}) \,=\, \mathbf{0}\Big\}\,, \tag{4.16}$$

which yields the discrete analogue of (3.43) with constant $\beta_{1,S}^{\mathtt{d}}$. In fact, given $\boldsymbol{\tau}_{S,h}\in\widetilde{\mathbb{H}}_{h,0}(\Omega_S)$ such that $\boldsymbol{\tau}_{S,h}^{\mathtt{d}}\neq\mathbf{0}$, we realize, thanks to (**H.3**), that $\mathbf{r}_{S,h} := -\boldsymbol{\tau}_{S,h}^{\mathtt{d}}\in\mathbb{L}_{\mathrm{tr},h}^2(\Omega_S)$, and hence, along with the inf-sup condition from (**H.4**), we deduce the discrete version of (3.40) holds, that is, the existence of positive constants $\beta_i^{\mathtt{d}}$, $i\in\{1,2\}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{r}}_h\in\widetilde{\mathbf{X}}_h\\ \vec{\mathbf{r}}_h\neq\mathbf{0}}} \frac{[b_i(\vec{\mathbf{r}}_h),\vec{\boldsymbol{\tau}}_h]}{\|\vec{\mathbf{r}}_h\|_{\mathbf{X}}} \,\geq\, \beta_i^{\mathtt{d}}\,\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{Y}} \quad \forall\,\vec{\boldsymbol{\tau}}_h\in\overline{\mathbf{Y}}_h\,.$$

Furthermore, we remark that, similarly to the analyses in the proofs of Lemmas 3.6 and 3.7, (4.13) (cf. (**H.5**)) is a sufficient condition for the discrete version of (3.49), whereas (4.14) and (4.15) (cf.

(**H.6**)) are equivalent to the discrete version of (3.57). We denote the constants involved in these discrete inf-sup conditions by $\boldsymbol{\beta}_{\mathtt{d}}$ and $\widetilde{\beta}_{\mathtt{d}}$, respectively.

Thus, having $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ satisfied for the present discrete scheme the hypotheses of Theorem 3.1 with constants $\widetilde{\alpha}_{\mathtt{d}}$ and $\widetilde{\beta}_{\mathtt{d}}$, we conclude, similarly to the continuous case, the existence of a positive constant $\alpha_{\mathbf{P},\mathtt{d}}$, depending on $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, and $\|\widetilde{\mathbf{A}}\|$, and hence independent of h, such that

$$\sup_{\substack{((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\\ ((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\eta}}_h,\vec{\boldsymbol{z}}_h,\vec{\mathbf{s}}_h),(\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h,\vec{\boldsymbol{q}}_h)]}{\|((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\|_{\mathbb{H}\times\mathbf{Q}}} \geq \alpha_{\mathbf{P},\mathtt{d}}\,\|(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\eta}}_h,\vec{\boldsymbol{z}}_h,\vec{\mathbf{s}}_h)\|_{\mathbb{H}\times\mathbf{Q}}\,, \tag{4.17}$$

for all $((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$, and thus, for each $\mathbf{w}_{S,h}\in\mathbf{L}^4_h(\Omega_S)$ such that $\|\mathbf{w}_{S,h}\|_{0,4;\Omega_S}\leq\dfrac{\alpha_{\mathbf{P},\mathtt{d}}}{2\rho}$, there holds

$$\begin{aligned}&\sup_{\substack{((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\\ ((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\eta}}_h,\vec{\boldsymbol{z}}_h,\vec{\mathbf{s}}_h),(\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h,\vec{\boldsymbol{q}}_h)]\,+\,b(\mathbf{w}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h})}{\|(\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h,\vec{\boldsymbol{q}}_h)\|_{\mathbb{H}\times\mathbf{Q}}}\\ &\quad\geq\frac{\alpha_{\mathbf{P},\mathtt{d}}}{2}\,\|(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\eta}}_h,\vec{\boldsymbol{z}}_h,\vec{\mathbf{s}}_h)\|_{\mathbb{H}\times\mathbf{Q}}\qquad\forall\,((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\,.\end{aligned} \tag{4.18}$$

According to the above, we are now in a position to present the discrete analogues of Theorem 3.8, Lemma 3.9, and Theorem 3.11, whose proofs follow almost verbatim to those for the continuous case, and hence only some remarks are provided. We begin with the well-posedness of (4.7), which is the same as establishing that $\mathbf{T}_h$ is well-defined.

Lemma 4.2. *For each $\mathbf{w}_{S,h}\in\mathbf{L}^4_h(\Omega_S)$ such that $\|\mathbf{w}_{S,h}\|\leq\dfrac{\alpha_{\mathbf{P},\mathtt{d}}}{2\rho}$, there exists a unique solution $((\vec{\mathbf{t}}_h,\vec{\boldsymbol{\sigma}}_h,\vec{\mathbf{u}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}\times\mathbf{Q}$ to (4.7), and hence we can define $\mathbf{T}_h(\mathbf{w}_{S,h})=\mathbf{u}_{S,h}\in\mathbf{L}^4_h(\Omega_S)$. Moreover, there holds*

$$\begin{aligned}\|\mathbf{T}_h(\mathbf{w}_{S,h})\|_{0,4;\Omega_S}\,&=\,\|\mathbf{u}_{S,h}\|_{0,4;\Omega_S}\leq\|(\vec{\mathbf{t}}_h,\vec{\boldsymbol{\sigma}}_h,\vec{\mathbf{u}}_h,\vec{\boldsymbol{p}}_h)\|_{\mathbb{H}\times\mathbf{Q}}\\ &\leq\,\frac{2}{\alpha_{\mathbf{P},\mathtt{d}}}\Big\{\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S}\,+\,\|\mathbf{f}_S\|_{0,4/3;\Omega_S}\,+\,\|f_D\|_{0,\Omega_D}\Big\}\,.\end{aligned} \tag{4.19}$$

Proof. Given $\mathbf{w}_{S,h}$ as indicated, and bearing in mind (4.18), it suffices to apply the discrete version of the Banach–Nečas–Babuška Theorem (cf. [19, Theorem 2.22]) and its corresponding *a priori* error estimate. □

We continue with the discrete analogue of Lemma 3.9, that is the result ensuring that $\mathbf{T}_h$ maps a ball of $\mathbf{L}^4_h(\Omega_S)$ into itself.

Lemma 4.3. *Let W_h be the ball*

$$\mathrm{W}_h\,:=\,\left\{\mathbf{w}_{S,h}\in\mathbf{L}^4_h(\Omega_S):\quad\|\mathbf{w}_{S,h}\|_{0,4;\Omega_S}\,\leq\,\frac{\alpha_{\mathbf{P},\mathtt{d}}}{2\rho}\right\}\,,$$

and assume that

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S}\,+\,\|\mathbf{f}_S\|_{0,4/3;\Omega_S}+\|f_D\|_{0,\Omega_D}\,\leq\,\frac{\alpha^2_{\mathbf{P},\mathtt{d}}}{4\rho}\,. \tag{4.20}$$

Then, there holds $\mathbf{T}_h(\mathrm{W}_h)\subseteq\mathrm{W}_h$.

Proof. It follows straightforwardly from (4.19) and (4.20). □

The discrete analogue of Theorem 3.11, that is the unique solvability of (4.8), and hence, equivalently that of (4.5), is stated next.

Theorem 4.4. *Assume that*

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} \leq \frac{\alpha_{\mathbf{P},\mathtt{d}}^2}{4\rho}\,.$$

Then, the operator $\mathbf{T}_h$ *has a unique fixed-point* $\mathbf{u}_{S,h} \in \mathrm{W}_h$. *Equivalently, problem* (4.5) *has a unique solution* $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \vec{\boldsymbol{p}}_h) \in \mathbb{H}_h \times \mathbf{Q}$ *with* $\mathbf{u}_{S,h} \in \mathrm{W}_h$. *Moreover, there holds*

$$\|(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{p}}_h)\|_{\mathbb{H}\times\mathbf{Q}} \leq \frac{2}{\alpha_{\mathbf{P},\mathtt{d}}} \Big\{ \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} \Big\}\,. \tag{4.21}$$

Proof. Similarly to the proof of Theorem 3.11, it reduces to employ (3.32), (4.7), (4.18) and (4.19) to prove that $\mathbf{T}_h : \mathrm{W}_h \to \mathrm{W}_h$ is a contraction, and then apply the Banach fixed-point theorem. □

We end this section by providing sufficient conditions for (4.12) and the particular case arising from (4.13) when $\mathbf{v}_{S,h} = \mathbf{0}$, that is for the existence of positive constants $\beta_{1,\Sigma}^{\mathtt{d}}$ and $\boldsymbol{\beta}_{S,2}^{\mathtt{d}}$, such that

$$\sup_{\substack{\mathbf{v}_{D,h} \in \widetilde{\mathbf{H}}_{h,0}(\Omega_D) \\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{\langle \mathbf{v}_{D,h} \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{\mathrm{div};\Omega_D}} \geq \beta_{1,\Sigma}^{\mathtt{d}} \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \in \Lambda_h^D(\Sigma)\,, \quad \text{and} \tag{4.22}$$

$$\sup_{\substack{\boldsymbol{\tau}_{S,h} \in \widetilde{\mathbb{H}}_h(\Omega_S) \\ \boldsymbol{\tau}_{S,h} \neq \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_{S,h}\mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}_{4/3};\Omega_S}} \geq \boldsymbol{\beta}_{S,2}^{\mathtt{d}} \|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma} \qquad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^S(\Sigma)\,, \tag{4.23}$$

where $\widetilde{\mathbb{H}}_h(\Omega_S) := \Big\{ \boldsymbol{\tau}_{S,h} \in \mathbb{H}_h(\Omega_S) : \quad \mathbf{div}(\boldsymbol{\tau}_{S,h}) = \mathbf{0} \Big\}$. In this regard, we first notice that the above inequalities, which deal with how the normal components of elements of $\widetilde{\mathbf{H}}_{h,0}(\Omega_D)$ and $\widetilde{\mathbb{H}}_h(\Omega_S)$ are tested against $\Lambda_h^D(\Sigma)$ and $\boldsymbol{\Lambda}_h^S(\Sigma)$, respectively, are shown below to be related to the eventual existence of a stable discrete lifting of the normal traces on Σ. Indeed, in order to establish (4.22) and (4.23), it suffices to prove that for each $* \in \{D, S\}$ there exists a positive constant $\beta_{*,\Sigma}^{\mathtt{d}}$, such that

$$\sup_{\substack{\mathbf{v}_h \in \widetilde{\mathbf{H}}_h(\Omega_*) \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{\langle \mathbf{v}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_h\|_{\mathrm{div};\Omega_*}} \geq \beta_{*,\Sigma}^{\mathtt{d}} \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi \in \Lambda_h^*(\Sigma)\,, \tag{4.24}$$

where

$$\widetilde{\mathbf{H}}_h(\Omega_D) := \Big\{ \mathbf{v}_h \in \mathbf{H}_{h,0}(\Omega_D) : \quad \mathrm{div}(\mathbf{v}_h) \in \mathrm{P}_0(\Omega_D) \Big\}, \quad \text{and}$$
$$\widetilde{\mathbf{H}}_h(\Omega_S) := \Big\{ \mathbf{v}_h \in \mathbf{H}_h(\Omega_S) : \quad \mathrm{div}(\mathbf{v}_h) = 0 \Big\}.$$

Next, for each $* \in \{D, S\}$ we define

$$\Phi_h^*(\Sigma) := \Big\{ \mathbf{v}_h \cdot \mathbf{n}|_\Sigma : \quad \mathbf{v}_h \in \widetilde{\mathbf{H}}_h(\Omega_*) \Big\}\,, \tag{4.25}$$

and assume that the linear operator $\mathbf{v}_h \to \mathbf{v}_h \cdot \mathbf{n}$ from $\widetilde{\mathbf{H}}_h(\Omega_*)$ to $\Phi_h^*(\Sigma)$ has a uniformly bounded right inverse, which means that there exists a linear operator $\mathcal{L}_h^* : \Phi_h^*(\Sigma) \to \widetilde{\mathbf{H}}_h(\Omega_*)$ and a constant $c_* > 0$, independent of h, such that

$$\begin{aligned} \|\mathcal{L}_h^*(\phi_h)\|_{\mathrm{div};\Omega_*} &\leq c_* \|\phi_h\|_{-1/2,\Sigma}\,, \quad \text{and} \\ \mathcal{L}_h^*(\phi_h) \cdot \mathbf{n} &= \phi_h \quad \text{on} \quad \Sigma \quad \forall\, \phi_h \in \Phi_h^*(\Sigma)\,. \end{aligned} \tag{4.26}$$

Such a uniformly bounded right inverse $\mathcal{L}_h^*$ of the normal trace will henceforth be referred to as a stable discrete lifting to Ω_*. Note that by [18], existence of $\mathcal{L}_h^*$ satisfying (4.26) is equivalent to the existence of a Scott–Zhang type linear and uniformly bounded operator $\pi_h^* : \mathbf{H}(\mathrm{div};\Omega_*) \to \widetilde{\mathbf{H}}_h(\Omega_*)$, such that

$$\pi_h^*(\mathbf{v}_h) = \mathbf{v}_h \quad \forall\, \mathbf{v}_h \in \widetilde{\mathbf{H}}_h(\Omega_*)\,, \quad \text{and} \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma \Longrightarrow (\pi_h^*(\mathbf{v}))\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma\,.$$

The following lemma, taken from [26, Lemma 4.2], reduces (4.24) to the inherited interaction between the elements of $\Phi_h^*(\Sigma)$ and $\Lambda_h^*(\Sigma)$.

Lemma 4.5. *Assume that there exists a stable discrete lifting to Ω_*. Then (4.24) is equivalent to the existence of a positive constant $\beta_*^{\mathtt{d}}$, independent of h, such that*

$$\sup_{\substack{\phi_h\in\Phi_h^*(\Sigma)\\ \phi_h\neq\mathbf{0}}} \frac{\langle\phi_h,\xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \,\geq\, \beta_*^{\mathtt{d}}\,\|\xi_h\|_{1/2,\Sigma} \qquad \forall\,\xi_h\in\Lambda_h^*(\Sigma)\,. \tag{4.27}$$

We have thus proved that the existence of stable discrete liftings to Ω_S and Ω_D together with the inf-sup condition (4.27) constitute sufficient conditions for (4.24) to hold. In this respect, we find it important to emphasize that (4.27) deals exclusively with spaces of functions defined on Σ.

4.3 *A priori* error analysis

In this section we consider finite element subspaces satisfying the assumptions specified in Section 4.2.2, and derive the Céa estimate for the Galerkin error

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,=\, \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} + \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{Y}} + \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Z}} + \|\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}_h\|_{\mathbf{Q}}\,,$$

where $\underline{\mathbf{t}} := (\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}},\vec{\boldsymbol{p}}) \in \mathbb{H}\times\mathbf{Q}$ and $\underline{\mathbf{t}}_h := (\vec{\mathbf{t}}_h,\vec{\boldsymbol{\sigma}}_h,\vec{\mathbf{u}}_h,\vec{\boldsymbol{p}}_h) \in \mathbb{H}_h\times\mathbf{Q}_h$ are the unique solutions of (3.17) and (4.5) respectively, with $\mathbf{u}_S\in\mathrm{W}$ and $\mathbf{u}_{S,h}\in\mathrm{W}_h$. In what follows, given a subspace Z_h of an arbitrary Banach space $\big(Z,\|\cdot\|_Z\big)$, we set

$$\mathrm{dist}\big(z,Z_h\big) \,:=\, \inf_{z_h\in Z_h}\|z-z_h\|_Z \qquad \forall\,z\in Z\,.$$

We begin by observing from (3.16) that for each $\underline{\mathbf{r}}_h := ((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h) \in \mathbb{H}_h\times\mathbf{Q}_h$ there holds

$$[\mathbf{P}(\underline{\mathbf{t}}),\underline{\mathbf{r}}_h] + b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h}) = [\mathrm{H},\underline{\mathbf{r}}_h]\,,$$

which combined with (4.5), yields for each $\underline{\mathbf{r}}_h\in\mathbb{H}_h\times\mathbf{Q}_h$

$$[\mathbf{P}(\underline{\mathbf{t}}-\underline{\mathbf{t}}_h),\underline{\mathbf{r}}_h] = b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) - b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h})\,. \tag{4.28}$$

Now, the triangle inequality gives for each $\underline{\boldsymbol{\zeta}}_h\in\mathbb{H}_h\times\mathbf{Q}_h$

$$\|\underline{\mathbf{t}}-\underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,\leq\, \|\underline{\mathbf{t}}-\underline{\boldsymbol{\zeta}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,+\, \|\underline{\boldsymbol{\zeta}}_h-\underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}}\,, \tag{4.29}$$

and then, applying (4.17) to $\underline{\boldsymbol{\zeta}}_h-\underline{\mathbf{t}}_h$, subtracting and adding $\underline{\mathbf{t}}$ in the first component of $\mathbf{P}$, using the boundedness of $\mathbf{P}$ with constant $\|\mathbf{P}\|$, and employing the identity (4.28), we find that

$$\begin{aligned}\alpha_{\mathbf{P},\mathtt{d}}\,\|\underline{\boldsymbol{\zeta}}_h-\underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,&\leq\, \sup_{\substack{\underline{\mathbf{r}}_h\in\mathbb{H}_h\times\mathbf{Q}_h\\ \underline{\mathbf{r}}_h\neq\mathbf{0}}}\frac{[\mathbf{P}(\underline{\boldsymbol{\zeta}}_h-\underline{\mathbf{t}}_h),\underline{\mathbf{r}}_h]}{\|\underline{\mathbf{r}}_h\|_{\mathbb{H}\times\mathbf{Q}}}\\ &\leq\, \|\mathbf{P}\|\,\|\underline{\mathbf{t}}-\underline{\boldsymbol{\zeta}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,+\, \sup_{\substack{\underline{\mathbf{r}}_h\in\mathbb{H}_h\times\mathbf{Q}_h\\ \underline{\mathbf{r}}_h\neq\mathbf{0}}}\frac{[\mathbf{P}(\underline{\mathbf{t}}-\underline{\mathbf{t}}_h),\underline{\mathbf{r}}_h]}{\|\underline{\mathbf{r}}_h\|_{\mathbb{H}\times\mathbf{Q}}}\\ &\leq\, \|\mathbf{P}\|\,\|\underline{\mathbf{t}}-\underline{\boldsymbol{\zeta}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,+\, \sup_{\substack{\underline{\mathbf{r}}_h\in\mathbb{H}_h\times\mathbf{Q}_h\\ \underline{\mathbf{r}}_h\neq\mathbf{0}}}\frac{b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) - b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h})}{\|\underline{\mathbf{r}}_h\|_{\mathbb{H}}\times\mathbf{Q}}\,.\end{aligned} \tag{4.30}$$

In this way, replacing the bound for $\|\underline{\boldsymbol{\zeta}}_h - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}}$ that arises from (4.30) back into (4.29), and taking infimum with respect to $\underline{\mathbf{r}}_h \in \mathbb{H}_h \times \mathbf{Q}_h$ we deduce that

$$\begin{aligned} \|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,&\le\, \left(1 + \frac{\|\mathbf{P}\|}{\alpha_{\mathbf{P},\mathtt{d}}}\right) \operatorname{dist}(\underline{\mathbf{t}}, \mathbb{H}_h \times \mathbf{Q}_h) \\ &+ \frac{1}{\alpha_{\mathbf{P},\mathtt{d}}} \sup_{\substack{\underline{\mathbf{r}}_h\in\mathbb{H}_h\times\mathbf{Q}_h \\ \underline{\mathbf{r}}_h\neq\mathbf{0}}} \frac{b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) - b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h})}{\|\underline{\mathbf{r}}_h\|_{\mathbb{H}\times\mathbf{Q}}}\,, \end{aligned} \tag{4.31}$$

which basically constitutes the Strang-type estimate for the joint setting formed by (3.17) and (4.5). Next, in order to estimate the consistency term given by the supremum in (4.31), we subtract and add $\mathbf{u}_S$ in the second component of $b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h})$, and then invoke the boundedness property of b (3.32), and the *a priori* estimates (3.71) and (4.21) for $\|\mathbf{u}_S\|_{0,4;\Omega_S}$ and $\|\mathbf{u}_{S,h}\|_{0,4;\Omega_S}$, respectively, thanks to all of which we obtain

$$\begin{aligned} b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) - b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h}) \,&=\, b(\mathbf{u}_{S,h};\mathbf{u}_{S,h}-\mathbf{u}_S,\mathbf{r}_{S,h}) \,+\, b(\mathbf{u}_{S,h}-\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h}) \\ \le\, \frac{4\rho}{\overline{\alpha}_{\mathbf{P}}} &\left\{\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \,+\, \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \,+\, \|f_D\|_{0,\Omega_D}\right\} \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,4;\Omega_S}\, \|\mathbf{r}_{S,h}\|_{0,\Omega_S}\,, \end{aligned} \tag{4.32}$$

where $\overline{\alpha}_{\mathbf{P}} \,:=\, \min\big\{\alpha_{\mathbf{P}}, \alpha_{\mathbf{P},\mathtt{d}}\big\}$. Hence, replacing (4.31) in (4.32), we conclude that

$$\begin{aligned} \|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,&\le\, \left(1 + \frac{\|\mathbf{P}\|}{\alpha_{\mathbf{P},\mathtt{d}}}\right) \operatorname{dist}(\underline{\mathbf{t}}, \mathbb{H}_h \times \mathbf{Q}_h) \\ &+ \frac{4\rho}{\overline{\alpha}_{\mathbf{P}}^2} \left\{\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \,+\, \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \,+\, \|f_D\|_{0,\Omega_D}\right\} \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,4;\Omega_S}\,. \end{aligned} \tag{4.33}$$

We are then in position to state the following result.

Theorem 4.6. *Assume that for some $\delta \in (0,1)$ there holds*

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \,+\, \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \,+\, \|f_D\|_{0,\Omega_D} \,\le\, \frac{\delta\,\alpha_{\mathbf{P},\mathtt{d}}^2}{4\rho}\,. \tag{4.34}$$

Then, there exists a positive constant $C_{\mathtt{d}}$, depending only on $\|\mathbf{P}\|$, $\alpha_{\mathbf{P},\mathtt{d}}$, and δ, and hence independent of h, such that

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,\le\, C_{\mathtt{d}} \operatorname{dist}(\underline{\mathbf{t}}, \mathbb{H}_h \times \mathbf{Q}_h)\,. \tag{4.35}$$

Proof. It suffices to use (4.34) in (4.33), which yields (4.35) with $C_{\mathtt{d}} \,:=\, (1-\delta)^{-1}\big(1 + \|\mathbf{P}\|/\alpha_{\mathbf{P},\mathtt{d}}\big)$. □

In particular, taking $\delta = 1/2$, we get $C_{\mathtt{d}} := 2\big(1 + \|\mathbf{P}\|/\alpha_{\mathbf{P},\mathtt{d}}\big)$ in the proof of Lemma 4.6, and (4.34) becomes

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \,+\, \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \,+\, \|f_D\|_{0,\Omega_D} \,\le\, \frac{\alpha_{\mathbf{P},\mathtt{d}}^2}{8\rho}\,. \tag{4.36}$$

We end this section by remarking that (2.4) and (3.7) suggest the following postprocessed approximation for the pressure p_S

$$p_{S,h} \,:=\, -\frac{1}{n}\operatorname{tr}\big(\boldsymbol{\sigma}_{S,h} \,+\, (\mathbf{u}_{S,h}\otimes\mathbf{u}_{S,h})\big) \,-\, l_h \quad \text{in} \quad \Omega_S\,, \tag{4.37}$$

where

$$l_h \,:=\, -\frac{1}{n\,|\Omega_S|}\int_{\Omega_S} \operatorname{tr}(\boldsymbol{\sigma}_{S,h})\,.$$

Then, applying the Cauchy–Schwarz inequality, performing some algebraic manipulations, and employing the *a priori* bounds for $\|\mathbf{u}_S\|_{0,4;\Omega_S}$ and $\|\mathbf{u}_{S,h}\|_{0,4;\Omega_S}$, we deduce the existence of a positive constant C, depending on data, but independent of h, such that

$$\|p - p_h\|_{0,\Omega_S} \le C\Big\{\|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{0,\Omega} + \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,4;\Omega_S}\Big\}. \tag{4.38}$$

Thus, combining (4.35) and (4.38), we conclude the existence of a positive constant $\widetilde{C}_{\mathtt{d}}$, independent of h, such that

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} + \|p - p_h\|_{0,\Omega_S} \le \widetilde{C}_{\mathtt{d}}\,\mathrm{dist}(\underline{\mathbf{t}}, \mathbb{H}_h \times \mathbf{Q}_h). \tag{4.39}$$

5 Specific finite element subspaces

In what follows we proceed similarly to [26] (see also [8]) and specify discrete spaces satisfying the hypotheses (**H.0**) up to (**H.6**) in 2D and 3D, thus ensuring the well-posedness of the Galerkin scheme (4.5). Their approximation properties and associated rates of convergence are also established.

5.1 Preliminaries

We begin by letting $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ be respective triangulations of the domains Ω_S and Ω_D, which are formed by shape-regular triangles (in R^2) or tetrahedra (in R^3) of diameter h_T, and assume that they match in Σ so that $\mathcal{T}_h^S \cup \mathcal{T}_h^D$ is a triangulation of $\Omega_S \cup \Sigma \cup \Omega_D$. We also let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^S$ (or $\mathcal{T}_h^D$). Then, given $T \in \mathcal{T}_h^S \cup \mathcal{T}_h{}^D$, we let $\mathrm{P}_0(T)$ be the space of polynomials of degree $= 0$ defined on T, whose vector and tensor versions are denoted by $\mathbf{P}_0(T) := [\mathrm{P}_0(T)]^n$ and $\mathbb{P}_0(T) := [\mathrm{P}_0(T)]^{n\times n}$, respectively. Next, we define the corresponding local Raviart-Thomas spaces of order 0 as

$$\mathbf{RT}_0(T) := \mathbf{P}_0(T) \oplus \mathrm{P}_0(T)\,\mathbf{x}$$

and its associated tensor counterpart $\mathbb{RT}_0(T)$, where $\mathbf{x}$ is a generic vector in $\mathbf{R} := \mathrm{R}^n$. In turn, given $* \in \{S, D\}$, we let $\mathrm{P}_0(\mathcal{T}_h^*)$, $\mathbf{P}_0(\mathcal{T}_h^*)$ and $\mathbf{RT}_0(\mathcal{T}_h{}^*)$ be the global versions of $\mathrm{P}_0(T)$, $\mathbf{P}_0(T)$, $\mathbb{P}_0(T)$, $\mathbf{RT}_0(T)$ and $\mathbb{RT}_0(T)$, respectively, that is

$$\begin{aligned}
\mathrm{P}_0(\mathcal{T}_h{}^*) &:= \big\{\mathbf{v}_h \in \mathrm{L}^2(\Omega_*) : \quad \mathbf{v}_h|_T \in \mathrm{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\big\}\,,\\
\mathbf{P}_0(\mathcal{T}_h{}^*) &:= \big\{\boldsymbol{\tau}_h \in \mathbf{L}^2(\Omega_*) : \quad \boldsymbol{\tau}_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\big\}\,,\\
\mathbb{P}_0(\mathcal{T}_h{}^*) &:= \big\{\boldsymbol{\tau}_h \in \mathbb{L}^2(\Omega_*) : \quad \boldsymbol{\tau}_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\big\}\,,\\
\mathbf{RT}_0(\mathcal{T}_h{}^*) &:= \{\boldsymbol{\tau}_h \in \mathbf{H}(\mathbf{div};\Omega_*) : \quad \boldsymbol{\tau}_h|_T \in \mathbf{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\}\,,\\
\mathbb{RT}_0(\mathcal{T}_h{}^*) &:= \{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega_*) : \quad \boldsymbol{\tau}_h|_T \in \mathbb{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\}\,.
\end{aligned}$$

Then, we introduce the corresponding discrete subspaces in (4.1) as

$$\mathrm{L}^2_h(\Omega_*) := \mathrm{P}_0(\mathcal{T}_h{}^*)\,, \quad \mathbf{H}_h(\Omega_*) := \mathbf{RT}_0(\mathcal{T}_h{}^*)\,, \quad \text{and} \quad \mathbf{L}^4_h(\Omega_S) := \mathbf{L}^4(\Omega_S) \cap \mathbf{P}_0(\mathcal{T}_h^S)\,, \tag{5.1}$$

so that the associated global spaces $\mathbb{L}^2_{\mathrm{tr},h}(\Omega_S)$, $\mathbf{H}_{h,0}(\Omega_D)$, $\mathbb{H}_h(\Omega_S)$, $\mathbb{H}_{h,0}(\Omega_S)$, and $\mathrm{L}^2_{h,0}(\Omega_D)$, are defined according to (4.2). The interface spaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$ will be specified later on by separating the 2D and 3D cases.

Next, for the verification of the hypotheses introduced in Section 4.2.2, we first realize that (**H.0**), (**H.1**), and (**H.2**) follow straightforwardly from the definitions in (5.1). In turn, regarding (**H.3**), we now recall that the divergence free tensors of $\mathbb{RT}_0(\mathcal{T}_h)$ are contained in $\mathbb{P}_0(\mathcal{T}_h)$ (cf. [22, Lemma

3.6]), so that, invoking (4.16), we deduce that $\widetilde{\mathbb{H}}_{h,0}(\Omega_S) \subseteq \mathbb{P}_0(\mathcal{T}_h)$. In this way, noting that certainly $\mathrm{tr}(\boldsymbol{\tau}_h^{\mathtt{d}}) = 0$ for all $\boldsymbol{\tau}_h \in \widetilde{\mathbb{H}}_{h,0}(\Omega_S)$, we find that $\widetilde{\mathbb{H}}^{\mathtt{d}}_{h,0}(\Omega_S) \subseteq \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_0(\mathcal{T}_h) = \mathbb{L}^2_{\mathrm{tr},h}(\Omega)$, thus confirming the occurrence of (**H.3**).

We now turn partially to (**H.5**) and (**H.6**) and establish first an inequality aiming to accomplish (4.13), and then the discrete inf-sup condition (4.14). More precisely, we have the following results taken from [14] and [22], respectively.

Lemma 5.1. *There exists a positive constant $\boldsymbol{\beta}^{\mathtt{d}}_{S,1}$, independent of h, such that*

$$\sup_{\substack{\boldsymbol{\tau}_{S,h} \in \mathbb{H}_{h,0}(\Omega_S) \\ \boldsymbol{\tau}_{S,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_S} \mathbf{v}_{S,h} \cdot \mathbf{div}(\boldsymbol{\tau}_{S,h})}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}_{4/3};\Omega_S}} \geq \boldsymbol{\beta}^{\mathtt{d}}_{S,1} \, \|\mathbf{v}_h\|_{0,4;\Omega_S} \qquad \forall\, \mathbf{v}_{S,h} \in \mathbf{L}^4_h(\Omega_S)\,. \tag{5.2}$$

Proof. See [14, Lemma 6.1]. We just stress that it is mainly based on the introduction of a suitable auxiliary boundary value problem, and the utilization of the elliptic regularity result provided by [20, Corollary 1]. □

Lemma 5.2. *There exists a positive constant $\widetilde{\beta}^{\mathtt{d}}_D$, independent of h, such that*

$$\sup_{\substack{\mathbf{v}_{D,h} \in \mathbf{H}_{h,0}(\Omega_D) \\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_D} q_{D,h} \, \mathrm{div}(\mathbf{v}_{D,h})}{\|\mathbf{v}_{D,h}\|_{\mathrm{div},\Omega_D}} \geq \widetilde{\beta}_D \, \|q_{D,h}\|_{0,\Omega_D} \qquad \forall\, q_{D,h} \in \mathrm{L}^2_{h,0}(\Omega_D)\,. \tag{5.3}$$

Proof. We refer to [22, Chapter IV, Section 4.2] for full details. It basically reduces to the verification of the hypotheses of Fortin's lemma (cf. [22, Lemma 2.6]), which makes use of an elliptic regularity result in convex domains, and the main properties of the Raviart-Thomas interpolation operator. □

We complete the accomplishment of the hypothesis (**H.6**) by remarking that the existence of $\boldsymbol{\psi}_{0,\mathtt{d}} \in \mathbf{H}^{1/2}_{00}(\Sigma)$ satisfying (4.15) is guaranteed at the beginning of [26, Section 5.3]. In particular, this holds if the sequence of subspaces $\{\boldsymbol{\Lambda}^S_h(\Sigma)\}_{h>0}$ is nested, which is confirmed below when defining $\Lambda^S_h(\Sigma)$. Thus, $\boldsymbol{\psi}_{0,\mathtt{d}}$ can be constructed as indicated in the proof of Lemma 3.7. A similar procedure applies to the 3D case.

5.2 The spaces $\Lambda^S_h(\Sigma)$ and $\Lambda^D_h(\Sigma)$ and the remaining hypotheses in 2D

We now introduce the particular subspaces $\Lambda^S_h(\Sigma)$ and $\Lambda^D_h(\Sigma)$ in 2D by following the simplest approach suggested in [26]. Indeed, we first assume, without loss of generality, that the number of edges of Σ_h is even, and let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. Since Σ_h is inherited from the interior triangulations, it is automatically of bounded variation, which means that ratio of lengths of adjacent edges is bounded, and, therefore, so is Σ_{2h}. Now, if the number of edges of Σ_h were odd, we simply reduce it to the even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this reduced partition. In this way, denoting by x_0 and x_N the extreme points of Σ, we set

$$\begin{aligned} \Lambda^S_h(\Sigma) &:= \Big\{ \xi_h \in C(\Sigma) : \quad \xi|_e \in \mathrm{P}_1(e) \quad \forall\, \text{edge } e \in \Sigma_{2h}, \quad \xi_h(x_0) = \xi_h(x_N) = 0 \Big\}, \\ \Lambda^D_h(\Sigma) &:= \Big\{ \xi_h \in C(\Sigma) : \quad \xi_h|_e \in \mathrm{P}_1(e) \quad \forall \text{ edge } e \in \Sigma_{2h} \Big\}. \end{aligned} \tag{5.4}$$

We now aim to establish the discrete inf-sup conditions (4.22) (or (4.12)) and (4.23) by applying Lemma 4.5. To this end, we suppose from now on that $\{\mathcal{T}_h^S\}_{h>0}$ and $\{\mathcal{T}_h^D\}_{h>0}$ are quasi-uniform in a neighborhood of Σ. More precisely, we assume that there is an open neighborhood of Σ, say Ω_Σ, with Lipschitz-continuous boundary $\partial\Omega_\Sigma$, such that the elements intersecting that region are roughly of the same size. In other words, defining

$$\mathcal{T}_{h,\Sigma} := \Big\{ T \in \mathcal{T}_h^S \cup \mathcal{T}_h^D : \quad T \cap \Omega_\Sigma \neq \emptyset \Big\}, \tag{5.5}$$

there exists a positive c, independent of h, such that

$$\max_{T\in\mathcal{T}_{h,\Sigma}} h_T \,\leq\, c \min_{T\in\mathcal{T}_{h,\Sigma}} h_T\,. \tag{5.6}$$

Under this quasi-uniformity condition, it was proved in [26, Lemma 5.1] that there exist stable discrete lifting operators $\mathcal{L}_h^*$ to Ω_*, $* \in \{S, D\}$, satisfying (4.26). Moreover, as a consequence of this result, it is easy to see that both $\Phi_h^S(\Sigma)$ and $\Phi_h^D(\Sigma)$ (cf. (4.25)) coincide with

$$\Phi_h(\Sigma) := \Big\{ \phi_h \in \mathrm{L}^2(\Sigma) : \quad \phi_h|_e \in \mathrm{P}_0(e) \quad \forall \text{ edge } e \in \Sigma_h \Big\}\,. \tag{5.7}$$

Hence, a straightforward application of Lemma 4.5 implies that, in order to conclude (4.24), which in turn yields (4.22) and (4.23), it suffices to show (4.27). In fact, this latter result, taken from [26], is stated as follows.

Lemma 5.3. *There exists a positive constant $\beta_\Sigma^{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\phi_h\in\Phi_h(\Sigma)\\ \phi_h\neq 0}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \,\geq\, \beta_\Sigma^{\mathtt{d}}\, \|\xi_h\|_{1/2,\Sigma} \quad \forall\, \xi_h \in \Lambda_h^S(\Sigma) \cup \Lambda_h^D(\Sigma)\,.$$

Proof. See [26, Lemma 5.2] for details. □

As previously remarked, Lemma 5.3 yields, in particular, the verification of (4.22), which is the same as (4.12), and thus (**H.4**) is accomplished. Similarly, having as well (4.23), a suitable combination of this inequality with the discrete inf-sup condition provided by Lemma 5.1 leads to (**H.5**), that is to (4.13), with a constant $\boldsymbol{\beta}_S^{\mathtt{d}}$ depending only on $\boldsymbol{\beta}_{S,1}^{\mathtt{d}}$ (cf. Lemma 5.1) and $\boldsymbol{\beta}_{S,2}^{\mathtt{d}}$ (cf. (4.23)).

5.3 The spaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$ and the remaining hypotheses in 3D

In order to set the particular subspaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$ in the 3D case, we need to introduce an independent triangulation $\Sigma_{\widehat{h}}$ of Σ, made up of triangles K of diameter $\widehat{h}_K$, so that we set the meshsize $\widehat{h} := \max\{\widehat{h}_K : \quad K \in \Sigma_{\widehat{h}}\}$. Then, denoting by $\partial\Sigma$ the polygonal boundary of Σ, we define

$$\Lambda_{\widehat{h}}^S(\Sigma) := \Big\{ \xi_{\widehat{h}} \in C(\Sigma) : \quad \xi_{\widehat{h}}|_K \in \mathrm{P}_1(K) \quad \forall\, K \in \Sigma_{\widehat{h}}, \quad \xi_{\widehat{h}} = 0 \quad \text{on} \quad \partial\Sigma \Big\},$$
$$\Lambda_{\widehat{h}}^D(\Sigma) := \Big\{ \xi_{\widehat{h}} \in C(\Sigma) : \quad \xi_{\widehat{h}}|_K \in \mathrm{P}_1(K) \quad \forall\, K \in \Sigma_{\widehat{h}} \Big\}\,.$$

Next, as in Section 5.2, we assume here that the families $\{\mathcal{T}_h^S\}_{h>0}$ and $\{\mathcal{T}_h^D\}_{h>0}$ are quasi-uniform as well in a neighborhood of Σ. Hence, proceeding similarly to the proof of [26, Lemma 5.1], it was proved in [1, Lemma 4.4] that there exist stable discrete lifting operators $\mathcal{L}_h^*$ to Ω_*, $* \in \{S, D\}$, satisfying the 3D version of (4.26). Moreover, since Σ_h is the partition of Σ inherited from $\mathcal{T}_h^S$ (or $\mathcal{T}_h^D$), made up of triangles K of diameter h_K, we set the respective meshsize $h_\Sigma := \max\{h_K : \quad K \in \Sigma_h\}$, and observe, as for the 2D case, that both $\Phi_h^S(\Sigma)$ and $\Phi_h^D(\Sigma)$ (cf. (4.25)) coincide with the 3D version of (5.7), that is

$$\Phi_h(\Sigma) := \Big\{\phi_h \in \mathrm{L}^2(\Sigma) : \quad \phi_h|_K \in \mathrm{P}_0(K) \quad \forall\, \text{triangle } K \in \Sigma_h\Big\}. \tag{5.8}$$

Consequently, applying again Lemma 4.5 we conclude, by means of (4.24), that (4.22) and (4.23) follow from the 3D version of (4.27), which is stated below.

Lemma 5.4. *There exist positive constants $\beta_\Sigma^{\mathrm{d}}$ and C_0, independent of h, such that for all $h_\Sigma \le C_0\,\widehat{h}$ there holds*

$$\sup_{\substack{\phi_h \in \Phi_h(\Sigma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \xi_{\widehat{h}}\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \;\ge\; \beta_\Sigma^{\mathrm{d}}\, \|\xi_{\widehat{h}}\|_{1/2,\Sigma} \quad \forall\, \xi_{\widehat{h}} \in \Lambda_{\widehat{h}}^S(\Sigma) \cup \Lambda_{\widehat{h}}^D(\Sigma)\,.$$

Proof. We refer to [1, Lemma 4.5] for full details (see also part of the proof of [23, Lemma 7.5]). □

The discussion regarding the consequent accomplishment of (**H.4**) and (**H.5**) in the present 3D case is analogous to the one given at the end of Section 5.2, the only difference being now the incorporation of the restriction $h_\Sigma \le C_0\,\widehat{h}$ in the respective statements.

5.4 The rates of convergence

Here we provide the rates of convergence of the Galerkin scheme (4.4) with the specific finite element subspaces introduced in Sections 5.1, 5.2, and 5.3. For this purpose, we collect next the respective approximation properties (cf. [19], [22]) under the notational convention that in 2D, $\widehat{h}$, $\Lambda_{\widehat{h}}^D(\Sigma)$, and $\Lambda_{\widehat{h}}^S(\Sigma)$ mean h, $\Lambda_h^D(\Sigma)$, and $\Lambda_h^S(\Sigma)$, respectively:

$\Big(\mathbf{AP}_h^{\mathbf{t}_S}\Big)$ there exists a positive constant C, independent of h, such that for each $\varrho \in [0,1]$, and for each $\mathbf{r}_S \in \mathbb{H}^\varrho(\Omega_S) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, there holds

$$\mathrm{dist}\big(\mathbf{r}_S, \mathbb{L}^2_{\mathrm{tr},h}(\Omega_S)\big) \;\le\; C\, h^\varrho\, \|\mathbf{r}_S\|_{\varrho,\Omega_S}\,,$$

$\big(\mathbf{AP}_h^{\mathbf{u}_D}\big)$ there exists a positive constant C, independent of h, such that for each $\varrho \in (0,1]$, and for each $\mathbf{v}_D \in \mathbf{H}^\varrho(\Omega_D) \cap \mathbf{H}_0(\mathrm{div};\Omega_D)$ with $\mathrm{div}(\mathbf{v}_D) \in \mathrm{H}^\rho(\Omega_D)$, there holds

$$\mathrm{dist}\big(\mathbf{v}_D, \mathbf{H}_{h,0}(\Omega_D)\big) \;\le\; C\, h^\varrho \Big\{\|\mathbf{v}_D\|_{\varrho,\Omega_D} \,+\, \|\mathrm{div}(\mathbf{v}_D)\|_{\varrho,\Omega_D}\Big\}\,,$$

$\big(\mathbf{AP}_h^{\boldsymbol{\sigma}_S}\big)$ there exists a positive constant C, independent of h, such that for each $\varrho \in (0,1]$, and for each $\boldsymbol{\tau}_S \in \mathbb{H}^\varrho(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)$ with $\mathbf{div}(\boldsymbol{\tau}_S) \in \mathbf{W}^{\varrho,4/3}(\Omega_S)$, there holds

$$\mathrm{dist}\big(\boldsymbol{\tau}_S, \mathbb{H}_{h,0}(\Omega_S)\big) \;\le\; C\, h^\varrho \Big\{\|\boldsymbol{\tau}_S\|_{\varrho,\Omega_S} \,+\, \|\mathbf{div}(\boldsymbol{\tau}_S)\|_{\varrho,4/3;\Omega_S}\Big\}\,,$$

$\Big(\mathbf{AP}_{\widehat{h}}^{\lambda}\Big)$ there exists a positive constant C, independent of h and $\widehat{h}$, such that for each $\varrho \in [0,1]$, and for each $\xi \in \mathrm{H}^{1/2+\varrho}(\Sigma)$, there holds

$$\mathrm{dist}\big(\xi, \Lambda_{\widehat{h}}^D(\Sigma)\big) \;\le\; C\, \widehat{h}^\varrho\, \|\xi\|_{1/2+\varrho,\Sigma}\,,$$

$\left(\mathbf{AP}_h^{\mathbf{u}_S}\right)$ there exists a positive constant C, independent of h, such that for each $\varrho \in [0,1]$, and for each $\mathbf{v}_S \in \mathbf{W}^{\varrho,4}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{v}_S, \mathbf{L}_h^4(\Omega_S)\big) \,\leq\, C\, h^{\varrho}\, \|\mathbf{v}_S\|_{\varrho,4;\Omega_S}\,,$$

$\left(\mathbf{AP}_{\widehat{h}}^{\boldsymbol{\varphi}}\right)$ there exists a positive constant C, independent of h and $\widehat{h}$, such that for each $\varrho \in [0,1]$, and for each $\boldsymbol{\psi} \in \mathbf{H}^{1/2+\varrho}(\Sigma) \cap \mathbf{H}_{00}^{1/2}(\Sigma)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\psi}, \boldsymbol{\Lambda}_{\widehat{h}}^S(\Sigma)\big) \,\leq\, C\, \widehat{h}^{\varrho}\, \|\boldsymbol{\psi}\|_{1/2+\varrho,\Sigma}\,,$$

$\left(\mathbf{AP}_h^{p_D}\right)$ there exists a positive constant C, independent of h, such that for each $\varrho \in [0,1]$, and for each $q_D \in \mathrm{H}^{\varrho}(\Omega_D) \,\cap\, \mathrm{L}_0^2(\Omega_D)$, there holds

$$\operatorname{dist}\big(q_D, \mathrm{L}_{h,0}^2(\Omega_D)\big) \,\leq\, C\, h^{\varrho}\, \|q_D\|_{\varrho,\Omega_D}\,.$$

The rates of convergence of (4.4) are now established by the following theorem.

Theorem 5.5. *Let* $((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}) \in \mathbb{H} \times \mathbf{Q}$ *and* $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h,)\vec{\boldsymbol{p}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ *be the unique solutions of* (3.9) (*or* (3.17)) *and* (4.4) (*or* (4.5)), *with* $\mathbf{u}_S \in \mathrm{W}$ *and* $\mathbf{u}_{S,h} \in \mathrm{W}_h$, *whose existences are guaranteed by Theorems* 3.11 *and* 4.4, *respectively. In turn, let* p *and* p_h *given by* (2.4) *and* (4.37), *respectively. Assume the hypotheses of Theorem* 4.6, *and that there exists* $\varrho \in (0,1]$ *such that* $\mathbf{t}_S \in \mathbb{H}^{\varrho}(\Omega_S) \cap \mathbb{L}_{\mathrm{tr}}^2(\Omega_S)$, $\mathbf{u}_D \in \mathbf{H}^{\varrho}(\Omega_D) \cap \mathbf{H}_0(\mathrm{div};\Omega_D)$, $\mathrm{div}(\mathbf{u}_D) \in \mathrm{H}^{\varrho}(\Omega_D)$, $\boldsymbol{\sigma}_S \in \mathbb{H}^{\varrho}(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)$, $\mathbf{div}(\boldsymbol{\sigma}_S) \in \mathbf{W}^{\varrho,4/3}(\Omega_S)$, $\lambda \in \mathrm{H}^{1/2+\varrho}(\Sigma)$, $\mathbf{u}_S \in \mathbf{W}^{\varrho,4}(\Omega_S)$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\varrho}(\Sigma) \cap \mathbf{H}_{00}^{1/2}(\Sigma)$, *and* $p_D \in \mathrm{H}^{\varrho}(\Omega_D) \cap \mathrm{L}_0^2(\Omega_D)$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\begin{aligned}
&\|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{p}}_h)\|_{\mathbb{H}\times\mathbf{Q}} \,+\, \|p_S - p_{S,h}\|_{0,\Omega_S} \\
&\quad\leq\, C\Big\{ h^{\varrho}\Big(\|\mathbf{t}_S\|_{\varrho,\Omega_S} + \|\mathbf{u}_D\|_{\varrho,\Omega_D} + \|\mathrm{div}(\mathbf{u}_D)\|_{\varrho,\Omega_D} + \|\boldsymbol{\sigma}_S\|_{\varrho,\Omega_S} + \|\mathbf{div}(\boldsymbol{\sigma}_S)\|_{\varrho,4/3;\Omega_S} \\
&\quad\quad + \|\mathbf{u}_S\|_{\varrho,4;\Omega_S} + \|p_D\|_{\varrho,\Omega_D}\Big) \,+\, \widehat{h}^{\varrho}\Big(\|\lambda\|_{1/2+\varrho,\Sigma} \,+\, \|\boldsymbol{\varphi}\|_{1/2+\varrho,\Sigma}\Big)\Big\}\,.
\end{aligned}$$

Proof. It follows straightforwardly from the Céa estimate (4.39) and the approximation properties $\left(\mathbf{AP}_h^{\mathbf{t}_S}\right)$, $\left(\mathbf{AP}_h^{\mathbf{u}_D}\right)$, $\left(\mathbf{AP}_h^{\boldsymbol{\sigma}_S}\right)$, $\left(\mathbf{AP}_{\widehat{h}}^{\lambda}\right)$, $\left(\mathbf{AP}_h^{\mathbf{u}_S}\right)$, $\left(\mathbf{AP}_{\widehat{h}}^{\boldsymbol{\varphi}}\right)$ and $\left(\mathbf{AP}_h^{p_D}\right)$. □

6 Computational results

In this section we present numerical results that illustrate the behavior of the Galerkin scheme (4.4). The computational implementation is based on a `FreeFem++` code (cf. [29]) and the use of the direct linear solvers UMFPACK (cf. [15]). The iterative method comes straightforwardly from the discrete fixed-point strategy along with a Newton-type method. Then, as a stopping criteria, we finish the algorithm when the relative error between two consecutive iterations of the complete coefficient vector **coeff** is small enough, that is

$$\frac{\|\mathbf{coeff}^{\,m+1} - \mathbf{coeff}^{\,m}\|_{l^2}}{\|\mathbf{coeff}^{\,m+1}\|_{l^2}} \,\leq\, \mathtt{tol}\,,$$

where $\|\cdot\|_{l^2}$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{dof}}$ with `dof` denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{L}_{\mathrm{tr},h}^2(\Omega_S)$, $\mathbb{H}_{h,0}(\Omega_S)$, $\mathbf{L}_h^4(\Omega_S)$, $\mathbf{H}_{h,0}(\mathrm{div};\Omega_D)$, $\boldsymbol{\Lambda}_{\widehat{h}}^S(\Sigma)$,

$\Lambda^D_{\widehat{h}}(\Sigma)$, and $L^2_{h,0}(\Omega_D)$. Subsequently, errors are defined as follows:

$$\mathsf{e}(\mathbf{t}_S) := \|\mathbf{t}_S - \mathbf{t}_{S,h}\|_{0,\Omega_S}, \quad \mathsf{e}(\boldsymbol{\sigma}_S) := \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{\mathbf{div}_{4/3};\Omega_S}, \quad \mathsf{e}(\mathbf{u}_S) := \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,4;\Omega_S},$$
$$\mathsf{e}(\mathbf{u}_D) := \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{\mathrm{div};\Omega_D}, \quad \mathsf{e}(\lambda) := \|\lambda - \lambda_{\widehat{h}}\|_{1/2,\Sigma}, \quad \mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_{\widehat{h}}\|_{1/2,00;\Sigma},$$
$$\mathsf{e}(p_D) := \|p_D - p_{D,h}\|_{0,\Omega_D}.$$

Again, hereafter, $\widehat{h}$, $\Lambda^D_{\widehat{h}}(\Sigma)$, and $\Lambda^S_{\widehat{h}}(\Sigma)$ mean h, $\Lambda^D_h(\Sigma)$, and $\Lambda^S_h(\Sigma)$, respectively, in 2D. Notice that, for ease of computation, and owing to the fact that $H^{1/2}(\Sigma)$ is the interpolation space with index $1/2$ between $H^1(\Sigma)$ and $L^2(\Sigma)$, the interface norm $\|\lambda - \lambda_{\widehat{h}}\|_{1/2,\Sigma}$ is replaced by $\|\lambda - \lambda_{\widehat{h}}\|_{(0,1),\Sigma}$, where

$$\|\xi\|_{(0,1),\Sigma} := \|\xi\|^{1/2}_{0,\Sigma}\,\|\xi\|^{1/2}_{1,\Sigma} \quad \forall \xi \in H^1(\Sigma).$$

Similarly, the interface norm $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_{\widehat{h}}\|_{1/2,00;\Sigma}$ is replaced by $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{(0,1),\Sigma}$. In turn, convergence rates are set as

$$r(\star) := \frac{\log(\mathsf{e}(\star)/\mathsf{e}'(\star))}{\log(h/h')}, \quad \forall \star \in \{\mathbf{t}_S, \boldsymbol{\sigma}_S, \mathbf{u}_S, \mathbf{u}_D, \boldsymbol{\varphi}, \lambda, p_D\},$$

where e and e' denote errors computed on two consecutive meshes of sizes h and h', respectively. In addition, we refer to the number of degrees of freedom and the number of Newton iterations as `dof` and `iter`, respectively.

Example 1: Tombstone-shaped domain. In our first example, a minor modification of [10, Example 1], we consider a porous unit square, coupled with a semi-disk-shaped fluid domain, that is,

$$\Omega_D := (-0.5, 0.5)^2 \quad \text{and} \quad \Omega_S := \Big\{(x_1, x_2): \quad x_1^2 + (x_2 - 0.5)^2 < 0.25, \quad x_2 > 0.5\Big\}.$$

We set the model parameters

$$\mathbf{K} := 10^{-3}\,\mathbb{I}, \quad \rho := 1, \quad \omega_1 := 1.0,$$

and choose the data $\mathbf{f}_S$, $\mathbf{g}_S$, and f_D such that the variable viscosity is defined as

$$\mu(\nabla \mathbf{u}_S) := 2 + \frac{1}{1 + |\nabla \mathbf{u}_S|},$$

where the exact solution in the domain $\Omega := \Omega_S \cup \Sigma \cup \Omega_D$ is given by the smooth functions

$$p_S(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2), \quad \mathbf{u}_S(\mathbf{x}) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_S,$$
$$p_D(\mathbf{x}) = \cos(\pi x_1)\exp(x_2 - 0.5), \quad \text{and} \quad \mathbf{u}_D(\mathbf{x}) = -\mathbf{K}\,\nabla p_D(\mathbf{x}) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_D.$$

Notice that $\mathbf{u}_S$, being the **curl** of a smooth function, satisfies the incompressibility condition, and also $\mathbf{u}_S \cdot \mathbf{n} = 0$ on Γ_D. Table 6 shows the convergence history for a sequence of quasi-uniform mesh refinements, including the resulting number of Newton iterations. According to the polynomial degree employed, the respective sets of finite element subspaces are denoted $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_1$ and $\mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_1$, for the fluid and the porous medium, respectively. This example confirms the theoretical rate of convergence $\mathcal{O}(h)$ provided by Theorem 5.5 with $\varrho = 1$. In addition, the aforementioned number of Newton iterations required to reach the convergence criterion based on the residuals with a tolerance of $1e-8$, was equal to 4 in all runs. Finally, samples of approximate solutions are shown in Figure 6.1.

$\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_1$ and $\mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_1$									
$e(\mathbf{t}_S)$	$r(\mathbf{t}_S)$	$e(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(p_S)$	$r(p_S)$
$3.18e-01$	$*$	$1.75e+00$	$*$	$1.27e-01$	$*$	$3.24e-01$	$*$	$2.65e-01$	$*$
$1.63e-01$	1.08	$8.83e-01$	1.11	$6.21e-02$	1.15	$1.64e-01$	1.10	$1.26e-01$	1.21
$8.32e-02$	0.96	$4.46e-01$	0.98	$3.12e-02$	0.98	$8.28e-02$	0.98	$6.31e-02$	0.98
$4.16e-02$	1.05	$2.23e-01$	1.05	$1.57e-02$	1.05	$4.16e-02$	1.05	$3.24e-02$	1.01
$2.06e-02$	1.01	$1.10e-01$	1.02	$7.78e-03$	1.01	$2.08e-02$	1.00	$1.58e-02$	1.03
$1.04e-02$	1.08	$5.54e-02$	1.09	$3.89e-03$	1.10	$1.05e-02$	1.09	$7.78e-03$	1.08

$e(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$e(p_D)$	$r(p_D)$	$e(\lambda)$	$r(\lambda)$	dof	iter
$2.28e-04$	$*$	$5.23e-02$	$*$	$2.50e-01$	$*$	731	4
$1.06e-04$	1.23	$2.29e-02$	1.26	$1.26e-01$	1.02	2659	4
$4.25e-05$	1.36	$1.05e-02$	1.16	$4.99e-02$	1.38	10460	4
$2.00e-05$	1.08	$5.00e-03$	1.05	$2.33e-02$	1.09	41804	4
$9.94e-06$	1.58	$2.53e-03$	1.54	$1.19e-02$	1.52	167808	4
$4.95e-06$	0.93	$1.27e-03$	0.93	$5.79e-03$	0.97	660726	4

Table 6.1: Example 1, convergence history and Newton iteration count for the fully-mixed approximations of the Navier–Stokes/Darcy equations with variable viscosity, and convergence of the P_0-approximation of the postprocessed pressure field.

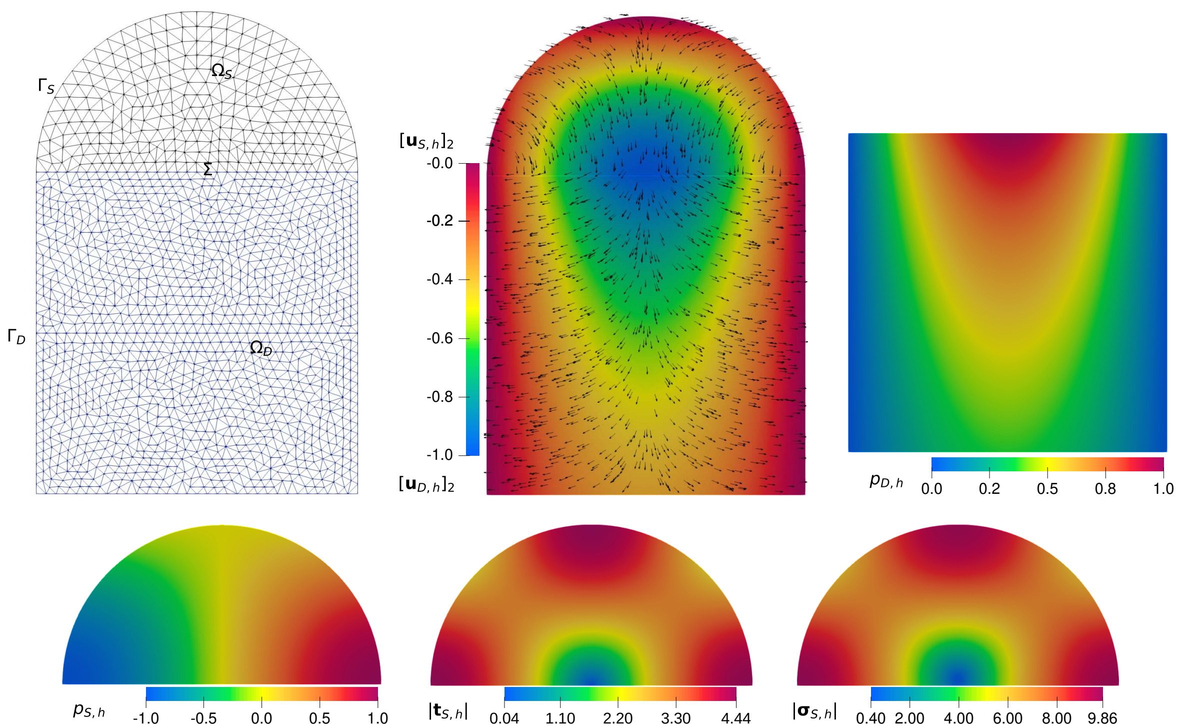

Figure 6.1: Example 1, domain configuration, approximated velocity component, Darcy pressure field, Navier—Stokes pressure field, spectral norm of the Navier—Stokes velocity gradient and pseudo-stress tensor.

Example 2: air flow through a filter. This example is similar to the one presented in [31, Section 4] (see also [11]). More precisely, we apply our mixed method to simulate air flow through a filter. To this end, we consider a two-dimensional channel with lenght 0.75 m and width 0.25 m which is partially blocked by a rectangular porous medium of length 0.25 m and width 0.2 m as shown in Figure 6.2, with boundaries $\Gamma_S = \Gamma_S^{\rm in} \cup \Gamma_S^{\rm top} \cup \Gamma_S^{\rm out} \cup \Gamma_S^{\rm bottom}$ and $\Gamma_D^{\rm bottom} := \Gamma_D$. The permeability tensor in the porous medium is given as

$$\mathbf{K} \,=\, \mathrm{R}(\theta) \begin{pmatrix} \frac{1}{\delta}\kappa & 0 \\ 0 & \kappa \end{pmatrix} \mathrm{R}^{-1}(\theta)\,, \qquad \text{with } \mathrm{R}(\theta) \,=\, \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

where the angle $\theta = -45°$, the anisotropy ratio $\delta = 100$, and $\kappa = 10^{-6}\, m^2$. In turn, $\rho = 1.225 \times 10^{-5}\,\mathrm{Mg/m^3}$, $\omega_1 = 1.0$, and the top and bottom of the domain are impermeable walls. The flow is driven with an inlet mean velocity of 0.25 m/s. The force terms $\mathbf{f}_S$ and f_D are set to zero. As motivated again by [10], the viscosity follows the Carreau law given by

$$\mu \,=\, 1.81 + 1.81\,(1 + |\mathbf{t}_S|^2)^{-1/2} \times 10^{-5}\ \mathrm{Pa\,s}\,, \tag{6.1}$$

whereas the boundary conditions are

$$\mathbf{u}_S \,=\, \Big[6\,\mathbf{u}_{\mathrm{in},S}\,\frac{x_2}{d}(1-\frac{x_2}{d}),0\Big] \quad \text{on} \quad \Gamma_S^{\rm in}\,, \quad \mathbf{u}_S \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_S^{\rm top} \cup \Gamma_S^{\rm bottom}\,,$$

$$\boldsymbol{\sigma}_S\mathbf{n} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_S^{\rm out}, \quad \mathbf{u}_D \cdot \mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma_D^{\rm bottom}\,,$$

with $\mathbf{u}_{\mathrm{in},S} = 0.25$ m/s and $d = 0.2$ m. We stress here that, because of the fully nonlinear character of μ (cf. (6.1)), which depends on the unknown fluid velocity gradient $\mathbf{t}_S := \nabla\mathbf{u}_S$, the use of the Newton method to solve the corresponding Galerkin scheme (4.4) implies linearizing not only the convective term given by the form b (cf. (3.11)), but also the one arising from the form a (cf. (3.10)). In addition, we remark that the analysis developed in the previous sections can be extended, with minor modifications, to the case of mixed boundary conditions considered in this example. Now, using again a sequence of quasi-uniform mesh refinements, we find that the number of Newton iterations required to reach the convergence criterion, based on the residuals with a tolerance of $1e-8$, is 7. In Fig. 6.2 we display various components of the computed solution. As we expected, the top-left panel shows an increment in air flow in the surrounding region above the filter. This is caused by the flow resistance in the porous medium. The effect of anisotropy is also evident, as the air flow that passes through the porous block aligns with the angle $\theta = -45°$. In other words, the flow follows the inclined principal direction of the permeability tensor. Furthermore, a continuous normal velocity is observed across all three interfaces, whereas the tangential velocity is discontinuous, especially at the interfaces with higher fluid velocity. This observation aligns with the continuity of flux and the BJS interface conditions. We also observe that the pressure drop is visible through the domain. Again, the effect of anisotropy is visible due to the inclined pressure drop in the porous domain. The pseudostress tensor $\boldsymbol{\sigma}_{S,h}$ is larger along the $\Gamma_S^{\rm in}$ boundary and zero at the $\Gamma_S^{\rm out}$ boundary, which is consistent with the boundary condition $\boldsymbol{\sigma}_S\,\mathbf{n} = \mathbf{0}$ on $\Gamma_S^{\rm out}$.

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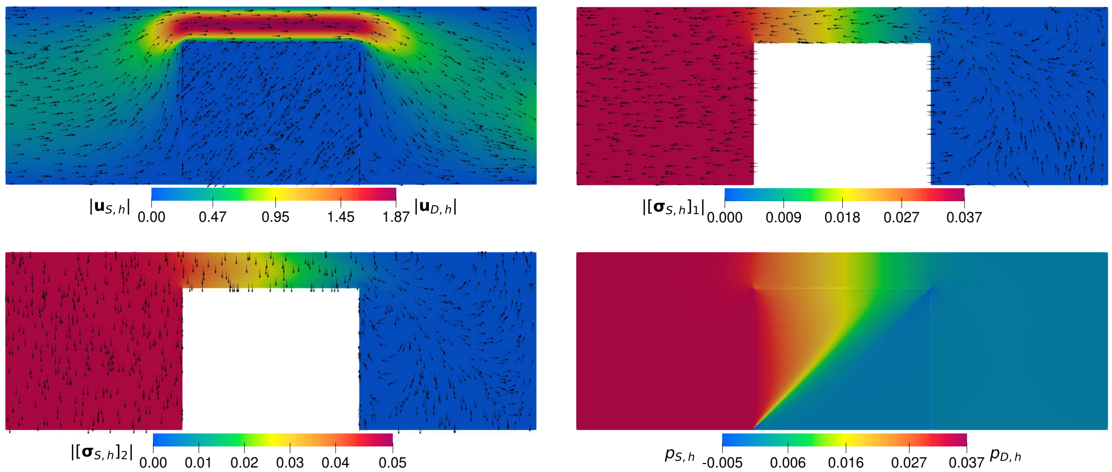

Figure 6.2: Example 2, approximated magnitude of the velocities (top-left), first rows (top-right) and second rows (bottom-left) of the pseudostress tensor with vector directions, and pressure fields (bottom-right).

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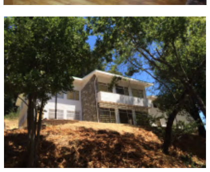

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