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## Centro de Investigación en Ingeniería Matemática (CI$^2$MA)

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# Optimal Control of Inter-Population Disease Spread via Reaction–Diffusion Models

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## Abstract

In this work, we study the control of a reaction–diffusion system modeling the spread of an infectious disease between two interacting populations, $H_1$ and $H_2$, within a shared spatial domain $\tilde{\Omega} = \Omega_1 \cap \Omega_2$. The model incorporates constant-coefficient spatial diffusion and excludes non-local, nonlinear, or cross-diffusion terms. The disease originates in population $H_1$ and is transmitted to $H_2$ through contact between infected individuals in $H_1$ and susceptible individuals in $H_2$. The transmission coefficient in $H_1$ is time-dependent and governed by control parameters $a = (\alpha, \gamma, t_c) \in Q \subset \mathbb{R}^3$, following an exponential decay. The objective is to minimize a cost functional associated with the attack rate and cumulative incidence in $H_2$. We establish the existence and uniqueness of solutions to the reaction–diffusion system and the associated optimal control problem. Using a Lagrangian framework, we derive the continuous gradient of the cost functional and prove the well-posedness of the adjoint system, along with the necessary optimality conditions. Numerical experiments illustrate how changes in the intensity ($\alpha$), rate ($\gamma$), and timing ($t_c$) of interventions in $H_1$ affect epidemic outcomes in $H_2$. Specifically, lower values of $\alpha$—corresponding to stronger intervention efficacy—lead to greater reductions in transmission in $H_1$ over time. The parameter $\gamma$ regulates how quickly interventions take effect, modeling delays in behavior change or intervention rollout. Our findings show that early and sustained control strategies in $H_1$ can substantially mitigate epidemic burden in $H_2$, even without direct interventions in $H_2$. This highlights the importance of targeting upstream sources of infection to achieve downstream public health benefits.

## 1. Introduction

In epidemiology, disease transmission is classically modeled using the framework proposed by William O. Kermack and Anderson G. McKendrick in 1927. [19], which consists of a set of ordinary differential equations that assumes that the population is divide into mutually exclusive groups (or classes) such as: susceptible, infected, and recovered. This model assumes that the population is homogeneously mixed and that the epidemic process is deterministic. However, in reality, individuals are spatially distributed and interact heterogeneously with both one another and their environment. There is considerable evidence that space can affect population dynamics ([9]) and these considerations can be represented, in a simple way, by a diffusion term.

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The Kermack–McKendrick model has evolved to incorporate additional compartments (e.g., exposed, asymptomatic, quarantined), demographic processes (e.g., births, deaths, migration), and more complex spatial dynamics beyond simple diffusion; heterogeneity of the population classifying groups by age, gender, ethnicity or social group; and other spatial dispersion behaviors more complex than simple diffusion such as convection, advection, taxis, chemotaxis or cross diffusion.

Mathematical models based on the Kermack-McKendrick compartmental model have been used to describe the spread of infectious diseases such as hantavirus, influenza A-H1N1, HIV, tuberculosis, cholera, hepatitis, Ebola, COVID-19, etc. Understanding the spatiotemporal dynamics of infectious diseases is essential for public health surveillance and the design of timely and effective control strategies. For example, in [7] the transmission of hantavirus in rodents is described using a spatio-temporal model that divides the population into susceptible-exposed-infected-recovered (SEIR), which distinguishes between male and female subpopulations, and describes it as a convection-diffusion-reaction system; or as in [8] where a SEIR spatio-temporal model of transmission of influenza A-H1N1 in 2009 in Chile is developed.

Strategies for controlling the spread of infectious diseases have been extensively studied in the context of epidemiological models using a variety of methodological approaches. For instance, Bichara et al. [5] propose an observer-based method, which involves constructing an auxiliary dynamical system to estimate parameters in an ODE model describing the transmission of diseases such as schistosomiasis. Alternatively, epidemic control via cost function minimization has been widely explored in ODE-based models, as discussed in works such as Borkar and Manjunath [6], Clancy [12], Howerton et al. [15], among others. Extensions of these techniques to PDE-based models, which incorporate spatial diffusion effects, can be found in studies by Chang et al. [10], Coronel et al. [13], Laaroussi and Rachik [20], and Zhou et al. [27].

In this study, we investigate the control of disease transmission dynamics between two interacting populations with overlapping spatial domains—motivated by real-world scenarios such as zoonotic spillover events or interspecies transmission in shared environments (e.g., livestock–wildlife interfaces, urban rodent–human interactions). In this framework, population $H_1$ represents a reservoir or primary host (e.g., rodents or livestock), where the disease originates and circulates, while population $H_2$ represents a secondary host or target population (e.g., humans), which is susceptible to spillover infection through contact in a shared region. This shared region, denoted as $\tilde{\Omega} = \Omega_1 \cap \Omega_2$, defines the spatial interface where interactions between the two populations occur, as illustrated in Figure 1. The state variables capture the spatiotemporal dynamics of susceptible, infected, and recovered individuals within each population.

The transmission coefficient in $H_1$ is modeled as a time-dependent function $\beta_{bb}(t)$ governed by the control parameters $a = (\alpha, \gamma, t_c)$. These parameters respectively reflect the intensity ($\alpha$), speed ($\gamma$), and timing ($t_c$) of public health interventions targeting $H_1$—such as hygiene promotion, reservoir population control, or isolation of infected individuals. The exponentially decaying structure of $\beta_{bb}(t)$ allows us to realistically capture gradual reductions in transmission following the implementation of interventions, acknowledging that such measures typically require time to achieve full effect.

Our aim is to quantify the impact of intervention timing, strength, and speed on key epidemiological outcomes in $H_2$. To this end, we formulate an optimal control problem that seeks to minimize a functional involving the cumulative incidence and attack rate in $H_2$, subject to the underlying reaction–diffusion system that governs disease transmission across both populations.

In a given time interval $[0, T]$ with $T > 0$, the model considers two independent populations $H_1$ and $H_2$ with total number of individuals $N_1, N_2 > 0$ respectively. The population $H_i$ is distributed over a bounded domain $\Omega_i \subset \mathbb{R}^l$ with a sufficiently smooth boundary $\partial\Omega_i$ ($i = 1, 2$ and $l = 1, 2, 3$), where $\Omega_1 \neq \Omega_2$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$. It is assumed that both populations interact in a region in common $\widetilde{\Omega} := \Omega_1 \cap \Omega_2$, with $\widetilde{\Omega} \neq \Omega_i$, $i = 1, 2$. The population $H_1$ is divided into two classes: susceptible and infected, while the population $H_2$ is divided into three classes: susceptible, infected and recovered. The disease begins in population $H_1$ and spreads to the population $H_2$ through contact between an individual infected population $H_1$ and a susceptible individual of the population $H_2$. In the population $H_1$ the disease will be described

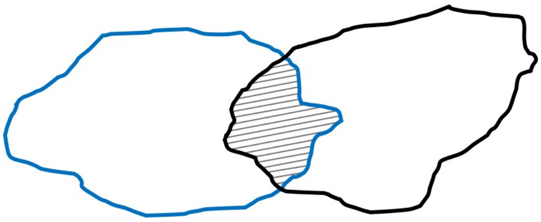

Figure 1: Intersection $\widetilde{\Omega}$ between region $\Omega_1$ (left) and $\Omega_2$ (right).

by means of the $SIS$ model with diffusion. Then, the model is given by the following system:

$$\begin{aligned}\partial_t u_1 &= d_u \Delta u_1 - \frac{\beta_{bb}(t,a)}{N_1} u_1 u_2 + \gamma_b u_2 \\ \partial_t u_2 &= d_u \Delta u_2 + \frac{\beta_{bb}(t,a)}{N_1} u_1 u_2 - \gamma_b u_2\end{aligned} \tag{1.1}$$

for $(x,t) \in \Omega_{1,T} := \Omega_1 \times (0,T)$.

$$\begin{aligned}\partial_t v_1 &= d_v \Delta v_1 - \frac{\beta_{hh}}{N_2} v_1 v_2 - \frac{\beta_{bh}}{N_2} v_1 u_2 \\ \partial_t v_2 &= d_v \Delta v_2 + \frac{\beta_{hh}}{N_2} v_1 v_2 + \frac{\beta_{bh}}{N_2} v_1 u_2 - \gamma_h v_2 \\ \partial_t v_3 &= d_v \Delta v_3 + \gamma_h v_2 \\ \partial_t c &= \frac{\beta_{hh}}{N_2} v_1 v_2\end{aligned} \tag{1.2}$$

for $(x,t) \in \Omega_{2,T} := \Omega_2 \times (0,T)$, where $\Delta$ is the Laplacian operator in $\Omega_i$ $(i = 1,2)$ and $d_u, d_v > 0$ are the constant diffusion coefficients. The state variables $u_1 := u_1(x,t)$ and $u_2 := u_2(x,t)$ represent the population densities for any $x$ and $t$ of the susceptible and infected classes from the population $H_1$ respectively, and the variables $v_1 := v_1(x,t)$, $v_2 := v_2(x,t)$ and $v_3 := v_3(x,t)$ represent the population densities of the susceptible, infected and recovered classes, respectively, of the population $H_2$ for any $x$ and $t$. The variable $c =: c(x,t)$ is considered, which represents the term of incidence of the disease in the population $H_2$. The requirement that the populations $H_i$ remains confined in $\Omega_i$ $(i = 1,2)$ for any time, can be translated into the following no-flow boundary condition:

$$\begin{cases}\nabla u_k(x,t) \cdot \eta_1 = 0, & \text{on } \Sigma_{1,T} := \partial\Omega_1 \times (0,T), \\ \nabla v_j(x,t) \cdot \eta_2 = 0, & \text{on } \Sigma_{2,T} := \partial\Omega_2 \times (0,T),\end{cases} \tag{1.3}$$

for $k = 1,2$ and $j = 1,2,3$, where $\eta_i$ is the normal vector along $\partial\Omega_i$ in the outward direction from $\Omega_i$ $(i = 1,2)$. Additionally, the initial conditions:

$$b(x,0) := b^0(x), \qquad b \in \{u_1, u_2, v_1, v_2, v_3, c\},$$

are nonnegative functions in their respective spatial domains.

The total population in each population is given by:

$$N_1 := \int_{\Omega_1} [u_1(x,t) + u_2(x,t)]dx, \qquad N_2 := \int_{\Omega_2} [v_1(x,t) + v_2(x,t) + v_3(x,t)]dx. \tag{1.4}$$

The recovery rate $0 < \gamma_r < 1$ of the population $H_i$ is given by $1/\gamma_r$ (if $i = 1$ then $r = b$, and if $i = 2$ then $r = h$). The transmission coefficient of the disease in $H_2$ $(0 < \beta_{hh} < 1)$ and the transmission coefficient between both populations ($0 < \beta_{bh} < 1$) will be considered constant; however, the transmission coefficient $\beta_{bb} = \beta_{bb}(t)$ in $H_1$ is assumed depending on $t$, and decreasing gradually from $\widetilde{\beta}_{bb}$ to $\alpha\widetilde{\beta}_{bb}$ according an exponential decay model (see [11]):

$$\beta_{bb}(t) = \begin{cases} \widetilde{\beta}_{bb}, & \text{si } t \leqslant t_c \\ \widetilde{\beta}_{bb}\big(\alpha + (1-\alpha)e^{-\gamma(t-t_c)}\big), & \text{si } t > t_c \end{cases} \tag{1.5}$$

where $0 < \widetilde{\beta}_{bb} < 1$ is the initial coefficient of disease trasmision in $H_1$, $t_c$ is the time at which interventions start, and $\gamma$ controls the rate of the transition from $\widetilde{\beta}_{bb}$ to $\alpha\widetilde{\beta}_{bb}$. The expression (1.5) aims to model the transmission rate and the impact of interventions to reduce said rate, such as hygiene measures, supervision, isolation of suspected cases in the controllable population $H_1$. This model was initially introduced by Chowell et al. [11], for the control of a disease with a non-instantaneous intervention impact. In our case, and in order to study the impact on the $H_2$ population from the interventions on the $H_1$ population, the objective is to minimize the functional

$$J(\mathbf{w}, a) := \frac{A}{|\Omega_2|}\int_{\Omega_2} v_3(x,T)dx + \frac{B}{|\Omega_2|}\int_{\Omega_2} c(x,T)dx + \varepsilon g(a) \tag{1.6}$$

where $a = (\alpha, \gamma, t_c) \in Q \subset (0,1) \times (0,M) \times (0,T)$; with $M > 0$ and $Q$ a closed bounded and convex set; $A$, $B$ and $\varepsilon$ are non-negative constants; and $g$ is a convex function in terms of $\alpha$, $\gamma$ and $t_c$. The term multiplied by $A$ represents the attack rate in humans and the term multiplied by $B$ represents the cumulative incidence in humans. Each of these terms are quantities desirable to minimize in order to control the spread of disease in humans. The term $\varepsilon g(a)$ corresponds to a Tikhonov-like regularization of the cost function in order to convex it in terms of the parameters (see [25]). On the other hand, this term of regularization $\varepsilon g(a)$ represents a practical restriction that considers that health and hygiene interventions cannot have an instantaneous impact on the reduction of the transmission rate, that is, the extreme cases $\alpha = 0$, $t_c = 0$, and $\gamma = M \to \infty$ cannot be feasible solutions.

**Remark 1.1.** *Given the intervention of the quantities $N_1$ and $N_2$ in this model, apparently a non-local PDE system is obtained. However, it is easy to see that when adding $u_1 + u_2$, and integrating in $\Omega_1$, and adding $v_1 + v_2 + v_3$ and integrating in $\Omega_2$, it is obtained from* (1.1), (1.2) *and* (1.3)*, that these quantities $N_1$ and $N_2$ remain constant independent of t, that is*

$$N_1 = \int_{\Omega_1} [u_1^0(x) + u_2^0(x)]dx, \qquad N_2 = \int_{\Omega_2} [v_1^0(x + v_2^0(x) + v_3^0(x)]dx.$$

The remainder of this paper is organized as follows. Section 2 establishes the existence and uniqueness of solutions to the proposed model using semigroup theory and functional analysis techniques. In Section 3, we formulate the optimal control problem and prove the existence of optimal solutions. Section 4 presents the continuous gradient derived via the Lagrangian framework and demonstrates the existence of solutions to the associated adjoint problem. Additionally, the first-order necessary optimality conditions for the control problem are derived. Finally, Section 5 provides numerical simulations, where the optimization problem is solved using MATLAB functions.

## 2. Existence and uniqueness

In this section, we will prove the existence and uniqueness of the solution of the reaction-diffusion system (1.1) - (1.2). Since the systems are coupled only by the term $(\beta_{bh}/N_2)v_1u_2$, first we will prove the existence and uniqueness of (1.1), and then we prove the wellposedness of (1.2). The function $u_1$ obtained from (1.1) will be extended appropriately to $\Omega_2 \setminus \widetilde{\Omega}$ and it will be a known function in (1.2). The proof of the existence and uniqueness is based on the procedure presented in [3] and [27].

We define the linear operator $A : D(A) \subset X \to X$, as $A = \begin{pmatrix} d_u\Delta & 0 \\ 0 & d_u\Delta \end{pmatrix}$, with $X = [L^2(\Omega_1)]^2$ and

$$D(A) := \left\{ \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in [H^2(\Omega_1)]^2 \;\middle|\; \nabla u_1 \cdot \eta_1 = \nabla u_2 \cdot \eta_1 = 0, \text{ on } \partial\Omega_1 \right\}.$$

**Theorem 2.1.** *For $a \in Q$ fixed, if $\mathbf{u}_0 = (u_1^0, u_2^0) \in D(A) \cap [L^\infty(\Omega_1)]^2$, with $u_j^0(x) \geqslant 0$, $j = 1, 2$, then the system* (1.1) *has a unique nonnegative solution $\mathbf{u} = (u_1, u_2) \in [H^1(0, T; L^2(\Omega_1))]^2$. Moreover, $u_j \in L^\infty(\Omega_{1,T}) \cap L^2(0, T; H^2(\Omega_1)) \cap L^\infty(0, T; H^1(\Omega_1))$, with $j = 1, 2$.*

*Proof.*

Let $a \in Q$ fixed, if $\mathbf{u}_0 = (u_1^0, u_2^0) \in D(A) \cap [L^\infty(\Omega_1)]^2$, with $u_j^0(x) \geqslant 0$, $j = 1, 2$. We define $f(t, u_1, u_2) = -\dfrac{\beta_{bb}(t, a)}{N_1} u_1 u_2 + \gamma_b u_2$, with $f : [0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, then (1.1) is rewritten as follows

$$\begin{cases} \partial_t u_1 = d_u \Delta u_1 + f(t, u_1, u_2), & \text{in } \Omega_{1,T} \\ \partial_t u_2 = d_u \Delta u_2 - f(t, u_1, u_2), & \text{in } \Omega_{1,T} \\ \nabla u_1 \cdot \eta_1 = \nabla u_2 \cdot \eta_1 = 0, & \text{on } \Sigma_{1,T} \\ u_1(x, 0) = u_1^0(x), \quad u_2(x, 0) = u_2^0(x), & \text{in } \Omega_1 \end{cases} \tag{2.1}$$

Problem (2.1) can be studied as an abstract nonhomogeneous reaction-diffusion system [21]:

$$\begin{cases} \dfrac{d}{dt}\mathbf{u}(t) = A\mathbf{u}(t) + F(t, \mathbf{u}(t)), & t \in [0, T] \\ \mathbf{u}(0) = \mathbf{u}_0, & \mathbf{u}_0 \in X \end{cases} \tag{2.2}$$

where $\mathbf{u}(t) \in X$ such that $[\mathbf{u}(t)](x) := (u_1(x, t), u_2(x, t))$, $\forall (x, t) \in \Omega_1 \times [0, T]$ and $F := (F_1, F_2)$ is a function $F : [0, T] \times X \longrightarrow X$ define as $[F_j(t, \varphi)](x) := \pm f(t, \varphi(x))$, $\forall (t, \varphi) \in [0, T] \times X$ and $x \in \Omega_1$.

The operator $A$ is an infinitesimal generator of a $C_0$ semigroup of contractions in $X$, $F$ is continuous and measurable in $t$ for $t \in [0, T]$ but it is not uniformly Lipschitz continuous in $u$ with respect to $t$; however, truncating $F$ is obtained a Lipschitz continuous function.

*A) The truncated problem*

The truncated problem associated with (2.14) is defined as:

$$\begin{cases} \dfrac{d}{dt}\mathbf{u}^N(t) = A\mathbf{u}^N(t) + F^N(t, \mathbf{u}^N(t)), & t \in [0, T] \\ \mathbf{u}^N(0) = \mathbf{u}_0, & \mathbf{u}_0 \in X \end{cases} \tag{2.3}$$

where $N > 0$ is large enough, $[\mathbf{u}^N(t)](x) := (u_1^N(x, t), u_2^N(x, t))$ for $(x, t) \in \Omega_1 \times [0, T]$ and $F^N := (F_1^N, F_2^N)$ is a function $F^N : [0, T] \times X \longrightarrow X$ such that $[F_j^N(t, \varphi_1, \varphi_2)](x) := \pm f^N(t, \varphi_1(x), \varphi_1(x))$ for $(t, \varphi_1, \varphi_2) \in [0, T] \times X$ and $x \in \Omega_1$, with $f^N$ defined by:

$$\begin{aligned} &f^N(t, u_1, u_2) := f(t, T^N u_1, T^N u_2) \\ &\text{where } \; T^N\varphi = \min\{\max\{\varphi, -N\}, N\}, \qquad \forall \varphi \in \mathbb{R}. \end{aligned} \tag{2.4}$$

consequently $F^N(t, u)$ is uniformly Lipschitz continuous in $u \in X$ with respect to $t \in [0, T]$ (see for example [18, Appendix B]), therefore [3, Theorem 2.1] implies that (2.3) admits a unique strong solution

$$\mathbf{u}^N \in H^1(0, T; X) \cap L^2(0, T; D(A)). \tag{2.5}$$

**Note:** We remark that to obtain the Lipschitz-continuity of $F^N$, it is enough to truncate either $u_1$ or $u_2$, but it is not necessary both. However, we choose to truncate $u_1$ and $u_2$ since it will be the last one for the proof of the $L^\infty$-estimate.

*B) Estimate in $L^\infty(0,T;H^1(\Omega_1))$*

Integrating the square of the first equation of (2.3), we obtain

$$\int_0^t\int_{\Omega_1}\left|\partial_s u_1^N(x,s)\right|^2\,dxds-2d_u\int_0^t\int_{\Omega_1}\partial_s u_1^N(x,s)\Delta u_1^N(x,s)\,dxds$$
$$+d_u^2\int_0^t\int_{\Omega_1}\left|\Delta u_1^N(x,s)\right|\,dxds=\int_0^t\int_{\Omega_1}\left|f_1^N(s,u_1^N(x,s),u_2^N(x,s))\right|^2\,dxds,$$

and then, using the Green's identity we have

$$\int_0^t\int_{\Omega_1}\left|f_1^N(s,u_1^N(x,s),u_2^N(x,s))\right|^2\,dxds+d_u\int_{\Omega_1}\left|\nabla u_1^0(x)\right|^2\,dx \tag{2.6}$$
$$=\int_0^t\int_{\Omega_1}\left|\partial_s u_1^N(x,s)\right|^2\,dxds+d_u\int_{\Omega_1}\left|\nabla u_1^N(x,t)\right|^2\,dx+d_u^2\int_0^t\int_{\Omega_1}\left|\Delta u_1^N(x,s)\right|\,dxds$$

On the other hand, by the Lipschitz continuity of $F^N(t,u)$ it get

$$\|f_1^N(s,u_1^N,u_2^N)\|_{L^2(\Omega_1)}\leqslant L\|u_1^N\|_{L^2(\Omega_1)}+L\|u_2^N\|_{L^2(\Omega_1)},$$

where $L>0$ is the Lipschitz continuity constant. Replacing, this last inequality in (2.6), we deduce from (2.5) that

$$d_u\int_{\Omega_1}\left|\nabla u_1^N(x,t)\right|^2\,dx\leqslant$$
$$L\|\mathbf{u}^N\|_{L^2(0,T;X)}+d_u\|u_1^0\|_{H^1(\Omega_1)}+\|u_1^N\|_{H^1(0,T;L^2(\Omega_1))}+d_u^2\|u_1^N\|_{L^2(0,T,H^2(\Omega_1))},$$

namely, $u_1^N\in L^\infty(0,T;H^1(\Omega_1))$. By the same procedure, it is proved that $u_2^N\in L^\infty(0,T;H^1(\Omega_1))$.

*C) Estimate in $L^\infty(\Omega_{1,T})$*

Let define

$$M=\max_{j=1,2}\{\|f_j^N\|_{L^\infty(Q)},\|u_j^0\|_{L^\infty(\Omega_1)}\}>0,$$

which it is well defined since $f_j^N$ is bounded by definition (2.4).

On the other hand, we define the linear operator $A_1:D(A_1)\subset X_1\to X_1$, as $A_1=d_u\Delta$, with $X_1=L^2(\Omega_1)$ and

$$D(A_1):=\left\{u\in H^2(\Omega_1)\middle|\nabla u\cdot\eta=0\right\}.$$

Let $\mathbf{u}^N\in H^1(0,T;X)\cap L^2(0,T;D(A))\cap L^\infty(0,T;H^1(\Omega_1))^2$ solution of (2.3) (fixed). Then $U_1^N(x,t)=u_1^N(x,t)-Mt-\|u_1^0\|_{L^\infty(\Omega_1)}$ satisfies the following problem:

$$\begin{cases}\dfrac{d}{dt}U_1^N(t)=A_1U_1^N(t)+F_1^N(t,u_1^N(t),u_2^N(t))-M, & t\in[0,T]\\ U_1^N(0)=u_1^0-\|u_1^0\|_{L^\infty(\Omega_1)}\end{cases} \tag{2.7}$$

From [3, Theorem 2.1], we know that the problem (2.7) have a unique solution $U_1^N\in H^1(0,T;X_1)\cap L^2(0,T;D(A_1))$ and

$$U_1^N(t)=e^{A_1t}\left(u_1^0-\|u_1^0\|_{L^\infty(\Omega_1)}\right)+\int_0^t e^{A_1(t-r)}\left(F_1^N(r,u_1^N(r),u_2^N(r))-M\right)dr.$$

Since $u_1^0-\|u_1^0\|_{L^\infty(\Omega_1)}\leqslant 0$ and $F_1^N(r,u_1^N(r),u_2^N(r))-M\leqslant 0$ then $U_1^N(t)\leqslant 0,\ \forall t\in[0,T]$, namely, $U_1^N(x,t)\leqslant 0$, for all $(x,t)\in\Omega_{1,T}$ and consequence $u_1^N(x,t)\leqslant Mt+\|u_1^0\|_{L^\infty(\Omega_1)}$. In the same way it is proved that $W_1^N(x,t) = u_1^N(x,t) + Mt + \|u_1^0\|_{L^\infty(\Omega_1)} \geqslant 0$ in $\Omega_{1,T}$. and then $u_1^N(x,t) \geqslant -Mt - \|u_1^0\|_{L^\infty(\Omega_1)}$, for all $(x,t) \in \Omega_{1,T}$. Proceding analogously with $u_2^N$, we obtain

$$|u_j^N(x,t)| \leqslant MT + \|u_j^0\|_{L^\infty(\Omega_1)}, \quad \text{for all } (x,t) \in \Omega_{1,T} \tag{2.8}$$

where $M$ only depends on $N$, thus $u_j^N(x,t) \in L^\infty(Q)$ $(j = 1,2)$.

*D) Nonnegativity*

Multiplying the second equation of (2.3) by $(u_2^N)^-(x,t) = -\min\{u_2^N(x,t), 0\}$, and integrating by parts on $\Omega_1$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_1} \left|(u_2^N)^-\right|^2 dx + d_u \int_{\Omega_1} \left|\nabla(u_2^N)^-\right|^2 dx = \int_{\Omega_1} -f^N(t,u_1^N,u_2^N)(u_2^N)^- \, dx.$$

Here $f^N(t,u_1^N,u_2^N) = -\left(\frac{\beta_{bb}(t,a)}{N_1}u_1^N - \gamma_b\right)u_2^N$ and from (2.8): $-f^N(t,u_1^N,u_2^N) \leqslant c_1^N u_2^N$, then

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_1} \left|(u_2^N)^-\right|^2 dx + d_u \int_{\Omega_1} \left|\nabla(u_2^N)^-\right|^2 dx \leqslant c_1^N \int_{\Omega_1} \left|(u_2^N)^-\right|^2 dx,$$

thus $(u_2^N)^-(x,t) = 0$, namely, $u_2^N(x,t) \geqslant 0, \ \forall (x,t) \in \Omega_{1,T}$.

To prove the nonnegativity of $u_1^N(x,t)$ consider the following auxiliar problem:

$$\begin{cases} \partial_t \widetilde{u}_1^N = d_u \Delta \widetilde{u}_1^N + \widetilde{f}_1^N(t,\widetilde{u}_1^N,u_2^N), & \text{in } \Omega_{1,T} \\ \nabla \widetilde{u}_1^N \cdot \eta_1 = 0, & \text{on } \Sigma_{1,T} \\ \widetilde{u}_1^N(x,0) = u_1^0(x), & \text{in } \Omega_1 \end{cases} \tag{2.9}$$

where $\widetilde{f}_1^N(t,\widetilde{u}_1^N,u_2^N) = -\frac{\beta_{bb}(t,a)}{N_1}u_2^N\widetilde{u}_1^N$ is uniformly Lipschitz continuous on $X$ with respect to $t \in [0,T]$ and $\widetilde{f}_1^N \leqslant f_1^N$ (see [18, appendix B]) and since $u_2^N(x,t) \in L^\infty(\Omega_{1,T})$, we have that $\widetilde{f}_1^N \leqslant c_2^N \widetilde{u}_1^N$ in $\Omega_{1,T}$. By [3, Theorem 2.1] the solution of problem (2.11) exists and proceeding as in the previous case, therefore

$$\frac{1}{2}\int_{\Omega_1} \left|(\widetilde{u}_1^N)^-\right|^2 dx \leqslant c_2^N \int_0^t \int_{\Omega_1} \left|(\widetilde{u}_1^N)^-\right|^2 dxds,$$

and then $(\widetilde{u}_1^N)^-(x,t) = 0$, that is

$$u_1^N(x,t) \geqslant \widetilde{u}_1^N(x,t) \geqslant 0 \quad \text{for all } (x,t) \in \Omega_{1,T}.$$

Finally, we choose $N > 2\max\{\|u_j^0\|_{L^\infty(\Omega_1)} : \ j = 1,2\}$, then there exists $\theta \in (0,T)$ such that

$$M\theta + \|u_j^0\|_{L^\infty(\Omega_1)} \leqslant \frac{N}{2}, \quad j = 1,2.$$

Using (2.8), we have $|u_j^N(x,t)| \leqslant N$ in $(x,t) \in \Omega_{1,\theta}$ $(j = 1,2)$, and then $F^N = F$ for all $t \in (0,\theta)$. We deduce that $(u_1^N, u_2^N)$ is mild solution of (2.1) in $\Omega_{1,\theta}$. By existence and uniqueness of Problem (2.3) and the $L^\infty$-estimate, we conclude that $u_j = u_j^N$, for $j = 1,2$, with $\theta = T$, and we conclude the result. ∎

**Theorem 2.2.** *The problem* (1.1) *is well-posedness. More precisely, the solution of* (1.1) *is unique and Lipschitz continuous with respect to its parameters $a \in Q$ and the initial condition. Moreover, if $\mathbf{u}^1$ and $\mathbf{u}^2$ are solutions of* (1.1)*, with initial condition $\mathbf{u}_0^1$ and $\mathbf{u}_0^2$ respectively, and with a set of parameters $a = a_1$ and $a = a_2$, respectively, then*

$$\|\mathbf{u}^1(t) - \mathbf{u}^2(t)\|_X \leqslant C\left(\|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_X + \int_0^t |\beta_{bb}(s,a_2) - \beta_{bb}(s,a_1)|^2 ds\right), \qquad 0 < t < T, \tag{2.10}$$

*where $C$ is a constant depending on $T$ and $L = \max_{|\mathbf{s}|\leqslant M}(\nabla_{\mathbf{u}}F(\mathbf{s}))$, with $M := \max\{\|\mathbf{u}^1\|_{L^\infty}, \|\mathbf{u}^2\|_{L^\infty}\}$.*

*Proof.* Let $\mathbf{u}^1 = (u_1^1, u_2^1) \in X$ the solution of (1.1) with $a_1$ and $\mathbf{u}_0^1$, and let $\mathbf{u}^2 = (v_1^2, v_2^2) \in X$ the solution of (1.1) with $a_2$ and $\mathbf{u}_0^2$, then:

$$\mathbf{u}^1(t) = S(t)\mathbf{u}_0^1 + \int_0^t S(t-s)F(s, a_1, \mathbf{u}^1(s))ds, \qquad t \in [0,T],$$
$$\mathbf{u}^2(t) = S(t)\mathbf{u}_0^2 + \int_0^t S(t-s)F(s, a_2, \mathbf{u}^2(s))ds, \qquad t \in [0,T].$$

Subtracting, taking the $X$-norm, and using the contraction property $\|S(t)\|_{\mathcal{L}(X)} \leqslant 1$, we have

$$\|\mathbf{u}^1(t) - \mathbf{u}^2(t)\|_X \leqslant \|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_X + \int_0^t \|F(r, a_1, \mathbf{u}^1(r)) - F(r, a_2, \mathbf{u}^2(r))\|_X dr$$

From the Lipschitz continuity of $F$ (see [18, Appendix B]):

$$\|\mathbf{u}^1(t) - \mathbf{u}^2(t)\|_X \leqslant \|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_X + L\int_0^t |\beta_{bb}(r, a_2) - \beta_{bb}(r, a_1)|^2 dr + \int_0^t L\|\mathbf{u}(r) - \boldsymbol{v}(r)\|_X dr$$

and the applying a generalized Gronwall Lemma (see [18, Appendix A]):

$$\|\mathbf{u}^1(t) - \mathbf{u}^2(t)\|_X \leqslant \|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_X + L\int_0^t |\beta_{bb}(r, a_2) - \beta_{bb}(r, a_1)|^2 dr$$
$$+ \int_0^t \left(\|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_X + L\int_0^s |\beta_{bb}(r, a_2) - \beta_{bb}(r, a_1)|^2 dr\right) Le^{L(t-s)} ds$$

Then, we deduce the uniqueness of the solution of (1.1) and the Lipschitz continuity respect of $a \in Q$.

■

Let $u_2$ solution of (1.1). [3, Theorem 2.1] and Theorem 2.1 tell us that $u_2 \in C(0,T; L^2(\Omega_1))$ and $u_2 \in L^\infty(\Omega_{1,T}) \cap H^1(0,T; L^2(\Omega_1))$, then we define a prolongation by zero of $u_2$ in $\Omega_2$ as

$$\overline{u}_2(x,t) = \begin{cases} u_2(x,t) & , \quad (x,t) \in \widetilde{\Omega} \times [0,T] \\ 0 & , \quad (x,t) \in (\Omega_2 \setminus \widetilde{\Omega}) \times [0,T] \end{cases} \tag{2.11}$$

with $\widetilde{\Omega} := \Omega_1 \cap \Omega_2$, therefore

$$\overline{u}_2 \in L^\infty(\Omega_{2,T}) \cap H^1(0,T; L^2(\Omega_2)) \cap C(0,T; L^2(\Omega_2)) \tag{2.12}$$

Also, let

$$g_1(\overline{u}_2, \boldsymbol{v}) := \left(-\frac{\beta_{hh}}{N_2}v_2 - \frac{\beta_{bh}}{N_2}\overline{u}_2\right)v_1, \quad g_2(\overline{u}_2, \boldsymbol{v}) := \left(\frac{\beta_{hh}}{N_2}v_1 - \gamma_h\right)v_2 + \frac{\beta_{bh}}{N_2}v_1\overline{u}_2, \quad g_3(\overline{u}_2, \boldsymbol{v}) := \gamma_h v_2,$$

for all $\overline{u}_2 \in \mathbb{R}$ and $\boldsymbol{v} := (v_1, v_2, v_3) \in \mathbb{R}^3$, and consider the linear operator $B : D(B) \subset Y \to Y$ defined as $B = \begin{pmatrix} d_v\Delta & 0 & 0 \\ 0 & d_v\Delta & 0 \\ 0 & 0 & d_v\Delta \end{pmatrix}$, with $Y = [L^2(\Omega_2)]^3$ and

$$D(B) := \left\{ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in [H^2(\Omega_1)]^3 \;\middle|\; \nabla v_1 \cdot \eta_2 = \nabla v_2 \cdot \eta_2 = \nabla v_3 \cdot \eta_2 = 0, \text{ on } \partial\Omega_2 \right\}.$$

**Theorem 2.3.** *Let $\mathbf{u} \in [H^1(0,T; L^2(\Omega_1))]^2$ a mild solution of (1.1) with the hypothesis of Theorem 2.1. If $\boldsymbol{v}_0 = (v_1^0, v_2^0, v_3^0) \in D(B) \cap [L^\infty(\Omega_2)]^3$ and $c^0 \in L^\infty(\Omega_2)$ with $v_j^0(x) \geqslant 0$ ($j = 1,2$) and $c^0(x) \geqslant 0$, then the system (1.2) has a unique nonnegative solution $\boldsymbol{v} = (v_1, v_2, v_3, c) \in [H^1(0,T; L^2(\Omega_2))]^4$. Moreover, $v_j, c \in L^\infty(\Omega_{2,T})$ and $v_j \in L^2(0,T; H^2(\Omega_2)) \cap L^\infty(0,T; H^1(\Omega_2))$ where $j = 1,2,3$.*

*Proof.* The proof is very similar to the Theorem 2.1. A part of the system (1.2) can be rewritten as follows

$$\begin{cases} \partial_t v_1 = d_v \Delta v_1 + g_1(\overline{u}_2, v_1, v_2, v_3), & \text{in } \Omega_{2,T} \\ \partial_t v_2 = d_v \Delta v_2 + g_2(\overline{u}_2, v_1, v_2, v_3), & \text{in } \Omega_{2,T} \\ \partial_t v_3 = d_v \Delta v_3 + g_3(\overline{u}_2, v_1, v_2, v_3), & \text{in } \Omega_{2,T} \\ \nabla v_1 \cdot \eta_2 = \nabla v_2 \cdot \eta_2 = \nabla v_3 \cdot \eta_2 = 0, & \text{on } \Sigma_{2,T} \\ v_1(x,0) = v_1^0(x), \quad v_2(x,0) = v_2^0(x), \quad v_3(x,0) = v_3^0(x), & \text{in } \Omega_2 \end{cases} \tag{2.13}$$

Like the proof of the Theorem 2.1, we study (2.13) as an abstract nonhomogeneous reaction-diffusion system [21]:

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{v}(t) = B\boldsymbol{v}(t) + G(t, \boldsymbol{v}(t)), & t \in [0,T] \\ \boldsymbol{v}(0) = \boldsymbol{v}_0, & \boldsymbol{v}_0 \in Y \end{cases} \tag{2.14}$$

where $\boldsymbol{v}(t) \in Y$ such that $[\boldsymbol{v}(t)](x) := (v_1(x,t), v_2(x,t), v_3(x,t)),\ \forall\ (x,t) \in \Omega_2 \times [0,T]$ and $G := (G_1, G_2, G_3)$ is a function $G : [0,T] \times Y \longrightarrow Y$ define as $[G_j(t,\varphi)](x) := g_j(\overline{u}_2(x,t), \varphi(x,t)),\ \forall\ (t,\varphi) \in [0,T] \times Y$ and $x \in \Omega_2$.

The operator $B$ is an infinitesimal generator of a $C_0$ semigroup of contractions in $Y$, and $G$ is well-defined, continuous and measurable in $t$ for $t \in [0,T]$ due to (2.12). Regarding the lack of Lipschitz continuity of $G$, we use the same truncation argument as the Theorem (2.1), defining $G^N(t,\varphi) = G(t, T^N\varphi)$, with $T^N$ defined in (2.4). We observe that $G^N$ is continuous Lipschitz continuous thanks to the property (2.12).

The rest of the proof, that is $L^\infty(\Omega_{2,T})$ and $L^\infty(0,T; H^1(\Omega_2))$ estimates, nonnegativity, existence and uniqueness of the solution $\boldsymbol{v}^N$ of the truncated problem, and the fact that $G = G^N$ and $\boldsymbol{v} = \boldsymbol{v}^N$ for $N$ large enough, is equivalent to the proof of the Theorem 2.1.

Finally, $c$ is the result of solving a cauchy problem and the spaces where it belongs are results of the properties of $v_1$ and $v_2$. ■

**Theorem 2.4.** *Let $u_2$ in $H^1(0,T;L^2(\Omega_1))$ taken from the solution of* (1.1) *under the hypotheses of Theorem 2.1 and $\overline{u}_2$ in $H^1(0,T;L^2(\Omega_2))$ defined by* (2.11)*. Then, the problem* (1.2) *is well-posedness. More precisely, the solution of* (1.2) *is unique and Lipschitz continuous with respect to its parameters $a \in Q$ and the initial condition. Moreover, if $\boldsymbol{v}^i := (v_1^i, v_2^i, v_3^i)$ with $i = 1,2$ are solutions of* (1.2)*, with initial condition $\boldsymbol{v}_0^1$ and $\boldsymbol{v}_0^2$ respectively, and with a set of parameters $a = a_1$ and $a = a_2$, respectively, then*

$$\|\boldsymbol{v}^1(t) - \boldsymbol{v}^2(t)\|_Y \leqslant C\left(\|\boldsymbol{v}_0^1 - \boldsymbol{v}_0^2\|_Y + \int_0^t \|\overline{u}_2^1(r) - \overline{u}_2^2(r)\|_{L^2(\Omega_2)} dr\right), \qquad 0 < t < T, \tag{2.15}$$

*where $C$ is a constant depending on $T$ and $L = \max\limits_{|s| \leqslant M_1,\ |t| \leqslant M_2} \{\nabla_{\mathbf{u}} F(s), \nabla_{\boldsymbol{v}} G(t)\}$, with*
*$M_1 := \max\{\|\mathbf{u}^1\|_{L^\infty}, \|\mathbf{u}^2\|_{L^\infty}\}$ and $M_2 := \max\{\|\boldsymbol{v}^1\|_{L^\infty}, \|\boldsymbol{v}^2\|_{L^\infty}\}$*

**Remark 2.1.** *The dependence of $\boldsymbol{v}$ with respect to $a$ is obviously indirect and is obtained through $u_2$: $\boldsymbol{v}(a) = \boldsymbol{v}(\overline{u}_2(a))$. Also, due to the explicit form of c, its dependence on $a$ is obtained through $v_1$ and $v_2$.*

*Proof of Theorem 2.4.* The proof is similar to that of Theorem 2.2, obtaining an inequality indirectly through $u_2$. More precisely, let $\mathbf{u}^1 = (u_1^1, u_2^1) \in X$ the solution of (1.1) with $a_1$, $\mathbf{u}^2 = (u_1^2, u_2^2) \in X$ the solution of (1.1) with $a_2$, $\boldsymbol{v}^1 = (v_1^1, v_2^1) \in X$ the solution of (1.2) with $\boldsymbol{v}_0^1$ and $\overline{u}_2^1$, $\boldsymbol{v}^2 = (v_1^2, v_2^2) \in X$ the solution of (1.2) with $\boldsymbol{v}_0^2$ and $\overline{u}_2^2$, then we obtain

$$\|\boldsymbol{v}^1(t) - \boldsymbol{v}^2(t)\|_Y \leqslant \|\boldsymbol{v}_0^1 - \boldsymbol{v}_0^2\|_Y + \int_0^t \|\overline{u}_2^1(r) - \overline{u}_2^2(r)\|_{L^2(\Omega_2)} dr$$

$$+\int_0^t \left( \|\boldsymbol{v}_0^1 - \boldsymbol{v}_0^2\|_Y + \int_0^s \|\overline{u}_2^1(r) - \overline{u}_2^2(r)\|_{L^2(\Omega_2)} dr \right) Le^{L(t-s)} ds$$

The proof concludes using the Lipschitz continuity of $\overline{u}_2$ with respect to $a$ which is deduced from Theorem 2.2. ■

## 3. Existence of optimal solution

As mentioned, it is desired to decrease the impact of the disease on the human population by minimizing the attack rate in humans ($v_3$) and the cumulative incidence in humans ($c$), through controlling the disease transmission coefficient ($\beta_{bb}(t,a)$) in the disease transmitting population, this translates into minimizing the functional (1.6), i.e, solve the following optimization problem

$$\begin{array}{ll} \min & J(w(\cdot,T),a) := \dfrac{1}{|\Omega_2|} \displaystyle\int_{\Omega_2} \{Av_3(x,T) + Bc(x,T)\} \, dx + \varepsilon g(a) \\ (w,a) \in W_{ad} \times Q & \\ \text{subject to:} & w \text{ satisfy (1.1) and (1.2)} \end{array} \tag{3.1}$$

here $w := (u_1, u_2, v_1, v_2, v_3, c)$, $J : [L^\infty(\Omega_{1,T}) \cap L^2(0,T; L^2(\Omega_1))]^2 \times [L^\infty(\Omega_{2,T}) \cap L^2(0,T; L^2(\Omega_2))]^4 \times \mathbb{R}^3 \to \mathbb{R}$, $g(a) := (1-\alpha)^2 + \gamma^2$ and $W_{ad} := [W_1]^2 \times [W_2]^4$ where

$$W_j := \left\{ \mathbf{w} \in L^\infty(\Omega_{j,T}) \cap L^2(0,T; L^2(\Omega_j)) \middle| \, \mathbf{w}(x,t) \geq 0, \ (x,t) \in \Omega_{j,T} \right\}$$

By Theorem 2.2 and Theorem 2.4 the solution of (1.1) and (1.2) depends on the parameter $a$ hence $w = w(a)$, therefore (3.1) can be formulated in an equivalent way by

$$\begin{array}{ll} \min & J(a) \\ a \in Q & \\ \text{s. t:} & w := w(a) \text{ satisfy (1.1) and (1.2)} \end{array} \tag{3.2}$$

**Theorem 3.1.** *Under the assumptions of Theorems* (2.1) *and* (2.3)*. If $J \geq 0$ then the optimization problem* (3.2) *has at least one optimal solution $a^* \in Q$.*

**Remark 3.1.** *The non-negativity of $J$ is satisfied since $w \in W_{ad}$ by Theorems* (2.1) *and* (2.3)*, and if $A, B$ and $\varepsilon$ are chosen appropriately.*

*Proof of Theorem 3.1.*

Let

$$\begin{aligned} f(t,a,u_1,u_2) &= -\frac{\beta_{bb}(t,a)}{N_1} u_1 u_2 + \gamma_b u_2 \\ g_1(\overline{u}_2, v_1, v_2, v_3) &= \left( -\frac{\beta_{hh}}{N_2} v_2 - \frac{\beta_{bh}}{N_2} \overline{u}_2 \right) v_1 \\ g_2(\overline{u}_2, v_1, v_2, v_3) &= \left( \frac{\beta_{hh}}{N_2} v_1 - \gamma_h \right) v_2 + \frac{\beta_{bh}}{N_2} v_1 \overline{u}_2 \\ g_3(\overline{u}_2, v_1, v_2, v_3) &= \gamma_h v_2 \\ h(\overline{u}_2, v_1, v_2, v_3) &= \frac{\beta_{hh}}{N_2} v_1 v_2 \end{aligned}$$

Since $J \geq 0$ and $Q$ is nonempty, the infimun

$$J^* := \inf_{a \in Q} J(w, a)$$

exists and hence it is possible to find a minimizing sequence $(a_k) \subseteq Q$ such that

$$J^* := \lim_{k \to \infty} J(w_k, a_k)$$

The sequence $a_k$ is bounded because $Q \subseteq \mathbb{R}^3$ is bounded, then exists a convergent subsequence $a_{k_n} \to a^* \in \mathbb{R}^3$ and how $Q$ is closed then $a^* \in Q$.

On the other hand, by Theorem (2.1) the solution of (1.1) is

$$u^i(t) = S_i(t)u_0^i + (-1)^{1+i} \int_0^t S_i(t-r) f(r, a, u_1, u_2)\, dr \quad , \quad t \in [0, T] \quad , \quad i = 1, 2.$$

and by Theorem (2.2) the solution of (1.2) is

$$v_j(t) = T_j(t)v_j^0 + \int_0^t T_j(t-r) g_j(\overline{u}_2, v_1, v_2, v_3)\, dr \quad , \quad t \in [0, T] \quad , \quad j = 1, 2, 3$$

and

$$c(t) = c^0 + \int_0^t h(\overline{u}_2, v_1, v_2, v_3)\, dr \quad , \quad t \in [0, T]$$

here $S_i$ is the $C_0$ semigroup of contractions on $L^2(\Omega_1)$ generated by the operator $A$ used on Theorem (2.1) ($i = 1, 2$) and $T_j$ is the $C_0$ semigroup of contractions on $L^2(\Omega_2)$ generated by the operator $B$ used on Theorem (2.2) ($j = 1, 2, 3$), then is posible to define a sequence on $W_{ad}$:

$$u_i^{k_n}(t) = S_i(t)u_i^0 + (-1)^{1+i} \int_0^t S_i(t-r) f(r, a_{k_n}, u_1^{k_n}, u_2^{k_n}) dr \quad , \quad t \in [0, T] \quad , \quad i = 1, 2.$$

$$v_j^{k_n}(t) = T_j(t)v_j^0 + \int_0^t T_j(t-r) g_j(\overline{u}_2^{k_n}, v_1^{k_n}, v_2^{k_n}, v_3^{k_n}) dr \quad , \quad t \in [0, T] \quad , \quad j = 1, 2, 3$$

$$c^{k_n}(t) = c^0 + \int_0^t h(\overline{u}_2^{k_n}, v_1^{k_n}, v_2^{k_n}, v_3^{k_n}) dr \quad , \quad t \in [0, T]$$

Let $t \in [0, T]$ and defined the solution of (1.1) with $a = a^*$ as

$$u_2^{a^*}(t) = S_2(t)u_2^0 - \int_0^t S_2(t-r) f(r, a^*, u_1^{a^*}, u_2^{a^*}) dr$$

then proceeding as the Theorem (2.2) and using proporties of $f$ is obtained

$$||u_2^{a^*}(t) - u_2^{k_n}(t)||_{L^2(\Omega_1)} \leqslant C \int_0^T |\beta_{bb}(s, a^*) - \beta_{bb}(s, a_{k_n})|^2 ds$$
$$+ C \int_0^T \left( \int_0^s |\beta_{bb}(\tau, a^*) - \beta_{bb}(\tau, a_{k_n})|^2 d\tau \right) e^{C(T-s)}\, ds$$

where $C > 0$ is independent on $a_{k_n}$ (see [18, Appendix B]). By continuity of $\beta_{bb}(t, a)$ we conclude that $u_2^{k_n}(t) \to u_2^{a^*}(t)$ in $L^2(\Omega_1)$ and $\overline{u}_2^{k_n}(t) \to \overline{u}_2^{a^*}(t)$ in $L^2(\Omega_2)$ by definition of $\overline{u}_2$. In addition, if $v_j^{a^*}$ ($j = 1, 2, 3$) is the solution of (1.2) with $a = a^*$ then proceeding as the Theorem (2.4) and using properties of $g_j$ $(j = 1, 2, 3)$ is obtained

$$||v_j^{a^*}(t) - v_j^{k_n}(t)||_{L^2(\Omega_2)} \leqslant L \int_0^T ||\overline{u}_2^{a^*}(r) - \overline{u}_2^{k_n}(r)||_{L^2(\Omega_2)} dr$$
$$+ \int_0^T \left( L \int_0^s ||\overline{u}_2^{a^*}(r) - \overline{u}_2^{k_n}(r)||_{L^2(\Omega_2)} dr \right) L e^{L(T-s)} ds$$

and

$$||v_3^{a^*}(t) - v_3^{k_n}(t)||_{L^2(\Omega_2)} \leq \gamma_h \int_0^T ||v_2^{a^*}(r) - v_2^{k_n}(r)||_{L^2(\Omega_2)} dr$$

where $j = 1, 2$ and $L > 0$ is independent on $a_{k_n}$ (see [18, Appendix B]), therefore $v_j^{k_n}(t) \to v_j^{a^*}(t)$ $(j = 1, 2, 3)$ in $L^2(\Omega_2)$.

Finally, $J(w, a)$ can be written as $J(w, a) = H(w, a) + \varepsilon g(a)$ where $g$ is a continuos function on $a$ and $H(w, a) := \dfrac{1}{|\Omega_2|} \displaystyle\int_{\Omega_2} \{A v_3(x, T) + B c(x, T)\} dx$ satisfies

$$\begin{aligned} |H(w_{a^*}, a^*) - H(w_{k_n}, a_{k_n})| &\leq \frac{1}{|\Omega_2|} \left\{ A \int_{\Omega_2} \left| v_3^{a^*}(x, T) - v_3^{k_n}(x, T) \right| dx \right. \\ &\quad \left. + B \int_{\Omega_2} \left| c^{a^*}(x, T) - c^{k_n}(x, T) \right| dx \right\} \end{aligned}$$

then by Hölder inequality

$$\begin{aligned} &\leq M \left\{ A ||v_3^{a^*}(T) - v_3^{k_n}(T)||_{L^2(\Omega_2)} \right. \\ &\quad \left. + B \int_{\Omega_2} |c^{a^*}(x, T) - c^{k_n}(x, T)| dx \right\} \end{aligned}$$

$M > 0$ and by

$$\int_{\Omega_2} |c^{a^*}(x, T) - c^{k_n}(x, T)| dx \leq N \left( \int_0^T \left\| v_1^{a^*}(r) - v_1^{k_n}(r) \right\|_{L^2(\Omega_2)} dr \right.$$
$$\left. + \int_0^T \left\| v_2^{a^*}(r) - v_2^{k_n}(r) \right\|_{L^2(\Omega_2)} dr \right)$$

$N > 0$ (see [18, Appendix B]), we conclude $J(w_{k_n}(\cdot, T), a_{k_n}) \to J(w_{a^*}(\cdot, T), a^*)$, i.e $J^* = J(w_{a^*}(\cdot, T), a^*)$ ($a^*$ is the optimal).

■

## 4. Calculus of the gradient of the cost function, adjoint problem and necessary optimality condition

Through this section we calculated the continuous gradient of the cost function using the Lagrangian formulation associated to the optimization problem (3.1). This procedure is based on [14], [16], [4] and [17], and generates a system of partial differential equations called adjoint problem, then we prove the existence of its solution and we deduce a necessary condition of optimality for our problem according to [13].

### *4.1. Calculus of the gradient and adjoint problem*

The Lagrangian associated to the optimization problem (3.1) is $\mathcal{L} : W \times Q \times Z^* \to \mathbb{R}$ defined as:

$$\mathcal{L}(w, a, p) := J(w(\cdot, T), a) + E(w, a, p) \tag{4.1}$$

where $w = (u_1, u_2, v_1, v_2, v_3, c)$, $Q \subset \mathbb{R}^3$, $Z^*$ denotes the dual space of $Z := [L^2(0,T; H^1(\Omega_1)^*)]^2 \times [L^2(0,T; H^1(\Omega_2)^*)]^4$, $W := [W_1]^2 \times [W_2]^4$, $W_1 := \{u \mid u \in L^2(0,T; L^2(\Omega_1)), \partial_t u \in L^2(0,T; L^2(\Omega_1)^*)\}$, $W_2 := \{v \mid v \in L^2(0,T; L^2(\Omega_2)), \partial_t v \in L^2(0,T; L^2(\Omega_2)^*)\}$, and by Theorems (2.1) and (2.3), we conclude that $w \in W$. Also, $E(w, a, \lambda) := \langle e(w, a), \lambda \rangle_{Z,Z^*}$ is the variational formulation of the state problem (1.1)-(1.2) considering fixed initial conditions.

Based on [16] and [4], from (4.1) we get a explicit expression for the derivative of the cost function respect to the parameter $a$:

$$\frac{dJ}{da}(w(\cdot,T), a) = \left\langle \frac{\partial \mathcal{L}}{\partial w}(w, a, p), \frac{dw}{da} \right\rangle + \frac{d\mathcal{L}}{da}(w, a, p) \tag{4.2}$$

where it is required that $\frac{\partial \mathcal{L}}{\partial w}(w, a, p) = 0$, i.e, $\frac{\partial J}{\partial w}(w(\cdot,T), a) + \frac{\partial E}{\partial w}(w, a, p) = 0$. Here the variational formulation is

$$\begin{aligned}
E(w, a, p) \quad &:= \quad \langle e(w, a), p \rangle_{Z,Z^*} \\
&= \quad \sum_{i=1}^{2} \left\{ \int_0^T \langle \partial_t u_i(t), p_i(t) \rangle_{H_1^{1*}, H_1^1} dt + \int_0^T \int_{\Omega_1} d_u \, \nabla u_i \nabla p_i \, dx \, dt \right. \\
&\qquad \left. -(-1)^{i+1} \int_0^T \int_{\Omega_1} f(t, a, u_1, u_2) p_i \, dx \, dt \right\} \\
&\qquad + \sum_{j=1}^{3} \left\{ \int_0^T \langle \partial_t v_j(t), p_{j+2}(t) \rangle_{H_2^{1*}, H_2^1} dt + \int_0^T \int_{\Omega_2} d_v \, \nabla v_j \nabla p_{j+2} \, dx \, dt \right. \\
&\qquad \left. - \int_0^T \int_{\Omega_2} g_j(\overline{u}_2, v_1, v_2, v_3, c) p_{j+2} \, dx \, dt \right\} \\
&\qquad + \int_0^T \langle \partial_t c(t), p_6(t) \rangle_{H_2^{1*}, H_2^1} dt - \int_0^T \int_{\Omega_2} h(\overline{u}_2, v_1, v_2, v_3, c) p_6 \, dx \, dt
\end{aligned}$$

with $H_i^1 := H^1(\Omega_i)$, $H_i^{1^*}$ as the dual space of $H_i^1$,

$$\begin{aligned}
f(t, a, u_1, u_2) \quad &= \quad -\frac{\beta_{bb}(t, a)}{N_1} u_1 u_2 + \gamma_b u_2 \\
g_1(\overline{u}_2, v_1, v_2, v_3) \quad &= \quad \left( -\frac{\beta_{hh}}{N_2} v_2 - \frac{\beta_{bh}}{N_2} \overline{u}_2 \right) v_1 \\
g_2(\overline{u}_2, v_1, v_2, v_3) \quad &= \quad \left( \frac{\beta_{hh}}{N_2} v_1 - \gamma_h \right) v_2 + \frac{\beta_{bh}}{N_2} v_1 \overline{u}_2 \\
g_3(\overline{u}_2, v_1, v_2, v_3) \quad &= \quad \gamma_h v_2 \\
h(\overline{u}_2, v_1, v_2, v_3) \quad &= \quad \frac{\beta_{hh}}{N_2} v_1 v_2
\end{aligned}$$

where $\overline{u}_2$ is defined as (2.11), and $p = (p_m)_{m=1}^6$ is the vector of the Lagrange multipliers with $p_i \in L^2(0,T; H^1(\Omega_1))$, $p_j \in L^2(0,T; H^1(\Omega_2))$ for $i = 1, 2$ and $j = 3, 4, 5, 6$, respectively.

By Green's identity $\int_{\Omega_i} \nabla u \nabla p \, dx = \int_{\partial\Omega_i} u \nabla p \cdot \eta \, dS - \int_{\Omega_i} u \Delta p \, dx$ and by the embedding $H^1(\Omega_i) \hookrightarrow L^2(\Omega_i) \hookrightarrow H^1(\Omega_i)^*$:

$$(u(T), p(T))_{L^2} - (u(0), p(0))_{L^2} = \int_0^T \langle \partial_t u(t), p(t) \rangle_{H_i^{1*}, H_i^1} dt + \int_0^T \langle \partial_t p(t), u(t) \rangle_{H_i^{1*}, H_i^1} dt$$

we get

$$
\begin{aligned}
E(w,a,p) \quad = \quad & \sum_{i=1}^{2}\Bigg\{-\int_0^T \langle \partial_t p_i(t), u_i(t)\rangle_{H_1^{1*},H_1^1} dt + \int_0^T \left(\int_{\partial\Omega_1} d_u\ u_i\ \nabla p_i\cdot\eta_1\ dS\right) dt \\
& -\int_0^T \left(\int_{\Omega_1} d_u\ u_i\ \Delta p_i\ dx\right) dt - (-1)^{i+1}\int_0^T\int_{\Omega_1} f(t,a,u_1,u_2)p_i\ dx\ dt \\
& -\int_{\Omega_1} u_i(0)p_i(0)\ dx + \int_{\Omega_1} u_i(T)p_i(T)\ dx\Bigg\} \\
& +\sum_{j=1}^{3}\Bigg\{-\int_0^T \langle \partial_t p_{j+2}(t), v_j(t)\rangle_{H_2^{1*},H_2^1} dt + \int_0^T \left(\int_{\partial\Omega_2} d_v\ v_j\ \nabla p_{j+2}\cdot\eta_2\ dS\right) dt \\
& -\int_0^T \left(\int_{\Omega_2} d_v\ v_j\ \Delta p_{j+2}\ dx\right) dt - \int_0^T\int_{\Omega_2} g_j(\overline{u}_2,v_1,v_2,v_3,c)p_{j+2}\ dx\ dt \\
& -\int_{\Omega_2} v_j(0)p_{j+2}(0)\ dx + \int_{\Omega_2} v_j(T)p_{j+2}(T)\ dx\Bigg\} \\
& -\int_0^T \langle \partial_t p_6(t), c(t)\rangle_{H_2^{1*},H_2^1} dt - \int_0^T\int_{\Omega_2} h(\overline{u}_2,v_1,v_2,v_3,c)p_6\ dx\ dt \\
& -\int_{\Omega_2} c(0)p_6(0)\ dx + \int_{\Omega_2} c(T)p_6(T)\ dx
\end{aligned}
$$

and derivating (4.1) in an arbitrary direction, for example $\delta u_2 \in W_1$ we obtain

$$
\left\langle \frac{\partial \mathcal{L}}{\partial u_2}(w,a,p), \delta u_2\right\rangle \quad = \quad \left\langle \frac{\partial J}{\partial u_2}(w(\cdot,T),a) + \frac{\partial E}{\partial u_2}(w,a,p), \delta u_2\right\rangle
$$

here

$$
\begin{aligned}
\left\langle \frac{\partial E}{\partial u_2}(w,a,p), \delta u_2\right\rangle \quad = \quad & -\int_0^T \langle \partial_t p_2(t), \delta u_2(t)\rangle_{H_1^{1*},H_1^1} dt + \int_0^T \left(\int_{\partial\Omega_1} d_u\ \delta u_2\ \nabla p_2\cdot\eta_1\ dS\right) dt \\
& -\int_0^T \left(\int_{\Omega_1} d_u\ \delta u_2\ \Delta p_2\ dx\right) dt - \int_0^T\int_{\Omega_1} \frac{\partial f}{\partial u_2}(t,a,u_1,u_2)p_1\delta u_2\ dx\ dt \\
& +\int_0^T\int_{\Omega_1} \frac{\partial f}{\partial u_2}(t,a,u_1,u_2)p_2\delta u_2\ dx\ dt \\
& -\int_0^T\int_{\Omega_2} \frac{\partial g_1}{\partial u_2}(\overline{u}_2,v_1,v_2,v_3,c)p_3\delta u_2\ dx\ dt \\
& -\int_0^T\int_{\Omega_2} \frac{\partial g_2}{\partial u_2}(\overline{u}_2,v_1,v_2,v_3,c)p_4\delta u_2\ dx\ dt \\
& -\int_{\Omega_1} \delta u_2(0)p_2(0)\ dx + \int_{\Omega_1} \delta u_2(T)p_2(T)\ dx
\end{aligned}
$$

such as $u_2(0) = u_2^0(x)$ is fixed then $\delta u_2(0) = 0$. Also, the definition of $\overline{u}_2$ implies that $\delta u_2$ participates on $\widetilde{\Omega}$, then $g_1$ and $g_2$ are redefined on $\widetilde{\Omega} \subset \Omega_1$ through the extension by zero of $v_1$, $p_3$ and $p_4$ on $\Omega_1 \setminus \widetilde{\Omega}$

$$
\begin{aligned}
= & -\int_0^T \int_{\Omega_1} \left\{ \partial_t p_2 + d_u \,\Delta p_2 - \left( -\frac{\beta_{bb}(t,a)}{N_1} u_1 + \gamma_b \right)(p_2 - p_1) \right. \\
& \left. + \frac{\beta_{bh}}{N_2} \overline{v}_1 (\overline{p}_4 - \overline{p}_3) \right\} \delta u_2 \; dx \; dt \\
& + \int_0^T \left( \int_{\partial\Omega_1} \delta u_2 \; d_u \; \nabla p_2 \cdot \eta_1 \; dS \right) dt \\
& + \int_{\Omega_1} \delta u_2(T) p_2(T) \; dx
\end{aligned}
$$

consequently

$$
\begin{aligned}
\left\langle \frac{\partial \mathcal{L}(w,a,p)}{\partial u_2}, \delta u_2 \right\rangle = & \frac{\partial J(w(\cdot,T),a)}{\partial u_2} \delta u_2(T) \\
& -\int_0^T \int_{\Omega_1} \left\{ \partial_t p_2 + d_u \,\Delta p_2 - \left( -\frac{\beta_{bb}(t,a)}{N_1} u_1 + \gamma_b \right)(p_2 - p_1) \right. \\
& \left. + \frac{\beta_{bh}}{N_2} \overline{v}_1 (\overline{p}_4 - \overline{p}_3) \right\} \delta u_2 \; dx \; dt \\
& + \int_0^T \left( \int_{\partial\Omega_1} \delta u_2 \; d_u \; \nabla p_2 \cdot \eta_1 \; dS \right) dt \\
& + \int_{\Omega_1} \delta u_2(T) p_2(T) \; dx
\end{aligned}
$$

where by definition of $J$ we get $\dfrac{\partial J(w(\cdot,T),a)}{\partial u_2} = 0$, and as was mentioned we wish to find $p := (p_m)_{m=1}^6$ such that

$$
\frac{\partial \mathcal{L}(w,a,p)}{\partial u_2} = 0
$$

therefore

$$
\int_0^T \int_{\Omega_1} \left\{ \partial_t p_2 + d_u \,\Delta p_2 - \left( -\frac{\beta_{bb}(t,a)}{N_1} u_1 + \gamma_b \right)(p_2 - p_1) + \frac{\beta_{bh}}{N_2} \overline{v}_1 (\overline{p}_4 - \overline{p}_3) \right\} \; dx \; dt = 0
$$

$$
\int_0^T \int_{\partial\Omega_1} \nabla p_2 \cdot \eta_1 \; dS dt = 0
$$

$$
\int_{\Omega_1} p_2(T) \; dx = 0
$$

consequently $p_2$ satisfies the following partial differential equation (adjoint equation):

$$
\begin{aligned}
& \partial_t p_2 = -d_u \Delta p_2 + \left( -\frac{\beta_{bb}(t,a)}{N_1} u_1 + \gamma_b \right)(p_2 - p_1) - \frac{\beta_{bh}}{N_2} \overline{v}_1 (\overline{p}_4 - \overline{p}_3) & , \quad \text{in } \Omega_{1,T} \\
& \nabla p_2 \cdot \eta_1 = 0 & , \quad \text{on } \sum_{1,T} \\
& p_2(x,T) = 0 & , \quad \text{in } \Omega_1
\end{aligned}
$$

Proceeding analogously for the remaining directions, we generates the following coupled system of partial differential equations called the adjoint problem:

$$
\begin{cases}
\partial_t p_6 = 0 & \text{,in } \Omega_{2,T} \\
p_6(x,T) = p_6^T(x) & \text{,in } \Omega_2
\end{cases}
\tag{4.3}
$$

$$
\begin{cases}
\partial_t p_5 = -d_v \Delta p_5 & ,\text{in } \Omega_{2,T} \\
\partial_t p_4 = -d_v \Delta p_4 - \dfrac{\beta_{hh}}{N_2} v_1 \left(p_6 + p_4 - p_3\right) - \gamma_h \left(p_5 - p_4\right) & ,\text{in } \Omega_{2,T} \\
\partial_t p_3 = -d_v \Delta p_3 - \dfrac{\beta_{hh}}{N_2} v_2 \left(p_6 + p_4 - p_3\right) - \dfrac{\beta_{bh}}{N_2} \overline{u}_2 \left(p_4 - p_3\right) & ,\text{in } \Omega_{2,T} \\
\nabla p_j \cdot \eta_2 = 0 & ,\text{on } \sum\limits_{2,T} \text{, for } j = 3,4,5. \\
p_j(x,T) = p_j^T(x) & ,\text{in } \Omega_2 \text{ , for } j = 3,4,5.
\end{cases}
\tag{4.4}
$$

with $p_6^T(x) = -\dfrac{B}{|\Omega_2|}$, $p_5^T(x) = -\dfrac{A}{|\Omega_2|}$, $p_4^T(x) = p_3^T(x) = 0$ for all $x \in \Omega_2$. And in $\Omega_1$ we get

$$
\begin{cases}
\partial_t p_2 = -d_u \Delta p_2 - \left(\dfrac{\beta_{bb}(t,a)}{N_1} u_1 - \gamma_b\right)\left(p_2 - p_1\right) - \dfrac{\beta_{bh}}{N_2} \overline{v}_1 \left(\overline{p}_4 - \overline{p}_3\right) & ,\text{ in } \Omega_{1,T} \\
\partial_t p_1 = -d_u \Delta p_1 - \dfrac{\beta_{bb}(t,a)}{N_1} u_2 \left(p_2 - p_1\right) & ,\text{ in } \Omega_{1,T} \\
\nabla p_j \cdot \eta_1 = 0 & ,\text{ on } \sum\limits_{1,T} \text{ , for } j = 1,2. \\
p_j(x,T) = 0 & ,\text{ in } \Omega_1 \text{ , for } j = 1,2.
\end{cases}
\tag{4.5}
$$

where $\overline{v}_1, \overline{p}_4, \overline{p}_3$ is the prolongation by zero on $\Omega_1 \setminus \widetilde{\Omega}$.

**Theorem 4.1.** *Assume that the hypothesis of* (2.1) *and* (2.3) *are satisfied and consider that* $u_1, u_2, v_1, v_2, v_3, c$ *is the solution of* (1.1) - (1.2)*. Then the adjoint problem presented* (4.3) - (4.4) - (4.5) *admits a unique strong solution* $p_i \in H^1([0,T]; L^2(\Omega_1))$ *(*$i = 1,2$*) and* $p_j \in H^1([0,T]; L^2(\Omega_2))$ *(*$j = 3,4,5,6$*).*

*Proof.* From equation (4.3) we get $p_6(x,t) = -\dfrac{B}{|\Omega_2|}$ for all $(x,t) \in \Omega_{2,T}$, then $p_6$ is a known function on (4.4). Consider the following change of variable $s = T - t$ with $t \in [0,T]$ and the following change of function $q_j(x,s) = p_j(x, T-s)$ where $p_j(x,T-s) = p_j(x,t)$ for $(x,t) \in \Omega_{2,T}$ ($j = 3,4,5$), hence the system (4.4) is equivalent to

$$
\begin{cases}
\partial_s q_5 = d_v \Delta q_5 & ,\text{in } \Omega_{2,T} \\
\partial_s q_4 = d_v \Delta q_4 - \dfrac{\beta_{hh}}{N_2} v_1 \left(q_6 + p_4 - q_3\right) - \gamma_h \left(q_5 - q_4\right) & ,\text{in } \Omega_{2,T} \\
\partial_s q_3 = d_v \Delta q_3 - \dfrac{\beta_{hh}}{N_2} v_2 \left(q_6 + p_4 - q_3\right) - \dfrac{\beta_{bh}}{N_2} \overline{u}_2 \left(q_4 - q_3\right) & ,\text{in } \Omega_{2,T} \\
\nabla q_j \cdot \eta_2 = 0 & ,\text{on } \sum\limits_{2,T} \text{, for } j = 3,4,5. \\
q_j(x,0) = q_j^T(x) & ,\text{in } \Omega_2 \text{ , for } j = 3,4,5.
\end{cases}
\tag{4.6}
$$

then proceeding as Theorem (2.3) (with the same operator $B$) we obtained that the system (4.6) is equivalent to

$$
\begin{cases}
\dfrac{d}{ds}\mathbf{q}(s) = B\mathbf{q}(s) + K(s, \mathbf{q}(s)) & , s \in [0,T] \\
\mathbf{q}(0) = \mathbf{q}_0 & , \mathbf{q}_0 \in X
\end{cases}
\tag{4.7}
$$

where $X = [L^2(\Omega_2)]^3$, $\mathbf{q}_0 = (p_{j+2}^T(x))_{j=1}^3 \in D(B) \cap [L^\infty(\Omega_2)]^3$, $\mathbf{q}(s) \in X$ is such that $[\mathbf{q}(s)](x) := (q_3(x,s), q_4(x,s), q_5(x,s))$ for $(x,s) \in \Omega_2 \times [0,T]$ and $K := (K_j)_{j=1}^3$ is defined by

$$[K_1(s,\varphi_3,\varphi_4,\varphi_5)](x) := 0$$
$$[K_2(s,\varphi_3,\varphi_4,\varphi_5)](x) := -\frac{\beta_{hh}}{N_2}v_1\left(p_6+\varphi_4(x)-\varphi_3(x)\right) - \gamma_h\left(\varphi_5(x)-\varphi_4(x)\right)$$
$$[K_3(s,\varphi_3,\varphi_4,\varphi_5)](x) := -\frac{\beta_{hh}}{N_2}v_2\left(p_6+\varphi_4(x)-\varphi_3(x)\right) - \frac{\beta_{bh}}{N_2}\overline{u}_2\left(\varphi_4(x)-\varphi_3(x)\right)$$

for $(s,\varphi) \in [0,T]\times X$ and $x \in \Omega_2$.

Also, $K : [0,T]\times X \longrightarrow X$ is well-posed because $v_1, v_2, \overline{u}_2 \in L^2(\Omega_2)$, is measurable in $t$ and is lineal (then Lipschitz continuous), hence (4.7) satisfy the hypothesis of the Theorem [3, Theorem 2.1] and consequently it admits a unique strong solution $q_i \in H^1([0,T];L^2(\Omega_2))$ $(j = 3,4,5)$ wich implies that (4.4) has solution.

Now, let $i \in \{3,4\}$, $\widetilde{\Omega} := \Omega_1 \cap \Omega_2$ and

$$\overline{p}_i(x,t) = \begin{cases} p_i(x,t) & , \quad (x,t) \in \widetilde{\Omega}\times[0,T] \\ 0 & , \quad (x,t) \in (\Omega_1 \setminus \widetilde{\Omega})\times[0,T] \end{cases}$$

then $\overline{p}_i \in H^1([0,T];L^2(\Omega_1))$. (4.5) is reaction-diffusion system with a linear function on the right-side, we can proceeding as above but this time using the operator $A$ defined on Theorem (2.1), then the system admits a unique strong solution $p_i \in H^1([0,T];L^2(\Omega_1))$ $(i = 1,2)$.

■

Finally, in view of $w := w(a)$ then the cost function can be rewritten as $\widetilde{J}(a) := J(w(\cdot,T),a)$, also consider (4.2) and $p$ solution of adjoint problem, the gradient of the cost function with respect to our control variable $a$ is

$$\begin{aligned} \frac{d\widetilde{J}(a)}{da} &= \frac{d\mathcal{L}}{da}(w,a,p) = \frac{\partial J}{\partial a}(w(\cdot,T),a) + \frac{\partial E}{\partial a}(w,a,p) \\ &= \varepsilon\nabla g(a) - \int_0^T \nabla\beta_{bb}(t,a)\left(\int_{\Omega_1}\frac{u_1u_2}{N_1}(p_2-p_1)dx\right)dt \end{aligned} \tag{4.8}$$

with $p_i$ and $u_i$ solution of the adjoint problem and direct problem on $\Omega_1$ $(i = 1,2)$, respectively.

### 4.2. *Necessary optimality condition*

**Theorem 4.2.** *Assume the hypothesis of Theorems* (2.1), (2.3), (3.1) *and* (4.1)*, and let* $w^* = (u_1^*, u_2^*, v_1^*, v_2^*, v_3^*, c^*)$ *the solution of* (1.1)-(1.2) *for* $a = a^*$*. If* $(w^*, a^*)$ *is a local solution of the optimization problem* (3.2) *then the following inequality:*

$$\varepsilon\nabla g(a^*) - \int_0^T \nabla\beta_{bb}(t,a^*)\left(\int_{\Omega_1}\frac{u_1^*u_2^*}{N_1}(p_2-p_1)dx\right)dt \geq 0$$

*is satisfied, where* $p_1$ *and* $p_2$ *are the solutions of the adjoint problem* (4.5) *on* $a^*$*.*

*Proof.* Let $\epsilon > 0$, $a^*$ an optimal solution of (3.2) and $a_0 \in Q$. Defined $a^\epsilon = (1-\epsilon)a^* + \epsilon a_0 \in Q$, $w^\epsilon = (u_1^\epsilon, u_2^\epsilon, v_1^\epsilon, v_2^\epsilon, v_3^\epsilon, c^\epsilon)$ as the solution of (1.1)-(1.2) for $a = a^\epsilon$, $w^* = (u_1^*, u_2^*, v_1^*, v_2^*, v_3^*, c^*)$ as the solution of (1.1)-(1.2) for $a = a^*$, and introduce the notation

$$z_i^\epsilon = \frac{u_i^\epsilon - u_i^*}{\epsilon}\ ,\quad i = 1,2. \quad ; \quad z_j^\epsilon = \frac{v_j^\epsilon - v_j^*}{\epsilon}\ ,\quad j = 3,4,5. \quad ; \quad z_6^\epsilon = \frac{c^\epsilon - c^*}{\epsilon}$$

As $a^*$ is an optimal then $J(w^*(\cdot,T),a^*) \leq J(w^\epsilon(\cdot,T),a^\epsilon)$ for all $\epsilon > 0$. Dividing by $\epsilon > 0$ and taking the limit $\epsilon \to 0$:

$$\frac{A}{|\Omega_2|}\int_{\Omega_2} z_3(x,T)dx + \frac{B}{|\Omega_2|}\int_{\Omega_2} z_6(x,T)dx + \varepsilon\nabla g(a^*) \geq 0 \tag{4.9}$$

Substracting system (1.1)-(1.2) for $a^*$ from (1.1)-(1.2) for $a^\epsilon$ and dividing by $\epsilon$ it is obtained

$$\begin{cases} \partial_t z_1^\epsilon = d_u\Delta z_1^\epsilon - \left(\dfrac{\beta_{bb}(t,a^\epsilon)-\beta_{bb}(t,a^*)}{\epsilon}\right)\dfrac{u_1^*u_2^*}{N_1} - \dfrac{\beta_{bb}(t,a^\epsilon)}{N_1}u_1^*z_2^\epsilon & \\ \qquad -\dfrac{\beta_{bb}(t,a^\epsilon)}{N_1}u_2^\epsilon z_1^\epsilon + \gamma_b z_2^\epsilon & \text{, in } \Omega_{1,T} \\ \partial_t z_2^\epsilon = d_u\Delta z_2^\epsilon + \left(\dfrac{\beta_{bb}(t,a^\epsilon)-\beta_{bb}(t,a^*)}{\epsilon}\right)\dfrac{u_1^*u_2^*}{N_1} + \dfrac{\beta_{bb}(t,a^\epsilon)}{N_1}u_1^*z_2^\epsilon & \\ \qquad +\dfrac{\beta_{bb}(t,a^\epsilon)}{N_1}u_2^\epsilon z_1^\epsilon - \gamma_b z_2^\epsilon & \text{, in } \Omega_{1,T} \\ \nabla z_1^\epsilon\cdot\eta_1 = \nabla z_2^\epsilon\cdot\eta_1 = 0 & \text{, on } \sum_{1,T} \\ z_1^\epsilon(x,0) = z_2^\epsilon(x,0) = 0 & \text{, in } \Omega_1 \end{cases} \tag{4.10}$$

$$\begin{cases} \partial_t z_3^\epsilon = d_v\Delta z_3^\epsilon - \dfrac{\beta_{hh}}{N_2}\left(v_2^\epsilon z_3^\epsilon + v_1^* z_4^\epsilon\right) - \dfrac{\beta_{bh}}{N_2}\left(\overline{u}_2^\epsilon z_3^\epsilon + v_1^*\overline{z}_2^\epsilon\right) & \text{, in } \Omega_{2,T} \\ \partial_t z_4^\epsilon = d_v\Delta z_4^\epsilon + \dfrac{\beta_{hh}}{N_2}\left(v_2^\epsilon z_3^\epsilon + v_1^* z_4^\epsilon\right) + \dfrac{\beta_{bh}}{N_2}\left(\overline{u}_2^\epsilon z_3^\epsilon + v_1^*\overline{z}_2^\epsilon\right) - \gamma_h z_4^\epsilon & \text{, in } \Omega_{2,T} \\ \partial_t z_5^\epsilon = d_v\Delta z_5^\epsilon + \gamma_h z_4^\epsilon & \text{, in } \Omega_{2,T} \\ \partial_t z_6 = \dfrac{\beta_{hh}}{N_2}\left(v_2^\epsilon z_3^\epsilon + v_1^* z_4^\epsilon\right) & \text{,in } \Omega_{2,T} \\ \nabla z_3^\epsilon\cdot\eta_2 = \nabla z_4^\epsilon\cdot\eta_2 = \nabla z_5^\epsilon\cdot\eta_2 = 0 & \text{, on } \sum_{2,T} \\ z_j^\epsilon(x,0) = 0 \;\; (j=3,\ldots,6) & \text{, in } \Omega_2 \end{cases} \tag{4.11}$$

here $\overline{u}_2^\epsilon$ and $\overline{z}_2$ denote the prolongation by zero of $u_2^\epsilon$ and $z_2$ on $\Omega_2 \setminus \widetilde{\Omega}$. Passing to the limit formally as $\epsilon \to 0$ in (4.10) - (4.11):

$$\begin{cases} \partial_t z_1 = d_u\Delta z_1 - \nabla\beta_{bb}(t,a^*)\dfrac{u_1^*u_2^*}{N_1} - \dfrac{\beta_{bb}(t,a^*)}{N_1}u_1^*z_2 - \dfrac{\beta_{bb}(t,a^*)}{N_1}u_2^*z_1 + \gamma_b z_2 & \text{, in } \Omega_{1,T} \\ \partial_t z_2 = d_u\Delta z_2 + \nabla\beta_{bb}(t,a^*)\dfrac{u_1^*u_2^*}{N_1} + \dfrac{\beta_{bb}(t,a^*)}{N_1}u_1^*z_2 + \dfrac{\beta_{bb}(t,a^*)}{N_1}u_2^*z_1 - \gamma_b z_2 & \text{, in } \Omega_{1,T} \\ \nabla z_1\cdot\eta_1 = \nabla z_2\cdot\eta_1 = 0 & \text{, on } \sum_{1,T} \\ z_1(x,0) = z_2(x,0) = 0 & \text{, in } \Omega_1 \end{cases} \tag{4.12}$$

$$
\begin{cases}
\partial_t z_3 = d_v \Delta z_3 - \dfrac{\beta_{hh}}{N_2}\left(v_2^* z_3 + v_1^* z_4\right) - \dfrac{\beta_{bh}}{N_2}\left(\overline{u}_2^* z_3 + v_1^* \overline{z}_2\right) & \text{, in } \Omega_{2,T} \\
\partial_t z_4 = d_v \Delta z_4 + \dfrac{\beta_{hh}}{N_2}\left(v_2^* z_3 + v_1^* z_4\right) + \dfrac{\beta_{bh}}{N_2}\left(\overline{u}_2^* z_3 + v_1^* \overline{z}_2\right) - \gamma_h z_4 & \text{, in } \Omega_{2,T} \\
\partial_t z_5 = d_v \Delta z_5 + \gamma_h z_4 & \text{, in } \Omega_{2,T} \\
\partial_t z_6 = \dfrac{\beta_{hh}}{N_2}\left(v_2^* z_3 + v_1^* z_4\right) & \text{,in } \Omega_{2,T} \\
\nabla z_3 \cdot \eta_2 = \nabla z_4 \cdot \eta_2 = \nabla z_5 \cdot \eta_2 = 0 & \text{, on } \sum_{2,T} \\
z_j(x,0) = 0 \;\; (j = 3, \dots, 6) & \text{, in } \Omega_2
\end{cases}
\tag{4.13}
$$

Using (4.4) - (4.5) on $a^*$ and (4.12) - (4.13):

$$
\begin{cases}
p_1 \partial_t z_1 + z_1 \partial_t p_1 = & d_u(p_1 \Delta z_1 - z_1 \Delta p_1) - \nabla \beta_{bb}(t,a^*) \dfrac{u_1^* u_2^*}{N_1} p_1 - \dfrac{\beta_{bb}(t,a^*)}{N_1} u_1^* z_2 p_1 \\
& -\dfrac{\beta_{bb}(t,a^*)}{N_1} u_2^* z_1 p_2 + \gamma_b z_2 p_1 \\
p_2 \partial_t z_2 + z_2 \partial_t p_2 = & d_u(p_2 \Delta z_2 - z_2 \Delta p_2) + \nabla \beta_{bb}(t,a^*) \dfrac{u_1^* u_2^*}{N_1} p_2 + \dfrac{\beta_{bb}(t,a^*)}{N_1} u_2^* z_1 p_2 \\
& +\dfrac{\beta_{bb}(t,a^*)}{N_1} u_1^* z_2 p_1 - \gamma_b z_2 p_1 - \dfrac{\beta_{bh}}{N_1} \overline{v}_1^* z_2 (\overline{p}_4 - \overline{p}_3) \\
p_3 \partial_t z_3 + z_3 \partial_t p_3 = & d_v(p_3 \Delta z_3 - z_3 \Delta p_3) - \dfrac{\beta_{hh}}{N_2} v_1^* z_4 p_3 - \dfrac{\beta_{bh}}{N_2} v_1^* \overline{z}_2 p_3 \\
& -\dfrac{\beta_{hh}}{N_2} v_2^* z_3 \left(p_4 + p_6\right) - \dfrac{\beta_{bh}}{N_2} \overline{u}_2^* z_3 p_4 \\
p_4 \partial_t z_4 + z_4 \partial_t p_4 = & d_v(p_4 \Delta z_4 - z_4 \Delta p_4) + \dfrac{\beta_{hh}}{N_2} v_2^* z_3 p_4 + \dfrac{\beta_{bh}}{N_2} \overline{u}_2^* z_3 p_4 + \dfrac{\beta_{bh}}{N_2} v_1^* \overline{z}_2 p_4 \\
& -\dfrac{\beta_{hh}}{N_2} v_1^* z_4 \left(p_6 - p_3\right) - \gamma_h z_4 p_5 \\
p_5 \partial_t z_5 + z_5 \partial_t p_5 = & d_v(p_5 \Delta z_5 - z_5 \Delta p_5) + \gamma_h z_4 p_5 \\
p_6 \partial_t z_6 + z_6 \partial_t p_6 = & \dfrac{\beta_{hh}}{N_2} p_6 \left(v_2^* z_3 + v_1^* z_4\right)
\end{cases}
\tag{4.14}
$$

Therefore

$$
\sum_{i=1}^{2} \frac{\partial}{\partial_t}\left(p_i \cdot z_i\right) = d_u \sum_{i=1}^{2} (p_i \Delta z_i - z_i \Delta p_i) + \nabla \beta_{bb}(t,a^*) \frac{u_1^* u_2^*}{N_1}(p_2 - p_1) - \frac{\beta_{bh}}{N_1} \overline{v}_1^* z_2 (\overline{p}_4 - \overline{p}_3) \tag{4.15}
$$

and

$$
\sum_{i=3}^{6} \frac{\partial}{\partial_t}\left(p_i \cdot z_i\right) = \quad d_v \sum_{i=3}^{5} (p_i \Delta z_i - z_i \Delta p_i) + \frac{\beta_{bh}}{N_1} v_1^* \overline{z}_2 (p_4 - p_3) \tag{4.16}
$$

Integrating (4.15) over $\Omega_{1,T}$ and (4.16) over $\Omega_{2,T}$, and using the boundary conditions of the direct and adjoint problems:

$$\sum_{i=1}^{2}\int_0^T\int_{\Omega_1}\frac{\partial}{\partial_t}(p_i\cdot z_i)\,dxdt = \int_0^T\int_{\Omega_1}\nabla\beta_{bb}(t,a^*)\frac{u_1^*u_2^*}{N_1}(p_2-p_1)dxdt$$
$$-\int_0^T\int_{\Omega_1}\frac{\beta_{bh}}{N_1}\overline{v}_1^* z_2(\overline{p}_4-\overline{p}_3)dxdt$$

and

$$\sum_{i=3}^{6}\int_0^T\int_{\Omega_2}\frac{\partial}{\partial_t}(p_i\cdot z_i)\,dxdt = \int_0^T\int_{\Omega_2}\frac{\beta_{bh}}{N_1}v_1^*\overline{z}_2(p_4-p_3)dxdt$$

remember $\overline{p}_3$ and $\overline{p}_4$ denote de prolongation by zero of $p_3$ and $p_4$ on $\Omega_1\setminus\widetilde{\Omega}$, then

$$\sum_{i=1}^{2}\int_0^T\int_{\Omega_1}\frac{\partial}{\partial_t}(p_i\cdot z_i)\,dxdt = \int_0^T\int_{\Omega_1}\nabla\beta_{bb}(t,a^*)\frac{u_1^*u_2^*}{N_1}(p_2-p_1)dxdt$$
$$-\int_0^T\int_{\widetilde{\Omega}}\frac{\beta_{bh}}{N_1}v_1^* z_2(p_4-p_3)dxdt$$

and

$$\sum_{i=3}^{6}\int_0^T\int_{\Omega_2}\frac{\partial}{\partial_t}(p_i\cdot z_i)\,dxdt = \int_0^T\int_{\widetilde{\Omega}}\frac{\beta_{bh}}{N_1}v_1^* z_2(p_4-p_3)dxdt$$

integrating the left-side and using the boundary conditions on time from (4.4), (4.5), (4.10) and (4.11), it get

$$0 = \int_0^T\int_{\Omega_1}\nabla\beta_{bb}(t,a^*)\frac{u_1^*u_2^*}{N_1}(p_2-p_1)dxdt$$
$$-\int_0^T\int_{\widetilde{\Omega}}\frac{\beta_{bh}}{N_1}v_1^* z_2(p_4-p_3)dxdt$$

and from

$$-\frac{A}{|\Omega_2|}\int_{\Omega_2}z_3(x,T)dx-\frac{B}{|\Omega_2|}\int_{\Omega_2}z_6(x,T)dx = \int_0^T\int_{\widetilde{\Omega}}\frac{\beta_{bh}}{N_1}v_1^* z_2(p_4-p_3)dxdt$$

implies that

$$-\frac{A}{|\Omega_2|}\int_{\Omega_2}z_3(x,T)dx-\frac{B}{|\Omega_2|}\int_{\Omega_2}z_6(x,T)dx = \int_0^T\nabla\beta_{bb}(t,a^*)\left(\int_{\Omega_1}\frac{u_1^*u_2^*}{N_1}(p_2-p_1)dx\right)dt$$

finally replacing on (4.9):

$$\varepsilon\nabla g(a^*)-\int_0^T\nabla\beta_{bb}(t,a^*)\left(\int_{\Omega_1}\frac{u_1^*u_2^*}{N_1}(p_2-p_1)dx\right)dt\geq 0$$

■

## 5. Numerical Examples

In the next subsections we report some numerical examples considering one dimensional spatial space. First, we present the dependence of the coefficient $\beta_{bb}(t,a)$ on $\alpha$, $\gamma$ and $t_c$, and the (1.1)-(1.2) solution's. Then, we show the cost function by choosing the value of $t_c$ as a fixed parameter and report an numerical experiment to find its minimum using MATLAB's function *fmincon*.

### 5.1. Numerical Implementation

The numerical simulations were implemented in MATLAB using finite difference schemes to approximate the reaction–diffusion system. Specifically, we discretized the one-dimensional spatial domain $\Omega = [0, L]$ into a uniform grid of $n = 100$ equally spaced points, yielding a spatial step size $\Delta x = L/n$. The Laplacian operator was approximated using a standard second-order central difference scheme. Temporal integration was performed using an explicit Euler method with a fixed time step $\Delta t = 0.01$, which was chosen to ensure numerical stability in accordance with the Courant–Friedrichs–Lewy (CFL) condition.

The optimization routine to minimize the cost functional $J(w, a)$ was implemented using a gradient-based approach, leveraging the continuous gradient derived from the adjoint system. We employed a projected gradient descent algorithm with backtracking line search to satisfy the box constraints on the control parameters $a = (\alpha, \gamma, t_c) \in Q \subset \mathbb{R}^3$. Convergence was assessed by monitoring the relative change in the objective function between successive iterations, with a stopping criterion of $10^{-6}$. All simulations were run on a standard desktop computer.

### 5.2. $\beta_{bb}(t,a)$ *and* (1.1)-(1.2) *solution's:*

According to the definition of (1.5) and its associated explanation, here we present some example how the coefficient is affected when the parameters change. In the graph on the left (figure 2) set some values for the parameter $a = (\alpha, \gamma, t_c)$ and in the graph on the right (figure 2) we present the way in which the coefficient varies when $\alpha$ (upper graph),$\gamma$ (center) and $t_c$ (lower graph) changes.

We solve (1.1)-(1.2) equations by applying an explicit finite difference scheme with spatial domain $\Omega_1 = [0, 100]$ and $\Omega_2 = [50, 150]$ where populations interact in $\widetilde{\Omega} = [50, 100]$, time domain $t \in [0, 300]$, $N_1 = N_2 = 100$ thousand individuals in each population, diffusion coefficient $d_u = d_v = 0.8$, recovery rate $\gamma_b = \gamma_h = 0.1$, initial coefficient of disease transmision in $H_1$ $\widetilde{\beta}_{bb} = 0.3$, control parameters $a = (0.2, 0.7, 100)$ which means that from time 100, $\widetilde{\beta}_{bb}$ decays by $80\%$ and does so at a rate of 0.7, transmision coefficient between both populations $\beta_{bh} = 0.3$ and coefficient of disease transmision in $H_2$ $\beta_{hh} = 0.2$.

The disease initiates in the $H_1$ population in $[0, 25]$ region of the $\Omega_1$ domain, so the initial conditions are:

$$u_1(x,0) = \begin{cases} 99.99 & , \quad x \in [0,25] \\ 100 & , \quad x \in ]25,100] \end{cases} , \; u_2(x,0) = \begin{cases} 0.01 & , \quad x \in [0,25] \\ 0 & , \quad x \in ]25,100] \end{cases}$$

$$v_1(x,0) = 100, \, v_2(x,0) = v_3(x,0) = c(x,0) = 0, \, x \in [50,150].$$

also we consider a discretization $\Delta x = 0.2$ and $\Delta t = 0.008$ so as to satisfy the associated CFL condition.

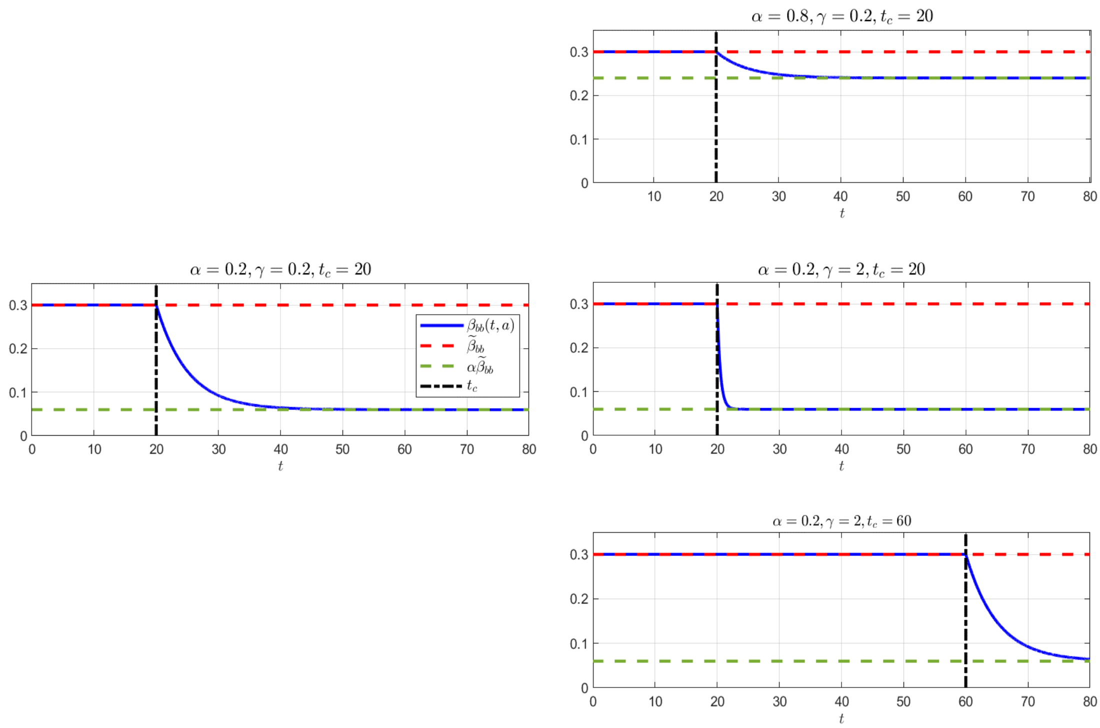


Figure 2: Transmission coefficient in population $H_2$ ($\beta_{bb}(t,a)$) with $\widetilde{\beta}_{bb} = 0.3$.

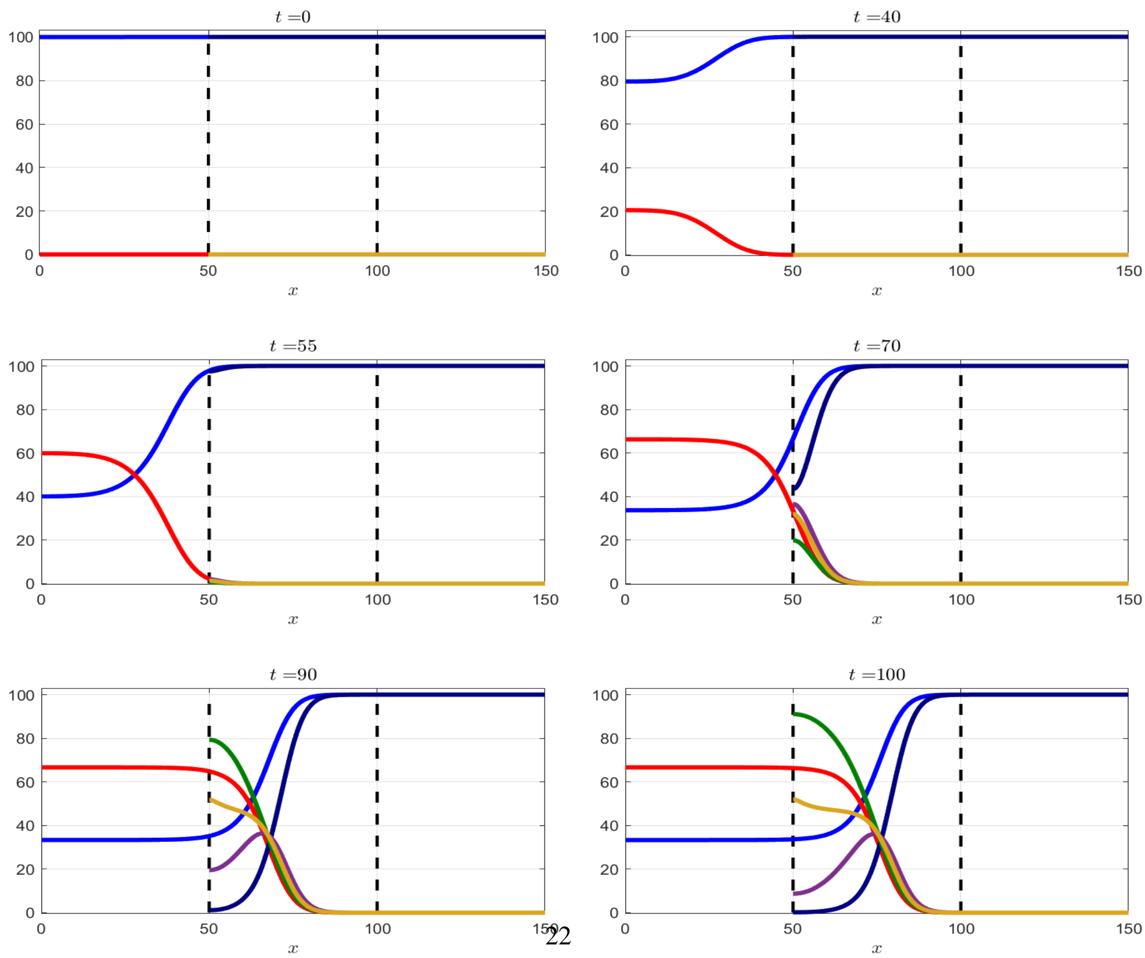




Figure 3: (1.1)-(1.2) solution's (part 1).

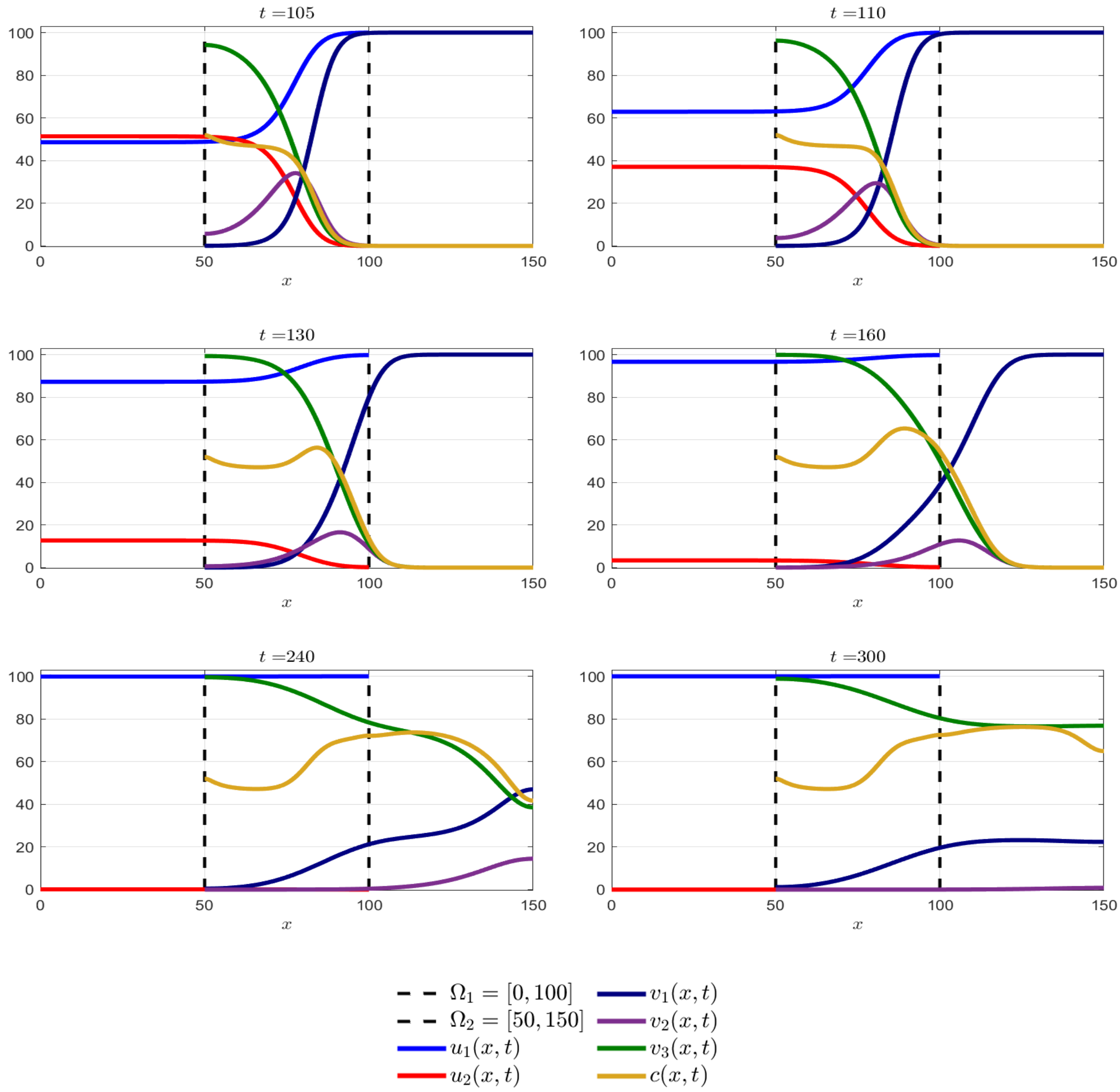


Figure 4: (1.1)-(1.2) solution's (part 2) and legend.

### 5.3. *Cost function and optimization with MATLAB's function:*

To ensure the existence of an optimum for (3.1), the cost function must be non-negative within the closed, bounded, and convex domain $Q$. Moreover, for the uniqueness of the optimum, the cost function should exhibit convexity either throughout $Q$ or within a specific subregion. Theorems (2.1) and (2.3) establish the non-negativity of $\displaystyle\int_{\Omega_2} v_3(x, T)\, dx$ and $\displaystyle\int_{\Omega_2} c(x, T)\, dx$, and assuming that the parameters $A$, $B$, and $\epsilon$ are non-negative, the cost function is guaranteed to remain non-negative.

Numerical simulations were performed to study the behavior of the cost function for various values of $a = (\alpha, \gamma, t_c) \in [0.05, 0.6] \times [0.05, 0.6] \times [70, 160]$ and different combinations of $A$, $B$, and $\epsilon$ (see [18, Chapter 7]). These experiments allowed to choose $A = 300$, $B = 50$, $\epsilon = 800$ and $t_c = 120$, to generate the required convexity within a subregion of $Q := [0.05, 0.6] \times [0.05, 0.6]$. To simplify the optimization process and ensure the continuity of the total derivative (4.8), the parameter $t_c$ is fixed as a constant. This choice is justified by the discontinuity of the partial derivative $\frac{\partial \beta_{bb}}{\partial t_c}(t,a)$ with respect to $t$ (see [18, Appendix B]). By keeping $t_c$ constant, the search for parameters that yield the desired properties of the cost function is significantly simplified. Consequently, the cost function depends solely on $a = (\alpha, \gamma)$, reducing the dimensionality of the parameter space.

As a result of this adjustment, a non-negative and convex cost function is achieved within a subregion $Q$, as illustrated in Figure 5.

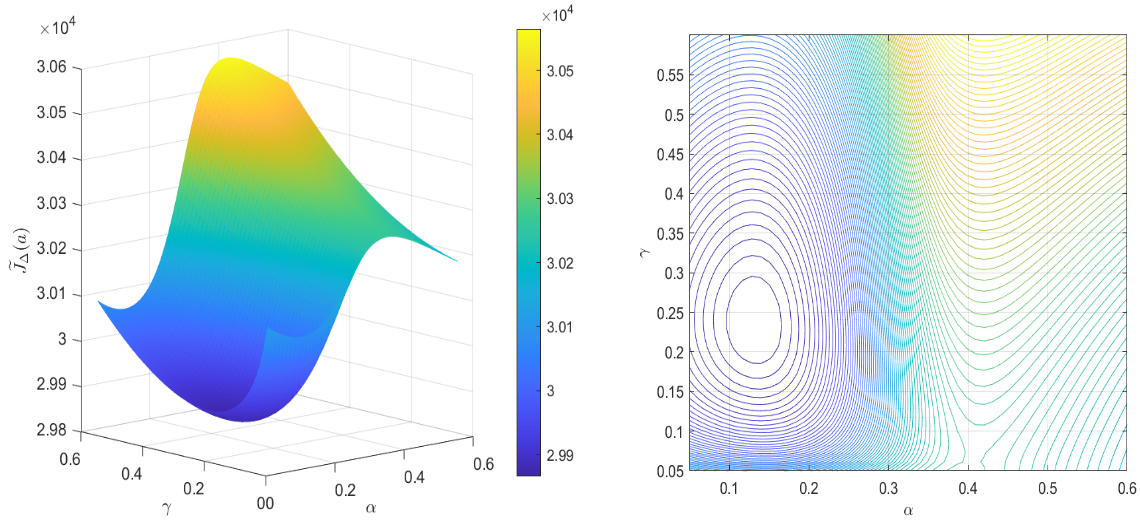


Figure 5: Cost function.

**Remark 5.1.** *The criterion for selecting the parameters $A$, $B$, $\varepsilon$ and $t_c$ in the cost function obeys mainly due to obtaining non-negativity and some partial convexity properties, however, the choice of them are it may be governed more strongly by epidemiological criteria.*

Thus, the optimization problem (3.1) is reformulated as finding $a = (\alpha, \gamma) \in \widetilde{Q}$ that minimizes the functional

$$\widetilde{J}(a) = \frac{1}{100}\int_{50}^{150} \{300 v_3(x, 300) + 50 c(x, 300)\}\, dx + 800\left((1-\alpha)^2 + \gamma^2\right),$$

where the search domain is restricted to $\widetilde{Q} := [0.07, 0.35] \times [0.07, 0.5]$ due to the structure of $\widetilde{J}(a)$. To solve this optimization problem, the function *fmincon* is employed, using lower and upper bounds for the variables as $[0.07, 0.35]$ and $[0.07, 0.5]$, respectively. The algorithm configuration is set to *sqp* (Sequential Quadratic Programming) with tolerance parameters *OptimalityTolerance*, *StepTolerance*, and *Constraint-Tolerance* all set to $10^{-8}$, and the initial point $a_0 = (0.1, 0.45)$ is used for the first run.

With these settings, the algorithm converges to the optimal point $a^* = (0.1329, 0.2365)$, achieving a cost function value $\widetilde{J}(a) = 29866.2$ in 6 iterations, and to verify consistency, the optimization is repeated with a starting point $a_0 = (0.35, 0.5)$, yielding identical results. The outcomes of these particular experiment are summarized in Table 1 and can be interpreted as, if it is decided to act in at time $t_c = 120$, minimizing the cost function requires the initial transmission coefficient of the population $H_1$ to decay to $13.29\%$ of its initial capacity, with a decay rate of $0.2365$.

## 6. Biological interpretation and real-world context.

Our results offer important biological and epidemiological insights into the indirect control of infectious diseases across interacting populations [23]. Specifically, we demonstrate that interventions in the reservoir

| Starting Point | Iter | J-count | J(a) | Max Constraints | Line search steplength | Directional derivative | First-order optimality | Optimal Point |
|---|---|---|---|---|---|---|---|---|
| $(0.1, 0.45)$ | 6 | 28 | 29866.2 | -0.06291 | 1 | -0.36 | 0.0288 | $(0.1329, 0.2365)$ |
| $(0.35, 0.5)$ | 8 | 31 | 29866.2 | -0.06291 | 1 | -0.302 | 0.0271 | $(0.1329, 0.2365)$ |

Table 1: Optimization results using the function *fmincon*.

population $H_1$—such as reducing the transmission intensity ($\alpha$), accelerating the speed of behavioral uptake or intervention rollout ($\gamma$), and initiating interventions earlier ($t_c$)—can substantially lower both the attack rate and cumulative incidence in the secondary population $H_2$, even in the absence of direct interventions in $H_2$. This finding has practical implications for zoonotic diseases such as hantavirus or leptospirosis, where human infections often arise from contact with infected rodents or livestock [24]. For instance, rapid deployment of rodent control or sanitation efforts in high-risk areas may delay or reduce the risk of human outbreaks [22, 26]. Similarly, at wildlife–livestock interfaces, preemptive vaccination or habitat modification in wildlife reservoirs may offer downstream protection to domestic animals. These insights highlight the potential benefits of targeting upstream drivers of transmission and provide a quantitative framework for evaluating such strategies under spatial and temporal constraints.

## 7. Conclusions

This work investigates an optimal control problem for a reaction–diffusion epidemic model describing the interaction between two host populations. The primary objective was to minimize the attack rate and cumulative incidence on $H_2$ through optimal intervention strategies targeting the transmission coefficient in the population where the disease originates $H_1$. Using semigroup theory and functional analysis, we established the existence and uniqueness of solutions for the controlled system. Furthermore, the optimal control problem was formulated, and the existence of optimal solutions was rigorously demonstrated. The first-order necessary optimality conditions were derived using the adjoint system and the Lagrangian formalism.

Numerical simulations were conducted to illustrate how the spread of the disease propagates between populations in a one-dimensional spatial domain and how controlling the transmission coefficient in the $H_1$ population impacts disease development in the $H_2$ population. While the optimization problem was solved using MATLAB functions in this work, an alternative approach involves employing a discrete formulation to compute the discrete gradient via the Lagrangian framework. This discrete gradient could then be applied in conjunction with the steepest descent method to determine the optimal solution, similar to the methodologies presented in [13], [16], and [4].

This study is subject to several modeling assumptions that, while mathematically tractable, may limit the generalizability of our results. First, we assume that the total population sizes $N_1$ and $N_2$ remain constant over time, neglecting demographic processes such as births, deaths unrelated to the disease, and migration. This simplification is appropriate for short-term outbreak dynamics but may not capture long-term endemic behavior. Second, the spatial diffusion terms for both populations are assumed to be isotropic and governed by constant coefficients, which does not account for directional movement (e.g., advection, preferential pathways) or heterogeneous landscapes. Third, we model control interventions in $H_1$ through a smooth, exponentially decaying transmission coefficient $\beta_{bb}(t)$, characterized by a fixed structure across space and time. While this approach allows for rigorous analysis, real-world interventions often vary spatially and may exhibit discontinuities or delays that are not captured in the current formulation. Future extensions could incorporate spatially heterogeneous control strategies, stochastic effects, or dynamic boundary conditions to better reflect ecological complexity and operational constraints.

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## Data Availability

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

## Conflict of interest

There is no potential conflict of interest.

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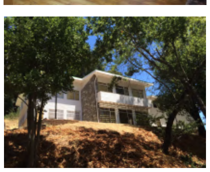

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