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A Transient Eddy Current Problem via Potential Formulation with Current Excitation

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Abstract

The aim of this work is to propose a transient eddy current model that incorporates input current intensities. We extend the classical $\boldsymbol{A}, V-\boldsymbol{A}$ potential formulation from the time-harmonic setting to the transient regime with nonlocal source conditions. We prove the existence and uniqueness of the solution to the corresponding continuous variational problem. Furthermore, we develop a fully discrete scheme based on the backward Euler method for time discretization and nodal finite elements for spatial approximation. The resulting discrete problem is shown to be well-posed, and optimal error estimates are derived. Numerical experiments are presented to validate our theoretical findings.

Mathematics Subject Classifications (1991): 65N30; 35K65; 78M10

Key words: Transient eddy current models, input current intensities, degenerate parabolic problems, a priori error estimates, finite element method, potential formulation.

1 Introduction

The eddy current model, widely used in science and engineering, is derived from Maxwell's equations by neglecting the displacement current term [9]. The well-known magnetic vector potential formulation, including the $\mathbf{A}$ and $V-\mathbf{A}$ approaches, has been extensively studied in the time-harmonic regime [11]. Potential-based formulations for eddy current problems have been proven effective in practice, and are widely used in commercial software [7]. More recently, in [3], a time-harmonic formulation based on the magnetic vector potential has been studied under voltage and current excitations. One approach incorporates a scalar electric potential in the conductor, while another relies solely on the vector potential, enforced through Coulomb-type gauge conditions. However, the transient counterpart of this formulation, particularly with current source excitation, remains relatively unexplored.

Several transient eddy current formulations have been proposed in recent years to address voltage and current excitation problems. These include magnetic field formulations where the input current is imposed via Lagrange multipliers [5], mixed formulation using the primitive of the electric field and a Lagrange multiplier, allowing current or voltage excitation without magnetic scalar potentials [6], and approaches based on a current vector representation [12, 8].

In this paper, we consider two potentials: a magnetic vector potential $\mathbf{A}$ for the magnetic field, and a scalar function v, representing the time primitive of the electric scalar potential in the conducting domain. We introduce a variational formulation that exhibits the structure of a degenerate parabolic problem [13, Chapter 3], and we prove that the Gårding inequality holds, thereby ensuring the well-posedness of the continuous formulation. A fully discrete scheme is then proposed, combining the backward Euler method for time discretization with nodal finite elements for spatial approximation. Following ideas proposed in [2], we show that this scheme admits a unique solution at each time step of the resulting elliptic problem. Projection operators onto the discrete finite element subspaces are defined, and quasi-optimal error estimates are derived. Finally, by choosing a sufficiently small time step Δt, we confirm the convergence of the method at each mesh step, along with the expected orders of convergence.

The outline of the paper is as follows. In Section 2, we introduce the eddy current problem with input current intensities. Section 3 presents the potential formulation under consideration. In Section 4, we derive the corresponding variational problem and prove its well-posedness. In Section 5, we establish error estimates for a standard finite element method used to solve the problem numerically. Finally, in Section 6, we present numerical results that confirm the theoretical findings.

2 Statement of the problem

Let $\Omega_{\mathrm{c}} \subseteq \mathbb{R}^3$ be the space occupied by the conductor material with boundary Γ_{C}. Let $\Omega \subseteq \mathbb{R}^3$ be a computational domain with boundary Γ, which is an open and bounded set such that $\overline{\Omega}_{\mathrm{c}} \subset \Omega$. We suppose that both, Ω and Ω_{c} are Lipschitz domains and we denote by $\boldsymbol{n}$ and $\boldsymbol{n}_{\mathrm{c}}$ the outward unit normal vectors to Ω and Ω_{c}, respectively. Let $\Omega_{\mathrm{d}} := \Omega \setminus \overline{\Omega}_{\mathrm{c}}$ be the domain occupied by the dielectric material, which includes the support of the source current $\boldsymbol{J}_{\mathrm{d}}$. The transient eddy current model reads (see, for instance, [9])

$$\operatorname{\mathbf{curl}} \boldsymbol{H} = \sigma \boldsymbol{E} \quad \text{in } \Omega \times [0,T], \qquad \frac{\partial(\mu \boldsymbol{H})}{\partial t} + \operatorname{\mathbf{curl}} \boldsymbol{E} = \mathbf{0} \quad \text{in } \Omega_{\mathrm{c}} \times [0,T], \tag{1}$$

$$\operatorname{\mathbf{curl}} \boldsymbol{H} = \boldsymbol{J}_{\mathrm{d}} \quad \text{in } \Omega_{\mathrm{d}} \times [0,T], \quad \operatorname{div}(\mu \boldsymbol{H}) = 0 \quad \text{in } \Omega \times [0,T], \quad \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}) \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{H}|_{\Omega_{\mathrm{c}}} \times \boldsymbol{n}_{\mathrm{c}} = \boldsymbol{H}|_{\Omega_{\mathrm{d}}} \times \boldsymbol{n}_{\mathrm{c}} \quad \text{in } \Gamma_{\mathrm{C}} \times [0,T], \quad \boldsymbol{H} \times \boldsymbol{n} = \mathbf{0} \quad \text{in } \Gamma \times [0,T], \tag{3}$$

where we have used standard notations in electromagnetism: $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{J}$ the current density, $\boldsymbol{B}$ the magnetic induction and $\boldsymbol{E}$ is the electric field. The magnetic permeability μ and conductivity σ are bounded functions satisfying:

$$\begin{aligned} &\sigma_1 \geq \sigma(\boldsymbol{x}) \geq \sigma_0 > 0 \quad \text{a.e. in } \Omega_{\mathrm{c}} \qquad \text{and} \qquad \sigma(\boldsymbol{x}) = 0 \quad \text{a.e. in } \Omega_{\mathrm{d}}, \\ &\mu_1 \geq \mu(\boldsymbol{x}) \geq \mu_0 > 0 \quad \text{a.e. in } \Omega_{\mathrm{c}} \qquad \text{and} \qquad \mu(\boldsymbol{x}) = \mu_0 \quad \text{a.e. in } \Omega_{\mathrm{d}}. \end{aligned} \tag{4}$$

In a similar manner as in [6] we consider sources provided by external circuits, namely input current intensities. For that reason it is necessary to assume two types of the conductor domain: internal conductors and inductors. An internal conductor Ω_{c}^k satisfies $\partial\Omega \cap \partial\Omega_{\mathrm{c}}^k = \emptyset$. On the other hand, an inductor domain Ω_{c}^m goes through the boundary of Ω, its boundary Γ_{C}^m is not empty, and it is split as $\Gamma_{\mathrm{C}}^m = \Gamma_{\mathrm{E}}^m \cup \Gamma_{\mathrm{J}}^m$, where Γ_{E}^m and Γ_{J}^m are respectively the current input surface and the current exit surface of the inductor domain Ω_{c}^m. We assume that there are M inductor domains $\{\Omega_{\mathrm{c}}^m\}_{m=1}^M$, which are connected and mutually disjoint, and set $\Gamma_{\mathrm{J}} := \cup_{m=1}^M \Gamma_{\mathrm{J}}^m$, $\Gamma_{\mathrm{E}} := \cup_{m=1}^M \Gamma_{\mathrm{E}}^m$. The internal conductors will be $\{\Omega_{\mathrm{c}}^m\}_{m=M+1}^{\tilde{M}}$. Thus, $\Omega_{\mathrm{c}} = \bigcup_{m=1}^{\tilde{M}} \Omega_{\mathrm{c}}^m$ is the conductor region , while the insulator region is defined by $\Omega_{\mathrm{d}} := \Omega \setminus \overline{\Omega}_{\mathrm{c}}$. (see Figure 1 $M = 2$; $\tilde{M} = 3$). We impose the intensities of the

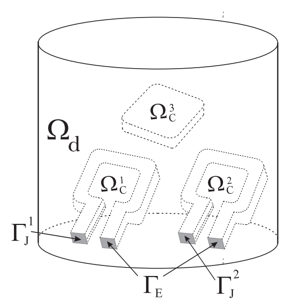

Figure 1: Sketch of the domain

input current as follows

$$\int_{\Gamma_{\mathrm{J}}^m} \sigma \boldsymbol{E} \cdot \boldsymbol{n} = I_m \quad \text{in} \quad [0,T], \quad m = 1, \cdots, M \tag{5}$$

where I_m is the current intensity through the surface Γ_{J}^m with $m = 1, \cdots, M$. Following the lines of [10] to complete the model, it is necessary to consider the following boundary conditions

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } [0,T] \times \Gamma_{\mathrm{E}}, \quad \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } [0,T] \times \Gamma_{\mathrm{J}}, \quad \text{and} \quad \mu \boldsymbol{H} \cdot \boldsymbol{n} = 0 \text{ on } [0,T] \times \partial\Omega\,. \tag{6}$$

3 A potential formulation

We are going to start this section by recalling a (strong) classical formulation of the eddy current problem in terms of two potentials: a magnetic vector potential $\boldsymbol{A}$ and an electric scalar potential V. We refer to [7] for a more detailed discussion. Next, we will introduce a variational formulation in the presence of electric ports, based in the strong problem.

It is well-known that from (2), we can find a unique vector potential function $\boldsymbol{A} : [0,T] \times \Omega \to \mathbb{R}^3$ satisfying

$$\mu \boldsymbol{H} = \mathbf{curl}\, \boldsymbol{A}, \quad \operatorname{div} \boldsymbol{A} = 0 \qquad \text{in } [0,T] \times \Omega; \quad \boldsymbol{A} \cdot \boldsymbol{n} = 0 \qquad \text{on } [0,T] \times \Gamma\,. \tag{7}$$

We also use equation (1) and look for a scalar potential $V : \Omega_{\mathrm{c}} \times [0,T] \to \mathbb{R}^3$ satisfying

$$\boldsymbol{E} = \frac{\partial \boldsymbol{A}}{\partial t} + \nabla \frac{\partial v}{\partial t} \qquad \text{in } [0,T] \times \Omega_{\mathrm{c}}\,. \tag{8}$$

where $v = \int_0^t V(\boldsymbol{x}, s) ds$. From the first two conditions of (6) and the last condition of (7) we obtain

$$\nabla_\Gamma v = \boldsymbol{n} \times v \times \boldsymbol{n} = -\boldsymbol{n} \times \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{J}},$$

which implies that v must be constant on each connected component of Γ_{J} and Γ_{E}. Without loss of generality we suppose that $v = 0$ on each Γ_{E} and constant on each Γ_{J}.

We impose the gauge condition (divergence-free) by adding a penalization term ν^*, which is a suitable average of ν in Ω (see [11]). Consequently, the original eddy current equations (see (1)-(3)) in terms of potentials $\boldsymbol{A}$ and v can read as follows:

$$\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla \frac{\partial v}{\partial t}\right) + \mathbf{curl}\left(\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{A}\right) - \nu^* \mathrm{grad}\, \mathrm{div}\, \boldsymbol{A} = \boldsymbol{0} \qquad \text{in } [0,T] \times \Omega_{\mathrm{c}}, \tag{9}$$

$$\mathrm{div}\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla \frac{\partial v}{\partial t}\right) = 0 \qquad \text{in } [0,T] \times \Omega_{\mathrm{d}}, \tag{10}$$

$$\mathbf{curl}\left(\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{A}\right) - \nu^*\, \mathrm{grad}\, \mathrm{div}\, \boldsymbol{A} = \boldsymbol{J}_{\mathrm{d}} \qquad \text{in } [0,T] \times \Omega_{\mathrm{d}}, \tag{11}$$

and satisfying

$$\boldsymbol{A} \cdot \boldsymbol{n} = 0 \qquad \text{on } [0,T] \times \Gamma, \tag{12}$$

$$\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla \frac{\partial v}{\partial t}\right) \cdot \boldsymbol{n}_{\mathrm{c}} = 0 \qquad \text{on } [0,T] \times \Gamma_{\mathrm{c}} \setminus (\Gamma_{\mathrm{E}}^n \cap \Gamma_{\mathrm{J}}^n) \quad n = 1, \cdots, M, \tag{13}$$

$$\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla \frac{\partial v}{\partial t}\right) \cdot \boldsymbol{n}_{\mathrm{c}} = 0 \qquad \text{on } [0,T] \times \Gamma_{\mathrm{c}}, \quad n = M+1, \cdots, \tilde{M}, \tag{14}$$

$$\left(\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{A}\right)\Big|_{\Omega_{\mathrm{c}}} \times \boldsymbol{n}_{\mathrm{c}} = \left(\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{A}\right)\Big|_{\Omega_{\mathrm{d}}} \times \boldsymbol{n}_{\mathrm{c}} \qquad \text{on } [0,T] \times \Gamma_{\mathrm{c}}, \tag{15}$$

$$\int_{\Gamma_{\mathrm{J}}^m} \sigma \boldsymbol{E} \cdot \boldsymbol{n} = I_m \quad \text{in} \quad [0,T], \quad m = 1, \cdots, M, \tag{16}$$

$$\boldsymbol{A}\big|_{\Omega_{\mathrm{c}}} \times \boldsymbol{n}_{\mathrm{c}} = \boldsymbol{A}\big|_{\Omega_{\mathrm{d}}} \times \boldsymbol{n}_{\mathrm{c}} \qquad \text{on } \Gamma_{\mathrm{c}} \times [0,T], \tag{17}$$

$$\boldsymbol{A}\big|_{\Omega_{\mathrm{c}}} \cdot \boldsymbol{n}_{\mathrm{c}} = \boldsymbol{A}\big|_{\Omega_{\mathrm{d}}} \cdot \boldsymbol{n}_{\mathrm{c}} \qquad \text{on } \Gamma_{\mathrm{c}} \times [0,T], \tag{18}$$

$$\boldsymbol{A}(\boldsymbol{x}, 0) = \boldsymbol{A}_0(\boldsymbol{x}) \qquad \text{a.e. } \boldsymbol{x} \in \Omega, \tag{19}$$

$$v(\boldsymbol{x}, 0) = 0 \qquad \text{a.e. } \boldsymbol{x} \in \Omega_{\mathrm{c}}, \tag{20}$$

$$\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{A} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{on } \Gamma \times [0,T]. \tag{21}$$

We introduce the following spaces

$$X := \mathbf{H}(\mathbf{curl}; \Omega) \cap \mathbf{H}_0(\mathrm{div}; \Omega)$$

$$\mathrm{H}^1_\sharp(\Omega_{\mathrm{c}}) := \left\{ u \in \mathrm{H}^1_{\Gamma_{\mathrm{E}}}(\Omega_{\mathrm{c}}) : \; u|_{\Gamma_{\mathrm{J}}^m} = \text{constant} \quad m = 1, \ldots, M \quad \int_{\Omega_{\mathrm{c}}^m} u = 0, \quad m = M+1, \ldots, \tilde{M} \right\}$$

The space X is endowed with the graph standard norm

$$\|\boldsymbol{Z}\|_X := \left(\|\boldsymbol{Z}\|^2_{0,\Omega} + \|\,\mathrm{div}\, \boldsymbol{Z}\|^2_{0,\Omega} + \|\,\mathbf{curl}\, \boldsymbol{Z}\|^2_{0,\Omega} \right)^{1/2}.$$

Furthermore, it is well-known that as a consequence of the generalized Poincaré inequality the semi-norm $|\cdot|_{\mathrm{H}^1(\Omega_c)} := \|\nabla(\cdot)\|_{0,\Omega_c}$ is a norm on $\mathrm{H}^1_\sharp(\Omega_c)$ which is equivalent to the $\mathrm{H}^1(\Omega_c)$-norm.

By using standard arguments of integration by parts, we can obtain the following variational formulation: Given $\boldsymbol{J}_d \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ and $I_n \in \mathrm{H}^2(0,T)$ $n = 1,\ldots,n$, find $(\boldsymbol{A}, v) \in \mathrm{L}^2(0,T; X \times \mathrm{H}^1_\sharp(\Omega_c)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_c)^3 \times \mathrm{H}^1_\sharp(\Omega_c))$ such that

$$\begin{aligned} &\frac{d}{dt}\left(\boldsymbol{A}+\nabla v, \boldsymbol{Z}+\nabla w\right)_{\sigma,\Omega_c} + \mathcal{A}(\boldsymbol{A},\boldsymbol{Z}) = \int_\Omega \boldsymbol{J}_d\cdot\boldsymbol{Z} - \sum_{m=1}^{M} w|_{\Gamma_J^m} I_m(t) \qquad \forall(\boldsymbol{Z},w)\in X\times\mathrm{H}^1_\sharp(\Omega_c), \\ &\qquad \boldsymbol{A}(\cdot,0) = \boldsymbol{A}_0 \text{ in } \Omega, \qquad v(\cdot,0) = 0 \text{ in } \Omega_c, \end{aligned} \tag{22}$$

where the following notations have been used

$$\mathcal{A}(\boldsymbol{A},\boldsymbol{Z}) = \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{Z} + \nu^*\int_\Omega(\operatorname{div}\boldsymbol{A})(\operatorname{div}\boldsymbol{Z}) \quad \text{and} \quad (\boldsymbol{u},\boldsymbol{w})_{\sigma,\Omega_c} := \int_{\Omega_c}\sigma\boldsymbol{u}\cdot\boldsymbol{w}.$$

Next result will be useful

Lemma 1 *There exists $C_1 > 0$, such that*

$$\int_\Omega |\boldsymbol{Z}|^2 \le C_1\left\{\int_\Omega |\mathbf{curl}\,\boldsymbol{Z}|^2 + \int_\Omega |\operatorname{div}\boldsymbol{Z}|^2\right\} \qquad \forall \boldsymbol{Z}\in X. \tag{23}$$

Moreover, there holds

$$\frac{1}{1+C_1}\|\boldsymbol{Z}\|_X^2 \le \|\,\mathbf{curl}\,\boldsymbol{Z}\|_{0,\Omega}^2 + \|\operatorname{div}\boldsymbol{Z}\|_{0,\Omega}^2 \le \|\boldsymbol{Z}\|_X^2 \quad \forall\, \boldsymbol{Z}\in X\,.$$

Proof. We refer to [4, Corollary 3.16]. □

Remark 2 *Taking into account properties of μ and σ, we can ensure the existence of two positive constants κ_0, κ_1, independent of the physical parameters, such that*

$$\frac{\kappa_0}{1+C_1}\|\boldsymbol{Z}\|_X^2 \le \kappa_0\left(\|\,\mathbf{curl}\,\boldsymbol{Z}\|_{0,\Omega}^2 + \|\operatorname{div}\boldsymbol{Z}\|_{0,\Omega}^2\right) \le \mathcal{A}(\boldsymbol{Z},\boldsymbol{Z}) \le \kappa_1\left(\|\,\mathbf{curl}\,\boldsymbol{Z}\|_{0,\Omega}^2 + \|\operatorname{div}\boldsymbol{Z}\|_{0,\Omega}^2\right) \le \kappa_1\|\boldsymbol{Z}\|_X^2$$

for any $\boldsymbol{Z}\in X$. We can check that $\kappa_0 = \min\{\mu_1^{-1},\nu^*\}$, and $\kappa_1 = \max\{\mu_0^{-1},\nu^*\}$.

Theorem 3 *The problem* (22) *admits a unique solution.*

Proof. First, we notice that the problem (22) is a degenerate parabolic problem. Therefore, in order to establish its existence and uniqueness, we need to show the conditions given in [13, Proposition III.3.2 and III.3.3]. We only verify the Gårding-type inequality holds true for $(\boldsymbol{Z}, w) \in X\times\mathrm{H}^1_\sharp(\Omega_c)$, since the other assumptions are verified straightforwardly. First of all, for any $\boldsymbol{Z}\in X$ and $w\in\mathrm{H}^1_\sharp(\Omega_c)$ we have, after taking into account Remark 2

$$(\boldsymbol{Z}+\nabla w, \boldsymbol{Z}+\nabla w)_{\sigma,\Omega_c} + \mathcal{A}(\boldsymbol{Z},\boldsymbol{Z}) \ge \sigma_0\int_{\Omega_c}|\boldsymbol{Z}+\nabla w|^2 + \frac{\kappa_0}{1+C_1}\|\boldsymbol{Z}\|_X^2\,. \tag{24}$$

Now, setting $C := \min\{\sigma_0, \frac{\kappa_0}{1+C_1}\}$, we obtain

$$\begin{aligned}
&(\boldsymbol{Z} + \nabla w, \boldsymbol{Z} + \nabla w)_{\sigma,\Omega_c} + \mathcal{A}(\boldsymbol{Z}, \boldsymbol{Z}) \\
&\quad \geq C\left[\int_{\Omega_c} |\boldsymbol{Z} + \nabla w|^2 + \frac{1}{2}\int_{\Omega} |\boldsymbol{Z}|^2 + \frac{1}{2}\|\boldsymbol{Z}\|_X^2\right] \\
&\quad \geq \frac{C}{2}\left[\int_{\Omega_c} |\boldsymbol{Z} + \nabla w|^2 + \int_{\Omega_c} |\boldsymbol{Z}|^2 + \|\boldsymbol{Z}\|_X^2\right].
\end{aligned}$$

Having in mind that

$$\int_{\Omega_c} |\boldsymbol{Z} + \nabla w|^2 + \int_{\Omega_c} |\boldsymbol{Z}|^2 = 2\int_{\Omega_c} |\boldsymbol{Z}|^2 + 2\int_{\Omega_c} \boldsymbol{Z}\cdot\nabla w + \int_{\Omega_c} |\nabla w|^2 \,,$$

and invoking Young's inequality

$$-2\boldsymbol{Z}\cdot\nabla w \leq 2\,|\boldsymbol{Z}|\,|\nabla w| \leq \frac{1}{\delta}\,|\boldsymbol{Z}|^2 + \delta\,|\nabla w|^2 \qquad \forall\delta > 0,$$

it follows that

$$(\boldsymbol{Z} + \nabla w, \boldsymbol{Z} + \nabla w)_{\sigma,\Omega_c} + \mathcal{A}(\boldsymbol{Z}, \boldsymbol{Z}) \geq \frac{C}{2}\left[\left(2 - \frac{1}{\delta}\right)\int_{\Omega_c} |\boldsymbol{Z}|^2 + (1-\delta)\int_{\Omega_c} |\nabla w|^2 + \|\boldsymbol{Z}\|_X^2\right]. \tag{25}$$

Consequently, by taking $1/2 < \delta < 1$, the proof is established. □

4 A fully discrete scheme

In what follows we assume that Ω and Ω_c are Lipschitz polyhedra (we recall that Ω is simply-connected). Let $\{\mathcal{T}_h\}_h$ be a regular family of tetrahedral meshes of $\overline{\Omega}$ such that each element $K \in \mathcal{T}_h$ is contained either in $\overline{\Omega}_c$ or in $\overline{\Omega}_d$. As usual, h stands for the largest diameter of tetrahedra K in $\mathcal{T}_h$. We consider the following finite element spaces:

$$\begin{aligned}
X_h &:= \left\{\boldsymbol{Z}_h \in X :\, \boldsymbol{Z}_h\big|_K \in \mathbb{P}_1(K)^3 \;\; \forall K \in \mathcal{T}_h \text{ with } K \subset \overline{\Omega}\right\}, \\
M_h &:= \left\{w_h \in \mathrm{H}^1_\sharp(\Omega_c) :\, w_h\big|_K \in \mathbb{P}_1(K) \;\; \forall K \in \mathcal{T}_h \text{ with } K \subset \overline{\Omega}_c\right\}.
\end{aligned}$$

We consider a uniform partition $\{t_n := n\Delta t :\; n = 0, \dots, N\}$ of $[0, T]$ with a step size $\Delta t := \frac{T}{N}$. For any finite sequence $\{\theta^n :\; n = 0, \cdots, N\}$, let

$$\bar{\partial}\theta^n := \frac{\theta^n - \theta^{n-1}}{\Delta t}, \qquad n = 1, 2, \dots, N.$$

The fully-discrete version of Problem (22) reads as follows:

Find $(\boldsymbol{A}_h^n, v_h^n) \in X_h \times M_h$, $n = 1, 2, \dots, N$ such that for any $(\boldsymbol{Z}_h, w_h) \in X_h \times M_h$:

$$\left(\bar{\partial}\boldsymbol{A}_h^n + \nabla\bar{\partial}v_h^n, \boldsymbol{Z}_h + \nabla w_h\right)_{\sigma,\Omega_c} + \mathcal{A}(\boldsymbol{A}_h^n, \boldsymbol{Z}_h) = (\boldsymbol{J}_d(t_n), \boldsymbol{Z}_h)_{0,\Omega} - \Delta t\sum_{m=1}^{M} w_h|_{\Gamma_J^m} I_m(t_n) \tag{26}$$

$$\boldsymbol{A}_h^0 = \boldsymbol{A}_{0,h}, \qquad v_h^0 = 0, \tag{27}$$

where $\boldsymbol{A}_{0,h} \in X_h$ is a suitable approximation of $\boldsymbol{A}_0$ to obtain optimal error estimates.

In order to prove that Problem (26)–(27) has a unique solution, we first notice that at each iteration step we need to find $(\boldsymbol{A}_h^n, v_h^n) \in X_h \times M_h$ such that for any $(\boldsymbol{Z}_h, w_h) \in X_h \times M_h$

$$(\boldsymbol{A}_h^n + \nabla v_h^n, \boldsymbol{Z}_h + \nabla w_h)_{\sigma,\Omega_{\mathrm{c}}} + \Delta t\, \mathcal{A}(\boldsymbol{A}_h^n, \boldsymbol{Z}_h) = F_n(\boldsymbol{Z}_h, w_h), \tag{28}$$

where

$$F_n(\boldsymbol{Z}_h, w_h) = \Delta t \int_\Omega \boldsymbol{J}_{\mathrm{d}}(t_n) \cdot \boldsymbol{Z}_h - \Delta t \sum_{m=1}^{M} w_h|_{\Gamma_{\mathrm{J}}^m} I_m(t_n) + \left(\boldsymbol{A}_h^{n-1} + \nabla v_h^{n-1}, \boldsymbol{Z}_h + \nabla w_h\right)_{\sigma,\Omega_{\mathrm{c}}}.$$

If Δt is sufficiently small the bilinear form given by $(\boldsymbol{A}_h^n + \nabla v_h^n, \boldsymbol{Z} + \nabla w_h)_{\sigma,\Omega_{\mathrm{c}}} + \Delta t\, \mathcal{A}(\boldsymbol{A}_h^n, \boldsymbol{Z})$ is coercive on $X_h \times M_h$. Thus, invoking Lax-Milgram's theorem, the problem (28) has a unique solution at each time step.

5 Error estimates

In this section we will prove error estimates for our fully-discrete scheme. To this end, we consider the projection operators $P_h : X \to X_h$ and $Q_h : \mathrm{H}^1_\sharp(\Omega_{\mathrm{c}}) \to M_h$ defined as follows

$$\text{Given } \boldsymbol{Z} \in X \;:\; P_h\boldsymbol{Z} \in X_h : \qquad (P_h\boldsymbol{Z} - \boldsymbol{Z}, \boldsymbol{Y})_X = 0 \quad \forall \boldsymbol{Y} \in X_h\,,$$

and

$$\text{given } u \in \mathrm{H}^1_\sharp(\Omega_{\mathrm{c}}) \;:\; Q_h u \in M_h : \qquad (\nabla Q_h u - \nabla u, \nabla w)_{\sigma,\Omega_c} = 0 \quad \forall w \in M_h\,.$$

We remark that P_h and Q_h are well defined. Besides, we have the following estimates

Lemma 4 *There exist positive constants C_1 and C_2, independent of h:*

$$\|\boldsymbol{Z} - P_h\boldsymbol{Z}\|_X \leq C_1 \inf_{\boldsymbol{Y}_h \in X_h} \|\boldsymbol{Z} - \boldsymbol{Y}_h\|_X \qquad \forall \boldsymbol{Z} \in X,$$
$$\|u - Q_h u\|_{\mathrm{H}^1_\sharp(\Omega_{\mathrm{c}})} \leq C_2 \inf_{w_h \in M_h} \|u - w_h\|_{\mathrm{H}^1_\sharp(\Omega_{\mathrm{c}})} \qquad \forall u \in \mathrm{H}^1_\sharp(\Omega_{\mathrm{c}}).$$

Now, in order to derive a priori error, we introduce the following notations:

$$\boldsymbol{\rho}_1^n := \boldsymbol{A}(t_n) - P_h\boldsymbol{A}(t_n), \quad \boldsymbol{\delta}_1^n := P_h\boldsymbol{A}(t_n) - \boldsymbol{A}_h^n, \quad \boldsymbol{\tau}_1^n := \bar{\partial}\boldsymbol{A}(t_n) - \partial_t\boldsymbol{A}(t_n)$$
$$\rho_2^n := v(t_n) - Q_h v(t_n), \quad \delta_2^n := Q_h v(t_n) - v_h^n, \quad \tau_2^n := \bar{\partial} v(t_n) - \partial_t v(t_n).$$

and

$$\boldsymbol{e}_1^n := \boldsymbol{A}(t_n) - \boldsymbol{A}_h^n, \quad e_2^n := v(t_n) - v_h^n. \tag{29}$$

Besides, for sufficiently smooth $\boldsymbol{A}$ and v there holds

$$\sum_{n=1}^{N} \|\boldsymbol{\tau}_1^n\|^2_{0,\Omega_{\mathrm{c}}} \leq \; \Delta t \int_0^T \|\partial_{tt}\boldsymbol{A}(t)\|^2_{0,\Omega_{\mathrm{c}}}\, dt\,, \quad \sum_{n=1}^{N} \|\nabla\tau_2^n\|^2_{0,\Omega_{\mathrm{c}}} \leq \Delta t \int_0^T \|\partial_{tt}\nabla(v(t))\|^2_{0,\Omega_{\mathrm{c}}}\, dt. \tag{30}$$

From here on we assume that Δt is small enough, for example $\Delta t \leq \dfrac{1}{2}$.

Lemma 5 *There exists a positive constant C, independent of h and Δt, such that for each $n = 1,\dots,N$*

$$\begin{aligned}
&\|\boldsymbol{\delta}_1^n\|_X^2 + \|\delta_2^n\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})} + \Delta t\sum_{k=1}^n \|\boldsymbol{\delta}_1^k\|_X^2 + \Delta t\sum_{k=1}^n \|\bar\partial\boldsymbol{\delta}_1^k + \nabla\bar\partial\delta_2^k\|^2_{0,\Omega_\mathrm{c}} \\
&\le C\left\{\|\boldsymbol{A}_0 - \boldsymbol{A}_{0,h}\|_X^2 + \|\boldsymbol{\rho}_1^0\|_X^2 + \|\boldsymbol{\rho}_1^n\|_X^2 + \Delta t\sum_{k=1}^n\left[\|\bar\partial\boldsymbol{\rho}_1^k\|_X^2 + \|\boldsymbol{\rho}_1^k\|_X^2 + \|\bar\partial\rho_2^k\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})} + \|\boldsymbol{\tau}_1^k\|^2_{0,\Omega_\mathrm{c}} + \|\nabla\tau_2^k\|^2_{0,\Omega_\mathrm{c}}\right]\right\}.
\end{aligned} \tag{31}$$

Proof. The proof of the lemma follows from standard arguments applied for degenerate parabolic problems (see, for instance, [1, Lemma 1]). For completeness purposes, we describe it. Let $1 \le n \le N$ and $1 \le k \le n$. It is straightforward to show that

$$\begin{aligned}
&(\bar\partial\boldsymbol{\delta}_1^k + \nabla\bar\partial\delta_2^k, \boldsymbol{Z} + \nabla w)_{\sigma,\Omega_\mathrm{c}} + \mathcal{A}(\boldsymbol{\delta}_1^k, \boldsymbol{Z}) \\
&\quad = -(\bar\partial\boldsymbol{\rho}_1^k + \nabla\bar\partial\rho_2^k, \boldsymbol{Z} + \nabla w)_{\sigma,\Omega_\mathrm{c}} - \mathcal{A}(\boldsymbol{\rho}_1^k, \boldsymbol{Z}) + (\boldsymbol{\tau}_1^k + \nabla\tau_2^k, \boldsymbol{Z} + \nabla w)_{\sigma,\Omega_\mathrm{c}},
\end{aligned} \tag{32}$$

for any $(\boldsymbol{Z}, w) \in X_h \times M_h$.

Step 1: Taking $(\boldsymbol{Z}, w) = (\boldsymbol{\delta}_1^k, \delta_2^k)$ in (32) and applying Cauchy-Schwarz inequality, as well as the following estimate

$$(\bar\partial\boldsymbol{\delta}_1^k + \nabla\bar\partial\delta_2^k, \boldsymbol{\delta}_1^k + \nabla\delta_2^k,)_{\sigma,\Omega_\mathrm{c}} \ge \frac{1}{2\Delta t}\left[\|\boldsymbol{\delta}_1^k + \nabla\delta_2^k\|^2_{\sigma,\Omega_\mathrm{c}} - \|\boldsymbol{\delta}_1^{k-1} + \nabla\delta_2^{k-1}\|^2_{\sigma,\Omega_\mathrm{c}}\right],$$

we can obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}\left[\|\boldsymbol{\delta}_1^k + \nabla\delta_2^k\|^2_{\sigma,\Omega_\mathrm{c}} - \|\boldsymbol{\delta}_1^{k-1} + \nabla\delta_2^{k-1}\|^2_{\sigma,\Omega_\mathrm{c}}\right] + \frac{1}{2}\mathcal{A}(\boldsymbol{\delta}_1^k, \boldsymbol{\delta}_1^k) \\
&\le \frac{1}{2}\|\boldsymbol{\delta}_1^k + \nabla\delta_2^k\|^2_{\sigma,\Omega_\mathrm{c}} + \|\bar\partial\boldsymbol{\rho}_1^k + \nabla\bar\partial\rho_2^k\|^2_{\sigma,\Omega_\mathrm{c}} + \|\boldsymbol{\tau}_1^k + \nabla\tau_2^k\|^2_{\sigma,\Omega_\mathrm{c}} + \frac{1}{2}\mathcal{A}(\boldsymbol{\rho}_1^k, \boldsymbol{\rho}_1^k)
\end{aligned}$$

Next, after summing from $k = 1$ to $k = n$, we have

$$\begin{aligned}
&\|\boldsymbol{\delta}_1^n + \nabla\delta_2^n\|^2_{\sigma,\Omega_\mathrm{c}} + (\Delta t)\sum_{k=1}^n \mathcal{A}(\boldsymbol{\delta}_1^k, \boldsymbol{\delta}_1^k) \le \|\boldsymbol{\delta}_1^0 + \nabla\delta_2^0\|^2_{\sigma,\Omega_\mathrm{c}} \\
&+ 2(\Delta t)\sum_{k=1}^n\left[\frac{1}{2}\|\boldsymbol{\delta}_1^k + \nabla\delta_2^k\|^2_{\sigma,\Omega_\mathrm{c}} + \|\bar\partial\boldsymbol{\rho}_1^k + \nabla\bar\partial\rho_2^k\|^2_{\sigma,\Omega_\mathrm{c}} + \|\boldsymbol{\tau}_1^k + \nabla\tau_2^k\|^2_{\sigma,\Omega_\mathrm{c}} + \frac{1}{2}\mathcal{A}(\boldsymbol{\rho}_1^k, \boldsymbol{\rho}_1^k)\right]
\end{aligned} \tag{33}$$

Since $\Delta t \le \frac{1}{2}$, we deduce

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{\delta}_1^n + \nabla\delta_2^n\|^2_{\sigma,\Omega_\mathrm{c}} + (\Delta t)\sum_{k=1}^n \mathcal{A}(\boldsymbol{\delta}_1^k, \boldsymbol{\delta}_1^k) \le \|\boldsymbol{\delta}_1^0 + \nabla\delta_2^0\|^2_{\sigma,\Omega_\mathrm{c}} \\
&+ (\Delta t)\sum_{k=1}^{n-1}\|\boldsymbol{\delta}_1^k + \nabla\delta_2^k\|^2_{\sigma,\Omega_\mathrm{c}} + (2\Delta t)\sum_{k=1}^n\left[\|\bar\partial\boldsymbol{\rho}_1^k + \nabla\bar\partial\rho_2^k\|^2_{\sigma,\Omega_\mathrm{c}} + \|\boldsymbol{\tau}_1^k + \nabla\tau_2^k\|^2_{\sigma,\Omega_\mathrm{c}} + \frac{1}{2}\mathcal{A}(\boldsymbol{\rho}_1^k, \boldsymbol{\rho}_1^k)\right]
\end{aligned} \tag{34}$$

Now, after invoking discrete Gronwall's Lemma, we establish for each $n \in \{1, ..., N\}$

$$
\begin{aligned}
&\frac{1}{2}\|\boldsymbol{\delta}_1^n + \nabla\delta_2^n\|^2_{\sigma,\Omega_c} \\
&\leq \exp(2T)\left(\|\boldsymbol{\delta}_1^0 + \nabla\delta_2^0\|^2_{\sigma,\Omega_c} + (2\Delta t)\sum_{k=1}^{n}\left[\|\bar{\partial}\boldsymbol{\rho}_1^k + \nabla\bar{\partial}\rho_2^k\|^2_{\sigma,\Omega_c} + \|\boldsymbol{\tau}_1^k + \nabla\tau_2^k\|^2_{\sigma,\Omega_c} + \frac{1}{2}\mathcal{A}(\boldsymbol{\rho}_1^k, \boldsymbol{\rho}_1^k)\right]\right)
\end{aligned}
\tag{35}
$$

Using (35), we infer that

$$
\begin{aligned}
&(\Delta t)\sum_{k=1}^{n}\mathcal{A}(\boldsymbol{\delta}_1^k, \boldsymbol{\delta}_1^k) \\
&\leq\ (2T\exp(2T) + 1)\left(\|\boldsymbol{\delta}_1^0 + \nabla\delta_2^0\|^2_{\sigma,\Omega_c} + (2\Delta t)\sum_{k=1}^{n}\left[\|\bar{\partial}\boldsymbol{\rho}_1^k + \nabla\bar{\partial}\rho_2^k\|^2_{\sigma,\Omega_c} + \|\boldsymbol{\tau}_1^k + \nabla\tau_2^k\|^2_{\sigma,\Omega_c} + \frac{1}{2}\mathcal{A}(\boldsymbol{\rho}_1^k, \boldsymbol{\rho}_1^k)\right]\right)
\end{aligned}
\tag{36}
$$

Step 2: Now we take $(\boldsymbol{Z}, w) := (\bar{\partial}\boldsymbol{\delta}_1^k, \bar{\partial}\delta_2^k)$ in (32) and obtain, after invoking Cauchy-Schwarz' and Young's inequalities,

$$
\|\bar{\partial}\boldsymbol{\delta}_1^k + \nabla\bar{\partial}\delta_2^k\|^2_{\sigma,\Omega_c} + \mathcal{A}(\boldsymbol{\delta}_1^k, \bar{\partial}\boldsymbol{\delta}_1^k) \leq\ \|\bar{\partial}\boldsymbol{\rho}_1^k + \nabla\bar{\partial}\rho_2^k\|^2_{\sigma,\Omega_c} + \|\boldsymbol{\tau}_1^k + \nabla\tau_2^k\|^2_{\sigma,\Omega_c} + \frac{1}{2}\|\bar{\partial}\boldsymbol{\delta}_1^k + \nabla\bar{\partial}\delta_2^k\|^2_{\sigma,\Omega_c} - \mathcal{A}(\boldsymbol{\rho}_1^k, \bar{\partial}\boldsymbol{\delta}_1^k)
\tag{37}
$$

Taking into account the relations

$$
\begin{aligned}
\mathcal{A}(\boldsymbol{\rho}_1^k, \bar{\partial}\boldsymbol{\delta}_1^k) &= \frac{1}{\Delta t}\left[\mathcal{A}(\boldsymbol{\rho}_1^k, \boldsymbol{\delta}_1^k) - \mathcal{A}(\boldsymbol{\rho}_1^{k-1}, \boldsymbol{\delta}_1^{k-1})\right] - \mathcal{A}(\bar{\partial}\boldsymbol{\rho}_1^k, \boldsymbol{\delta}_1^{k-1})\,, \\
\mathcal{A}(\boldsymbol{\delta}_1^k, \bar{\partial}\boldsymbol{\delta}_1^k) &\geq \frac{1}{2\Delta t}\left[\mathcal{A}(\boldsymbol{\delta}_1^k, \boldsymbol{\delta}_1^k) - \mathcal{A}(\boldsymbol{\delta}_1^{k-1}, \boldsymbol{\delta}_1^{k-1})\right]\,,
\end{aligned}
$$

(37) yields to

$$
\begin{aligned}
&\frac{1}{2}\|\bar{\partial}\boldsymbol{\delta}_1^k + \nabla\bar{\partial}\delta_2^k\|^2_{\sigma,\Omega_c} + \frac{1}{2\Delta t}\left[\mathcal{A}(\boldsymbol{\delta}_1^k, \boldsymbol{\delta}_1^k) - \mathcal{A}(\boldsymbol{\delta}_1^{k-1}, \boldsymbol{\delta}_1^{k-1})\right] \\
&\leq\ \|\bar{\partial}\boldsymbol{\rho}_1^k + \nabla\bar{\partial}\rho_2^k\|^2_{\sigma,\Omega_c} + \|\boldsymbol{\tau}_1^k + \nabla\tau_2^k\|^2_{\sigma,\Omega_c} \\
&- \frac{1}{\Delta t}\left[\mathcal{A}(\boldsymbol{\rho}_1^k, \boldsymbol{\delta}_1^k) - \mathcal{A}(\boldsymbol{\rho}_1^{k-1}, \boldsymbol{\delta}_1^{k-1})\right] + \frac{1}{2}\mathcal{A}(\bar{\partial}\boldsymbol{\rho}_1^k, \bar{\partial}\boldsymbol{\rho}_1^k) + \frac{1}{2}\mathcal{A}(\boldsymbol{\delta}_1^{k-1}, \boldsymbol{\delta}_1^{k-1})\,.
\end{aligned}
\tag{38}
$$

Now, summing from $k = 1$ to $k = n$ in (38), we obtain

$$
\begin{aligned}
&(\Delta t)\sum_{k=1}^{n}\|\bar{\partial}\boldsymbol{\delta}_1^k + \nabla\bar{\partial}\delta_2^k\|^2_{\sigma,\Omega_c} + \left[\mathcal{A}(\boldsymbol{\delta}_1^n, \boldsymbol{\delta}_1^n) - \mathcal{A}(\boldsymbol{\delta}_1^0, \boldsymbol{\delta}_1^0)\right] \leq\ -2\left[\mathcal{A}(\boldsymbol{\rho}_1^n, \boldsymbol{\delta}_1^n) - \mathcal{A}(\boldsymbol{\rho}_1^0, \boldsymbol{\delta}_1^0)\right] \\
&+ 2(\Delta t)\sum_{k=1}^{n}\left[\|\bar{\partial}\boldsymbol{\rho}_1^k + \nabla\bar{\partial}\rho_2^k\|^2_{\sigma,\Omega_c} + \|\boldsymbol{\tau}_1^k + \nabla\tau_2^k\|^2_{\sigma,\Omega_c} + \mathcal{A}(\bar{\partial}\boldsymbol{\rho}_1^k, \bar{\partial}\boldsymbol{\rho}_1^k) + \mathcal{A}(\boldsymbol{\delta}_1^{k-1}, \boldsymbol{\delta}_1^{k-1})\right]
\end{aligned}
\tag{39}
$$

Now, thanks to Young's inequality, we have

$$-\mathcal{A}(\boldsymbol{\rho}_1^n,\boldsymbol{\delta}_1^n)\leq\ \mathcal{A}(\boldsymbol{\rho}_1^n,\boldsymbol{\rho}_1^n)+\frac{1}{4}\mathcal{A}(\boldsymbol{\delta}_1^n,\boldsymbol{\delta}_1^n)\,,$$

which helps us, together with the assumption $\Delta t\leq\dfrac{1}{2}$, to infer from (39)

$$\begin{aligned}&(\Delta t)\sum_{k=1}^{n}\|\bar\partial\boldsymbol{\delta}_1^k+\nabla\bar\partial\delta_2^k\|^2_{\sigma,\Omega_\mathrm{c}}+\frac{1}{2}\mathcal{A}(\boldsymbol{\delta}_1^n,\boldsymbol{\delta}_1^n)\leq\ 2\mathcal{A}(\boldsymbol{\rho}_1^0,\boldsymbol{\delta}_1^0)+\mathcal{A}(\boldsymbol{\delta}_1^0,\boldsymbol{\delta}_1^0)\\&+2(\Delta t)\sum_{k=1}^{n}\Big[\|\bar\partial\boldsymbol{\rho}_1^k+\nabla\bar\partial\rho_2^k\|^2_{\sigma,\Omega_\mathrm{c}}+\|\boldsymbol{\tau}_1^k+\nabla\tau_2^k\|^2_{\sigma,\Omega_\mathrm{c}}+\mathcal{A}(\bar\partial\boldsymbol{\rho}_1^k,\bar\partial\boldsymbol{\rho}_1^k)+\mathcal{A}(\boldsymbol{\delta}_1^k,\boldsymbol{\delta}_1^k)\Big]\,.\end{aligned}\tag{40}$$

Invoking (36), to bound the sum involving $\mathcal{A}(\boldsymbol{\delta}_1^k,\boldsymbol{\delta}_1^k)$ on the right hand side in (40), we deduce for each $n\in\{1,...,N\}$

$$\begin{aligned}&(\Delta t)\sum_{k=1}^{n}\|\bar\partial\boldsymbol{\delta}_1^k+\nabla\bar\partial\delta_2^k\|^2_{\sigma,\Omega_\mathrm{c}}+\frac{1}{2}\mathcal{A}(\boldsymbol{\delta}_1^n,\boldsymbol{\delta}_1^n)\leq\ 2\mathcal{A}(\boldsymbol{\rho}_1^0,\boldsymbol{\delta}_1^0)+\mathcal{A}(\boldsymbol{\delta}_1^0,\boldsymbol{\delta}_1^0)\\&+2(\Delta t)\sum_{k=1}^{n}\Big[\|\bar\partial\boldsymbol{\rho}_1^k+\nabla\bar\partial\rho_2^k\|^2_{\sigma,\Omega_\mathrm{c}}+\|\boldsymbol{\tau}_1^k+\nabla\tau_2^k\|^2_{\sigma,\Omega_\mathrm{c}}+\mathcal{A}(\bar\partial\boldsymbol{\rho}_1^k,\bar\partial\boldsymbol{\rho}_1^k)\Big]\\&+2(2T\exp(2T)+1)\left(\|\boldsymbol{\delta}_1^0+\nabla\delta_2^0\|^2_{\sigma,\Omega_\mathrm{c}}+(2\Delta t)\sum_{k=1}^{n}\Big[\|\bar\partial\boldsymbol{\rho}_1^k+\nabla\bar\partial\rho_2^k\|^2_{\sigma,\Omega_\mathrm{c}}+\|\boldsymbol{\tau}_1^k+\nabla\tau_2^k\|^2_{\sigma,\Omega_\mathrm{c}}+\frac{1}{2}\mathcal{A}(\boldsymbol{\rho}_1^k,\boldsymbol{\rho}_1^k)\Big]\right)\,.\end{aligned}\tag{41}$$

Finally, noticing that

$$\|\boldsymbol{\delta}_1^0\|^2_X=\|P_h\boldsymbol{A}(t_0)-\boldsymbol{A}_{0,h}\|^2_X\leq2\Big(\|\boldsymbol{\rho}_1^0\|^2_X+\|\boldsymbol{A}_0-\boldsymbol{A}_{0,h}\|^2_X\Big)\,,$$

(31) is established, after combining (35), (36) and (41), and invoking Lemma 1, Remark 2, as well as Gårding inequality (25).

□

Theorem 6 *If $\boldsymbol{A}\in\mathrm{H}^1(0,T;X)\cap\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega))$ and $v\in\mathrm{H}^1(0,T;\mathrm{H}^1_\sharp(\Omega_\mathrm{c}))\cap\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_\mathrm{c}))$, then there exists a constant $C>0$, independent of h and Δt, such that*

$$\begin{aligned}&\max_{1\leq n\leq N}\Big[\|\boldsymbol{e}_1^n\|^2_X+\|e_2^n\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})}\Big]+\Delta t\sum_{n=1}^{N}\|\boldsymbol{e}_1^n\|^2_X+\Delta t\sum_{n=1}^{N}\|\bar\partial\boldsymbol{e}_1^n+\nabla\bar\partial e_2^n\|^2_{0,\Omega_\mathrm{c}}\\&\leq C\Bigg\{\|\boldsymbol{A}_0-\boldsymbol{A}_{0,h}\|^2_X+\max_{1\leq n\leq N}\Big[\inf_{\boldsymbol{Z}\in X_h}\|\boldsymbol{A}(t_n)-\boldsymbol{Z}\|^2_X+\inf_{w\in M_h}\|v(t_n)-w\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})}\Big]\\&\quad+\Delta t\sum_{n=1}^{N}\inf_{\boldsymbol{Z}\in X_h}\|\boldsymbol{A}(t_n)-\boldsymbol{Z}\|^2_X+\int_0^T\Big[\inf_{\boldsymbol{Z}\in X_h}\|\partial_t\boldsymbol{A}(t)-\boldsymbol{Z}\|^2_X\Big]\,dt\\&\quad+\int_0^T\Big[\inf_{w\in M_h}\|\partial_t v(t)-w\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})}\Big]\,dt+(\Delta t)^2\int_0^T\big[\|\partial_{tt}\boldsymbol{A}(t)\|^2_{0,\Omega_\mathrm{c}}+\|\partial_{tt}\nabla(v(t))\|^2_{0,\Omega_\mathrm{c}}\big]\,dt\Bigg\}\,.\end{aligned}$$

Proof. Taking into account the definition for the errors given in (29), and after applying triangle inequality, we obtain

$$\max_{1\le n\le N}\left[\|\boldsymbol{e}_1^n\|_X^2+\|e_2^n\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})}\right]+\Delta t\sum_{n=1}^N\|\boldsymbol{e}_1^n\|_X^2+\Delta t\sum_{n=1}^N\|\bar\partial\boldsymbol{e}_1^n+\nabla\bar\partial e_2^n\|^2_{0,\Omega_\mathrm{c}}$$
$$\le C\left\{\|\boldsymbol{A}_0-\boldsymbol{A}_{0,h}\|_X^2+\max_{1\le n\le N}\left[\|\boldsymbol{\rho}_1^n\|_X^2+\|\rho_2^n\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})}\right]+\Delta t\sum_{n=1}^N\left[\|\bar\partial\boldsymbol{\rho}_1^n\|_X^2\right.\right. \tag{42}$$
$$\left.\left.+\|\boldsymbol{\rho}_1^n\|_X^2+\|\bar\partial\rho_2^n\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})}+\|\boldsymbol{\tau}_1^n\|^2_{0,\Omega_\mathrm{c}}+\|\nabla\tau_2^n\|^2_{0,\Omega_\mathrm{c}}\right]\right\}.$$

Besides, $\partial_t\left(P_h\boldsymbol{A}(t)\right)=P_h\left(\partial_t\boldsymbol{A}(t)\right)$ and $\partial_t\left(Q_hv(t)\right)=Q_h\left(\partial_tv(t)\right)$. On the other hand, we have

$$\sum_{n=1}^N\|\bar\partial\boldsymbol{\rho}_1^n\|_X^2\le\frac{C}{\Delta t}\int_0^T\left[\inf_{\boldsymbol{Z}\in X_h}\|\partial_t\boldsymbol{A}(t)-\boldsymbol{Z}\|_X^2\right]dt,\ \sum_{n=1}^N\|\bar\partial\rho_2^n\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})}\le\frac{C}{\Delta t}\int_0^T\left[\inf_{u\in M_h}\|\partial_tv(t)-u\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})}\right]dt. \tag{43}$$

Finally, the result follows from using the last inequalities, Lemma 4 and (30). □

Corollary 7 *Let's assume that* $\boldsymbol{A}\in\mathrm{H}^1(0,T;X\cap\mathrm{H}^{1+s}(\Omega)^3)\cap\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega))$ *and* $v\in\mathrm{H}^1(0,T;\mathrm{H}^1_\sharp(\Omega_\mathrm{c})\cap\mathrm{H}^{1+s}(\Omega_\mathrm{c}))\cap\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_\mathrm{c}))$*, for some* $0<s<1$*. If* $\boldsymbol{A}_{0,h}=\boldsymbol{\Pi}_h(\boldsymbol{A}_0)$*, where* $\boldsymbol{\Pi}_h:X\cap\mathrm{H}^{1+s}(\Omega)^3\to X_h$ *is the Lagrange interpolant, then there exists a positive constant* C*, independent of* h *and* Δt*, such that*

$$\max_{1\le n\le N}\|\boldsymbol{e}_1^n\|_X^2+\max_{1\le n\le N}\|e_2^n\|^2_{\mathrm{H}^1_\sharp(\Omega_\mathrm{c})}+\Delta t\sum_{n=1}^N\|\boldsymbol{e}_1^n\|_X^2+\Delta t\sum_{n=1}^N\|\bar\partial\boldsymbol{e}_1^n+\nabla\bar\partial e_2^n\|^2_{0,\Omega_\mathrm{c}}$$
$$\le Ch^{2s}\left(\max_{0\le n\le N}\|\boldsymbol{A}(t_n)\|^2_{1+s,\Omega}+\max_{1\le n\le N}\|v(t_n)\|^2_{1+s,\Omega_\mathrm{c}}+\int_0^T\|\partial_t\boldsymbol{A}(t)\|^2_{1+s,\Omega}dt+\int_0^T\|\partial_tv(t)\|^2_{1+s,\Omega_\mathrm{c}}dt\right)$$
$$+C(\Delta t)^2\left(\int_0^T\|\partial_{tt}\boldsymbol{A}(t)\|^2_{0,\Omega_\mathrm{c}}dt+\int_0^T\|\partial_{tt}\nabla(v(t))\|^2_{0,\Omega_\mathrm{c}}dt\right).$$

Proof. Let $\Pi_h:\mathrm{H}^1_\sharp(\Omega_\mathrm{c})\cap\mathrm{H}^{1+s}(\Omega_\mathrm{c})\to M_h$ be the standard scalar finite element Lagrange interpolant. The result is a direct consequence of the well-known approximation properties of $\boldsymbol{\Pi}_h$, Π_h and Theorem 6. □

Remark 8 *At each time step* $t=t_k$ *we can approximate the eddy current* $\boldsymbol{E}(\boldsymbol{x},t_k)$ *and the magnetic field* $\boldsymbol{H}(\boldsymbol{x},t_k)$ *by means of* $\boldsymbol{E}_h^k=-\sigma\bar\partial\boldsymbol{A}_h^k-\sigma\bar\partial\nabla v_h^k$ *and* $\mu\boldsymbol{H}_h^k=\mathbf{curl}\,\boldsymbol{A}_h^k$*. Thus, the Corollary 7 yields the following error estimates*

$$\Delta t\sum_{n=1}^N\|\sigma\boldsymbol{E}(t_n)-\left(-\sigma\bar\partial\boldsymbol{A}_h^n-\sigma\bar\partial\nabla v_h^n\right)\|^2_{0,\Omega_\mathrm{c}}\le\mathcal{O}(h^{2s}+(\Delta t)^2)$$

$$\max_{1\le n\le N}\|\mu\boldsymbol{H}(t_n)-\mathbf{curl}\,\boldsymbol{A}_h^n\|^2_{0,\Omega}+\Delta t\sum_{n=1}^N\|\mu\boldsymbol{H}(t_n)-\mathbf{curl}\,\boldsymbol{A}_h^n\|^2_{0,\Omega}\le\mathcal{O}(h^{2s}+(\Delta t)^2).$$

6 Numerical result

In this section, we present numerical results obtained using a MATLAB code that implements a problem with a known analytical solution. Specifically, we approximate the solution of the following source problem

$$\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \frac{\partial v}{\partial t}\right) + \mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{A}) - \operatorname{grad}\operatorname{div}\boldsymbol{A} = \boldsymbol{f_1} \qquad \text{in } [0,T]\times\Omega, \tag{44}$$

$$\operatorname{div}\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \frac{\partial v}{\partial t}\right) = f_2 \qquad \text{in } [0,T]\times\Omega_{\mathrm{d}}, \tag{45}$$

$$\boldsymbol{A}\cdot\boldsymbol{n} = 0 \quad v = 0 \qquad \text{on } [0,T]\times\Gamma, \tag{46}$$

$$\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \frac{\partial v}{\partial t}\right)\cdot\boldsymbol{n}_{\mathrm{c}} = \boldsymbol{g}\cdot\boldsymbol{n}_{\mathrm{c}} \qquad \text{on } [0,T]\times\Gamma_{\mathrm{c}} \tag{47}$$

$$(\mathbf{curl}\,\boldsymbol{A})\big|_{\Omega_{\mathrm{c}}}\times\boldsymbol{n}_{\mathrm{c}} = (\mathbf{curl}\,\boldsymbol{A})\big|_{\Omega_{\mathrm{d}}}\times\boldsymbol{n}_{\mathrm{c}} \qquad \text{on } [0,T]\times\Gamma_{\mathrm{c}}, \tag{48}$$

$$\boldsymbol{A}\big|_{\Omega_{\mathrm{c}}}\times\boldsymbol{n}_{\mathrm{c}} = \boldsymbol{A}\big|_{\Omega_{\mathrm{d}}}\times\boldsymbol{n}_{\mathrm{c}} \qquad \text{on } \Gamma_{\mathrm{c}}\times[0,T], \tag{49}$$

$$\boldsymbol{A}\big|_{\Omega_{\mathrm{c}}}\cdot\boldsymbol{n}_{\mathrm{c}} = \boldsymbol{A}\big|_{\Omega_{\mathrm{d}}}\cdot\boldsymbol{n}_{\mathrm{c}} \qquad \text{on } \Gamma_{\mathrm{c}}\times[0,T], \tag{50}$$

$$\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{n} = \boldsymbol{f_3}\times\boldsymbol{n} \qquad \text{on } \Gamma\times[0,T]. \tag{51}$$

In this case, we consider $\Omega = (0,1)^3$, $\Omega_{\mathrm{c}} = (0.2, 0.8)^3$, and $\Omega_{\mathrm{d}} = \Omega\setminus\overline{\Omega_{\mathrm{c}}}$. The data $\boldsymbol{f_1}$,f_2, $\boldsymbol{g}$ and $\boldsymbol{f_3}$ have been chosen so that the analytical solution is

$$\boldsymbol{A}(x_1,x_2,x_3,t) = \sin(\pi t)\begin{bmatrix} 0 \\ \pi \mathrm{e}^{x_1}\cos(\pi x_3)\sin(\pi x_2) \\ -\pi \mathrm{e}^{x_1}\cos(\pi x_2)\sin(\pi x_3)\end{bmatrix} \qquad v(x_1,x_2,x_3,t) = t^2\sin(x_1)\sin(x_2)\sin(x_3).$$

In this example, we used to corroborate the convergence rate in the three-dimensional domain. Thus, we take $T = 0.5s$ and the time-step $\Delta t = 10^{-3}s$. The time step is sufficiently small, so that the time discretization error is not affected.

The convergence for a set of quasi-uniform mesh refinements is shown in Table 1. The method achieves optimal convergence with order $\mathcal{O}(h)$, confirming the theoretical optimal rates provided by Corollary 7.

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Table 1: Mesh sizes, percentage errors and rates of convergences for source problem (44)-(51).

	$100\frac{\max_{1\le n\le N}\|\boldsymbol{A}(t_n)-\boldsymbol{A}_h^n\|_X^2}{\max_{1\le n\le N}\|\boldsymbol{A}(t_n)\|_X^2}$		$100\frac{\sqrt{\Delta t}\left\{\sum_{n=1}^N\|\,\mathbf{curl}\,\boldsymbol{A}(t_n)-\mathbf{curl}\,\boldsymbol{A}_h^n\|_{0,\Omega}^2\right\}^{1/2}}{\sqrt{\Delta t}\left\{\sum_{n=1}^N\|\,\mathbf{curl}\,\boldsymbol{A}(t_n)\|_{0,\Omega}^2\right\}^{1/2}}$		$100\frac{\max_{1\le n\le N}\|v(t_n)-v_h^n\|_{\mathrm{H}^1_\sharp(\Omega_c)}^2}{\max_{1\le n\le N}\|v(t_n)\|_{\mathrm{H}^1_\sharp(\Omega_c)}^2}$	
h	error	rate	error	rate	error	rate
0.3464	15.313	$--$	26.898	$--$	39.119	$--$
0.1732	4.232	1.8551	13.828	0.9599	20.124	0.9590
0.8660	1.105	1.9356	7.097	0.9621	10.076	0.9980
0.4330	0.279	1.9838	3.571	0.9907	5.010	1.0080

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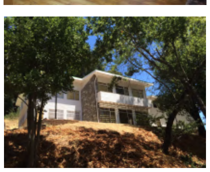

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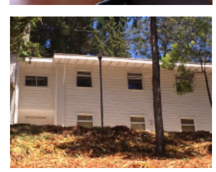
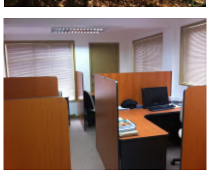

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