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Abstract

In this work, we construct a dynamically consistent second-order nonstandard finite difference (NSFD) scheme to numerically solve a generalized eco-epidemiological predator-prey model. We prove that the proposed scheme preserves some essential qualitative features of the generalized model. These features are equilibrium points, stability, and positivity of populations, which are satisfied regardless of the time step size, i.e., the method is unconditionally stable, making this scheme a very attractive numerical method. The design of the scheme relies on the usual nonlocal approximation of the right-hand side function while the nonstandard denominator functions are defined depending not only on the time step size but also on the state variables. We prove that the NSFD scheme is convergent with the desired order. The proposed methodology can be used to design other second-order NSFD numerical schemes for other mathematical models similar to the prey-predator model presented in this paper. Finally, we present numerical examples that support the mathematical analysis and show the advantages of the constructed NSFD schemes.

Keywords: nonstandard finite difference methods, second order, eco-epidemiological models, qualitative dynamics preserving numerical scheme, positivity, stability

1. Introduction

1.1. Scope

Many predator-prey mathematical models have been proposed in the scientific literature. In particular, there has been a recent increase in interest in eco-epidemiological predator-prey models. These eco-epidemiological predator-prey models integrate ecological interactions between predators and prey with the dynamics of infectious diseases affecting one or both populations [1, 2, 3, 4]. These models are in some way extensions of the classical predator-prey frameworks, such as the Lotka–Volterra equations, where disease transmission is incorporated into the model. Oftentimes, the prey population is divided into susceptible and infected subgroups. Depending on the disease, the prey infected population becomes more vulnerable to predation or has a reduced reproductive rate due to the disease. Analogously, some mathematical models have considered the possibility that predators may contract the disease through the consumption of infected prey. Incorporation of a disease into the prey-predator models can affect the dynamics of the prey and predator populations [5, 6, 3]. Thus, these mathematical models that incorporate a disease are crucial for understanding the complex feedback between disease ecology and food web dynamics. In addition, these mathematical models can better approximate the real-world situation. Analysis of these models can also provide relevant information for wildlife management, conservation, and zoonotic disease prevention by highlighting how epidemics might influence the coexistence or extinction of species [7, 8, 9].

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Most of the eco-epidemiological predator-prey models are nonlinear and based on differential equations. Their analytical solutions cannot be obtained. Thus, we need to rely on numerical schemes to numerically solve these models. Numerical schemes are essential for solving mathematical models based on ordinary differential equations (ODEs), especially nonlinear models, since most of them cannot be solved analytically. These schemes provide approximate solutions that enable us to simulate complex dynamical systems such as population dynamics in different fields [10, 11]. Numerical solutions allow us to validate theoretical predictions, perform sensitivity analyses, and guide decision-making in real-world problems where analytical solutions are unavailable or intractable. In this work, we construct nonstandard finite difference (NSFD) schemes to numerically solve a generalized eco-epidemiological predator-prey model.

NSFD schemes are numerical methods designed to preserve the essential qualitative features of mathematical models based on differential equations [12]. Oftentimes, these features are equilibrium points, stability, positivity of populations, conservation laws, and boundedness [13, 12, 14, 15]. More importantly, these features are satisfied regardless of the time step size, making the NSFD schemes a very attractive numerical method. For example, the positivity of populations is especially important in epidemic models such as SIR and SEIR models, where negative population values are biologically unrealistic. Oftentimes, the NSFD schemes use nonlocal approximations or modified denominator functions to ensure that the features of the continuous models are also satisfied by the discrete model or numerical solution. Thus, NSFD methods are highly suitable for long-term simulations, particularly for stiff mathematical models where classical finite difference schemes, such as Euler and Runge–Kutta methods, may fail [12, 16, 17, 18]. Thus, with NSFD methods, we can obtain an accurate solution using larger time steps and capture the processes described by the continuous models.

1.2. Related work

In this work, we adopt similar ideas of Kojouharov et. al. [19, 20] and Hoang et. al. [21, 14, 22] to develop second-order NSFD schemes for different generalized eco-epidemiological predator-prey models. In particular, we implement these NSFD schemes for two specific models in order to show their reliability and accuracy. Moreover, these NSFD schemes are designed to always generate positive solutions, since the models deal with populations that must be non-negative [23, 17]. Numerical schemes that generate positive solutions are critically important when modeling real-world phenomena where the variables must remain non-negative, such as population sizes or financial quantities [16, 24, 25]. Ensuring positivity preserves the integrity and realism of the simulation, especially over long time intervals or in systems sensitive to initial conditions. Standard numerical methods may fail to guarantee positivity, particularly for stiff problems, which can cause numerical instabilities or spurious solutions [13, 12, 26]. Therefore, designing schemes that inherently preserve positivity, such as NSFD methods or specially adapted Runge-Kutta schemes, is essential to maintain the qualitative features of the original ODE model and to ensure reliable and meaningful solutions [27, 28].

1.3. Outline of the paper

This article is structured as follows. In Section 2, we present the generalized eco-epidemiological predator-prey mathematical model with some preliminary results that are useful for the construction of the NSFD schemes. Section 3 is devoted to the design of a second-order NSFD scheme for the general model, the proof of results about preservation of positivity and equilibria, and a proof of convergence of the method. In Section 4, we present the stability analysis of the NSFD scheme for two particular models with respect to their equilibrium points. Section 5 is devoted to the numerical results that support the theoretical results and show the reliability of the constructed NSFD schemes. Finally, in Section 6, we present the conclusions of our research and potential future work.

2. Generalized eco-epidemiological predator-prey model

Let us consider a predator-prey ecosystem where X denotes the population of the prey and Y the population of the predators. We assume that an infection exists within the prey population; thus, we divide this population into two subpopulations: susceptible and infected, denoted by S and I, respectively. Thus, at any arbitrary time t we have

$X(t) = S(t) + I(t)$. We suppose that the prey population has a logistic growth in the absence of infected prey. The ordinary differential equation system that describes the ecosystem is

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1 - \frac{S+I}{k}\right) - \lambda SI - \mathcal{H}(S,Y)S, \\ \frac{dI}{dt} &= \lambda SI - \mathcal{G}(I,Y)I - \mu I, \\ \frac{dY}{dt} &= Y(\mathcal{R}(S) - \mathcal{F}(S,I,Y)), \end{aligned} \tag{2.1}$$

which are defined on $\Omega = \{(S, I, Y) \in \mathbb{R}^3 : S \geq 0, I \geq 0, Y \geq 0\}$ with initial conditions

$$S(0) = S_0, \quad I(0) = I_0, \quad Y(0) = Y_0. \tag{2.2}$$

Here, r is an intrinsic birth rate constant, k is the carrying capacity of the system, λ is the disease transmission coefficient, μ the per capita death rate of infected prey, and $\mathcal{H}, \mathcal{G} : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \mathbb{R}$, $\mathcal{R} : \overline{\mathbb{R}}_+ \to \mathbb{R}$, and $\mathcal{F} : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \mathbb{R}$, where $\overline{\mathbb{R}}_+ = [0, +\infty)$, are functional responses that satisfy the following assumptions.

Assumption 2.1. *The functional responses satisfy the following assumptions*

(A1) $\mathcal{H}(S,Y) \geq 0$, $\mathcal{G}(I,Y) \geq 0$, $\mathcal{R}(S) \geq 0$, $\mathcal{F}(S,I,Y) \geq 0$, *for all* $(S, I, Y) \in \Omega$.

(A2) $\mathcal{H}, \mathcal{G}, \mathcal{R}, \mathcal{F}$ *are of class* C^2.

Models that can be represented in the form (2.1) can be found in [29, 30, 31]; however, in this work, we will apply our discretization method to the following cases:

- A predator–prey model with linear response functions proposed by Bairagi et. al. in [32] with

$$\begin{aligned} \mathcal{H}(S,Y) &= \alpha S, \qquad & \mathcal{R}(S) &= \theta\alpha S, \\ \mathcal{G}(I,Y) &= \beta I, & \mathcal{F}(S,I,Y) &= \delta + \theta\beta I, \end{aligned} \tag{2.3}$$

  where α is the attack rate, β is the attack rate on infected prey, θ is the conversion efficiency of predators, and δ is the predator mortality.

- A Leslie–Gower predator–prey model with disease in prey incorporating a prey refuge studied by S. Sharma and G.P. Samanta in [33], with

$$\begin{aligned} \mathcal{H}(S,Y) &= 0, & \mathcal{R}(S) &= \delta, \\ \mathcal{G}(I,Y) &= \frac{\alpha(1-\nu)Y}{a + (1-\nu)I}, \quad & \mathcal{F}(S,I,Y) &= \frac{\eta Y}{a + (1-\nu)I}, \end{aligned} \tag{2.4}$$

  where a is the half saturation constant for infected prey population in absence of refuge, νI is the capacity of a refuge at time t, where $0 < \nu < 1$. This leaves $(1 - \nu)I$ of the infected prey available to the predator. Finally, η is the density-dependent mortality of the predator.

We will point out a few basic yet significant properties of the solutions for system (2.1), with $\mathcal{R}, \mathcal{G}, \mathcal{H}, \mathcal{F}$ given by (2.3) or (2.4) under Assumption 2.1. Since the right-hand side of the system (2.1) is completely continuous and locally Lipschitzian, the solution of (2.1) with initial conditions (2.2) exists and is unique in a maximal interval $J = [0, \omega_+)$. We introduce the solution $Z(t) = (S(t), I(t), Y(t))$ defined on J. According to Theorem 3.1 in [32] and Lemma 3.1 in [33], the solutions of (2.1) are uniformly bounded with the response functions given by (2.3) and (2.4). Since the vector field associated with system (2.1) is of class C^∞, and following the ideas in the proof of Proposition 3.1 in [34], one can establish the following result.

Theorem 1. *The flow of* (2.1) *is positively complete, i.e.,* $\omega_+ = \infty$.

In addition, it is easy to check that the solutions of the continuous model remain in Ω for every initial data in Ω, as explained in the following Theorem.

Theorem 2. *The set Ω is positively invariant under the flow of* (2.1).

Proof. Let $t \in J$ be given and $Z(0) = (S_0, I_0, Y_0) \in \Omega$. Integrating on the system (2.1) and using the fact that $S_0, I_0, Y_0 \geq 0$, we get

$$S(t) = S_0 \exp\left[\int_0^t \left(r\left(1 - \frac{S(\theta) + I(\theta)}{k}\right) - \lambda I(\theta) - \mathcal{H}(S(\theta), Y(\theta))\right) d\theta\right] \geq 0$$
$$I(t) = I_0 \exp\left[\int_0^t (\lambda S(\theta) - \mathcal{G}(I(\theta), Y(\theta)) - \mu)\, d\theta\right] \geq 0$$
$$Y(t) = Y_0 \exp\left[\int_0^t (\mathcal{R}(S(\theta)) - \mathcal{F}(S(\theta), I(\theta), Y(\theta)))\, d\theta\right] \geq 0$$

thus $Z(t) \in \Omega$, which completes the proof. □

3. Design of a second-order NSFD scheme

When we study biological models, one fundamental characteristic that we must guarantee is the positivity of the solutions, but normally this is challenging and generally, the classical numerical schemes to solve differential equations such as Euler and Runge-Kutta 4 method do not preserve these properties and sometimes fail and generate oscillations, bifurcations, chaos, and incorrect stable states [35]. One of the modern techniques to avoid the problems mentioned above is using the nonstandard finite difference schemes (NSFD) proposed by Ronald Mickens [36, 13, 12]. This NSFD methodology designs numerical schemes that preserve important properties, such as the positivity and stability of equilibrium points, while being unconditionally stable. We start by discussing the design and construction of these schemes.

3.1. Discretization

To describe the NSFD discretization approach, let us consider a general Cauchy problem of the form

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(\boldsymbol{x}), \text{ in } [0, T] \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0, \end{aligned} \tag{3.1}$$

where T is a positive real number, $\boldsymbol{x} = (x_1, x_2, \dots, x_N)^{\mathrm{T}} : [0, T] \to \mathbb{R}^N$, and the function $\boldsymbol{f} = (f_1, f_2, \dots, f_n)^{\mathrm{T}} : \mathbb{R}^N \to \mathbb{R}^N$ is differentiable at $\boldsymbol{x}_0 \in \mathbb{R}^n$. First, we discretize the computational domain $[0, T]$ by partitioning $t^n = n\Delta t$, where $\Delta t = T/M$ is the step size, and M is a fixed positive integer. A numerical scheme with step size Δt that approximates the solution $\boldsymbol{x}(t^n)$ of the system (3.1) can be written as

$$\mathcal{D}_{\Delta t}(\boldsymbol{x}_n) = \boldsymbol{F}_{\Delta t}(\boldsymbol{f}; \boldsymbol{x}_n), \tag{3.2}$$

where, $\mathcal{D}_{\Delta t}(\boldsymbol{x}_n) \approx \left.\frac{d\boldsymbol{x}}{dt}\right|_{t=t^n}$, $\boldsymbol{x}_n \approx \boldsymbol{x}(t^n)$, and $\boldsymbol{F}_{\Delta t}(\boldsymbol{f}; \boldsymbol{x}_n)$ approximate the right side of the system (3.1).

Definition 1. *The numerical scheme given by (3.2) is called a NonStandard Finite Difference (NSFD) scheme if at least one of the following conditions is satisfied,*

1. $\mathcal{D}_{\Delta t}(\boldsymbol{x}_n) = \frac{\boldsymbol{x}_{n+1} - \boldsymbol{x}_n}{\varphi(\Delta t)}$, *where φ is a non-negative real valued function which satisfies,*

$$\varphi(\Delta t) = \Delta t + O(\Delta t^2).$$

Some examples of this kind of functions are:

$$\varphi(\Delta t) = \frac{1 - e^{-\lambda \Delta t}}{\lambda},\ \lambda > 0,\ \ \varphi(\Delta t) = e^{\Delta t} - 1, \textit{see [13].} \tag{3.3}$$

2. *$\boldsymbol{F}_{\Delta t}(\boldsymbol{f};\boldsymbol{x}_n) = \boldsymbol{G}(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}, \Delta t)$, where $\boldsymbol{G}(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}, \Delta t)$ is a non-local approximation of the right side of the system (3.1).*

Remark 1. *The Definition 1 suggests that the denominator function φ should be the same for all components of $\mathcal{D}_{\Delta t}(\boldsymbol{x}_n)$; however, one can choose different functions φ_i for each $i = 1, \ldots, N$ and still have stable numerical schemes.*

Now, by using the definition (1) we design a numerical scheme to approximate numerically the solutions of the system (2.1). We denote by S^n, I^n and Y^n the approximations $S(t^n), I(t^n)$, and $Y(t^n)$ respectively, for $n = 0, 1, 2...$ and by $\boldsymbol{f} = (f_1, f_2, f_3)^{\mathrm{T}}$ the right-hand side function in (2.1). We proceed as follows:

1. Implement the non-local approximations on the right-hand side of (2.1) with weights w_i [21]:

$$
\begin{aligned}
f_1(S, I, Y) &\approx rS^n - \frac{r}{k}(S^n + I^n)S^{n+1} - \lambda S^{n+1}I^n - \mathcal{H}(S^n, Y^n)S^{n+1} + w_1S^n - w_1S^{n+1},\\
f_2(S, I, Y) &\approx \lambda S^nI^n - \mathcal{G}(I^n, Y^n)I^{n+1} - \mu I^{n+1} + w_2I^n - w_2I^{n+1},\\
f_3(S, I, Y) &\approx \mathcal{R}(S^n)Y^n - \mathcal{F}(S^n, I^n, Y^n)Y^{n+1} + w_3Y^n - w_3Y^{n+1}.
\end{aligned}
\tag{3.4}
$$

2. The first order derivatives of the system (2.1) are approximate by:

$$
\begin{aligned}
\frac{dS}{dt} &\approx \frac{S^{n+1} - S^n}{\varphi_1(\Delta t, S^n, I^n, Y^n)},\\
\frac{dI}{dt} &\approx \frac{I^{n+1} - I^n}{\varphi_2(\Delta t, S^n, I^n, Y^n)},\\
\frac{dY}{dt} &\approx \frac{Y^{n+1} - Y^n}{\varphi_3(\Delta t, S^n, I^n, Y^n)},
\end{aligned}
\tag{3.5}
$$

   where the denominator functions φ_i for $i = 1, 2, 3$ are specified later.

Hence, the system (2.1) can be discretized as

$$
\begin{aligned}
\frac{S^{n+1} - S^n}{\varphi_1(\Delta t, S^n, I^n, Y^n)} &= rS^n - \frac{r}{k}(S^n + I^n)S^{n+1} - \lambda S^{n+1}I^n - \mathcal{H}(S^n, Y^n)S^{n+1} + w_1S^n - w_1S^{n+1},\\
\frac{I^{n+1} - I^n}{\varphi_2(\Delta t, S^n, I^n, Y^n)} &= \lambda S^nI^n - \mathcal{G}(I^n, Y^n)I^{n+1} - \mu I^{n+1} + w_2I^n - w_2I^{n+1},\\
\frac{Y^{n+1} - Y^n}{\varphi_3(\Delta t, S^n, I^n, Y^n)} &= \mathcal{R}(S^n)Y^n - \mathcal{F}(S^n, I^n, Y^n)Y^{n+1} + w_3Y^n - w_3Y^{n+1}.
\end{aligned}
\tag{3.6}
$$

Remark 2. *It is well known that the choice of denominator functions like* (3.3) *leads to first-order schemes [37]. Notice in* (3.5) *that the denominator functions depend not only on Δt but also on the approximations of the state variables: S^n, I^n, Y^n. This modification allows us to obtain a higher order of convergence in the NSFD method, which will be described in the next subsection.*

3.2. Properties of the numerical scheme

Theorem 3 (Positivity)**.** *Let $w_1, w_2, w_3 \in \mathbb{R}$ and satisfy $w_1, w_2, w_3 \geq 0$. Then, the NSFD method* (3.6) *is dynamically consistent with respect to the positivity of the model* (2.1) *for all the values of the step size Δt, that is, $S^n, I^n, Y^n \geq 0$ for all $n \geq 1$ whenever $S^0, I^0, Y^0 \geq 0$.*

Proof. We proceed by induction on n. By hypothesis $S^0, I^0, Y^0 \geq 0$, so let us assume that $S^n, I^n, Y^n \geq 0$. Rearranging the scheme (3.6) in explicit form we obtain that,

$$
\begin{aligned}
S^{n+1} &= \frac{S^n + r\varphi_1(\Delta t, S^n, I^n, Y^n)S^n + w_1\varphi_1(\Delta t, S^n, I^n, Y^n)S^n}{1 + \varphi_1(\Delta t, S^n, I^n, Y^n)\Big(\frac{r}{k}(S^n + I^n) + \lambda I^n + \mathcal{H}(S^n, Y^n) + w_1\Big)},\\
I^{n+1} &= \frac{I^n + \lambda\varphi_2(\Delta t, S^n, I^n, Y^n)S^nI^n + w_2\varphi_2(\Delta t, S^n, I^n, Y^n)I^n}{1 + \varphi_2(\Delta t, S^n, I^n, Y^n)\Big(\mathcal{G}(I^n, Y^n) + \mu + w_2\Big)},\\
Y^{n+1} &= \frac{Y^n + \varphi_3(\Delta t, S^n, I^n, Y^n)\mathcal{R}(S^n)Y^n + w_3\varphi_3(\Delta t, S^n, I^n, Y^n)Y^n}{1 + \varphi_3(\Delta t, S^n, I^n, Y^n)\Big(\mathcal{F}(S^n, I^n, Y^n) + w_3\Big)}.
\end{aligned}
\tag{3.7}
$$

Therefore, using the non-negativity of the functions $\mathcal{H}, \mathcal{G}, \mathcal{R}, \mathcal{F}$, and φ_i, we deduce that $S^{n+1}, I^{n+1}, Y^{n+1} \geq 0$. This concludes the proof. $\square$

Theorem 4 (Equilibrium points)**.** *Let $w_1, w_2, w_3 \in \mathbb{R}$ and satisfy $w_1, w_2, w_3 \geq 0$. Then, the NSFD method* (3.6) *is dynamically consistent with respect to the equilibrium points of the model* (2.1) *for all the values of the step size $\Delta t \geq 0$.*

Proof. In order to find the equilibrium points of the NSFD scheme (3.6), we need to solve the system $S^{n+1} = S^n, I^{n+1} = I^n, Y^{n+1} = Y^n$. First, we start by expressing the system (3.7) in the form

$$\begin{aligned} S^{n+1} &= S^n + g_1(\Delta t, S^n, I^n, Y^n) f_1(S^n, I^n, Y^n), \\ I^{n+1} &= I^n + g_2(\Delta t, S^n, I^n, Y^n) f_2(S^n, I^n, Y^n), \\ Y^{n+1} &= Y^n + g_3(\Delta t, S^n, I^n, Y^n) f_3(S^n, I^n, Y^n), \end{aligned} \tag{3.8}$$

where,

$$\begin{aligned} g_1(\Delta t, S^n, I^n, Y^n) &:= \frac{\varphi_1(\Delta t, S^n, I^n, Y^n)}{1 + \varphi_1(\Delta t, S^n, I^n, Y^n)\Big(\frac{r}{k}(S^n + I^n) + \lambda I^n + \mathcal{H}(S^n, Y^n) + w_1\Big)}, \\ g_2(\Delta t, S^n, I^n, Y^n) &:= \frac{\varphi_2(\Delta t, S^n, I^n, Y^n)}{1 + \varphi_2(\Delta t, S^n, I^n, Y^n)\Big(\mathcal{G}(I^n, Y^n) + \mu + w_2\Big)}, \\ g_3(\Delta t, S^n, I^n, Y^n) &:= \frac{\varphi_3(\Delta t, S^n, I^n, Y^n)}{1 + \varphi_3(\Delta t, S^n, I^n, Y^n)\Big(\mathcal{F}(S^n, I^n, Y^n) + w_3\Big)}. \end{aligned} \tag{3.9}$$

Then, in light of (3.8) we get

$$\begin{aligned} g_1(\Delta t, S^n, I^n, Y^n) f_1(S^n, I^n, Y^n) = 0, \\ g_2(\Delta t, S^n, I^n, Y^n) f_2(S^n, I^n, Y^n) = 0, \\ g_3(\Delta t, S^n, I^n, Y^n) f_3(S^n, I^n, Y^n) = 0. \end{aligned} \tag{3.10}$$

Considering the positivity of functions $g_i(\Delta t, S^n, I^n, Y^n)$ for $i = 1, 2, 3$, it follows that $f_i(S^n, I^n, Y^n) = 0$ for $i = 1, 2, 3$. Therefore, the NSFD method (3.6) and the continuous-time model (2.1) have the same sets of equilibria. $\square$

In order to simplify the notation in the proof of the following theorem, let us introduce the following auxiliary functions appearing in the denominators of expressions in the right-hand side of (3.7):

$$\begin{aligned} \psi_1(S, I, Y) &:= \frac{r}{k}(S + I) + \lambda I + \mathcal{H}(S, Y) + w_1, \\ \psi_2(S, I, Y) &:= \mathcal{G}(I, Y) + \mu + w_2, \\ \psi_3(S, I, Y) &:= \mathcal{F}(S, I, Y) + w_3, \end{aligned} \tag{3.11}$$

defined for all $(S, I, Y) \in \Omega$. With all of this set, we can follow the ideas of Appendix B. in [37] and Theorem 5 in [21] to show that the proposed NSFD scheme is convergent with order 2.

Theorem 5 (Convergence)**.** *Let $\varphi_1(\Delta t, S, I, Y), \varphi_2(\Delta t, S, I, Y)$, and $\varphi_3(\Delta t, S, I, Y)$ be functions satisfying the following conditions*

$$\begin{aligned} \frac{\partial^2 \varphi_1}{\partial \Delta t^2}(0, S, I, Y) &= 2\psi_1(S, I, Y) + \frac{\partial f_1(S, I, Y)}{\partial S} + \frac{\partial f_1(S, I, Y)}{\partial I}\frac{f_2(S, I, Y)}{f_1(S, I, Y)} + \frac{\partial f_1(S, I, Y)}{\partial Y}\frac{f_3(S, I, Y)}{f_1(S, I, Y)}, \\ \frac{\partial^2 \varphi_2}{\partial \Delta t^2}(0, S, I, Y) &= 2\psi_2(S, I, Y) + \frac{\partial f_2(S, I, Y)}{\partial S}\frac{f_1(S, I, Y)}{f_2(S, I, Y)} + \frac{\partial f_2(S, I, Y)}{\partial I} + \frac{\partial f_2(S, I, Y)}{\partial Y}\frac{f_3(S, I, Y)}{f_2(S, I, Y)}, \\ \frac{\partial^2 \varphi_3}{\partial \Delta t^2}(0, S, I, Y) &= 2\psi_3(S, I, Y) + \frac{\partial f_3(S, I, Y)}{\partial S}\frac{f_1(S, I, Y)}{f_3(S, I, Y)} + \frac{\partial f_3(S, I, Y)}{\partial I}\frac{f_2(S, I, Y)}{f_3(S, I, Y)} + \frac{\partial f_3(S, I, Y)}{\partial Y}, \end{aligned} \tag{3.12}$$

for all $(S, I, Y) \in \Omega$ and $f_i(S, I, Y) \neq 0$, $i = 1, 2, 3$. Then, NSFD method (3.6) *is convergent of order* 2.

Proof. First, we apply Taylor's theorem to the components of the solution $S(t)$, $I(t)$, and $Y(t)$ to obtain

$$
\begin{aligned}
S(t^{n+1}) &= S(t^n) + \Delta t S'(t^n) + \frac{\Delta t^2}{2} S''(t^n) + \tau_S^n \\
&= S(t^n) + \Delta t f_1(S(t^n), I(t^n), Y(t^n)) + \frac{\Delta t^2}{2} \frac{\partial f_1(S(t^n), I(t^n), Y(t^n))}{\partial t} + \tau_S^n, \\
I(t^{n+1}) &= I(t^n) + \Delta t I'(t^n) + \frac{\Delta t^2}{2} I''(t^n) + \tau_I^n \\
&= I(t^n) + \Delta t f_2(S(t^n), I(t^n), Y(t^n)) + \frac{\Delta t^2}{2} \frac{\partial f_2(S(t^n), I(t^n), Y(t^n))}{\partial t} + \tau_I^n, \\
Y(t^{n+1}) &= Y(t^n) + \Delta t Y'(t^n) + \frac{\Delta t^2}{2} Y''(t^n) + \tau_Y^n \\
&= Y(t^n) + \Delta t f_3(S(t^n), I(t^n), Y(t^n)) + \frac{\Delta t^2}{2} \frac{\partial f_3(S(t^n), I(t^n), Y(t^n))}{\partial t} + \tau_Y^n,
\end{aligned}
\tag{3.13}
$$

where

$$
\tau_S^n := \frac{\Delta t^3}{6} \frac{d^3 S}{dt^3}(t^n + \theta_S \Delta t),\ \tau_I^n := \frac{\Delta t^3}{6} \frac{d^3 I}{dt^3}(t^n + \theta_I \Delta t),\ \tau_Y^n := \frac{\Delta t^3}{6} \frac{d^3 Y}{dt^3}(t^n + \theta_Y \Delta t), \tag{3.14}
$$

with $\theta_S, \theta_I, \theta_Y \in [0,1]$, are the local truncation errors. If $\tau^n = (\tau_S^n, \tau_I^n, \tau_Y^n)^{\mathrm{T}}$, then the norm of the local truncation error is

$$
\|\tau^n\|_\infty = \sup\left\{ \left| \frac{\Delta t^3}{6} \frac{d^3 S}{dt^3}(t^n + \theta_S \Delta t) \right|, \left| \frac{\Delta t^3}{6} \frac{d^3 I}{dt^3}(t^n + \theta_I \Delta t) \right|, \left| \frac{\Delta t^3}{6} \frac{d^3 Y}{dt^3}(t^n + \theta_Y \Delta t) \right| \right\}.
$$

By Assumptions 2.1, we have that the right-hand side functions are of class $C^2([0,T], \mathbb{R}^3)$, so it follows that the solution is of class $C^3([0,T], \mathbb{R}^3)$ with bounded derivatives over $[0,T]$. Therefore, we have the following bound for the local truncation error:

$$
\|\tau^n\|_\infty \leq \frac{1}{6} M \Delta t^3. \tag{3.15}
$$

We now employ the notation $(F_1(\Delta t, S^n, I^n, Y^n), F_2(\Delta t, S^n, I^n, Y^n), F_3(\Delta t, S^n, I^n, Y^n))^{\mathrm{T}}$ for the right-hand side of the scheme (3.7). It follows from (3.8) that

$$
F_1(0, S, I, Y) = S, \quad F_2(0, S, I, Y) = I, \quad F_3(0, S, I, Y) = Y. \tag{3.16}
$$

In addition, we observe that the first partial derivative of F_1 with respect to Δt is given by

$$
\begin{aligned}
\frac{\partial F_1(\Delta t, S, I, Y)}{\partial \Delta t} &= \frac{\partial}{\partial \Delta t}(S + g_1(\Delta t, S, I, Y) f_1(S, I, Y)) \\
&= \left[\frac{\partial \varphi_1(\Delta t, S, I, Y)}{\partial \Delta t} \left(\frac{1}{1 + \varphi_1(\Delta t, S, I, Y)\psi_1(S, I, Y)} \right) \right. \\
&\quad \left. + \varphi_1(\Delta t, S, I, Y) \left(\frac{ -\dfrac{\partial \varphi_1(\Delta t, S, I, Y)}{\partial \Delta t} \psi_1(S, I, Y)}{(1 + \varphi_1(\Delta t, S, I, Y)\psi_1(S, I, Y))^2} \right) \right] f_1(S, I, Y).
\end{aligned}
\tag{3.17}
$$

Then, taking into account that $\varphi_1(0, S, I, Y) = 0$ and $\frac{\partial \varphi_1(0,S,I,Y)}{\partial \Delta t} = 1$, it follows that

$$
\frac{\partial F_1(0, S, I, Y)}{\partial \Delta t} = f_1(S, I, Y). \tag{3.18}
$$

Following the same lines, one gets the following result:

$$
\frac{\partial F_2(0, S, I, Y)}{\partial \Delta t} = f_2(S, I, Y), \quad \frac{\partial F_3(0, S, I, Y)}{\partial \Delta t} = f_3(S, I, Y). \tag{3.19}
$$

Now, from (3.17) we can compute the second derivative of F_1 by

$$\begin{aligned}\frac{\partial^2 F_1(\Delta t, S, I, Y)}{\partial \Delta t^2} = \Bigg[&\frac{\partial^2 \varphi_1(\Delta t, S, I, Y)}{\partial \Delta t^2}\left(\frac{1}{1+\varphi_1(\Delta t, S, I, Y)\psi_1(S, I, Y)}\right)\\ &-2\frac{\partial \varphi_1(\Delta t, S, I, Y)}{\partial \Delta t}\frac{\psi_1(S, I, Y)}{(1+\varphi_1(\Delta t, S, I, Y)\psi_1(S, I, Y))^2}\\ &+\varphi_1(\Delta t, S, I, Y)\frac{\partial}{\partial \Delta t}\left(\frac{-\frac{\partial \varphi_1(\Delta t, S, I, Y)}{\partial \Delta t}\psi_1(S, I, Y)}{(1+\varphi_1(\Delta t, S, I, Y)\psi_1(S, I, Y))^2}\right)\Bigg] f_1(S, I, Y).\end{aligned}$$

Again we use the facts $\varphi_1(0, S, I, Y) = 0$ and $\frac{\partial \varphi_1(0,S,I,Y)}{\partial \Delta t} = 1$, to obtain

$$\frac{\partial^2 F_1(0, S, I, Y)}{\partial \Delta t^2} = f_1(S, I, Y)\left[\frac{\partial^2 \varphi_1(0, S, I, Y)}{\partial \Delta t^2} - 2\psi_1(S, I, Y)\right]. \tag{3.20}$$

By employing the same arguments, we can prove that

$$\begin{aligned}\frac{\partial^2 F_2(0, S, I, Y)}{\partial \Delta t^2} &= f_2(S, I, Y)\left[\frac{\partial^2 \varphi_2(0, S, I, Y)}{\partial \Delta t^2} - 2\psi_2(S, I, Y)\right],\\ \frac{\partial^2 F_3(0, S, I, Y)}{\partial \Delta t^2} &= f_3(S, I, Y)\left[\frac{\partial^2 \varphi_3(0, S, I, Y)}{\partial \Delta t^2} - 2\psi_3(S, I, Y)\right].\end{aligned} \tag{3.21}$$

Using the Taylor expansion and combining (3.16)-(3.21), we obtain that

$$\begin{aligned}S^{n+1} &= F_1(0, S^n, I^n, Y^n) + \Delta t\frac{\partial F_1(0, S^n, I^n, Y^n)}{\partial \Delta t} + \frac{\Delta t^2}{2}\frac{\partial^2 F_1(0, S^n, I^n, Y^n)}{\partial \Delta t^2} + O(\Delta t^3)\\ &= S^n + \Delta t f_1(S^n, I^n, Y^n) + \frac{\Delta t^2}{2} f_1(S^n, I^n, Y^n)\left[\frac{\partial^2 \varphi_1(0, S^n, I^n, Y^n)}{\partial \Delta t^2} - 2\psi_1(S^n, I^n, Y^n)\right] + O(\Delta t^3),\\ I^{n+1} &= F_2(0, S^n, I^n, Y^n) + \Delta t\frac{\partial F_2(0, S^n, I^n, Y^n)}{\partial \Delta t} + \frac{\Delta t^2}{2}\frac{\partial^2 F_2(0, S^n, I^n, Y^n)}{\partial \Delta t^2} + O(\Delta t^3)\\ &= I^n + \Delta t f_2(S^n, I^n, Y^n) + \frac{\Delta t^2}{2} f_2(S^n, I^n, Y^n)\left[\frac{\partial^2 \varphi_2(0, S^n, I^n, Y^n)}{\partial \Delta t^2} - 2\psi_2(S^n, I^n, Y^n)\right] + O(\Delta t^3),\\ Y^{n+1} &= F_3(0, S^n, I^n, Y^n) + \Delta t\frac{\partial F_3(0, S^n, I^n, Y^n)}{\partial \Delta t} + \frac{\Delta t^2}{2}\frac{\partial^2 F_3(0, S^n, I^n, Y^n)}{\partial \Delta t^2} + O(\Delta t^3)\\ &= Y^n + \Delta t f_3(S^n, I^n, Y^n) + \frac{\Delta t^2}{2} f_3(S^n, I^n, Y^n)\left[\frac{\partial^2 \varphi_3(0, S^n, I^n, Y^n)}{\partial \Delta t^2} - 2\psi_3(S^n, I^n, Y^n)\right] + O(\Delta t^3).\end{aligned} \tag{3.22}$$

Now, if $X^n = (S^n, I^n, Y^n)^{\mathrm{T}}$ and $X(t^n) = (S(t^n), I(t^n), Y(t^n))^{\mathrm{T}}$, then from (3.12), (3.13), (3.15), and (3.22) we infer that the difference equation for the error $e^{n+1} := X^{n+1} - X(t^{n+1})$ is

$$e^{n+1} := e^n + \Delta t\begin{pmatrix} f_1(S^n, I^n, Y^n) - f_1(S(t^n), I(t^n), Y(t^n))\\ f_2(S^n, I^n, Y^n) - f_2(S(t^n), I(t^n), Y(t^n))\\ f_3(S^n, I^n, Y^n) - f_3(S(t^n), I(t^n), Y(t^n))\end{pmatrix} + \frac{\Delta t^2}{2}\begin{pmatrix}\frac{\partial f_1}{\partial t}(S^n, I^n, Y^n) - \frac{\partial f_1}{\partial t}(S(t^n), I(t^n), Y(t^n))\\ \frac{\partial f_2}{\partial t}(S^n, I^n, Y^n) - \frac{\partial f_2}{\partial t}(S(t^n), I(t^n), Y(t^n))\\ \frac{\partial f_3}{\partial t}(S^n, I^n, Y^n) - \frac{\partial f_3}{\partial t}(S(t^n), I(t^n), Y(t^n))\end{pmatrix} + \tilde{M}\Delta t^3,$$

for some constant $\tilde{M} > 0$ independent of Δt. Then, if $L = \max(L_S, L_I, L_Y)$ and $\tilde{L} = \max(\tilde{L}_S, \tilde{L}_I, \tilde{L}_Y)$, where L_j and $\tilde{L}_j$, are the Lipschitz constants for f_j and $\partial_t f_j$, for $j \in \{S, I, Y\}$, respectively, then we get the following bound for the local truncation error

$$\|e^{n+1}\|_\infty \le \|e^n\|_\infty + \left(\Delta t L + \frac{\Delta t^2}{2}\tilde{L}\right)\|e^n\|_\infty + \tilde{M}\Delta t^3 = \left(1 + \Delta t L + \frac{\Delta t^2}{2}\tilde{L}\right)\|e^n\|_\infty + \tilde{M}\Delta t^3. \tag{3.23}$$

Then, by applying the discrete Gronwall inequality (see Lemma 2, in Appendix A) we arrive at

$$\|e^n\|_\infty \le e^{n\left(L\Delta t+\frac{\Delta t^2}{2}\tilde{L}\right)}\|e^0\|_\infty + e^{n\left(L\Delta t+\frac{\Delta t^2}{2}\tilde{L}\right)}(n\tilde{M}\Delta t^3) \le e^{\left(LT+\frac{T^2}{2}\tilde{L}\right)}\|e^0\|_\infty + e^{\left(LT+\frac{T^2}{2}\tilde{L}\right)}\tilde{M}T\Delta t^2. \tag{3.24}$$

Hence we conclude that,

$$\|X^n - X(t^n)\|_\infty \le e^{\left(LT+\frac{LT^2}{2}\right)}\|X^0 - X(0)\|_\infty + e^{\left(LT+\frac{T^2}{2}\tilde{L}\right)}T\tilde{M}\Delta t^2 \le O(\|X^0 - X(0)\|_\infty) + O(\Delta t^2). \tag{3.25}$$

Thus, the scheme (3.7) is convergent of order 2. □

Remark 3. *Let us define*

$$\begin{aligned}
\tau_1(S,I,Y) &:= 2\psi_1(S,I,Y) + \frac{\partial f_1(S,I,Y)}{\partial S} + \frac{\partial f_1(S,I,Y)}{\partial I}\frac{f_2(S,I,Y)}{f_1(S,I,Y)} + \frac{\partial f_1(S,I,Y)}{\partial Y}\frac{f_3(S,I,Y)}{f_1(S,I,Y)},\\
\tau_2(S,I,Y) &:= 2\psi_2(S,I,Y) + \frac{\partial f_2(S,I,Y)}{\partial S}\frac{f_1(S,I,Y)}{f_2(S,I,Y)} + \frac{\partial f_2(S,I,Y)}{\partial I} + \frac{\partial f_2(S,I,Y)}{\partial Y}\frac{f_3(S,I,Y)}{f_2(S,I,Y)},\\
\tau_3(S,I,Y) &:= 2\psi_3(S,I,Y) + \frac{\partial f_3(S,I,Y)}{\partial S}\frac{f_1(S,I,Y)}{f_3(S,I,Y)} + \frac{\partial f_3(S,I,Y)}{\partial I}\frac{f_2(S,I,Y)}{f_3(S,I,Y)} + \frac{\partial f_3(S,I,Y)}{\partial Y},
\end{aligned} \tag{3.26}$$

for all $(S,I,Y) \in \Omega$, where ψ_i are those defined in (3.11). *Therefore the denominator functions can be chosen as*

$$\varphi_i(\Delta t, S, I, Y) = \begin{cases} \dfrac{e^{\tau_i(S,I,Y)\Delta t} - 1}{\tau_i(S,I,Y)}, & \text{if } \tau_i(S,I,Y) \neq 0\\ \Delta t, & \text{if } \tau_i(S,I,Y) = 0. \end{cases} \tag{3.27}$$

The functions (3.27) *satisfies not only* (3.12) *but $\varphi_i(\Delta t, S, I, Y) = \Delta t^2 + O(\Delta t^3)$, as $\Delta t \to 0$ and $\varphi_i(\Delta t, S, I, Y) > 0$, for all $\Delta t > 0$ and $(S,I,Y) \in \Omega$.*

4. Stability of equilibrium points

There are several mathematical models that can be put in the form of (2.1). To fix ideas, we will focus on the two specific eco-epidemiological prey-predator models with functional responses given by (2.3) and (2.4). The main difference between these models is the choice of the functional responses, in the sense that for the first model these functional responses are linear, while for the second they are nonlinear. In addition to the dynamical properties preserved by the NSFD scheme (3.7) obtained in Theorem 3.2, we can analytically show the consistency of the NSFD scheme with the continuous model in terms of the local stability of certain equilibrium points.

4.1. NSFD scheme for Bairagi et. al. model [32]

The scaled version of the system of ODEs modeling the ecosystem is as follows,

$$\begin{aligned}
\frac{dS}{dt} &= bS\,(1 - (S + I)) - SI - m_1SY,\\
\frac{dI}{dt} &= SI - dIY - eI,\\
\frac{dY}{dt} &= -\theta dIY - gY + \theta m_1SY,
\end{aligned} \tag{4.1}$$

with equilibrium points given by

(*i*) The trivial equilibrium $\mathcal{E}_0 = (0,0,0)$.

(*ii*) The axial equilibrium $\mathcal{E}_1 = (1,0,0)$.

(*iii*) The planar equilibria $\mathcal{E}_2 = \left(e, \frac{b(1-e)}{b+1}, 0\right)$ and $\mathcal{E}_3 = \left(\frac{g}{\theta m_1}, 0, \frac{b(\theta m_1 - g)}{\theta m_1^2}\right)$.

(iv) The interior equilibrium $\mathcal{E}^* = (S^*, I^*, Y^*)$, where $S^* = \frac{bd\theta + m_1 e\theta + bg + g}{\theta(bd + bm_1 + 2m_2)}$, $I^* = \frac{m_1\theta S^* - g}{d\theta}$, and $Y^* = \frac{S^* - e}{d}$.

Remark 4. *The equilibria $\mathcal{E}_0$ and $\mathcal{E}_1$ exist for all parameter values, the equilibrium $\mathcal{E}_2$ exists if $e < 1$, the equilibrium $\mathcal{E}_3$ exists if $m_1 > \frac{g}{\theta}$ and the interior equilibrium $\mathcal{E}_4$ exists if*

$$e < 1, \quad m_1 > \frac{g}{\theta}, \quad \max\left(e, \frac{g}{m_1\theta}\right) < S^* < \frac{g + d\theta}{\theta m_1 + d}. \tag{4.2}$$

The final form of the NSFD scheme for this model is described by the following algorithm:

Algorithm 1 Second Order NSFD scheme for Model (4.1)

Input: $b, m_1, e, d, \theta, g, w_1, w_2, w_3, T, \Delta t, S^0, I^0, Y^0$

$S^n \leftarrow S^0, \;\; I^n \leftarrow I^0, \;\; Y^n \leftarrow Y^0$

$M \leftarrow \lceil \frac{T}{\Delta t} \rceil$

for $n = 1, \ldots, M-1$ **do**

compute $\varphi_i(\Delta t, S^n, I^n, Y^n)$, $i = 1, 2, 3$ from (3.27) given $\Delta t, S^n, I^n, Y^n$

$$S^{n+1} \leftarrow \frac{S^n + b\varphi_1(\Delta t, S^n, I^n, Y^n)S^n + w_1\varphi_1(\Delta t, S^n, I^n, Y^n)S^n}{1 + \varphi_1(\Delta t, S^n, I^n, Y^n)\Big(b(S^n + I^n) + I^n + m_1 Y^n + w_1\Big)}$$

$$I^{n+1} \leftarrow \frac{I^n + \varphi_2(\Delta t, S^n, I^n, Y^n)S^n I^n + w_2\varphi_2(\Delta t, S^n, I^n, Y^n)I^n}{1 + \varphi_2(\Delta t, S^n, I^n, Y^n)\Big(dY^n + e + w_2\Big)}$$

$$Y^{n+1} \leftarrow \frac{Y^n + \varphi_3(\Delta t, S^n, I^n, Y^n)\theta m_1 S^n Y^n + w_3\varphi_3(\Delta t, S^n, I^n, Y^n)Y^n}{1 + \varphi_3(\Delta t, S^n, I^n, Y^n)\Big(\theta d I^n + g + w_3\Big)}$$

end for

Output: $\{(S^1, I^1, Y^1), \ldots, (S^M, I^M, Y^M)\}$

Now, we address the stability of the equilibrium points of this discrete system.

Theorem 6. *Let $w_1, w_2, w_3 \in \mathbb{R}$ satisfying $w_1 \geq 0$, $w_2 \geq 1$ and $w_3 \geq \theta m_1$. Then, the NSFD method given by Algorithm 1 is dynamically consistent with respect to the local stability of the equilibrium $\mathcal{E}_i$, $i = 0, \ldots, 4$ of model* (4.1) *for all the values of the step size Δt, that is:*

(i) *The trivial equilibrium $\mathcal{E}_0$ is unstable.*

(ii) *The axial equilibrium $\mathcal{E}_1$ is locally asymptotically stable if $e > 1$ and $m_1 < \frac{g}{\theta}$.*

(iii) *The disease-free equilibrium $\mathcal{E}_2$ is locally asymptotically stable if $e < 1$ and $m_1 < \frac{1}{e\theta}[g + \frac{bd\theta(1-e)}{b+1}]$.*

(iv) *The predator-free equilibrium $\mathcal{E}_3$ is locally asymptotically stable if $m_1 > \frac{g}{e\theta}$ or $m_1 > \frac{g}{\theta}$ according as $e < 1$ or $e > 1$.*

(v) *The interior equilibrium $\mathcal{E}_4$ is unstable.*

Proof. Let us use the notation $\varphi_{i,\ell}(\Delta t) := \varphi_i(\Delta t, \mathcal{E}_\ell)$, where $i = 1, 2, 3$, and $\ell = 0, \ldots, 4$, for denoting the evaluation of the functions φ_i at the equilibrium points $\mathcal{E}_\ell$. We discuss the stability of each equilibrium point:

(i) The Jacobian matrix of the system (4.1) at $\mathcal{E}_0$ is

$$\mathcal{J}^C(\mathcal{E}_0) = \begin{pmatrix} b & 0 & 0 \\ 0 & -e & 0 \\ 0 & 0 & -g \end{pmatrix}.$$

Then by Lemma 3, in Appendix A we get

$$\mathcal{J}^D(\mathcal{E}_0) = \begin{pmatrix} 1 + \frac{b\varphi_{1,0}(\Delta t)}{1+\varphi_{1,0}(\Delta t)w_1} & 0 & 0 \\ 0 & 1 - \frac{e\varphi_{2,0}(\Delta t)}{1+\varphi_{2,0}(\Delta t)(e+w_2)} & 0 \\ 0 & 0 & 1 - \frac{g\varphi_{3,0}(\Delta t)}{1+\varphi_{3,0}(\Delta t)(g+w_3)} \end{pmatrix}.$$

Clearly, $\xi_1 = 1 + \frac{b\varphi_{1,0}(\Delta t)}{1+\varphi_{1,0}(\Delta t)w_1} > 1$ is an eigenvalue of $\mathcal{J}^D(\mathcal{E}_0)$. Thus, by Lemma 4, in Appendix A it follows that $\mathcal{E}_0$ is unstable for the discrete system. *(ii)* The Jacobian matrix of the system (4.1) at $\mathcal{E}_1$ is

$$\mathcal{J}^D(\mathcal{E}_1) = \begin{pmatrix} 1 - \frac{b\varphi_{1,1}(\Delta t)}{1+\varphi_{1,1}(\Delta t)(b+w_1)} & -\frac{(1+b)\varphi_{1,1}(\Delta t)}{1+\varphi_{1,1}(\Delta t)(b+w_1)} & -\frac{m_1\varphi_{1,1}(\Delta t)}{1+\varphi_{1,1}(\Delta t)(b+w_1)} \\ 0 & \frac{1+\varphi_{2,1}(\Delta t)(1+w_2)}{1+\varphi_{2,1}(\Delta t)(e+w_2)} & 0 \\ 0 & 0 & 1 + \frac{\varphi_{3,1}(\Delta t)(\theta m_1 - g)}{1+\varphi_{3,1}(\Delta t)(g+w_3)} \end{pmatrix}.$$

The characteristic polynomial of $\mathcal{J}^D(\mathcal{E}_1)$ can be written as

$$P_{\mathcal{E}_1}(\xi) = \left(1 - \xi - \frac{b\varphi_{1,1}(\Delta t)}{1 + \varphi_{1,1}(\Delta t)(b + w_1)}\right)\left(-\xi + \frac{1 + \varphi_{2,1}(\Delta t)(1 + w_2)}{1 + \varphi_{2,1}(\Delta t)(e + w_2)}\right)\left(1 - \xi + \frac{\varphi_{3,1}(\Delta t)(\theta m_1 - g)}{1 + \varphi_{3,1}(\Delta t)(g + w_3)}\right).$$

Thus, the eigenvalues of $\mathcal{J}^D(\mathcal{E}_1)$ are

$$\xi_1^{(1)} = 1 - \frac{b\varphi_{1,1}(\Delta t)}{1 + \varphi_{1,1}(\Delta t)(b + w_1)}, \; \xi_2^{(1)} = \frac{1 + \varphi_{2,1}(\Delta t)(1 + w_2)}{1 + \varphi_{2,1}(\Delta t)(e + w_2)}, \; \xi_3^{(1)} = 1 - \frac{\varphi_{3,1}(\Delta t)(g - \theta m_1)}{1 + \varphi_{3,1}(\Delta t)(g + w_3)}.$$

Clearly, we have that $0 < \xi_1^{(1)} < 1$. The condition $e > 1$ implies that $0 < \xi_2^{(1)} < 1$ and the condition $m_1 < \frac{g}{\theta}$, i.e $g > m_1\theta$ produces $0 < \xi_3^{(1)} < 1$. Therefore, all the eigenvalues of $\mathcal{J}^D(\mathcal{E}_1)$ belong to the open unit disk. In light of Lemma 4, in Appendix A we have that the axial equilibrium $\mathcal{E}_1$ is locally asymptotically stable if $e > 1$ and $m_1 < \frac{g}{\theta}$. *(iii)* The Jacobian matrix of system (4.1) at $\mathcal{E}_2$ is

$$\mathcal{J}^D(\mathcal{E}_2) = \begin{pmatrix} 1 - \frac{be\varphi_{1,2}(\Delta t)}{1+\varphi_{1,2}(\Delta t)(b+w_1)} & -\frac{(1+b)e\varphi_{1,2}(\Delta t)}{1+\varphi_{1,2}(\Delta t)(b+w_1)} & -\frac{em_1\varphi_{1,2}(\Delta t)}{1+\varphi_{1,2}(\Delta t)(b+w_1)} \\ \frac{b(1-e)\varphi_{2,2}(\Delta t)}{(1+b)(1+\varphi_{2,2}(\Delta t))(e+w_2)} & 1 & \frac{bd(1-e)\varphi_{2,2}(\Delta t)}{(1+b)(1+\varphi_{2,2}(\Delta t)(e+w_2))} \\ 0 & 0 & \frac{(1+b)(1+\varphi_{3,2}(\Delta t)(e\theta m_1+w_3)}{(1+b)(1+g\varphi_{3,2}(\Delta t))+\varphi_{3,2}(\Delta t)(bd\theta(1-e)+(1+b)w_3)} \end{pmatrix}.$$

The characteristic polynomial of $\mathcal{J}^D(\mathcal{E}_2)$ can be written as

$$P_{\mathcal{E}_2}(\xi) = Q_{\mathcal{E}_2}(\xi)\left(\frac{(1 + b)(1 + \varphi_{3,2}(\Delta t)(e\theta m_1 + w_3)}{(1 + b)(1 + g\varphi_{3,2}(\Delta t)) + \varphi_{3,2}(\Delta t)(bd\theta(1 - e) + (1 + b)w_3)} - \xi\right),$$

where $Q_{\mathcal{E}_2}$ is a second-order polynomial of the form $Q_{\mathcal{E}_2}(\xi) = \xi^2 + a_1^{(2)}\xi + a_0^{(2)}$, where

$$a_0^{(2)} = 1 - \frac{be\varphi_{1,2}(\Delta t)}{1 + \varphi_{1,2}(\Delta t)(b + w_1)} + \frac{be(1 - e)\varphi_{1,2}(\Delta t)\varphi_{2,2}(\Delta t)}{(1 + \varphi_{1,2}(\Delta t)(b + w_1))(1 + \varphi_{2,2}(\Delta t)(e + w_2))},$$
$$a_1^{(2)} = \frac{be\varphi_{1,2}(\Delta t)}{1 + \varphi_{1,2}(\Delta t)(b + w_1)} - 2.$$

We notice that

$$\xi_1^{(2)} = \frac{(1 + b)(1 + \varphi_{3,2}(\Delta t)(e\theta m_1 + w_3)}{(1 + b)(1 + g\varphi_{3,2}(\Delta t)) + \varphi_{3,2}(\Delta t)(bd\theta(1 - e) + (1 + b)w_3)},$$

is an eigenvalue of $\mathcal{J}^D(\mathcal{E}_2)$. The condition $e < 1$ implies that $\xi_1^{(2)} > 0$. We rewrite $\xi_1^{(2)}$ as

$$\xi_1^{(2)} = \frac{1 + b + \varphi_{3,2}(\Delta t)((1 + b)e\theta m_1 + (1 + b)w_3)}{1 + b + \varphi_{3,2}(\Delta t)((b + 1)g + bd\theta(1 - e) + (1 + b)w_3)}.$$

In this way we can see that condition $m_1 < \frac{1}{e\theta}[g + \frac{bd\theta(1-e)}{b+1}]$, i.e $(1+b)e\theta m_1 < (b+1)g + bd\theta(1-e)$ implies that $\xi_1^{(2)} < 1$.

Now let us analyze the second order polynomial $Q_{\mathcal{E}_2}$. We claim that the roots $\xi_2^{(2)}$ and $\xi_3^{(2)}$ of $Q_{\mathcal{E}_2}$ are in the unit disk. To prove this, we check all the conditions of Lemma 5, in Appendix A. First, the condition $e < 1$ implies $a_0^{(2)} > 0$. In addition, as $w_2 \geq 1$, we get

$$\frac{be(1 - e)\varphi_{1,2}(\Delta t)\varphi_{2,2}(\Delta t)}{(1 + \varphi_{1,2}(\Delta t)(b + w_1))(1 + \varphi_{2,2}(\Delta t)(e + w_2))} < \frac{be(1 - e)\varphi_{1,2}(\Delta t)\varphi_{2,2}(\Delta t)}{(1 + \varphi_{1,2}(\Delta t)(b + w_1))\varphi_{2,2}(\Delta t)w_2} < \frac{be\varphi_{1,2}(\Delta t)}{1 + \varphi_{1,2}(\Delta t)(b + w_1)}.$$

Hence

$$a_0^{(2)} < 1 - \frac{be\varphi_{1,2}(\Delta t)}{1+\varphi_{1,2}(\Delta t)(b+w_1)} + \frac{be\varphi_{1,2}(\Delta t)}{1+\varphi_{1,2}(\Delta t)(b+w_1)} = 1.$$

So we have $|a_0^{(2)}| < 1$. On the other hand, we observe

$$1 + a_1^{(2)} + a_0^{(2)} = \frac{be(1-e)\varphi_{1,2}(\Delta t)\varphi_{2,2}(\Delta t)}{(1+\varphi_{1,2}(\Delta t)(b+w_1))(1+\varphi_{2,2}(\Delta t)(e+w_2))} > 0,$$

and

$$1 - a_1^{(2)} + a_0^{(2)} = 2\left(2 - \frac{be\varphi_{1,2}(\Delta t)}{1+\varphi_{1,2}(\Delta t)(b+w_1)}\right) + \frac{be(1-e)\varphi_{1,2}(\Delta t)\varphi_{2,2}(\Delta t)}{(1+\varphi_{1,2}(\Delta t)(b+w_1))(1+\varphi_{2,2}(\Delta t)(e+w_2))} > 0.$$

Therefore, from the analysis above we get that $|\xi_j^{(2)}| < 1$, for $j = 1,2,3$. By using Lemma 4, in Appendix A we conclude that the disease-free equilibrium $\mathcal{E}_2$ is locally asymptotically stable if $e < 1$ and $m_1 < \frac{1}{e\theta}[g + \frac{bd\theta(1-e)}{b+1}]$. (iv) The Jacobian matrix of system (4.1) at $\mathcal{E}_3$ is

$$\mathcal{J}^D(\mathcal{E}_3) = \begin{pmatrix} 1 - \frac{bg\varphi_{1,3}(\Delta t)}{1+\varphi_{1,3}(\Delta t)(b+w_1)} & -\frac{(1+b)g\varphi_{1,3}(\Delta t)}{\theta m_1(1+\varphi_{1,3}(\Delta t)(b+w_1))} & -\frac{g\varphi_{1,3}(\Delta t)}{\theta(1+\varphi_{1,3}(\Delta t)(b+w_1))} \\ 0 & \frac{m_1(g\varphi_{2,3}(\Delta t)+m_1\theta(1+\varphi_{2,3}(\Delta t)w_2))}{-bdg\varphi_{2,3}(\Delta t)+\theta m_1(bd\varphi_{2,3}(\Delta t)+m_1(1+(e+w_2)\varphi_{2,3}(\Delta t)))} & 0 \\ \frac{b\varphi_{3,3}(\Delta t)(\theta m_1-g)}{m_1(1+\varphi_{3,3}(\Delta t)(g+w_3))} & \frac{bd\varphi_{3,3}(\Delta t)(g-\theta m_1)}{m_1^2(1+\varphi_{3,3}(\Delta t)(g+w_3))} & 1 \end{pmatrix}.$$

The characteristic polynomial of $\mathcal{J}^D(\mathcal{E}_3)$ can be written as

$$P_{\mathcal{E}_3}(\xi) = Q_{\mathcal{E}_3}(\xi)\left(\frac{m_1(g\varphi_{2,3}(\Delta t) + m_1\theta(1+\varphi_{2,3}(\Delta t)w_2))}{-bdg\varphi_{2,3}(\Delta t) + \theta m_1(bd\varphi_{2,3}(\Delta t) + m_1(1+(e+w_2)\varphi_{2,3}(\Delta t)))} - \xi\right),$$

where $Q_{\mathcal{E}_3}$ is a second order polynomial of the form $Q_{\mathcal{E}_3}(\xi) = \xi^2 + a_1^{(3)}\xi + a_0^{(3)}$, where

$$\begin{aligned} a_0^{(3)} &= 1 + \frac{bg\varphi_{1,3}(\Delta t)\varphi_{3,3}(\Delta t)}{(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))} - \frac{bg\varphi_{1,3}(\Delta t)}{\theta m_1(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))} \\ &\quad - \frac{2bg^2\varphi_{1,3}(\Delta t)\varphi_{3,3}(\Delta t)}{\theta m_1(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))} - \frac{bg\varphi_{1,3}(\Delta t)\varphi_{3,3}(\Delta t)w_3}{\theta m_1(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))}, \\ a_1^{(3)} &= -2 + \frac{bg\varphi_{1,3}(\Delta t)}{\theta m_1(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))} + \frac{bg^2\varphi_{1,3}(\Delta t)\varphi_{3,3}(\Delta t)}{\theta m_1(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))} \\ &\quad + \frac{bg\varphi_{1,3}(\Delta t)\varphi_{3,3}(\Delta t)w_3}{\theta m_1(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))}. \end{aligned}$$

After some algebraic manipulations, one gets that

$$a_0^{(3)} = \frac{b\varphi_{1,3}(\Delta t)(1+\varphi_{3,3}(\Delta t)(2g+w_3))(\theta m_1 - g) + \theta m_1(1+\varphi_{1,3}(\Delta t)w_1)(1+\varphi_{3,3}(\Delta t)(g+w_3))}{\theta m_1(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))}.$$

So, the condition $m_1 > \frac{g}{e\theta}$ or $m_1 > \frac{g}{\theta}$ according as $e < 1$ or $e > 1$, implies in any case that $\theta m_1 - g > 0$ and therefore we get that $a_0^{(3)} > 0$. In addition, by using the restriction over the weight $w_3 \geq \theta m_1$ we observe that

$$\begin{aligned} a_0^{(3)} &< 1 + \frac{bg\varphi_{1,3}(\Delta t)\varphi_{3,3}(\Delta t)}{(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))} - \frac{bg\varphi_{1,3}(\Delta t)\varphi_{3,3}(\Delta t)w_3}{\theta m_1(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))} \\ &= 1 - \frac{bg\varphi_{1,3}(\Delta t)\varphi_{3,3}(\Delta t)}{1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))}\left(\frac{w_3}{\theta m_1} - 1\right) < 1. \end{aligned}$$

In this way we see that $|a_0^{(3)}| < 1$. On the other hand, after some computations and by using again the condition $\theta m_1 - g > 0$ one gets that

$$1 + a_1^{(3)} + a_0^{(3)} = \frac{bg\varphi_{1,3}(\Delta t)\varphi_{3,3}(\Delta t)(\theta m_1 - g)}{(1+\varphi_{1,3}(\Delta t)(b+w_1))(1+\varphi_{3,3}(\Delta t)(g+w_3))} > 0.$$

Finally, we observe that

$$
\begin{aligned}
1 - a_1^{(3)} + a_0^{(3)} &= \frac{bg\varphi_{1,3}(\Delta t)(2 + 3g\varphi_{3,3}(\Delta t) + 2w_3\varphi_{3,3}(\Delta t))(\theta m_1 - g) + 2\theta m_1(2 + b\varphi_{1,3}(\Delta t) + w_1\varphi_{1,3}(\Delta t))(1 + g\varphi_{3,3}(\Delta t) + w_3\varphi_{3,3}(\Delta t))}{\theta m_1(1 + \varphi_{1,3}(\Delta t)(b + w_1))(1 + \varphi_{3,3}(\Delta t)(g + w_3))} \\
&> 0.
\end{aligned}
$$

Therefore all the conditions of Lemma 5, in Appendix A are fulfilled, so we get that $|\xi_j^{(2)}| < 1$, for $j = 1, 2, 3$. By using Lemma 4, Appendix A we conclude that the disease-free equilibrium $\mathcal{E}_3$ is locally asymptotically stable if $m_1 > \frac{g}{e\theta}$ or $m_1 > \frac{g}{\theta}$ according as $e < 1$ or $e > 1$. (v) The Jacobian matrix of system (4.1) at $\mathcal{E}_4$ is $\mathcal{J}^D(\mathcal{E}_4) = (\mathcal{J}_{ij}^D(\mathcal{E}_4))_{i,j=1}^3$, where

$$
\begin{aligned}
\mathcal{J}_{11}^D(\mathcal{E}_4) &= 1 - \frac{b\varphi_{1,4}(\Delta t)(g(b + 1) + bd\theta + e\theta m_1)}{(\theta(bd + (2 + b)m_1))(1 + \varphi_{1,4}(\Delta t)(b + w_1))}, \\
\mathcal{J}_{12}^D(\mathcal{E}_4) &= -\frac{(1 + b)\varphi_{1,4}(\Delta t)(g(b + 1) + bd\theta + e\theta m_1)}{(\theta(bd + (2 + b)m_1))(1 + \varphi_{1,4}(\Delta t)(b + w_1))}, \\
\mathcal{J}_{13}^D(\mathcal{E}_4) &= -\frac{m_1\varphi_{1,4}(\Delta t)(g(b + 1) + bd\theta + e\theta m_1)}{(\theta(bd + (2 + b)m_1))(1 + \varphi_{1,4}(\Delta t)(b + w_1))}, \\
\mathcal{J}_{21}^D(\mathcal{E}_4) &= \frac{\varphi_{2,4}(\Delta t)(m_1(e\theta m_1 + bd\theta - g) - bdg)}{d(b + 1)g\varphi_{2,4}(\Delta t) + bd\theta(1 + \varphi_{2,4}(\Delta t)) + bd\theta w_2\varphi_{2,4}(\Delta t)\theta m_1(2 + b + e\varphi_{2,4}(\Delta t) + (b + 2)\varphi_{2,4}(\Delta t)w_2)}, \\
\mathcal{J}_{22}^D(\mathcal{E}_4) &= 1, \\
\mathcal{J}_{23}^D(\mathcal{E}_4) &= \frac{\varphi_{2,4}(\Delta t))(bdg + m_1(g - bd\theta e\theta m_1))}{(1 + b)g\varphi_{2,4}(\Delta t)) + bd(1 + \varphi_{2,4}(\Delta t)))\theta + bdw_2\theta\varphi_{2,4}(\Delta t)) + \theta m_1(2 + b + e\varphi_{2,4}(\Delta t)) + (b + 2)w_2\varphi_{2,4}(\Delta t)))}, \\
\mathcal{J}_{31}^D(\mathcal{E}_4) &= \frac{\varphi_{3,4}(\Delta t)m_1((1 + b)g - bd(e - 1)\theta - (b + 1)e\theta m_1)}{d(bd + m_1(2 + b + \varphi_{3,4}(\Delta t)(g + bg + bd + e\theta m_1)) + \varphi_{3,4}(\Delta t)(bd + (b + 2)m_1)w_3)}, \\
\mathcal{J}_{32}^D(\mathcal{E}_4) &= \frac{\varphi_{3,4}(\Delta t)(-(b + 1)g + bd(e - 1)\theta + (1 + b)e\theta m_1)}{bd + m_1(2 + b + \varphi_{3,4}(\Delta t)(g + bg + bd + e\theta m_1)) + w_3\varphi_{3,4}(\Delta t)(bd + (b + 2)m_1)}, \\
\mathcal{J}_{33}^D(\mathcal{E}_4) &= 1.
\end{aligned}
$$

The characteristic polynomial of $\mathcal{J}^D(\mathcal{E}_4)$ can be written as a third order polynomial $P_{\mathcal{E}_4}(\xi) = \xi^3 + a_2^{(4)}\xi^2 + a_1^{(4)}\xi + a_0^{(4)}$, where the coefficients are long expressions that can be found in detail in Appendix B. After many algebraic manipulations, one can show that the term $A := 1 + a_0^{(4)} + a_1^{(4)} + a_2^{(4)}$ can be expressed as

$$
\begin{aligned}
A &= ((1 + b)g - bd(e - 1)\theta - (1 + b)e\theta m_1)(bdg + m_1(g - bd\theta - e\theta m_1))\widetilde{A} \\
&= ((1 + b)(g - e\theta m_1) + bd(1 - e)\theta)(bd(g - \theta m_1) + m_1(g - e\theta m_1))\widetilde{A},
\end{aligned} \tag{4.3}
$$

where $\widetilde{A}$ is a positive constant depending only on the parameters of the model and the denominator functions φ_i. From the existence conditions for $\mathcal{E}_4$ given by (4.2) in Remark 4 we have that,

$$
S^* > e, \quad S^* > \frac{g}{m_1\theta},
$$

which are equivalent to,

$$
(1 + b)(g - e\theta m_1) + bd\theta(1 - e) > 0, \quad bd(\theta m_1 - g) + m_1(e\theta m_1 - g) > 0. \tag{4.4}
$$

In the light of (4.3) and (4.4), we get that $A < 0$. In this way, we see that the first condition in Lemma 6, in Appendix A is violated and hence we have that there is a root ξ_0 of polynomial $P_{\mathcal{E}_4}$ such that $|\xi_0| \geq 1$, i.e. $\mathcal{E}_4$ is unstable. $\square$

4.2. NSFD scheme for Sharma-Samanta model [33]

The system of ODEs modeling the ecosystem with the incorporation of a prey refuge reads

$$
\begin{aligned}
\frac{dS}{dt} &= rS\left(1-\frac{S+I}{k}\right)-\lambda SI,\\
\frac{dI}{dt} &= \lambda SI-\frac{\alpha(1-\nu)YI}{a+(1-\nu)I}-\mu I,\\
\frac{dY}{dt} &= Y\left(\delta-\frac{\eta Y}{a+(1-\nu)I}\right),
\end{aligned}
\tag{4.5}
$$

We will focus on two equilibrium points which have interest from the biological point of view; these are the predator-free equilibrium and the disease-free equilibrium. In [33] it is shown that these points are given by

(i) $\mathcal{E}_1 = (S_1, I_1, 0)$, where $S_1 = \frac{\mu}{\lambda}$, $I_1 = \frac{r\lambda - b\mu}{\lambda(b+\lambda)}$, with $b = r/k$.

(ii) $\mathcal{E}_2 = (S_2, 0, Y_2)$, where $S_2 = \frac{r}{b}$, $Y_2 = \frac{a\delta}{\eta}$,

respectively.

Remark 5. *The equilibria $\mathcal{E}_1$ exist for all parameter values and the equilibrium $\mathcal{E}_2$ exists if $r > \frac{b\delta}{\lambda}$.*

The final form of the NSFD scheme for this model is described by the following algorithm:

Algorithm 2 Second-order NSFD scheme for Model (4.5)

Input: $b, \lambda, \mu, \nu, \alpha, a, \eta, w_1, w_2, w_3, T, \Delta t, S^0, I^0, Y^0$

$S^n \leftarrow S^0,\ \ I^n \leftarrow I^0,\ \ Y^n \leftarrow Y^0$

$M \leftarrow \lceil \frac{T}{\Delta t} \rceil$

for $n = 1, \ldots, M-1$ **do**

compute $\varphi_i(\Delta t, S^n, I^n, Y^n)$, $i = 1, 2, 3$ from (3.27) given $\Delta t, S^n, I^n, Y^n$

$$S^{n+1} \leftarrow \frac{S^n + b\varphi_1(\Delta t, S^n, I^n, Y^n)S^n + w_1\varphi_1(\Delta t, S^n, I^n, Y^n)S^n}{1+\varphi_1(\Delta t, S^n, I^n, Y^n)\Big(b(S^n+I^n)+\lambda I^n + w_1\Big)}$$

$$I^{n+1} \leftarrow \frac{I^n + \varphi_2(\Delta t, S^n, I^n, Y^n)\lambda S^n I^n + w_2\varphi_2(\Delta t, S^n, I^n, Y^n)I^n}{1+\varphi_2(\Delta t, S^n, I^n, Y^n)\Big(\frac{\alpha(1-\nu)Y^n}{a+(1-\nu)I^n}+\mu+w_2\Big)}$$

$$Y^{n+1} \leftarrow \frac{Y^n + \varphi_3(\Delta t, S^n, I^n, Y^n)\delta Y^n + w_3\varphi_3(\Delta t, S^n, I^n, Y^n)Y^n}{1+\varphi_3(\Delta t, S^n, I^n, Y^n)\Big(\frac{\eta Y^n}{a+(1-\nu)I^n}+w_3\Big)}$$

end for

Output: $\{(S^1, I^1, Y^1), \ldots, (S^M, I^M, Y^M)\}$

If what follows, we show that our numerical scheme preserves the stability conditions for these two equilibria.

Theorem 7. *Let $w_1, w_2, w_3 \in \mathbb{R}$ satisfying $w_1, w_2, w_3 \geq 0$. Then, the NSFD given by Algorithm 2 is dynamically consistent with respect to the stability of the equilibrium $\mathcal{E}_i$, $i = 1, 2$ of the model* (2.1) *for all the values of the step size Δt, that is:*

(i) The predator-free equilibrium $\mathcal{E}_1$ exists if $r > \frac{b\mu}{\lambda}$ and it is unstable.

(ii) The disease-free equilibrium $\mathcal{E}_2$ exists if $r < \frac{b\eta\mu+\alpha(1-\nu)b\delta}{\lambda\eta}$ and is locally asymptotically stable.

Proof. Let us use the notation $\varphi_{i,\ell}(\Delta t) := \varphi_i(\Delta t, \mathcal{E}_\ell)$, where $i = 1, 2, 3$, and $\ell = 1, 2$, for denoting the evaluation of functions φ_i at the equilibrium points $\mathcal{E}_\ell$.

(i) The Jacobian matrix of system (4.5) at $\mathcal{E}_1$ is

$$\mathcal{J}^C(\mathcal{E}_1) = \begin{pmatrix} -bS_1 & -(b+\lambda)S_1 & 0 \\ \lambda I_1 & 0 & -\dfrac{\alpha(1-\nu)I_1}{a+(1-\nu)I_1} \\ 0 & 0 & \delta \end{pmatrix}.$$

By employing Lemma 3, in Appendix A and after some algebraic manipulations we get that $\mathcal{J}^D(\mathcal{E}_1) = (\mathcal{J}^D_{ij}(\mathcal{E}_1))^3_{i,j=1}$, where

$$
\begin{aligned}
\mathcal{J}^D_{11}(\mathcal{E}_1) &= 1 - \frac{b\mu\varphi_{1,1}(\Delta t)}{\lambda(1+\varphi_{1,1}(\Delta t)(r+w_1))},\\
\mathcal{J}^D_{12}(\mathcal{E}_1) &= -\frac{\mu\varphi_{1,1}(\Delta t)(b+\lambda)}{\lambda\left(1+\varphi_{1,1}(\Delta t)(r+w_1)\right)},\\
\mathcal{J}^D_{13}(\mathcal{E}_1) &= 0,\\
\mathcal{J}^D_{21}(\mathcal{E}_1) &= \frac{\varphi_{2,1}(\Delta t)(\lambda r - b\mu)}{(b+\lambda)\left(1+\varphi_{2,1}(\Delta t)(\mu+w_2)\right)},\\
\mathcal{J}^D_{22}(\mathcal{E}_1) &= 1,\\
\mathcal{J}^D_{23}(\mathcal{E}_1) &= -\frac{\alpha(1-\nu)\varphi_{2,1}(\Delta t)(\lambda r - b\mu)}{\left(1+\varphi_{2,1}(\Delta t)(\mu+w_2)\right)\left(a\lambda(b+\lambda)+(1-\nu)(\lambda r - b\mu)\right)},\\
\mathcal{J}^D_{31}(\mathcal{E}_1) &= 0,\\
\mathcal{J}^D_{32}(\mathcal{E}_1) &= 0,\\
\mathcal{J}^D_{33}(\mathcal{E}_1) &= 1 + \frac{\delta\varphi_{3,1}(\Delta t)}{\varphi_{3,1}(\Delta t)w_3 + 1}.
\end{aligned}
$$

The characteristic polynomial of $\mathcal{J}^D(\mathcal{E}_1)$ can be written as

$$
P_{\mathcal{E}_1}(\xi) = Q_{\mathcal{E}_1}(\xi)\left(1 + \frac{\delta\varphi_{3,1}(\Delta t)}{\varphi_{3,1}(\Delta t)w_3+1} - \xi\right),
$$

where $Q_{\mathcal{E}_1}$ is a second order polynomial of the form $Q_{\mathcal{E}_1}(\xi) = \xi^2 + b^{(1)}_1\xi + b^{(1)}_0$, where

$$
\begin{aligned}
b^{(1)}_0 &= 1 - \frac{\mu\varphi_{1,1}(\Delta t)\left(2b\mu\varphi_{2,1}(\Delta t) + b\varphi_{2,1}(\Delta t)w_2 + b - \lambda\varphi_{2,1}(\Delta t)r\right)}{\lambda\left(\varphi_{1,1}(\Delta t)r + \varphi_{1,1}(\Delta t)w_1 + 1\right)\left(\mu\varphi_{2,1}(\Delta t) + \varphi_{2,1}(\Delta t)w_2 + 1\right)},\\
b^{(1)}_1 &= \frac{b\mu\varphi_{1,1}(\Delta t)}{\lambda + \lambda\varphi_{1,1}(\Delta t)r + \lambda\varphi_{1,1}(\Delta t)w_1} - 2.
\end{aligned}
$$

Hence we have that $\xi_1 = 1 + \frac{\delta\varphi_{3,1}(\Delta t)}{\varphi_{3,1}(\Delta t)w_3+1} > 1$ is an eigenvalue of $\mathcal{J}^D(\mathcal{E}_1)$. Therefore, by Lemma 4, in Appendix A it follows that $\mathcal{E}_1$ is unstable for the NSFD system given by Algorithm 2. (*ii*) By employing Lemma 3, in Appendix A and after some algebraic manipulations we get

$$
\mathcal{J}^D(\mathcal{E}_2) = \begin{pmatrix}
1 - \frac{\varphi_{1,2}(\Delta t)r}{\varphi_{1,2}(\Delta t)(r+w_1)+1} & -\frac{\varphi_{1,2}(\Delta t)r(b+\lambda)}{b\left(\varphi_{1,2}(\Delta t)r+\varphi_{1,2}(\Delta t)w_1+1\right)} & 0\\
0 & \frac{\eta\left(b\varphi_{2,2}(\Delta t)w_2+b+\lambda\varphi_{2,2}(\Delta t)r\right)}{b\left(\eta+\alpha\varphi_{2,2}(\Delta t)(\delta-\delta\nu)+\eta\mu\varphi_{2,2}(\Delta t)+\eta\varphi_{2,2}(\Delta t)w_2\right)} & 0\\
0 & \frac{\delta^2(1-\nu)\varphi_{3,2}(\Delta t)}{\eta+\eta\varphi_{3,2}(\Delta t)(\delta+w_3)} & 1 - \frac{\delta\varphi_{3,2}(\Delta t)}{\varphi_{3,2}(\Delta t)(\delta+w_3)+1}
\end{pmatrix}.
$$

The characteristic polynomial of $\mathcal{J}^D(\mathcal{E}_2)$ can be written as $P_{\mathcal{E}_2}(\xi) = (\xi_1 - \xi)(\xi_2 - \xi)(\xi_3 - \xi)$, where

$$
\xi_1 = 1 - \frac{\varphi_{1,2}(\Delta t)r}{\varphi_{1,2}(\Delta t)\,(r+w_1)+1},\quad \xi_2 = 1 - \frac{\delta\varphi_{3,2}(\Delta t)}{\varphi_{3,2}(\Delta t)\,(\delta+w_3)+1},\quad \xi_3 = \frac{\eta\left(b\varphi_{2,2}(\Delta t)w_2 + b + \lambda\varphi_{2,2}(\Delta t)r\right)}{b\left(\eta + \alpha\varphi_{2,2}(\Delta t)(\delta(1-\nu)+\eta\mu+\eta w_2)\right)}.
$$

Therefore, the eigenvalues of $\mathcal{J}^D(\mathcal{E}_2)$ are ξ_i, $i = 1,2,3$. Now we need to check that $|\xi_i| < 1$, whenever $r < \frac{b\eta\mu+\alpha(1-\nu)b\delta}{\lambda\eta}$, for $i = 1,2,3$. To this end, we first observe that $\xi_1, \xi_2 < 1$, and $\xi_3 > 0$, as $\nu < 1$. In addition,

$$
\begin{aligned}
1 - \xi_1 &= \frac{\varphi_{1,2}(\Delta t)r}{\varphi_{1,2}(\Delta t)\,(r+w_1)+1} < \frac{\varphi_{1,2}(\Delta t)r}{\varphi_{1,2}(\Delta t)r} = 1,\\
1 - \xi_2 &= \frac{\delta\varphi_{3,2}(\Delta t)}{\varphi_{3,2}(\Delta t)\,(\delta+w_3)+1} < \frac{\delta\varphi_{3,2}(\Delta t)}{\delta\varphi_{3,2}(\Delta t)} = 1,
\end{aligned}
$$

so $\xi_1, \xi_2 > 0$. Moreover, we notice

$$1 - \xi_3 = \frac{\varphi_{2,2}(\Delta t)(\alpha b(\delta - \delta\nu) + b\eta\mu + \eta\lambda(-r))}{b\left(\eta + \alpha\varphi_{2,2}(\Delta t)(\delta(1-\nu) + \eta\mu + \eta w_2)\right)} = \frac{\eta\lambda\varphi_{2,2}(\Delta t)\left(\frac{\alpha b\delta(1-\nu)+b\eta\mu}{\eta\lambda} - r\right)}{b\left(\eta + \alpha\varphi_{2,2}(\Delta t)(\delta(1-\nu) + \eta\mu + \eta w_2)\right)}$$

and by using $r < \frac{b\eta\mu+\alpha(1-\nu)b\delta}{\lambda\eta}$ and $0 < \nu < 1$ we get that $1 - \xi_3 > 0$, i.e. $\xi_3 < 1$. From all of the above we have that $|\xi_i| < 1$, for $i = 1, 2, 3$. Thus, $\mathcal{E}_2$ is locally asymptotically stable. □

We can also establish that the NSFD scheme (3.7) preserves the global stability of the disease-free equilibria of the Sharma–Samanta model (4.5). To this end, we first prove the following lemma concerning the boundedness of the sequence S^n.

Lemma 1. *Let $w_1, w_2, w_3 \in \mathbb{R}$ satisfying $w_1, w_2, w_3 \geq 0$. Then the sequence of susceptible prey $\{S^n\}$ is bounded above by k, that is*

$$\limsup_{n\to\infty} S^n \leq k. \tag{4.6}$$

Proof. Let us use the following notations $g_i^n := g_i(S^n, I^n, Y^n)$, $f_i^n = f_i(S^n, I^n, Y^n)$, $\varphi_i^n := \varphi_i(\Delta t, S^n, I^n, Y^n)$, for $i = 1, 2, 3$. We consider the two following cases for S^0, namely, $S^0 \leq k$ and $S^0 > k$.

Cases 1: $S^0 \leq k$. We proceed by induction on n. Suppose that $S^n \leq k$, then from second equation in (3.7) we have that

$$S^{n+1} \leq \frac{(1 + \varphi_1^n(r + w_1))S^n}{1 + \varphi_1^n(\frac{r}{k}S^n + w_1)} := \Psi(S^n),$$

where $\Psi(z) := \frac{(1+\varphi_1^n(r+w_1))z}{1+\varphi_1^n(\frac{r}{k}z+w_1)}$. Notice that,

$$\Psi'(z) = \frac{(1 + \varphi_1^n(r + w_1))(1 + \varphi_1^n(\frac{r}{k}z + w_1)) - (1 + \varphi_1^n(r + w_1))z(\varphi_1^n(\frac{r}{k}))}{(1 + \varphi_1^n(\frac{r}{k}z + w_1))^2} = \frac{(1 + \varphi_1^n(r + w_1))(1 + \varphi_1^n w_1)}{(1 + \varphi_1^n(\frac{r}{k}z + w_1))^2}. \tag{4.7}$$

Thus, Ψ is an increasing function. So, it follows that $S^{n+1} \leq \Psi(S^n) \leq \Psi(k) = k$. Therefore, we can conclude that $S^n \leq k$, for all $n \geq 0$.

Cases 2: $S^0 > k$. We consider two sub-cases:

- *Cases 1.1: There exists $N \in \mathbb{N}$ such that $S^N \leq k$.* In this case we can proceed by induction on $n \geq N$ as in Case 1, to get $S^n \leq k$, for all $n \geq N$.
- *Cases 1.2: $S^n > k$, for every $k \in \mathbb{N}$.* From first equation in (3.8) we obtain,

$$S^{n+1} = S^n + g_1^n f_1^n = S^n + g_1^n\left(rS^n\left(1 - \frac{S^n + I^n}{k}\right) - \lambda S^n I^n\right) \leq S^n + g_1^n r S^n\left(1 - \frac{S^n}{k}\right).$$

  As $g_1^n > 0$ and $S^n > k$, it follows that $S^{n+1} \leq S^n$, for all $n \geq 0$.

From all of the above cases we can conclude that $\limsup_{n\to\infty} S^n \leq k$. □

Theorem 8. *Let $w_1, w_2, w_3 \in \mathbb{R}$ and satisfying $w_1, w_2, w_3 \geq 0$. Then, the NSFD method given by Algorithm 2 is dynamically consistent with respect to the global stability of the equilibrium $\mathcal{E}_2$ of the model* (4.5) *for all the values of the step size Δt, that is, if $r < \frac{b\mu}{\lambda}$, then $\mathcal{E}_2$ exists and is globally asymptotically stable.*

Proof. To simplify the notation during the proof, let us use the notation $\varphi_i = \varphi_i(\Delta t, S^n, I^n, Y^n)$, for $i = 1, 2, 3$. We first consider the Lyapunov function $\mathcal{L}_1(n) := I^n$, then by using Lemma 1 we notice from (3.6) that

$$\begin{aligned}\Delta\mathcal{L}_1(n) := \mathcal{L}_1(n+1) - \mathcal{L}_1(n) = I^{n+1} - I^n &= \varphi_2\left(\lambda S^n I^n - \mathcal{G}(I^n, Y^n)I^{n+1} - \mu I^{n+1} + w_2 I^n - w_2 I^{n+1}\right)\\ &< \varphi_2(\lambda k I^n - \mu I^{n+1} - w_2\Delta\mathcal{L}_1(n))\\ &\leq \varphi_2(\mu I^n - \mu I^{n+1} - w_2\Delta\mathcal{L}_1(n))\\ &\leq -\varphi_2(\mu + w_2)\Delta\mathcal{L}_1(n),\end{aligned}$$

from which we get $\Delta\mathcal{L}_1(n) < 0$. Therefore, $\lim_{n\to\infty} I^n = 0$. This allows us to study the reduced system,

$$\begin{aligned} S^{n+1} &= \frac{S^n + \varphi_1 r S^n + \varphi_1 w_1 S^n}{1 + \varphi_1\left(\frac{r}{k} S^n + w_1\right)}, \\ Y^{n+1} &= \frac{Y^n + \varphi_3 \delta Y^n + \varphi_3 w_3 Y^n}{1 + \varphi_3\left(\frac{\eta}{a} Y^n + w_3\right)}. \end{aligned} \tag{4.8}$$

Following the proof of Lemma 1 we can establish that,

$$\limsup_{n\to\infty} Y^n \le \frac{a\delta}{\eta}. \tag{4.9}$$

Let us define the Lyapunov function

$$\mathcal{L}_2(n) := S_2 \Psi\left(\frac{S_2}{S^n}\right) + Y_2 \Psi\left(\frac{Y_2}{Y^n}\right), \tag{4.10}$$

where $\Psi(x) = x - 1 - \ln(x)$, $x \in \mathbb{R}^+$. Then by using the inequality $\ln x \le x - 1$, (3.6), (4.6), and (4.9) we have

$$\begin{aligned} \Delta\mathcal{L}_2(n) &:= (S^{n+1} - S^n) + S_2 \ln\left(\frac{S^n}{S^{n+1}}\right) + (Y^{n+1} - Y^n) + S_2 \ln\left(\frac{Y^n}{Y^{n+1}}\right) \\ &\le (S^{n+1} - S^n) + S_2\left(\frac{S^n}{S^{n+1}} - 1\right) + (Y^{n+1} - Y^n) + Y_2\left(\frac{Y^n}{Y^{n+1}} - 1\right) \\ &= (S^{n+1} - S^n)\left(1 - \frac{S_2}{S^{n+1}}\right) + (Y^{n+1} - Y^n)\left(1 - \frac{Y_2}{Y^{n+1}}\right) \\ &= \frac{rS^n}{1 + w_1}\left(1 - \frac{S^{n+1}}{k}\right)\left(1 - \frac{S_2}{S^{n+1}}\right) + \frac{\delta Y^n}{1 + w_2}\left(1 - \frac{Y^{n+1}}{(\frac{a\delta}{\eta})}\right)\left(1 - \frac{Y_2}{Y^{n+1}}\right) \\ &= -\frac{bS^n S^{n+1}}{1 + w_1}\left(1 - \frac{S_2}{S^{n+1}}\right)^2 - \frac{(\frac{\eta}{a}) Y^n Y^{n+1}}{1 + w_2}\left(1 - \frac{Y_2}{Y^{n+1}}\right)^2. \end{aligned}$$

We have $\mathcal{L}_2(n+1) - \mathcal{L}_2(n) \le 0$, for every $n \ge 0$. Then $\{\mathcal{L}_2(n)\}$ is a monotone decreasing sequence. Being $\mathcal{L}_2(n) \ge 0$, we know that there exists a limit $\lim_{n\to\infty} \mathcal{L}_2(n) \ge 0$. Thus, $\lim_{n\to\infty}(\mathcal{L}_2(n+1) - \mathcal{L}_2(n)) = 0$, from which it follows that $\lim_{n\to\infty} S^{n+1} = S_2$ and $\lim_{n\to\infty} Y^{n+1} = Y_2$.

By the analysis above, we can conclude that $\mathcal{E}_2$ is globally asymptotically stable if $r < \frac{b\mu}{\lambda}$. □

5. Numerical examples

This section is devoted to the numerical results that support the theoretical results. In addition, numerical examples show the advantages of the constructed second-order NSFD schemes (NSFD-2) given in Algorithm 1 for mathematical model (4.1), and Algorithm 2 for mathematical model (4.5). We compare the numerical approximations with respect to the approximations obtained with a classical first-order NSFD scheme (NSFD-1) with denominator functions given by $\varphi_i(\Delta t) = \Delta t$, $i = 1, 2, 3$ and the classical first-order Euler scheme and second-order Runge-Kutta scheme (RK-2).

5.1. Example 1. Convergence tests.

To evaluate the accuracy of the constructed second-order numerical schemes, we compute the absolute error and the convergence rate at the final time T of each numerical example. To do so, we consider the following sequence of nodes $M_\ell = 2^\ell \cdot 10$, which defines a sequence of time-steps $\Delta t_\ell = T/M_\ell$, $\ell = 0, \ldots, 5$. To compare the accuracy of the solutions, we compute a reference solution using a classical fourth-order Runge-Kutta (RK-4) scheme with $M_{\text{ref}} := 2^8 \cdot 10 = 2560$ nodes. At each level ℓ, we compute the ℓ^∞ norm between the solution vector $(S^n_\ell, I^n_\ell, Y^n_\ell)^{\mathrm{T}}$ and the interpolated reference solution in the grid of level ℓ, denoted by $(S^n_{\text{ref},\ell}, I^n_{\text{ref},\ell}, Y^n_{\text{ref},\ell})^{\mathrm{T}}$, $n = 0, \ldots, M_\ell$, i.e. we compute the ℓ^∞-error in $[0, T]$ by using the following expression

$$\text{error}_\ell(T) := \max\left(\max_{n\in\mathbb{Z}_{M_\ell}} |S^n_\ell - S^n_{\text{ref},\ell}|, \max_{n\in\mathbb{Z}_{M_\ell}} |I^n_\ell - I^n_{\text{ref},\ell}|, \max_{n\in\mathbb{Z}_{M_\ell}} |Y^n_\ell - Y^n_{\text{ref},\ell}|\right), \quad \ell = 0, \ldots, 5, \tag{5.1}$$

and the convergence rates were calculated with the formula

$$\theta_\ell := \log_2(\mathrm{error}_{\ell-1}/\mathrm{error}_\ell), \quad \ell = 1, \ldots, 5. \tag{5.2}$$

5.1.1. Test 1. Accuracy test for Model (4.1).

In this test, we consider an example proposed in [32] to examine the local stability of the disease-free equilibrium $\mathcal{E}_2$ of model (4.1). We use the following values of the parameters $r = 3, k = 45, a = 15, \alpha = 0.004, \lambda = 0.003, \beta = 0.05, \nu = 0.24, \theta = 0.4, \delta = 0.09$, with the initial condition $S^0 = 30, I^0 = 10, Y^0 = 15$. The weights of the second-order scheme are chosen according to the analysis performed in Theorem 6 for the equilibrium $\mathcal{E}_2$. In particular, we set $w_2 = 1$. As there are no restrictions for the other weights with respect to the stability of equilibrium of $\mathcal{E}_2$, we fix $w_1 = w_3 = 0$.

In Table 1 we show the approximate error at the simulation time $T = 1$ for different discretizations $M_\ell = 2^\ell \cdot 10$ with $\ell = 0, 1, \ldots, 5$. In addition, we compute and show the convergence rate. Numerical results confirm the second-order accuracy of the NSFD-2 scheme in contrast to the first-order of the Euler and NSFD-1 schemes. In particular, the performance of the NSFD-2 is compared with the RK-2 scheme, which can be observed in Figure 1 (left), while in Figure 1 (right) we display the ℓ^∞-error $\mathrm{error}_6(t)$ for $0 \le t \le 10$. It can be observed that the NSFD-2 scheme converges to the equilibrium point faster than the NSFD-1 or Euler schemes.

Table 1: Example 1. Test 1: Comparison of ℓ^∞-error, convergence rate θ_ℓ, and CPU time [s] at simulation time $T = 1$ for different discretizations $M_\ell = 2^\ell \cdot 10$ with $\ell = 0, 1, \ldots, 5$.

M_ℓ	$\mathrm{error}_\ell(T)$	θ_ℓ	cpu [s]	$\mathrm{error}_\ell(T)$	θ_ℓ	cpu [s]	$\mathrm{error}_\ell(T)$	θ_ℓ	cpu [s]	$\mathrm{error}_\ell(T)$	θ_ℓ	cpu [s]
	NSFD-1			Euler			NSFD-2			RK-2		
10	8.65e-02	⋆	1.19e-04	2.11e-02	⋆	1.20e-04	6.07e-04	⋆	2.39e-04	7.45e-05	⋆	1.88e-04
20	4.34e-02	0.995	1.24e-04	1.05e-02	1.004	1.81e-04	1.51e-04	2.007	4.16e-04	1.84e-05	2.016	3.02e-04
40	2.17e-02	0.997	1.75e-04	5.27e-03	1.002	2.89e-04	3.76e-05	2.004	7.11e-04	4.57e-06	2.008	5.62e-04
80	1.08e-02	0.998	3.23e-04	2.63e-03	1.001	6.27e-04	9.40e-06	2.002	1.54e-03	1.14e-06	2.004	1.03e-03
160	5.43e-03	0.999	5.66e-04	1.31e-03	1.000	1.08e-03	2.34e-06	2.001	2.87e-03	2.84e-07	2.002	2.23e-03
320	2.72e-03	1.000	1.19e-03	6.58e-04	1.000	2.03e-03	5.86e-07	2.000	5.81e-03	7.11e-08	2.001	4.44e-03

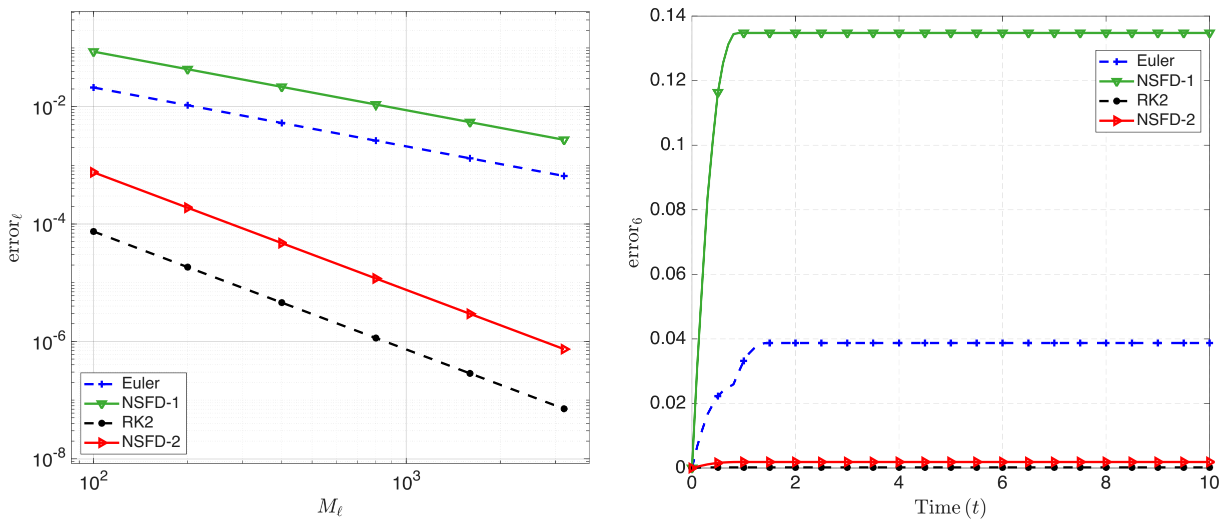

Figure 1: Example 1. Test 1. Plot of the $\mathrm{error}_l(T)$ against $M_\ell = 2^\ell \cdot 10$ with $\ell = 0, 1, \ldots, 5$, for each schemes tested as function of M (left). Plot of $\mathrm{error}_6(t)$ for $0 \le t \le 10$ (right).

5.1.2. Test 2. Accuracy test for Model (4.5).

In this test, we consider an example proposed in [33] to examine the local stability of the disease-free equilibrium $\mathcal{E}_2$ of the Model (4.5), with parameters $r = 1, b = 0.3, \lambda = 0.09, \alpha = 1.4, \mu = 0.6, \delta = 0.2, \eta = 0.4, \nu = 0.42, a = 0.2$ with initial condition $S^0 = 10, I^0 = 5, Y^0 = 3$. According to the analysis performed in Theorem 7 for the equilibrium $\mathcal{E}_2$, there are no restrictions (must be nonnegative) for the weights with respect to the stability of equilibrium of $\mathcal{E}_2$. Thus, we fix $w_1 = w_2 = w_3 = 0$.

In Table 2 we show the approximate error at the simulation time $T = 1$ for different discretizations $M_\ell = 2^\ell \cdot 10$ with $\ell = 0, 1, \dots, 5$. In addition, we show the convergence rate. Numerical results confirm the second-order approximation of the NSFD-2 method in contrast with the first-order of Euler and NSFD-1 numerical schemes. Figure 2 (left) shows the performance of the NSFD-2 and RK-2 numerical schemes. Figure 2 (right) depicts the ℓ^∞-error $\mathrm{error}_6(t)$ for $0 \le t \le 10$, where we can observe that the NSFD-2 scheme converges to the equilibrium point faster than the NSFD-1 or Euler schemes.

Table 2: Example 1. Test 2: Comparison of ℓ^∞-error, convergence rate θ_ℓ, and CPU time [s] at simulation time $T = 1$ for different discretizations $M_\ell = 2^\ell \cdot 10$ with $\ell = 0, 1, \dots, 5$.

M_ℓ	error_ℓ	θ_ℓ	cpu [s]	error_ℓ	θ_ℓ	cpu [s]	error_ℓ	θ_ℓ	cpu [s]	error_ℓ	θ_ℓ	cpu [s]
	NSFD-1			Euler			NSFD-2			RK-2		
10	4.63e-02	⋆	1.11e-04	7.95e-02	⋆	1.53e-04	1.64e-04	⋆	2.37e-04	1.64e-03	⋆	1.88e-04
20	2.32e-02	0.994	1.63e-04	3.93e-03	1.017	2.33e-04	4.03e-05	2.003	4.36e-04	4.01e-04	2.033	3.08e-04
40	1.16e-02	0.996	1.80e-04	1.95e-02	1.008	2.96e-04	9.96e-06	2.015	7.24e-04	9.91e-05	2.016	5.68e-04
80	5.84e-02	0.998	3.22e-04	9.74e-03	1.004	5.67e-04	2.47e-06	2.007	1.41e-03	2.46e-06	2.008	1.10e-03
160	2.92e-03	0.999	6.13e-04	4.86e-03	1.002	1.09e-03	6.17e-07	2.003	2.78e-03	6.14e-06	2.004	2.18e-03
320	1.46e-03	1.000	1.23e-03	2.43e-03	1.001	2.44e-03	1.54e-07	2.002	5.67e-03	1.53e-06	2.002	4.82e-03

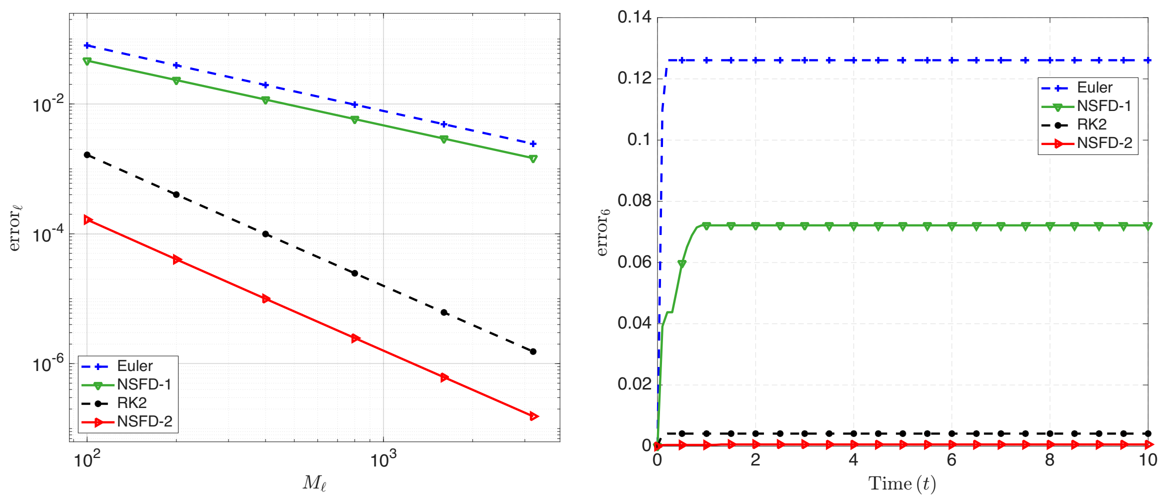

Figure 2: Example 1, Test 2. Plot of $\mathrm{error}_l(T)$ against $M_\ell = 2^\ell \cdot 10$ with $\ell = 0, 1, \dots, 5$, for each schemes tested as function of M (left). Plot of ℓ^∞-error $\mathrm{error}_6(t)$ for $0 \le t \le 10$ (right).

5.2. *Example 2. Stability with respect to the time step size Δt*

In this numerical test, we study the stability of the proposed second-order NSFD scheme with respect to the time step size Δt. To this end, we consider the model (4.5) with a numerical example proposed in [33] where the solution converges to the interior equilibrium

$$\mathcal{E}^* = (S^*, I^*, Y^*) = \left(\frac{\alpha(1-\nu)\delta + \eta\mu}{\lambda\eta}, \frac{r\lambda\eta - b(\alpha(1-\nu)\delta + \eta\mu}{\lambda\eta(b+\lambda)}, \frac{\delta}{\eta}\left(a + \frac{b(1-\nu)(\alpha(1-\mu)\delta + \eta\mu)}{\lambda\eta(b+\lambda)}\right)\right).$$

The proof of the stability of this point with respect to the NSFD-2 scheme given by Algorithm 2 can be done following the lines of (iv) in Theorem 6 and using Lemma 6 in the Appendix A. For the test, we set $r = 5.5$, $\nu = 0.15$ and the rest of the parameters and initial condition are the same as in Test 2 of Example 1.

Figure 3 displays the numerical solutions of model (4.5) for each component $S(t), I(t)$ and $Y(t)$ for $0 \leq t \leq 100$ computed with NSFD-2, Euler, and RK-2 schemes for different step sizes $\Delta t \in \{0.1, 0.5, 1\}$. For $\Delta t = 0.1$, we can observe in Figure 3 (left column) that all the numerical schemes describe the correct dynamics. However, for $\Delta t \in \{0.5, 1\}$ we can observe in Figure 3 (middle and right columns, respectively) some spurious oscillations in the numerical solutions obtained with the Euler and RK-2 schemes. These results corroborate the unconditional stability of the proposed NSFD-2 numerical scheme. Figure 4 shows the numerical solutions of model (4.5) using different phase portraits for the components $S(t), I(t)$ and $Y(t)$ for $0 \leq t \leq 100$ computed with the Euler and NSFD-2 schemes for different step sizes $\Delta t \in \{0.1, 0.5\}$. For $\Delta t = 0.1$, we can observe in Figure 4 (left column) that both numerical schemes describe the correct dynamics. However, for $\Delta t = 0.5$, we can observe in Figure 4 (right column) some spurious oscillations in the numerical solution obtained with the Euler numerical scheme.

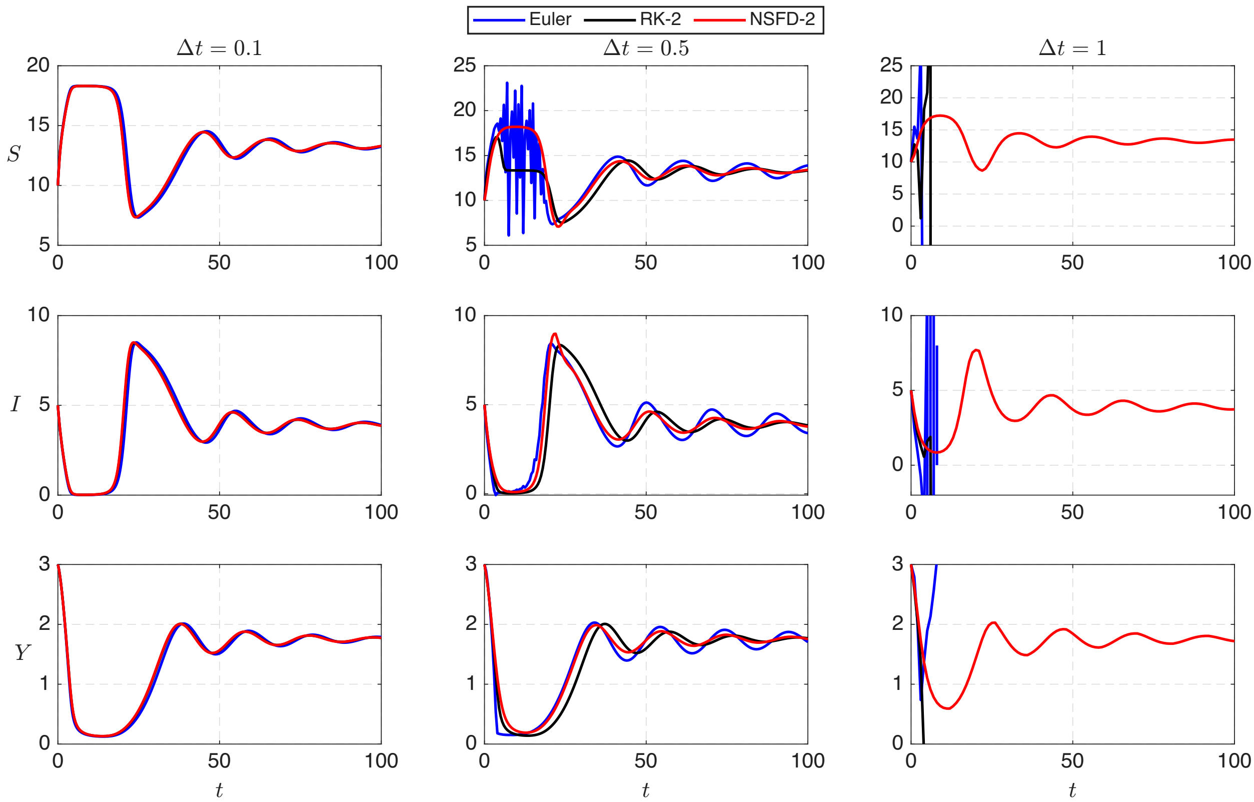

Figure 3: Example 2: Numerical solutions for the three populations $S(t), I(t)$ and $Y(t)$ for model (4.5), with $0 \leq t \leq 100$ computed with Euler, RK-2 and NSFD-2 numerical schemes with step sizes $\Delta t \in \{0.1, 0.5, 1\}$.

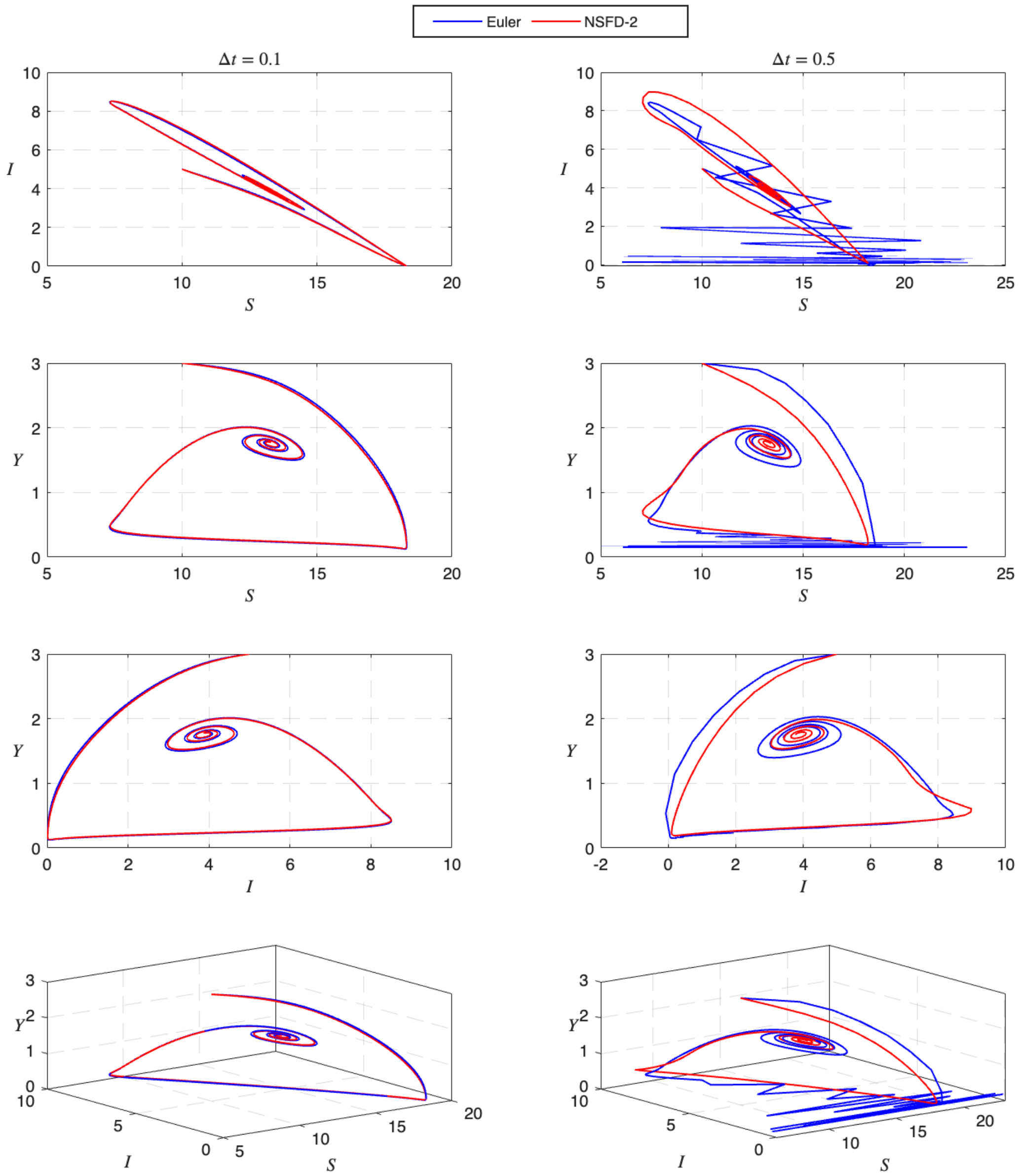

Figure 4: Example 2. Comparison of the numerical solutions (different phase portraits) of model (4.5) generated by Euler and the second-order NSFD numerical schemes with $\Delta t = 0.1$ (left column) and $\Delta t = 0.5$ (right column).

6. Conclusions

The main contribution of this paper is the design and construction of an unconditionally-stable, dynamically consistent second-order nonstandard finite difference (NSFD) scheme to numerically solve a generalized eco-epidemiological predator-prey model. We have proved that the proposed scheme preserves some essential qualitative features of the generalized model, which are described in Theorem 3, Theorem 4 and Theorem 5. There are several mathematical models that can be put in the form of the studied generalized eco-epidemiological predator-prey model. In particular, in this work, we have considered two specific models studied in [32] and [33], for which we have constructed the corresponding second-order NSFD schemes detailed in Algorithm 1 and Algorithm 2, respectively. In both cases, in Section 4 we have studied the consistency of these numerical schemes with respect to the local stability of the continuous models. The constructed numerical scheme and mathematical approach could be applied to a wide range of eco-epidemiological models of the form (2.1). For instance, for the mathematical models presented in [29, 30, 31]. The numerical results presented in this work support the mathematical analysis and show, in several numerical tests, the advantages of the constructed NSFD schemes. Future works in this direction aim to extend our results to high-order nonstandard finite difference schemes and to address convection-diffusion-reaction systems.

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Appendix A. Additional results.

Lemma 2 (Gronwall inequality)**.** *Let $\{u_n\}$ be a sequence of real numbers such that*

$$u_{n+1} \leq (1+\alpha)u_n + b_{n+1}, \quad \forall\, n \geq 0, \tag{A.1}$$

where $\alpha > 0$ and $\{b_n\}$ is a sequence of positive real numbers. Then

$$u_n \leq e^{n\alpha}u_0 + e^{n\alpha}\sum_{k=1}^{n} b_k, \quad \forall\, n \geq 0. \tag{A.2}$$

Proof. See [38], p. 229. □

Lemma 3. *Let $\mathcal{E} \in \mathbb{R}^3$ be an equilibrium point, and $\mathcal{J}^D(\mathcal{E})$ the Jacobian matrix evaluated at $\mathcal{E}$ for the discrete system. Then, the following identity holds*

$$\mathcal{J}^D_{ij}(\mathcal{E}) = \delta_{ij} + g_i(\mathcal{E})\mathcal{J}^C_{ij}(\mathcal{E}), \tag{A.3}$$

for all i, j, where $\mathcal{J}^C(\mathcal{E})$ is the Jacobian matrix of the continuous problem evaluated at $\mathcal{E}$.

Proof. If $\mathcal{E}$ is an equilibrium point, then $f_i(\mathcal{E}) = 0$, for $i = 1, 2, 3$. So, from equations (3.8) we obtain

$$\mathcal{J}^D_{ij}(\mathcal{E}) = \delta_{ij} + g_i(\mathcal{E})\frac{\partial f_i}{\partial x_j}(\mathcal{E}) + \frac{\partial g_i}{\partial x_j}(\mathcal{E})f_i(\mathcal{E}) = \delta_{ij} + g_i(\mathcal{E})\mathcal{J}^C_{ij}(\mathcal{E}),$$

where we have used the notation $x_1 = S, x_2 = I, x_3 = Y$. This concludes the proof. □

Lemma 4. *Consider the nonlinear system $X_{t+1} = \Psi(X_t)$, where $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism with a fixed point, X_0. Then a steady-state equilibrium, X_0, is locally asymptotically stable if and only if the moduli of all eigenvalues of the Jacobian matrix, $\mathcal{J}^D(X_0)$, are smaller than one.*

Proof. See [39]. □

Lemma 5 (Jury's criterion)**.** *The solutions of the quadratic equation $x^2 - ax + b = 0$ satisfy $|x_i| < 1, i = 1, 2$ if and only if all of the following conditions are fulfilled:*

1. $|b| < 1$,

2. $1 + a + b > 0$,

3. $1 - a + b > 0$.

Proof. See [40] $\square$

Lemma 6. *The solutions of the cubic equation $x^3 + bx^2 + cx + d = 0$ satisfy $|x_i| < 1, i = 1, 2, 3$ if and only if all of the following conditions are fulfilled:*

1. $a_i > 0$, $i = 0, \dots, 3$,

2. $a_2 a_1 > a_3 a_0$,

where $a_3 = 1 + b + c + d$, $a_2 = 3 + b - c - 3d$, $a_1 = 3 - b - c - 3d$, and $a_0 = 1 - b - c + d$.

Proof. If $p(x) = x^3 + bx^2 + cx + d$, then the result is a direct consequence of applying Routh-Hurwitz criterion to the polynomial

$$q(x) = (x-1)^3 p\left(\frac{x+1}{x-1}\right) = (1 + b + c + d)x^3 + (3 + b - c - 3d)x^2 + (3 - b - c - 3d)x + (1 - b - c + d).$$

$\square$

Appendix B. Coefficients of polynomial $P_{\mathcal{E}_4}$ in Theorem 6.

To simplify the notation, let us set $\varphi_i := \varphi_i(\Delta t, \mathcal{E}_4)$, $i = 1, 2, 3$. Then the polynomial $P_{\mathcal{E}_4}(\xi) = \xi^3 + a_2^{(4)}\xi^2 + a_1^{(4)}\xi + a_0^{(4)}$ in Theorem 6 has coefficients given by:

$$\begin{aligned}
a_0^{(4)} = {} & \frac{b\varphi_1\left(bg + g + bd\theta + e\theta m_1\right)}{\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + w_1\varphi_1 + 1\right)} \\
& - \frac{(b+1)\varphi_1\varphi_2\left(bg + g + bd\theta + e\theta m_1\right)\left(m_1\left(-g + bd\theta + e\theta m_1\right) - bdg\right)}{d\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + w_1\varphi_1 + 1\right)\left((b+1)g\varphi_2 + bd\theta w_2\varphi_2 + bd(\varphi_2 + 1)\theta + \theta m_1\left(b + e\varphi_2 + (b+2)\varphi_2 w_2 + 2\right)\right)} \\
& - \frac{\varphi_1\varphi_3 m_1^2\left(bg + g + bd\theta + e\theta m_1\right)\left((b+1)g - bd(e-1)\theta - (b+1)e\theta m_1\right)}{d\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + w_1\varphi_1 + 1\right)\left(bd + m_1\left(g\varphi_3 b + d\varphi_3\theta b + b + g\varphi_3 + e\varphi_3\theta m_1 + 2\right) + \varphi_3\left(bd + (b+2)m_1\right)w_3\right)} \\
& + \frac{\varphi_2\varphi_3\left(-((b+1)g) + bd(e-1)\theta + (b+1)e\theta m_1\right)\left(bdg + m_1\left(g - bd\theta - e\theta m_1\right)\right)}{\left((b+1)g\varphi_2 + bd\theta w_2\varphi_2 + bd(\varphi_2 + 1)\theta + \theta m_1\left(b + e\varphi_2 + (b+2)\varphi_2 w_2 + 2\right)\right)\left(bd + m_1\left(g\varphi_3 b + d\varphi_3\theta b + b + g\varphi_3 + e\varphi_3\theta m_1 + 2\right) + \varphi_3\left(bd + (b+2)m_1\right)w_3\right)} \\
& - \frac{b\varphi_1\varphi_2\varphi_3\left(bg + g + bd\theta + e\theta m_1\right)\left(-((b+1)g) + bd(e-1)\theta + (b+1)e\theta m_1\right)\left(bdg + m_1\left(g - bd\theta - e\theta m_1\right)\right)}{\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + w_1\varphi_1 + 1\right)\left((b+1)g\varphi_2 + bd\theta w_2\varphi_2 + bd(\varphi_2 + 1)\theta + \theta m_1\left(b + e\varphi_2 + (b+2)\varphi_2 w_2 + 2\right)\right)\left(bd + m_1\left(g\varphi_3 b + d\varphi_3\theta b + b + g\varphi_3 + e\varphi_3\theta m_1 + 2\right) + \varphi_3\left(bd + (b+2)m_1\right)w_3\right)} \\
& + \frac{\varphi_1\varphi_2\varphi_3 m_1\left(bg + g + bd\theta + e\theta m_1\right)\left(-((b+1)g) + bd(e-1)\theta + (b+1)e\theta m_1\right)\left(m_1\left(-g + bd\theta + e\theta m_1\right) - bdg\right)}{d\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + w_1\varphi_1 + 1\right)\left((b+1)g\varphi_2 + bd\theta w_2\varphi_2 + bd(\varphi_2 + 1)\theta + \theta m_1\left(b + e\varphi_2 + (b+2)\varphi_2 w_2 + 2\right)\right)\left(bd + m_1\left(g\varphi_3 b + d\varphi_3\theta b + b + g\varphi_3 + e\varphi_3\theta m_1 + 2\right) + \varphi_3\left(bd + (b+2)m_1\right)w_3\right)} \\
& + \frac{(b+1)\varphi_1\varphi_2\varphi_3 m_1\left(bg + g + bd\theta + e\theta m_1\right)\left((b+1)g - bd(e-1)\theta - (b+1)e\theta m_1\right)\left(bdg + m_1\left(g - bd\theta - e\theta m_1\right)\right)}{d\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + w_1\varphi_1 + 1\right)\left((b+1)g\varphi_2 + bd\theta w_2\varphi_2 + bd(\varphi_2 + 1)\theta + \theta m_1\left(b + e\varphi_2 + (b+2)\varphi_2 w_2 + 2\right)\right)\left(bd + m_1\left(g\varphi_3 b + d\varphi_3\theta b + b + g\varphi_3 + e\varphi_3\theta m_1 + 2\right) + \varphi_3\left(bd + (b+2)m_1\right)w_3\right)} \\
a_1^{(4)} = {} & -\frac{(-b-1)\varphi_1\varphi_2\left(bd\theta + bg + e\theta m_1 + g\right)\left(m_1\left(bd\theta + e\theta m_1 - g\right) - bdg\right)}{d\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + \varphi_1 w_1 + 1\right)\left(bd\theta(\varphi_2 + 1) + bd\theta\varphi_2 w_2 + \theta m_1\left((b+2)\varphi_2 w_2 + b + e\varphi_2 + 2\right) + (b+1)g\varphi_2\right)}, \\
& + \frac{m_1^2\varphi_1\varphi_3\left(bd\theta + bg + e\theta m_1 + g\right)\left(-bd(e-1)\theta - (b+1)e\theta m_1 + (b+1)g\right)}{d\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + \varphi_1 w_1 + 1\right)\left(m_1\left(bd\theta\varphi_3 + bg\varphi_3 + b + e\theta m_1\varphi_3 + g\varphi_3 + 2\right) + \varphi_3 w_3\left(bd + (b+2)m_1\right) + bd\right)} - \frac{2b\varphi_1\left(bd\theta + bg + e\theta m_1 + g\right)}{\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + \varphi_1 w_1 + 1\right)} \\
& - \frac{\varphi_2\varphi_3\left(bd(e-1)\theta + (b+1)e\theta m_1 - ((b+1)g)\right)\left(m_1\left(-bd\theta - e\theta m_1 + g\right) + bdg\right)}{\left(bd\theta(\varphi_2 + 1) + bd\theta\varphi_2 w_2 + \theta m_1\left((b+2)\varphi_2 w_2 + b + e\varphi_2 + 2\right) + (b+1)g\varphi_2\right)\left(m_1\left(bd\theta\varphi_3 + bg\varphi_3 + b + e\theta m_1\varphi_3 + g\varphi_3 + 2\right) + \varphi_3 w_3\left(bd + (b+2)m_1\right) + bd\right)} + 3, \\
a_2^{(4)} = {} & \frac{b\varphi_1\left(bd\theta + bg + e\theta m_1 + g\right)}{\theta\left(bd + (b+2)m_1\right)\left(b\varphi_1 + \varphi_1 w_1 + 1\right)} - 3.
\end{aligned}$$

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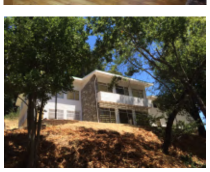
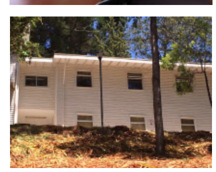

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