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Banach spaces-based fully mixed finite element methods for the n-dimensional Boussinesq problem with temperature-dependent parameters*

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Abstract

In this paper, we introduce and analyze a family of mixed finite element methods for the numerical solution of heat-driven flows with temperature-dependent parameters, modeled by a generalization of the stationary Boussinesq equations. Our approach relies on a reformulation of the governing equations in terms of the velocity, strain-rate tensor, vorticity, stress, pseudoheat, temperature, and its gradient. The pressure is eliminated from the system using the incompressibility constraint and can be subsequently recovered through a postprocessing formula involving the stress and velocity fields. Then, the resulting continuous formulation consists of a Banach spaces-based nonlinearly perturbed coupled system of twofold saddle point operator equations. By introducing suitable linearizations of the corresponding variational equations, we establish the unique solvability of the continuous problem through a fixed-point strategy. This analysis combines the Banach–Nečas–Babuška and Babuška–Brezzi theories in Banach spaces with the Banach fixed-point theorem, under an extraregularity assumption on the aforementioned linear systems and a smallness assumption on the data. Adopting an analogous approach for the associated Galerkin scheme, and under suitable hypotheses on the finite element subspaces employed, we establish existence of a discrete solution by applying the Brouwer fixed-point theorem and the discrete versions of the Banach–Nečas–Babuška and Babuška–Brezzi theories. Furthermore, the error analysis is carried out under appropriate assumptions on the data, and by employing similar arguments to those yielding Strang-type estimates. Finally, several numerical experiments are presented to illustrate the performance of the proposed scheme and to confirm the convergence rates predicted by the theoretical analysis.

Key words: generalized Boussinesq problem, temperature-dependent parameters, Banach spaces, fully mixed finite element method, a priori error analysis.

Mathematics Subject Classifications (2020): 65N30, 65N12, 65N15, 65N99, 76M25, 76S05

1 Introduction

Various types of free convection arise in both natural and industrial contexts, including mantle convection, stratified oceanic flows, and onboard cooling systems for electronic devices. These processes

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are typically modeled by coupling the continuity and momentum equations (Navier–Stokes) with the energy equation under the Boussinesq approximation, where the fluid density is assumed constant except in the buoyancy term, where its linear dependence on temperature is retained. However, in many physically relevant scenarios, other fluid properties, particularly viscosity and thermal conductivity, also exhibit strong temperature dependence (see, e.g. [5]). This is a key feature, for instance, in geophysical processes such as mantle convection and magma dynamics, in the flow of heavy oils and nanofluids, and in advanced thermal systems including electronic cooling, solar thermal collectors, and microfluidic applications. Accurately modeling such phenomena requires considering generalized Boussinesq equations with temperature-dependent coefficients, which leads to additional mathematical and numerical challenges. In this context, several numerical strategies have been proposed for generalized Boussinesq equations with temperature-dependent parameters. These include standard finite element methods (FEM) [8, 22, 24, 26], as well as mixed and augmented-mixed FEM formulations [1, 2, 3, 4, 23]. More recently, a virtual element method was developed in [6] for the Boussinesq system with temperature-dependent viscosity. Additionally, finite element methods based on primal formulations for generalized Boussinesq equations were studied in [24, 23]. Subsequently, the mixed-primal formulation initially introduced in [14] for the Boussinesq system with constant parameters was extended in [4] to the two-dimensional case with temperature-dependent viscosity, and further generalized to the n-dimensional setting in [1]. Similarly, the augmented fully mixed numerical scheme originally proposed in [16] for the Boussinesq model with constant parameters was extended to the n-dimensional problem with temperature-dependent parameters in [3].

One of the main advantages of the mixed formulations studied in [4, 1, 3] is that they allow for the direct recovery of additional variables of physical interest beyond the original unknowns. However, to ensure the well-posedness of the continuous and discrete problems, and to derive optimal convergence estimates, redundant Galerkin-type terms are incorporated into these formulations. Such an approach inevitably leads to denser matrices and increased computational costs, thus motivating the development of alternative formulations that retain the mathematical and numerical advantages but minimize computational complexity. In this context, a new Banach spaces-based fully mixed formulation was introduced in [12] for the Boussinesq model with constant parameters. This formulation, based on previous works [10, 14], enables the use of standard inf-sup stable finite element spaces suitable for mixed problems without resorting to the augmentation procedure involving the aforementioned redundant Galerkin-type terms introduced in [16]. In addition, the method introduced in [12] retains the advantages of the formulations studied in [1, 3, 4], namely, the direct recovery of additional variables of physical interest beyond the original unknowns. Furthermore, this formulation allows for the exact conservation of important physical quantities such as momentum and thermal energy.

Motivated by the preceding discussion, in this work we extend the results presented in [1, 3, 4] by utilizing the Banach spaces-based framework employed in [12]. Specifically, we propose and analyze a new fully mixed finite element method for the numerical approximation of the Boussinesq model with temperature-dependent parameters. Our approach relies on rewriting the governing equations as a first-order system, in which, analogously to [3], the pseudostress tensor, strain-rate tensor, vorticity, and velocity, together with the pseudoheat vector, temperature, and temperature gradient, constitute the primary unknowns. By multiplying the resulting equations with suitable test functions and employing standard integration by parts formulas, we derive a fully mixed variational formulation posed in appropriate Banach spaces. Concerning the discrete scheme, the variational formulation is discretized using generic finite-dimensional subspaces. We then establish suitable hypotheses ensuring existence of solution of the discrete problem and the validity of the associated Céa's estimate. Finally, we introduce specific finite element spaces satisfying these hypotheses, which allow us to derive optimal convergence rates. These discrete spaces are then employed to validate the theoretical results through a series of numerical experiments, which confirm the predicted convergence behavior and demonstrate the effectiveness of the proposed method.

The remainder of this work is organized as follows. In Section 2, we introduce the model problem, define the auxiliary variables required for the formulation, and derive the continuous fully mixed variational problem. Section 3 is devoted to the solvability analysis of the continuous problem, where we employ a fixed-point strategy combined with additional regularity assumptions on certain auxiliary linearized problems and a smallness assumption on the data. The corresponding Galerkin scheme is presented in Section 4, where, under suitable assumptions on generic discrete spaces and using analogous arguments to those in Section 3, we prove the existence of a discrete solution. Next, in Section 5, we derive a priori error estimates for the proposed Galerkin method, and Section 6 is dedicated to the introduction of specific finite element subspaces that satisfy the assumptions from Section 4. Finally, in Section 7, we report numerical results that demonstrate the performance of the method and confirm the theoretical rates of convergence established in Section 6.

1.1 Preliminaries

Let us denote by $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$ a given bounded domain with polyhedral boundary Γ, and denote by $\mathbf{n}$ the outward unit normal vector on Γ. Standard notations will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$, with $p \in [1,\infty]$ and Sobolev spaces $\mathrm{W}^{r,p}(\Omega)$ with $r \geqslant 0$, endowed with the norms $\|\cdot\|_{0,p,\Omega}$ and $\|\cdot\|_{r,p,\Omega}$, respectively. Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$ and if $p = 2$, we write $\mathrm{H}^r(\Omega)$ in place of $\mathrm{W}^{r,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{r,\Omega}$, respectively. We also write $|\cdot|_{r,\Omega}$ for the H^r-seminorm. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the spaces of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. With $\langle\cdot,\cdot\rangle$ we denote the corresponding product of duality between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. By $\mathbf{S}$ and $\mathbb{S}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space S. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$ we set the gradient, divergence and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\boldsymbol{\tau}_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and } \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I}, \tag{1.1}$$

where $\mathbb{I}$ denotes the identity tensor in $\mathrm{R}^{n\times n}$. On the other hand, given $t \geqslant \frac{2n}{n+2}$, we introduce the non-standard Banach spaces

$$\mathbf{H}(\operatorname{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \operatorname{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\Big\}$$

and

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\},$$

equipped with the norms

$$\|\boldsymbol{\tau}\|_{\operatorname{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\operatorname{div}(\boldsymbol{\tau})\|_{0,t;\Omega}, \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\operatorname{div}_t,\Omega)$$

and

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega}, \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t,\Omega),$$

respectively. Then, proceeding as in [19, eq. (1.43), Section 1.3.4], it is easy to show that for each $t \geqslant \frac{2n}{n+2}$, there holds

$$\langle \boldsymbol{\tau} \cdot \mathbf{n}, v \rangle \,=\, \int_\Omega \Big\{ \boldsymbol{\tau} \cdot \nabla v \,+\, v \operatorname{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_t; \Omega) \times \mathrm{H}^1(\Omega)\,, \tag{1.2a}$$

$$\langle \boldsymbol{\tau} \mathbf{n}, \mathbf{v} \rangle \,=\, \int_\Omega \Big\{ \boldsymbol{\tau} \cdot \boldsymbol{\nabla} \mathbf{v} \,+\, \mathbf{v} \,\mathbf{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t; \Omega) \times \mathbf{H}^1(\Omega)\,. \tag{1.2b}$$

Note that the above constraint on t guarantees that $\mathrm{H}^1(\Omega)$ is embedded in $\mathrm{L}^{t'}(\Omega)$, where t' is the Hölder conjugate of t. As a consequence of it, one deduces from (1.2a) the existence of a positive constant C_t, depending on t (or, equivalently, on t') and Ω, such that

$$\|\boldsymbol{\tau} \cdot \mathbf{n}\|_{-1/2,\Gamma} \,\leqslant\, C_t \,\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t; \Omega)\,. \tag{1.3}$$

Analogously, it follows from (1.2b) that

$$\|\boldsymbol{\tau}\, \mathbf{n}\|_{-1/2,\Gamma} \,\leqslant\, C_t \,\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t; \Omega)\,. \tag{1.4}$$

In the sequel, for the case $t = 2$ we simply employ the notation $\mathbf{H}(\mathrm{div}; \Omega)$ and $\mathbb{H}(\mathbf{div}; \Omega)$, respectively.

2 The continuous formulation

2.1 The model problem

Let $\Omega \subseteq \mathrm{R}^n$ be a bounded domain, $n \in \{2, 3\}$, with Lipschitz-boundary Γ. The boundary of this domain is divided into two portions Γ_D and Γ_N, such that $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \Gamma$ and $|\Gamma_D| > 0$. The model consists in a system of equations where the incompressible Navier–Stokes equation is coupled with the heat equation through convective and buoyancy terms, the latter typically acting in opposite direction to gravity. More precisely, we are interested in the following system of equations

$$-\mathbf{div}(\mu(\varphi)\,\mathbf{e}(\mathbf{u})) + (\nabla \mathbf{u})\mathbf{u} + \nabla p - \varphi\,\mathbf{g} \,=\, \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.1a}$$

$$\mathrm{div}(\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega, \tag{2.1b}$$

$$-\mathrm{div}\big(\kappa(\varphi)\,\nabla\varphi\big) + \mathbf{u} \cdot \nabla\varphi \,=\, f \quad \text{in} \quad \Omega, \tag{2.1c}$$

$$\mathbf{u} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma, \tag{2.1d}$$

$$\varphi \,=\, \varphi_D \quad \text{on} \quad \Gamma_D, \tag{2.1e}$$

$$\kappa(\varphi)\nabla\varphi \cdot \mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma_N, \tag{2.1f}$$

$$\int_\Omega p \,=\, 0\,, \tag{2.1g}$$

where $\mathbf{u}$, p and φ represent the velocity, pressure and temperature of the fluid, respectively, $\mathbf{g}$ is an external force per unit mass, f is a source term, φ_D is a prescribed temperature on Γ_D and

$$\mathbf{e}(\mathbf{u}) \,=\, \frac{1}{2}\big(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{t}}\big)\,.$$

The right spaces to which $\mathbf{g}$, f and φ_D belong, will be specified throughout the forthcoming analysis. In turn, $\mu, \kappa : \mathrm{R} \to \mathrm{R}^+$ are the temperature-dependent viscosity and thermal conductivity functions, respectively, which are assumed to be bounded from above and below by positive constants, that is

$$\mu_1 \,\leqslant\, \mu(w) \,\leqslant\, \mu_2 \quad \text{and} \quad \kappa_1 \,\leqslant\, \kappa(w) \,\leqslant\, \kappa_2 \qquad \forall\, w \in \mathrm{R}\,, \tag{2.2}$$

with $\mu_1, \mu_2, \kappa_1, \kappa_2 > 0$. In what follows, we assume that these functions are Lipschitz continuous, that is, there exist positive constants $\mathcal{L}_\mu$ and $\mathcal{L}_\kappa$, such that

$$\big|\mu(w) - \mu(v)\big| \leqslant \mathcal{L}_\mu \,|w - v| \quad \text{and} \quad \big|\kappa(w) - \kappa(v)\big| \leqslant \mathcal{L}_\kappa \,|w - v| \qquad \forall\, w, v \in \mathrm{R}\,. \tag{2.3}$$

Notice that equation (2.1g), is incorporated in the system to ensure uniqueness of the pressure.

Now, since we are interested in employing a fully mixed variational formulation for the coupled system (2.1a)-(2.1g), we first adopt the approach from [3] for the fluid equations and introduce the vorticity, strain and pseudostress tensors as further unknowns, given respectively by

$$\boldsymbol{\gamma} := \frac{1}{2}\big(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\big) \quad \text{in} \quad \Omega\,, \qquad \mathbf{t} := \mathbf{e}(\mathbf{u}) = \nabla\mathbf{u} - \boldsymbol{\gamma}\,, \quad \text{and}$$

$$\boldsymbol{\sigma} := \mu(\varphi)\,\mathbf{t} - \mathbf{u}\otimes\mathbf{u} - p\,\mathbb{I} \quad \text{in} \quad \Omega\,.$$

In particular, applying the tensor trace (cf. (1.1)) to $\mathbf{t}$ and $\boldsymbol{\sigma}$, and utilizing the incompressibility condition (2.1b), one arrives at

$$\mathrm{tr}(\mathbf{t}) = 0 \quad \text{in}\,\Omega \quad \text{and} \quad p = -\frac{1}{n}\,\mathrm{tr}\big(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}\big) \quad \text{in} \quad \Omega\,.$$

In this way, one can eliminate the pressure from the system (2.1a)–(2.1g) and rewrite it in terms of $\boldsymbol{\sigma}$, $\mathbf{t}$, $\boldsymbol{\gamma}$, $\mathbf{u}$ and φ, as follows

$$\begin{gathered}
\mathbf{t} + \boldsymbol{\gamma} = \boldsymbol{\nabla}\mathbf{u} \quad \text{in} \quad \Omega\,, \qquad \mu(\varphi)\,\mathbf{t} - (\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} = \boldsymbol{\sigma}^{\mathtt{d}} \quad \text{in} \quad \Omega\,,\\
-\,\mathbf{div}(\boldsymbol{\sigma}) - \varphi\mathbf{g} = \mathbf{0} \quad \text{in} \quad \Omega\,, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}} \quad \text{in} \quad \Omega\,,\\
\boldsymbol{\gamma} = -\boldsymbol{\gamma}^{\mathtt{t}} \quad \text{in} \quad \Omega\,, \quad \mathbf{t} = \mathbf{t}^{\mathtt{t}} \quad \text{in} \quad \Omega\,, \quad \mathrm{tr}(\mathbf{t}) = 0 \quad \text{in} \quad \Omega\,,\\
\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma \quad \text{and} \quad \int_\Omega \mathrm{tr}\big(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}\big) = 0\,.
\end{gathered} \tag{2.4}$$

Notice that the fourth and fifth equations of (2.4) are included in the system to ensure the skew-symmetry and symmetry of $\boldsymbol{\gamma}$ and $\mathbf{t}$, respectively, whereas the incompressibility of the fluid is now imposed through the sixth equation of (2.4). In addition, let us observe that last equation of (2.4) is equivalent to (2.1g).

Next, for the remaining equations (2.1c), (2.1e) and (2.1f), we follow the approach from [16, 3] and introduce, as additional unknowns, the gradient of the temperature and the pseudoheat vector field, given respectively by

$$\boldsymbol{\zeta} := \nabla\varphi \quad \text{and} \quad \boldsymbol{\rho} := \kappa(\varphi)\,\boldsymbol{\zeta} - \varphi\,\mathbf{u} \quad \text{in}\,\Omega\,,$$

so that we obtain the equivalent system

$$\begin{gathered}
\boldsymbol{\zeta} = \nabla\varphi \quad \text{in} \quad \Omega\,, \qquad \kappa(\varphi)\,\boldsymbol{\zeta} - \varphi\,\mathbf{u} = \boldsymbol{\rho} \quad \text{in} \quad \Omega\,, \qquad -\,\mathrm{div}(\boldsymbol{\rho}) = f \quad \text{in} \quad \Omega\,,\\
\varphi = \varphi_D \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \boldsymbol{\rho}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_N\,,
\end{gathered} \tag{2.5}$$

where (2.1f) has been converted to $\boldsymbol{\rho}\cdot\mathbf{n} = 0$ on Γ_N thanks to the no-slip condition $\mathbf{u} = \mathbf{0}$ on Γ.

2.2 The fully mixed variational formulation

In this section we introduce the variational formulation for the system given by (2.4) and (2.5). We begin with the set of equations (2.4) by introducing, as suggested by the properties satisfied by the unknown $\mathbf{t}$, the space

$$\mathbb{L}^2_{\mathtt{tr}}(\Omega) := \Big\{\mathbf{s}\in\mathbb{L}^2(\Omega): \quad \mathbf{s} - \mathbf{s}^{\mathtt{t}} = \mathbf{0} \quad \text{and} \quad \mathrm{tr}(\mathbf{s}) = 0\Big\}\,,$$

to which $\mathbf{t}$ clearly belongs. Then, testing the second equation in the first row of (2.4) with $\mathbf{s} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, and using that $\mathbf{r}^{\mathtt{d}} : \mathbf{s} = \mathbf{r} : \mathbf{s}$, for any tensor $\mathbf{r}$, we formally obtain

$$\int_\Omega \mu(\varphi)\, \mathbf{t} : \mathbf{s} - \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \mathbf{s} - \int_\Omega \boldsymbol{\sigma} : \mathbf{s} \;=\; 0 \qquad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega)\,. \tag{2.6}$$

The boundedness of μ (cf. (2.2)) guarantees that the first term of (2.6) is well-defined, whereas the third one is as well if $\boldsymbol{\sigma}$ belongs to $\mathbb{L}^2(\Omega)$. In turn, simple applications of the Cauchy-Schwarz inequality show that the second term makes sense if $\mathbf{u} \in \mathbf{L}^4(\Omega)$. Thus, it is reasonable to assume, at least at first instance, that actually $\mathbf{u} \in \mathbf{H}^1(\Omega)$, which is certainly embedded in $\mathbf{L}^4(\Omega)$. In this way, proceeding similarly to [9, 12], we now test the first equation in the first row of (2.4) with $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, so that applying the integration by parts formula (1.2b), with $t = 4/3$, and employing the boundary condition $\mathbf{u} = \mathbf{0}$ on Γ, we arrive at

$$\int_\Omega \mathbf{t} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} + \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \;=\; 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)\,. \tag{2.7}$$

The first term of (2.7) is certainly well-defined since $\mathbf{t}$ and $\boldsymbol{\tau}$ belong to $\mathbb{L}^2(\Omega)$, and the second is as well if $\boldsymbol{\gamma}$, which must satisfy $\boldsymbol{\gamma} = -\boldsymbol{\gamma}^{\mathtt{t}}$, is sought in the space

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) \;:=\; \Big\{ \boldsymbol{\omega} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\omega} = -\boldsymbol{\omega}^{\mathtt{t}} \Big\}\,.$$

In addition, the third term of (2.7) makes sense for $\mathbf{u}$ again in $\mathbf{L}^4(\Omega)$ thanks to the Hölder inequality, thus explaining the previous choice of $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ for $\boldsymbol{\tau}$. Furthermore, we impose the symmetry of $\boldsymbol{\sigma}$ through

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\omega} \;=\; 0 \qquad \forall\, \boldsymbol{\omega} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)\,.$$

Finally, adding the constraint $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$, the first equation in the second row of (2.4) is weakly imposed as follows:

$$\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) + \int_\Omega \varphi\, \mathbf{g} \cdot \mathbf{v} \;=\; 0 \qquad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,, \tag{2.8}$$

and hence, an appropriate space for $\boldsymbol{\sigma}$ is the same as for $\boldsymbol{\tau}$, that is $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$. Regarding the respective spaces for φ and $\mathbf{g}$, which appear in the second term of the left-hand side of (2.8), they will become clear next when deriving the variational formulation for the system (2.5). Indeed, testing the second equation in the first row of (2.5) with $\boldsymbol{\xi} \in \mathbf{L}^2(\Omega)$, we formally obtain

$$\int_\Omega \kappa(\varphi)\, \boldsymbol{\zeta} \cdot \boldsymbol{\xi} - \int_\Omega \varphi\, \mathbf{u} \cdot \boldsymbol{\xi} - \int_\Omega \boldsymbol{\rho} \cdot \boldsymbol{\xi} \;=\; 0 \qquad \forall\, \boldsymbol{\xi} \in \mathbf{L}^2(\Omega)\,,$$

from which, assuming that $\boldsymbol{\zeta}$ and $\boldsymbol{\rho}$ are originally sought in $\mathbf{L}^2(\Omega)$, we observe that its first and third terms are well-defined, in particular the first one thanks also to the boundedness of κ (cf. (2.2)). Regarding the second term, and knowing already that $\mathbf{u} \in \mathbf{L}^4(\Omega)$, it follows again by Hölder's inequality that it makes sense if we look for φ in $\mathrm{L}^4(\Omega)$. Moreover, proceeding similarly as for (2.7), we actually assume now that $\varphi \in \mathrm{H}^1(\Omega)$ and that the Dirichlet datum $\varphi_D \in \mathrm{H}^{1/2}(\Omega)$, so that employing the integration by parts formula (1.2a) with $t = 4/3$, and making use of the boundary condition $\varphi|_\Gamma = \varphi_D$, the testing of the first equation in the first row of (2.5) against $\boldsymbol{\eta} \in \mathbf{H}_N(\mathrm{div}_{4/3};\Omega)$, yields

$$\int_\Omega \boldsymbol{\zeta} \cdot \boldsymbol{\eta} + \int_\Omega \varphi\, \mathrm{div}(\boldsymbol{\eta}) \;=\; \langle \boldsymbol{\eta} \cdot \mathbf{n}, \varphi_D \rangle_{\Gamma_D} \qquad \forall\, \boldsymbol{\eta} \in \mathbf{H}_N(\mathrm{div}_{4/3};\Omega)\,, \tag{2.9}$$

where

$$\mathbf{H}_N(\mathrm{div}_{4/3};\Omega) \,=\, \Big\{\boldsymbol{\eta}\in\mathbf{H}(\mathrm{div}_{4/3};\Omega): \quad \boldsymbol{\eta}\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma_N\Big\}\,.$$

Note that the second term of (2.9) makes sense for φ again in $\mathrm{L}^4(\Omega)$. Then, going back to the second term on the left-hand side of (2.8), we realize that it is well-defined if the datum $\mathbf{g}$ is assumed to belong to $\mathbf{L}^2(\Omega)$. We conclude the derivation of the variational formulation by adding to $\boldsymbol{\rho}$ the condition $\mathrm{div}(\boldsymbol{\rho})\in\mathrm{L}^{4/3}(\Omega)$, which, together with the boundary condition $\boldsymbol{\rho}\cdot\mathbf{n}=0$ on Γ_N, turns this unknown to be looked for in $\mathbf{H}_N(\mathrm{div}_{4/3};\Omega)$. Hence, assuming from now on that $f\in\mathrm{L}^{4/3}(\Omega)$, the third equation in the first row of (2.5) is imposed weakly through

$$\int_\Omega \psi\,\mathrm{div}(\boldsymbol{\rho}) \,=\, -\int_\Omega f\,\psi \qquad \forall\,\psi\in\mathrm{L}^4(\Omega)\,. \tag{2.10}$$

In light of the above, the variational problem associated with the system (2.4)–(2.5) reads as follows: Find $\mathbf{t}\in\mathbb{L}^2_{\mathtt{tr}}(\Omega)$, $\boldsymbol{\sigma}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, $\mathbf{u}\in\mathbf{L}^4(\Omega)$, $\boldsymbol{\gamma}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, $\boldsymbol{\zeta}\in\mathbf{L}^2(\Omega)$, $\boldsymbol{\rho}\in\mathbf{H}_N(\mathrm{div}_{4/3};\Omega)$, and $\varphi\in\mathrm{L}^4(\Omega)$, such that (2.6)–(2.10) hold, with

$$\int_\Omega \mathrm{tr}\big(\boldsymbol{\sigma}+\mathbf{u}\otimes\mathbf{u}\big) = 0. \tag{2.11}$$

However, following the approach in [11], we observe that due to the constraint (2.11) and the decomposition (see, e.g., [9], [19])

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\oplus\mathbb{R}\mathbb{I},$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) := \left\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega): \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\right\},$$

any solution $\boldsymbol{\sigma}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ to the system can be written as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\mathbb{I}$, where $\boldsymbol{\sigma}_0\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and

$$c \,:=\, -\frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}\big(\mathbf{u}\otimes\mathbf{u}\big)\,.$$

Then, noting that $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_0^{\mathtt{d}}$ and $\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{div}(\boldsymbol{\sigma}_0)$, equations (2.6) and (2.8) can be equivalently expressed in terms of $\boldsymbol{\sigma}_0$ without altering their meaning. Accordingly, in what follows, we omit the constraint (2.11) and study the system (2.6)–(2.10) with $\boldsymbol{\sigma}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, where the subscript 0 in $\boldsymbol{\sigma}$ has been dropped for the sake of simplicity of notation.

3 The continuous solvability analysis

In this section, we analyze the solvability of the system (2.6)–(2.10) by combining the Babuška-Brezzi theory and the classical Banach–Nečas–Babuška theorem, both in Banach spaces, with a fixed-point strategy.

3.1 Preliminaries

We begin by rewriting the equations (2.6) up to (2.10) in terms of suitable bilinear forms. Indeed, we first introduce the spaces:

$$\mathcal{H} \,:=\, \mathbb{L}^2_{\mathtt{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \qquad \mathcal{Q} \,:=\, \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)\,,$$

$$\widetilde{\mathcal{H}} \,:=\, \mathbf{L}^2(\Omega)\times\mathbf{H}_N(\mathrm{div}_{4/3};\Omega)\,, \quad \text{and} \quad \widetilde{\mathcal{Q}} \,:=\, \mathrm{L}^4(\Omega)\,,$$

which are endowed with the respective norms

$$\|\vec{\mathbf{s}}\|_{\mathcal{H}} := \|\mathbf{s}\|_{0,\Omega} + \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}\,, \qquad \|\vec{\mathbf{w}}\|_{\mathcal{Q}} := \|\mathbf{w}\|_{0,4;\Omega} + \|\boldsymbol{\omega}\|_{0,\Omega}\,,$$
$$\|\vec{\boldsymbol{\xi}}\|_{\widetilde{\mathcal{H}}} := \|\boldsymbol{\xi}\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{\mathrm{div}_{4/3};\Omega}\,, \quad \text{and} \quad \|\psi\|_{\widetilde{\mathcal{Q}}} := \|\psi\|_{0,4;\Omega}\,,$$

for all $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}) \in \mathcal{H}$, $\vec{\mathbf{w}} := (\mathbf{w}, \boldsymbol{\omega}) \in \mathcal{Q}$, $\vec{\boldsymbol{\xi}} := (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \widetilde{\mathcal{H}}$, and $\psi \in \widetilde{\mathcal{Q}}$. Then, it is easily seen that the system given by the equations (2.6) up to (2.10) can be reformulated as the following nonlinearly perturbed coupled system of twofold saddle point formulations: Find $\vec{\mathbf{t}} = (\mathbf{t}, \boldsymbol{\sigma}) \in \mathcal{H}$, $\vec{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\gamma}) \in \mathcal{Q}$, $\vec{\boldsymbol{\zeta}} = (\boldsymbol{\zeta}, \boldsymbol{\rho}) \in \widetilde{\mathcal{H}}$, and $\varphi \in \widetilde{\mathcal{Q}}$, such that

$$\begin{aligned} A_\varphi(\vec{\mathbf{t}}, \vec{\mathbf{s}}) + B(\vec{\mathbf{s}}, \vec{\mathbf{u}}) + C_{\mathbf{u}}(\mathbf{u}, \mathbf{s}) &= 0 && \forall\, \vec{\mathbf{s}} \in \mathcal{H}\,, \\ B(\vec{\mathbf{t}}, \vec{\mathbf{w}}) &= G_\varphi(\vec{\mathbf{w}}) && \forall\, \vec{\mathbf{w}} \in \mathcal{Q}\,, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \widetilde{A}_\varphi(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\xi}}) + \widetilde{B}(\vec{\boldsymbol{\xi}}, \varphi) + \widetilde{C}_{\mathbf{u}}(\varphi, \boldsymbol{\xi}) &= \widetilde{F}(\vec{\boldsymbol{\xi}}) && \forall\, \vec{\boldsymbol{\xi}} \in \widetilde{\mathcal{H}}\,, \\ \widetilde{B}(\vec{\boldsymbol{\zeta}}, \psi) &= \widetilde{G}(\psi) && \forall\, \psi \in \widetilde{\mathcal{Q}}\,, \end{aligned} \tag{3.2}$$

where, for each $\phi \in \mathrm{L}^4(\Omega)$, $A_\phi : \mathcal{H} \times \mathcal{H} \to \mathrm{R}$ and $\widetilde{A}_\phi : \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}} \to \mathrm{R}$ are defined by

$$A_\phi(\vec{\mathbf{t}}, \vec{\mathbf{s}}) := a_\phi(\mathbf{t}, \mathbf{s}) + b(\mathbf{s}, \boldsymbol{\sigma}) + b(\mathbf{t}, \boldsymbol{\tau}) \quad \text{and} \quad \widetilde{A}_\phi(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\xi}}) := \widetilde{a}_\phi(\boldsymbol{\zeta}, \boldsymbol{\xi}) + \widetilde{b}(\boldsymbol{\xi}, \boldsymbol{\rho}) + \widetilde{b}(\boldsymbol{\zeta}, \boldsymbol{\eta})\,, \tag{3.3}$$

with $a_\phi : \mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbb{L}^2_{\mathtt{tr}}(\Omega) \to \mathrm{R}$, $b : \mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \to \mathrm{R}$, $\widetilde{a}_\phi : \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \to \mathrm{R}$, and $\widetilde{b} : \mathbf{L}^2(\Omega) \times \mathbf{H}_N(\mathrm{div}_{4/3}; \Omega) \to \mathrm{R}$, given by

$$a_\phi(\mathbf{t}, \mathbf{s}) := \int_\Omega \mu(\phi)\, \mathbf{t} : \mathbf{s}\,, \qquad b(\mathbf{s}, \boldsymbol{\tau}) := -\int_\Omega \mathbf{s} : \boldsymbol{\tau}\,,$$
$$\widetilde{a}_\phi(\boldsymbol{\zeta}, \boldsymbol{\xi}) := \int_\Omega \kappa(\phi)\, \boldsymbol{\zeta} \cdot \boldsymbol{\xi}\,, \qquad \widetilde{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) := -\int_\Omega \boldsymbol{\xi} \cdot \boldsymbol{\eta}\,.$$

Note here that the saddle point structure of both A_ϕ and $\widetilde{A}_\phi$ explains the twofold concept employed to refer to (3.1) and (3.2). In turn, $B : \mathcal{H} \times \mathcal{Q} \to \mathrm{R}$ and $\widetilde{B} : \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}} \to \mathrm{R}$ are defined as

$$B((\mathbf{s}, \boldsymbol{\tau}), (\mathbf{v}, \boldsymbol{\omega})) := -\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) - \int_\Omega \boldsymbol{\omega} : \boldsymbol{\tau} \quad \text{and} \quad \widetilde{B}((\boldsymbol{\xi}, \boldsymbol{\eta}), \psi) := -\int_\Omega \psi\, \mathrm{div}(\boldsymbol{\eta})\,, \tag{3.4}$$

whereas $C_{\mathbf{w}} : \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathtt{tr}}(\Omega) \to \mathrm{R}$ and $\widetilde{C}_{\mathbf{w}} : \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \to \mathrm{R}$ reduce, for each $\mathbf{w} \in \mathbf{L}^4(\Omega)$, to

$$C_{\mathbf{w}}(\mathbf{v}, \mathbf{s}) := -\int_\Omega (\mathbf{w} \otimes \mathbf{v}) : \mathbf{s} \quad \text{and} \quad \widetilde{C}_{\mathbf{w}}(\psi, \boldsymbol{\xi}) := -\int_\Omega \psi(\mathbf{w} \cdot \boldsymbol{\xi})\,.$$

We now remark that, although $\widetilde{C}_{\mathbf{u}}$ is actually a bilinear form, the fact that $C_{\mathbf{u}}(\mathbf{u}, \mathbf{s})$ is nonlinear in $\mathbf{u}$ explains the nonlinearly perturbed concept utilized before. Finally, the functionals G_ϕ, for each $\phi \in \mathrm{L}^4(\Omega)$, $\widetilde{F} : \widetilde{\mathcal{H}} \to \mathrm{R}$, and $\widetilde{G} : \widetilde{\mathcal{Q}} \to \mathrm{R}$, are defined by

$$G_\phi(\vec{\mathbf{w}}) := -\int_\Omega \phi\, \mathbf{g} \cdot \mathbf{w}\,, \qquad \widetilde{F}(\vec{\boldsymbol{\xi}}) := -\langle \boldsymbol{\eta} \cdot \mathbf{n}, \varphi_D \rangle_{\Gamma_D}\,,$$
$$\text{and} \qquad \widetilde{G}(\psi) := \int_\Omega f \psi\,.$$

3.2 The fixed-point strategy

We now introduce the fixed-point strategy to analyze the solvability of the coupled system (3.1)-(3.2). We begin by introducing the auxiliary operator $\mathbf{S} : \mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega) \to \mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega)$ defined by

$$\mathbf{S}(\mathbf{z},\phi) \;=\; (\mathbf{S}_1(\mathbf{z},\phi),\mathbf{S}_2(\mathbf{z},\phi)) \;:=\; (\mathbf{t},\mathbf{u}) \qquad \forall\,(\mathbf{z},\phi)\in \mathbf{L}^4(\Omega)\times \mathrm{L}^4(\Omega)\,, \tag{3.5}$$

where $\mathbf{t}\in\mathbb{L}^2_{\mathtt{tr}}(\Omega)$ and $\mathbf{u}\in\mathbf{L}^4(\Omega)$ are the first and third components, respectively, of the unique solution (to be confirmed below) of the linearized problem arising from (3.1) when replacing A_φ, $C_{\mathbf{u}}$, and G_φ by A_ϕ, $C_{\mathbf{z}}$, and G_ϕ, respectively, that is: Find $(\vec{\mathbf{t}},\vec{\mathbf{u}}) = ((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{u},\boldsymbol{\gamma}))\in\mathcal{H}\times\mathcal{Q}$, such that

$$\begin{aligned} &A_\phi(\vec{\mathbf{t}},\vec{\mathbf{s}}) + B(\vec{\mathbf{s}},\vec{\mathbf{u}}) + C_{\mathbf{z}}(\mathbf{u},\mathbf{s}) &&= 0 &&\forall\,\vec{\mathbf{s}} = (\mathbf{s},\boldsymbol{\tau})\in\mathcal{H},\\ &B(\vec{\mathbf{t}},\vec{\mathbf{v}}) &&= G_\phi(\vec{\mathbf{v}}) &&\forall\,\vec{\mathbf{v}} = (\mathbf{v},\boldsymbol{\omega})\in\mathcal{Q}\,. \end{aligned} \tag{3.6}$$

In turn, we let $\mathbf{T} : \mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)\to\mathbf{L}^2(\Omega)\times\mathrm{L}^4(\Omega)$ be the operator given by

$$\mathbf{T}(\mathbf{z},\phi) \;=\; (\mathbf{T}_1(\mathbf{z},\phi),\mathbf{T}_2(\mathbf{z},\phi)) \;:=\; (\boldsymbol{\zeta},\varphi) \quad \forall\,(\mathbf{z},\phi)\in\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)\,, \tag{3.7}$$

where $\boldsymbol{\zeta}\in\mathbf{L}^2(\Omega)$ and $\varphi\in\mathrm{L}^4(\Omega)$ are the first and third components, respectively, of the unique solution (to be confirmed below) of the linearized problem arising from (3.2) when replacing $\widetilde{A}_\varphi$ and $\widetilde{C}_{\mathbf{u}}$ by $\widetilde{A}_\phi$ and $\widetilde{C}_{\mathbf{z}}$, respectively, that is: Find $(\vec{\boldsymbol{\zeta}},\varphi) = ((\boldsymbol{\zeta},\boldsymbol{\rho}),\varphi)\in\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}$, such that

$$\begin{aligned} &\widetilde{A}_\phi(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\xi}}) + \widetilde{B}(\vec{\boldsymbol{\xi}},\varphi) + \widetilde{C}_{\mathbf{z}}(\varphi,\boldsymbol{\xi}) &&= \widetilde{F}(\vec{\boldsymbol{\xi}}) &&\forall\,\vec{\boldsymbol{\xi}} = (\boldsymbol{\xi},\boldsymbol{\eta})\in\widetilde{\mathcal{H}},\\ &\widetilde{B}(\vec{\boldsymbol{\zeta}},\psi) &&= \widetilde{G}(\psi) &&\forall\,\psi\in\widetilde{\mathcal{Q}}\,. \end{aligned} \tag{3.8}$$

Thus, we let $\mathbf{J} : \mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)\to\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$ be the operator defined by

$$\mathbf{J}(\mathbf{z},\phi) \;=\; \big(\mathbf{S}_2(\mathbf{z},\mathbf{T}_2(\mathbf{z},\phi)),\mathbf{T}_2(\mathbf{z},\phi)\big) \qquad \forall\,(\mathbf{z},\phi)\in\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)\,, \tag{3.9}$$

and realize that solving (3.1)-(3.2) is equivalent to seeking a fixed-point of $\mathbf{J}$, that is: Find $(\mathbf{u},\varphi)\in\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$, such that

$$\mathbf{J}(\mathbf{u},\varphi) \;=\; (\mathbf{u},\varphi)\,. \tag{3.10}$$

In this way, in what follows we focus on proving that $\mathbf{J}$ possesses a unique fixed-point, for which we begin by establishing in the following section that $\mathbf{S}$ and $\mathbf{T}$, and hence $\mathbf{J}$, are well defined.

3.3 Well-definiteness of S, T and J

In order to prove that $\mathbf{S}$ and $\mathbf{T}$ are well defined, equivalently that the linear problems (3.6) and (3.8) are well-posed, we resort to the Banach–Nečas–Babuška Theorem and the classical Babuška–Brezzi theory, which are recalled next in the setting of Banach spaces (cf. [18, Theorems 2.6 and 2.34]).

Theorem 3.1. *Let* H *and* Q *be Banach spaces such that Q is reflexive, and let $A : H\times Q \longrightarrow \mathrm{R}$ be a bounded bilinear form. Assume that*

i) *there exists $\alpha>0$ such that*

$$\sup_{\substack{v\in Q\\ v\neq 0}} \frac{A(w,v)}{\|v\|_Q} \;\geqslant\; \alpha\,\|w\|_H \qquad \forall\,w\in H\,, \tag{3.11}$$

ii) *there holds*

$$\sup_{w \in H} A(w,v) \;>\; 0 \qquad \forall\, v \in Q,\; v \neq 0\,. \tag{3.12}$$

Then, for each $F \in Q'$ *there exists a unique* $u \in H$ *such that*

$$A(u,v) \;=\; F(v) \qquad \forall\, v \in Q\,, \tag{3.13}$$

and the following a priori estimate holds

$$\|u\|_H \;\leqslant\; \frac{1}{\alpha}\,\|F\|_{Q'}\,. \tag{3.14}$$

Moreover, i) *and* ii) *are also necessary conditions for the well-posedness of* (3.13).

Theorem 3.2. *Let* H *and* Q *be reflexive Banach spaces, and let* $a : \mathrm{H} \times \mathrm{H} \longrightarrow \mathrm{R}$ *and* $b : \mathrm{H} \times \mathrm{Q} \longrightarrow \mathrm{R}$ *be bounded bilinear forms with boundedness constants denoted by* $\|a\|$ *and* $\|b\|$*, respectively. In addition, let* V *be the null space of the operator induced by* b*, which reduces to*

$$\mathrm{V} \;:=\; \Big\{\tau \in \mathrm{H} : \quad b(\tau, v) = 0 \quad \forall\, v \in Q\Big\}\,,$$

and assume that

i) *there exists* $\alpha > 0$ *such that*

$$\sup_{\substack{\tau \in \mathrm{V}\\ \tau \neq 0}} \frac{a(\zeta,\tau)}{\|\tau\|_{\mathrm{H}}} \;\geqslant\; \alpha\,\|\zeta\|_{\mathrm{H}} \qquad \forall\, \zeta \in \mathrm{V}\,, \tag{3.15}$$

ii) *there holds*

$$\sup_{\tau \in \mathrm{V}} a(\tau,\zeta) > 0 \qquad \forall\, \zeta \in \mathrm{V}, \quad \zeta \neq 0\,, \tag{3.16}$$

iii) *there exists* β *such that*

$$\sup_{\substack{\tau \in \mathrm{H}\\ \tau \neq 0}} \frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}} \;\geqslant\; \beta\,\|v\|_{\mathrm{Q}} \qquad \forall\, v \in \mathrm{Q}\,.$$

Then, for each pair $(F,G) \in \mathrm{H}' \times \mathrm{Q}'$ *there exists a unique* $(\sigma,u) \in \mathrm{H} \times \mathrm{Q}$ *such that*

$$\begin{aligned} a(\sigma,\tau) \;+\; b(\tau,u) \;&=\; F(\tau) \qquad \forall\, \tau \in \mathrm{H}\,,\\ b(\sigma,v) \;&=\; G(v) \qquad \forall\, v \in \mathrm{Q}\,, \end{aligned} \tag{3.17}$$

and the following a priori estimates hold

$$\begin{aligned} \|\sigma\| &\leqslant \frac{1}{\alpha}\,\|F\|_{\mathrm{H}'} + \frac{1}{\beta}\Big(1 + \frac{\|a\|}{\alpha}\Big)\|G\|_{\mathrm{Q}'}\,,\\ \|u\| &\leqslant \frac{1}{\beta}\,\Big(1 + \frac{\|a\|}{\alpha}\Big)\|F\|_{\mathrm{H}'} \;+\; \frac{\|a\|}{\beta^2}\Big(1 + \frac{\|a\|}{\alpha}\Big)\|G\|_{\mathrm{Q}'}\,. \end{aligned} \tag{3.18}$$

Moreover, i)*,* ii)*, and* iii) *are also necessary conditions for the well-posedness of* (3.17).

We stress here that (3.18) is equivalent to the following global inf-sup condition for (3.17):

$$\sup_{\substack{(\tau,v)\in \mathrm{H}\times\mathrm{Q}\\ (\tau,v)\neq 0}} \frac{a(\zeta,\tau)+b(\tau,w)+b(\zeta,v)}{\|(\tau,v)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \widetilde{\alpha}\,\|(\zeta,w)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\zeta,w)\in \mathrm{H}\times\mathrm{Q}\,, \tag{3.19}$$

where $\widetilde{\alpha}>0$ is a constant depending only on α, β, and $\|a\|$ (as it follows from (3.18)).

In what follows we prove first that the formulations arising from (3.6) and (3.8) after dropping the perturbations $C_{\mathbf{z}}$ and $\widetilde{C}_{\mathbf{z}}$, satisfy the hypotheses of Theorem 3.2. Secondly, we show that the full problems (3.6) and (3.8) satisfy the hypotheses of Theorem 3.1. We begin with the stability properties of the bilinear forms and functionals involved. Indeed, given $\phi\in\widetilde{\mathcal{Q}}$ and $\mathbf{w}\in\mathbf{L}^4(\Omega)$, we employ (2.2) and Hölder's inequality to deduce that

$$\begin{gathered}
|a_\phi(\mathbf{t},\mathbf{s})| \leqslant \mu_2\,\|\mathbf{t}\|_{0,\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,, \qquad |b(\mathbf{s},\boldsymbol{\sigma})| \leqslant \|\mathbf{s}\|_{0,\Omega}\|\,\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3},\Omega}\,,\\
|B(\vec{\mathbf{s}},\vec{\mathbf{v}})| \leqslant \|(\mathbf{s},\boldsymbol{\tau})\|_{\mathcal{H}}\,\|(\mathbf{v},\boldsymbol{\omega})\|_{\mathcal{Q}}\,, \qquad |C_{\mathbf{w}}(\mathbf{v},\mathbf{s})| \leqslant \|\mathbf{w}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,,\\
|G_\phi(\vec{\mathbf{v}})| \leqslant \|\phi\|_{0,4;\Omega}\,\|\mathbf{g}\|_{0,\Omega}\,\|\vec{\mathbf{v}}\|_{\mathcal{Q}}\,,
\end{gathered} \tag{3.20}$$

for all $\vec{\mathbf{t}}=(\mathbf{t},\boldsymbol{\sigma})$, $\vec{\mathbf{s}}=(\mathbf{s},\boldsymbol{\tau})\in\mathcal{H}$, for all $\vec{\mathbf{v}}=(\mathbf{v},\boldsymbol{\omega})\in\mathcal{Q}$, and

$$\begin{gathered}
|\widetilde{a}_\phi(\boldsymbol{\zeta},\boldsymbol{\xi})| \leqslant \kappa_2\,\|\boldsymbol{\zeta}\|_{0,\Omega}\,\|\boldsymbol{\xi}\|_{0,\Omega}\,, \qquad |\widetilde{b}(\boldsymbol{\xi},\boldsymbol{\rho})| \leqslant \|\boldsymbol{\xi}\|_{0,\Omega}\,\|\boldsymbol{\rho}\|_{\mathrm{div}_{4/3},\Omega}\,,\\
|\widetilde{B}(\vec{\boldsymbol{\xi}},\psi)| \leqslant \|\vec{\boldsymbol{\xi}}\|_{\widetilde{\mathcal{H}}}\,\|\psi\|_{0,4;\Omega}\,, \qquad |\widetilde{C}_{\mathbf{w}}(\psi,\boldsymbol{\xi})| \leqslant \|\mathbf{w}\|_{0,4;\Omega}\,\|\psi\|_{0,4;\Omega}\,\|\boldsymbol{\xi}\|_{0,\Omega}\,,\\
|\widetilde{F}(\vec{\boldsymbol{\xi}})| \leqslant C_D\,\|\varphi_D\|_{1/2,\Gamma_D}\,\|\vec{\boldsymbol{\xi}}\|_{\widetilde{\mathcal{H}}}\,, \qquad |\widetilde{G}(\psi)| \leqslant \|f\|_{0,4/3,\Omega}\,\|\psi\|_{0,4;\Omega}\,,
\end{gathered} \tag{3.21}$$

for all $\vec{\boldsymbol{\zeta}}=(\boldsymbol{\zeta},\boldsymbol{\rho})$, $\vec{\boldsymbol{\xi}}=(\boldsymbol{\xi},\boldsymbol{\eta})\in\widetilde{\mathcal{H}}$, for all $\psi\in\widetilde{\mathcal{Q}}$. Note here that the constant C_D in the boundedness of $\widetilde{F}$ is given by the norm of a suitable continuous extension $E_D:\mathrm{H}^{1/2}(\Gamma_D)\to\mathrm{H}^{1/2}(\Gamma)$ times $C_{4/3}$ (cf. (1.3)). In turn, we know from [21, Lemma 3.4] and [12, Lemma 3.1] that there exist positive constant β and $\widetilde{\beta}$, such that the bilinear forms B and $\widetilde{B}$ satisfy continuous inf-sup conditions in $\mathcal{H}\times\mathcal{Q}$ and $\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}$, respectively, that is

$$\sup_{\vec{\mathbf{s}}\in\mathcal{H}\backslash\mathbf{0}} \frac{B(\vec{\mathbf{s}},\vec{\mathbf{v}})}{\|(\mathbf{s},\boldsymbol{\tau})\|_{\mathcal{H}}} \geqslant \beta\,\|\vec{\mathbf{v}}\|_{\mathcal{Q}} \qquad \forall\,\vec{\mathbf{v}}\in\mathcal{Q}\,, \tag{3.22}$$

and

$$\sup_{\vec{\boldsymbol{\xi}}\in\widetilde{\mathcal{H}}\backslash\mathbf{0}} \frac{\widetilde{B}(\vec{\boldsymbol{\xi}},\psi)}{\|\vec{\boldsymbol{\xi}}\|_{\widetilde{\mathcal{H}}}} \geqslant \widetilde{\beta}\,\|\psi\|_{0,4;\Omega} \qquad \forall\,\psi\in\widetilde{\mathcal{Q}}\,. \tag{3.23}$$

Now, let $\mathcal{V}$ be the kernel of B, that is

$$\mathcal{V} := \Big\{(\mathbf{s},\boldsymbol{\tau})\in\mathcal{H}: \quad B((\mathbf{s},\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\omega}))=0 \quad \forall\,(\mathbf{v},\boldsymbol{\omega})\in\mathcal{Q}\Big\}\,,$$

which, according to the definition of B (cf. (3.4)), can be characterized as

$$\mathcal{V} = \mathbb{L}^2_{\mathtt{tr}}(\Omega)\,\times\,\mathcal{V}_0\,,$$

with

$$\mathcal{V}_0 := \Big\{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega): \quad \mathbf{div}(\boldsymbol{\tau})=\mathbf{0} \quad \text{and} \quad \boldsymbol{\tau}=\boldsymbol{\tau}^t \quad \text{in} \quad \Omega\Big\}\,.$$

Since the formulation arising from (3.6) after dropping $C_{\mathbf{z}}$ has a saddle-point structure, we proceed next to verify that, given $\phi\in\mathcal{Q}$, the bilinear form A_ϕ satisfies the assumptions (3.15) and (3.16) required by Theorem 3.2. However, bearing in mind that A_ϕ itself exhibits the same structure in terms of the bilinear forms a_ϕ and b (see (3.3)), it suffices to prove, according to the equivalence between (3.18) and (3.19), that a_ϕ and b satisfy the assumptions of Theorem 3.2 on $\mathcal{V} := \mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathcal{V}_0$. We first observe, thanks to (2.2), that there holds

$$a_\phi(\mathbf{s},\mathbf{s}) \,\geqslant\, \alpha_1 \, \|\mathbf{s}\|^2_{0,\Omega} \qquad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega)\,, \tag{3.24}$$

with $\alpha_1 = \mu_1$, which, noting that the null space of $b|_{\mathcal{V}}$ is certainly a subspace of $\mathbb{L}^2_{\mathtt{tr}}(\Omega)$, easily implies that a_ϕ satisfies assumptions i) and ii) of Theorem 3.2. In turn, we know from [20, Lemma 3.3] that there exists a positive constant β_1 such that

$$\sup_{\mathbf{s}\in\mathbb{L}^2_{\mathtt{tr}}(\Omega)\backslash\{\mathbf{0}\}} \frac{b(\mathbf{s},\boldsymbol{\tau})}{\|\mathbf{s}\|_{0,\Omega}} \,\geqslant\, \beta_1 \, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3},\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathcal{V}_0\,, \tag{3.25}$$

which proves that b satisfies assumption iii) of Theorem 3.2. In this way, having shown that a_ϕ and b satisfy the hypotheses of Theorem 3.2 on the product space $\mathcal{V} := \mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathcal{V}_0$, we conclude, because of the equivalence between (3.18) and (3.19), that the bilinear form A_ϕ satisfies the inf-sup condition (3.15) (cf. hypothesis i) of Theorem 3.2), that is

$$\sup_{\vec{\mathbf{s}}\in\mathbb{L}^2_{\mathtt{tr}}(\Omega)\times\mathcal{V}_0\backslash\{\mathbf{0}\}} \frac{A_\phi(\vec{\mathbf{t}},\vec{\mathbf{s}})}{\|\vec{\mathbf{s}}\|_{\mathcal{H}}} \,\geqslant\, \alpha \, \|\vec{\mathbf{t}}\|_{\mathcal{H}} \qquad \forall\, \vec{\mathbf{t}} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathcal{V}_0\,, \tag{3.26}$$

with a positive constant α depending only on α_1, β_1, and μ_2 (since $\|a_\phi\| \leqslant \mu_2$, as seen from (3.20)). Moreover, the symmetry of a_ϕ directly implies that A_ϕ is symmetric as well, which allows us to deduce from (3.26) that A_ϕ also satisfies (3.16). Furthermore, for $\widetilde{A}$ we proceed analogously as for A, so that we first observe that the kernel $\widetilde{\mathcal{V}}$ of $\widetilde{B}$ (cf. (3.4)) can be characterized as

$$\widetilde{\mathcal{V}} \,:=\, \mathbf{L}^2(\Omega) \times \widetilde{\mathcal{V}}_0\,,$$

with

$$\widetilde{\mathcal{V}}_0 \,:=\, \Big\{ \boldsymbol{\eta} \in \mathbf{H}_N(\mathrm{div};\Omega) : \quad \mathrm{div}(\boldsymbol{\eta}) = 0 \quad \text{in} \quad \Omega \Big\}\,.$$

Then, similarly as for (3.24), we use again (2.2) to deduce now that

$$\widetilde{a}_\phi(\boldsymbol{\xi},\boldsymbol{\xi}) \,\geqslant\, \widetilde{\alpha}_1 \, \|\boldsymbol{\xi}\|^2_{0,\Omega} \qquad \forall\, \boldsymbol{\xi} \in \mathbf{L}^2(\Omega)\,, \tag{3.27}$$

with $\widetilde{\alpha}_1 = \kappa_1$, which implies that $\widetilde{a}_\phi$ satisfies assumptions i) and ii) of Theorem 3.2. In addition, given $\boldsymbol{\eta} \in \widetilde{\mathcal{V}}_0\backslash\{\mathbf{0}\}$, we readily find that

$$\sup_{\boldsymbol{\xi}\in\mathbf{L}^2(\Omega)\backslash\{\mathbf{0}\}} \frac{\widetilde{b}(\boldsymbol{\xi},\boldsymbol{\eta})}{\|\boldsymbol{\xi}\|_{0,\Omega}} \,\geqslant\, \frac{\widetilde{b}(-\boldsymbol{\eta},\boldsymbol{\eta})}{\|-\boldsymbol{\eta}\|_{0,\Omega}} = \|\boldsymbol{\eta}\|_{0,\Omega} = \widetilde{\beta}_1 \, \|\boldsymbol{\eta}\|_{\mathrm{div}_{4/3},\Omega}\,, \tag{3.28}$$

with $\widetilde{\beta}_1 = 1$, thus showing that $\widetilde{b}$ satisfies assumption iii) of Theorem 3.2. Hence, as for the deduction of (3.26), we now arrive at

$$\sup_{\vec{\boldsymbol{\xi}}\in\mathbf{L}^2(\Omega)\times\widetilde{\mathcal{V}}_0\backslash\{\mathbf{0}\}} \frac{\widetilde{A}_\phi(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\xi}})}{\|\vec{\boldsymbol{\xi}}\|_{\widetilde{\mathcal{H}}}} \,\geqslant\, \widetilde{\alpha} \, \|\vec{\boldsymbol{\zeta}}\|_{\widetilde{\mathcal{H}}} \qquad \forall\, \vec{\boldsymbol{\zeta}} \in \mathbf{L}^2(\Omega) \times \widetilde{\mathcal{V}}_0\,, \tag{3.29}$$

with a positive constant $\widetilde{\alpha}$ depending only on $\widetilde{\alpha}_1$, $\widetilde{\beta}_1$, and κ_2 (since $\|\widetilde{a}_\phi\| \leqslant \kappa_2$, as seen from (3.21)), which along with the symmetry of $\widetilde{A}_\phi$, proves that $\widetilde{A}_\phi$ satisfies both (3.15) and (3.16).

We now aim to establish the well-definiteness of operator $\mathbf{S}$, equivalently that (3.6) is well-posed. To this end, we begin by noticing that, given $(\mathbf{z}, \phi) \in \mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega)$, this problem can be rewritten as: Find $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) = ((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})) \in \mathcal{H} \times \mathcal{Q}$, such that

$$\mathcal{A}_\phi((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})) + C_{\mathbf{z}}(\mathbf{u}, \mathbf{s}) = G_\phi(\vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) = ((\mathbf{s}, \boldsymbol{\tau}), (\mathbf{v}, \boldsymbol{\omega})) \in \mathcal{H} \times \mathcal{Q}\,, \tag{3.30}$$

where $\mathcal{A}_\phi : (\mathcal{H} \times \mathcal{Q}) \times (\mathcal{H} \times \mathcal{Q}) \to \mathrm{R}$ is the bilinear form defined by

$$\mathcal{A}_\phi((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})) := A_\phi(\vec{\mathbf{t}}, \vec{\mathbf{s}}) + B(\vec{\mathbf{s}}, \vec{\mathbf{u}}) + B(\vec{\mathbf{t}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{t}}, \vec{\mathbf{u}}),\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathcal{H} \times \mathcal{Q}\,. \tag{3.31}$$

According to the above, we now proceed analogously to the proof of [9, Theorem 3.7] and aim to show that (3.30) satisfies the hypotheses of Theorem 3.1. Indeed, the boundedness of the bilinear form on the left hand-side of (3.30) follows straightforwardly from the stability properties provided by (3.20), which yield $\|A_\phi\| \leqslant \max\{1, \mu_2\}$, $\|\mathcal{A}_\phi\| \leqslant \max\{1, \mu_2\}$, and $\|C_{\mathbf{z}}\| \leqslant \|\mathbf{z}\|_{0,4;\Omega}$. In addition, since A_ϕ and B satisfy the inf-sup conditions required by Theorem 3.2, we deduce that there exists a positive constant ϑ, depending on α (cf. (3.26)), β (cf. (3.22)), and μ_2 (because of the above established bound for $\|A_\phi\|$), such that (cf. (3.19))

$$\sup_{(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in (\mathcal{H} \times \mathcal{Q}) \backslash \{\mathbf{0}\}} \frac{\mathcal{A}_\phi((\vec{\mathbf{r}}, \vec{\mathbf{w}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}}))}{\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathcal{H} \times \mathcal{Q}}} \geqslant \vartheta\, \|(\vec{\mathbf{r}}, \vec{\mathbf{w}})\|_{\mathcal{H} \times \mathcal{Q}} \qquad \forall\, (\vec{\mathbf{r}}, \vec{\mathbf{w}}) \in \mathcal{H} \times \mathcal{Q}\,, \tag{3.32}$$

thanks to which, using the boundedness of $C_{\mathbf{z}}$, and performing some algebraic manipulations, we get

$$\sup_{(\vec{\mathbf{s}}, \vec{\mathbf{v}}) = ((\mathbf{s}, \boldsymbol{\tau}), (\mathbf{v}, \boldsymbol{\omega})) \in (\mathcal{H} \times \mathcal{Q}) \backslash \{\mathbf{0}\}} \frac{\mathcal{A}_\phi((\vec{\mathbf{r}}, \vec{\mathbf{w}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})) + C_{\mathbf{z}}(\mathbf{w}, \mathbf{s})}{\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathcal{H} \times \mathcal{Q}}} \geqslant (\vartheta - \|\mathbf{z}\|_{0,4;\Omega})\, \|(\vec{\mathbf{r}}, \vec{\mathbf{w}})\|_{\mathcal{H} \times \mathcal{Q}}\,, \tag{3.33}$$

for all $(\vec{\mathbf{r}}, \vec{\mathbf{w}}) = ((\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{w}, \boldsymbol{\chi})) \in \mathcal{H} \times \mathcal{Q}$. Thus, assuming now that $\mathbf{z}$ is chosen such that $\|\mathbf{z}\|_{0,4;\Omega} \leqslant \frac{\vartheta}{2}$, the foregoing inequality yields

$$\sup_{(\vec{\mathbf{s}}, \vec{\mathbf{v}}) = ((\mathbf{s}, \boldsymbol{\tau}), (\mathbf{v}, \boldsymbol{\omega})) \in (\mathcal{H} \times \mathcal{Q}) \backslash \{\mathbf{0}\}} \frac{\mathcal{A}_\phi((\vec{\mathbf{r}}, \vec{\mathbf{w}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})) + C_{\mathbf{z}}(\mathbf{w}, \mathbf{s})}{\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathcal{H} \times \mathcal{Q}}} \geqslant \frac{\vartheta}{2}\, \|(\vec{\mathbf{r}}, \vec{\mathbf{w}})\|_{\mathcal{H} \times \mathcal{Q}}\,, \tag{3.34}$$

for all $(\vec{\mathbf{r}}, \vec{\mathbf{w}}) = ((\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{w}, \boldsymbol{\chi})) \in \mathcal{H} \times \mathcal{Q}$. In addition, as in the proof of [9, Theorem 3.7], estimate (3.34) and the symmetry of $\mathcal{A}_\phi$ readily imply that there holds

$$\sup_{(\vec{\mathbf{r}}, \vec{\mathbf{w}}) = ((\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{w}, \boldsymbol{\chi})) \in (\mathcal{H} \times \mathcal{Q})} \mathcal{A}_\phi((\vec{\mathbf{r}}, \vec{\mathbf{w}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})) + C_{\mathbf{z}}(\mathbf{w}, \mathbf{s}) > 0, \tag{3.35}$$

for all $(\vec{\mathbf{s}}, \vec{\mathbf{v}}) = ((\mathbf{s}, \boldsymbol{\tau}), (\mathbf{v}, \boldsymbol{\omega})) \in (\mathcal{H} \times \mathcal{Q}) \backslash \{\mathbf{0}\}$.

Summarizing, we have basically demonstrated the following result.

Lemma 3.3. *Given $(\mathbf{z}, \phi) \in \mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega)$ such that $\|\mathbf{z}\|_{0,4;\Omega} \leqslant \frac{\vartheta}{2}$, problem (3.30) (equivalently (3.6)) has a unique solution $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) = ((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})) \in \mathcal{H} \times \mathcal{Q}$, and hence one can define $\mathbf{S}(\mathbf{z}, \phi) = (\mathbf{t}, \mathbf{u})$. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending only on ϑ, such that*

$$\|\mathbf{S}(\mathbf{z}, \phi)\|_{\mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{L}^4(\Omega)} \leqslant \|(\vec{\mathbf{t}}, \vec{\mathbf{u}})\|_{\mathcal{H} \times \mathcal{Q}} \leqslant C_{\mathbf{S}}\, \|\phi\|_{0,4;\Omega}\, \|\mathbf{g}\|_{0,\Omega}\,. \tag{3.36}$$

Proof. Thanks to the previous discussion, the unique solvability of (3.30) (equivalently (3.6)) follows from a straightforward application of Theorem 3.1. In turn, (3.14) and the boundedness of G_ϕ (cf. (3.20)) imply the a priori estimate

$$\|(\mathbf{t}, \boldsymbol{\sigma})\|_{\mathcal{H}} + \|(\mathbf{u}, \boldsymbol{\gamma})\|_{\mathcal{Q}} \leqslant \frac{2}{\vartheta}\, \|\phi\|_{0,4;\Omega}\, \|\mathbf{g}\|_{0,\Omega}\,,$$

which yields (3.36) with $C_{\mathbf{S}} := \frac{2}{\vartheta}$ and concludes the proof. □

Now we turn to prove the well-definiteness of operator $\mathbf{T}$, equivalently that (3.8) is well-posed. Analogously to the previous analysis, given $(\mathbf{z},\phi) \in \mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega)$, we rewrite this problem as: Find $(\vec{\boldsymbol{\zeta}},\varphi) = ((\boldsymbol{\zeta},\boldsymbol{\rho}),\varphi) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}$, such that

$$\widetilde{\mathcal{A}}_\phi((\vec{\boldsymbol{\zeta}},\varphi),(\vec{\boldsymbol{\xi}},\psi)) + \widetilde{C}_{\mathbf{z}}(\varphi,\boldsymbol{\xi}) = \widetilde{F}(\vec{\boldsymbol{\xi}}) + \widetilde{G}(\psi) \qquad \forall\, (\vec{\boldsymbol{\xi}},\psi) = ((\boldsymbol{\xi},\boldsymbol{\eta}),\psi) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}, \tag{3.37}$$

where $\widetilde{\mathcal{A}}_\phi : (\widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}) \times (\widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}) \to \mathrm{R}$ is the bilinear form defined by

$$\widetilde{\mathcal{A}}_\phi((\vec{\boldsymbol{\zeta}},\varphi),(\vec{\boldsymbol{\xi}},\psi)) := \widetilde{A}_\phi(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\xi}}) + \widetilde{B}(\vec{\boldsymbol{\xi}},\varphi) + \widetilde{B}(\vec{\boldsymbol{\zeta}},\psi) \qquad \forall\, (\vec{\boldsymbol{\zeta}},\varphi),(\vec{\boldsymbol{\xi}},\psi) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}, \tag{3.38}$$

and proceed next to prove that (3.37) satisfies the hypotheses of Theorem 3.1. Indeed, the boundedness of the left hand-side of (3.37) follows from (3.21) with $\|\widetilde{\mathcal{A}}_\phi\| \leqslant \max\{1,\kappa_2\}$ and $\|\widetilde{C}_{\mathbf{z}}\| \leqslant \|\mathbf{z}\|_{0,4;\Omega}$. In turn, having $\widetilde{A}_\phi$ and $\widetilde{B}$ satisfied the inf-sup conditions required by Theorem 3.2 (cf. (3.29), (3.23)), and noting from (3.21) again that $\|\widetilde{A}_\phi\| \leqslant \max\{1,\kappa_2\}$, we deduce the existence of a positive constant $\widetilde{\vartheta}$, depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, and κ_2, such that

$$\sup_{(\vec{\boldsymbol{\xi}},\psi)=((\boldsymbol{\xi},\boldsymbol{\eta}),\psi)\in(\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}})\setminus\{\mathbf{0}\}} \frac{\widetilde{\mathcal{A}}_\phi((\vec{\boldsymbol{\varsigma}},\theta),(\vec{\boldsymbol{\xi}},\psi))}{\|(\vec{\boldsymbol{\xi}},\psi)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}}} \geqslant \widetilde{\vartheta}\,\|(\vec{\boldsymbol{\varsigma}},\theta)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}} \qquad \forall\,(\vec{\boldsymbol{\varsigma}},\theta) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}. \tag{3.39}$$

Then, similarly as for the derivation of (3.34) and (3.35), it follows from (3.39) and the boundedness of $\widetilde{C}_{\mathbf{z}}$ that, under the assumption $\|\mathbf{z}\|_{0,4;\Omega} \leqslant \frac{\widetilde{\vartheta}}{2}$, there hold

$$\sup_{(\vec{\boldsymbol{\xi}},\psi)=((\boldsymbol{\xi},\boldsymbol{\eta}),\psi)\in(\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}})\setminus\{\mathbf{0}\}} \frac{\widetilde{\mathcal{A}}_\phi((\vec{\boldsymbol{\varsigma}},\theta),(\vec{\boldsymbol{\xi}},\psi)) + \widetilde{C}_{\mathbf{z}}(\theta,\boldsymbol{\xi})}{\|(\vec{\boldsymbol{\xi}},\psi)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}}} \geqslant \frac{\widetilde{\vartheta}}{2}\,\|(\vec{\boldsymbol{\varsigma}},\theta)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}} \qquad \forall\,(\vec{\boldsymbol{\varsigma}},\theta) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}, \tag{3.40}$$

and

$$\sup_{(\vec{\boldsymbol{\varsigma}},\theta)\in(\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}})} \widetilde{\mathcal{A}}_\phi((\vec{\boldsymbol{\varsigma}},\theta),(\vec{\boldsymbol{\xi}},\psi)) + \widetilde{C}_{\mathbf{z}}(\theta,\boldsymbol{\xi}) > 0 \qquad \forall\,(\vec{\boldsymbol{\xi}},\psi) = ((\boldsymbol{\xi},\boldsymbol{\eta}),\psi) \in (\widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}})\setminus\{\mathbf{0}\},$$

where the latter makes use certainly of the symmetry of $\widetilde{\mathcal{A}}_\phi$.

We are thus in position to establish the following result.

Lemma 3.4. *Given $(\mathbf{z},\phi) \in \mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega)$ such that $\|\mathbf{z}\|_{0,4;\Omega} \leqslant \frac{\widetilde{\vartheta}}{2}$, problem (3.37) (equivalently (3.8)) has a unique solution $(\vec{\boldsymbol{\zeta}},\varphi) = ((\boldsymbol{\zeta},\boldsymbol{\rho}),\varphi) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}$, and hence one can define $\mathbf{T}(\mathbf{z},\phi) = (\boldsymbol{\zeta},\varphi)$. Moreover, there exists a positive constant $C_{\mathbf{T}}$, depending only on $\widetilde{\vartheta}$ and C_D (cf. (3.21)), such that*

$$\|\mathbf{T}(\mathbf{z},\phi)\|_{\mathbf{L}^2(\Omega)\times\mathrm{L}^4(\Omega)} \leqslant \|(\vec{\boldsymbol{\zeta}},\varphi)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}} \leqslant C_{\mathbf{T}}\left\{\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\right\}. \tag{3.41}$$

Proof. Similarly as for the proof of Lemma 3.3, the unique solvability of (3.37) (equivalently (3.8)) follows from a direct application of Theorem 3.1. Then, (3.14) along with the estimates for $\|\widetilde{F}\|$ and $\|\widetilde{G}\|$ (cf. (3.21)) give

$$\|(\boldsymbol{\zeta},\boldsymbol{\rho})\|_{\widetilde{\mathcal{H}}} + \|\varphi\|_{0,4;\Omega} \leqslant \frac{2}{\widetilde{\vartheta}}\left\{C_D\,\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\right\},$$

which yields (3.41) with $C_{\mathbf{T}} := \frac{2}{\widetilde{\vartheta}}\max\{C_D,1\}$ and ends the proof. □

Having verified that $\mathbf{S}$ and $\mathbf{T}$ are well-defined, it follows that $\mathbf{J}$ is as well. More precisely, we have the following result.

Lemma 3.5. *For each* $(\mathbf{z},\phi) \in \mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega)$ *such that* $\|\mathbf{z}\|_{0,4;\Omega} \leqslant \min\left\{\frac{\widetilde{\vartheta}}{2}, \frac{\vartheta}{2}\right\}$*, one can define* $\mathbf{J}(\mathbf{z},\phi) = \big(\mathbf{J}_1(\mathbf{z},\phi), \mathbf{J}_2(\mathbf{z},\phi)\big) \in \mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega)$*. Moreover, there exist positive constants* $C_{\mathbf{J},1}$ *and* $C_{\mathbf{J},2}$*, depending only on* $C_{\mathbf{S}}$ *and* $C_{\mathbf{T}}$*, such that*

$$\|\mathbf{J}_1(\mathbf{z},\phi)\|_{0,4;\Omega} \leqslant C_{\mathbf{J},1}\, \|\mathbf{g}\|_{0,\Omega} \left\{ \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \right\}, \tag{3.42}$$

and

$$\|\mathbf{J}_2(\mathbf{z},\phi)\|_{0,4;\Omega} \leqslant C_{\mathbf{J},2} \left\{ \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \right\}. \tag{3.43}$$

Proof. Given $(\mathbf{z},\phi)$ as indicated, the well-definiteness of $\mathbf{J}(\mathbf{z},\phi)$ follows straightforwardly from (3.9), the assumption on $\|\mathbf{z}\|_{0,4;\Omega}$, and Lemmas 3.3 and 3.4. Moreover, employing (3.36) and (3.41), we find that

$$\begin{aligned} \|\mathbf{J}_1(\mathbf{z},\phi)\|_{0,4;\Omega} &= \|\mathbf{S}_2(\mathbf{z},\mathbf{T}_2(\mathbf{z},\phi))\|_{0,4;\Omega} \leqslant C_{\mathbf{S}}\, \|\mathbf{g}\|_{0,\Omega}\, \|\mathbf{T}_2(\mathbf{z},\phi)\|_{0,4;\Omega} \\ &\leqslant C_{\mathbf{S}}\, \|\mathbf{g}\|_{0,\Omega}\, C_{\mathbf{T}} \left\{ \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \right\}, \end{aligned}$$

and

$$\|\mathbf{J}_2(\mathbf{z},\phi)\|_{0,4;\Omega} = \|\mathbf{T}_2(\mathbf{z},\phi)\|_{0,4;\Omega} \leqslant C_{\mathbf{T}} \left\{ \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \right\},$$

which yields (3.42) and (3.43) with $C_{\mathbf{J},1} := C_{\mathbf{S}}\, C_{\mathbf{T}}$ and $C_{\mathbf{J},2} := C_{\mathbf{T}}$, respectively. □

3.4 Existence and uniqueness of solution

We now address the unique solvability of our coupled problem (3.1)–(3.2) by proving, via the classical Banach theorem, that the equivalent fixed-point equation (3.10) admits a unique solution. To this end, from now on we choose any $\lambda > 0$ such that

$$\lambda \leqslant \min\left\{ \frac{\vartheta}{2}, \frac{\widetilde{\vartheta}}{2} \right\}, \tag{3.44}$$

and introduce the closed cylinder

$$\mathbf{W}(\lambda) := \Big\{ (\mathbf{z},\phi) \in \mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega) : \quad \|\mathbf{z}\|_{0,4;\Omega} \leqslant \lambda \Big\}. \tag{3.45}$$

The following lemma establishes a condition under which $\mathbf{J}$ maps $\mathbf{W}(\lambda)$ into itself.

Lemma 3.6. *Assume that the data are sufficiently small so that*

$$C_{\mathbf{J},1}\, \|\mathbf{g}\|_{0,\Omega} \left\{ \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \right\} \leqslant \lambda. \tag{3.46}$$

Then, there holds $\mathbf{J}\big(\mathbf{W}(\lambda)\big) \subseteq \mathbf{W}(\lambda)$*.*

Proof. The result is a direct consequence of (3.45) and estimate (3.42). □

Next, we aim to show that the operators $\mathbf{S}$ and $\mathbf{T}$, and hence $\mathbf{J}$, are Lipschitz continuous. For this purpose, we require additional regularity hypotheses on the solutions to the problems defining these operators. Specifically, from now on we assume that the data are a bit more regular than usual, namely

$$\varphi_D \in \mathrm{H}^{1/2+\epsilon}(\Gamma) \quad \text{and} \quad f \in \mathrm{W}^{\epsilon,4/3}(\Omega), \quad \text{for some } \epsilon \in [n/4, 1),$$

and that there hold the following:

(**RH.1**) for each $(\mathbf{z},\phi)\in\mathbf{W}(\lambda)$, the solution $((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{u},\boldsymbol{\gamma}))\in\mathcal{H}\times\mathcal{Q}$ of problem (3.6) satisfies $\mathbf{t}\in\mathbb{L}^2_{\mathtt{tr}}(\Omega)\cap\mathbb{H}^\epsilon(\Omega)$, $\boldsymbol{\sigma}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\cap\mathbb{H}^\epsilon(\Omega)$, $\mathbf{u}\in\mathbf{W}^{\epsilon,4}(\Omega)$, $\boldsymbol{\gamma}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega)\cap\mathbb{H}^\epsilon(\Omega)$, and there exists a positive constant C_ϵ, such that

$$\|\mathbf{t}\|_{\epsilon,\Omega}+\|\boldsymbol{\sigma}\|_{\epsilon,\Omega}+\|\mathbf{u}\|_{\epsilon,4;\Omega}+\|\boldsymbol{\gamma}\|_{\epsilon,\Omega}\,\leqslant\,C_\epsilon\,\|\mathbf{g}\|_{0,\Omega}\,\|\phi\|_{0,4;\Omega}\,,\tag{3.47}$$

(**RH.2**) for each $(\mathbf{z},\phi)\in\mathbf{W}(\lambda)$, the solution $((\boldsymbol{\zeta},\boldsymbol{\rho}),\varphi)\in\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}$ of problem (3.8) satisfies $\boldsymbol{\zeta}\in\mathbf{H}^\epsilon(\Omega)$, $\boldsymbol{\rho}\in\mathbf{H}_N(\mathrm{div}_{4/3};\Omega)\cap\mathbf{H}^\epsilon(\Omega)$, $\varphi\in\mathrm{W}^{\epsilon,4}(\Omega)$, and there exists a positive constant $\widetilde{C}_\epsilon$, such that

$$\|\boldsymbol{\zeta}\|_{\epsilon,\Omega}+\|\boldsymbol{\rho}\|_{\epsilon,\Omega}+\|\varphi\|_{\epsilon,4;\Omega}\,\leqslant\,\widetilde{C}_\epsilon\left(\|\varphi_D\|_{1/2+\epsilon,\Gamma_D}+\|f\|_{\epsilon,4/3,\Omega}\right).\tag{3.48}$$

We stress here that for the specified range of ϵ, the following continuous embeddings hold:

$$\mathbb{H}^\epsilon(\Omega)\subseteq\mathbb{L}^{\epsilon^*}(\Omega)\quad\text{and}\quad\mathbf{H}^\epsilon(\Omega)\subseteq\mathbf{L}^{\epsilon^*}(\Omega)\,,\tag{3.49}$$

with continuous injection operators $\mathbf{i}_\epsilon:\mathbb{H}^\epsilon(\Omega)\to\mathbb{L}^{\epsilon^*}(\Omega)$ and $\mathrm{i}_\epsilon:\mathbf{H}^\epsilon(\Omega)\to\mathbf{L}^{\epsilon^*}(\Omega)$, where $\epsilon^*=\dfrac{2n}{n-2\epsilon}$ (see e.g. [25, Theorem 1.3.4]). In turn, noting that $n/\epsilon\leqslant 4$, we let $\mathrm{i}_{4,\epsilon}:\mathrm{L}^4(\Omega)\to\mathrm{L}^{n/\epsilon}(\Omega)$ be the respective continuous injection operator.

The following result establishes the Lipschitz continuity of $\mathbf{S}$.

Lemma 3.7. *There exists a positive constant $\mathcal{L}_{\mathbf{S}}$, depending only on ϑ, $\mathcal{L}_\mu$, $\|\mathrm{i}_{4,\epsilon}\|$, $\|\mathbf{i}_\epsilon\|$, $C_{\mathbf{S}}$, and C_ϵ, such that*

$$\begin{aligned}&\|\mathbf{S}(\mathbf{z},\phi)-\mathbf{S}(\underline{\mathbf{z}},\underline{\phi})\|_{\mathbb{L}^2_{\mathtt{tr}}(\Omega)\times\mathbf{L}^4(\Omega)}\\&\qquad\leqslant\,\mathcal{L}_{\mathbf{S}}\,\|\mathbf{g}\|_{0,\Omega}\left\{\|\phi\|_{0,4;\Omega}\,\|\mathbf{z}-\underline{\mathbf{z}}\|_{0,4;\Omega}\,+\,\left(\|\phi\|_{0,4;\Omega}+1\right)\|\phi-\underline{\phi}\|_{0,4;\Omega}\right\},\end{aligned}\tag{3.50}$$

for all $(\mathbf{z},\phi)$, $(\underline{\mathbf{z}},\underline{\phi})\in\mathbf{W}(\lambda)$.

Proof. Given $(\mathbf{z},\phi)$, $(\underline{\mathbf{z}},\underline{\phi})\in\mathbf{W}(\lambda)$, we let $(\mathbf{t},\mathbf{u})=\mathbf{S}(\mathbf{z},\phi)$ and $(\underline{\mathbf{t}},\underline{\mathbf{u}})=\mathbf{S}(\underline{\mathbf{z}},\underline{\phi})$, where $(\vec{\mathbf{t}},\vec{\mathbf{u}})=((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{u},\boldsymbol{\gamma}))\in\mathcal{H}\times\mathcal{Q}$ and $(\underline{\vec{\mathbf{t}}},\underline{\vec{\mathbf{u}}})=((\underline{\mathbf{t}},\underline{\boldsymbol{\sigma}}),(\underline{\mathbf{u}},\underline{\boldsymbol{\gamma}}))\in\mathcal{H}\times\mathcal{Q}$ are the unique solutions, according to Lemma 3.3, of the respective problem (3.6), or equivalently (3.30), that is

$$\begin{aligned}\mathcal{A}_\phi((\vec{\mathbf{t}},\vec{\mathbf{u}}),(\vec{\mathbf{s}},\vec{\mathbf{v}}))\,+\,C_{\mathbf{z}}(\mathbf{u},\mathbf{s})=G_\phi(\vec{\mathbf{v}})\qquad\forall\,(\vec{\mathbf{s}},\vec{\mathbf{v}})=((\mathbf{s},\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\omega}))\in\mathcal{H}\times\mathcal{Q}\,,\quad\text{and}\\\mathcal{A}_{\underline{\phi}}((\underline{\vec{\mathbf{t}}},\underline{\vec{\mathbf{u}}}),(\vec{\mathbf{s}},\vec{\mathbf{v}}))\,+\,C_{\underline{\mathbf{z}}}(\underline{\mathbf{u}},\mathbf{s})=G_{\underline{\phi}}(\vec{\mathbf{v}})\qquad\forall\,(\vec{\mathbf{s}},\vec{\mathbf{v}})=((\mathbf{s},\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\omega}))\in\mathcal{H}\times\mathcal{Q}\,.\end{aligned}\tag{3.51}$$

Next, in order to bound $\|\mathbf{S}(\mathbf{z},\phi)-\mathbf{S}(\underline{\mathbf{z}},\underline{\phi})\|=\|(\mathbf{t},\mathbf{u})-(\underline{\mathbf{t}},\underline{\mathbf{u}})\|$, we apply the inf-sup condition (3.34) with $(\underline{\mathbf{z}},\underline{\phi})$ to $(\vec{\mathbf{r}},\vec{\mathbf{w}})=(\vec{\mathbf{t}},\vec{\mathbf{u}})-(\underline{\vec{\mathbf{t}}},\underline{\vec{\mathbf{u}}})$, thus yielding

$$\frac{\vartheta}{2}\,\|(\vec{\mathbf{t}}-\underline{\vec{\mathbf{t}}},\vec{\mathbf{u}}-\underline{\vec{\mathbf{u}}})\|_{\mathcal{H}\times\mathcal{Q}}\,\leqslant\sup_{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in(\mathcal{H}\times\mathcal{Q})\backslash\{\mathbf{0}\}}\frac{\mathcal{A}_{\underline{\phi}}((\vec{\mathbf{t}}-\underline{\vec{\mathbf{t}}},\vec{\mathbf{u}}-\underline{\vec{\mathbf{u}}}),(\vec{\mathbf{s}},\vec{\mathbf{v}}))\,+\,C_{\underline{\mathbf{z}}}(\mathbf{u}-\underline{\mathbf{u}},\mathbf{s})}{\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathcal{H}\times\mathcal{Q}}}\,.\tag{3.52}$$

Now, subtracting the equations from (3.51) and adding/subtracting suitable terms, we find that

$$\mathcal{A}_{\underline{\phi}}((\vec{\mathbf{t}}-\underline{\vec{\mathbf{t}}},\vec{\mathbf{u}}-\underline{\vec{\mathbf{u}}}),(\vec{\mathbf{s}},\vec{\mathbf{v}}))\,+\,C_{\underline{\mathbf{z}}}(\mathbf{u}-\underline{\mathbf{u}},\mathbf{s})=G_{\phi-\underline{\phi}}(\vec{\mathbf{v}})-C_{\mathbf{z}-\underline{\mathbf{z}}}(\mathbf{u},\mathbf{s})-\left(a_\phi(\mathbf{t},\mathbf{s})-a_{\underline{\phi}}(\mathbf{t},\mathbf{s})\right),\tag{3.53}$$

for all $(\vec{\mathbf{s}},\vec{\mathbf{v}})=((\mathbf{s},\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\omega}))\in\mathcal{H}\times\mathcal{Q}$. For the last term in the right-hand side of (3.53) we notice that the Lipschitz-continuity of μ (cf. (2.3)) and the Hölder inequality imply that

$$|a_\phi(\mathbf{t},\mathbf{s})-a_{\underline{\phi}}(\mathbf{t},\mathbf{s})|:=\left|\int_\Omega(\mu(\phi)-\mu(\underline{\phi}))\,\mathbf{t}:\mathbf{s}\right|\,\leqslant\,\mathcal{L}_\mu\,\|\phi-\underline{\phi}\|_{0,2l;\Omega}\,\|\mathbf{t}\|_{0,2l';\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,,\tag{3.54}$$

where $l, l' \in (1,+\infty)$ are conjugate to each other. In particular, choosing l' such that $2l' = \epsilon^*$, we get $2l = n/\epsilon$, and thus $\mathrm{L}^4(\Omega)$ is continuously embedded into $\mathrm{L}^{2l}(\Omega) = \mathrm{L}^{n/\epsilon}(\Omega)$. In this way, using additionally the continuity of $\mathbf{i}_\epsilon : \mathbb{H}^\epsilon(\Omega) \to \mathbb{L}^{\epsilon^*}(\Omega)$ along with the regularity estimate (3.47), inequality (3.54) implies

$$
\begin{aligned}
|a_\phi(\mathbf{t},\mathbf{s}) - a_{\underline{\phi}}(\mathbf{t},\mathbf{s})| \,&\leqslant\, \mathcal{L}_\mu \,\|\phi - \underline{\phi}\|_{0,\frac{n}{\epsilon};\Omega}\,\|\mathbf{t}\|_{0,\epsilon^*;\Omega}\,\|\mathbf{s}\|_{0,\Omega} \\
&\leqslant\, \mathcal{L}_\mu \,\|\mathrm{i}_{4,\epsilon}\|\,\|\phi - \underline{\phi}\|_{0,4;\Omega}\,\|\mathbf{t}\|_{0,\epsilon^*;\Omega}\,\|\mathbf{s}\|_{0,\Omega} \\
&\leqslant\, \mathcal{L}_\mu \,\|\mathrm{i}_{4,\epsilon}\|\,\|\phi - \underline{\phi}\|_{0,4;\Omega}\,\|\mathbf{i}_\epsilon\|\,\|\mathbf{t}\|_{\epsilon,\Omega}\,\|\mathbf{s}\|_{0,\Omega} \\
&\leqslant\, \mathcal{L}_\mu \,\|\mathrm{i}_{4,\epsilon}\|\,\|\mathbf{i}_\epsilon\|\,C_\epsilon\,\|\phi\|_{0,4;\Omega}\,\|\mathbf{g}\|_{0,\Omega}\,\|\phi - \underline{\phi}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,.
\end{aligned}
\tag{3.55}
$$

On the other hand, for the first and second terms on the right-hand side of (3.53), we first apply the boundedness of G_ϕ and $C_\mathbf{w}$ (cf. (3.20)), and then the bound for $\|\mathbf{u}\| = \|\mathbf{S}_2(\mathbf{z},\phi)\|$ provided by (3.36), to deduce that

$$
|G_{\phi-\underline{\phi}}(\vec{\mathbf{v}})| \,\leqslant\, \|\mathbf{g}\|_{0,\Omega}\,\|\phi - \underline{\phi}\|_{0,4;\Omega}\,\|\vec{\mathbf{v}}\|_{\mathcal{Q}}\,,
\tag{3.56}
$$

and

$$
\begin{aligned}
|C_{\mathbf{z}-\underline{\mathbf{z}}}(\mathbf{u},\mathbf{s})| \,&\leqslant\, \|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{z} - \underline{\mathbf{z}}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega} \\
&\leqslant\, C_\mathbf{S}\,\|\phi\|_{0,4;\Omega}\,\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{z} - \underline{\mathbf{z}}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,.
\end{aligned}
\tag{3.57}
$$

In this way, employing (3.55), (3.56), and (3.57) to bound (3.53), and then replacing the resulting estimate back into (3.52), we easily arrive at (3.50) with a positive constant $\mathcal{L}_\mathbf{S}$ as indicated. □

Now we turn to analyze the Lipschitz continuity of $\mathbf{T}$. This is addressed in the following lemma.

Lemma 3.8. *There exists a positive constant $\mathcal{L}_\mathbf{T}$, depending only on $\widetilde{\vartheta}$, $\mathcal{L}_\kappa$, $\|\mathrm{i}_{4,\epsilon}\|$, $\|\mathrm{i}_\epsilon\|$, $C_\mathbf{T}$, and $\widetilde{C}_\epsilon$, such that*

$$
\begin{aligned}
\|\mathbf{T}(\mathbf{z},\phi) - \mathbf{T}(\underline{\mathbf{z}},\underline{\phi})\|_{\mathbf{L}^2(\Omega)\times\mathrm{L}^4(\Omega)} \,&\leqslant\, \mathcal{L}_\mathbf{T}\Big\{\big(\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3;\Omega}\big)\,\|\mathbf{z} - \underline{\mathbf{z}}\|_{0,4;\Omega} \\
&\quad + \big(\|\varphi_D\|_{1/2+\epsilon,\Gamma_D} + \|f\|_{\epsilon,4/3;\Omega}\big)\,\|\phi - \underline{\phi}\|_{0,4;\Omega}\Big\},
\end{aligned}
\tag{3.58}
$$

for all $(\mathbf{z},\phi)\,,(\underline{\mathbf{z}},\underline{\phi}) \in \mathbf{W}(\lambda)$.

Proof. We proceed similarly to the proof of Lemma 3.7. Indeed, given $(\mathbf{z},\phi)\,,(\underline{\mathbf{z}},\underline{\phi}) \in \mathbf{W}(\lambda)$, we first let $(\boldsymbol{\zeta},\varphi) = \mathbf{T}(\mathbf{z},\phi)$ and $(\underline{\boldsymbol{\zeta}},\underline{\varphi}) = \mathbf{T}(\underline{\mathbf{z}},\underline{\phi})$, where $(\vec{\boldsymbol{\zeta}},\varphi) = ((\boldsymbol{\zeta},\boldsymbol{\rho}),\varphi) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}$ and $(\underline{\vec{\boldsymbol{\zeta}}},\underline{\varphi}) = ((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\rho}}),\underline{\varphi}) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}$ are the unique solutions, as guaranteed by Lemma 3.4, of the respective problem (3.7), or equivalently (3.37), that is

$$
\begin{aligned}
\widetilde{\mathcal{A}}_\phi((\vec{\boldsymbol{\zeta}},\varphi),(\vec{\boldsymbol{\xi}},\psi)) + \widetilde{C}_\mathbf{z}(\varphi,\boldsymbol{\xi}) = \widetilde{F}(\vec{\boldsymbol{\xi}}) + \widetilde{G}(\psi) \qquad \forall\,(\vec{\boldsymbol{\xi}},\psi) = ((\boldsymbol{\xi},\boldsymbol{\eta}),\psi) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}\,, \quad \text{and} \\
\widetilde{\mathcal{A}}_{\underline{\phi}}((\underline{\vec{\boldsymbol{\zeta}}},\underline{\varphi}),(\vec{\boldsymbol{\xi}},\psi)) + \widetilde{C}_{\underline{\mathbf{z}}}(\underline{\varphi},\boldsymbol{\xi}) = \widetilde{F}(\vec{\boldsymbol{\xi}}) + \widetilde{G}(\psi) \qquad \forall\,(\vec{\boldsymbol{\xi}},\psi) = ((\boldsymbol{\xi},\boldsymbol{\eta}),\psi) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}\,.
\end{aligned}
\tag{3.59}
$$

We then apply the inf-sup condition (3.40) with $(\underline{\mathbf{z}},\underline{\phi})$ to $(\vec{\varsigma},\theta) = (\vec{\boldsymbol{\zeta}},\varphi) - (\underline{\vec{\boldsymbol{\zeta}}},\underline{\varphi})$, thus obtaining

$$
\frac{\widetilde{\vartheta}}{2}\,\|(\vec{\boldsymbol{\zeta}} - \underline{\vec{\boldsymbol{\zeta}}},\varphi - \underline{\varphi})\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}} \,\leqslant \sup_{(\vec{\boldsymbol{\xi}},\psi)\in(\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}})\backslash\{\mathbf{0}\}} \frac{\widetilde{\mathcal{A}}_{\underline{\phi}}((\vec{\boldsymbol{\zeta}} - \underline{\vec{\boldsymbol{\zeta}}},\varphi - \underline{\varphi}),(\vec{\boldsymbol{\xi}},\psi)) + \widetilde{C}_\mathbf{z}(\varphi - \underline{\varphi},\boldsymbol{\xi})}{\|(\vec{\boldsymbol{\xi}},\psi)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}}}\,.
\tag{3.60}
$$

In turn, the numerator within the supremum arises after subtracting the equations from (3.59), namely

$$
\widetilde{\mathcal{A}}_{\underline{\phi}}((\vec{\boldsymbol{\zeta}} - \underline{\vec{\boldsymbol{\zeta}}},\varphi - \underline{\varphi}),(\vec{\boldsymbol{\xi}},\psi)) + \widetilde{C}_{\underline{\mathbf{z}}}(\varphi - \underline{\varphi},\boldsymbol{\xi}) = -\,\widetilde{C}_{\mathbf{z}-\underline{\mathbf{z}}}(\varphi,\boldsymbol{\xi}) - (\widetilde{a}_\phi(\boldsymbol{\zeta},\boldsymbol{\xi}) - \widetilde{a}_{\underline{\phi}}(\boldsymbol{\zeta},\boldsymbol{\xi})),
\tag{3.61}
$$

for all $(\vec{\boldsymbol{\xi}}, \psi) = ((\boldsymbol{\xi}, \boldsymbol{\eta}), \psi) \in \widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}$, so that, using the boundedness of $\widetilde{C}_{\mathbf{w}}(\cdot,\cdot)$ (cf. (3.21)), the Lipschitz-continuity of κ (cf. (2.3)), and Hölder's inequality, it follows from (3.60) and (3.61) that

$$\begin{aligned}\frac{\widetilde{\vartheta}}{2}\,\|(\vec{\boldsymbol{\zeta}} - \underline{\vec{\boldsymbol{\zeta}}}, \varphi - \underline{\varphi})\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}} &\leqslant \|\varphi\|_{0,4;\Omega}\,\|\mathbf{z} - \underline{\mathbf{z}}\|_{0,4;\Omega} \,+\, \mathcal{L}_\kappa\,\|\phi - \underline{\phi}\|_{0,2l;\Omega}\,\|\boldsymbol{\zeta}\|_{0,2l';\Omega}\,,\\ &= \|\mathbf{T}_2(\mathbf{z},\phi)\|_{0,4;\Omega}\,\|\mathbf{z} - \underline{\mathbf{z}}\|_{0,4;\Omega} \,+\, \mathcal{L}_\kappa\,\|\phi - \underline{\phi}\|_{0,2l;\Omega}\,\|\boldsymbol{\zeta}\|_{0,2l';\Omega}\,,\end{aligned} \tag{3.62}$$

where $l,\, l' \in (1,\infty)$ are conjugate to each other. Then, similarly to the proof of Lemma 3.7, we choose l' such that $2l' = \epsilon^*$, which yields $2\ell = n/\epsilon$. Thus, employing now the bound for $\|\mathbf{T}_2(\mathbf{z},\phi)\|_{0,4;\Omega}$ provided by (3.41), the continuous injections $\mathrm{i}_{4,\epsilon}$ and i_ϵ, and the regularity estimate (3.48), we conclude from (3.62) the desired result. We omit further details. □

Having derived the Lipschitz continuity of $\mathbf{S}$ and $\mathbf{T}$ we now establish the same property for the fixed point operator $\mathbf{J}$ in the closed cylinder $\mathbf{W}(\lambda)$ (cf. (3.45)).

Lemma 3.9. *There exists a positive constant $\mathcal{L}_{\mathbf{J}}$, depending only on $C_{\mathbf{T}}$ (cf. (3.41)), $\mathcal{L}_{\mathbf{S}}$ (cf. (3.50)), and $\mathcal{L}_{\mathbf{T}}$ (cf. (3.58)), such that, denoting*

$$\mathtt{D}(\mathbf{g},\varphi_D,f) \,:=\, \|\mathbf{g}\|_{0,\Omega}\left(\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} + 1\right) + 1\,,$$

there holds

$$\begin{aligned}\|\mathbf{J}(\mathbf{z},\phi) - \mathbf{J}(\underline{\mathbf{z}},\underline{\phi})\|_{\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)} \,\leqslant\, \mathcal{L}_{\mathbf{J}}\,\mathtt{D}(\mathbf{g},\varphi_D,f)\Big\{&\left(\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\right)\|\mathbf{z} - \underline{\mathbf{z}}\|_{0,4;\Omega}\\ &+ \left(\|\varphi_D\|_{1/2+\epsilon,\Gamma_D} + \|f\|_{\epsilon,4/3;\Omega}\right)\|\phi - \underline{\phi}\|_{0,4;\Omega}\Big\}\,,\end{aligned} \tag{3.63}$$

for all $(\mathbf{z},\phi)\,,(\underline{\mathbf{z}},\underline{\phi}) \in \mathbf{W}(\lambda)$. Moreover, letting $\bar{C}_\epsilon$ be the positive constant such that

$$\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \,\leqslant\, \bar{C}_\epsilon\left(\|\varphi_D\|_{1/2+\epsilon,\Gamma_D} + \|f\|_{\epsilon,4/3;\Omega}\right), \tag{3.64}$$

the inequality (3.63) simplifies to

$$\begin{aligned}&\|\mathbf{J}(\mathbf{z},\phi) - \mathbf{J}(\underline{\mathbf{z}},\underline{\phi})\|_{\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)}\\ &\quad\leqslant\, \bar{\mathcal{L}}_{\mathbf{J}}\,\mathtt{D}(\mathbf{g},\varphi_D,f)\left(\|\varphi_D\|_{1/2+\epsilon,\Gamma_D} + \|f\|_{\epsilon,4/3;\Omega}\right)\|(\mathbf{z},\phi) - (\underline{\mathbf{z}},\underline{\phi})\|_{\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)}\Big\}\,,\end{aligned} \tag{3.65}$$

for all $(\mathbf{z},\phi)\,,(\underline{\mathbf{z}},\underline{\phi}) \in \mathbf{W}(\lambda)$, where $\bar{\mathcal{L}}_{\mathbf{J}} \,=\, \mathcal{L}_{\mathbf{J}}\,\max\left\{\bar{C}_\epsilon, 1\right\}$.

Proof. Given $(\mathbf{z},\phi)$, $(\underline{\mathbf{z}},\underline{\phi}) \in \mathbf{W}(\lambda)$, we first observe from the definition of $\mathbf{J}$ (cf. (3.9)) that

$$\begin{aligned}\|\mathbf{J}(\mathbf{z},\phi) - \mathbf{J}(\underline{\mathbf{z}},\underline{\phi})\|_{\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)} \,&=\, \|\mathbf{S}_2(\mathbf{z},\mathbf{T}_2(\mathbf{z},\phi)) - \mathbf{S}_2(\underline{\mathbf{z}},\mathbf{T}_2(\underline{\mathbf{z}},\underline{\phi}))\|_{0,4;\Omega}\\ &\quad+\quad \|\mathbf{T}_2(\mathbf{z},\phi) - \mathbf{T}_2(\underline{\mathbf{z}},\underline{\phi})\|_{0,4;\Omega}\,.\end{aligned} \tag{3.66}$$

Next, applying the Lipschitz continuity of $\mathbf{S}$ (cf. (3.50)) to the first term on the right hand-side of (3.66), and adding the resulting expression to the second one, we obtain

$$\begin{aligned}\|\mathbf{J}(\mathbf{z},\phi) - \mathbf{J}(\underline{\mathbf{z}},\underline{\phi})\|_{\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)} \,&\leqslant\, \mathcal{L}_{\mathbf{S}}\,\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{T}_2(\mathbf{z},\phi)\|_{0,4;\Omega}\,\|\mathbf{z} - \underline{\mathbf{z}}\|_{0,4;\Omega}\\ &+ \Big\{\mathcal{L}_{\mathbf{S}}\,\|\mathbf{g}\|_{0,\Omega}\left(\|\mathbf{T}_2(\mathbf{z},\phi)\|_{0,4;\Omega} + 1\right) + 1\Big\}\,\|\mathbf{T}_2(\mathbf{z},\phi) - \mathbf{T}_2(\underline{\mathbf{z}},\underline{\phi})\|_{\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)}\,.\end{aligned}$$

Then, bounding $\|\mathbf{T}_2(\mathbf{z},\phi)\|_{0,4;\Omega}$ and $\|\mathbf{T}_2(\mathbf{z},\phi) - \mathbf{T}_2(\underline{\mathbf{z}},\underline{\phi})\|_{\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)}$ according, respectively, to (3.41) and the Lipschitz continuity of $\mathbf{T}$ (cf. (3.58)), and performing suitable algebraic arrangements, we arrive at the required inequality (3.63) with $\mathcal{L}_{\mathbf{J}}$ and $\mathtt{D}(\mathbf{g},\varphi_D,f)$ as indicated. Finally, it is easy to see that (3.65) follows from (3.63) and (3.64). □

We are now in position to state the main result of this section.

Theorem 3.10. *Assume that the data satisfy* (3.46) *and*

$$\bar{\mathcal{L}}_{\mathbf{J}}\, \mathtt{D}(\mathbf{g},\varphi_D,f)\left(\|\varphi_D\|_{1/2+\epsilon,\Gamma_D} + \|f\|_{\epsilon,4/3;\Omega}\right) \,<\, 1. \tag{3.67}$$

Then, $\mathbf{J}$ *has a unique fixed point* $(\mathbf{u},\varphi)\in\mathbf{W}(\lambda)$. *Equivalently, the coupled problem* (3.1)–(3.2) *has a unique solution* $((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{u},\boldsymbol{\gamma}))\in\mathcal{H}\times\mathcal{Q}$ *and* $((\boldsymbol{\zeta},\boldsymbol{\rho}),\varphi)\in\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}$. *Moreover, there hold the following a priori estimates*

$$\begin{gathered}\|(\mathbf{t},\boldsymbol{\sigma})\|_{\mathcal{H}} + \|(\mathbf{u},\boldsymbol{\gamma})\|_{\mathcal{Q}} \,\leqslant\, C_{\mathbf{S}}\, C_{\mathbf{T}}\,\|\mathbf{g}\|_{0,\Omega}\Big\{\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\Big\}\,, \quad \textit{and}\\ \|(\boldsymbol{\zeta},\boldsymbol{\rho})\|_{\widetilde{\mathcal{H}}} + \|\varphi\|_{0,4;\Omega} \,\leqslant\, C_{\mathbf{T}}\,\Big\{\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\Big\}\,.\end{gathered} \tag{3.68}$$

Proof. We begin by recalling from Lemma 3.6 that assumption (3.46) ensures that $\mathbf{J}$ maps $\mathbf{W}(\lambda)$ into itself. Then, using the Lipschitz continuity of $\mathbf{J}$ established in (3.65), along with assumption (3.67) and a straightforward application of the classical Banach theorem, we deduce that the fixed-point equation (3.10) admits a unique solution $(\mathbf{u},\varphi)\in\mathbf{W}(\lambda)$. Consequently, by the equivalence between the coupled system (3.1)–(3.2) and (3.10), we conclude the well-posedness of the former. Finally, the second estimate in (3.68) follows directly from (3.41), whereas the first one is consequence of (3.36) and the second one. □

4 The Galerkin scheme

In this section, we introduce the Galerkin discretization of the fully mixed formulation (3.1)–(3.2), and analyze its solvability by employing the discrete analogue of the fixed-point strategy introduced in Section 3.2, along with the discrete versions of Theorems 3.2 and 3.1 In what follows, we first describe the discrete scheme on generic finite-dimensional spaces and state the assumptions that guarantee its well-posedness. Later on, in Sections 5 and 6 we provide the associated a priori error analysis and exhibit concrete choices of finite element subspaces that satisfy those assumptions.

4.1 Preliminaries

We let $\{\mathcal{T}_h\}_{h>0}$ be a sequence of partitions of Ω into triangles T (when $n=2$) or tetrahedra T (when $n=3$) with diameter denoted by h_T, and set, as usual, $h:=\max\big\{h_T\,:\,T\in\mathcal{T}_h\big\}$. Then, we consider generic finite-dimensional subspaces

$$\begin{gathered}\mathbb{L}^{\mathbf{t}}_{h,\mathtt{tr}}\,\subseteq\,\mathbb{L}^2_{\mathtt{tr}}(\Omega)\,,\quad \mathbb{H}^{\boldsymbol{\sigma}}_h\,\subseteq\,\mathbb{H}(\mathbf{div}_{4/3};\Omega)\,,\quad \mathbf{L}^{\mathbf{u}}_h\,\subseteq\,\mathbf{L}^4(\Omega)\,,\quad \mathbb{L}^{\gamma}_{h,\mathtt{skew}}\,\subseteq\,\mathbb{L}^2_{\mathtt{skew}}(\Omega)\,,\\ \mathbf{L}^{\boldsymbol{\zeta}}_h\,\subseteq\,\mathbf{L}^2(\Omega)\,,\quad \mathbf{H}^{\boldsymbol{\rho}}_h\,\subseteq\,\mathbf{H}(\mathrm{div}_{4/3};\Omega)\,,\quad \widetilde{\mathcal{Q}}_h\,\subseteq\,\mathrm{L}^4(\Omega)\,,\end{gathered} \tag{4.1}$$

and assume first that

($\mathbf{H.0}$) $\mathbb{H}^{\boldsymbol{\sigma}}_h$ contains the tensors with constant coefficients.

In particular, it follows from ($\mathbf{H.0}$) that $\mathbb{I}\in\mathbb{H}^{\boldsymbol{\sigma}}_h$ for all h, which implies the decomposition

$$\mathbb{H}^{\boldsymbol{\sigma}}_h \,=\, \mathbb{H}^{\boldsymbol{\sigma}}_{h,0}\oplus\mathbb{R}\mathbb{I}\,,$$

where

$$\mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \,:=\, \Big\{\boldsymbol{\tau}\in\mathbb{H}^{\boldsymbol{\sigma}}_h:\quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\Big\}\,\subseteq\,\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,.$$

Then, defining the global spaces

$$\mathcal{H}_h := \mathbb{L}^{\mathbf{t}}_{h,\mathtt{tr}} \times \mathbb{H}^{\boldsymbol{\sigma}}_{h,0}\,, \quad \mathcal{Q}_h := \mathbf{L}^{\mathbf{u}}_h \times \mathbb{L}^{\gamma}_{h,\mathtt{skew}}\,, \quad \mathbf{H}^{\boldsymbol{\rho}}_{h,N} := \mathbf{H}^{\boldsymbol{\rho}}_h \cap \mathbf{H}_N(\mathrm{div}_{4/3};\Omega)\,,$$
$$\text{and} \qquad \widetilde{\mathcal{H}}_h := \mathbf{L}^{\boldsymbol{\zeta}}_h \times \mathbf{H}^{\boldsymbol{\rho}}_{h,N}\,,$$

the Galerkin scheme associated with (3.1)–(3.2) reads: Find $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) = ((\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{u}_h, \gamma_h)) \in \mathcal{H}_h \times \mathcal{Q}_h$ and $(\vec{\boldsymbol{\zeta}}_h, \varphi_h) = ((\boldsymbol{\zeta}_h, \boldsymbol{\rho}_h), \varphi_h) \in \widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h$, such that

$$\begin{aligned} A_{\varphi_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) + B(\vec{\mathbf{s}}_h, \vec{\mathbf{u}}_h) + C_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{s}_h) &= 0 && \forall\, \vec{\mathbf{s}}_h = (\mathbf{s}_h, \boldsymbol{\tau}_h) \in \mathcal{H}_h\,, \\ B(\vec{\mathbf{t}}_h, \vec{\mathbf{v}}_h) &= G_{\varphi_h}(\vec{\mathbf{v}}_h) && \forall\, \vec{\mathbf{v}}_h = (\mathbf{v}_h, \boldsymbol{\omega}_h) \in \mathcal{Q}_h\,, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \widetilde{A}_{\varphi_h}(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\xi}}_h) + \widetilde{B}(\vec{\boldsymbol{\xi}}_h, \varphi_h) + \widetilde{C}_{\mathbf{u}_h}(\varphi_h, \boldsymbol{\xi}_h) &= \widetilde{F}(\vec{\boldsymbol{\xi}}_h) && \forall\, \vec{\boldsymbol{\xi}}_h = (\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) \in \widetilde{\mathcal{H}}_h\,, \\ \widetilde{B}(\vec{\boldsymbol{\zeta}}_h, \psi_h) &= \widetilde{G}(\psi_h) && \forall\, \psi_h \in \widetilde{\mathcal{Q}}_h\,. \end{aligned} \tag{4.3}$$

4.2 The discrete fixed-point strategy

Analogously to the analysis in Section 3.2, we now introduce an equivalent fixed-point equation. Indeed, we first let $\mathbf{S}_h : \mathbf{L}^{\mathbf{u}}_h \times \widetilde{\mathcal{Q}}_h \to \mathbb{L}^{\mathbf{t}}_{h,\mathtt{tr}} \times \mathbf{L}^{\mathbf{u}}_h$ be the discrete analogue of $\mathbf{S}$ (cf. (3.5)), which is defined as

$$\mathbf{S}_h(\mathbf{z}_h, \phi_h) = (\mathbf{S}_{1,h}(\mathbf{z}_h, \phi_h), \mathbf{S}_{2,h}(\mathbf{z}_h, \phi_h)) := (\mathbf{t}_h, \mathbf{u}_h) \quad \forall\, (\mathbf{z}_h, \phi_h) \in \mathbf{L}^{\mathbf{u}}_h \times \widetilde{\mathcal{Q}}_h\,, \tag{4.4}$$

where $\mathbf{t}_h \in \mathbb{L}^{\mathbf{t}}_{h,\mathtt{tr}}$ and $\mathbf{u}_h \in \mathbf{L}^{\mathbf{u}}_h$ are the first and third components, respectively, of the unique solution (to be confirmed below) of the linearized problem arisig from (4.2) after replacing A_{φ_h}, $C_{\mathbf{u}_h}$, and G_{φ_h} by A_{ϕ_h}, $C_{\mathbf{z}_h}$, and G_{ϕ_h}, respectively, that is: Find $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) = ((\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{u}_h, \gamma_h)) \in \mathcal{H}_h \times \mathcal{Q}_h$, such that

$$\begin{aligned} A_{\phi_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) + B(\vec{\mathbf{s}}_h, \vec{\mathbf{u}}_h) + C_{\mathbf{z}_h}(\mathbf{u}_h, \mathbf{s}_h) &= 0 && \forall\, \vec{\mathbf{s}}_h = (\mathbf{s}_h, \boldsymbol{\tau}_h) \in \mathcal{H}_h\,, \\ B(\vec{\mathbf{t}}_h, \vec{\mathbf{v}}_h) &= G_{\phi_h}(\vec{\mathbf{v}}_h) && \forall\, \vec{\mathbf{v}}_h = (\mathbf{v}_h, \boldsymbol{\omega}_h) \in \mathcal{Q}_h\,. \end{aligned} \tag{4.5}$$

Similarly, we let $\mathbf{T}_h : \mathbf{L}^{\mathbf{u}}_h \times \widetilde{\mathcal{Q}}_h \to \mathbf{L}^{\mathbf{u}}_h \times \widetilde{\mathcal{Q}}_h$ be the discrete analogue of $\mathbf{T}$ (cf. (3.7)), which is given by

$$\mathbf{T}_h(\mathbf{z}_h, \phi_h) = (\mathbf{T}_{1,h}(\mathbf{z}_h, \phi_h), \mathbf{T}_{2,h}(\mathbf{z}_h, \phi_h)) := (\boldsymbol{\zeta}_h, \varphi_h) \quad \forall\, (\mathbf{z}_h, \phi_h) \in \mathbf{L}^{\mathbf{u}}_h \times \widetilde{\mathcal{Q}}_h\,, \tag{4.6}$$

where $\boldsymbol{\zeta}_h \in \mathbf{L}^{\boldsymbol{\zeta}}_h$ and $\varphi_h \in \widetilde{\mathcal{Q}}_h$ are the first and third components, respectively, of the unique solution (to be confirmed below) of the linearized problem arising from (4.3) when replacing $\widetilde{A}_{\varphi_h}$ and $\widetilde{C}_{\mathbf{u}_h}$ by $\widetilde{A}_{\phi_h}$ and $\widetilde{C}_{\mathbf{z}_h}$, respectively, that is: Find $(\vec{\boldsymbol{\zeta}}_h, \varphi_h) = ((\boldsymbol{\zeta}_h, \boldsymbol{\rho}_h), \varphi_h) \in \widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h$, such that

$$\begin{aligned} \widetilde{A}_{\phi_h}(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\xi}}_h) + \widetilde{B}(\vec{\boldsymbol{\xi}}_h, \varphi_h) + \widetilde{C}_{\mathbf{z}_h}(\varphi_h, \boldsymbol{\xi}_h) &= \widetilde{F}(\vec{\boldsymbol{\xi}}_h) && \forall\, \vec{\boldsymbol{\xi}}_h = (\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) \in \widetilde{\mathcal{H}}_h\,, \\ \widetilde{B}(\vec{\boldsymbol{\zeta}}_h, \psi_h) &= \widetilde{G}(\psi_h) && \forall\, \psi_h \in \widetilde{\mathcal{Q}}_h\,. \end{aligned} \tag{4.7}$$

Then we let $\mathbf{J}_h : \mathbf{L}^{\mathbf{u}}_h \times \widetilde{\mathcal{Q}}_h \to \mathbf{L}^{\mathbf{u}}_h \times \widetilde{\mathcal{Q}}_h$ be the discrete analogue of $\mathbf{J}$ (cf. (3.9)), which is defined by

$$\mathbf{J}_h(\mathbf{z}_h, \phi_h) := \big(\mathbf{S}_{2,h}(\mathbf{z}_h, \mathbf{T}_{2,h}(\mathbf{z}_h, \phi_h)), \mathbf{T}_{2,h}(\mathbf{z}_h, \phi_h)\big) \qquad \forall\, (\mathbf{z}_h, \phi_h) \in \mathbf{L}^{\mathbf{u}}_h \times \widetilde{\mathcal{Q}}_h\,,$$

and observe that solving (4.2)–(4.3) is equivalent to seeking a fixed-point of $\mathbf{J}_h$, that is: Find $(\mathbf{u}_h, \varphi_h) \in \mathbf{L}^{\mathbf{u}}_h \times \widetilde{\mathcal{Q}}_h$ such that

$$\mathbf{J}_h(\mathbf{u}_h, \varphi_h) = (\mathbf{u}_h, \varphi_h)\,. \tag{4.8}$$

4.3 Well-definiteness of $\mathbf{S}_h$, $\mathbf{T}_h$ and $\mathbf{J}_h$

In this section we employ the discrete counterparts of the Babuška-Brezzi theory (cf. [18, Proposition 2.42]) and the classical Banach–Nečas–Babuška theorem (cf. [18, Theorem 2.22]) to show that $\mathbf{S}_h$ and $\mathbf{T}_h$, and hence $\mathbf{T}_h$, are well-defined. To this end, we introduce suitable assumptions on the discrete subspaces that ensure the well-posedness of the linear problems (4.5) and (4.7), which are equivalent to the well-posedness of $\mathbf{S}_h$ and $\mathbf{T}_h$, respectively. Our approach consists of adapting to the discrete setting the arguments used in the analysis of the continuous problem, particularly those in the proofs of Lemmas 3.3 and 3.4.

We begin by stating some assumptions needed to prove later on that (4.5) is well-posed:

(**H.1**) $\mathbf{div}(\mathbb{H}_h^{\boldsymbol{\sigma}}) \subseteq \mathbf{L}_h^{\mathbf{u}}$,

(**H.2**) there exists a positive constant $\beta_{\rm d}$, independent of h, such that

$$\sup_{\vec{\mathbf{s}}_h \in \mathcal{H}_h \backslash \mathbf{0}} \frac{B(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathcal{H}}} \geqslant \beta_{\rm d} \|\vec{\mathbf{v}}_h\|_{\mathcal{Q}} \qquad \forall\, \vec{\mathbf{v}}_h \in \mathcal{Q}_h\,. \tag{4.9}$$

Then, denoting by $\mathcal{V}_h$ the discrete kernel of B, that is

$$\mathcal{V}_h \,:=\, \Big\{ (\mathbf{s}_h, \boldsymbol{\tau}_h) \in \mathcal{H}_{h,0} : \quad B((\mathbf{s}_h, \boldsymbol{\tau}_h), (\mathbf{v}_h, \boldsymbol{\omega}_h)) = 0 \quad \forall\, (\mathbf{v}_h, \boldsymbol{\omega}_h) \in \mathcal{Q}_h \Big\}\,,$$

and using (**H.1**), we readily find that

$$\mathcal{V}_h \,=\, \mathbb{L}_{h,\mathtt{tr}} \times \mathcal{V}_{h,0}\,,$$

with

$$\mathcal{V}_{h,0} \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0} \quad \text{in} \quad \Omega \quad \text{and} \quad \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\omega}_h = 0 \quad \forall\, \boldsymbol{\omega}_h \in \mathbb{L}_{h,\mathtt{skew}}^{\gamma} \Big\}\,. \tag{4.10}$$

It follows, thanks to (2.2), that for any $\phi_h \in \widetilde{\mathcal{Q}}_h$ there holds

$$a_{\phi_h}(\mathbf{s}_h, \mathbf{s}_h) \,\geqslant\, \alpha_{1,\rm d}\, \|\mathbf{s}_h\|_{0,\Omega}^2 \qquad \forall\, \mathbf{s}_h \in \mathbb{L}_{h,\mathtt{tr}}\,, \tag{4.11}$$

with $\alpha_{1,\rm d} \,=\, \alpha_1 \,=\, \mu_1$, which proves that a_{ϕ_h} satisfies the discrete inf-sup condition required by [18, Proposition 2.42, eq. (2.35)]. Next, in order to show that b satisfies the accompanying inf-sup condition given by [18, Proposition 2.42, eq. (2.36)], which corresponds to the discrete counterpart of (3.25), we assume that

(**H.3**) $(\mathcal{V}_{h,0})^{\mathtt{d}} \,:=\, \Big\{ \boldsymbol{\tau}_h^{\mathtt{d}} : \quad \boldsymbol{\tau}_h \in \mathcal{V}_{h,0} \Big\} \,\subseteq\, \mathbb{L}_{h,\mathtt{tr}}^{\mathbf{t}}$.

In this way, employing (**H.3**) and [14, Lemma 3.1], and proceeding analogously to [20, Lemma 3.3], we deduce that the bilinear form b satisfies the aforementioned condition, namely

$$\sup_{\mathbf{s}_h \in \mathbb{L}_{h,\mathtt{tr}} \backslash \{\mathbf{0}\}} \frac{b(\mathbf{s}_h, \boldsymbol{\tau}_h)}{\|\mathbf{s}_h\|_{0,\Omega}} \,\geqslant\, \beta_{1,\rm d}\, \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3},\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathcal{V}_{h,0}\,, \tag{4.12}$$

with a positive constant $\beta_{1,\rm d}$, independent of h. Thus, having a_{ϕ_h} and b satisfied the hypotheses of [18, Proposition 2.42], we deduce, thanks to the discrete version of the equivalence between (3.18) and (3.19), that there exists a positive constant $\alpha_{\rm d}$, depending only on $\alpha_{1,\rm d}$, $\beta_{1,\rm d}$, and μ_2 (since $\|a_{\phi_h}\| \,\leqslant\, \mu_2$), and hence independent of h, such that the discrete analogue of (3.26) holds, that is

$$\sup_{(\mathbf{s}_h, \boldsymbol{\tau}_h) \in \mathbb{L}_{h,\mathtt{tr}} \times \mathcal{V}_{h,0} \backslash \{\mathbf{0}\}} \frac{A_{\phi_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathcal{H}}} \,\geqslant\, \alpha_{\rm d}\, \|\vec{\mathbf{t}}_h\|_{\mathcal{H}} \qquad \forall\, \vec{\mathbf{t}}_h \in \mathbb{L}_{h,\mathtt{tr}}^{\mathbf{t}} \times \mathcal{V}_{h,0}\,. \tag{4.13}$$

As previously announced, we now address the well-posedness of (4.5) by adopting the discrete version of the analysis yielding Lemma 3.3. To this end, we first observe that, given $(\mathbf{z}_h,\phi_h)\in\mathbf{L}_h^{\mathbf{u}}\times\widetilde{\mathcal{Q}}_h$, the aforementioned problem can be reformulated as: Find $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)=((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\gamma_h))\in\mathcal{H}_h\times\mathcal{Q}_h$, such that

$$\mathcal{A}_{\phi_h}((\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h))\,+\,C_{\mathbf{z}_h}(\mathbf{u}_h,\mathbf{s}_h)\,=\,G_{\phi_h}(\vec{\mathbf{v}}_h)\,,\tag{4.14}$$

for all $(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)=((\mathbf{s}_h,\boldsymbol{\tau}_h),(\mathbf{v}_h,\boldsymbol{\omega}_h))\in\mathcal{H}_h\times\mathcal{Q}_h$, where $\mathcal{A}_{\phi_h}$ is the bilinear form defined in (3.31), so that we now aim to show that (4.14) satisfies the hypotheses of [18, Theorem 2.22]. Indeed, the boundedness of the bilinear form and functional involved was already established in the continuous analysis (cf. Section 3.3). Next, since A_{ϕ_h} and B satisfy the hypotheses of [18, Proposition 2.42], we deduce the existence of a positive constant $\vartheta_{\rm d}$, depending only on $\alpha_{\rm d}$ (cf. (4.13)), $\beta_{\rm d}$ (cf. (4.9)), and μ_2, such that

$$\sup_{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\in(\mathcal{H}_h\times\mathcal{Q}_h)\backslash\{\mathbf{0}\}}\frac{\mathcal{A}_{\phi_h}((\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h))}{\|(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\|_{\mathcal{H}\times\mathcal{Q}}}\,\geqslant\,\vartheta_{\rm d}\,\|(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathcal{H}\times\mathcal{Q}}\qquad\forall\,(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\in\mathcal{H}_h\times\mathcal{Q}_h\,.\tag{4.15}$$

Then, proceeding analogously as for the derivation of (3.32) and (3.33), which means employing (4.15) and the boundedness of $C_{\mathbf{z}_h}$ (cf. (3.20), we find that for each $\mathbf{z}_h\in\mathbf{L}_h^{\mathbf{u}}$ such that $\|\mathbf{z}_h\|_{0,4;\Omega}\leqslant\frac{\vartheta_{\rm d}}{2}$, there holds

$$\sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)=((\mathbf{s}_h,\boldsymbol{\tau}_h),(\mathbf{v}_h,\boldsymbol{\omega}_h))\\ \in(\mathcal{H}_h\times\mathcal{Q}_h)\backslash\{\mathbf{0}\}}}\frac{\mathcal{A}_{\phi_h}((\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h))\,+\,C_{\mathbf{z}_h}(\mathbf{w}_h,\mathbf{s}_h)}{\|(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\|_{\mathcal{H}\times\mathcal{Q}}}\,\geqslant\,\frac{\vartheta_{\rm d}}{2}\,\|(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathcal{H}\times\mathcal{Q}}\,,\tag{4.16}$$

for all $(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)=((\mathbf{r}_h,\boldsymbol{\varrho}_h),(\mathbf{w}_h,\boldsymbol{\chi}_h))\in\mathcal{H}_h\times\mathcal{Q}_h$, which constitutes the discrete analogue of (3.34).

We are now in position to establish the well-definedness of $\mathbf{S}_h$.

Lemma 4.1. *Given $(\mathbf{z}_h,\phi_h)\in\mathbf{L}_h^{\mathbf{u}}\times\widetilde{\mathcal{Q}}_h$ such that $\|\mathbf{z}_h\|_{0,4;\Omega}\leqslant\frac{\vartheta_{\rm d}}{2}$, problem (4.14), (equivalently (4.5)), has a unique solution $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)=((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\gamma_h))\in\mathcal{H}_h\times\mathcal{Q}_h$, and hence one can define $\mathbf{S}_h(\mathbf{z}_h,\phi_h)=(\mathbf{t}_h,\mathbf{u}_h)$. Moreover, there exists a positive constant $C_{\mathbf{S},\rm d}$, depending only on $\vartheta_{\rm d}$, and hence independent of h, such that*

$$\|\mathbf{S}_h(\mathbf{z}_h,\phi_h)\|_{\mathbb{L}^2_{\mathbf{tr}}(\Omega)\times\mathbf{L}^4(\Omega)}\,\leqslant\,\|(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathcal{H}\times\mathcal{Q}}\,\leqslant\,C_{\mathbf{S},\rm d}\,\|\phi_h\|_{0,4;\Omega}\,\|\mathbf{g}\|_{0,\Omega}\,.\tag{4.17}$$

Proof. The unique solvability of (4.14) readily follows from (4.16) and a straightforward application of [18, Theorem 2.22]. In turn, the corresponding a priori estimate provided in this later result, along with the boundedness of G_{ϕ_h} (cf. (3.20)), imply

$$\|(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathcal{H}\times\mathcal{Q}}\,\leqslant\,\frac{2}{\vartheta_d}\,\|\phi_h\|_{0,4;\Omega}\,\|\mathbf{g}\|_{0,\Omega}\,,$$

which yields (4.17) with $C_{\mathbf{S},\rm d}:=\frac{2}{\vartheta_d}$ and ends the proof. □

We continue with some hypotheses that are required to show that (4.7) is well-posed:

$(\widetilde{\mathbf{H.1}})$ $\mathbf{div}(\mathbf{H}_h^{\boldsymbol{\rho}})\,\subseteq\,\widetilde{\mathcal{Q}}_h\,,$

$(\widetilde{\mathbf{H.2}})$ there exists a positive constant $\widetilde{\beta}_{\rm d}$, independent of h, such that

$$\sup_{\vec{\boldsymbol{\xi}}_h\in\widetilde{\mathcal{H}}_h\backslash\mathbf{0}}\frac{\widetilde{B}(\vec{\boldsymbol{\xi}}_h,\psi_h)}{\|\vec{\boldsymbol{\xi}}_h\|_{\widetilde{\mathcal{H}}}}\,\geqslant\,\widetilde{\beta}_{\rm d}\,\|\psi_h\|_{0,4;\Omega}\qquad\forall\,\psi_h\in\widetilde{\mathcal{Q}}_h\,.\tag{4.18}$$

It is easily seen from $(\widetilde{\mathbf{H.1}})$ that the discrete kernel of $\widetilde{B}$, denoted by $\widetilde{\mathcal{V}}_h$, reduces to

$$\widetilde{\mathcal{V}}_h \,:=\, \mathbf{L}_h^{\boldsymbol{\zeta}} \times \widetilde{\mathcal{V}}_{h,0}\,,$$

with

$$\widetilde{\mathcal{V}}_{h,0} \,:=\, \Big\{ \boldsymbol{\eta}_h \in \mathbf{H}_{h,N}^{\boldsymbol{\rho}} : \quad \mathrm{div}(\boldsymbol{\eta}_h) = 0 \quad \text{in} \quad \Omega \Big\}\,. \tag{4.19}$$

Thus, according to (3.27), we know that for any $\phi_h \in \widetilde{\mathcal{Q}}_h$ there holds

$$\widetilde{a}_{\phi_h}(\boldsymbol{\xi}_h, \boldsymbol{\xi}_h) \,\geqslant\, \widetilde{\alpha}_{1,\mathrm{d}}\, \|\boldsymbol{\xi}_h\|_{0,\Omega}^2 \qquad \forall\, \boldsymbol{\xi}_h \in \mathbf{L}_h^{\boldsymbol{\zeta}}\,,$$

with $\widetilde{\alpha}_{1,\mathrm{d}} = \widetilde{\alpha}_1 = \kappa_1$, which shows that $\widetilde{a}_{\phi_h}$ satisfies the discrete inf-sup condition required by [18, Proposition 2.42, eq. (2.35)]. Next, we introduce the hypothesis

$(\widetilde{\mathbf{H.3}})$ $\widetilde{\mathcal{V}}_{h,0} \subseteq \mathbf{L}_h^{\boldsymbol{\zeta}}$,

thanks to which, given $\boldsymbol{\eta}_h \in \widetilde{\mathcal{V}}_{h,0} \backslash \{\mathbf{0}\}$, we can bound by below with $\boldsymbol{\xi}_h = -\boldsymbol{\eta}_h$ to obtain

$$\sup_{\boldsymbol{\xi}_h \in \mathbf{L}_h^{\boldsymbol{\zeta}} \backslash \{\mathbf{0}\}} \frac{\widetilde{b}(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h)}{\|\boldsymbol{\xi}\|_{0,\Omega}} \,\geqslant\, \frac{\widetilde{b}(-\boldsymbol{\eta}_h, \boldsymbol{\eta}_h)}{\|-\boldsymbol{\eta}_h\|_{0,\Omega}} = \|\boldsymbol{\eta}_h\|_{0,\Omega} = \widetilde{\beta}_{1,\mathrm{d}}\, \|\boldsymbol{\eta}_h\|_{\mathrm{div}_{4/3},\Omega}\,,$$

with $\widetilde{\beta}_{1,\mathrm{d}} = 1$, thus proving that $\widetilde{b}$ satisfies [18, Proposition 2.42, eq. (2.36)]. Consequently, having $\widetilde{a}_{\phi_h}$ and $\widetilde{b}$ satisfied the hypotheses of [18, Proposition 2.42] on $\widetilde{\mathcal{V}}_{h,0}$, we conclude, similarly as for the derivation of (4.13), the existence of a positive constant $\widetilde{\alpha}_{\mathrm{d}}$, depending only on $\widetilde{\alpha}_{1,\mathrm{d}}$, $\widetilde{\beta}_{1,\mathrm{d}}$, and κ_2, and hence independent of h, such that there holds the discrete counterpart of (3.29), that is

$$\sup_{\vec{\boldsymbol{\xi}}_h \in \mathbf{L}_h^{\boldsymbol{\zeta}} \times \widetilde{\mathcal{V}}_{h,0} \backslash \{\mathbf{0}\}} \frac{\widetilde{A}_{\phi_h}(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\xi}}_h)}{\|\vec{\boldsymbol{\xi}}_h\|_{\widetilde{\mathcal{H}}}} \,\geqslant\, \widetilde{\alpha}_{\mathrm{d}}\, \|\vec{\boldsymbol{\zeta}}_h\|_{\widetilde{\mathcal{H}}} \qquad \forall\, \vec{\boldsymbol{\zeta}}_h \in \mathbf{L}_h^{\boldsymbol{\zeta}} \times \widetilde{\mathcal{V}}_{h,0}\,. \tag{4.20}$$

We are ready now to establish the well-posedness of (4.7), for which, analogously as for that of (4.5), we adopt the discrete version of the analysis yielding Lemma 3.4. In fact, we begin by noticing that, given $(\mathbf{z}_h, \phi_h) \in \mathbf{L}_h^{\mathbf{u}} \times \widetilde{\mathcal{Q}}_h$, problem (4.7) can be reformulated, equivalently, as: Find $(\vec{\boldsymbol{\zeta}}_h, \varphi_h) = ((\boldsymbol{\zeta}_h, \boldsymbol{\rho}_h), \varphi_h) \in \widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h$ such that

$$\widetilde{\mathcal{A}}_{\phi_h}((\vec{\boldsymbol{\zeta}}_h, \varphi_h), (\vec{\boldsymbol{\xi}}_h, \psi_h)) + \widetilde{C}_{\mathbf{z}_h}(\varphi_h, \boldsymbol{\xi}_h) = \widetilde{F}(\vec{\boldsymbol{\xi}}_h) + \widetilde{G}(\psi_h)\,, \tag{4.21}$$

for all $(\vec{\boldsymbol{\xi}}_h, \psi_h) = ((\boldsymbol{\xi}_h, \boldsymbol{\eta}_h), \psi_h) \in \widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h$, where $\widetilde{\mathcal{A}}_{\phi_h}$ is the bilinear form defined in (3.38). In this way, having $\widetilde{A}_{\phi_h}$ and $\widetilde{B}$ satisfied the discrete inf-sup conditions required by [18, Proposition 2.42], namely (4.20) and (4.18), we deduce the existence of a positive constant $\widetilde{\vartheta}_{\mathrm{d}}$, depending only on $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{\mathrm{d}}$, and κ_2, and hence independent of h, such that

$$\sup_{(\vec{\boldsymbol{\xi}}_h, \psi_h) \in (\widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}) \backslash \{\mathbf{0}\}} \frac{\widetilde{\mathcal{A}}_{\phi_h}((\vec{\varsigma}_h, \theta_h), (\vec{\boldsymbol{\xi}}_h, \psi_h))}{\|(\vec{\boldsymbol{\xi}}_h, \psi_h)\|_{\widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}}} \,\geqslant\, \widetilde{\vartheta}_{\mathrm{d}}\, \|(\vec{\varsigma}_h, \theta_h)\|_{\widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}} \qquad \forall\, (\vec{\varsigma}_h, \theta_h) \in \widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h\,. \tag{4.22}$$

Then, similarly as for the derivation of (4.16), we now employ (4.22) and the boundedness of $\widetilde{C}_{\mathbf{z}}$ (cf. (3.21)), to conclude that, for each $\mathbf{z}_h \in \mathbf{L}_h^{\mathbf{u}}$ such that $\|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \frac{\widetilde{\vartheta}_{\mathrm{d}}}{2}$, there holds

$$\sup_{\substack{(\vec{\boldsymbol{\xi}}_h, \psi_h) = ((\boldsymbol{\xi}_h, \boldsymbol{\eta}_h), \psi_h) \\ \in (\widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h) \backslash \{\mathbf{0}\}}} \frac{\widetilde{\mathcal{A}}_{\phi_h}((\vec{\varsigma}_h, \theta_h), (\vec{\boldsymbol{\xi}}_h, \psi_h)) + \widetilde{C}_{\mathbf{z}_h}(\theta_h, \boldsymbol{\xi}_h)}{\|(\vec{\boldsymbol{\xi}}_h, \psi_h)\|_{\widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}}} \,\geqslant\, \frac{\widetilde{\vartheta}_{\mathrm{d}}}{2}\, \|(\vec{\varsigma}_h, \theta_h)\|_{\widetilde{\mathcal{H}} \times \widetilde{\mathcal{Q}}}\,, \tag{4.23}$$

for all $(\vec{\varsigma}_h, \theta_h) \in \widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h$, which constitutes the discrete counterpart of (3.40).

Consequently, the well-definiteness of $\mathbf{T}_h$ is established as follows.

Lemma 4.2. *Given* $(\mathbf{z}_h, \phi_h) \in \mathbf{L}_h^{\mathbf{u}} \times \widetilde{\mathcal{Q}}_h$ *such that* $\|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \frac{\widetilde{\vartheta}_d}{2}$, *problem* (4.21) (*equivalently* (4.7)) *has a unique solution* $(\vec{\boldsymbol{\zeta}}_h, \varphi_h) = ((\boldsymbol{\zeta}_h, \boldsymbol{\rho}_h), \varphi_h) \in \widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h$, *and hence one can define* $\mathbf{T}_h(\mathbf{z}_h, \phi_h) = (\boldsymbol{\zeta}_h, \varphi_h)$. *Moreover, there exists a positive constant* $C_{\mathbf{T},\mathrm{d}}$, *depending only on* $\widetilde{\vartheta}_\mathrm{d}$ *and* C_D (*cf.* (3.21)), *and hence independent of* h, *such that*

$$\|\mathbf{T}_h(\mathbf{z}_h, \phi_h)\|_{\mathbf{L}^2(\Omega)\times \mathrm{L}^4(\Omega)} \leqslant \|(\vec{\boldsymbol{\zeta}}_h, \varphi_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}} \leqslant C_{\mathbf{T},\mathrm{d}} \Big\{ \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \Big\} . \tag{4.24}$$

Proof. Thanks to (4.23), the unique solvability of (4.21) follows from a direct application of [18, Theorem 2.22]. Moreover, the a priori estimate provided by this result, along with the boundedness of $\widetilde{F}$ and $\widetilde{G}$ (cf. (3.21)), imply

$$\|(\boldsymbol{\zeta}_h, \boldsymbol{\rho}_h)\|_{\widetilde{\mathcal{H}}} + \|\varphi_h\|_{0,4;\Omega} \leqslant \frac{2}{\widetilde{\vartheta}_\mathrm{d}} \Big\{ C_D \, \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \Big\} ,$$

which yields (4.24) with $C_{\mathbf{T},\mathrm{d}} := \frac{2}{\widetilde{\vartheta}_\mathrm{d}} \max\big\{C_D, 1\big\}$ and completes the proof. $\square$

The following lemma establishes the well-definiteness of the operator $\mathbf{J}_h$. Being its proof analogous to that of Lemma 3.5, we omit further details.

Lemma 4.3. *For each* $(\mathbf{z}_h, \phi_h) \in \mathbf{L}_h^{\mathbf{u}} \times \widetilde{\mathcal{Q}}_h$ *such that* $\|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \min\Big\{\frac{\widetilde{\vartheta}_\mathrm{d}}{2}, \frac{\vartheta_\mathrm{d}}{2}\Big\}$, *one can define* $\mathbf{J}_h(\mathbf{z}_h, \phi_h) = \big(\mathbf{J}_{1,h}(\mathbf{z}_h, \phi_h), \mathbf{J}_{2,h}(\mathbf{z}_h, \phi_h)\big) \in \mathbf{L}_h^{\mathbf{u}} \times \widetilde{\mathcal{Q}}_h$. *Moreover, there exist positive constants* $\bar{C}_{\mathbf{J},1}$ *and* $\bar{C}_{\mathbf{J},2}$, *depending only on* $C_{\mathbf{S},\mathrm{d}}$ *and* $C_{\mathbf{T},\mathrm{d}}$, *and hence independent of* h, *such that*

$$\|\mathbf{J}_{1,h}(\mathbf{z}_h, \phi_h)\|_{0,4;\Omega} \leqslant \bar{C}_{\mathbf{J},1} \, \|\mathbf{g}\|_{0,\Omega} \Big\{ \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \Big\} , \tag{4.25}$$

and

$$\|\mathbf{J}_{2,h}(\mathbf{z}_h, \phi_h)\|_{0,4;\Omega} \leqslant \bar{C}_{\mathbf{J},2} \Big\{ \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \Big\} . \tag{4.26}$$

4.4 Existence of solution of the Galerkin scheme

In this section, we address the solvability of our discrete coupled system (4.2)–(4.3) by studying the equivalent fixed-point equation (4.8). In this regard, we stress in advance that, not being the regularity hypotheses **(RH.1)** and **(RH.2)** valid at the discrete level, we are only able to establish existence of solution by applying the well-known Brouwer fixed-point theorem. Similarly to the analysis of the fixed-point equation for the continuous problem (cf. Section 3.4), we begin by choosing $\lambda_\mathrm{d} > 0$ such that

$$\lambda_\mathrm{d} \leqslant \min\left\{ \frac{\vartheta_d}{2}, \frac{\widetilde{\vartheta}_d}{2} \right\} , \tag{4.27}$$

and define the closed cylinder

$$\mathbf{W}_h(\lambda_\mathrm{d}) := \Big\{ (\mathbf{z}_h, \phi_h) \in \mathbf{L}_h^{\mathbf{u}} \times \widetilde{\mathcal{Q}}_h : \quad \|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \lambda_\mathrm{d} \Big\} . \tag{4.28}$$

The following result establishes the discrete analogue of Lemma 3.6.

Lemma 4.4. *Assume that the data are sufficiently small so that*

$$\bar{C}_{\mathbf{J},1} \, \|\mathbf{g}\|_{0,\Omega} \Big\{ \|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega} \Big\} \leqslant \lambda_\mathrm{d} . \tag{4.29}$$

Then, there holds $\mathbf{J}_h\big(\mathbf{W}_h(\lambda_\mathrm{d})\big) \subseteq \mathbf{W}_h(\lambda_\mathrm{d})$.

Proof. The result is a direct consequence of estimate (4.25). □

We now aim to prove that $\mathbf{J}_h$ is continuous, for which it suffices to establish the same property for $\mathbf{S}_h$ and $\mathbf{T}_h$. In fact, we begin with the former by providing next the discrete analogue of Lemma 3.7.

Lemma 4.5. *There exists a positive constant $\mathcal{L}_{\mathbf{S},\mathrm{d}}$, depending only on ϑ_d, $\mathcal{L}_\mu$, and $C_{\mathbf{S},\mathrm{d}}$, and hence independent of h, such that*

$$\begin{aligned}\|\mathbf{S}_h(\mathbf{z}_h,\phi_h) - \mathbf{S}_h(\underline{\mathbf{z}}_h,\underline{\phi}_h)\|_{\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{L}^4(\Omega)} \,\leqslant\, \mathcal{L}_{\mathbf{S},d}\Big\{&\|\mathbf{g}\|_{0,\Omega}\,\|\phi_h\|_{0,4;\Omega}\,\|\mathbf{z}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega}\\ + \quad &\big(\|\mathbf{g}\|_{0,\Omega} + \|\mathbf{S}_{1,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}\big)\,\|\phi_h - \underline{\phi}_h\|_{0,4;\Omega}\Big\}\,,\end{aligned} \tag{4.30}$$

for all $(\mathbf{z}_h,\phi_h), (\underline{\mathbf{z}}_h,\,\underline{\phi}_h) \in \mathbf{W}_h(\lambda_d)$.

Proof. Given $(\mathbf{z}_h,\phi_h)$, $(\underline{\mathbf{z}}_h,\underline{\phi}_h) \in \mathbf{W}_h(\lambda_d)$, we let $(\mathbf{t}_h,\mathbf{u}_h) = \mathbf{S}_h(\mathbf{z}_h,\phi_h)$ and $(\underline{\mathbf{t}}_h,\underline{\mathbf{u}}_h) = \mathbf{S}_h(\underline{\mathbf{z}}_h,\underline{\phi}_h)$, where $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h) = ((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\boldsymbol{\gamma}_h)) \in \mathcal{H}_h\times\mathcal{Q}_h$ and $(\underline{\vec{\mathbf{t}}}_h,\underline{\vec{\mathbf{u}}}_h) = ((\underline{\mathbf{t}}_h,\underline{\boldsymbol{\sigma}}_h),(\underline{\mathbf{u}}_h,\underline{\boldsymbol{\gamma}}_h)) \in \mathcal{H}_h\times\mathcal{Q}_h$ are the unique solutions, according to Lemma 4.1, of the respective problem (4.14), equivalently (4.5). Next, applying the inf-sup condition (4.16) with $(\underline{\mathbf{z}}_h,\underline{\phi}_h)$ to $(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h) = (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h) - (\underline{\vec{\mathbf{t}}}_h,\underline{\vec{\mathbf{u}}}_h)$, we get

$$\begin{aligned}&\frac{\vartheta_{\mathrm{d}}}{2}\,\|(\vec{\mathbf{t}}_h - \underline{\vec{\mathbf{t}}}_h,\vec{\mathbf{u}}_h - \underline{\vec{\mathbf{u}}}_h)\|_{\mathcal{H}\times\mathcal{Q}}\\ &\leqslant \sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)=((\mathbf{s}_h,\boldsymbol{\tau}_h),(\mathbf{v}_h,\boldsymbol{\omega}_h))\\ \in(\mathcal{H}_h\times\mathcal{Q}_h)\backslash\{\mathbf{0}\}}} \frac{\mathcal{A}_{\underline{\phi}_h}((\vec{\mathbf{t}}_h - \underline{\vec{\mathbf{t}}}_h,\vec{\mathbf{u}}_h - \underline{\vec{\mathbf{u}}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)) \,+\, C_{\underline{\mathbf{z}}_h}(\mathbf{u}_h - \underline{\mathbf{u}}_h,\mathbf{s}_h)}{\|(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\|_{\mathcal{H}\times\mathcal{Q}}}\,,\end{aligned} \tag{4.31}$$

and proceeding analogously to the proof of Lemma 3.7 we find that

$$\begin{aligned}\mathcal{A}_{\underline{\phi}_h}((\vec{\mathbf{t}}_h - \underline{\vec{\mathbf{t}}}_h,\vec{\mathbf{u}}_h - \underline{\vec{\mathbf{u}}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)) \,+\, C_{\underline{\mathbf{z}}_h}(\mathbf{u}_h - \underline{\mathbf{u}}_h,\mathbf{s}_h) \,=\, G_{\phi_h-\underline{\phi}_h}(\vec{\mathbf{v}}_h)\\ -\,C_{\mathbf{z}_h-\underline{\mathbf{z}}_h}(\mathbf{u}_h,\mathbf{s}_h) \,-\, \big(a_{\phi_h}(\mathbf{t}_h,\mathbf{s}_h) - a_{\underline{\phi}_h}(\mathbf{t}_h,\mathbf{s}_h)\big)\,,\end{aligned} \tag{4.32}$$

for all $(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h) = ((\mathbf{s}_h,\boldsymbol{\tau}_h),(\mathbf{v}_h,\boldsymbol{\omega}_h)) \in \mathcal{H}_h\times\mathcal{Q}_h$. For the last term on the right-hand side of (4.32) we simply use the Lipschitz-continuity of μ (cf. (2.3)), Cauchy-Schwarz's inequality twice, and the fact that $\mathbf{t}_h = \mathbf{S}_{1,h}(\mathbf{z}_h,\phi_h)$, to obtain

$$\begin{aligned}\big|a_{\phi_h}(\mathbf{t}_h,\mathbf{s}_h) - a_{\underline{\phi}_h}(\mathbf{t}_h,\mathbf{s}_h)\big| \,&\leqslant\, \mathcal{L}_\mu\,\|\phi_h - \underline{\phi}_h\|_{0,4;\Omega}\,\|\mathbf{t}_h\|_{0,4;\Omega}\,\|\mathbf{s}_h\|_{0,\Omega}\,,\\ &=\, \mathcal{L}_\mu\,\|\mathbf{S}_{1,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}\,\|\phi_h - \underline{\phi}_h\|_{0,4;\Omega}\,\|\mathbf{s}_h\|_{0,\Omega}\,.\end{aligned} \tag{4.33}$$

In turn, for the first and second terms on the right-hand side of (4.32) we apply again the boundedness of G_ϕ and $C_{\mathbf{w}}$ (cf. (3.20)), along with the upper bound for $\|\mathbf{u}_h\|_{0,4;\Omega} = \|\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}$ provided by (4.17), to deduce that

$$|G_{\phi_h-\underline{\phi}_h}(\vec{\mathbf{v}}_h)| \,\leqslant\, \|\mathbf{g}\|_{0,\Omega}\,\|\phi_h - \underline{\phi}_h\|_{0,4;\Omega}\,\|\vec{\mathbf{v}}_h\|_{\mathcal{Q}}\,, \tag{4.34}$$

and

$$\begin{aligned}|C_{\mathbf{z}_h-\underline{\mathbf{z}}_h}(\mathbf{u},\mathbf{s})| \,&\leqslant\, \|\mathbf{u}_h\|_{0,4;\Omega}\,\|\mathbf{z}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega}\,\|\mathbf{s}_h\|_{0,\Omega}\\ &\leqslant\, C_{\mathbf{S},\mathrm{d}}\,\|\phi_h\|_{0,4;\Omega}\,\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{z}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega}\,\|\mathbf{s}_h\|_{0,\Omega}\,.\end{aligned} \tag{4.35}$$

Therefore, employing (4.33), (4.34), and (4.35) to bound (4.32), and then replacing the resulting estimate back into (4.31), we arrive at (4.30) with a positive constant $\mathcal{L}_{\mathbf{S},\mathrm{d}}$ as indicated. □

We continue with the discrete analogue of Lemma 3.8, thus yielding the continuity of $\mathbf{T}_h$.

Lemma 4.6. *There exists a positive constant $\mathcal{L}_{\mathbf{T},\mathrm{d}}$, depending only on $\widetilde{\vartheta}_{\mathrm{d}}$, $\mathcal{L}_\kappa$, and $C_{\mathbf{T},\mathrm{d}}$, such that*

$$\begin{aligned}\|\mathbf{T}_h(\mathbf{z}_h,\phi_h) - \mathbf{T}_h(\underline{\mathbf{z}}_h,\underline{\phi}_h)\|_{\mathbf{L}^2(\Omega)\times \mathrm{L}^4(\Omega)} \,\leqslant\, \mathcal{L}_{\mathbf{T},\mathrm{d}}\Big\{\big(\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\big)\,\|\mathbf{z}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega}\\ +\quad \|\mathbf{T}_{1,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}\,\|\phi_h - \underline{\phi}_h\|_{0,4;\Omega}\Big\}\,,\end{aligned} \tag{4.36}$$

for all $(\mathbf{z}_h,\phi_h)$, $(\underline{\mathbf{z}}_h,\underline{\phi}_h) \in \mathbf{W}_h(\lambda_{\mathrm{d}})$.

Proof. It follows similarly to the proof of Lemma 4.5. Indeed, given $(\mathbf{z}_h,\phi_h)$, $(\underline{\mathbf{z}}_h,\underline{\phi}_h) \in \mathbf{W}_h(\lambda_{\mathrm{d}})$, we let $\mathbf{T}(\mathbf{z}_h,\phi_h) = (\boldsymbol{\zeta}_h,\varphi_h)$ and $\mathbf{T}(\underline{\mathbf{z}}_h,\underline{\phi}_h) = (\underline{\boldsymbol{\zeta}}_h,\underline{\varphi}_h)$, where $(\vec{\boldsymbol{\zeta}}_h,\varphi_h) = ((\boldsymbol{\zeta}_h,\boldsymbol{\rho}_h),\varphi_h) \in \widetilde{\mathcal{H}}_h\times\widetilde{\mathcal{Q}}_h$ and $(\underline{\vec{\boldsymbol{\zeta}}}_h,\underline{\varphi}_h) = ((\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\rho}}_h),\underline{\varphi}_h) \in \widetilde{\mathcal{H}}_h\times\widetilde{\mathcal{Q}}_h$ are the unique solutions, as guaranteed by Lemma 4.2, of the respective problem (4.21), equivalently (4.7). Then, applying the inf-sup condition (4.23) with $(\underline{\mathbf{z}}_h,\underline{\phi}_h)$ to $(\vec{\varsigma}_h,\theta_h) = (\vec{\boldsymbol{\zeta}}_h,\varphi_h) - (\underline{\vec{\boldsymbol{\zeta}}}_h,\underline{\varphi}_h)$, we arrive at the discrete analogue of (3.60), whose numerator within the respective supremum becomes

$$\begin{aligned}&\widetilde{\mathcal{A}}_{\underline{\phi}_h}((\vec{\boldsymbol{\zeta}}_h - \underline{\vec{\boldsymbol{\zeta}}}_h,\varphi_h-\underline{\varphi}_h),(\vec{\boldsymbol{\xi}}_h,\psi_h)) + \widetilde{C}_{\underline{\mathbf{z}}_h}(\varphi_h - \underline{\varphi}_h,\boldsymbol{\xi}_h)\\ &=\; -\widetilde{C}_{\mathbf{z}_h-\underline{\mathbf{z}}_h}(\varphi_h,\boldsymbol{\xi}_h) \;-\; \big(\widetilde{a}_{\phi_h}(\boldsymbol{\zeta}_h,\boldsymbol{\xi}_h) - \widetilde{a}_{\underline{\phi}_h}(\boldsymbol{\zeta}_h,\boldsymbol{\xi}_h)\big),\end{aligned} \tag{4.37}$$

for all $(\vec{\boldsymbol{\xi}}_h,\psi_h) = ((\boldsymbol{\xi}_h,\boldsymbol{\eta}_h),\psi_h) \in \widetilde{\mathcal{H}}_h\times\widetilde{\mathcal{Q}}_h$. In turn, employing the boundedness of $\widetilde{C}_{\mathbf{w}}$ (cf. (3.21)) and the upper bound for $\|\varphi_h\|_{0,4;\Omega} = \|\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}$ provided by (4.24), we find that

$$|\widetilde{C}_{\mathbf{z}_h-\underline{\mathbf{z}}_h}(\varphi_h,\boldsymbol{\xi}_h)| \,\leqslant\, C_{\mathbf{T},\mathrm{d}}\big(\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\big)\,\|\mathbf{z} - \underline{\mathbf{z}}_h\|_{0,4;\Omega}\,\|\boldsymbol{\xi}_h\|_{0,\Omega}\,, \tag{4.38}$$

whereas applying the Lipschitz-continuity of κ (cf. (2.3)), Cauchy-Schwarz's inequality twice, and the fact that $\boldsymbol{\zeta}_h = \mathbf{T}_{1,h}(\mathbf{z}_h,\phi_h)$, we arrive at

$$\big|\widetilde{a}_{\phi_h}(\boldsymbol{\zeta}_h,\boldsymbol{\xi}_h) - \widetilde{a}_{\underline{\phi}_h}(\boldsymbol{\zeta}_h,\boldsymbol{\xi}_h)\big| \,\leqslant\, \mathcal{L}_\kappa\,\|\mathbf{T}_{1,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}\,\|\phi_h - \underline{\phi}_h\|_{0,4;\Omega}\,\|\boldsymbol{\xi}_h\|_{0,\Omega}\,. \tag{4.39}$$

In this way, using (4.38) and (4.39) to bound the expression from (4.37), and then replacing the resulting estimate in the aforementioned discrete counterpart of (3.60), we are lead to (4.36) with a positive constant $\mathcal{L}_{\mathbf{T},\mathrm{d}}$ as announced. □

Having established Lemmas 4.5 and 4.6, we are now in position to conclude the continuity of $\mathbf{J}_h$.

Lemma 4.7. *There exist positive a positive constant $\mathcal{L}_{\mathbf{J},\mathrm{d}}$, depending only on $\mathcal{L}_{\mathbf{S},\mathrm{d}}$, $\mathcal{L}_{\mathbf{T},\mathrm{d}}$, and $C_{\mathbf{T},\mathrm{d}}$, and hence independent of h, such that, denoting*

$$\mathtt{D}_h(\mathbf{g},\mathbf{z}_h,\phi_h) \,:=\, \|\mathbf{g}\|_{0,\Omega} \,+\, \|\mathbf{S}_{1,h}(\mathbf{z}_h,\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h))\|_{0,4;\Omega} \,+\, 1\,,$$

there holds

$$\begin{aligned}\|\mathbf{J}_h(\mathbf{z}_h,\phi_h) - \mathbf{J}_h(\underline{\mathbf{z}}_h,\underline{\phi}_h)\|_{\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)} \,\leqslant\, \mathcal{L}_{\mathbf{J},\mathrm{d}}\,\mathtt{D}_h(\mathbf{g},\mathbf{z}_h,\phi_h)\Big\{\|\mathbf{T}_{1,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}\,\|\phi_h - \underline{\phi}_h\|_{0,4;\Omega}\\ +\quad \big(\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\big)\,\|\mathbf{z}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega}\Big\}\,,\end{aligned} \tag{4.40}$$

for all $(\mathbf{z}_h,\phi_h)$, $(\underline{\mathbf{z}}_h,\underline{\phi}_h) \in \mathbf{W}_h(\lambda_{\mathrm{d}})$.

Proof. Given $(\mathbf{z}_h,\phi_h)$, $(\underline{\mathbf{z}}_h,\underline{\phi}_h) \in \mathbf{W}_h(\lambda_{\mathrm{d}})$, the discrete version of (3.66) (cf. proof of Lemma 3.9) becomes

$$\begin{aligned}\|\mathbf{J}_h(\mathbf{z}_h,\phi_h) - \mathbf{J}_h(\underline{\mathbf{z}}_h,\underline{\phi}_h)\|_{\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)} \,=\, \|\mathbf{S}_{2,h}(\mathbf{z}_h,\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h)) - \mathbf{S}_{2,h}(\underline{\mathbf{z}}_h,\mathbf{T}_{2,h}(\underline{\mathbf{z}}_h,\underline{\phi}_h))\|_{0,4;\Omega}\\ +\quad \|\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h) - \mathbf{T}_{2,h}(\underline{\mathbf{z}}_h,\underline{\phi}_h)\|_{0,4;\Omega}\,,\end{aligned}$$

from which, using (4.30) to estimate its first term on the right-hand side, we get

$$\begin{aligned} \|\mathbf{J}_h(\mathbf{z}_h,\phi_h) - \mathbf{J}_h(\underline{\mathbf{z}}_h,\underline{\phi}_h)\|_{\mathbf{L}^4(\Omega)\times \mathrm{L}^4(\Omega)} \,&\leqslant\, \mathcal{L}_{\mathbf{S},\mathrm{d}}\,\|\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}\,\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{z}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega} \\ + \Big\{\mathcal{L}_{\mathbf{S},\mathrm{d}}\Big(\|\mathbf{g}\|_{0,\Omega} + \|\mathbf{S}_{1,h}\big(\mathbf{z}_h,\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h)\big)\|_{0,4;\Omega}\Big) + 1\Big\}&\,\|\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h) - \mathbf{T}_{2,h}(\underline{\mathbf{z}}_h,\underline{\phi}_h)\|_{0,4;\Omega}\,. \end{aligned} \tag{4.41}$$

Then, bounding $\|\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}$ and $\|\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h)-\mathbf{T}_{2,h}(\underline{\mathbf{z}}_h,\underline{\phi}_h)\|_{0,4;\Omega}$ according to (4.24) and (4.36), respectively, incorporating the resulting estimates in (4.41), and performing minor manipulations, we obtain (4.40) and conclude the proof. □

The existence of solution of the fixed-point equation (4.8), equivalently of the coupled system (4.2)–(4.3), is established below as a consequence of Brouwer's theorem (cf. [13, Theorem 9.9-2]), whose statement is previously recalled next.

Theorem 4.8 (Brouwer Fixed-Point Theorem)**.** *Let V be a finite-dimensional normed vector space, and let $B \subset V$ be a nonempty, convex, compact subset. If $\mathcal{F} : B \to B$ is a continuous mapping, then $\mathcal{F}$ has at least one fixed point in B; that is, there exists $x \in B$ such that $\mathcal{F}(x) = x$.*

The main result of this section is then stated as follows

Theorem 4.9. *Assume that the data satisfy assumption* (4.29)*. Then, $\mathbf{J}_h$ has at least one fixed point $(\mathbf{u}_h,\varphi_h) \in \mathbf{W}_h(\lambda_\mathrm{d})$. Equivalently, the Galerkin scheme* (4.2)*–*(4.3) *has at least one solution $((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\gamma_h)) \in \mathcal{H}_h \times \mathcal{Q}_h$ and $((\boldsymbol{\zeta}_h,\boldsymbol{\rho}_h),\varphi_h) \in \widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h$. Moreover, there hold the following a priori estimates*

$$\begin{aligned} \|((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\gamma_h))\|_{\mathcal{H}\times\mathcal{Q}} \,&\leqslant\, C_{\mathbf{S},\mathrm{d}}\,C_{\mathbf{T},\mathrm{d}}\,\|\mathbf{g}\|_{0,\Omega}\Big\{\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\Big\}, \quad \textit{and} \\ \|((\boldsymbol{\zeta}_h,\boldsymbol{\rho}_h),\varphi_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}} \,&\leqslant\, C_{\mathbf{T},\mathrm{d}}\Big\{\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\Big\}\,. \end{aligned} \tag{4.42}$$

Proof. We recall from Lemma 4.4 that, under assumption (4.29), $\mathbf{J}_h$ maps $\mathbf{W}_h(\lambda_\mathrm{d})$ into itself. Then, bearing in mind Lemma 4.7, which establishes the continuity of $\mathbf{J}_h$, a straightforward application of Theorem 4.8 implies that the fixed-point equation (4.8) has a solution $(\mathbf{u}_h,\varphi_h) \in \mathbf{W}_h(\lambda_\mathrm{d})$. Thus, the equivalence between the coupled system (4.2)-(4.3) and (4.8) allows us to conclude the existence of a solution of (4.2)-(4.3). Finally, analogously as in the proof of Theorem 3.10, the second estimate in (4.42) follows from (4.24), whereas the first one is consequence of (4.17) and the second one. □

We conclude this section by noting that the absence of a suitable bound for $\|\mathbf{T}_{1,h}(\mathbf{z}_h,\phi_h)\|_{0,4;\Omega}$ and $\|\mathbf{S}_{1,h}(\mathbf{z}_h,\mathbf{T}_{2,h}(\mathbf{z}_h,\phi_h))\|_{0,4;\Omega}$, uniform-in-$h$, prevents us from using (4.40) to derive a contraction estimate. As a result, the Banach fixed-point theorem cannot be applied to guarantee, based on that inequality, uniqueness of the discrete solution for sufficiently small data.

5 A priori error analysis

Let $(\vec{\mathbf{t}},\vec{\mathbf{u}}) = ((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{u},\gamma)) \in \mathcal{H}\times\mathcal{Q}$ and $(\vec{\boldsymbol{\zeta}},\varphi) = ((\boldsymbol{\zeta},\boldsymbol{\rho}),\varphi) \in \widetilde{\mathcal{H}}\times\widetilde{Q}$ be the unique solution of the coupled system (3.1)-(3.2), with $(\mathbf{u},\varphi) \in \mathbf{W}(\lambda)$ solution of (3.10), and, for a given h, let $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h) = ((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\gamma_h)) \in \mathcal{H}_h\times\mathcal{Q}_h$ and $(\vec{\boldsymbol{\zeta}}_h,\varphi_h) = ((\boldsymbol{\zeta}_h,\boldsymbol{\rho}_h),\varphi_h) \in \widetilde{\mathcal{H}}_h\times\widetilde{Q}_h$ be a solution of the Galerkin scheme (4.2)-(4.3), with $(\mathbf{u}_h,\varphi_h) \in \mathbf{W}_h(\lambda_\mathrm{d})$ solution of (4.8). In this section we assume again (**RH.1**) and (**RH.2**) (cf. Section 3.4), and, using similar arguments to those yielding Strang-type estimates (see, e.g. [18, Lemma 2.27]), we derive the Céa error estimate for the global error

$$\|(\vec{\mathbf{t}},\vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathcal{H}\times\mathcal{Q}} \,+\, \|(\vec{\boldsymbol{\zeta}},\varphi) - (\vec{\boldsymbol{\zeta}}_h,\varphi_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{Q}}\,.$$

To this end, in what follows, given a subspace X_h of an arbitrary Banach space $(X, \|\cdot\|_X)$, we set

$$\mathrm{dist}(x, X_h) \,:=\, \inf_{x_h \in X_h} \|x - x_h\|_X \,.$$

We begin the analysis by noticing that the pair of continuous and discrete schemes formed by (3.1) and (4.2), respectively, can be rewritten as

$$\mathcal{A}_\varphi((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})) \,+\, C_{\mathbf{u}}(\mathbf{u}, \mathbf{s}) \,=\, G_\varphi(\vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathcal{H} \times \mathcal{Q}\,, \quad \text{and} \tag{5.1}$$

$$\mathcal{A}_{\varphi_h}((\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)) \,+\, C_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{s}_h) \,=\, G_{\varphi_h}(\vec{\mathbf{v}}_h) \qquad \forall\, (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathcal{H}_h \times \mathcal{Q}_h\,, \tag{5.2}$$

where $\mathcal{A}_\varphi$ and $\mathcal{A}_{\varphi_h}$ are defined according to (3.31). Next, since $\mathbf{u}_h \leqslant \lambda_{\mathrm{d}} \leqslant \frac{\vartheta_{\mathrm{d}}}{2}$, there holds the discrete inf-sup condition (4.16) with $\mathbf{z}_h = \mathbf{u}_h$ and $\phi_h = \varphi_h$, which, applied to $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) - (\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h)$, yields

$$\frac{\vartheta_{\mathrm{d}}}{2}\, \|(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) - (\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h)\|_{\mathcal{H}\times\mathcal{Q}} \,\leqslant \sup_{\substack{(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \\ \in (\mathcal{H}_h \times \mathcal{Q}_h)\backslash\{\mathbf{0}\}}} \frac{\mathcal{R}_h\big((\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big)}{\|(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\|_{\mathcal{H}\times\mathcal{Q}}}\,, \tag{5.3}$$

where

$$\mathcal{R}_h\big((\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) \,:=\, \mathcal{A}_{\varphi_h}((\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) - (\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)) \,+\, C_{\mathbf{u}_h}(\mathbf{u}_h - \mathbf{w}_h, \mathbf{s}_h) \tag{5.4}$$

for all $(\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h)$, $(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathcal{H}_h \times \mathcal{Q}_h$. Then, using (5.2), we realize from (5.4) that

$$\mathcal{R}_h\big((\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) \,=\, G_{\varphi_h}(\vec{\mathbf{v}}_h) - \mathcal{A}_{\varphi_h}((\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)) \,-\, C_{\mathbf{u}_h}(\mathbf{w}_h, \mathbf{s}_h)\,,$$

from which, subtracting and adding the continuous solution in the first components of $\mathcal{A}_{\varphi_h}$ and $C_{\mathbf{u}_h}$, and then incorporating the evaluation of (5.1) with $(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)$ into the resulting expression, we get

$$\begin{aligned} \mathcal{R}_h\big((\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) \,&=\, G_{\varphi_h}(\vec{\mathbf{v}}_h) - \mathcal{A}_{\varphi_h}((\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h) - (\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)) \,-\, C_{\mathbf{u}_h}(\mathbf{w}_h - \mathbf{u}, \mathbf{s}_h) \\ &+\, \big(\mathcal{A}_\varphi - \mathcal{A}_{\varphi_h}\big)((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \,+\, C_{\mathbf{u}-\mathbf{u}_h}(\mathbf{u}, \mathbf{s}_h) \,-\, G_\varphi(\vec{\mathbf{v}}_h)\,. \end{aligned} \tag{5.5}$$

Now, the boundedness properties from (3.20) and the fact that $\|\mathbf{u}_h\| \leqslant \lambda_{\mathrm{d}}$, allow us to deduce that

$$\begin{aligned} &|\mathcal{A}_{\varphi_h}((\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h) - (\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)) \,+\, C_{\mathbf{u}_h}(\mathbf{w}_h - \mathbf{u}, \mathbf{s}_h)| \\ &\leqslant\, c(\mu_2, \lambda_{\mathrm{d}})\, \|(\vec{\mathbf{t}}, \vec{\mathbf{u}}) - (\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h)\|_{\mathcal{H}\times\mathcal{Q}}\, \|(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h))\|_{\mathcal{H}\times\mathcal{Q}}\,, \end{aligned} \tag{5.6}$$

where $c(\mu_2, \lambda_{\mathrm{d}})$ is a positive constant depending only on μ_2 and λ_{d}, whereas, using additionally the bound for $\|\mathbf{u}\|_{0,4;\Omega}$ provided by the first row of (3.68), we obtain

$$|C_{\mathbf{u}-\mathbf{u}_h}(\mathbf{u}, \mathbf{s}_h)| \,\leqslant\, C_{\mathbf{S}}\, C_{\mathbf{T}}\, \|\mathbf{g}\|_{0,\Omega} \big\{\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3,\Omega}\big\}\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\, \|\mathbf{s}_h\|_{0,\Omega}\,. \tag{5.7}$$

In turn, proceeding similarly as for the derivation of (3.55) and (3.56), respectively, and utilizing the bound for $\|\varphi\|_{0,4;\Omega}$ provided by the second row of (3.68) in the first estimate below, we find that

$$\begin{aligned} \big|\big(\mathcal{A}_\varphi - \mathcal{A}_{\varphi_h}\big)((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big| \,&=\, |a_\varphi(\mathbf{t}, \mathbf{s}_h) - a_{\varphi_h}(\mathbf{t}, \mathbf{s}_h)| \\ &\leqslant\, \mathcal{L}_{\mathcal{A}}\, \big\{\|\varphi_D\|_{1/2,\Gamma_D} + \|f\|_{0,4/3;\Omega}\big\}\, \|\mathbf{g}\|_{0,\Omega}\, \|\varphi - \varphi_h\|_{0,4;\Omega}\, \|\mathbf{s}_h\|_{0,\Omega}\,, \end{aligned} \tag{5.8}$$

and

$$|G_{\varphi_h - \varphi}(\vec{\mathbf{v}}_h)| \,\leqslant\, \|\mathbf{g}\|_{0,\Omega}\, \|\varphi - \varphi_h\|_{0,4;\Omega}\, \|\vec{\mathbf{v}}_h\|_{\mathcal{Q}}\,, \tag{5.9}$$

where, bearing in mind (2.3), the embeddings indicated right after (3.49), the regularity estimate (3.47), and the second row of (3.68), there holds $\mathcal{L}_{\mathcal{A}} \,:=\, \mathcal{L}_\mu\, \|\mathrm{i}_{4,\epsilon}\|\, \|\mathbf{i}_\epsilon\|\, C_\epsilon\, C_{\mathbf{T}}$. Then, employing (5.6), (5.7), (5.8), and (5.9) to bound $|\mathcal{R}_h\big((\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\big)|$ from (5.5), replacing the resulting estimate in (5.3), thus yielding an upper bound of $\|(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)-(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathcal{H}\times\mathcal{Q}}$, and hence of $\|(\vec{\mathbf{t}},\vec{\mathbf{u}})-(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathcal{H}\times\mathcal{Q}}$ via the triangle inequality, and taking infimum with respect to $(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\in\mathcal{H}_h\times\mathcal{Q}_h$, we arrive at

$$\begin{aligned}\|(\vec{\mathbf{t}},\vec{\mathbf{u}})-(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathcal{H}\times\mathcal{Q}} &\leqslant \mathcal{C}_1\Big\{\mathrm{dist}\big((\vec{\mathbf{t}},\vec{\mathbf{u}}),\mathcal{H}_h\times\mathcal{Q}_h\big)\\ &+\quad \|\mathbf{g}\|_{0,\Omega}\left(\|\varphi_D\|_{1/2,\Gamma_D}+\|f\|_{0,4/3;\Omega}\right)\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\\ &+\quad \|\mathbf{g}\|_{0,\Omega}\left(\|\varphi_D\|_{1/2,\Gamma_D}+\|f\|_{0,4/3;\Omega}+1\right)\|\varphi-\varphi_h\|_{0,4;\Omega}\Big\},\end{aligned}\tag{5.10}$$

where $\mathcal{C}_1$ is a positive constant depending only on $\vartheta_{\rm d}$, μ_2, $\lambda_{\rm d}$, $C_{\mathbf{S}}$, $C_{\mathbf{T}}$, and $\mathcal{L}_{\mathcal{A}}$.

On the other hand, regarding the pair of continuous and discrete schemes formed by (3.2) and (4.3), we proceed similarly as above and observe first that they can be rewritten, respectively, as

$$\widetilde{\mathcal{A}}_\varphi((\vec{\boldsymbol{\zeta}},\varphi),(\vec{\boldsymbol{\xi}},\psi))+\widetilde{C}_{\mathbf{u}}(\varphi,\boldsymbol{\xi}) \;=\; \widetilde{F}(\vec{\boldsymbol{\xi}})+\widetilde{G}(\psi)\qquad\forall\,(\vec{\boldsymbol{\xi}},\psi)\in\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}},\quad\text{and}\tag{5.11}$$

$$\widetilde{\mathcal{A}}_{\varphi_h}((\vec{\boldsymbol{\zeta}}_h,\varphi_h),(\vec{\boldsymbol{\xi}}_h,\psi_h))+\widetilde{C}_{\mathbf{u}_h}(\varphi_h,\boldsymbol{\xi}_h) \;=\; \widetilde{F}(\vec{\boldsymbol{\xi}}_h)+\widetilde{G}(\psi_h)\qquad\forall\,(\vec{\boldsymbol{\xi}}_h,\psi_h)\in\widetilde{\mathcal{H}}_h\times\widetilde{\mathcal{Q}}_h,\tag{5.12}$$

where $\widetilde{\mathcal{A}}_\varphi$ and $\widetilde{\mathcal{A}}_{\varphi_h}$ are defined according to (3.38). Now, recalling that $\|\mathbf{u}_h\|_{0,4;\Omega}\leqslant\lambda_{\rm d}\leqslant\frac{\widetilde{\vartheta}_{\rm d}}{2}$, there holds the discrete inf-sup condition (4.23) with $\mathbf{z}_h=\mathbf{u}_h$ and $\phi_h=\varphi_h$, so that, similarly as for the deduction of (5.3), it follows that

$$\frac{\widetilde{\vartheta}_{\rm d}}{2}\,\|(\vec{\boldsymbol{\zeta}}_h,\varphi_h)-(\vec{\boldsymbol{\varsigma}}_h,\theta_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}}\leqslant\sup_{\substack{(\vec{\boldsymbol{\xi}}_h,\psi_h)\\ \in(\widetilde{\mathcal{H}}_h\times\widetilde{\mathcal{Q}}_h)\backslash\{\mathbf{0}\}}}\frac{\widetilde{\mathcal{R}}_h\big((\vec{\boldsymbol{\varsigma}}_h,\theta_h),(\vec{\boldsymbol{\xi}}_h,\psi_h)\big)}{\|(\vec{\boldsymbol{\xi}}_h,\psi_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}}},\tag{5.13}$$

where

$$\widetilde{\mathcal{R}}_h\big((\vec{\boldsymbol{\varsigma}}_h,\theta_h),(\vec{\boldsymbol{\xi}}_h,\psi_h)\big)\;=\;\widetilde{\mathcal{A}}_{\varphi_h}((\vec{\boldsymbol{\zeta}}_h,\varphi_h)-(\vec{\boldsymbol{\varsigma}}_h,\theta_h),(\vec{\boldsymbol{\xi}}_h,\psi_h))+\widetilde{C}_{\mathbf{u}_h}(\varphi_h-\theta_h,\boldsymbol{\xi}_h)$$

for all $(\vec{\boldsymbol{\varsigma}}_h,\theta_h)$, $(\vec{\boldsymbol{\xi}}_h,\psi_h)\in\widetilde{\mathcal{H}}_h\times\widetilde{\mathcal{Q}}_h$. Then, using analogue arguments to those employed to derive (5.5), which means resorting here to (5.11) and (5.12), we are able to show that

$$\begin{aligned}\widetilde{\mathcal{R}}_h\big((\vec{\boldsymbol{\varsigma}}_h,\theta_h),(\vec{\boldsymbol{\xi}}_h,\psi_h)\big)\;&=\;-\widetilde{\mathcal{A}}_{\varphi_h}((\vec{\boldsymbol{\varsigma}}_h,\theta_h)-(\vec{\boldsymbol{\zeta}},\varphi),(\vec{\boldsymbol{\xi}}_h,\psi_h))\;-\;\widetilde{C}_{\mathbf{u}_h}(\theta_h-\varphi,\boldsymbol{\xi}_h)\\ &+\quad\big(\widetilde{\mathcal{A}}_\varphi-\widetilde{\mathcal{A}}_{\varphi_h}\big)((\vec{\boldsymbol{\zeta}},\varphi),(\vec{\boldsymbol{\xi}}_h,\psi_h))\;+\;\widetilde{C}_{\mathbf{u}-\mathbf{u}_h}(\varphi,\boldsymbol{\xi}_h),\end{aligned}\tag{5.14}$$

so that, using the boundedness properties from (3.21) and the fact that $\|\mathbf{u}_h\|\leqslant\lambda_{\rm d}$, we first obtain

$$\begin{aligned}&|\widetilde{\mathcal{A}}_{\varphi_h}((\vec{\boldsymbol{\varsigma}}_h,\theta_h)-(\vec{\boldsymbol{\zeta}},\varphi),(\vec{\boldsymbol{\xi}}_h,\psi_h))\;+\;\widetilde{C}_{\mathbf{u}_h}(\theta_h-\varphi,\boldsymbol{\xi}_h)|\\ &\leqslant\;\widetilde{c}(\kappa_2,\lambda_{\rm d})\,\|(\vec{\boldsymbol{\zeta}},\varphi)-(\vec{\boldsymbol{\varsigma}}_h,\theta_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}}\,\|(\vec{\boldsymbol{\xi}}_h,\psi_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}},\end{aligned}\tag{5.15}$$

where $\widetilde{c}(\kappa_2,\lambda_{\rm d})$ is a positive constant depending only on κ_2 and $\lambda_{\rm d}$. In turn, proceeding as for the second term on the right-hand side of (3.62), or just analogously to the derivation of (3.55), which means in this case employing the Lipschitz-continuity of κ (cf. (2.3)), Hölder's inequality, the embeddings indicated right after (3.49), and the regularity estimate (3.48), we can prove that

$$\begin{aligned}\big|\big(\widetilde{\mathcal{A}}_\varphi-\widetilde{\mathcal{A}}_{\varphi_h}\big)((\vec{\boldsymbol{\zeta}},\varphi),(\vec{\boldsymbol{\xi}}_h,\psi_h))\big|\;&=\;\big|\big(\widetilde{a}_\varphi-\widetilde{a}_{\varphi_h}\big)(\boldsymbol{\zeta},\boldsymbol{\xi}_h)\big|\\ &\leqslant\;L_{\widetilde{\mathcal{A}}}\left\{\|\varphi_D\|_{1/2+\epsilon,\Gamma_D}+\|f\|_{\epsilon,4/3;\Omega}\right\}\|\varphi-\varphi_h\|_{0,4;\Omega}\,\|\boldsymbol{\xi}_h\|_{0,\Omega},\end{aligned}\tag{5.16}$$

where $\mathcal{L}_{\widetilde{\mathcal{A}}}:=\mathcal{L}_\mu\,\|\mathrm{i}_{4,\epsilon}\|\,\|\mathbf{i}_\epsilon\|\,\widetilde{C}_\epsilon$. Finally, the stability property of $\widetilde{C}_{\mathbf{w}}$ (cf. (3.21)), and the bound for $\|\varphi\|_{0,4;\Omega}$ provided by the second row of (3.68), imply

$$|\widetilde{C}_{\mathbf{u}-\mathbf{u}_h}(\varphi,\boldsymbol{\xi}_h)|\;\leqslant\;C_{\mathbf{T}}\left\{\|\varphi_D\|_{1/2,\Gamma_D}+\|f\|_{0,4/3,\Omega}\right\}\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,\|\boldsymbol{\xi}_h\|_{0,\Omega}\,.\tag{5.17}$$

In this way, employing (5.15), (5.16), and (5.17), to bound $|\widetilde{\mathcal{R}}_h\big((\vec{\varsigma}_h,\theta_h),(\vec{\boldsymbol{\xi}}_h,\psi_h)\big)|$ from (5.14), and then using the resulting estimate along with (5.13) and the triangle inequality, we are lead to

$$\begin{aligned}\|(\vec{\boldsymbol{\zeta}},\varphi)-(\vec{\boldsymbol{\zeta}}_h,\varphi_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}} &\leqslant \mathcal{C}_2\Big\{\mathrm{dist}\big((\vec{\boldsymbol{\zeta}},\varphi),\widetilde{\mathcal{H}}_h\times\widetilde{\mathcal{Q}}_h\big)\\ &+\ \big(\|\varphi_D\|_{1/2,\Gamma_D}+\|f\|_{0,4/3;\Omega}\big)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\\ &+\ \big(\|\varphi_D\|_{1/2+\epsilon,\Gamma_D}+\|f\|_{\epsilon,4/3;\Omega}\big)\,\|\varphi-\varphi_h\|_{0,4;\Omega}\Big\},\end{aligned}\tag{5.18}$$

where $\mathcal{C}_2$ is a positive constant depending only on ϑ_{d}, κ_2, λ_{d}, $C_{\mathbf{T}}$, and $\mathcal{L}_{\widetilde{\mathcal{A}}}$.

Having established the estimates (5.10) and (5.18), we now add them up and arrive at

$$\begin{aligned}&\|(\vec{\mathbf{t}},\vec{\mathbf{u}})-(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathcal{H}\times\mathcal{Q}}\ +\ \|(\vec{\boldsymbol{\zeta}},\varphi)-(\vec{\boldsymbol{\zeta}}_h,\varphi_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}}\\ &\quad\leqslant\ \mathcal{C}_3\Big\{\mathrm{dist}\big((\vec{\mathbf{t}},\vec{\mathbf{u}}),\mathcal{H}_h\times\mathcal{Q}_h\big)\ +\ \mathrm{dist}\big((\vec{\boldsymbol{\zeta}},\varphi),\widetilde{\mathcal{H}}_h\times\widetilde{\mathcal{Q}}_h\big)\\ &\quad+\ \big(\mathtt{E}(\mathbf{g},\varphi_D,f)+\widetilde{\mathtt{E}}(\varphi_D,f)\big)\,\big(\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}+\|\varphi-\varphi_h\|_{0,4;\Omega}\big)\Big\},\end{aligned}\tag{5.19}$$

where $\mathcal{C}_3$ is a positive constant depending only on $\mathcal{C}_1$, $\mathcal{C}_2$, and $\bar{C}_\epsilon$ (cf. (3.64)), and the data-depending expressions $\mathtt{E}(\mathbf{g},\varphi_D,f)$ and $\widetilde{\mathtt{E}}(\varphi_D,f)$ are given by

$$\begin{aligned}\mathtt{E}(\mathbf{g},\varphi_D,f)\ &:=\ \|\mathbf{g}\|_{0,\Omega}\,\big(\|\varphi_D\|_{1/2,\Gamma_D}+\|f\|_{0,4/3;\Omega}+1\big)\,,\quad\text{and}\\ \widetilde{\mathtt{E}}(\varphi_D,f)\ &:=\ \|\varphi_D\|_{1/2+\epsilon,\Gamma_D}+\|f\|_{\epsilon,4/3;\Omega}\,.\end{aligned}\tag{5.20}$$

We are now in position to establish the announced Céa error estimate.

Theorem 5.1. *In addition to the hypotheses of Theorems 3.10 and 4.9, assume that*

$$\mathcal{C}_3\Big\{\mathtt{E}(\mathbf{g},\varphi_D,f)\ +\ \widetilde{\mathtt{E}}(\varphi_D,f)\Big\}\ \leqslant\ \frac{1}{2}\,.\tag{5.21}$$

Then, there holds

$$\begin{aligned}&\|(\vec{\mathbf{t}},\vec{\mathbf{u}})-(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathcal{H}\times\mathcal{Q}}\ +\ \|(\vec{\boldsymbol{\zeta}},\varphi)-(\vec{\boldsymbol{\zeta}}_h,\varphi_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}}\\ &\quad\leqslant\ 2\,\mathcal{C}_3\Big\{\mathrm{dist}\big((\vec{\mathbf{t}},\vec{\mathbf{u}}),\mathcal{H}_h\times\mathcal{Q}_h\big)\ +\ \mathrm{dist}\big((\vec{\boldsymbol{\zeta}},\varphi),\widetilde{\mathcal{H}}_h\times\widetilde{\mathcal{Q}}_h\big)\Big\}\,.\end{aligned}\tag{5.22}$$

Proof. It follows straightforwardly from (5.19) and (5.21). □

6 Specific finite element subspaces

Here we give concrete examples of finite element subspaces (cf. (4.1)) satisfying the hypotheses (**H.0**)–(**H.3**) and ($\widetilde{\mathbf{H.1}}$)-($\widetilde{\mathbf{H.3}}$), and derive the associated theoretical rates of convergence for the Galerkin scheme (4.2)-(4.3). To that end, given an integer $k\geqslant 0$ and a subset $S\subseteq \mathrm{R}^n$, we first recall that $\mathrm{P}_k(S)$ and $\widetilde{\mathrm{P}}_k(S)$ denote the space of polynomial functions on S of degree $\leqslant k$ and $=k$, respectively. In addition, with the same notations and definitions from Section 4.1, we define for each $K\in\mathcal{T}_h$ the corresponding local Raviart–Thomas space of order k as

$$\mathbf{RT}_k(T)\ :=\ \mathbf{P}_k(T)\,\oplus\,\widetilde{\mathrm{P}}_k(T)\mathbf{x}\,,$$

where $\mathbf{P_k}(K) := [\mathrm{P}_k(K)]^n$, and $\mathbf{x}$ is the generic vector in R^n. Then, denoting by $\mathbb{P}_k(K)$ the tensor version of $\mathrm{P}_k(K)$, and recalling that $\mathbf{H}(\mathrm{div};\Omega) \subseteq \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ and $\mathrm{L}^2(\Omega) \subseteq \mathrm{L}^4(\Omega)$, we introduce the following finite element subspaces:

$$
\begin{aligned}
\mathbb{L}^{\mathbf{t}}_{h,\mathtt{tr}} &:= \Big\{ \mathbf{s}_h \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) : \quad \mathbf{s}_h|_K \in \mathbb{P}_k(K)\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\
\mathbb{H}^{\boldsymbol{\sigma}}_h &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \mathbf{c}^t \boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(T) \quad \forall\, \mathbf{c} \in \mathrm{R}^n\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\
\mathbf{L}^{\mathbf{u}}_h &:= \Big\{ \mathbf{v}_h \in \mathbf{L}^4(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K)\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\
\mathbb{L}^{\gamma}_{h,\mathtt{skew}} &:= \Big\{ \boldsymbol{\omega}_h \in \mathbb{L}^2_{\mathtt{skew}}(\Omega) : \quad \boldsymbol{\omega}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\
\mathbf{L}^{\zeta}_h &:= \Big\{ \boldsymbol{\xi}_h \in \mathbf{L}^2(\Omega) : \quad \boldsymbol{\xi}_h|_K \in \mathbf{P}_k(K)\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\
\mathbf{H}^{\boldsymbol{\rho}}_h &:= \Big\{ \boldsymbol{\eta}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \boldsymbol{\eta}_h|_K \in \mathbf{RT}_k(T)\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\
\widetilde{\mathcal{Q}}_h &:= \Big\{ \psi_h \in \mathrm{L}^4(\Omega) : \quad \psi_h|_K \in \mathrm{P}_k(K)\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,.
\end{aligned}
\tag{6.1}
$$

It is immediate from the definition of $\mathbb{H}^{\boldsymbol{\sigma}}_h$ that $\mathbb{P}_0(\Omega) \subseteq \mathbb{H}^{\boldsymbol{\sigma}}_h$, and thus assumption (**H.0**) is satisfied. Furthermore, from [19, Lemma 3.6] we have the inclusions $\mathbf{div}(\mathbb{H}^{\boldsymbol{\sigma}}_h) \subseteq \mathbf{L}^{\mathbf{u}}_h$ and $\mathrm{div}(\mathbf{H}^{\boldsymbol{\rho}}_h) \subseteq \widetilde{Q}_h$, which imply that hypotheses (**H.1**) and ($\widetilde{\mathbf{H.1}}$) hold. Next, we recall from [14, Lemma 4.3] that the inf-sup condition (4.9) holds on $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathbf{L}^{\mathbf{u}}_h$, which confirms (**H.2**). Similarly, assuming that there exists a convex domain B such that $\Omega \subseteq B$ and $\Gamma_N \subseteq \partial B$, it follows from [12, Lemma 4.1] that there holds the inf-sup condition (4.18), and hence ($\widetilde{\mathbf{H.2}}$) is satisfied. Finally, given $\boldsymbol{\tau}_h \in \mathcal{V}_{h,0}$, we know from the definition of $\mathcal{V}_{h,0}$ (cf. (4.10)) that $\mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0}$ in Ω, which implies (cf. proof of [19, Theorem 3.3]) that $\boldsymbol{\tau}_h|_K \in \mathbb{P}_k(K)$ for every $K \in \mathcal{T}_h$, and hence $\boldsymbol{\tau}^{\mathtt{d}}_h|_K \in \mathbb{P}_k(K)$ for every $K \in \mathcal{T}_h$ as well. It is clear then that $\boldsymbol{\tau}^{\mathtt{d}}_h \in \mathbb{L}^{\mathbf{t}}_{h,\mathtt{tr}}$, thus proving that (**H.3**) holds. An analogous argument applied to $\widetilde{\mathcal{V}}_{h,0}$ (cf. (4.19)) shows that ($\widetilde{\mathbf{H.3}}$) is also attained.

Now we turn to collecting the approximation properties of the finite element subspaces defined in (6.1), which basically follow from approximation properties of the Raviart–Thomas interpolator and of the orthogonal projector onto piecewise scalar, vector and tensor polynomials, in the corresponding L^p-norms, along with the estimates arising from the interpolation between Sobolev spaces (see, for instance, [14, Section 4.2] and [21, Section 4.4.3]). More precisely, for each space defined in (6.1), we have:

($\mathbf{AP}^{\mathbf{t}}_h$) there exists a constant $C > 0$, independent of h, such that for each $s \in [0, k+1]$, and for each $\mathbf{s} \in \mathbb{L}^2_{h,\mathtt{tr}}(\Omega) \cap \mathbb{H}^s(\Omega)$, there holds

$$\mathrm{dist}(\mathbf{s}, \mathbb{L}^{\mathbf{t}}_{h,\mathtt{tr}}) \,\leqslant\, C\, h^s\, \|\mathbf{s}\|_{s,\Omega}\,,$$

($\mathbf{AP}^{\boldsymbol{\sigma}}_h$) there exists a constant $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) \cap \mathbb{H}^s(\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{s,4/3}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau}, \mathbb{H}^{\boldsymbol{\sigma}}_h) \,\leqslant\, C\, h^s \Big\{ \|\boldsymbol{\tau}\|_{s,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{s,4/3;\Omega} \Big\}\,,$$

($\mathbf{AP}^{\mathbf{u}}_h$) there exists a constant $C > 0$, independent of h, such that for each $s \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{s,4}(\Omega)$, there holds

$$\mathrm{dist}(\mathbf{v}, \mathbf{L}^{\mathbf{u}}_h) \,\leqslant\, C\, h^s\, \|\mathbf{v}\|_{s,4;\Omega}\,,$$

($\mathbf{AP}_h^{\gamma}$) there exists a constant $C > 0$, independent of h, such that for each $s \in [0, k+1]$, and for each $\boldsymbol{\omega} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{H}^s(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\omega}, \mathbb{L}^{\gamma}_{h,\mathtt{skew}}) \,\leqslant\, C\, h^s\, \|\boldsymbol{\omega}\|_{s,\Omega}\,,$$

($\mathbf{AP}_h^{\zeta}$) there exists a constant $C > 0$, independent of h, such that for each $s \in [0, k+1]$, and for each $\boldsymbol{\xi} \in \mathbf{H}^s(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\xi}, \mathbf{L}^{\zeta}_h) \,\leqslant\, C\, h^s\, \|\boldsymbol{\xi}\|_{s,\Omega}\,,$$

($\mathbf{AP}_h^{\rho}$) there exists a constant $C > 0$, independent of h, such that for each $s \in [0, k+1]$, and for each $\boldsymbol{\eta} \in \mathbf{H}(\operatorname{div}_{4/3};\Omega) \cap \mathbf{H}^s(\Omega)$ with $\operatorname{div}(\boldsymbol{\eta}) \in \mathrm{W}^{s,4/3}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\eta}, \mathbf{H}^{\rho}_h) \,\leqslant\, C\, h^s \Big\{ \|\boldsymbol{\eta}\|_{s,\Omega} + \|\operatorname{div}(\boldsymbol{\eta})\|_{s,4/3;\Omega} \Big\}\,,$$

($\mathbf{AP}_h^{\varphi}$) there exists a constant $C > 0$, independent of h, such that for each $s \in [0, k+1]$, and for each $\psi \in \mathrm{L}^4(\Omega) \cap \mathrm{W}^{s,4}(\Omega)$, there holds

$$\operatorname{dist}(\psi, \widetilde{\mathcal{Q}}_h) \,\leqslant\, C\, h^s\, \|\psi\|_{s,4;\Omega}\,.$$

We are now in position to establish the theoretical rates of convergence of the discrete scheme (4.2)–(4.3) for the specific subspaces defined in (6.1).

Theorem 6.1. *Assume the hypotheses of Theorem 5.1 hold, and that there exists a convex domain B such that $\Omega \subseteq B$ and $\Gamma_N \subseteq \partial B$. In addition, let $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) = ((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})) \in \mathcal{H} \times \mathcal{Q}$ and $(\vec{\boldsymbol{\zeta}}, \varphi) = ((\boldsymbol{\zeta}, \boldsymbol{\rho}), \varphi) \in \widetilde{\mathcal{H}} \times \widetilde{Q}$ be the unique solution of the coupled system (3.1)-(3.2), and for a given $h > 0$, let $((\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h)) \in \mathcal{H}_h \times \mathcal{Q}_h$ and $((\boldsymbol{\zeta}_h, \boldsymbol{\rho}_h), \varphi_h) \in \widetilde{\mathcal{H}}_h \times \widetilde{\mathcal{Q}}_h$ be a solution of (4.2)–(4.3) for the specific finite element spaces defined in (6.1). Assume further that that there exists $s \in (0, k+1]$, such that $\mathbf{t} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) \cap \mathbb{H}^s(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \cap \mathbb{H}^s(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{s,4/3}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{s,4}(\Omega)$, $\boldsymbol{\gamma} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbf{H}^s(\Omega)$, $\boldsymbol{\zeta} \in \mathbf{H}^s(\Omega)$, $\boldsymbol{\rho} \in \mathbf{H}_N(\operatorname{div}_{4/3};\Omega) \cap \mathbf{H}^s(\Omega)$, $\operatorname{div}(\boldsymbol{\rho}) \in \mathrm{W}^{s,4/3}(\Omega)$, and $\varphi \in \mathrm{W}^{s,4}(\Omega)$. Then, there exists a positive constant C, independent of h, such that*

$$\begin{aligned}\|(\vec{\mathbf{t}} - \vec{\mathbf{t}}_h, \vec{\mathbf{u}} - \vec{\mathbf{u}}_h)\|_{\mathcal{H}\times\mathcal{Q}} \;+\; \|(\vec{\boldsymbol{\zeta}} - \vec{\boldsymbol{\zeta}}_h, \varphi - \varphi_h)\|_{\widetilde{\mathcal{H}}\times\widetilde{\mathcal{Q}}} \,\leqslant\, C\, h^s \Big\{ &\|\mathbf{t}\|_{s,\Omega} + \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{s,4/3;\Omega} \\ + \|\mathbf{u}\|_{s,4;\Omega} + \|\boldsymbol{\gamma}\|_{s,\Omega} + \|\boldsymbol{\zeta}\|_{s,\Omega} + \|\boldsymbol{\rho}\|_{s,\Omega} + &\|\operatorname{div}(\boldsymbol{\rho})\|_{s,4/3;\Omega} + \|\varphi\|_{s,4;\Omega} \Big\}\,.\end{aligned}$$

Proof. It follows directly from (5.22) and the above approximation properties. □

7 Numerical Results

We now present some numerical examples to illustrate the performance of the fully mixed finite element method for the problem (4.2)–(4.3), with the discrete subspaces specified in (6.1). The implementation of the method has been carried out using the open source finite element library FEniCS. The nonlinear problem is solved using a Newton-Raphson algorithm with a prescribed tolerance tol =1E-6 and the zero vector as the initial guess. Furthermore, the null mean value condition for $\operatorname{tr}(\boldsymbol{\sigma}_h)$ is enforced by means of a real-valued Lagrange multiplier. Finally, the individual errors associated with the principal unknowns are denoted and defined, as usual, by

$$\mathsf{e}(\mathbf{t}) \,:=\, \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}\,, \quad \mathsf{e}(\boldsymbol{\sigma}) \,:=\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}\,, \quad \mathsf{e}(\mathbf{u}) \,:=\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,,$$

$$\mathsf{e}(\boldsymbol{\zeta}) \,:=\, \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{0,\Omega}\,, \quad \mathsf{e}(\boldsymbol{\rho}) \,:=\, \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{\operatorname{div}_{4/3};\Omega}\,, \quad \text{and} \quad \mathsf{e}(\varphi) \,:=\, \|\varphi - \varphi_h\|_{0,4;\Omega}\,.$$

Finally, for each $\star \in \{\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\zeta}, \boldsymbol{\rho}, \varphi\}$, we let $\mathtt{r}(\star) := \dfrac{\log(\mathtt{e}(\star)/\mathtt{e}'(\star))}{\log(h/h')}$ be the experimental rates of convergence, where h and $h^{'}$ denote two consecutive mesh sizes with errors $\mathtt{e}(\star)$ and $\mathtt{e}^{'}(\star)$, respectively.

7.1 Example 1: accuracy verification

The convergence of the method is assessed in 2D and 3D. We consider the square $(-1,1)^2$ and unit cube $(0,1)^3$ domains, discretized into meshes that are successively refined. We fix $\mathbf{g} = (0,1)^{\mathtt{t}}$ (in 2D) and $\mathbf{g} = (0,0,1)^{\mathtt{t}}$ (in 3D) together with the viscosity and thermal conductivity $\mu(\varphi) = \exp(-0.25\,\varphi)$, $\kappa(\varphi) = \exp(0.25\,\varphi)$ (in 2D) and $\mu(\varphi) = \exp(-0.25\,\varphi)$, $\kappa(\varphi) = 1$ (in 3D). Then, we choose a boundary temperature φ_D and a forcing term f such that the exact solutions are

$$\mathbf{u}(x_1,x_2) := \begin{pmatrix} 2x_2\sin(\pi x_1)\sin(\pi x_2)(x_1^2-1) + \pi\sin(\pi x_1)\cos(\pi x_2)(x_1^2-1)(x_2^2-1) \\ -2x_1\sin(\pi x_1)\sin(\pi x_2)(x_2^2-1) - \pi\sin(\pi x_2)\cos(\pi x_1)(x_1^2-1)(x_2^2-1) \end{pmatrix},$$

$$p(x_1,x_2) := x_1^2 - x_2^2\,, \qquad \varphi(x_1,x_2) = (x_1^2-1)(x_2^2-1)\,,$$

and

$$\mathbf{u}(x_1,x_2,x_3) := \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix},$$

$$p(x_1,x_2) := \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)\,, \qquad \varphi(x_1,x_2) = 1 - \sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3)\,,$$

for the 2D and 3D cases, respectively. In Tables 7.1 and 7.2 we summarize the convergence history of the fully mixed finite element method (4.2)–(4.3) in 2D and 3D, respectively, using polynomial degrees $k \in \{0,1,2\}$, from which we realize that, as predicted by Theorem 6.1, the rate of convergence of order $O(h^{k+1})$ is attained by all the unknowns. Furthermore, in order to illustrate the accuracy of the discrete scheme, in Figures 7.1 and 7.2 we display some components of the approximate solution obtained with the polynomial degree $k = 0$ in 2D and 3D, respectively.

7.2 Example 2: natural convection in a square cavity

In a second example, we consider natural convection of a fluid in a square cavity with differently heated walls. This configuration has been extensively investigated under various boundary conditions (see, e.g., [7, 17, 27]). In particular, following [17], we recall the modified dimensionless formulation of the problem: find $(\mathbf{u}, p, \varphi)$ such that

$$\begin{aligned} -\mathrm{Pr}\ \mathbf{div}(2\mu(\varphi)\,\mathbf{e}(\mathbf{u})) + (\boldsymbol{\nabla}\mathbf{u})\mathbf{u} + \nabla p - \mathrm{Ra}\ \mathrm{Pr}\ \varphi\,\mathbf{g} &= \mathbf{0} \qquad \text{in } \Omega\,, \\ \mathrm{div}(\mathbf{u}) &= 0 \qquad \text{in } \Omega\,, \\ -\,\mathrm{div}(\kappa(\varphi)\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi &= 0 \qquad \text{in } \Omega\,, \end{aligned} \tag{7.1}$$

where Pr and Ra denote the Prandtl and Rayleigh numbers, respectively, defined as the ratio of momentum diffusivity to thermal diffusivity, and the ratio of buoyancy forces to viscous forces multiplied by the Prandtl number. Accordingly, the cavity is modeled as $\Omega = (0,1)^2$, with $\mathrm{Pr} = 0.5$ and $\mathrm{Ra} = 4000$. Moreover, the viscosity, thermal conductivity, and body force are specified as follows:

$$\mu(\varphi) = \exp(-\varphi), \qquad \kappa(\varphi) = \exp(\varphi) \quad \text{and} \quad \mathbf{g} = (0,1)^{\mathtt{t}}\,.$$

The boundary conditions are prescribed as in [17] (see also [2]), namely,

$$\mathbf{u}_D(x_1,x_2) = \mathbf{0} \quad \text{and} \quad \varphi_D(\mathbf{x}) = \frac{1}{2}(1-\cos(2\pi x_1))(1-x_2) \quad \text{on} \quad \Gamma\,.$$

Errors and rates of convergence for the fluid variables

k	h	e(**t**)	r(**t**)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(**u**)	r(**u**)	e(γ)	r(γ)
	0.500	5.365	-	7.280	-	1.321	-	6.025	-
	0.250	2.382	1.171	3.382	1.106	0.626	1.076	1.346	2.162
0	0.125	1.150	1.050	1.642	1.041	0.315	0.988	0.366	1.876
	0.062	0.568	1.016	0.813	1.013	0.158	0.996	0.101	1.850
	0.031	0.283	1.005	0.405	1.004	0.079	0.999	0.029	1.791
	0.500	1.212	-	1.738	-	0.332	-	0.815	-
	0.250	0.334	1.855	0.453	1.938	0.084	1.976	0.253	1.687
1	0.125	0.094	1.824	0.124	1.866	0.021	1.991	0.080	1.651
	0.062	0.025	1.915	0.032	1.927	0.0053	1.998	0.0232	1.792
	0.031	0.0063	1.970	0.0083	1.973	0.0013	1.999	0.0061	1.924
	0.500	0.683	-	0.958	-	0.0641	-	0.6640	-
	0.250	0.08697	2.974	0.1284	2.898	0.00774	3.049	0.0837	2.988
2	0.125	0.01085	3.002	0.01642	2.967	0.00095	3.026	0.01041	3.006
	0.062	0.00135	3.000	0.00206	2.989	0.000118	3.007	0.00129	3.002

Errors and rates of convergence for the heat variables

k	h	e($\boldsymbol{\zeta}$)	r($\boldsymbol{\zeta}$)	e($\boldsymbol{\rho}$)	r($\boldsymbol{\rho}$)	e(φ)	r(φ)
	0.500	0.839	-	1.244	-	0.201	-
	0.250	0.395	1.085	0.576	1.111	0.099	1.020
0	0.125	0.197	1.000	0.289	0.994	0.049	0.997
	0.062	0.098	0.999	0.144	0.998	0.024	0.999
	0.031	0.049	0.999	0.072	0.999	0.012	0.999
	0.500	0.141	-	0.280	-	0.020	-
	0.250	0.038	1.857	0.075	1.900	0.005	2.000
1	0.125	0.010	1.902	0.019	1.947	0.0012	2.012
	0.062	0.0026	1.967	0.0049	1.985	0.0003	2.004
	0.031	0.0006	1.988	0.0012	1.995	0.00007	2.001
	0.500	0.0327	-	0.06442	-	0.00149	-
	0.250	0.00426	2.940	0.00846	2.928	0.00013	3.432
2	0.125	0.00052	3.014	0.00105	3.004	0.0000152	3.180
	0.062	0.000065	3.004	0.0001316	3.001	0.00000184	3.051

Table 7.1: Example 1, degrees of freedom, errors and rates of convergence for the Galerkin scheme (4.2)–(4.3) in 2D employing the subspaces defined in (6.1) with $k \in \{0, 1, 2\}$

The numerical experiments are carried out with $k = 0$. In Fig. 7.3, we present the approximate solutions obtained using the lowest-order mixed scheme. The results are consistent with those reported in [2] and with the physical behavior expected from the problem, in agreement with [17].

7.3 Example 3: natural convection in non-convex geometry

In this study, we investigate steady-state natural convection within a two-dimensional cross-section of a shell-and-tube configuration. Specifically, the simulations are performed for $k = 0$, corresponding

Errors and rates of convergence for the fluid variables

k	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\gamma)$	$\mathtt{r}(\gamma)$
	0.866	1.790	-	1.668	-	0.444	-	1.341	-
	0.433	0.948	0.916	0.953	0.806	0.235	0.915	0.477	1.491
0	0.216	0.482	0.976	0.495	0.945	0.119	0.976	0.137	1.798
	0.144	0.323	0.985	0.331	0.991	0.080	0.993	0.073	1.540
	0.866	0.581	-	0.605	-	0.155	-	0.417	-
1	0.433	0.167	1.793	0.170	1.826	0.042	1.873	0.109	1.932
	0.216	0.046	1.856	0.046	1.880	0.010	1.967	0.038	1.942
	0.866	0.186	-	0.197	-	0.0436	-	0.1641	-
2	0.433	0.02826	2.719	0.03073	2.687	0.00600	2.864	0.01968	3.059
	0.216	0.00371	2.927	0.00399	2.944	0.000767	2.967	0.00216	3.187

Errors and rates of convergence for the heat variables

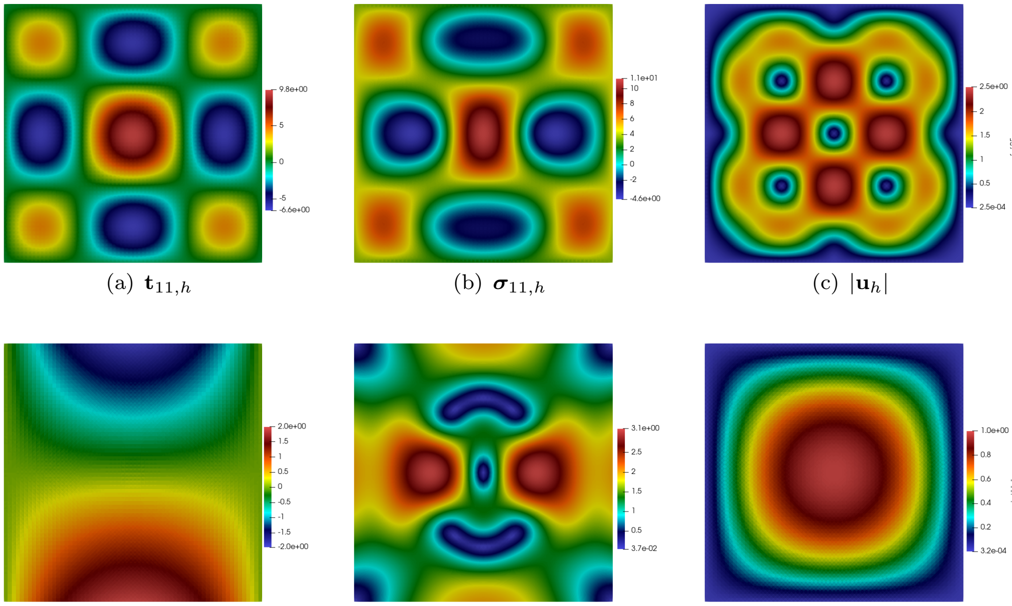
k	h	$\mathtt{e}(\boldsymbol{\zeta})$	$\mathtt{r}(\boldsymbol{\zeta})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\varphi)$	$\mathtt{r}(\varphi)$
	0.866	1.048	-	1.001	-	0.181	-
	0.433	0.585	0.839	0.549	0.865	0.096	0.913
0	0.216	0.304	0.942	0.284	0.949	0.048	0.979
	0.144	0.205	0.977	0.191	0.979	0.032	0.994
	0.866	0.376	-	0.325	-	0.064	-
1	0.433	0.107	1.806	0.091	1.828	0.017	1.884
	0.216	0.028	1.935	0.024	1.931	0.0044	1.971
	0.866	0.10681	-	0.08782	-	0.01795	-
2	0.433	0.01541	2.793	0.013276	2.725	0.002452	2.871
	0.216	0.002009	2.939	0.001755	2.919	0.000313	2.968

Table 7.2: Example 1, degrees of freedom, errors and rates of convergence for the Galerkin scheme (4.2)–(4.3) in 3D employing the subspaces defined in (6.1) with $k \in \{0, 1, 2\}$.

to a disk-shaped domain Ω, where problem (7.1) is solved under mixed boundary conditions for the energy equation, as described below. The geometry consists of two circular cavities, each with radius $\frac{1}{8}$. The right-hand inner cylinder is maintained at a hot temperature with $\varphi_D = 1$, while the left-hand inner cylinder is cooled to $\varphi_D = -1$. The outer shell is assumed to be adiabatic, which corresponds to the condition $\boldsymbol{\rho} \cdot \mathbf{n} = 0$ imposed on its surface. For the momentum equations, a no-slip boundary condition is prescribed along all boundaries. The Prandtl number is fixed at $\mathrm{Pr} = 1$, and the analysis is conducted for three Rayleigh number regimes:

$$\text{low } (\mathrm{Ra} = 1\mathrm{e}2)\,, \quad \text{medium } (\mathrm{Ra} = 1\mathrm{e}3)\,, \quad \text{and} \quad \text{high } (\mathrm{Ra} = 1\mathrm{e}4)\,. \tag{7.2}$$

Figure 7.4 displays, from left to right, the computed velocity magnitude, vorticity magnitude, and temperature distribution, obtained from numerical simulations for each of the above Rayleigh numbers (arranged from the first to the third columns).



(a) $\mathbf{t}_{11,h}$ (b) $\boldsymbol{\sigma}_{11,h}$ (c) $|\mathbf{u}_h|$

(d) $\boldsymbol{\zeta}_{1,h}$ (e) $|\boldsymbol{\rho}_h|$ (f) φ_h

Figure 7.1: Example 1, sample of 2D approximate solutions for the convergence test obtained using the polynomial degree $k = 0$ on the mesh with $h = 0.031$.

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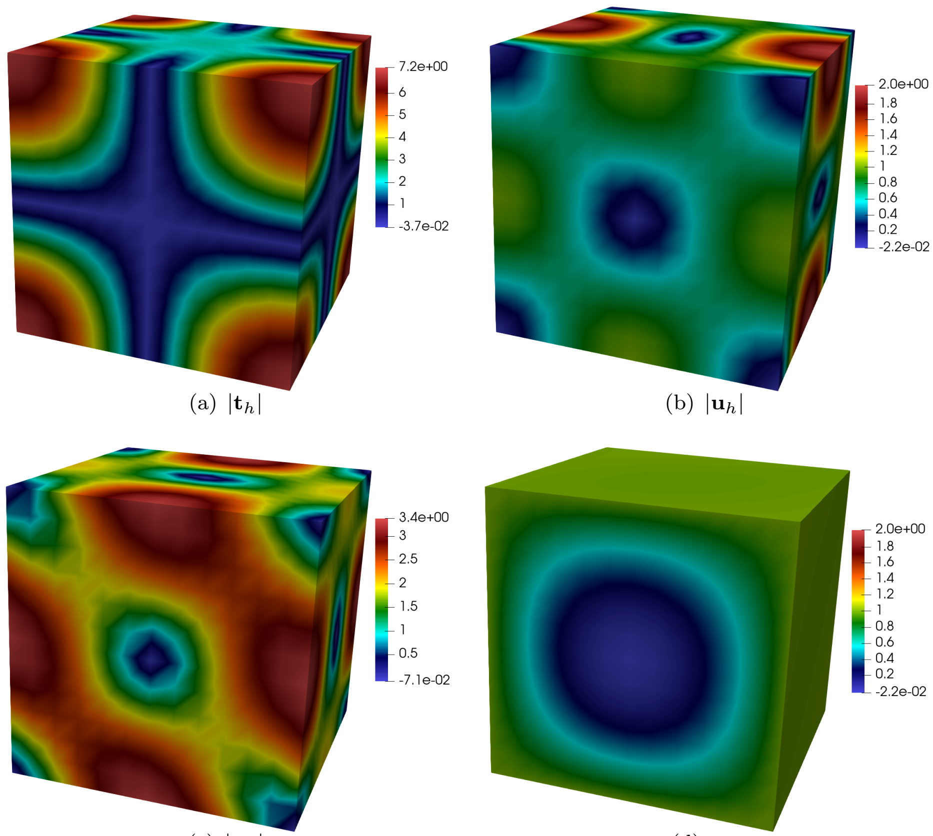

(a) $|\mathbf{t}_h|$ (b) $|\mathbf{u}_h|$

(c) $|\boldsymbol{\rho}_h|$ (d) φ_h

Figure 7.2: Example 1, sample of 3D approximate solutions for the convergence test, obtained using the polynomial degree $k = 0$ on the mesh with $h = 0.144$.

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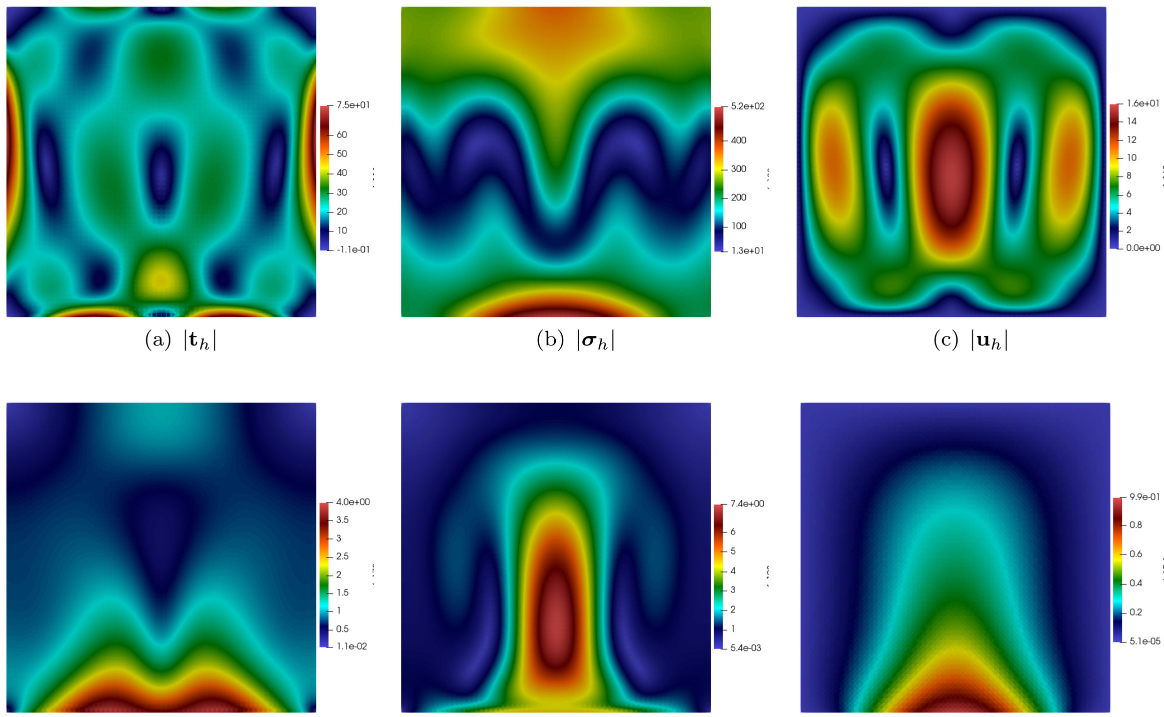

(a) $|\mathbf{t}_h|$ (b) $|\boldsymbol{\sigma}_h|$ (c) $|\mathbf{u}_h|$

(d) $|\boldsymbol{\zeta}_h|$ (e) $|\boldsymbol{\rho}_h|$ (f) φ_h

Figure 7.3: EXAMPLE 2, snapshots of strain rate, stress, and velocity (top row), and temperature gradient, heat flux, and temperature (bottom row), for $k = 0$ on a mesh with $h = 0.0303$.

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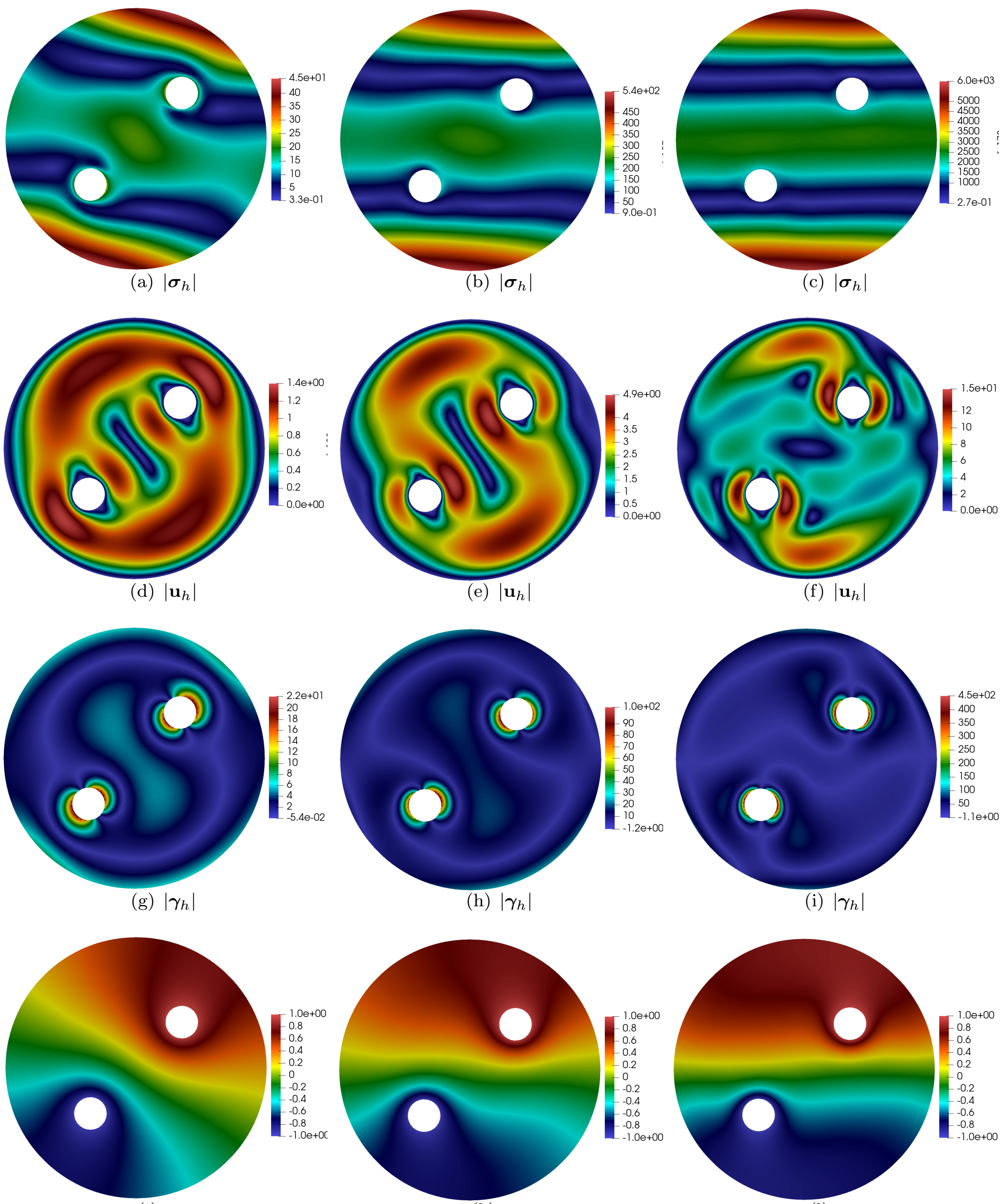

(a) $|\boldsymbol{\sigma}_h|$ (b) $|\boldsymbol{\sigma}_h|$ (c) $|\boldsymbol{\sigma}_h|$

(d) $|\mathbf{u}_h|$ (e) $|\mathbf{u}_h|$ (f) $|\mathbf{u}_h|$

(g) $|\boldsymbol{\gamma}_h|$ (h) $|\boldsymbol{\gamma}_h|$ (i) $|\boldsymbol{\gamma}_h|$

(j) φ_h (k) φ_h (l) φ_h

Figure 7.4: EXAMPLE 3, snapshots of stress, velocity and vorticity magnitudes, and temperature (top to bottom), for $k = 0$ and the Rayleigh numbers from (7.2) (left to right), on a mesh with $h = 0.0303$.

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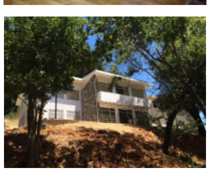

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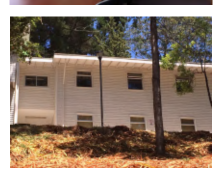

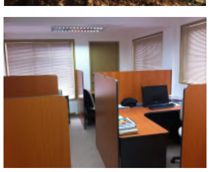

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