UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en
Ingeniería Matemática (CI^{2}MA)

A priori and a posteriori error analyses of a fully-mixed finite element method for the coupled Navier–Stokes/Darcy problem

Sergio Caucao, Ricardo Oyarzúa,
Segundo Villa-Fuentes

PREPRINT 2025-15

SERIE DE PRE-PUBLICACIONES

A priori and *a posteriori* error analyses of a fully-mixed finite element method for the coupled Navier–Stokes/Darcy problem *

Sergio Caucao[†] Ricardo Oyarzúa[‡] Segundo Villa–Fuentes[§]

Abstract

We propose and analyze a fully mixed finite element formulation for coupling free fluid flow with porous media flow, governed respectively by the Navier–Stokes and Darcy equations. The coupling is enforced through continuity of the normal velocity (mass conservation), balance of normal forces, and the Beavers–Joseph–Saffman law. For the Navier–Stokes region, we adopt a pseudostress–velocity–vorticity formulation in a Banach space setting, where the pseudostress tensor depends on the pressure as well as on the diffusive and convective terms of the equations, and the trace of the velocity on the interface is also included as an independent unknown. For the Darcy region, we employ the standard dual-mixed formulation, with velocity, pressure, and the trace of the latter on the interface as primary unknowns. The resulting scheme can be written as a nonlinear perturbation of a two-fold saddle-point problem. Well-posedness of both the continuous and discrete formulations is established under smallness assumptions on the data, by means of a fixed-point strategy combined with the Banach–Nečas–Babuška theorem and Banach's fixed-point theorem. These results hold for arbitrary finite element subspaces satisfying suitable stability conditions. Specific choices of finite element spaces are identified that fulfill these requirements, and we derive optimal-order *a priori* error estimates. In addition, we develop a reliable and efficient residual-based *a posteriori* error estimator for the proposed method. The proofs of reliability and efficiency rely on the global inf-sup condition, Helmholtz decompositions, inverse inequalities, and well-known properties of bubble functions. Several two-dimensional numerical experiments, with and without manufactured solutions, are presented to confirm the theoretical convergence rates and to illustrate the accuracy and flexibility of the method.

Key words: Navier–Stokes, Darcy, momentum conservation, mixed finite element method, Banach spaces, Arnold–Falk–Winther elements, Raviart–Thomas elements

Mathematics Subject Classifications (1991): 65N15, 65N30, 76D05, 76M10

*This work was partially supported by ANID-Chile through Centro de Modelamiento Matemático (FB210005), and Fondecyt project 1250937; by Universidad del Bío-Bío through VRIP project "Concurso Atracción de Investigadores Postdoctorales 2025" and by Grupo de Investigación en Análisis Numérico y Cálculo Científico (GIANuC^2), Universidad Católica de la Santísima Concepción.

[†]GIANuC^2 and Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile, email: scaucao@ucsc.cl.

[‡]GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: royarzua@ubiobio.cl.

[§]GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, email: svilla@ubiobio.cl.

1 Introduction

The coupling of free fluid flow with flow through porous media arises in a wide range of applications, including blood filtration through vessel walls, the interaction between rivers and their riverbeds, and enhanced oil recovery, among many others. From a modeling perspective, these processes are often described by a coupled system in which distinct flow regimes are governed by different physical laws, linked through interface conditions that ensure physical consistency. A classical example is the Stokes–Darcy (SD) system, where the Stokes equations describe the free-flow region and the Darcy law governs the porous-medium flow. The coupling is enforced through continuity of the normal velocity (ensuring mass conservation), balance of normal forces, and the Beavers–Joseph–Saffman (BJS) law for the tangential velocity. While this model is appropriate for slow flows, many practical situations involve higher Reynolds numbers in the free-flow region, where inertial effects become significant. In such cases, the Stokes equations are replaced by the Navier–Stokes equations, leading to the so-called Navier–Stokes/Darcy (NSD) coupled problem. The NSD system preserves the interface structure of the SD model but introduces additional analytical and numerical challenges due to the nonlinear convective term in the Navier–Stokes equations.

Over the past years, various numerical schemes have been developed for the approximation of both SD and NSD systems, including finite element and discontinuous Galerkin methods (see, for instance, [5, 13, 16, 22, 24, 25, 26, 28, 35, 36, 37, 40, 42, 47, 51] and the references therein). Most of these methods adopt the classical velocity-pressure formulation for the free-flow equations, coupled with either primal or dual-mixed formulations for the porous-medium equations, as in [5, 22, 24, 25, 26, 36, 42, 47, 51]. An alternative strategy was proposed in [37] (see also [13]) for the SD system, based on a nonstandard pseudostress–velocity formulation for the Stokes equations. This approach permits the use of the same family of finite elements in both subdomains, facilitates the computation of additional quantities of interest such as the velocity gradient and the vorticity, and preserves momentum at the discrete level. The method in [37] was later extended to the NSD system in [39], where, to address the nonlinearity, and inspired by [15], Galerkin-type terms arising from the constitutive and equilibrium equations were incorporated into the variational formulation. This modification ensures the well-posedness of both the continuous and discrete schemes.

More recently, [12] introduced a new approach for the pseudostress-velocity formulation of the Navier–Stokes problem, avoiding the inclusion of the aforementioned Galerkin-type terms. This strategy is based on considering Banach spaces for each unknown, instead of the classical Hilbert spaces, which allows one to establish well-posedness and to derive optimal convergence results using standard mixed finite element spaces, namely Raviart–Thomas elements for the pseudostress and discontinuous piecewise polynomials for the velocity. In this direction, [8] proposed a Banach spaces-based approach for a modified NSD system with variable viscosity, in which the gradient of the velocity is considered instead of the strain tensor in the Cauchy stress tensor, together with a modified BJS law on the interface. In that formulation, besides the pseudostress, velocities, and pressure, the velocity gradient is also introduced as an additional unknown to handle the variable viscosity.

Inspired by [12] and [13, 37], in this study we introduce and analyze a new formulation for the original NSD model studied in [25, 26]. The proposed approach incorporates, for the Navier–Stokes region, the pseudostress tensor, the velocity, the vorticity, and the trace of the velocity on the interface, all within the Banach space framework of [12]. For the Darcy region, we consider the velocity, the pressure, and the trace of the latter on the interface, as in [13, 37]. This leads to a double saddle-point structure with a perturbation arising from the convective term of the Navier–Stokes equations. A fixed-point strategy, combined with the Banach–Nečas–Babuška theorem and Banach's fixed-point theorem, is employed to prove, under smallness assumptions on the data, the well-posedness of both the continuous and discrete formulations, the latter for arbitrary finite element subspaces satisfying suitable stability conditions. We then identify specific finite element spaces that meet these requirements and derive optimal-order *a priori* error estimates. In addition, we propose a reliable and efficient residual-based *a posteriori* error estimator for the new numerical scheme. The proofs of reliability and efficiency follow the techniques developed in [11, 38], making use of the global inf-sup condition, Helmholtz decompositions, inverse inequalities, and well-known properties of bubble functions.

The rest of this work is organized as follows. The remainder of this section introduces the notation and functional spaces that will be used throughout the paper. In Section 2, we present the model problem and derive the fully mixed variational formulation, expressed as a nonlinear perturbation of a two-fold saddle-point problem in a Banach space framework. Section 2.3 is devoted to the well-posedness analysis of the continuous formulation. The corresponding Galerkin scheme is introduced and analyzed in Section 3 for generic finite dimensional subspaces, where the discrete counterpart of the continuous theory is applied to prove existence and uniqueness of solution, along with an *a priori* error estimate. In Section 4, we specify particular finite element spaces that satisfy the stability hypotheses from Section 3 and derive the corresponding theoretical convergence rates. In Section 5 we derive a reliable and efficient residual-based error *a posteriori* estimator and finally in Section 6, we present numerical experiments that illustrate the accuracy and flexibility of the proposed mixed finite element method, as well as the reliability and efficiency of the *a posteriori* error estimator. Finally, in Section 7 we give some concluding remarks.

Preliminaries

Let us denote by $\Omega \subseteq \mathbb{R}^n$, $n \in \{2,3\}$ a given bounded domain with polygonal/polyhedral boundary Γ. Standard notations will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$, with $p \in [1,\infty]$ and Sobolev spaces $\mathrm{W}^{r,p}(\Omega)$ with $r \geq 0$, endowed with the norms $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{r,p;\Omega}$, respectively. Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$ and if $p = 2$, we write $\mathrm{H}^r(\Omega)$ in place of $\mathrm{W}^{r,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{r,\Omega}$, respectively. We also write $|\cdot|_{r,\Omega}$ for the H^r-seminorm. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. With $\langle\cdot,\cdot\rangle_\Gamma$ we denote the corresponding duality product between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$. By $\mathbf{S}$ and $\mathbb{S}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space S, whereas S' denotes its dual space, whose norm is defined by $\|f\|_{S'} := \sup\limits_{0\neq s\in S} \dfrac{|f(s)|}{\|s\|_S}$. Unless otherwise stated, given any pair $(\cdot,\cdot)$ in a product space $X \times Y$, we denote $\|(\cdot,\cdot)\| := \|\cdot\|_X + \|\cdot\|_Y$. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$ we set the gradient, divergence and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^n \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}\,.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\boldsymbol{\tau}_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^n \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}\,,$$

where $\mathbb{I}$ is the identity tensor in $\mathbb{R}^{n\times n}$.

Additionally, we recall that the Hilbert space

$$\mathbf{H}(\mathrm{div};\Omega) := \Big\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\mathbf{v}) \in \mathrm{L}^2(\Omega) \Big\},$$

equipped with the usual norm $\|\mathbf{v}\|_{\mathrm{div};\Omega} := \Big(\|\mathbf{v}\|_{0,\Omega}^2 + \|\mathrm{div}(\mathbf{v})\|_{0,\Omega}^2 \Big)^{1/2}$ is standard in the realm of mixed problems. In the sequel we will make use of its tensor counterpart, and more generally, of the Banach space $\mathbb{H}(\mathbf{div}_p;\Omega)$ (for $p > 1$) defined by

$$\mathbb{H}(\mathbf{div}_p;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^p(\Omega) \Big\},$$

endowed with the norm $\|\boldsymbol{\tau}\|_{\mathbf{div}_p;\Omega} := \Big(\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,p;\Omega}^2 \Big)^{1/2}$.

Finally, we end this section by mentioning that, throughout the rest of the paper, we employ $\mathbf{0}$ to denote a generic null vector (or tensor), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 Continuous problem

In this section we introduce the model problem at the continuous level and we derive the corresponding weak formulation.

2.1 Model problem

In order to describe the geometry of the problem, we let Ω_{S} and Ω_{D} be two bounded and simply connected polygonal/polyhedral domains in $\mathbb{R}^n$, $n \in \{2,3\}$, such that $\partial\Omega_{\mathrm{S}} \cap \partial\Omega_{\mathrm{D}} = \Sigma \neq \emptyset$ and $\Omega_{\mathrm{S}} \cap \Omega_{\mathrm{D}} = \emptyset$. Then, let $\Gamma_{\mathrm{S}} := \partial\Omega_{\mathrm{S}} \backslash \bar{\Sigma}$, $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \backslash \bar{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ and Ω_{S} (and hence inward to Ω_{D} when seen on Σ). On Σ we also consider a set of unit tangent vectors, which is given by $\mathbf{t} = \mathbf{t}_1$ when $n = 2$ (see Figure 1 below), and $\{\mathbf{t}_1, \mathbf{t}_2\}$ when $n = 3$.

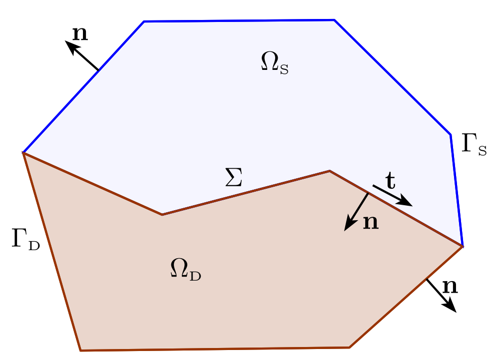


Figure 1: Sketch of a 2D geometry of the coupled Navier–Stokes/Darcy model

The mathematical model is characterized by two distinct sets of equations and a set of coupling terms, where the Navier–Stokes system in Ω_{S} and the Darcy model in Ω_{D} are linked through appropriate interface conditions on Σ. Specifically, within the free fluid domain Ω_{S}, the fluid motion is governed by the incompressible Navier–Stokes equations:

$$\begin{aligned} \mathbf{T}_{\rm S} \,=\, 2\,\nu\,\mathbf{e}(\mathbf{u}_{\rm S}) \,-\, p_{\rm S}\,\mathbb{I} \quad \text{in} \quad \Omega_{\rm S}\,, \quad \operatorname{div}(\mathbf{u}_{\rm S}) \,=\, 0 \quad \text{in} \quad \Omega_{\rm S}\,, \\ -\,\mathbf{div}(\mathbf{T}_{\rm S}) \,+\, \rho\,(\nabla\mathbf{u}_{\rm S})\,\mathbf{u}_{\rm S} \,=\, \mathbf{f}_{\rm S} \quad \text{in} \quad \Omega_{\rm S}\,, \quad \mathbf{u}_{\rm S} = \mathbf{0} \quad \text{on} \quad \Gamma_{\rm S}\,, \end{aligned} \tag{2.1}$$

where, $\mathbf{T}_{\rm S}$ is the Cauchy stress tensor, $\mathbf{u}_{\rm S}$ is the fluid velocity, and $p_{\rm S}$ is the pressure. In addition, $\mathbf{e}(\mathbf{u}_{\rm S}) := \frac{1}{2}\Big(\nabla\mathbf{u}_{\rm S} + (\nabla\mathbf{u}_{\rm S})^{\rm t}\Big)$ is strain rate tensor, $\nu > 0$ is the dynamic viscosity of the fluid, ρ is the density, and $\mathbf{f}_{\rm S}$ is a given external force.

In the porous medium $\Omega_{\rm D}$, we consider the following Darcy model:

$$\mathbf{K}^{-1}\mathbf{u}_{\rm D} + \nabla p_{\rm D} = \mathbf{f}_{\rm D} \quad \text{in}\ \Omega_{\rm D}\,, \quad \operatorname{div}(\mathbf{u}_{\rm D}) = 0 \quad \text{in}\ \Omega_{\rm D}\,, \quad \mathbf{u}_{\rm D}\cdot\mathbf{n} = 0 \quad \text{on}\ \Gamma_{\rm D}\,, \tag{2.2}$$

where $\mathbf{u}_{\rm D}$ is the Darcy velocity (specific discharge), $p_{\rm D}$ is the pressure, and $\mathbf{K} \in \mathbb{L}^\infty(\Omega_{\rm D})$ is a symmetric tensor representing the intrinsic permeability of the porous medium divided by the dynamic viscosity ν of the fluid, assumed to be uniformly positive definite. This implies that there exists a constant $C_{\mathbf{K}} > 0$, such that

$$\boldsymbol{\xi}\cdot\mathbf{K}^{-1}(\mathbf{x})\boldsymbol{\xi} \geq C_{\mathbf{K}}|\boldsymbol{\xi}|^2\,, \tag{2.3}$$

for almost all $\mathbf{x} \in \Omega_{\rm D}$, and for all $\boldsymbol{\xi} \in \mathbb{R}^n$. Additionally, $\mathbf{f}_{\rm D}$ is a given external force.

Finally, the transmission conditions on Σ are given by

$$\mathbf{u}_{\rm S}\cdot\mathbf{n} = \mathbf{u}_{\rm D}\cdot\mathbf{n} \quad \text{on}\ \Sigma \quad \text{and} \quad \mathbf{T}_{\rm S}\mathbf{n} + \sum_{i=1}^{n-1}\omega_i^{-1}(\mathbf{u}_{\rm S}\cdot\mathbf{t}_i)\mathbf{t}_i = -p_{\rm D}\mathbf{n} \quad \text{on}\ \Sigma\,, \tag{2.4}$$

where $\{\omega_1,\ldots,\omega_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally. The first condition in (2.4) corresponds to mass conservation on Σ, while the second condition can be decomposed into its normal and tangential components as follows:

$$(\mathbf{T}_{\rm S}\mathbf{n})\cdot\mathbf{n} = -p_{\rm D} \quad \text{and} \quad (\mathbf{T}_{\rm S}\mathbf{n})\cdot\mathbf{t}_i = -\omega_i(\mathbf{u}_{\rm S}\cdot\mathbf{t}_i), \quad i = 1,\ldots,n-1. \tag{2.5}$$

The first condition in (2.5) corresponds to the balance of normal forces, while the second, known as the Beavers–Joseph–Saffman law, establishes that the slip velocity along Σ is proportional to the shear stress along Σ, under the assumption, based on experimental evidence, that $\mathbf{u}_{\rm D}\cdot\mathbf{t}_i$, for $i = 1,\ldots,n-1$, is negligible. For further details on this interface condition, we refer the reader to [7, 45, 48].

2.2 Variational formulation

We now proceed with the derivation of the fully-mixed variational formulation for the coupled Navier–Stokes/Darcy problem described by the equations (2.1), (2.2), and (2.4). To achieve this, we proceed similarly to [39] (see also [19]) by introducing the pseudostress tensor:

$$\boldsymbol{\sigma}_{\rm S} := \mathbf{T}_{\rm S} - \rho\,(\mathbf{u}_{\rm S}\otimes\mathbf{u}_{\rm S}) = 2\,\nu\,\mathbf{e}(\mathbf{u}_{\rm S}) - p_{\rm S}\,\mathbb{I} - \rho\,(\mathbf{u}_{\rm S}\otimes\mathbf{u}_{\rm S})\,. \tag{2.6}$$

Observe that, due to the incompressibility condition $\operatorname{tr}(\mathbf{e}(\mathbf{u}_{\rm S})) = \operatorname{div}(\mathbf{u}_{\rm S}) = 0$ in $\Omega_{\rm S}$, the pressure can be expressed in terms of $\boldsymbol{\sigma}_{\rm S}$ and $\mathbf{u}_{\rm S}$ as follows:

$$p_{\rm S} = -\frac{1}{n}\Big\{\operatorname{tr}(\boldsymbol{\sigma}_{\rm S}) + \rho\operatorname{tr}(\mathbf{u}_{\rm S}\otimes\mathbf{u}_{\rm S})\Big\} \quad \text{in}\ \Omega_{\rm S}. \tag{2.7}$$

Additionally, the equilibrium equation (third equation of (2.1)) can be written now in terms of $\boldsymbol{\sigma}_{\rm S}$ as:

$$-\,\mathbf{div}(\boldsymbol{\sigma}_{\rm S}) = \mathbf{f}_{\rm S} \quad \text{in} \quad \Omega_{\rm S}\,. \tag{2.8}$$

In turn, let us additionally introduce the variable:

$$\boldsymbol{\gamma}_{\mathrm{S}} := \frac{1}{2}\left(\nabla \mathbf{u}_{\mathrm{S}} - (\nabla \mathbf{u}_{\mathrm{S}})^{\mathrm{t}}\right) \quad \text{in } \Omega_{\mathrm{S}}\,.$$

Noting that $\mathbf{e}(\mathbf{u}_{\mathrm{S}}) = \nabla \mathbf{u}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S}}$, from (2.6) and (2.7), is easy to see that $\boldsymbol{\sigma}_{\mathrm{S}}$, $\mathbf{u}_{\mathrm{S}}$, and $\boldsymbol{\gamma}_{\mathrm{S}}$ satisfy the following identity:

$$\frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}} = \nabla \mathbf{u}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S}} - \frac{\rho}{2\nu}(\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathrm{d}} \quad \text{in } \Omega_{\mathrm{S}}.$$

In this way, from the latter, (2.8), and recalling that $\boldsymbol{\sigma}_{\mathrm{S}}$ is a symmetric tensor, we observe that the Navier–Stokes system (2.1) can be rewritten equivalently in terms of $\boldsymbol{\sigma}_{\mathrm{S}}$, $\mathbf{u}_{\mathrm{S}}$ and $\boldsymbol{\gamma}_{\mathrm{S}}$ by means of the following first-order set of equations:

$$\begin{gathered} \frac{1}{2\nu}\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}} = \nabla \mathbf{u}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S}} - \frac{\rho}{2\nu}(\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathrm{d}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad \boldsymbol{\sigma}_{\mathrm{S}} = \boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{t}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \\ -\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S}}) = \mathbf{f}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}} \quad \text{and} \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}. \end{gathered} \tag{2.9}$$

In addition, the second equation in (2.4) can be equivalently rewritten in terms of $\boldsymbol{\sigma}_{\mathrm{S}}$, $\mathbf{u}_{\mathrm{S}}$ and p_{D} as follows:

$$\boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n} \,+\, \rho\,(\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})\mathbf{n} \,+\, \sum_{i=1}^{n-1} \omega_i^{-1}(\mathbf{u}_{\mathrm{S}} \cdot \mathbf{t}_i)\,\mathbf{t}_i \;=\; -\,p_{\mathrm{D}}\,\mathbf{n} \quad \text{on} \quad \Sigma\,. \tag{2.10}$$

Observe that, after applying (2.7), p_{S} has been eliminated from the system. However, this variable can be easily recovered by using (2.7).

According to the above, in what follows we make use of the equivalent system given by equations (2.2), (2.9), (2.10) and the first equation in (2.4), to derive the fully-mixed variational formulation. For simplicity, in what follows, for $\star \in \{\mathrm{S}, \mathrm{D}\}$ we denote

$$(v, w)_\star := \int_\star v\,w, \quad (\mathbf{v}, \mathbf{w})_\star := \int_\star \mathbf{v}\cdot\mathbf{w}, \quad \text{and} \quad (\boldsymbol{\tau}, \boldsymbol{\zeta})_\star := \int_\star \boldsymbol{\tau} : \boldsymbol{\zeta}.$$

We begin by proceeding similarly to [12] and [37] by testing the first equation of (2.9) by $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ and integrating by parts, and using the boundary condition $\mathbf{u}_{\mathrm{S}} = \mathbf{0}$ on Γ_{S}, to obtain

$$\frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}}, \boldsymbol{\tau}_{\mathrm{S}}^{\mathrm{d}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\varphi}\rangle_\Sigma + (\boldsymbol{\gamma}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{u}_{\mathrm{S}}, \mathbf{div}(\boldsymbol{\tau}_{\mathrm{S}}))_{\mathrm{S}} + \frac{\rho}{2\nu}((\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} = 0, \quad \forall \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega). \tag{2.11}$$

Above, $\boldsymbol{\varphi}$ represents the trace of $-\mathbf{u}_{\mathrm{S}}$ on Σ, that is $\boldsymbol{\varphi} := -\mathbf{u}_{\mathrm{S}}|_\Sigma \in \mathbf{H}_{00}^{1/2}(\Sigma) := [\mathrm{H}_{00}^{1/2}(\Sigma)]^n$, where

$$\mathrm{H}_{00}^{1/2}(\Sigma) \;:=\; \Big\{ v|_\Sigma : \quad v \in \mathrm{H}^1(\Omega_{\mathrm{S}}) \quad \text{and} \quad v = 0 \quad \text{on} \quad \Gamma_{\mathrm{S}} \Big\},$$

endowed with the norm $\|\psi\|_{1/2,00;\Sigma} := \|E_{0,\mathrm{S}}(\psi)\|_{1/2;\partial\Omega_{\mathrm{S}}}$, with $E_{0,\mathrm{S}} : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{L}^2(\partial\Omega_{\mathrm{S}})$, being the extension operator defined by

$$E_{0,\mathrm{S}}(\psi) \;=\; \begin{cases} \psi & \text{on } \Sigma, \\ 0 & \text{on } \Gamma_{\mathrm{S}}, \end{cases} \qquad \forall\, \psi \in \mathrm{H}^{1/2}(\Sigma)\,.$$

The introduction of the additional unknown $\boldsymbol{\varphi}$ is motivated by the fact that, since $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, equation (2.11) makes sense if $\mathbf{u}_{\mathrm{S}} \in \mathbf{L}^4(\Omega)$, a condition under which its trace cannot be well-defined. Additionally, as noted in [12, Lemma 3.5], it is established that $\boldsymbol{\tau}_{\mathrm{S}}\mathbf{n} \in \mathbf{H}^{-1/2}(\partial\Omega_{\mathrm{S}})$ for all $\boldsymbol{\tau}_{\mathrm{S}} \in$

$\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$, whose restriction to Σ can be formally identified with an element in $\mathbf{H}_{00}^{-1/2}(\Sigma) = [\mathbf{H}_{00}^{1/2}(\Sigma)]'$, also denoted by $\boldsymbol{\tau}_S\mathbf{n}$, through the identity

$$\langle \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\varphi}\rangle_\Sigma = \langle \boldsymbol{\tau}_S\mathbf{n}, \mathbf{E}_{0,S}(\boldsymbol{\varphi})\rangle_{\partial\Omega_S},$$

where $\mathbf{E}_{0,S}$ is the vector version of $E_{0,S}$. This ensures that the second term in (2.11) is well-defined for $\boldsymbol{\varphi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$.

Next, we impose weakly the second and third equations of (2.9) through the following equations

$$(\mathbf{v}_S, \mathbf{div}(\boldsymbol{\sigma}_S))_S \;=\; -(\mathbf{f}_S, \mathbf{v}_S)_S \quad \forall\, \mathbf{v}_S \in \mathbf{L}^4(\Omega) \quad \text{and} \quad (\boldsymbol{\sigma}_S, \boldsymbol{\eta}_S)_S \;=\; 0 \quad \forall\, \boldsymbol{\eta}_S \in \mathbb{L}^2_{\text{skew}}(\Omega_S)\,, \tag{2.12}$$

where

$$\mathbb{L}^2_{\text{skew}}(\Omega_S) := \Big\{ \boldsymbol{\eta}_S \in \mathbb{L}^2(\Omega_S) : \quad \boldsymbol{\eta}_S^{\rm t} = -\boldsymbol{\eta}_S \Big\}\,.$$

Finally, for (2.2), (2.10) and the first equation of (2.4), we proceed similarly to [13] and [37] by introducing the additional unknown $\lambda := p_D|_\Sigma \in \mathrm{H}^{1/2}(\Sigma)$, multiplying by suitable test functions and integrating by parts, in particular the first equation of (2.2), to obtain the equations:

$$\begin{aligned}
(\mathbf{K}^{-1}\mathbf{u}_D, \mathbf{v}_D)_D - \langle \mathbf{v}_D\cdot\mathbf{n}, \lambda\rangle_\Sigma - (p_D, \mathrm{div}(\mathbf{v}_D))_D &= (\mathbf{f}_D, \mathbf{v}_D)_D && \forall\, \mathbf{v}_D \in \mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D),\\
(q_D, \mathrm{div}(\mathbf{u}_D))_D &= 0 && \forall\, q_D \in \mathrm{L}^2(\Omega_D)\,,\\
-\langle \boldsymbol{\varphi}\cdot\mathbf{n}, \xi\rangle_\Sigma - \langle \mathbf{u}_D\cdot\mathbf{n}, \xi\rangle_\Sigma &= 0 && \forall\, \xi \in \mathrm{H}^{1/2}(\Sigma),\\
\langle \boldsymbol{\sigma}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma - \langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda\rangle_\Sigma + \rho\langle \boldsymbol{\varphi}\cdot\mathbf{n}, \boldsymbol{\varphi}\cdot\boldsymbol{\psi}\rangle_\Sigma &= 0 && \forall\, \boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma),
\end{aligned} \tag{2.13}$$

where

$$\langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} \;= \sum_{i=1}^{n-1} \omega_i^{-1} \langle \boldsymbol{\varphi}\cdot\mathbf{t}_i, \boldsymbol{\psi}\cdot\mathbf{t}_i\rangle_\Sigma\,.$$

From the above, we arrive at the initial variational formulation: Find $\boldsymbol{\sigma}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$, $\mathbf{u}_S \in \mathbf{L}^4(\Omega_S)$, $\boldsymbol{\gamma}_S \in \mathbb{L}^2_{\text{skew}}(\Omega_S)$, $\boldsymbol{\varphi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$, $\mathbf{u}_D \in \mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)$, $p_D \in \mathrm{L}^2(\Omega_D)$, and $\lambda \in \mathrm{H}^{1/2}(\Sigma)$, such that (2.11), (2.12), and (2.13) hold. However, it is important to note that if $(\boldsymbol{\sigma}_S, \mathbf{u}_S, \boldsymbol{\gamma}_S, \boldsymbol{\varphi}, \mathbf{u}_D, p_D, \lambda)$ is a solution of the variational problem, then for any $c \in \mathbb{R}$, $(\boldsymbol{\sigma}_S - c\,\mathbb{I}, \mathbf{u}_S, \boldsymbol{\gamma}_S, \boldsymbol{\varphi}, \mathbf{u}_D, p_D + c, \lambda + c)$ is also a solution. To avoid this non-uniqueness, we will seek the pressure p_D in the space

$$\mathrm{L}^2_0(\Omega_D) := \big\{ q_D \in \mathrm{L}^2(\Omega_D) : \quad (q_D, 1)_D = 0 \big\}\,.$$

Notice that the first and second equations of (2.13) hold for $p_D, q_D \in \mathrm{L}^2_0(\Omega_D)$.

2.3 Analysis of the continuous problem

In this section we study the coupled variational system introduced in the previous section. To that end, we first recall that the following decomposition holds (see, for instance, [10, 12, 30])

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega_S) \;=\; \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) \oplus \mathbb{R}\,\mathbb{I}, \tag{2.14}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) \;:=\; \Big\{ \boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S) : \quad (\mathrm{tr}(\boldsymbol{\tau}_S), 1)_S = 0 \Big\}.$$

Following a similar approach to [37] we use this decomposition to redefine the pseudostress tensor as $\boldsymbol{\sigma}_{\mathrm{S}} := \boldsymbol{\sigma}_{\mathrm{S}} + \ell\,\mathbb{I}$, with the new unknowns $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_{\mathrm{S}})$ and $\ell \in \mathbb{R}$, and to additionally rewrite (2.11) and the last equation of (2.13) equivalently, as

$$\begin{aligned}
\frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{S}}^{\mathrm{d}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma + (\boldsymbol{\gamma}_{\mathrm{S}},\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{u}_{\mathrm{S}},\mathbf{div}(\boldsymbol{\tau}_{\mathrm{S}}))_{\mathrm{S}} + \frac{\rho}{2\nu}((\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} &= 0\,,\\
\jmath\,\langle \boldsymbol{\varphi}\cdot\mathbf{n},1\rangle_\Sigma &= 0\,,\\
\langle \boldsymbol{\sigma}_{\mathrm{S}}\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma - \langle \boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_\Sigma + \ell\,\langle \boldsymbol{\psi}\cdot\mathbf{n},1\rangle_\Sigma + \rho\langle \boldsymbol{\varphi}\cdot\mathbf{n},\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\rangle_\Sigma &= 0\,,
\end{aligned}$$

for all $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_{\mathrm{S}})$, $\jmath \in \mathbb{R}$, and $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$. Considering the above, to study the system given by (2.11)–(2.13), in what follows we analyze the equivalent variational formulation: Find $((\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{u}}) \in \mathbb{H}\times\mathbb{Q}$, such that

$$\begin{aligned}
A((\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\varphi}}),(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})) + \mathbf{C}_{\boldsymbol{\varphi}}(\underline{\boldsymbol{\varphi}},\underline{\boldsymbol{\psi}}) + B((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{u}}) + \mathbf{C}_{\mathbf{u}_{\mathrm{S}}}(\underline{\mathbf{u}},\underline{\boldsymbol{\tau}}) &= (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \quad \forall\,(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}) \in \mathbb{H}\,,\\
B((\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{v}}) &= -(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \quad \forall\,\underline{\mathbf{v}} \in \mathbb{Q}\,,
\end{aligned} \tag{2.15}$$

where, for the sake of clarity in the exposition, the spaces, unknowns and test functions have been grouped as:

$$\begin{gathered}
\mathbf{X} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\,,\quad \mathbf{Y} := \mathbf{H}^{1/2}_{00}(\Sigma)\times \mathrm{H}^{1/2}(\Sigma)\times \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})\\
\mathbb{H} := \mathbf{X}\times\mathbf{Y} \quad\text{and}\quad \mathbb{Q} := \mathbf{L}^4(\Omega_{\mathrm{S}})\times \mathrm{L}^2_0(\Omega_{\mathrm{D}})\times\mathbb{R}\,,\\
\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}}) \in \mathbf{X}\,,\quad \underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi},\lambda,\boldsymbol{\gamma}_{\mathrm{S}}) \in \mathbf{Y}\,,\quad \underline{\mathbf{u}} := (\mathbf{u}_{\mathrm{S}},p_{\mathrm{D}},\ell) \in \mathbb{Q}\,,\\
\underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{X}\,,\quad \underline{\boldsymbol{\psi}} := (\boldsymbol{\psi},\xi,\boldsymbol{\eta}_{\mathrm{S}}) \in \mathbf{Y}\,,\quad \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}},q_{\mathrm{D}},\jmath) \in \mathbb{Q}\,,
\end{gathered}$$

where $\mathbf{X}$, $\mathbf{Y}$, $\mathbb{H}$ and $\mathbb{Q}$ are respectively endowed with the norms

$$\begin{gathered}
\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}} := \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}\,,\quad \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} := \|\boldsymbol{\psi}\|_{1/2,00;\Sigma} + \|\xi\|_{1/2;\Sigma} + \|\boldsymbol{\eta}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}\,,\\
\|(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}} := \|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}} + \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}}\,,\quad\text{and}\quad \|\underline{\mathbf{v}}\|_{\mathbb{Q}} := \|\mathbf{v}_{\mathrm{S}}\|_{0,4;\Omega_{\mathrm{S}}} + \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + |\jmath|\,,
\end{gathered}$$

and the bilinear forms $B:\mathbb{H}\times\mathbb{Q}\to\mathbb{R}$ and $A:\mathbb{H}\times\mathbb{H}\to\mathbb{R}$ are given by

$$B((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}}) := (\mathbf{v}_{\mathrm{S}},\mathbf{div}(\boldsymbol{\tau}_{\mathrm{S}}))_{\Omega_{\mathrm{S}}} - (q_{\mathrm{D}},\mathrm{div}(\mathbf{v}_{\mathrm{D}}))_{\Omega_{\mathrm{D}}} + \jmath\,\langle \boldsymbol{\psi}\cdot\mathbf{n},1\rangle_\Sigma\,, \tag{2.16}$$

and

$$A((\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\varphi}}),(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})) := \mathbf{a}(\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\tau}}) + \mathbf{b}(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\varphi}}) + \mathbf{b}(\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\psi}}) - \mathbf{c}(\underline{\boldsymbol{\varphi}},\underline{\boldsymbol{\psi}})\,, \tag{2.17}$$

with $\mathbf{a}:\mathbf{X}\times\mathbf{X}\to\mathbb{R}$, $\mathbf{b}:\mathbf{X}\times\mathbf{Y}\to\mathbb{R}$, and $\mathbf{c}:\mathbf{Y}\times\mathbf{Y}\to\mathbb{R}$ defined, respectively, by

$$\begin{aligned}
\mathbf{a}(\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\tau}}) &:= \frac{1}{2\,\nu}(\boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{S}}^{\mathrm{d}})_{\Omega_{\mathrm{S}}} + (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}}\,,\\
\mathbf{b}(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}) &:= -\langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_\Sigma + \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + (\boldsymbol{\tau}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}})_{\Omega_{\mathrm{S}}}\,,\\
\mathbf{c}(\underline{\boldsymbol{\varphi}},\underline{\boldsymbol{\psi}}) &:= \langle \boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_\Sigma - \langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_\Sigma\,,
\end{aligned} \tag{2.18}$$

whereas for each $(\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}) \in \mathbf{L}^4(\Omega_{\mathrm{S}})\times\mathbf{H}^{1/2}_{00}(\Sigma)$, $\mathbf{C}_{\boldsymbol{\phi}}:\mathbf{Y}\times\mathbf{Y}\to\mathbb{R}$ and $\mathbf{C}_{\mathbf{w}_{\mathrm{S}}}:\mathbb{Q}\times\mathbf{X}\to\mathbb{R}$ are the bilinear forms given by

$$\mathbf{C}_{\boldsymbol{\phi}}(\underline{\boldsymbol{\varphi}},\underline{\boldsymbol{\psi}}) := \rho\,\langle \boldsymbol{\phi}\cdot\mathbf{n},\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\rangle_\Sigma \quad\text{and}\quad \mathbf{C}_{\mathbf{w}_{\mathrm{S}}}(\underline{\mathbf{u}},\underline{\boldsymbol{\tau}}) := \frac{\rho}{2\,\nu}((\mathbf{w}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{S}})_{\Omega_{\mathrm{S}}}. \tag{2.19}$$

Now, due to the nonlinear nature of problem (2.15), arising from the presence of the forms $\mathbf{C}_{\boldsymbol{\phi}}$ and $\mathbf{C}_{\mathbf{w}_{\mathrm{S}}}$ in the system, the analysis will be based on a fixed-point strategy. More precisely, we let $\mathcal{J} : \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma) \to \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma)$ be the operator given by

$$\mathcal{J}(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi}) \,:=\, (\mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi}) \quad \forall\, (\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma)\,, \tag{2.20}$$

where, $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) = ((\underline{\boldsymbol{\sigma}}, (\boldsymbol{\varphi}, \lambda, \boldsymbol{\gamma}_{\mathrm{S}})), (\mathbf{u}_{\mathrm{S}}, p_{\mathrm{D}}, \ell)) \in \mathbb{H} \times \mathbb{Q}$ is the unique solution (to be confirmed below) of the linearized version of problem (2.15):

$$\begin{aligned} A((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), (\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}})) + \mathbf{C}_{\boldsymbol{\phi}}(\underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\psi}}) + B((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{u}}) + \mathbf{C}_{\mathbf{w}_{\mathrm{S}}}(\underline{\mathbf{u}}, \underline{\boldsymbol{\tau}}) \;&=\; (\mathbf{f}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \quad \forall\, (\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}) \in \mathbb{H}, \\ B((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{v}}) \;&=\; -(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \quad \forall\, \underline{\mathbf{v}} \in \mathbb{Q}, \end{aligned} \tag{2.21}$$

and realize that solving (2.15) is equivalent to the fixed-point problem: Find $(\mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi}) \in \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma)$, such that

$$\mathcal{J}(\mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi}) \,=\, (\mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi})\,. \tag{2.22}$$

In light of the above, we now focus on establishing suitable hypotheses under which problem (2.22) admits a unique solution. Before proceeding with this, we first study the well-definiteness of the operator $\mathcal{J}$.

2.3.1 Well-definiteness of the fixed-point operator

To prove that $\mathcal{J}$ is well-defined, in what follows we focus on analyzing the well-posedness of the linear problem (2.21). To that end, we fix $(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma)$, define the bilinear forms $\mathbf{A} : (\mathbb{H} \times \mathbb{Q}) \times (\mathbb{H} \times \mathbb{Q}) \to \mathbb{R}$ and $\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}} : \big(\mathbb{H} \times \mathbb{Q}\big) \times \big(\mathbb{H} \times \mathbb{Q}\big) \to \mathbb{R}$, given respectively by:

$$\mathbf{A}(((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})) \,:=\, A((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), (\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}})) \,+\, B((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{u}}) \,+\, B((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{v}})\,, \tag{2.23}$$

for all $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}$, and

$$\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}(((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})) \,:=\, \mathbf{A}(((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})) + \mathbf{C}_{\mathbf{w}_{\mathrm{S}}}(\underline{\mathbf{u}}, \underline{\boldsymbol{\tau}}) + \mathbf{C}_{\boldsymbol{\phi}}(\underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\psi}})\,, \tag{2.24}$$

for all $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}$, and observe that (2.21) can be rewritten as: Find $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) \in \mathbb{H} \times \mathbb{Q}$, such that

$$\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}(((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})) \,=\, \mathbf{F}((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \quad \forall\, ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}\,, \tag{2.25}$$

where $\mathbf{F} \in \big(\mathbb{H} \times \mathbb{Q}\big)'$ is defined by

$$\mathbf{F}((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \,:=\, (\mathbf{f}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}} - (\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} \quad \forall\, ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}\,. \tag{2.26}$$

Therefore, in what follows we apply the Banach–Nečas–Babuška theorem (see e.g. [27, Theorem 2.6]) to prove that for any sufficiently small $(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma)$, $\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}$, problem (2.25) admits a unique solution. More precisely, in what follows we prove that for any sufficiently small $(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathbf{H}^{1/2}_{00}(\Sigma)$, $\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}$ satisfies the following inf-sup conditions:

$$\sup_{\mathbf{0} \neq ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}} \frac{\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}(((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}))}{\|((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\|} \,\geq\, \vartheta\, \|((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}})\| \quad \forall\, ((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}}) \in \mathbb{H} \times \mathbb{Q}\,, \tag{2.27}$$

with $\vartheta > 0$ and

$$\sup_{((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}})\in\mathbb{H}\times\mathbb{Q}} \mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}(((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}}),((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})) \;>\; 0, \quad \forall\, \mathbf{0} \neq ((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}}) \in \mathbb{H}\times\mathbb{Q}\,. \tag{2.28}$$

To this end, we proceed similarly to [12] by first proving that $\mathbf{A}$ (see (2.23)) satisfies the hypotheses of the Banach–Nečas–Babuška theorem. However, since $\mathbf{A}$ has a saddle-point structure, it is sufficient, according to [27, Proposition 2.36], to show that the bilinear forms A and B fulfill the conditions of the classical Babuška–Brezzi theory (see [27, Theorem 2.34]). We begin by establishing the boundedness of the bilinear forms involved.

Let us first observe that the bilinear forms $\mathbf{a},\mathbf{b},\mathbf{c}$ (cf. (2.18)) that define the bilinear form A (cf. (2.17)), satisfy

$$\big|\mathbf{a}(\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\tau}})\big| \;\leq\; \|\mathbf{a}\|\|\boldsymbol{\sigma}_{\mathrm{S}}\|_{\mathbf{X}}\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{X}}, \quad \big|\mathbf{b}(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\big| \;\leq\; \|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}}, \quad \big|\mathbf{c}(\underline{\boldsymbol{\varphi}},\underline{\boldsymbol{\psi}})\big| \;\leq\; \|\mathbf{c}\|\,\|\underline{\boldsymbol{\varphi}}\|_{\mathbf{Y}}\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}},$$

where $\|\mathbf{a}\| \leq \max\big\{1/(2\nu), \|\mathbf{K}^{-1}\|_{0,\infty;\Omega_{\mathrm{D}}}\big\}$ and $\|\mathbf{c}\|$ is a positive constant that depends on ω_i, with $i\in\{1,\ldots,n-1\}$. Using these estimates, it is easy to see that A satisfies

$$\Big|A((\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\varphi}}),(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}))\Big| \;\leq\; \|A\|\,\|(\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\varphi}})\|_{\mathbb{H}}\|(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}}\,, \tag{2.29}$$

with $\|A\| \leq \max\big\{2, \|\mathbf{a}\|, \|\mathbf{c}\|\big\}$. Now for B (cf. (2.16)), one can readily deduce that

$$\Big|B((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\Big| \;\leq\; \|(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}}\|\underline{\mathbf{v}}\|_{\mathbb{Q}}\,. \tag{2.30}$$

We continue by proving the corresponding inf-sup condition for B (cf. (2.16)).

Lemma 2.1 *There exists $\beta > 0$, such that*

$$\sup_{\mathbf{0}\neq(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\in\mathbb{H}} \frac{B((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})}{\|(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}}} \;\geq\; \beta\,\|\underline{\mathbf{v}}\|_{\mathbb{Q}} \quad \forall\,\underline{\mathbf{v}}\in\mathbb{Q}\,. \tag{2.31}$$

Proof. Analogously to the proof of [13, Lemma 3.2] (see also [37, Lemma 3.6]), we observe that, due to the diagonal character of B (cf. (2.16)), to prove estimate (2.31) it suffices to verify that the following three independent inf-sup conditions hold:

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_{\mathrm{S}}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_{\mathrm{S}})} \frac{(\mathbf{v}_{\mathrm{S}},\mathbf{div}(\boldsymbol{\tau}_{\mathrm{S}}))_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{S}}}} \;\geq\; \beta_1\|\mathbf{v}_{\mathrm{S}}\|_{0,4;\Omega_{\mathrm{S}}} \quad \forall\,\mathbf{v}_{\mathrm{S}}\in\mathbf{L}^4(\Omega_{\mathrm{S}})\,, \tag{2.32}$$

$$\sup_{\mathbf{0}\neq\mathbf{v}_{\mathrm{D}}\in\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})} \frac{(q_{\mathrm{D}},\mathrm{div}(\mathbf{v}_{\mathrm{D}}))_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \;\geq\; \beta_2\,\|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \quad \forall\,q_{\mathrm{D}}\in\mathrm{L}^2_0(\Omega_{\mathrm{D}})\,, \tag{2.33}$$

and

$$\sup_{\mathbf{0}\neq\boldsymbol{\psi}\in\mathbf{H}^{1/2}_{00}(\Sigma)} \frac{\jmath\,\langle\boldsymbol{\psi}\cdot\mathbf{n},1\rangle_{\Sigma}}{\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}} \;\geq\; \beta_3\,|\jmath| \quad \forall\,\jmath\in\mathbb{R}\,, \tag{2.34}$$

with β_1, β_2, $\beta_3 > 0$.

For (2.32) we refer the reader to [12, Lemma 3.3], whereas the inf-sup conditions (2.33) and (2.34) can be found in [13, Lemma 3.2]. □

Now, we let $\mathbb{V}$ be the kernel of B (cf. (2.16)), that is

$$\mathbb{V} \;:=\; \Big\{(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\in\mathbb{H}: \quad B((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}}) = 0 \quad \forall\,\underline{\mathbf{v}}\in\mathbb{Q}\Big\}.$$

Using the definition of B, it is not difficult to see that $\mathbb{V}$ can be characterized as follows

$$\mathbb{V} = \widetilde{\mathbf{X}} \times \widetilde{\mathbf{Y}},$$

where

$$\widetilde{\mathbf{X}} := \widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S) \times \widetilde{\mathbf{H}}_{\Gamma_D}(\mathrm{div};\Omega_D) \quad \text{and} \quad \widetilde{\mathbf{Y}} := \widetilde{\mathbf{H}}_{00}^{1/2}(\Sigma) \times \mathrm{H}^{1/2}(\Sigma) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_S)\,,$$

with

$$\widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S) := \Big\{ \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) : \quad \mathbf{div}(\boldsymbol{\tau}_S) = \mathbf{0} \Big\}\,,$$

$$\widetilde{\mathbf{H}}_{\Gamma_D}(\mathrm{div};\Omega_D) := \Big\{ \mathbf{v}_D \in \mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D) : \quad \mathrm{div}(\mathbf{v}_D) \in \mathrm{P}_0(\Omega_D) \Big\}\,,$$

and

$$\widetilde{\mathbf{H}}_{00}^{1/2}(\Sigma) := \Big\{ \boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma) : \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma = 0 \Big\}\,.$$

Having characterized $\mathbb{V}$, now we turn to proving that A satisfies the hypotheses of the Banach–Nečas–Babuška theorem to conclude that A induces an invertible operator on $\mathbb{V}$. However, given that the bilinear form A exhibits a perturbed saddle-point structure (cf. (2.17)), we will achieve this by applying the following abstract result, which extends [27, Proposition 2.36] to the case of perturbed saddle-point problems.

Theorem 2.2 *Let X and Y be separable and reflexive Banach spaces and let $\tilde{X} \subseteq X$ and $\tilde{Y} \subseteq Y$ be subspaces of X and Y, respectively. In addition, let $a : X \times X \to \mathbb{R}$, $b : X \times Y \to \mathbb{R}$, and $c : Y \times Y \to \mathbb{R}$ be bounded bilinear satisfying*

(i) *there exists $\widetilde{\alpha} > 0$, such that $a(x,x) \,\geq\, \widetilde{\alpha}\, \|x\|_X^2 \quad \forall\, x \in \tilde{X}$,*

(ii) *there exists $\widetilde{\beta} > 0$, such that* $\displaystyle\sup_{0\neq x\in\tilde{X}} \frac{b(x,y)}{\|x\|_X} \,\geq\, \widetilde{\beta}\, \|y\|_Y \quad \forall\, y \in \tilde{Y}$,

(iii) $c(y,y) \,\geq\, 0 \quad \forall\, y \in \tilde{Y}$.

Then, the global bilinear form $\mathcal{A} : (X \times Y) \times (X \times Y) \to \mathbb{R}$ defined by

$$\mathcal{A}((x,y),(z,w)) \,:=\, a(x,z) + b(z,y) + b(x,w) - c(y,w) \quad \forall\, (x,y),(z,w) \in X \times Y\,,$$

satisfies the Banach–Nečas–Babuška conditions on $\tilde{X} \times \tilde{Y}$, namely:

(a) *There exists $\varrho > 0$, such that*

$$S_1(x,y) := \sup_{\mathbf{0}\neq(z,w)\in\tilde{X}\times\tilde{Y}} \frac{\mathcal{A}((x,y),(z,w))}{\|(z,w)\|} \,\geq\, \varrho\, \|(x,y)\| \quad \forall\, (x,y) \in \tilde{X} \times \tilde{Y}\,,$$

(b)

$$S_2(z,w) := \sup_{(x,y)\in\tilde{X}\times\tilde{Y}} \mathcal{A}((x,y),(z,w)) \,>\, 0, \quad \forall\, \mathbf{0} \neq (z,w) \in \tilde{X} \times \tilde{Y}\,.$$

Proof. We begin by proving (a). To that end, we let $(x,y) \in \tilde{X} \times \tilde{Y}$ and make use of the fact that a is a bounded and elliptic bilinear form on $\tilde{X}$, to deduce that there exists a unique $\hat{x} \in \tilde{X}$, such that

$$a(z,\hat{x}) = b(z,y), \quad \forall z \in \tilde{X}. \tag{2.35}$$

Notice that from (ii) and (2.35), y can be bound in terms of $\hat{x}$ as follows

$$\widetilde{\beta}\,\|y\|_Y \le \sup_{0\neq z\in\tilde{X}} \frac{b(z,y)}{\|z\|_X} = \sup_{0\neq z\in\tilde{X}} \frac{a(z,\hat{x})}{\|z\|_X} \le \|a\|\|\hat{x}\|_X,$$

that is

$$\|y\|_Y \le \widetilde{\beta}^{-1}\|a\|\|\hat{x}\|_X, \tag{2.36}$$

with $\|a\| > 0$ being the bounding constant of a. In addition, from (2.35) we observe that the following identities hold

$$a(\hat{x},\hat{x}) = b(\hat{x},y) \quad \text{and} \quad a(x,\hat{x}) = b(x,y). \tag{2.37}$$

In this way, combining (2.36), (2.37), (i) and (iii), we deduce that

$$\begin{aligned} S_1(x,y) \ge\ & \frac{\mathcal{A}((x,y),(\hat{x},-y))}{\|(\hat{x},-y)\|} = \frac{a(x,\hat{x}) + b(\hat{x},y) - b(x,y) + c(y,y)}{\|(\hat{x},y)\|} = \frac{a(\hat{x},\hat{x}) + c(y,y)}{\|\hat{x}\|_X + \|y\|_Y} \\ \ge\ & \frac{\widetilde{\alpha}\|\hat{x}\|_X^2}{\|\hat{x}\|_X + \|y\|_Y} \ge \frac{\widetilde{\alpha}\widetilde{\beta}}{\widetilde{\beta} + \|a\|}\|\hat{x}\|_X\,, \end{aligned}$$

which combined again with (2.36), implies

$$S_1(x,y) \ge \frac{\widetilde{\alpha}\widetilde{\beta}^2}{\|a\|\left(\widetilde{\beta} + \|a\|\right)}\|y\|_Y. \tag{2.38}$$

On the other hand, from (i) and the boundedness of b, we deduce that

$$S_1(x,y) \ge \frac{|\mathcal{A}((x,y),(x,0))|}{\|(x,0)\|} = \frac{|a(x,x) + b(x,y)|}{\|x\|_X} \ge \widetilde{\alpha}\|x\|_X - \|b\|\|y\|_Y, \tag{2.39}$$

with $\|b\| > 0$ being the bounding constant of b. Thus, combining (2.38) and (2.39), we obtain

$$\left(1 + \frac{\widetilde{\alpha}\widetilde{\beta}^2}{\|a\|\|b\|\left(\widetilde{\beta} + \|a\|\right)}\right) S_1(x,y) \ge \frac{\widetilde{\alpha}^2\widetilde{\beta}^2}{\|a\|\|b\|\left(\widetilde{\beta} + \|a\|\right)}\|x\|_X,$$

which implies

$$S_1(x,y) \ge \frac{\widetilde{\alpha}^2\widetilde{\beta}^2}{\|a\|\|b\|\left(\widetilde{\beta} + \|a\|\right) + \widetilde{\alpha}\widetilde{\beta}^2}\|x\|_X. \tag{2.40}$$

Therefore, from (2.38) and (2.40) we easily deduce (a).

Next, for (b) we let $(z,w) \neq \mathbf{0}$ in $\tilde{X}\times\tilde{Y}$, and proceed analogously as for (a) by employing (i) and (ii), to define $\hat{z} \neq 0$ in $\tilde{X}$, satisfying

$$a(\hat{z},v) = b(v,w),\ \forall\, v\in\tilde{X}, \quad \|w\|_Y \le \widetilde{\beta}^{-1}\|a\|\|\hat{z}\|_X, \quad a(\hat{z},\hat{z}) = b(\hat{z},w) \quad \text{and} \quad a(\hat{z},z) = b(z,w). \tag{2.41}$$

Then, noticing that

$$S_2(z,w) \ge \sup_{\substack{(x,y)\in\tilde{X}\times\tilde{Y}\\ \|(x,y)\|=1}} \mathcal{A}((x,y),(z,w)) = \sup_{\mathbf{0}\neq(x,y)\in\tilde{X}\times\tilde{Y}} \frac{\mathcal{A}((x,y),(z,w))}{\|(x,y)\|},$$

we make use of (2.41) and proceed analogously as for (2.38), to conclude that

$$S_2(z,w) \geq \sup_{\mathbf{0}\neq(x,y)\in\tilde{X}\times\tilde{Y}} \frac{\mathcal{A}((x,y),(z,w))}{\|(x,y)\|} \geq \frac{\mathcal{A}((\hat{z},-w),(z,w))}{\|(\hat{z},-w)\|} \geq \frac{\widetilde{\alpha}\widetilde{\beta}^2}{\|a\|\left(\widetilde{\beta}+\|a\|\right)}\|w\|_Y. \tag{2.42}$$

Then, noticing that $w\neq 0$, (b) can be easily deduced from (2.42), which concludes the proof. $\square$

Remark 2.3 *Given* $(z,w)\in\tilde{X}\times\tilde{Y}$*, and proceeding analogously as for* (2.39)*, we deduce that*

$$\sup_{\mathbf{0}\neq(x,y)\in\tilde{X}\times\tilde{Y}} \frac{\mathcal{A}((x,y),(z,w))}{\|(x,y)\|} \geq \frac{\mathcal{A}((z,0),(z,w))}{\|(z,0)\|} = \frac{|a(z,z)+b(z,w)|}{\|x\|_X} \geq \widetilde{\alpha}\|z\|_X - \|b\|\|w\|_Y,$$

which together with (2.42)*, implies*

$$\sup_{\mathbf{0}\neq(x,y)\in\tilde{X}\times\tilde{Y}} \frac{\mathcal{A}((x,y),(z,w))}{\|(x,y)\|} \geq \varrho\|(z,w)\|, \quad \forall (z,w)\in\tilde{X}\times\tilde{Y},$$

where $\varrho>0$ *is the same constant that satisfies* (a) *in Theorem 2.2. This estimate will be employed later on to prove* (2.28)*.*

Now we are in position of proving that A (cf. (2.17)) satisfies the hypotheses of the Banach–Nečas–Babuška theorem, or equivalently, that A induces an invertible operator on $\mathbb{V}$.

Lemma 2.4 *The bilinear form A satisfies the Banach–Nečas–Babuška conditions on $\mathbb{V}$, that is, there exists $\alpha_A>0$, such that*

$$\sup_{\mathbf{0}\neq(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\in\mathbb{V}} \frac{A((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}))}{\|(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}}} \;\geq\; \alpha_A\,\|(\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}})\|_{\mathbb{H}} \quad \forall\,(\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}})\in\mathbb{V}, \tag{2.43}$$

and

$$\sup_{(\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}})\in\mathbb{V}} A((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})) > 0 \quad \forall\,(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\in\mathbb{V}. \tag{2.44}$$

Proof. According to the above, it is clear that to prove (2.43) and (2.44) it suffices to prove that the bilinear forms $\mathbf{a},\mathbf{b}$, and $\mathbf{c}$ defining A (cf. (2.18)) satisfy the hypotheses of Theorem 2.2. To do that, we first observe that using [12, Lemma 3.2] and [37, Lemma 3.2], the following estimates hold

$$C_{\mathrm{S}}\,\|\boldsymbol{\tau}_{\mathrm{S}}\|^2_{\mathbf{div}_{4/3};\Omega_{\mathrm{S}}} \;\leq\; \|\boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \quad \forall\,\boldsymbol{\tau}_{\mathrm{S}}\in\widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_{\mathrm{S}}) \tag{2.45}$$

and

$$C_{\mathrm{D}}\,\|\mathbf{v}_{\mathrm{D}}\|^2_{\mathrm{div};\Omega_{\mathrm{D}}} \;\leq\; \|\mathbf{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}} \quad \forall\,\mathbf{v}_{\mathrm{D}}\in\widetilde{\mathbf{H}}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}), \tag{2.46}$$

with C_{S} and C_{D}, being positive constants. Then, from the definition of the bilinear forms $\mathbf{a}$ and $\mathbf{c}$ (cf. (2.18)), using (2.3), (2.45) and (2.46), we deduce that

$$\mathbf{a}(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\tau}}) \;=\; \frac{1}{2\,\nu}\,\|\boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + C_{\mathbf{K}}\,\|\mathbf{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}} \;\geq\; \alpha_{\mathbf{a}}\,\|\underline{\boldsymbol{\tau}}\|^2_{\mathbf{X}} \quad \forall\,\underline{\boldsymbol{\tau}}\in\widetilde{\mathbf{X}}, \tag{2.47}$$

where $\alpha_{\mathbf{a}} = \min\big\{C_{\mathrm{S}}/(2\,\nu), C_{\mathrm{D}}\,C_{\mathbf{K}}\big\}$, and

$$\mathbf{c}(\underline{\boldsymbol{\psi}},\underline{\boldsymbol{\psi}}) \;=\; \sum_{i=1}^{n-1}\omega_i^{-1}\|\boldsymbol{\psi}\cdot\mathbf{t}_i\|^2_{0,\Sigma} \;\geq\; 0 \quad \forall\,\underline{\boldsymbol{\psi}}\in\widetilde{\mathbf{Y}}. \tag{2.48}$$

In turn, a straightforward adaptation of [13, Lemma 3.3] allows us to obtain the inf-sup condition of $\mathbf{b}$ on $\mathbb{V}$:

$$\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\widetilde{\mathbf{X}}} \frac{\mathbf{b}(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \,\geq\, \beta_{\mathbf{b}}\,\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} \quad \forall\,\underline{\boldsymbol{\psi}}\in\widetilde{\mathbf{Y}}\,. \tag{2.49}$$

In particular, we observe that the inf-sup condition associated with the term $\langle\boldsymbol{\tau}_{\mathrm{S}}\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma$ in the definition of the bilinear form $\mathbf{b}$ (cf. (2.18)) follows exactly as in [33, Lemma 4.3] since $\boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}\in\mathbf{H}^{-1/2}(\partial\Omega_{\mathrm{S}})$ for all $\boldsymbol{\tau}_{\mathrm{S}}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_{\mathrm{S}})$. In this way, (2.47), (2.48), (2.49), and a straightforward application of Theorem 2.2, imply that A satisfies (2.43) and (2.44) with α_A depending only on $\alpha_{\mathbf{a}}$, $\beta_{\mathbf{b}}$, and $\|\mathbf{a}\|$, which concludes the proof. □

As previously announced, from Lemmas 2.1 and 2.4 we obtain that the bilinear form $\mathbf{A}$ (cf. (2.23)) satisfies the Banach–Nečas–Babuška conditions. More precisely, we have the following lemma.

Lemma 2.5

$$\sup_{\mathbf{0}\neq((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\in\mathbb{H}\times\mathbb{Q}} \frac{\mathbf{A}(((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}}),((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}}))}{\|((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\|} \,\geq\, \gamma\,\|((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}})\| \quad \forall\,((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}})\in\mathbb{H}\times\mathbb{Q}, \tag{2.50}$$

with

$$\gamma \,:=\, \frac{(\alpha_A+\beta^2+\|A\|)^2}{\alpha_A\,\beta^2}\,, \tag{2.51}$$

and

$$\sup_{((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}})\in\mathbb{H}\times\mathbb{Q}} \mathbf{A}(((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}}),((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})) \,>\, 0 \quad \forall\,((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\neq\mathbf{0} \quad in \quad \mathbb{H}\times\mathbb{Q}.$$

Proof. The proof follows from Lemmas 2.1 and 2.4 and a straightforward application of [27, Proposition 2.36]. We omit further details. □

Next, we establish the well-posedness of problem (2.21) or equivalently, the well-definiteness of the fixed-point operator $\mathscr{J}$. Before doing that we observe that, owing to the Hölder and Cauchy–Schwarz inequalities, the functional $\mathbf{F}$ ((2.26)) satisfies

$$\big|\mathbf{F}((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\big| \,\leq\, \big(\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}+\|\mathbf{f}_{\mathrm{S}}\|_{0,4/3;\Omega_{\mathrm{S}}}\big)\,\|((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\| \quad \forall\,((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\in\mathbb{H}\times\mathbb{Q}\,. \tag{2.52}$$

In addition, given $(\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi})\in\mathbf{L}^4(\Omega_{\mathrm{S}})\times\mathbf{H}^{1/2}_{00}(\Sigma)$, making use of the estimates

$$\big|((\mathbf{w}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}}\big| \leq n^{1/2}\|\mathbf{w}_{\mathrm{S}}\|_{0,4;\Omega_{\mathrm{S}}}\|\mathbf{u}_{\mathrm{S}}\|_{0,4;\Omega_{\mathrm{S}}}\|\boldsymbol{\tau}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}\,, \quad \forall\,\mathbf{w}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}}\in\mathbf{L}^4(\Omega_{\mathrm{S}}),\boldsymbol{\tau}_{\mathrm{S}}\in\mathbb{H}(\mathbf{div}_{4/3},\Omega_{\mathrm{S}}),$$

$$\big|\langle\boldsymbol{\phi}\cdot\mathbf{n},\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\rangle_\Sigma\big| \leq \|\mathbf{i}_\Sigma\|^3\,\|\boldsymbol{\phi}\|_{1/2,00;\Sigma}\,\|\boldsymbol{\varphi}\|_{1/2,00;\Sigma}\,\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}, \quad \forall\,\boldsymbol{\phi},\boldsymbol{\varphi},\boldsymbol{\psi}\in\mathbf{H}^{1/2}_{00}(\Sigma),$$

where $\mathbf{i}_\Sigma$ is the continuous injection from $\mathbf{H}^{1/2}(\Sigma)$ to $\mathbf{L}^3(\Sigma)$, we obtain

$$\Big|\mathbf{C}_{\mathbf{w}_{\mathrm{S}}}(\underline{\mathbf{u}},\underline{\boldsymbol{\tau}})+\mathbf{C}_{\boldsymbol{\phi}}(\underline{\boldsymbol{\varphi}},\underline{\boldsymbol{\psi}})\Big| \,\leq\, \|\mathbf{C}\|\|(\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi})\|\,\|(\mathbf{u}_{\mathrm{S}},\boldsymbol{\varphi})\|\|(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}}, \tag{2.53}$$

for all $\underline{\mathbf{u}}=(\mathbf{u}_{\mathrm{S}},p_{\mathrm{D}},\ell)\in\mathbb{Q}$, $\underline{\boldsymbol{\varphi}}:=(\boldsymbol{\varphi},\lambda,\boldsymbol{\gamma}_{\mathrm{S}})\in\mathbf{Y}$, $(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\in\mathbb{H}$, with $\|\mathbf{C}\|\leq\rho\,\max\Big\{\|\mathbf{i}_\Sigma\|^3,n^{1/2}/(2\nu)\Big\}$.

Now we are in position of establishing the well-posedness of problem (2.25).

Lemma 2.6 *Let* $(\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi})\in\mathbf{L}^4(\Omega_{\mathrm{S}})\times\mathbf{H}^{1/2}_{00}(\Sigma)$*, be such that*

$$\|\mathbf{C}\|\,\|(\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi})\| \,\leq\, \frac{\gamma}{2}, \tag{2.54}$$

where $\|\mathbf{C}\|$ *is the positive constants satisfying* (2.53) *and* γ *is given by* (2.51). *Then, there exists a unique* $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) \in \mathbb{H} \times \mathbb{Q}$*, solution to* (2.25)*, which satisfies*

$$\|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}})\| \leq \frac{2}{\gamma}\Big(\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{S}}\|_{0,4/3;\Omega_{\mathrm{S}}}\Big). \tag{2.55}$$

Proof. Given $(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathbf{H}_{00}^{1/2}(\Sigma)$ satisfying (2.54), in what follows we prove that $\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}$ satisfies the inf-sup conditions (2.27) and (2.28). To that end, we let $((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}}) \in \mathbb{H} \times \mathbb{Q}$ and combine estimate (2.50) with (2.53), to find that

$$\begin{aligned} &\sup_{\mathbf{0}\neq((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\in\mathbb{H}\times\mathbb{Q}} \frac{\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}(((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}))}{\|((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\|} \\ &= \sup_{\mathbf{0}\neq((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\in\mathbb{H}\times\mathbb{Q}} \frac{\mathbf{A}(((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})) + \mathbf{C}_{\mathbf{w}_{\mathrm{S}}}(\underline{\mathbf{z}}, \underline{\boldsymbol{\tau}}) + \mathbf{C}_{\boldsymbol{\phi}}(\underline{\boldsymbol{\mu}}, \underline{\boldsymbol{\psi}})}{\|((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\|} \\ &\geq \Big(\gamma - \|\mathbf{C}\|\,\|(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi})\|\Big)\,\|((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}})\|. \end{aligned} \tag{2.56}$$

This inequality with assumption (2.54) clearly imply that

$$\sup_{\mathbf{0}\neq((\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{v}})\in\mathbb{H}\times\mathbb{Q}} \frac{\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}(((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}))}{\|((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\|_{\mathbb{H}\times\mathbb{Q}}} \geq \frac{\gamma}{2}\|((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}})\|, \tag{2.57}$$

thus $\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}$ satisfies (2.27), with $\vartheta = \dfrac{\gamma}{2}$.

Now, for (2.28) we first employ Remark 2.3 to deduce that the bilinear form A satisfies

$$\sup_{\mathbf{0}\neq(\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}})\in\mathbb{V}} \frac{A((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), (\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}))}{\|(\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}})\|_{\mathbb{H}}} \geq \alpha_A\,\|(\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}})\|_{\mathbb{H}} \quad \forall\,(\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}) \in \mathbb{V},$$

where $\alpha_A > 0$ is the same constant satisfying (2.43). Then, from this inequality, the inf-sup condition (2.31) and [27, Proposition 2.36] we readily obtain that $\mathbf{A}$ also satisfies

$$\sup_{\mathbf{0}\neq((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}})\in\mathbb{H}\times\mathbb{Q}} \frac{\mathbf{A}(((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}))}{\|((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}})\|} \geq \gamma\,\|((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\| \quad \forall\,((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}. \tag{2.58}$$

In this way, from (2.53), (2.58) and similarly as for (2.56), we deduce that for $\mathbf{0} \neq ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}$, there hold

$$\begin{aligned} \sup_{((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}})\in\mathbb{H}\times\mathbb{Q}} \mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}(((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})) &\geq \sup_{\mathbf{0}\neq((\underline{\boldsymbol{\zeta}},\underline{\boldsymbol{\mu}}),\underline{\mathbf{z}})\in\mathbb{H}\times\mathbb{Q}} \frac{\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}(((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}}), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}))}{\|((\underline{\boldsymbol{\zeta}}, \underline{\boldsymbol{\mu}}), \underline{\mathbf{z}})\|}, \\ &\geq \Big(\gamma - \|\mathbf{C}\|\,\|(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi})\|\Big)\,\|((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\|, \end{aligned}$$

which combined with (2.54) and the fact that $((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \neq \mathbf{0}$, implies that $\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}$ satisfies (2.28).

Finally, if $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) \in \mathbb{H} \times \mathbb{Q}$, is the solution of (2.25) from (2.52) and (2.57) it is clear that (2.55) holds, which concludes the proof. □

2.4 Well-posedness of the continuous problem

Now we provide the main result of this section, namely, the existence and uniqueness of solution of problem (2.15). This result is established in the following theorem.

Theorem 2.7 *Let* $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$ *and* $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^{4/3}(\Omega_{\mathrm{S}})$ *and define the bounded set*

$$\mathbf{W} := \left\{ (\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathbf{H}_{00}^{1/2}(\Sigma) : \quad \|(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi})\| \,\leq\, \frac{2}{\gamma} \left(\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{S}}\|_{0,4/3;\Omega_{\mathrm{S}}} \right) \right\}, \tag{2.59}$$

where γ *is the constant defined in* (2.51)*. Assume that* $\mathbf{f}_{\mathrm{D}}$ *and* $\mathbf{f}_{\mathrm{S}}$ *satisfies*

$$\frac{4\,\|\mathbf{C}\|}{\gamma^2} \left(\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{S}}\|_{0,4/3;\Omega_{\mathrm{S}}} \right) \,<\, 1\,, \tag{2.60}$$

with $\|\mathbf{C}\|$ *satisfying* (2.53)*. Then, there exists a unique* $(\mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi}) \in \mathbf{W}$ *such that* $\mathcal{J}(\mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi}) = (\mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi})$*. Equivalently, there exists a unique* $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) \in \mathbb{H} \times \mathbb{Q}$ *solution to problem* (2.15)*. Moreover,* $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}})$ *satisfies*

$$\|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}})\| \,\leq\, \frac{2}{\gamma} \left(\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{S}}\|_{0,4/3;\Omega_{\mathrm{S}}} \right). \tag{2.61}$$

Proof. We begin by noting that, under assumption (2.60), any $(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi}) \in \mathbf{W}$ satisfies condition (2.54), thereby ensuring the existence of $\mathcal{J}(\mathbf{w}_{\mathrm{S}}, \boldsymbol{\phi})$. Consequently, let $(\mathbf{w}_{\mathrm{S},1}, \boldsymbol{\phi}_1)$, $(\mathbf{w}_{\mathrm{S},2}, \boldsymbol{\phi}_2) \in \mathbf{W}$ and $(\mathbf{u}_{\mathrm{S},1}, \boldsymbol{\varphi}_1)$, $(\mathbf{u}_{\mathrm{S},2}, \boldsymbol{\varphi}_2) \in \mathbf{W}$, be such that

$$\mathcal{J}(\mathbf{w}_{\mathrm{S},1}, \boldsymbol{\phi}_1) = (\mathbf{u}_{\mathrm{S},1}, \boldsymbol{\varphi}_1) \quad \text{and} \quad \mathcal{J}(\mathbf{w}_{\mathrm{S},2}, \boldsymbol{\phi}_2) = (\mathbf{u}_{\mathrm{S},2}, \boldsymbol{\varphi}_2)\,.$$

According to the definition of $\mathcal{J}$ (cf. (2.25)), it follows that there exist unique $((\underline{\boldsymbol{\sigma}}_i, \underline{\boldsymbol{\varphi}}_i), \underline{\mathbf{u}}_i) \in \mathbb{H} \times \mathbb{Q}$, with $i \in \{1, 2\}$, such that for all $((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}$, there hold

$$\mathbf{A}_{\mathbf{w}_{\mathrm{S},i}, \boldsymbol{\phi}_i}(((\underline{\boldsymbol{\sigma}}_i, \underline{\boldsymbol{\varphi}}_i), \underline{\mathbf{u}}_i), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})) \,=\, \mathbf{F}((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\,.$$

Then, subtracting the corresponding equations and making use of (2.24), we deduce that for all $((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}$, there holds

$$\mathbf{A}_{\mathbf{w}_{\mathrm{S},1}, \boldsymbol{\phi}_1}(((\underline{\boldsymbol{\sigma}}_1 - \underline{\boldsymbol{\sigma}}_2, \underline{\boldsymbol{\varphi}}_1 - \underline{\boldsymbol{\varphi}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})) \,=\, \mathbf{C}_{\mathbf{w}_{\mathrm{S},1} - \mathbf{w}_{\mathrm{S},2}}(\underline{\mathbf{u}}_2, \underline{\boldsymbol{\tau}}) + \mathbf{C}_{\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2}(\underline{\boldsymbol{\varphi}}_2, \underline{\boldsymbol{\psi}})$$

Therefore, recalling that $(\mathbf{w}_{\mathrm{S},1}, \boldsymbol{\phi}_1) \in \mathbf{W}$ satisfies (2.6), from the latter identity and the estimates (2.53) and (2.57), we obtain

$$\begin{aligned}
\frac{\gamma}{2}\, \|(\mathbf{u}_{\mathrm{S}1}, \boldsymbol{\varphi}_1) - (\mathbf{u}_{\mathrm{S}2}, \boldsymbol{\varphi}_2)\| \,&\leq\, \sup_{\mathbf{0} \neq ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}} \frac{\mathbf{A}_{\mathbf{w}_{\mathrm{S}1}, \boldsymbol{\phi}_1}(((\underline{\boldsymbol{\sigma}}_1 - \underline{\boldsymbol{\sigma}}_2, \underline{\boldsymbol{\varphi}}_1 - \underline{\boldsymbol{\varphi}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2), ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}))}{\|((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\|} \\
&= \sup_{\mathbf{0} \neq ((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}} \frac{\mathbf{C}_{\mathbf{w}_{\mathrm{S},1} - \mathbf{w}_{\mathrm{S},2}}(\underline{\mathbf{u}}_2, \underline{\boldsymbol{\tau}}) + \mathbf{C}_{\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2}(\underline{\boldsymbol{\varphi}}_2, \underline{\boldsymbol{\psi}})}{\|((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\|} \\
&\leq\, \|\mathbf{C}\|\, \|(\mathbf{u}_{\mathrm{S},2}, \boldsymbol{\varphi}_2)\|\, \|(\mathbf{w}_{\mathrm{S},1}, \boldsymbol{\phi}_1) - (\mathbf{w}_{\mathrm{S},2}, \boldsymbol{\phi}_2)\|\,,
\end{aligned}$$

which together with the fact that $(\mathbf{u}_{\mathrm{S},2}, \boldsymbol{\varphi}_2) \in \mathbf{W}$ (cf. (2.59)), implies

$$\begin{aligned}
\|\mathcal{J}(\mathbf{w}_{\mathrm{S},1}, \boldsymbol{\phi}_1) - \mathcal{J}(\mathbf{w}_{\mathrm{S},2}, \boldsymbol{\phi}_2)\| \,&=\, \|(\mathbf{u}_{\mathrm{S},1}, \boldsymbol{\varphi}_1) - (\mathbf{u}_{\mathrm{S},2}, \boldsymbol{\varphi}_2)\| \\
&\leq\, \frac{4\,\|\mathbf{C}\|}{\gamma^2} \left(\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{S}}\|_{0,4/3;\Omega_{\mathrm{S}}} \right) \|(\mathbf{w}_{\mathrm{S},1}, \boldsymbol{\phi}_1) - (\mathbf{w}_{\mathrm{S},2}, \boldsymbol{\phi}_2)\|\,.
\end{aligned}$$

The latter, together with the assumption (2.60) and the Banach fixed-point theorem implies that $\mathcal{J}$ has a unique fixed-point in $\mathbf{W}$, which equivalently implies that there exists a unique $((\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{u}}) \in \mathbb{H}\times\mathbb{Q}$ satisfying (2.15).

Finally, estimate (2.61) follows from the fact that $((\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{u}}) \in \mathbb{H}\times\mathbb{Q}$ can be seen as the solution of (2.21), with $(\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}) = (\mathbf{u}_{\mathrm{S}},\boldsymbol{\varphi})$, thus $((\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{u}})$ satisfies (2.55). □

3 Galerkin scheme

In this section we introduce the Galerkin scheme associated to problem (2.15) and analyze its well-posedness by establishing suitable assumptions on the discrete subspaces involved.

3.1 Preliminaries

We begin by selecting a set of arbitrary discrete subspaces, namely

$$\begin{gathered}\mathbb{H}^{\mathrm{S}}_h \subseteq \mathbb{H}(\mathbf{div}_{4/3};\Omega_{\mathrm{S}})\,, \quad \mathbf{H}^{\mathrm{D}}_h \subseteq \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\,, \quad \mathbf{L}^{\mathrm{S}}_h \subseteq \mathbf{L}^4(\Omega_{\mathrm{S}})\,, \quad \mathrm{L}^{\mathrm{D}}_h \subseteq \mathrm{L}^2(\Omega_{\mathrm{D}})\,, \\ \boldsymbol{\Lambda}^{\mathrm{S}}_h \subseteq \mathbf{H}^{1/2}_{00}(\Sigma)\,, \quad \Lambda^{\mathrm{D}}_h \subseteq \mathrm{H}^{1/2}(\Sigma)\,, \quad \mathbb{S}_h \subseteq \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})\,.\end{gathered} \tag{3.1}$$

Then, letting

$$\mathbb{H}^{\mathrm{S}}_{h,0} := \mathbb{H}^{\mathrm{S}}_h \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_{\mathrm{S}}) \quad \text{and} \quad \mathrm{L}^{\mathrm{D}}_{h,0} := \mathrm{L}^{\mathrm{D}}_h \cap \mathrm{L}^2_0(\Omega_{\mathrm{D}})\,, \tag{3.2}$$

and grouping the discrete spaces, unknowns and test functions as follows

$$\begin{gathered}\mathbf{X}_h := \mathbb{H}^{\mathrm{S}}_{h,0} \times \mathbf{H}^{\mathrm{D}}_h\,, \quad \mathbf{Y}_h := \boldsymbol{\Lambda}^{\mathrm{S}}_h \times \Lambda^{\mathrm{D}}_h \times \mathbb{S}_h \\ \mathbb{H}_h := \mathbf{X}_h \times \mathbf{Y}_h \quad \text{and} \quad \mathbb{Q}_h := \mathbf{L}^{\mathrm{S}}_h \times \mathrm{L}^{\mathrm{D}}_{h,0} \times \mathbb{R}\,, \\ \underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{\mathrm{S},h},\mathbf{u}_{\mathrm{D},h}) \in \mathbf{X}_h\,, \quad \underline{\boldsymbol{\varphi}}_h := (\boldsymbol{\varphi}_h,\lambda_h,\boldsymbol{\gamma}_{\mathrm{S},h}) \in \mathbf{Y}_h\,, \quad \underline{\mathbf{u}}_h := (\mathbf{u}_{\mathrm{S},h},p_{\mathrm{D},h},\ell_h) \in \mathbb{Q}_h\,, \\ \underline{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{\mathrm{S},h},\mathbf{v}_{\mathrm{D},h}) \in \mathbf{X}_h\,, \quad \underline{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h,\xi_h,\boldsymbol{\eta}_{\mathrm{S},h}) \in \mathbf{Y}_h\,, \quad \underline{\mathbf{v}}_h := (\mathbf{v}_{\mathrm{S},h},q_{\mathrm{D},h},\jmath_h) \in \mathbb{Q}_h\,,\end{gathered} \tag{3.3}$$

the Galerkin scheme associated with (2.15) reads: Find $((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\begin{aligned} A((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h)) + \mathbf{C}_{\boldsymbol{\varphi}_h}(\underline{\boldsymbol{\varphi}}_h,\underline{\boldsymbol{\psi}}_h) + B((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{u}}_h) + \mathbf{C}_{\mathbf{u}_{\mathrm{S},h}}(\underline{\mathbf{u}}_h,\underline{\boldsymbol{\tau}}_h) &= (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D},h})_{\mathrm{D}}\,, \\ B((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{v}}_h) &= -(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S},h})_{\mathrm{S}}\,, \end{aligned} \tag{3.4}$$

for all $((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$.

Next, we proceed analogously to [37, Section 4] (see also [13], [39], [19]) and derive suitable hypotheses on the spaces (3.1) ensuring the well-posedness of (3.4). We begin by noticing that, in order to have meaningful subspaces $\mathbb{H}^{\mathrm{S}}_{h,0}$ and $\mathrm{L}^{\mathrm{D}}_{h,0}$ we need to be able to eliminate multiples of the identity matrix and constant polynomials from $\mathbb{H}^{\mathrm{S}}_h$ and $\mathrm{L}^{\mathrm{D}}_h$, respectively. This requirement is certainly satisfied if we assume:

(H.0) $[\mathrm{P}_0(\Omega_{\mathrm{S}})]^{n\times n} \subseteq \mathbb{H}^{\mathrm{S}}_h$ and $\mathrm{P}_0(\Omega_{\mathrm{D}}) \subseteq \mathrm{L}^{\mathrm{D}}_h$.

In particular, it follows that $\mathbb{I} \in \mathbb{H}^{\mathrm{S}}_h$ for all h, which implies that the following decomposition holds

$$\mathbb{H}^{\mathrm{S}}_h = \mathbb{H}^{\mathrm{S}}_{h,0} \oplus \mathrm{P}_0(\Omega_{\mathrm{S}})\mathbb{I}\,.$$

Now, following the same diagonal argument utilized in the proof of Lemma 2.1, we observe that the discrete version of the inf-sup condition (2.31) of B (cf. (2.16)), i.e., there exists $\beta_{\mathtt{d},B} > 0$ such that

$$\sup_{\mathbf{0}\neq(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h)\in\mathbb{H}_h} \frac{B((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h)}{\|(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h)\|_{\mathbb{H}}} \,\geq\, \beta_{\mathtt{d}}\,\|\underline{\mathbf{v}}_h\|_{\mathbb{Q}} \quad \forall\,\underline{\mathbf{v}}_h\in\mathbb{Q}_h\,, \tag{3.5}$$

holds if we assume:

($\mathbf{H}$.**1**) There exist $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}} > 0$, independent of h, and there exists $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}_{00}(\Sigma)$, such that

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_{\mathrm{S},h}\in\mathbb{H}^{\mathrm{S}}_{h,0}} \frac{(\mathbf{v}_{\mathrm{S},h},\mathbf{div}(\boldsymbol{\tau}_{\mathrm{S},h}))_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{S}}}} \,\geq\, \beta_{1,\mathtt{d}}\,\|\mathbf{v}_{\mathrm{S},h}\|_{0,4;\Omega_{\mathrm{S}}} \quad \forall\,\mathbf{v}_{\mathrm{S},h}\in\mathbf{L}^{\mathrm{S}}_h\,, \tag{3.6}$$

$$\sup_{\mathbf{0}\neq\mathbf{v}_{\mathrm{D},h}\in\mathbf{H}^{\mathrm{D}}_h} \frac{(q_{\mathrm{D},h},\mathrm{div}(\mathbf{v}_{\mathrm{D},h}))_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \,\geq\, \beta_{2,\mathtt{d}}\,\|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \quad \forall\,q_{\mathrm{D},h}\in\mathrm{L}^{\mathrm{D}}_{h,0}\,, \tag{3.7}$$

$$\boldsymbol{\psi}_0\in\boldsymbol{\Lambda}^{\mathrm{S}}_h \quad \forall h \quad \text{and} \quad \langle\boldsymbol{\psi}_0\cdot\mathbf{n},1\rangle_\Sigma\neq 0\,. \tag{3.8}$$

In particular, note that (3.8) implies the inf-sup condition

$$\sup_{\mathbf{0}\neq\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}^{\mathrm{S}}_h} \frac{\jmath_h\,\langle\boldsymbol{\psi}_h\cdot\mathbf{n},1\rangle_\Sigma}{\|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma}} \,\geq\, \beta_{3,\mathtt{d}}\,|\jmath_h| \quad \forall\,\jmath_h\in\mathbb{R}\,.$$

We now look at the discrete kernel of B, which is defined by

$$\mathbb{V}_h \,:=\, \Big\{(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h)\in\mathbb{H}_h: \quad B((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h) \,=\, 0 \quad \forall\,\underline{\mathbf{v}}_h\in\mathbb{Q}_h\Big\}\,.$$

In order to have a more explicit definition of $\mathbb{V}_h$, we introduce the following assumption:

($\mathbf{H}$.**2**) $\mathbf{div}(\mathbb{H}^{\mathrm{S}}_h)\subseteq\mathbf{L}^{\mathrm{S}}_h$ and $\mathrm{div}(\mathbf{H}^{\mathrm{D}}_h)\subseteq\mathrm{L}^{\mathrm{D}}_h$.

It follows from ($\mathbf{H}$.**2**) and the definition of B (cf. (2.16)), that $\mathbb{V}_h=\widetilde{\mathbf{X}}_h\times\widetilde{\mathbf{Y}}_h$, where

$$\widetilde{\mathbf{X}}_h \,:=\, \widetilde{\mathbb{H}}^{\mathrm{S}}_{h,0}\times\widetilde{\mathbf{H}}^{\mathrm{D}}_h \quad \text{and} \quad \widetilde{\mathbf{Y}}_h \,:=\, \widetilde{\boldsymbol{\Lambda}}^{\mathrm{S}}_h\times\Lambda^{\mathrm{D}}_h\times\mathbb{S}_h\,,$$

with

$$\widetilde{\mathbb{H}}^{\mathrm{S}}_{h,0} \,:=\, \Big\{\boldsymbol{\tau}_{\mathrm{S},h}\in\mathbb{H}^{\mathrm{S}}_{h,0}: \quad \mathbf{div}(\boldsymbol{\tau}_{\mathrm{S},h}) \,=\, \mathbf{0}\Big\}, \quad \widetilde{\mathbf{H}}^{\mathrm{D}}_h \,:=\, \Big\{\mathbf{v}_{\mathrm{D},h}\in\mathbf{H}^{\mathrm{D}}_h: \quad \mathrm{div}(\mathbf{v}_{\mathrm{D},h})\in\mathrm{P}_0(\Omega_{\mathrm{D}})\Big\}\,,$$
$$\text{and} \quad \widetilde{\boldsymbol{\Lambda}}^{\mathrm{S}}_h \,:=\, \Big\{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}^{\mathrm{S}}_h: \quad \langle\boldsymbol{\psi}_h\cdot\mathbf{n},1\rangle_\Sigma \,=\, 0\Big\}\,.$$

In particular, it readily follows that $\mathbb{V}_h\subset\mathbb{V}$. In addition, defining the subspace

$$\widetilde{\mathbb{H}}^{\mathrm{S}}_h \,:=\, \Big\{\boldsymbol{\tau}_{\mathrm{S},h}\in\mathbb{H}^{\mathrm{S}}_h: \quad \mathbf{div}(\boldsymbol{\tau}_{\mathrm{S},h}) \,=\, \mathbf{0}\Big\}\,,$$

we observe that the discrete version of the inf-sup condition (2.49) of $\mathbf{b}$ (cf. (2.18)), holds if we assume:

($\mathbf{H}$.**3**) There exist $\beta^{\mathrm{S}}_{\Sigma,\mathtt{d}},\beta^{\mathrm{D}}_{\Sigma,\mathtt{d}}>0$, independent of h, such that

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_{\mathrm{S},h}\in\widetilde{\mathbb{H}}^{\mathrm{S}}_h} \frac{\langle\boldsymbol{\tau}_{\mathrm{S},h}\mathbf{n},\boldsymbol{\psi}_h\rangle_\Sigma+(\boldsymbol{\tau}_{\mathrm{S},h},\boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{S}}}} \,\geq\, \beta^{\mathrm{S}}_{\Sigma,\mathtt{d}}\Big(\|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma}+\|\boldsymbol{\eta}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}\Big)\,, \tag{3.9}$$

for all $(\boldsymbol{\eta}_{\mathrm{S},h}, \boldsymbol{\psi}_h) \in \mathbb{S}_h \times \widetilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}$, and

$$\sup_{\mathbf{0}\neq\mathbf{v}_{\mathrm{D},h}\in\widetilde{\mathbf{H}}_h^{\mathrm{D}}} \frac{\langle \mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \,\geq\, \beta_{\Sigma,\mathtt{d}}^{\mathrm{D}}\,\|\xi_h\|_{1/2,\Sigma} \quad \forall\,\xi_h \in \Lambda_h^{\mathrm{D}}\,. \tag{3.10}$$

In particular, given $(\boldsymbol{\eta}_{\mathrm{S},h}, \boldsymbol{\psi}_h) \in \mathbb{S}_h \times \widetilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}$, we observe that (3.9) and the fact that $\langle \boldsymbol{\psi}_h\cdot\mathbf{n}, 1\rangle_\Sigma = 0$, imply

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_{\mathrm{S},h}\in\widetilde{\mathbb{H}}_{h,0}^{\mathrm{S}}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma + (\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{S}}}} \,\geq\, \beta_{\Sigma,\mathtt{d}}^{\mathrm{S}}\left(\|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma} + \|\boldsymbol{\eta}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}\right).$$

We observe that proceeding as in Lemma 2.4, employing hypotheses (**H.0**)-(**H.3**) and Theorem 2.2, we can obtain the following discrete inf-sup condition for A.

Lemma 3.1 *Assume that the hypotheses* (**H.0**), (**H.1**), (**H.2**) *and* (**H.3**) *hold. Then, there exists a constant $\alpha_{A,\mathtt{d}} > 0$ depending only on $\alpha_{\mathbf{a}}$, $\beta_{\mathbf{b},\mathtt{d}}$, and $\|A\|$, such that*

$$\sup_{\mathbf{0}\neq(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h)\in\mathbb{V}_h} \frac{A((\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\mu}}_h),(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h))}{\|(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h)\|_{\mathbb{H}}} \,\geq\, \alpha_{A,\mathtt{d}}\,\|(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\mu}}_h)\|_{\mathbb{H}} \quad \forall\,(\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\mu}}_h) \in \mathbb{V}_h\,. \tag{3.11}$$

We end this section by noting that (2.29), (2.30), (3.5) (cf. hypothesis (**H.1**)), (3.11), [27, Proposition 2.36] and similar arguments to the ones employed to derive (2.50), we have that the bilinear form $\mathbf{A}$ (cf. (2.23)) satisfies the following discrete inf-sup condition:

$$\sup_{\mathbf{0}\neq((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h)\in\mathbb{H}_h\times\mathbb{Q}_h} \frac{\mathbf{A}(((\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\mu}}_h),\underline{\mathbf{z}}_h),((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h))}{\|((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h)\|} \,\geq\, \gamma_{\mathtt{d}}\,\|((\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\mu}}_h),\underline{\mathbf{z}}_h)\|\,, \tag{3.12}$$

for all $((\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\mu}}_h),\underline{\mathbf{z}}_h) \in \mathbb{H}_h\times\mathbb{Q}_h$, where

$$\gamma_{\mathtt{d}} \,:=\, \frac{(\alpha_{A,\mathtt{d}} + \beta_{\mathtt{d}}^2 + \|A\|)^2}{\alpha_{A,\mathtt{d}}\,\beta_{\mathtt{d}}^2}\,. \tag{3.13}$$

3.2 Solvability analysis of the discrete problem

Now, let $\mathscr{J}_{\mathtt{d}} : \mathbf{L}_h^{\mathrm{S}} \times \boldsymbol{\Lambda}_h^{\mathrm{S}} \to \mathbf{L}_h^{\mathrm{S}} \times \boldsymbol{\Lambda}_h^{\mathrm{S}}$ the discrete version of the fixed-point operator (2.20), given by

$$\mathscr{J}_{\mathtt{d}}(\mathbf{w}_{\mathrm{S},h}, \boldsymbol{\phi}_h) \,=\, (\mathbf{u}_{\mathrm{S},h}, \boldsymbol{\varphi}_h) \quad \forall\,(\mathbf{w}_{\mathrm{S},h}, \boldsymbol{\phi}_h) \in \mathbf{L}_h^{\mathrm{S}} \times \boldsymbol{\Lambda}_h^{\mathrm{S}}\,,$$

where, $((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h) = ((\underline{\boldsymbol{\sigma}}_h,(\boldsymbol{\varphi}_h,\lambda_h,\boldsymbol{\gamma}_{\mathrm{S},h})),(\mathbf{u}_{\mathrm{S},h},p_{\mathrm{D},h},\ell_h)) \in \mathbb{H}_h\times\mathbb{Q}_h$ is the unique solution (to be confirmed below) of the linearized version of problem (3.4):

$$\begin{aligned}
A((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h)) + \mathbf{C}_{\boldsymbol{\phi}_h}(\underline{\boldsymbol{\varphi}}_h,\underline{\boldsymbol{\psi}}_h) + B((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{u}}_h) + \mathbf{C}_{\mathbf{w}_{\mathrm{S},h}}(\underline{\mathbf{u}}_h,\underline{\boldsymbol{\tau}}_h) \;&=\; (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D},h})_{\mathrm{D}}\,,\\
B((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{v}}_h) \;&=\; -(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S},h})_{\mathrm{S}}\,,
\end{aligned} \tag{3.14}$$

for all $((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h) \in \mathbb{H}_h\times\mathbb{Q}_h$. Equivalently, and similarly to (2.25), we deduce that (3.14) can be rewritten as: Find $((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h) \in \mathbb{H}_h\times\mathbb{Q}_h$, such that

$$\mathbf{A}_{\mathbf{w}_{\mathrm{S},h},\boldsymbol{\phi}_h}(((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h),((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h)) \,=\, \mathbf{F}((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h) \quad \forall\,((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h) \in \mathbb{H}_h\times\mathbb{Q}_h\,, \tag{3.15}$$

where the functional $\mathbf{F}$ is defined as in (2.26), whereas the bilinear form $\mathbf{A}_{\mathbf{w}_{S,h},\boldsymbol{\phi}_h}$ is the bilinear form given by (2.24) (with $(\mathbf{w}_{S,h},\boldsymbol{\phi}_h)$ instead of $(\mathbf{w}_S,\boldsymbol{\phi})$). Therefore, solving (3.4) is equivalent to seeking a fixed point of the operator $\mathscr{J}_{\mathtt{d}}$, that is: Find $(\mathbf{u}_{S,h},\boldsymbol{\varphi}_h)\in\mathbf{L}_h^S\times\boldsymbol{\Lambda}_h^S$ such that

$$\mathscr{J}_{\mathtt{d}}(\mathbf{u}_{S,h},\boldsymbol{\varphi}_h) \,=\, (\mathbf{u}_{S,h},\boldsymbol{\varphi}_h)\,. \tag{3.16}$$

The following lemma stablishes the well-definiteness of $\mathscr{J}_{\mathtt{d}}$.

Lemma 3.2 *Assume that the hypotheses* (**H.0**)–(**H.3**) *hold. Let* $(\mathbf{w}_{S,h},\boldsymbol{\phi}_h)\in\mathbf{L}_h^S\times\boldsymbol{\Lambda}_h^S$*, be such that*

$$\|\mathbf{C}\|\,\|(\mathbf{w}_{S,h},\boldsymbol{\phi}_h)\|\,\leq\,\frac{\gamma_{\mathtt{d}}}{2}, \tag{3.17}$$

where $\|\mathbf{C}\|$ *is the positive constants satisfying* (2.53) *and* $\gamma_{\mathtt{d}}$ *is given by* (3.13)*. Then, there exists a unique* $((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h)\in\mathbb{H}_h\times\mathbb{Q}_h$*, solution to* (3.15)*, which satisfies*

$$\|((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h)\|\,\leq\,\frac{2}{\gamma_{\mathtt{d}}}\Big(\|\mathbf{f}_D\|_{0,\Omega_D}+\|\mathbf{f}_S\|_{0,4/3;\Omega_S}\Big)\,. \tag{3.18}$$

Proof. Given $(\mathbf{w}_{S,h},\boldsymbol{\phi}_h)\in\mathbf{L}_h^S\times\boldsymbol{\Lambda}_h^S$ we proceed analogously to the proof of Lemma 2.6 and utilize (2.53), (3.12) and (3.17) to deduce that $\mathbf{A}_{\mathbf{w}_{S,h},\boldsymbol{\phi}_h}$ (cf. (2.24)) satisfies the discrete inf-sup condition

$$\sup_{\mathbf{0}\neq((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h)\in\mathbb{H}_h\times\mathbb{Q}_h}\frac{\mathbf{A}_{\mathbf{w}_{S,h},\boldsymbol{\phi}_h}(((\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\mu}}_h),\underline{\mathbf{z}}_h),((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h))}{\|((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h)\|}\,\geq\,\frac{\gamma_{\mathtt{d}}}{2}\,\|((\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\mu}}_h),\underline{\mathbf{z}}_h)\|\,, \tag{3.19}$$

for all $((\underline{\boldsymbol{\zeta}}_h,\underline{\boldsymbol{\mu}}_h),\underline{\mathbf{z}}_h)\in\mathbb{H}_h\times\mathbb{Q}_h$ and $\gamma_{\mathtt{d}}$ defined in (3.13). Therefore, owing to the fact that for finite dimensional linear problems, surjectivity and injectivity are equivalent, from (3.19) and the Banach–Nečas–Babuška theorem we obtain that there exists a unique $((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h)\in\mathbb{H}_h\times\mathbb{Q}_h$ satisfying (3.15) and hence $\mathscr{J}_{\mathtt{d}}(\mathbf{w}_{S,h},\boldsymbol{\phi}_h)=(\mathbf{u}_{S,h},\boldsymbol{\varphi}_h)$ is well defined. Finally, if $((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h)\in\mathbb{H}_h\times\mathbb{Q}_h$, is the solution of (3.15) from (2.52) and (3.19) it is clear that (3.18) holds, which concludes the proof. □

The following theorem provides the main result of this section, namely, existence and uniqueness of solution of the fixed-point problem (3.16), or equivalently, the well-posedness of problem (3.4).

Theorem 3.3 *Assume that the hypotheses* (**H.0**)–(**H.3**) *hold, let us define the bounded set*

$$\mathbf{W}_{\mathtt{d}}\,:=\,\Big\{(\mathbf{w}_{S,h},\boldsymbol{\phi}_h)\in\mathbf{L}_h^S\times\boldsymbol{\Lambda}_h^S:\quad\|(\mathbf{w}_{S,h},\boldsymbol{\phi}_h)\|\,\leq\,\frac{2}{\gamma_{\mathtt{d}}}\Big(\|\mathbf{f}_D\|_{0,\Omega_D}+\|\mathbf{f}_S\|_{0,4/3;\Omega_S}\Big)\Big\}\,. \tag{3.20}$$

Assume in addition that

$$\frac{4\,\|\mathbf{C}\|}{\gamma_{\mathtt{d}}^2}\Big(\|\mathbf{f}_D\|_{0,\Omega_D}+\|\mathbf{f}_S\|_{0,4/3;\Omega_S}\Big)\,<\,1\,, \tag{3.21}$$

with $\|\mathbf{C}\|$ *and* $\gamma_{\mathtt{d}}$ *satisfying* (2.53) *and* (3.13)*, respectively. Then, the operator* $\mathscr{J}_{\mathtt{d}}$ *has a unique fixed-point* $(\mathbf{u}_{S,h},\boldsymbol{\varphi}_h)\in\mathbf{W}_{\mathtt{d}}$*. Equivalently, the problem* (3.4) *has a unique solution* $((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h)\in\mathbb{H}_h\times\mathbb{Q}_h$*. Moreover, there hold*

$$\|((\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{u}}_h)\|\,\leq\,\frac{2}{\gamma_{\mathtt{d}}}\Big(\|\mathbf{f}_D\|_{0,\Omega_D}+\|\mathbf{f}_S\|_{0,4/3;\Omega_S}\Big)\,. \tag{3.22}$$

Proof. First we observe that, as in its continuous counterpart in Theorem 2.7, assumption (3.21) ensures the well-definedness of $\mathscr{J}_{\mathtt{d}}$. Next, adapting the arguments employed in Theorem 2.7 to the present discrete setting, we can obtain the following estimate

$$\|\mathscr{J}_{\mathtt{d}}(\mathbf{w}_{S1},\boldsymbol{\phi}_1)-\mathscr{J}_{\mathtt{d}}(\mathbf{w}_{S2},\boldsymbol{\phi}_2)\|\,\leq\,\frac{4\,\|\mathbf{C}\|}{\gamma_{\mathtt{d}}^2}\Big(\|\mathbf{f}_D\|_{0,\Omega_D}+\|\mathbf{f}_S\|_{0,4/3;\Omega_S}\Big)\|(\mathbf{w}_{S1},\boldsymbol{\phi}_1)-(\mathbf{w}_{S2},\boldsymbol{\phi}_2)\|\,,$$

for all $(\mathbf{w}_{S1},\boldsymbol{\phi}_1),(\mathbf{w}_{S2},\boldsymbol{\phi}_2)\in\mathbf{W}_{\mathtt{d}}$. In this way, using estimate (3.21) we obtain that $\mathscr{J}_{\mathtt{d}}$ is a contraction mapping on $\mathbf{W}_{\mathtt{d}}$, thus problem (3.16), or equivalently (3.4) is well-posed. Finally, analogously to (3.18) we can obtain (3.22), which concludes the proof. □

3.3 A priori error analysis

In this section we establish the corresponding Céa estimate of the Galerkin scheme (3.4). To that end, and in order to simplify the subsequent analysis, we write $\mathbf{e}_{\underline{\boldsymbol{\sigma}}} = \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h$, $\mathbf{e}_{\underline{\boldsymbol{\varphi}}} = \underline{\boldsymbol{\varphi}} - \underline{\boldsymbol{\varphi}}_h$ and $\mathbf{e}_{\underline{\mathbf{u}}} = \underline{\mathbf{u}} - \underline{\mathbf{u}}_h$. Next, given arbitrary $((\widehat{\underline{\boldsymbol{\tau}}}_h, \widehat{\underline{\boldsymbol{\psi}}}_h), \widehat{\underline{\mathbf{v}}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$, we decompose the errors into

$$\mathbf{e}_{\underline{\boldsymbol{\sigma}}} = \boldsymbol{\xi}_{\underline{\boldsymbol{\sigma}}} + \boldsymbol{\chi}_{\underline{\boldsymbol{\sigma}}}, \quad \mathbf{e}_{\underline{\boldsymbol{\varphi}}} = \boldsymbol{\xi}_{\underline{\boldsymbol{\varphi}}} + \boldsymbol{\chi}_{\underline{\boldsymbol{\varphi}}}, \quad \text{and} \quad \mathbf{e}_{\underline{\mathbf{u}}} = \boldsymbol{\xi}_{\underline{\mathbf{u}}} + \boldsymbol{\chi}_{\underline{\mathbf{u}}}\,, \tag{3.23}$$

with

$$\begin{gathered} \boldsymbol{\xi}_{\underline{\boldsymbol{\sigma}}} = \underline{\boldsymbol{\sigma}} - \widehat{\underline{\boldsymbol{\tau}}}_h\,, \quad \boldsymbol{\xi}_{\underline{\boldsymbol{\varphi}}} = \underline{\boldsymbol{\varphi}} - \widehat{\underline{\boldsymbol{\psi}}}_h\,, \quad \boldsymbol{\xi}_{\underline{\mathbf{u}}} = \underline{\mathbf{u}} - \widehat{\underline{\mathbf{v}}}_h\,, \\ \boldsymbol{\chi}_{\underline{\boldsymbol{\sigma}}} = \widehat{\underline{\boldsymbol{\tau}}}_h - \underline{\boldsymbol{\sigma}}_h\,, \quad \boldsymbol{\chi}_{\underline{\boldsymbol{\varphi}}} = \widehat{\underline{\boldsymbol{\psi}}}_h - \underline{\boldsymbol{\varphi}}_h\,, \quad \boldsymbol{\chi}_{\underline{\mathbf{u}}} = \widehat{\underline{\mathbf{v}}}_h - \underline{\mathbf{u}}_h\,. \end{gathered} \tag{3.24}$$

Consequently, subtracting (2.15) and (3.4), and using the definition of the bilinear form $\mathbf{A}$ (cf. (2.23)), we deduce the Galerkin orthogonality property:

$$\mathbf{A}(((\mathbf{e}_{\underline{\boldsymbol{\sigma}}}, \mathbf{e}_{\underline{\boldsymbol{\varphi}}}), \mathbf{e}_{\underline{\mathbf{u}}}), ((\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{v}}_h)) + \mathbf{C}_{\mathbf{u}_{\mathrm{S}}}(\underline{\mathbf{u}}, \underline{\boldsymbol{\tau}}_h) - \mathbf{C}_{\mathbf{u}_{\mathrm{S},h}}(\underline{\mathbf{u}}_h, \underline{\boldsymbol{\tau}}_h) + \mathbf{C}_{\boldsymbol{\varphi}}(\underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\psi}}_h) - \mathbf{C}_{\boldsymbol{\varphi}_h}(\underline{\boldsymbol{\varphi}}_h, \underline{\boldsymbol{\psi}}_h) \;=\; 0\,, \tag{3.25}$$

for all $((\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$.

We now provide the main result of this section.

Theorem 3.4 *In addition to the hypotheses of Theorem 2.7 and Theorem 3.3, assume that*

$$\frac{4\,\|\mathbf{C}\|}{\gamma\,\gamma_{\mathtt{d}}}\Big(\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\mathbf{f}_{\mathrm{S}}\|_{0,4/3;\Omega_{\mathrm{S}}}\Big) \;\leq\; \frac{1}{2}\,, \tag{3.26}$$

with γ *and* $\gamma_{\mathtt{d}}$ *defined in* (2.51) *and* (3.13), *respectively. Let* $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) \in \mathbb{H} \times \mathbb{Q}$ *and* $((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ *be the unique solutions of* (2.15) *and* (3.4), *respectively. Then, there exists* $C > 0$, *independent of* h *and the continuous and discrete solutions, such that*

$$\|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) - ((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h)\| \;\leq\; C \inf_{((\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h} \|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) - ((\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{v}}_h)\|\,. \tag{3.27}$$

Proof. First, for sake of simplicity and according to the notation (3.24), we denote $(\boldsymbol{\xi}_{\mathbf{u}_{\mathrm{S}}}, \boldsymbol{\xi}_{\boldsymbol{\varphi}}) = (\mathbf{u}_{\mathrm{S}} - \widehat{\mathbf{v}}_{\mathrm{S},h}, \boldsymbol{\varphi} - \widehat{\boldsymbol{\psi}}_h)$ and $(\boldsymbol{\chi}_{\mathbf{u}_{\mathrm{S}}}, \boldsymbol{\chi}_{\boldsymbol{\varphi}}) = (\widehat{\mathbf{v}}_{\mathrm{S},h} - \mathbf{u}_{\mathrm{S},h}, \widehat{\boldsymbol{\psi}}_h - \boldsymbol{\varphi}_h)$. Next, using (3.23), (3.25), the definition of the bilinear form $\mathbf{A}_{\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi}}$ (cf. (2.24)), and simple computations, we easily deduce that

$$\begin{aligned} \mathbf{A}_{\mathbf{u}_{\mathrm{S},h},\boldsymbol{\varphi}_h}(((\boldsymbol{\chi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\chi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\chi}_{\underline{\mathbf{u}}}), ((\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{v}}_h)) &\;= -A((\boldsymbol{\xi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\xi}_{\underline{\boldsymbol{\varphi}}}), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h)) - B((\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h), \boldsymbol{\xi}_{\underline{\mathbf{u}}}) \\ - B((\boldsymbol{\xi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\xi}_{\underline{\boldsymbol{\varphi}}}), \underline{\mathbf{v}}_h) - \mathbf{C}_{\boldsymbol{\xi}_{\mathbf{u}_{\mathrm{S}}}}(\underline{\mathbf{u}}, \underline{\boldsymbol{\tau}}_h) &- \mathbf{C}_{\boldsymbol{\xi}_{\boldsymbol{\varphi}}}(\underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\psi}}_h) - \mathbf{C}_{\mathbf{u}_{\mathrm{S},h}}(\boldsymbol{\xi}_{\underline{\mathbf{u}}}, \underline{\boldsymbol{\tau}}_h) \\ - \mathbf{C}_{\boldsymbol{\varphi}_h}(\boldsymbol{\xi}_{\underline{\boldsymbol{\varphi}}}, \underline{\boldsymbol{\psi}}_h) - \mathbf{C}_{\boldsymbol{\chi}_{\mathbf{u}_{\mathrm{S}}}}(\underline{\mathbf{u}}, \underline{\boldsymbol{\tau}}_h) &- \mathbf{C}_{\boldsymbol{\chi}_{\boldsymbol{\varphi}}}(\underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\psi}}_h)\,, \end{aligned}$$

for all $((\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$. Then, since $(\mathbf{u}_{\mathrm{S},h}, \boldsymbol{\varphi}_h) \in \mathbf{W}_{\mathtt{d}}$, we use the discrete inf-sup condition (3.19), the continuity properties of A and B (cf. (2.29) and (2.30)), together with estimate (2.53), and simple algebraic computation, to obtain

$$\begin{aligned} \frac{\gamma_{\mathtt{d}}}{2}\,\|((\boldsymbol{\chi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\chi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\chi}_{\underline{\mathbf{u}}})\| \;&\leq\; \Big\{1 + \|A\| + \|\mathbf{C}\|\,\big(\|(\mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi})\| + \|(\mathbf{u}_{\mathrm{S},h}, \boldsymbol{\varphi}_h)\|\big)\Big\}\,\|((\boldsymbol{\xi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\xi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\xi}_{\underline{\mathbf{u}}})\| \\ &+ \|\mathbf{C}\|\,\|(\mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi})\|\,\|((\boldsymbol{\chi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\chi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\chi}_{\underline{\mathbf{u}}})\|\,, \end{aligned}$$

which, together with the facts that $(\mathbf{u}_{\rm S}, \boldsymbol{\varphi}) \in \mathbf{W}$ (cf. (2.59)) and $(\mathbf{u}_{{\rm S},h}, \boldsymbol{\varphi}_h) \in \mathbf{W}_{\mathtt{d}}$ (cf. (3.20)), we get

$$\|((\boldsymbol{\chi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\chi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\chi}_{\underline{\mathbf{u}}})\| \,\le\, \widetilde{C}\, \|((\boldsymbol{\xi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\xi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\xi}_{\underline{\mathbf{u}}})\| \,+\, \frac{4\,\|\mathbf{C}\|}{\gamma\,\gamma_{\mathtt{d}}} \Big(\|\mathbf{f}_{\rm D}\|_{0,\Omega_{\rm D}} + \|\mathbf{f}_{\rm S}\|_{0,4/3;\Omega_{\rm S}} \Big) \|((\boldsymbol{\chi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\chi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\chi}_{\underline{\mathbf{u}}})\| \,,$$

with $\widetilde{C}$ depending on the constants γ, $\gamma_{\mathtt{d}}$, $\|\mathbf{C}\|$, $\|A\|$, and data $\mathbf{f}_{\rm D}, \mathbf{f}_{\rm S}$. Thus, the foregoing inequality in conjunction with the data assumption (3.26), yields

$$\|((\boldsymbol{\chi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\chi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\chi}_{\underline{\mathbf{u}}})\| \,\le\, 2\,\widetilde{C}\, \|((\boldsymbol{\xi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\xi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\xi}_{\underline{\mathbf{u}}})\| \,. \tag{3.28}$$

Finally, from (3.23), (3.28) and the triangle inequality we obtain

$$\|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) - ((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h)\| \,\le\, (1 + 2\,\widetilde{C})\, \|((\boldsymbol{\xi}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\xi}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\xi}_{\underline{\mathbf{u}}})\| \,,$$

which, combined to the fact that $(\widehat{\underline{\boldsymbol{\tau}}}_h, \widehat{\underline{\boldsymbol{\psi}}}_h) \in \mathbb{H}_h$ and $\widehat{\underline{\mathbf{v}}}_h \in \mathbb{Q}_h$ are arbitrary, concludes the proof. $\square$

4 Particular choices of discrete subspaces

We now introduce specific discrete spaces satisfying hypotheses (**H.0**), (**H.1**), (**H.2**), and (**H.3**) in 2D and 3D. To this end, we let $\mathcal{T}_h^{\rm S}$ and $\mathcal{T}_h^{\rm D}$ be respective triangulations of the domains $\Omega_{\rm S}$ and $\Omega_{\rm D}$, which are formed by shape-regular triangles T when $n = 2$ (or tetrahedra when $n = 3$). Assume that these triangulations match in Σ, so that $\mathcal{T}_h^{\rm S} \cup \mathcal{T}_h^{\rm D}$ is a triangulation of $\Omega_{\rm S} \cup \Sigma \cup \Omega_{\rm D}$. Let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\rm S}$ (or $\mathcal{T}_h^{\rm D}$). We let $h_{\rm S} := \max\big\{h_T \,:\, T \in \mathcal{T}_h^{\rm S}\big\}$, $h_{\rm D} := \max\big\{h_T \,:\, T \in \mathcal{T}_h^{\rm D}\big\}$ and $h := \max\big\{h_{\rm S}, h_{\rm D}\big\}$. Furthermore, given an integer $k \ge 0$ and $T \in \mathcal{T}_h^{\rm S} \cup \mathcal{T}_h^{\rm D}$, we let ${\rm P}_k(T)$ be the space of polynomials of degree $\le k$ defined on T, whose vector and tensor versions are denoted $[{\rm P}_k(T)]^n$ and $[{\rm P}_k(T)]^{n\times n}$, respectively. In addition, for each $T \in \mathcal{T}_h^{\rm S} \cup \mathcal{T}_h^{\rm D}$ we consider the local Raviart–Thomas space of lower order as

$$\mathbf{RT}_0(T) := [{\rm P}_0(T)]^n \oplus {\rm P}_0(T)\,\mathbf{x} \,.$$

where $\mathbf{x} := (x_1, \dots, x_n)^{\rm t}$ is a generic vector of $\mathbb{R}^n$.

4.1 AFW + Raviart–Thomas in 2D

We define the discrete subspaces in (3.1) as follows:

$$\begin{aligned}
\mathbb{H}_h^{\rm S} &:= \Big\{ \boldsymbol{\tau}_{{\rm S},h} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_{\rm S}) : \quad \boldsymbol{\tau}_{{\rm S},h}|_T \in [{\rm P}_1(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h^{\rm S} \Big\} \,, \\
\mathbf{H}_h^{\rm D} &:= \Big\{ \mathbf{v}_{{\rm D},h} \in \mathbf{H}_{\Gamma_{\rm D}}({\rm div};\Omega_{\rm D}) : \quad \mathbf{v}_{{\rm D},h}|_T \in \mathbf{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\rm D} \Big\} \,, \\
\mathbb{S}_h &:= \Big\{ \boldsymbol{\eta}_{{\rm S},h} \in \mathbb{L}^2_{\rm skew}(\Omega_{\rm S}) : \quad \boldsymbol{\eta}_{{\rm S},h}|_T \in [{\rm P}_0(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h^{\rm S} \Big\} \,, \\
\mathbf{L}_h^{\rm S} &:= \Big\{ \mathbf{v}_{{\rm S},h} \in \mathbf{L}^4(\Omega_{\rm S}) : \quad \mathbf{v}_{{\rm S},h}|_T \in [{\rm P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h^{\rm S} \Big\} \,, \\
{\rm L}_h^{\rm D} &:= \Big\{ q_{{\rm D},h} \in {\rm L}^2(\Omega_{\rm D}) : \quad q_{{\rm D},h}|_T \in {\rm P}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\rm D} \Big\} \,.
\end{aligned} \tag{4.1}$$

Note that $\mathbb{H}_h^{\rm S} \times \mathbf{L}_h^{\rm S} \times \mathbb{S}_h$ constitutes the lowest order mixed finite element approximation of the linear elasticity problem introduced by Arnold, Falk and Winther in [2]. In turn, $\mathbf{H}_h^{\rm D} \times \mathbf{L}_h^{\rm D}$ is the Raviart–Thomas stable element of lowest order for the mixed formulation of the Poisson problem (see for instance [10]).

Now, we turn to define the finite dimensional subspaces $\mathbf{\Lambda}_h^{\rm S} \subseteq \mathbf{H}_{00}^{1/2}(\Sigma)$ and $\Lambda_h^{\rm D} \subseteq {\rm H}^{1/2}(\Sigma)$. For this purpose, let us assume that the number of edges of Σ_h is even and let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h (if the number of edges of Σ_h is odd, we simply reduce to the even case by joining any pair of two adjacent elements and then construct Σ_{2h} from this reduced partition). In this way, denoting by x_0 and x_N the extreme points of Σ, we define

$$\begin{aligned} \mathbf{\Lambda}_h^{\rm S} &:= \Big\{ \boldsymbol{\psi}_h \in [C(\Sigma)]^2 : \quad \boldsymbol{\psi}_h|_e \in [{\rm P}_1(e)]^2 \quad \forall\, e \in \Sigma_{2h}, \quad \boldsymbol{\psi}_h(x_0) = \boldsymbol{\psi}_h(x_N) = 0 \Big\}, \\ \Lambda_h^{\rm D} &:= \Big\{ \xi_h \in C(\Sigma) : \quad \xi_h|_e \in {\rm P}_1(e) \quad \forall\, e \in \Sigma_{2h} \Big\}. \end{aligned} \tag{4.2}$$

In what follow we verify that the discrete spaces $\mathbb{H}_h$ and $\mathbb{Q}_h$ defined by the combination of (3.1), (3.2), (3.3), (4.1) and (4.2), satisfy hypotheses (**H.0**)–(**H.3**).

First, observe that $\mathbb{H}_h^{\rm S}$, $\mathbf{H}_h^{\rm D}$, $\mathbf{L}_h^{\rm S}$ and ${\rm L}_h^{\rm D}$ clearly satisfy (**H.0**) and (**H.2**). In turn, the proof of (3.6) follows exactly the same steps as [12, Lemma 4.3] and using the properties of BDM interpolant (see [9, Proposition 2.5.1]), besides that, it is well known that the discrete inf-sup condition (3.7) holds (see for instance [10, Chapter IV]). In addition, the existence of $\boldsymbol{\psi}_0 \in \mathbf{\Lambda}_h^{\rm S}$ satisfying (3.8) follows as explained in [37, Section 3.2], whence hypothesis (**H.1**) holds. On the other hand, regarding the discrete inf-sup condition for $\mathbf{b}$ (cf. (**H.3**)), we proceed similarly to [13, Section 3.4.1]. Indeed, from [46, Theorem A.1] and [37, Lemmas 4.2 and 5.2] we obtain (3.10), and the inf-sup condition

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_{{\rm S},h}\in\widetilde{\mathbb{H}}_h^{\rm S}} \frac{\langle \boldsymbol{\tau}_{{\rm S},h}\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma}{\|\boldsymbol{\tau}_{\rm S}\|_{\mathbf{div}_{4/3},\Omega_{\rm S}}} \;\geq\; C_\Sigma^{\rm S}\, \|\boldsymbol{\psi}_h\|_{1/2,\Sigma} \quad \forall\, \boldsymbol{\psi}_h \in \mathbf{\Lambda}_h^{\rm S}, \tag{4.3}$$

with $C_\Sigma^{\rm S} > 0$, independent of h, which yields

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_{{\rm S},h}\in\widetilde{\mathbb{H}}_h^{\rm S}} \frac{\langle \boldsymbol{\tau}_{{\rm S},h}\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma + (\boldsymbol{\tau}_{{\rm S},h}, \boldsymbol{\eta}_{{\rm S},h})_{\rm S}}{\|\boldsymbol{\tau}_{{\rm S},h}\|_{\mathbf{div}_{4/3},\Omega_{\rm S}}} \;\geq\; C_\Sigma^{\rm S}\, \|\boldsymbol{\psi}\|_{1/2,00;\Sigma} - \|\boldsymbol{\eta}_{{\rm S},h}\|_{0,\Omega_{\rm S}}, \tag{4.4}$$

for all $(\boldsymbol{\eta}_{{\rm S},h}, \boldsymbol{\psi}_h) \in \mathbb{S}_h \times \widetilde{\mathbf{\Lambda}}_h^{\rm S}$. In turn, defining

$$\begin{aligned} \widehat{\mathbb{H}}_h^{\rm S} := \Big\{ \boldsymbol{\tau}_{{\rm S},h} \in \mathbb{H}(\mathbf{div};\Omega_{\rm S}) : \quad & \mathbf{c}^{\rm t}\boldsymbol{\tau}_{{\rm S},h}|_T \in [{\rm P}_1(T)]^2 \quad \forall\, \mathbf{c} \in \mathbb{R}^n \quad \forall\, T \in \mathcal{T}_h^{\rm S}, \\ & \mathbf{div}(\boldsymbol{\tau}_{{\rm S},h}) = \mathbf{0} \quad \text{in} \quad \Omega_{\rm S} \quad \text{and} \quad \boldsymbol{\tau}_{{\rm S},h}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Sigma \Big\}, \end{aligned}$$

we recall from [3, Theorem 11.9] that there exists $C_{\sf skew}^{\rm S} > 0$, independent of h, such that

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_{{\rm S},h}\in\widehat{\mathbb{H}}_h^{\rm S}} \frac{(\boldsymbol{\tau}_{{\rm S},h}, \boldsymbol{\eta}_{{\rm S},h})_{\rm S}}{\|\boldsymbol{\tau}_{{\rm S},h}\|_{\mathbf{div},\Omega_{\rm S}}} \;\geq\; C_{\sf skew}^{\rm S}\, \|\boldsymbol{\eta}_{{\rm S},h}\|_{0,\Omega_{\rm S}}.$$

Thus, using that $\widehat{\mathbb{H}}_h^{\rm S} \subseteq \widetilde{\mathbb{H}}_h^{\rm S}$, we get

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_{{\rm S},h}\in\widetilde{\mathbb{H}}_h^{\rm S}} \frac{\langle \boldsymbol{\tau}_{{\rm S},h}\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma + (\boldsymbol{\tau}_{{\rm S},h}, \boldsymbol{\eta}_{{\rm S},h})_{\rm S}}{\|\boldsymbol{\tau}_{{\rm S},h}\|_{\mathbf{div}_{4/3};\Omega_{\rm S}}} \;\geq\; C_{\sf skew}^{\rm S}\, \|\boldsymbol{\eta}_{{\rm S},h}\|_{0,\Omega_{\rm S}}. \tag{4.5}$$

Finally, combining (4.4) and (4.5), we obtain (3.9) with $\beta_{\Sigma,\mathtt{d}}^{\rm S} = C_{\sf skew}^{\rm S}/2\, \min\big\{1, C_\Sigma^{\rm S}/(1 + C_{\sf skew}^{\rm S})\big\}$, concluding the proof of (**H.3**).

4.2 AFW + Raviart–Thomas in 3D

Let us now define the discrete subspaces in (3.1) as follows:

$$
\begin{aligned}
\mathbb{H}_h^{\mathrm{S}} &:= \Big\{ \boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_{\mathrm{S}}) : \quad \boldsymbol{\tau}_{\mathrm{S},h}|_T \in [\mathrm{P}_1(T)]^{3\times 3} \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \Big\}, \\
\mathbf{H}_h^{\mathrm{D}} &:= \Big\{ \mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) : \quad \mathbf{v}_{\mathrm{D},h}|_T \in \mathbf{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}} \Big\}, \\
\mathbb{S}_h &:= \Big\{ \boldsymbol{\eta}_{\mathrm{S},h} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) : \quad \boldsymbol{\eta}_{\mathrm{S},h}|_T \in [\mathrm{P}_0(T)]^{3\times 3} \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \Big\}, \\
\mathbf{L}_h^{\mathrm{S}} &:= \Big\{ \mathbf{v}_{\mathrm{S},h} \in \mathbf{L}^4(\Omega_{\mathrm{S}}) : \quad \mathbf{v}_{\mathrm{S},h}|_T \in [\mathrm{P}_0(T)]^{3} \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \Big\}, \\
\mathrm{L}_h^{\mathrm{D}} &:= \Big\{ q_{\mathrm{D},h} \in \mathrm{L}^2(\Omega_{\mathrm{D}}) : \quad q_{\mathrm{D},h}|_T \in \mathrm{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}} \Big\}.
\end{aligned}
\tag{4.6}
$$

Notice that these finite element subspaces are the 3D version of the ones defined in (4.1), considering that the vector and tensor fields live now in $\mathbb{R}^3$ and $\mathbb{R}^{3\times 3}$, respectively.

Now, in order to define the discrete spaces $\boldsymbol{\Lambda}_h^{\mathrm{S}} \subseteq \mathbf{H}_{00}^{1/2}(\Sigma)$ and $\Lambda_h^{\mathrm{D}} \subseteq \mathrm{H}^{1/2}(\Sigma)$ for the unknowns on the interface Σ, we proceed as in [13, Section 3.4.2], and introduce an independent triangulation $\Sigma_{\widehat{h}}$ of Σ, by triangles K of diameter $\widehat{h}_K$ and define $\widehat{h}_\Sigma := \max\{\widehat{h}_K : \ K \in \Sigma_{\widehat{h}}\}$. Then, denoting by $\partial\Sigma$ the polygonal boundary of Σ, we define

$$
\begin{aligned}
\boldsymbol{\Lambda}_h^{\mathrm{S}} &:= \Big\{ \boldsymbol{\psi}_h \in [C(\Sigma)]^3 : \quad \boldsymbol{\psi}_h|_K \in [\mathrm{P}_1(K)]^3 \quad \forall\, K \in \Sigma_{\widehat{h}}, \quad \boldsymbol{\psi}_h = 0 \text{ on } \partial\Sigma \Big\}, \\
\Lambda_h^{\mathrm{D}} &:= \Big\{ \xi_h \in C(\Sigma) : \quad \xi_h|_K \in \mathrm{P}_1(K) \quad \forall\, K \in \Sigma_{\widehat{h}} \Big\}.
\end{aligned}
\tag{4.7}
$$

In this way, we define the discrete spaces $\mathbb{H}_h$ and $\mathbb{Q}_h$ by combining (3.1), (3.2), (3.3), (4.6) and (4.7).

Next, for the verification of the required hypotheses for the corresponding discrete analysis, we first observe that the same arguments from the 2D case imply the verification of (**H.0**), (**H.1**) and (**H.2**) in 3D. However, for the inf-sup conditions in (**H.3**), we proceed in the same way as in [13, Section 3.4.2], we let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$), formed by triangles of diameter h_K, and define $h_\Sigma := \max\{h_K : \ K \in \Sigma_h\}$. Then, defining the set of normal traces of $\widetilde{\mathbf{H}}_h^{\mathrm{D}}$ and $\widetilde{\mathbb{H}}_h^{\mathrm{S}}$ as in (4.7) (considering triangles instead of edges), utilizing [31, Lemma 7.5] (see also [46, Theorem A.1 and Remark A.2]), assuming that the mesh is quasi-uniform in a neighborhood of the interface, and employing inverse inequalities on Σ, we can construct the stable discrete lifting of the normal traces of $\widetilde{\mathbf{H}}_h^{\mathrm{D}}$ and $\widetilde{\mathbb{H}}_h^{\mathrm{S}}$, and obtain that there exists $C_0 \in (0,1)$ such that for each pair $(h_\Sigma, \widehat{h}_\Sigma)$ verifying $h_\Sigma \le C_0\, \widehat{h}_\Sigma$, (3.9) and (3.10) are satisfied. From the above, we can obtain the 3D version of (3.10) and (4.3), and the discrete inf-sup condition for $\mathbf{b}$ is satisfied. According to this, we obtain (**H.3**).

4.3 Rate of convergence

Now, for both cases 2D and 3D domains, we establish the theoretical rates of convergence of our discrete scheme (3.4). To that end, we first recall from [2], [10], [30], [41], and [14, Section 3.1] (see also [23, Section 5.5]), the approximation properties of the finite element subspaces involved, which are named after the unknowns to which they are applied later on.

($\mathbf{AP}_h^{\boldsymbol{\sigma}_{\mathrm{S}}}$) For each $\delta \in (0,1]$ and for each $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_{\mathrm{S}}) \cap \mathbb{H}^\delta(\Omega_{\mathrm{S}})$ with $\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}} \in \mathbf{W}^{\delta,4/3}(\Omega_{\mathrm{S}})$, there holds

$$
\operatorname{dist}\big(\boldsymbol{\tau}_{\mathrm{S}}, \mathbb{H}_{h,0}^{\mathrm{S}}\big) := \inf_{\boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_{h,0}^{\mathrm{S}}} \|\boldsymbol{\tau}_{\mathrm{S}} - \boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{S}}} \le C\, h^\delta \Big\{ \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{div}(\boldsymbol{\tau}_{\mathrm{S}})\|_{\delta,4/3;\Omega_{\mathrm{S}}} \Big\}.
$$

$(\mathbf{AP}_h^{\mathbf{u}_\mathrm{S}})$ For each $\delta \in [0,1]$ and for each $\mathbf{v}_\mathrm{S} \in \mathbf{W}^{\delta,4}(\Omega_\mathrm{S})$, there holds

$$\operatorname{dist}\big(\mathbf{v}_\mathrm{S}, \mathbf{L}_h^\mathrm{S}\big) \,:=\, \inf_{\mathbf{v}_{\mathrm{S},h}\in \mathbf{L}_h^\mathrm{S}} \|\mathbf{v}_\mathrm{S} - \mathbf{v}_{\mathrm{S},h}\|_{0,4;\Omega_\mathrm{S}} \,\le\, C\, h^\delta\, \|\mathbf{v}_\mathrm{S}\|_{\delta,4;\Omega_\mathrm{S}}\,.$$

$(\mathbf{AP}_h^{\boldsymbol{\gamma}_\mathrm{S}})$ For each $\delta \in [0,1]$ and for each $\boldsymbol{\eta}_\mathrm{S} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega_\mathrm{S}) \cap \mathbb{H}^\delta(\Omega_\mathrm{S})$, there holds

$$\operatorname{dist}\big(\boldsymbol{\eta}_\mathrm{S}, \mathbb{S}_h\big) \,:=\, \inf_{\boldsymbol{\eta}_{\mathrm{S},h}\in \mathbb{S}_h} \|\boldsymbol{\eta}_\mathrm{S} - \boldsymbol{\eta}_{\mathrm{S},h}\|_{0,\Omega_\mathrm{S}} \,\le\, C\, h^\delta\, \|\boldsymbol{\eta}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}}\,.$$

$(\mathbf{AP}_h^{\mathbf{u}_\mathrm{D}})$ For each $\delta \in (0,1]$ and for each $\mathbf{v}_\mathrm{D} \in \mathbf{H}_{\Gamma_\mathrm{D}}(\operatorname{div};\Omega_\mathrm{D}) \cap \mathbf{H}^\delta(\Omega_\mathrm{D})$ with $\operatorname{div}\mathbf{v}_\mathrm{D} \in \mathrm{H}^\delta(\Omega_\mathrm{D})$, there holds

$$\operatorname{dist}\big(\mathbf{v}_\mathrm{D}, \mathbf{H}_h^\mathrm{D}\big) \,:=\, \inf_{\mathbf{v}_{\mathrm{D},h}\in \mathbf{H}_h^\mathrm{D}} \|\mathbf{v}_\mathrm{D} - \mathbf{v}_{\mathrm{D},h}\|_{\operatorname{div};\Omega_\mathrm{D}} \,\le\, C\, h^\delta \Big\{\|\mathbf{v}_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}} + \|\operatorname{div}(\mathbf{v}_\mathrm{D})\|_{\delta,\Omega_\mathrm{D}}\Big\}\,.$$

$(\mathbf{AP}_h^{\boldsymbol{\varphi}})$ For each $\delta \in (0,1]$ and for each $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma) \cap \mathbf{H}^{1/2+\delta}(\Sigma)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\psi}, \boldsymbol{\Lambda}_h^\mathrm{S}\big) \,:=\, \inf_{\boldsymbol{\psi}_h\in \boldsymbol{\Lambda}_h^\mathrm{S}} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1/2,00;\Sigma} \,\le\, C\, h^\delta\, \|\boldsymbol{\psi}\|_{1/2+\delta,\Sigma}\,.$$

$(\mathbf{AP}_h^{\lambda})$ For each $\delta \in (0,1]$ and for each $\xi \in \mathrm{H}^{1/2+\delta}(\Sigma)$, there holds

$$\operatorname{dist}\big(\xi, \Lambda_h^\mathrm{D}\big) \,:=\, \inf_{\xi_h\in \Lambda_h^\mathrm{D}} \|\xi - \xi_h\|_{1/2,\Sigma} \,\le\, C\, h^\delta\, \|\xi\|_{1/2+\delta,\Sigma}\,.$$

$(\mathbf{AP}_h^{p_\mathrm{D}})$ For each $\delta \in [0,1]$ and for each $q_\mathrm{D} \in \mathrm{L}^2_0(\Omega_\mathrm{D}) \cap \mathrm{H}^\delta(\Omega_\mathrm{D})$, there holds

$$\operatorname{dist}\big(q_\mathrm{D}, \mathrm{L}_{h,0}^\mathrm{D}\big) \,:=\, \inf_{q_{\mathrm{D},h}\in \mathrm{L}_{h,0}^\mathrm{D}} \|q_\mathrm{D} - q_{\mathrm{D},h}\|_{0,\Omega_\mathrm{D}} \,\le\, C\, h^\delta\, \|q_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}}\,.$$

Then, we establish the theoretical rate of convergence of the Galerkin scheme (3.4) for the particular choices of discrete subspaces (4.1)–(4.2) and (4.6)–(4.7) in 2D and 3D, respectively.

Theorem 4.1 *Assume that the hypotheses of Theorem 3.4 hold. Let* $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) \in \mathbb{H} \times \mathbb{Q}$ *and* $((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ *be the unique solutions of the continuous and discrete problems* (2.15) *and* (3.4)*, respectively. Assume that there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{W}^{\delta,4/3}(\Omega_\mathrm{S})$, $\mathbf{u}_\mathrm{S} \in \mathbf{W}^{\delta,4}(\Omega_\mathrm{S})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\boldsymbol{\gamma}_\mathrm{S} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$, $\mathbf{u}_\mathrm{D} \in \mathbf{H}^\delta(\Omega_\mathrm{D})$, $\operatorname{div}\mathbf{u}_\mathrm{D} \in \mathrm{H}^\delta(\Omega_\mathrm{D})$, $p_\mathrm{D} \in \mathrm{H}^\delta(\Omega_\mathrm{D})$, *and* $\lambda \in \mathrm{H}^{1/2+\delta}(\Sigma)$*. Then, there exists* $C_{rate} > 0$*, independent of* h*, such that*

$$\begin{aligned}\|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) - ((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h)\| \,\le\, C_{rate}\, h^\delta \Big\{&\|\boldsymbol{\sigma}\|_{\delta,\Omega_\mathrm{S}} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\delta,4/3;\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{S}\|_{\delta,4;\Omega_\mathrm{S}} \\ &+ \|\boldsymbol{\varphi}\|_{1/2+\delta,\Sigma} + \|\boldsymbol{\gamma}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}} + \|\operatorname{div}(\mathbf{u}_\mathrm{D})\|_{\delta,\Omega_\mathrm{D}} + \|p_\mathrm{D}\|_{1+\delta,\Omega_\mathrm{D}} + \|\lambda\|_{1/2+\delta,\Sigma}\Big\}\,.\end{aligned}$$

Proof. The result follows from a direct application of Theorem 3.4 and the approximation properties of the discrete subspaces. Further details are omitted. □

Remark 4.2 *As an alternative to the discrete spaces* (4.1) *and* (4.6) *for the 2D and 3D cases, respectively, one may consider the PEERS + Raviart–Thomas spaces (see [1] for PEERS elements), which also satisfy hypotheses* $(\mathbf{H.0})$–$(\mathbf{H.3})$*. Consequently, problem* (3.4) *is well-posed and satisfies* (3.27)*. Moreover, under regularity assumptions on the solution similar to those in Theorem 4.1, an optimal rate of convergence can be obtained.*

5 A posteriori error Analysis

In this section, we focus on the two-dimensional case and develop a reliable and efficient residual-based *a posteriori* error estimate for the finite element scheme (3.4), using the discrete spaces introduced in (4.1)–(4.2). The extension to the three-dimensional case should be quite straightforward. We begin with some notations. Similarly to [13], for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$. Let $\mathcal{E}(T)$ be the set of edges of T, and denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, subdivided as follows:

$$\mathcal{E}_h := \mathcal{E}_h(\Gamma_{\mathrm{S}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{D}}) \cup \mathcal{E}_h(\Omega_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{D}}) \cup \mathcal{E}_h(\Sigma),$$

where $\mathcal{E}(\Gamma_\star) := \{e \in \mathcal{E}_h : \quad e \subseteq \Gamma_\star\}$, $\mathcal{E}(\Omega_\star) := \{e \in \mathcal{E}_h : \quad e \subseteq \Omega_\star\}$, for each $\star \in \{\mathrm{S}, \mathrm{D}\}$, and $\mathcal{E}(\Sigma) := \{e \in \mathcal{E}_h : \quad e \subseteq \Sigma\}$. Note that $\mathcal{E}(\Sigma)$ is the set of edges defining the partition Σ_h. Analogously, we let $\mathcal{E}_{2h}(\Sigma)$ be the set of double edges defining the partition Σ_{2h}. In what follows, h_e stands for the diameter of a given edge $e \in \mathcal{E}_h \cup \mathcal{E}_{2h}(\Sigma)$. Now, let $q \in [L^2(\Omega_\star)]^m$, with $m \in \{1,2\}$, such that $q|_T \in [C(T)]^m$ for each $T \in \mathcal{T}_h^\star$. Then, given $e \in \mathcal{E}_h(\Omega_\star)$, we denote by $[q]$ the jump of q across e, that is $[q] := (q|_{T'})|_e - (q|_{T''})|_e$, where T' and T'' are the triangles of $\mathcal{T}_h^\star$ having e as an edge. Also, we fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\mathtt{t}}$ to the edge e and let $\mathbf{t}_e := (-n_2, n_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Hence, given $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$ such that $\mathbf{v}|_T \in [C(T)]^2$ and $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$, respectively, for each $T \in \mathcal{T}_h^\star$, we let $[\![\mathbf{v}\cdot\mathbf{t}_e]\!]$ and $[\![\boldsymbol{\tau}\,\mathbf{t}_e]\!]$ be the tangential jumps of $\mathbf{v}$ and $\boldsymbol{\tau}$, across e, that is $[\![\mathbf{v}\cdot\mathbf{t}_e]\!] := \{(\mathbf{v}|_{T'})|_e - (\mathbf{v}|_{T''})|_e\}\cdot\mathbf{t}_e$ and $[\![\boldsymbol{\tau}\,\mathbf{t}_e]\!] := \{(\boldsymbol{\tau}|_{T'})|_e - (\boldsymbol{\tau}|_{T''})|_e\}\,\mathbf{t}_e$, respectively. From now on, when no confusion arises, we will simply write $\mathbf{t}$ and $\mathbf{n}$ instead of $\mathbf{t}_e$ and $\mathbf{n}_e$, respectively. Finally, for any suffiently smooth scalar, vector and tensors fields q, $\mathbf{v} := (v_1, v_2)^{\mathtt{t}}$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we let

$$\mathbf{curl}(\mathbf{v}) := \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix}, \qquad \mathrm{curl}(q) := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right)^{\mathtt{t}},$$

$$\mathrm{rot}(\mathbf{v}) := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \text{and} \quad \mathbf{rot}(\boldsymbol{\tau}) := \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2}\right)^{\mathtt{t}}.$$

Next, for the sake of simplicity, in this section, we replace the formulation (2.15) with the equivalent one arising from the utilization of the decomposition (2.14). In other words, we eliminate the explicit unknown $\ell \in \mathbb{R}$ and treat it implicitly by redefining the stress $\boldsymbol{\sigma}$ as an unknown in $\mathbb{H}(\mathbf{div}_{4/3};\Omega_{\mathrm{S}})$. In this way, defining

$$\begin{gathered}\mathbf{X} := \mathbb{H}(\mathbf{div}_{4/3};\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}), \quad \mathbf{Y} := \mathbf{H}_{00}^{1/2}(\Sigma) \times \mathrm{H}^{1/2}(\Sigma) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \\ \mathbb{H} := \mathbf{X} \times \mathbf{Y} \quad \text{and} \quad \mathbb{Q} := \mathbf{L}^4(\Omega_{\mathrm{S}}) \times \mathrm{L}^2_0(\Omega_{\mathrm{D}}), \\ \underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}) \in \mathbf{X}, \quad \underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi}, \lambda, \boldsymbol{\gamma}_{\mathrm{S}}) \in \mathbf{Y}, \quad \underline{\mathbf{u}} := (\mathbf{u}_{\mathrm{S}}, p_{\mathrm{D}}) \in \mathbb{Q}, \\ \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{X}, \quad \underline{\boldsymbol{\psi}} := (\boldsymbol{\psi}, \xi, \boldsymbol{\eta}_{\mathrm{S}}) \in \mathbf{Y}, \quad \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}) \in \mathbb{Q},\end{gathered}$$

the mixed formulation reduces to: Find $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) \in \mathbb{H} \times \mathbb{Q}$, such that

$$\begin{aligned} A((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), (\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}})) + \mathbf{C}_{\boldsymbol{\varphi}}(\underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\psi}}) + B((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{u}}) + \mathbf{C}_{\mathbf{u}_{\mathrm{S}}}(\underline{\mathbf{u}}, \underline{\boldsymbol{\tau}}) &= (\mathbf{f}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \quad \forall\, (\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}) \in \mathbb{H}, \\ B((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{v}}) &= -(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \quad \forall\, \underline{\mathbf{v}} \in \mathbb{Q}, \end{aligned} \tag{5.1}$$

where the form B has been redefined by omitting the last term in (2.16), that is, $B((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) := (\mathbf{v}_{\mathrm{S}}, \mathbf{div}(\boldsymbol{\tau}_{\mathrm{S}}))_{\mathrm{S}} - (q_{\mathrm{D}}, \mathrm{div}(\mathbf{v}_{\mathrm{D}}))_{\mathrm{D}}$, for all $((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}$. As a result, letting $\mathrm{L}^{\mathrm{D}}_{h,0} := \mathrm{L}^{\mathrm{D}}_h \cap \mathrm{L}^2_0(\Omega_{\mathrm{D}})$, and grouping the discrete spaces, unknowns and test functions as follows

$$\begin{gathered}
\mathbf{X}_h := \mathbb{H}_h^{\mathrm{S}} \times \mathbf{H}_h^{\mathrm{D}}, \quad \mathbf{Y}_h := \mathbf{\Lambda}_h^{\mathrm{S}} \times \Lambda_h^{\mathrm{D}} \times \mathbb{S}_h, \\
\mathbb{H}_h := \mathbf{X}_h \times \mathbf{Y}_h \quad \text{and} \quad \mathbb{Q}_h := \mathbf{L}_h^{\mathrm{S}} \times \mathrm{L}_{h,0}^{\mathrm{D}}, \\
\underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\varphi}}_h := (\boldsymbol{\varphi}_h, \lambda_h, \boldsymbol{\gamma}_{\mathrm{S},h}) \in \mathbf{Y}_h, \quad \underline{\mathbf{u}}_h := (\mathbf{u}_{\mathrm{S},h}, p_{\mathrm{D},h}) \in \mathbb{Q}_h, \\
\underline{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h, \xi_h, \boldsymbol{\eta}_{\mathrm{S},h}) \in \mathbf{Y}_h, \quad \underline{\mathbf{v}}_h := (\mathbf{v}_{\mathrm{S},h}, q_{\mathrm{D},h}) \in \mathbb{Q}_h,
\end{gathered}$$

the corresponding discrete problem is defined as follows: Find $((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\begin{aligned}
A((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h)) + \mathbf{C}_{\boldsymbol{\varphi}_h}(\underline{\boldsymbol{\varphi}}_h, \underline{\boldsymbol{\psi}}_h) + B((\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{u}}_h) + \mathbf{C}_{\mathbf{u}_{\mathrm{S},h}}(\underline{\mathbf{u}}_h, \underline{\boldsymbol{\tau}}_h) &= (\mathbf{f}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D},h})_{\mathrm{D}}, \\
B((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{v}}_h) &= -(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S},h})_{\mathrm{S}},
\end{aligned} \tag{5.2}$$

for all $((\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$.

We recall that $\mathbb{H}_h$ and $\mathbb{Q}_h$ are defined based on the discrete spaces introduced in (4.1)-(4.2). In addition, owing to the equivalence between (2.15) and (5.1), as well as between (3.4) and (5.2), it is evident that both problems are well-posed and satisfy the respective continuous dependence inequalities.

Let $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) \in \mathbb{H} \times \mathbb{Q}$ and $((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ be the unique solutions of the continuous and discrete problems (5.1) and (5.2), respectively. Then, our global *a posteriori* error estimator is defined by:

$$\Theta := \left\{ \sum_{T \in \mathcal{T}_h^{\mathrm{S}}} \Theta_{\mathrm{S},T}^2 + \sum_{T \in \mathcal{T}_h^{\mathrm{D}}} \Theta_{\mathrm{D},T}^2 \right\}^{1/2} + \left\{ \sum_{T \in \mathcal{T}_h^{\mathrm{S}}} \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,4/3;T}^{4/3} \right\}^{3/4}, \tag{5.3}$$

where, for each $T \in \mathcal{T}_h^{\mathrm{S}}$, the local error indicator is defined as follows:

$$\begin{aligned}
\Theta_{\mathrm{S},T}^2 := \ & h_T \left\| \boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu} \big(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h})\big)^{\mathrm{d}} \right\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e^{1/2} \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,4;e}^2 \\
& + \|\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{t}}\|_{0,T}^2 + h_T^2 \left\| \mathbf{rot} \left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu} \big(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h})\big)^{\mathrm{d}} \right) \right\|_{0,T}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_{\mathrm{S}})} h_e \left\| \left[\!\left[\left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu} \big(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h})\big)^{\mathrm{d}} \right) \mathbf{t} \right]\!\right] \right\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e \left\| \left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu} \big(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h})\big)^{\mathrm{d}} \right) \mathbf{t} \right\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \left\| \left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu} (\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h}))^{\mathrm{d}} \right) \mathbf{t} + \boldsymbol{\varphi}_h' \right\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} - \omega^{-1}(\boldsymbol{\varphi}_h \cdot \mathbf{t})\mathbf{t} + \lambda_h \mathbf{n} + \rho(\boldsymbol{\varphi}_h \otimes \boldsymbol{\varphi}_h)\mathbf{n}\|_{0,e}^2,
\end{aligned} \tag{5.4}$$

and for each $T \in \mathcal{T}_h^{\mathrm{D}}$:

$$\begin{aligned}\Theta_{\mathrm{D},T}^2 \,:=\, & \|\mathrm{div}(\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 \,+\, h_T^2 \left\|\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\right\|_{0,T}^2 \,+\, h_T^2 \left\|\mathrm{rot}\left(\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\right)\right\|_{0,T}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{D}})} h_e \left\| [\![\left(\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\right)\cdot\mathbf{t}]\!] \right\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \left\{ \|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \,+\, \left\|\left(\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\right)\cdot\mathbf{t} - \lambda_h'\right\|_{0,e}^2 \,+\, \|(\mathbf{u}_{\mathrm{D},h} + \boldsymbol{\varphi}_h)\cdot\mathbf{n}\|_{0,e}^2 \right\}.\end{aligned} \tag{5.5}$$

Hereafter, ω denotes the sole frictional constant in (2.4), while the expressions $\boldsymbol{\varphi}_h'$ and λ_h' represent the tangential derivatives of $\boldsymbol{\varphi}_h$ and λ_h, respectively, along the interface Σ.

5.1 Reliability of the a posteriori error estimator

We start by recalling that the continuous dependence result provided in (2.61), after adapting the continuous spaces accordingly, is equivalent to the global inf-sup condition for the continuous formulation (5.1). Subsequently, by applying this estimate to the error $((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) - ((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h) \in \mathbb{H}\times\mathbb{Q}$, using the methodology outlined in [11, Lemma 5.2] together with the small data assumption (3.26), we deduce the existence of a constant $C_{glob} > 0$, independent of h, such that:

$$\|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) - ((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h)\| \,\leq\, C_{glob} \sup_{\mathbf{0}\neq((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\in\mathbb{H}\times\mathbb{Q}} \frac{\mathcal{R}((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})}{\|((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}})\|}, \tag{5.6}$$

where $\mathcal{R} : \mathbb{H}\times\mathbb{Q} \to \mathbb{R}$ is the residual functional

$$\begin{aligned}\mathcal{R}((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) := & A((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}) - (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}})) \,+\, B((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{u}} - \underline{\mathbf{u}}_h) \\ & + B((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}) - (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{v}}) + \mathbf{C}_{\boldsymbol{\varphi}}(\underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\psi}}) + \mathbf{C}_{\mathbf{u}_{\mathrm{S}}}(\underline{\mathbf{u}}, \underline{\boldsymbol{\tau}}) - \mathbf{C}_{\boldsymbol{\varphi}_h}(\underline{\boldsymbol{\varphi}}_h, \underline{\boldsymbol{\psi}}) - \mathbf{C}_{\mathbf{u}_{\mathrm{S},h}}(\underline{\mathbf{u}}_h, \underline{\boldsymbol{\tau}}),\end{aligned}$$

for all $((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H}\times\mathbb{Q}$.

In turn, according to (5.1), (5.2), the definition of the forms A, B, $\mathbf{C}_{\boldsymbol{\phi}}$, $\mathbf{C}_{\mathbf{w}_{\mathrm{S}}}$, and using the fact that $\mathrm{div}(\mathbf{u}_{\mathrm{D},h}) = 0$ in Ω_{D}, we find that, for any $((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) \in \mathbb{H}\times\mathbb{Q}$, there holds

$$\mathcal{R}((\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{v}}) := \mathcal{R}_1(\boldsymbol{\tau}_{\mathrm{S}}) \,+\, \mathcal{R}_2(\mathbf{v}_{\mathrm{D}}) \,+\, \mathcal{R}_3(\boldsymbol{\eta}_{\mathrm{S}}) \,+\, \mathcal{R}_4(\mathbf{v}_{\mathrm{S}}) \,+\, \mathcal{R}_5(\boldsymbol{\psi}) \,+\, \mathcal{R}_6(\xi),$$

where

$$\begin{aligned}\mathcal{R}_1(\boldsymbol{\tau}_{\mathrm{S}}) &= -\langle\boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma - \left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}\left(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho\,(\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})\right)^{\mathrm{d}}, \boldsymbol{\tau}_{\mathrm{S}}\right)_{\mathrm{S}} - (\mathbf{u}_{\mathrm{S},h}, \mathbf{div}(\boldsymbol{\tau}_{\mathrm{S}}))_{\mathrm{S}}\,, \\ \mathcal{R}_2(\mathbf{v}_{\mathrm{D}}) &= (\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma + (p_{\mathrm{D},h}, \mathrm{div}(\mathbf{v}_{\mathrm{D}}))_{\mathrm{D}}\,, \\ \mathcal{R}_3(\boldsymbol{\eta}_{\mathrm{S}}) &= \frac{1}{2}(\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{t}}, \boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}}\,, \\ \mathcal{R}_4(\mathbf{v}_{\mathrm{S}}) &= -(\mathbf{f}_{\mathrm{S}} + \mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S}})_{\mathrm{S}}\,, \\ \mathcal{R}_5(\boldsymbol{\psi}) &= -\langle\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} - \omega^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t} + \lambda_h\mathbf{n} + \rho\,(\boldsymbol{\varphi}_h\otimes\boldsymbol{\varphi}_h)\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma\,, \\ \mathcal{R}_6(\xi) &= \langle\boldsymbol{\varphi}_h\cdot\mathbf{n} + \mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n}, \xi\rangle_\Sigma\,.\end{aligned}$$

The main result of this section is stated in the following theorem.

Theorem 5.1 *Assume that there exists a convex domain $\mathcal{B}$ such that $\Omega_{\mathrm{D}} \subseteq \mathcal{B}$ and $\Gamma_{\mathrm{D}} \subseteq \partial\mathcal{B}$. Assume further that the hypotheses of Theorem 4.1 hold. Then, there exist $C_{rel} > 0$, independent of h, such that*

$$\|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) - ((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h)\| \leq C_{rel}\, \Theta\,.$$

Proof. Similarly to [13, Section 5.1], the supremum in (5.6) can be bounded in terms of R_i, with $i \in \{1, \ldots, 6\}$, as

$$\begin{aligned}\|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) - ((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h)\| \;\leq\; & C\Big(\|\mathcal{R}_1\|_{\mathbb{H}(\mathbf{div}_{4/3};\Omega_{\mathrm{S}})'} + \|\mathcal{R}_2\|_{\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})'} + \|\mathcal{R}_3\|_{\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})'} \\ & + \|\mathcal{R}_4\|_{\mathbf{L}^4(\Omega_{\mathrm{S}})'} + \|\mathcal{R}_5\|_{\mathbf{H}^{1/2}_{00}(\Sigma)'} + \|\mathcal{R}_6\|_{\mathrm{H}^{1/2}(\Sigma)'} \Big)\,.\end{aligned}$$

Then, we need to provide suitable upper bounds for $\mathcal{R}_i$, with $i \in \{1, \ldots, 6\}$. First, to estimate $\|\mathcal{R}_1\|_{\mathbb{H}(\mathbf{div}_{4/3};\Omega_{\mathrm{S}})'}$, we note that the BDM interpolant satisfies both [11, Lemma 4.2] and the properties stated in [11, Lemma 4.1]. Therefore, by following the same steps as in the proof of [11, Lemma 5.6], but using the BDM interpolant instead of the Raviart–Thomas interpolant, and applying the properties of the Clément interpolant, the desired estimate follows (see also [20, Section 5.1] and [38, Section 3.1] for similar results in Hilbert spaces). The proof of the upper bound for $\|\mathcal{R}_2\|_{\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})}$ follows the same lines as in [21, Lemma 3.6], under the assumption that there exists a convex domain $\mathcal{B}$ such that $\Omega_{\mathrm{D}} \subseteq \mathcal{B}$ and $\Gamma_{\mathrm{D}} \subseteq \partial\mathcal{B}$. The corresponding estimate for $\mathcal{R}_3$ and $\mathcal{R}_4$, follow from a straightforward application of Cauchy–Schwarz and Hölder inequalities (see [13, Section 5.1]), whereas the estimates for the terms acting on the interface Σ, $\mathcal{R}_5$ and $\mathcal{R}_6$, can be obtained by adapting the arguments from the proof of [38, Lemma 3.2]. We omit further details. □

5.2 Local efficiency of the a posteriori error estimator

We now aim to establish the efficiency estimate of Θ (cf. (5.3)). For this purpose, we will make extensive use of the original system of equations given by (2.2), (2.4), (2.9), and (2.10). These equations can be derived from the mixed continuous formulation (2.15) by choosing suitable test functions and integrating by parts backwardly the corresponding equations. The following theorem is the main result of this section.

Theorem 5.2 *There exist $C_{eff} > 0$, independent of h, such that*

$$C_{eff}\, \Theta \leq \|((\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{u}}) - ((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{u}}_h)\|\,. \tag{5.7}$$

We begin with the estimates for the zero order terms appearing in the definition of Θ and $\Theta_{\mathrm{S},T}$ (cf. (5.3), (5.4)).

Lemma 5.3 *For each $T \in \mathcal{T}_h^{\mathrm{S}}$ there hold*

$$\|\mathbf{f}_{\mathrm{S}} + \mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,4/3;T} \leq \|\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,4/3;T}$$

$$\text{and} \quad \left\|\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathrm{t}}\right\|_{0,T} \;\leq\; 2\,\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}\,.$$

Proof. It suffices to recall that $\mathbf{f}_{\mathrm{S}} = -\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S}})$ and $\boldsymbol{\sigma}_{\mathrm{S}} = \boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{t}}$ in Ω_{S} (cf. (2.9)). Further details are omitted. □

We now estimate the terms defined on $\Omega_{\mathrm{S}} \cup \Sigma$.

Lemma 5.4 *There exist positive constants $C_i > 0$, $i \in \{1, \ldots, 7\}$, independent of h, such that*

$$
\begin{aligned}
& h_T^{1/2} \left\| \boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h}))^{\mathrm{d}} \right\|_{0,T} \\
& \quad \le C_1 \Big\{ (1 + h_T^{1/2}) \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,4;T} + h_T^{1/2} \big\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\big\|_{0,T} + h_T^{1/2} \big\|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\big\|_{0,T} \Big\} \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}},
\end{aligned} \tag{5.8a}
$$

$$
h_e^{1/4} \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,4;e} \le C_2 \Big\{ (1 + h_T^{1/2}) \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,4;T} + h_T^{1/2} \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e} \Big\}, \tag{5.8b}
$$

for all $e \in \mathcal{E}_h(\Sigma)$, where T is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge,

$$
\begin{aligned}
& h_T \left\| \mathbf{rot} \left(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h}))^{\mathrm{d}} \right) \right\|_{0,T} \\
& \qquad \le C_3 \Big\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,4;T} + \big\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\big\|_{0,T} + \big\|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\big\|_{0,T} \Big\} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{S}},
\end{aligned} \tag{5.8c}
$$

$$
\begin{aligned}
& h_e^{1/2} \left\| \left[\!\left[\Big(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h}))^{\mathrm{d}} \Big)\mathbf{t} \right]\!\right] \right\|_{0,e} \\
& \qquad \le C_4 \Big\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,4;\omega_e} + \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,\omega_e} + \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,\omega_e} \Big\},
\end{aligned} \tag{5.8d}
$$

for all $e \in \mathcal{E}_h(\Omega_{\mathrm{S}})$, where the set ω_e is given by $\omega_e := \cup\big\{T' \in \mathcal{T}_h^{\mathrm{S}} : \quad e \in \mathcal{E}(T')\big\}$,

$$
\begin{aligned}
& h_e^{1/2} \left\| \Big(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h}))^{\mathrm{d}} \Big)\mathbf{t} \right\|_{0,e} \\
& \qquad \le C_5 \Big\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,4;T_e} + \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e} + \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T_e} \Big\},
\end{aligned} \tag{5.8e}
$$

for all $e \in \mathcal{E}_h(\Sigma)$, where T_e is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge,

$$
\begin{aligned}
& \sum_{e \in \mathcal{E}_h(\Sigma)} h_e \left\| \Big(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h}))^{\mathrm{d}} \Big)\mathbf{t} + \boldsymbol{\varphi}_h' \right\|_{0,e}^2 \\
& \quad \le C_6 \sum_{e \in \mathcal{E}_h(\Sigma)} \Big\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,4;T_e}^2 + \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e}^2 + \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T_e}^2 \Big\} + \big\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\big\|_{1/2,00;\Sigma}^2,
\end{aligned} \tag{5.8f}
$$

for all $e \in \mathcal{E}_h(\Sigma)$, where T_e is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge, and

$$
\begin{aligned}
& \sum_{e \in \mathcal{E}_h(\Sigma)} h_e \big\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} - \omega^{-1}(\boldsymbol{\varphi}_h \cdot \mathbf{t})\mathbf{t} + \lambda_h \mathbf{n} + \rho(\boldsymbol{\varphi}_h \otimes \boldsymbol{\varphi}_h)\mathbf{n}\big\|_{0,e}^2 \le C_7 \sum_{e \in \mathcal{E}_h(\Sigma)} \Big\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e}^2 \\
& \qquad + h_T^2 \|\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,T_e}^2 + h_e \big\|\lambda - \lambda_h\big\|_{0,e}^2 \Big\} + h \big\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\big\|_{1/2,00;\Sigma}^2,
\end{aligned} \tag{5.8g}
$$

for all $e \in \mathcal{E}_h(\Sigma)$, where T_e is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge.

Proof. Estimates (5.8a), (5.8b) are derived from slight modifications of [11, Lemmas 5.10 and 5.11], respectively. In particular, for (5.8b) we use the fact that $\boldsymbol{\varphi} = -\mathbf{u}_{\mathrm{S}}|_\Sigma$ and [11, eq. (5.20)], to deduce that

$$
\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,4;e} \,\le\, C\, h_e^{-1/4} \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e} \,\le\, C\, h^{-1/4} \Big(\|\mathbf{u}_{\mathrm{S}}|_\Sigma - \mathbf{u}_{\mathrm{S},h}\|_{0,e} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e} \Big).
$$

Thus, bounding $\|\mathbf{u}_{\mathrm{S}}|_\Sigma - \mathbf{u}_{\mathrm{S},h}\|_{0,e}$ as in [11, eqs. (5.21)–(5.23)], from the above we conclude (5.8b).

On the other hand, for (5.8c), (5.8d) and (5.8e) we first observe that the following estimate holds

$$\|(\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}}-(\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})^{\mathrm{d}}\|_{0,T}\leq\|\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h}\|_{0,T}.$$

Then, a similar algebraic manipulation to that used in [12, Corollary 4.10], combined with Hölder's inequality, estimates (2.61) and (3.22), and the smallness assumption on the data, yields

$$\big\|\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h}\big\|_{0,T}\;\leq\;\big(\|\mathbf{u}_{\mathrm{S}}\|_{0,4;T}+\|\mathbf{u}_{\mathrm{S},h}\|_{0,4;T}\big)\|\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\|_{0,4;T}\;\leq\;C\,\|\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\|_{0,4;T}\,,\tag{5.9}$$

with $C>0$ independent of h. In this way, defining $\boldsymbol{\zeta}:=\boldsymbol{\gamma}_{\mathrm{S}}+\frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S}}+\rho(\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}}))^{\mathrm{d}}=\nabla\mathbf{u}_{\mathrm{S}}$ and $\boldsymbol{\zeta}_h:=\boldsymbol{\gamma}_{\mathrm{S},h}+\frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S},h}+\rho(\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h}))^{\mathrm{d}}$, we apply [34, Lemmas 4.9, 4.10, and 4.15] to $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}_h$, make use of (5.9) and proceed analogously to [11, Lemma 5.12], to obtain (5.8c), (5.8d) and (5.8e). Now, to deduce (5.8f), defining $\boldsymbol{\vartheta}_h:=\boldsymbol{\gamma}_{\mathrm{S},h}+\frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S},h}+\rho(\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h}))^{\mathrm{d}}$ and $\boldsymbol{\vartheta}:=\boldsymbol{\gamma}_{\mathrm{S}}+\frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S}}+\rho(\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}}))^{\mathrm{d}}=\mathbf{0}$, proceeding similarly as in [29, Lemma 5.7] and employing (5.9), we have that

$$\begin{aligned}\sum_{e\in\mathcal{E}_h(\Sigma)}h_e\|\boldsymbol{\vartheta}_h\mathbf{t}_e-\boldsymbol{\varphi}_h'\|_{0,e}^2&\leq C\Big\{\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,\Sigma}^2+\sum_{e\in\mathcal{E}_h(\Sigma)}\|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_h\|_{0;T_e}^2\Big\}\\&\leq C\Big\{\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,\Sigma}+\sum_{e\in\mathcal{E}_h(\Sigma)}\big\{\|\mathbf{u}_{\mathrm{S}}-\mathbf{u}_{\mathrm{S},h}\|_{0,4;T_e}^2+\|\boldsymbol{\sigma}_{\mathrm{S}}-\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e}^2+\|\boldsymbol{\gamma}_{\mathrm{S}}-\boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T_e}^2\big\}\Big\},\end{aligned}$$

which implies (5.8f).

Finally, for (5.8g) we proceed analogously to [38, Lemma 3.16], to obtain

$$\begin{aligned}h_e\,\big\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n}-\omega^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}+\lambda_h\mathbf{n}+\rho(\boldsymbol{\varphi}_h\otimes\boldsymbol{\varphi}_h)\mathbf{n}\big\|_{0,e}^2&\leq C\Big\{\|\boldsymbol{\sigma}_{\mathrm{S}}-\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e}^2+h_T^2\|\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S}}-\boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,T_e}^2\\&+h_e\big\|\lambda-\lambda_h\big\|_{0,e}^2+h_e\big\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\big\|_{0,e}^2+h_e\big\|\boldsymbol{\varphi}\otimes\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\otimes\boldsymbol{\varphi}_h\big\|_{0,e}^2\Big\}.\end{aligned}\tag{5.10}$$

In turn, similarly as for (5.9), we observe that for the last term in (5.10) we have

$$\|\boldsymbol{\varphi}\otimes\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\otimes\boldsymbol{\varphi}_h\big\|_{0,e}^2\leq 2\big(\|\boldsymbol{\varphi}\|_{0,4;e}^2+\|\boldsymbol{\varphi}_h\|_{0,4;e}^2\big)\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,4;e}^2,\tag{5.11}$$

which implies

$$\begin{aligned}\sum_{e\in\mathcal{E}_h(\Sigma)}h_e\|\boldsymbol{\varphi}\otimes\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\otimes\boldsymbol{\varphi}_h\big\|_{0,e}^2&\leq 2h\sum_{e\in\mathcal{E}_h(\Sigma)}\big(\|\boldsymbol{\varphi}\|_{0,4;e}^2+\|\boldsymbol{\varphi}_h\|_{0,4;e}^2\big)\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,4;e}^2\\&\leq 2\sqrt{2}h\Big(\sum_{e\in\mathcal{E}_h(\Sigma)}\big(\|\boldsymbol{\varphi}\|_{0,4;e}^4+\|\boldsymbol{\varphi}_h\|_{0,4;e}^4\big)\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h(\Sigma)}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,4;e}^4\Big)^{1/2}\\&=2\sqrt{2}h\big(\|\boldsymbol{\varphi}\|_{0,4;\Sigma}^2+\|\boldsymbol{\varphi}_h\|_{0,4;\Sigma}^2\big)\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,4;\Sigma}^2.\end{aligned}\tag{5.12}$$

In addition, it is easy to see that

$$\sum_{e\in\mathcal{E}_h(\Sigma)}h_e\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2\;\leq\;h\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,\Sigma}^2\;\leq\;C\,h\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}^2\,.\tag{5.13}$$

In this way, from (5.10), (5.12), (5.13), the Sobolev embedding $\mathbf{H}_{00}^{1/2}(\Sigma)\hookrightarrow\mathbf{L}^4(\Sigma)$, and the fact that $\|\boldsymbol{\varphi}\|_{1/2,00;\Sigma}$ and $\|\boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}$ are bounded by data, we obtain (5.8g). □

We now estimate the terms defined on $\Omega_{\mathrm{D}}\cup\Sigma$.

Lemma 5.5 *There exist positive constants $\widehat{C}_i > 0$, $i \in \{1, \ldots, 6\}$, independent of h, such that*

$$h_T \left\|\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\right\|_{0,T} \leq \widehat{C}_1\Big\{\left\|p_{\mathrm{D}} - p_{\mathrm{D},h}\right\|_{0,T} + h_T\left\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\right\|_{0,T}\Big\} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}, \tag{5.14a}$$

$$h_T \left\|\mathrm{rot}\left(\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\right)\right\|_{0,T} \leq \widehat{C}_2\left\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\right\|_{0,T} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}, \tag{5.14b}$$

$$h_e^{1/2} \left\| [\![(\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}) \cdot \mathbf{t}]\!] \right\|_{0,e} \leq \widehat{C}_3\left\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\right\|_{0,\omega_e}, \tag{5.14c}$$

for all $e \in \mathcal{E}_h(\Omega_{\mathrm{D}})$, where the set ω_e is given by $\omega_e := \cup\{T' \in \mathcal{T}_h^{\mathrm{D}} : \quad e \in \mathcal{E}(T')\}$,

$$h_e^{1/2} \left\|p_{\mathrm{D},h} - \lambda_h\right\|_{0,e} \leq \widehat{C}_4\Big\{\left\|p_{\mathrm{D}} - p_{\mathrm{D},h}\right\|_{0,T} + h_T \left\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\right\|_{0,T} + h_e^{1/2} \left\|\lambda - \lambda_h\right\|_{0,e}\Big\}, \tag{5.14d}$$

for all $e \in \mathcal{E}_h(\Sigma)$, where T is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge,

$$\sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \left\|\left(\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\right) \cdot \mathbf{t} - \lambda_h'\right\|_{0,e}^2 \leq \widehat{C}_5\Big\{\sum_{e \in \mathcal{E}_h(\Sigma)} \left\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\right\|_{0,T_e}^2 + \left\|\lambda - \lambda_h\right\|_{1/2,\Sigma}^2\Big\}, \tag{5.14e}$$

where given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge, and

$$h_e^{1/2} \left\|(\mathbf{u}_{\mathrm{D},h} + \boldsymbol{\varphi}_h) \cdot \mathbf{n}\right\|_{0,e} \leq \widehat{C}_6\Big\{\left\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\right\|_{0,T_e} + h_T\left\|\mathrm{div}(\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h})\right\|_{0,T_e} + h_e^{1/2} \left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\right\|_{0,e}\Big\}, \tag{5.14f}$$

for all $e \in \mathcal{E}_h(\Sigma)$, where T_e is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge.

Proof. For (5.14a) we refer to [17, Lemma 6.3]. The estimation provided by (5.14b) aligns with [6, Lemma 4.3]. The proof of (5.14c) follows directly from [17, Lemma 6.2]. However, for (5.14d) we refer to [4, Lemma 4.12]. The proofs of (5.14e) can be obtained from slight modifications of the proof of [29, Lemma 5.7]. The estimation given by (5.14f) corresponds to [38, Lemma 3.15]. □

Notice that the estimates (5.8f), (5.8g) and (5.14e) in the preceding lemmas are the sole ones offering non-local bounds. However, with additional regularity assumptions on λ and $\boldsymbol{\varphi}$, we can instead provide the following local bounds.

Lemma 5.6 *There exist $C_1 > 0$, such that for each $e \in \mathcal{E}_h(\Sigma)$ there hold*

$$\begin{aligned} h_e \left\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} - \omega^{-1}(\boldsymbol{\varphi}_h \cdot \mathbf{t})\mathbf{t} + \lambda_h\mathbf{n} + \rho(\boldsymbol{\varphi}_h \otimes \boldsymbol{\varphi}_h)\mathbf{n}\right\|_{0,e}^2 &\leq C_1\Big\{\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e}^2 \\ &+ h_T^2\|\mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,T_e}^2 + h_e\left\|\lambda - \lambda_h\right\|_{0,e}^2 + h_e\left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\right\|_{0,e}^2 + h_e\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,4;e}^2\Big\}. \end{aligned} \tag{5.15a}$$

In addition, if $\lambda|_e \in \mathrm{H}^1(e)$ and $\boldsymbol{\varphi}|_e \in \mathbf{H}^1(e)$, for each $e \in \mathcal{E}_h(\Sigma)$. Then there exist $C_2\, C_3 > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$ there hold

$$\begin{aligned} h_e^{1/2} \left\|\Big(\boldsymbol{\gamma}_{\mathrm{S},h} + \frac{1}{2\nu}(\boldsymbol{\sigma}_{\mathrm{S},h} + \rho(\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h}))^{\mathrm{d}}\Big)\mathbf{t} + \boldsymbol{\varphi}_h'\right\|_{0,e} &\leq C_2\Big\{\|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,4;T_e} \\ &+ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e} + \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T_e} + h_e^{1/2}\left\|\boldsymbol{\varphi}' - \boldsymbol{\varphi}_h'\right\|_{0,e}\Big\}, \end{aligned} \tag{5.15b}$$

$$h_e^{1/2} \left\|\left(\mathbf{f}_{\mathrm{D}} - \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\right) \cdot \mathbf{t} - \lambda_h'\right\|_{0,e} \leq C_3\Big\{\left\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\right\|_{0,T_e} + h_e^{1/2} \left\|\lambda' - \lambda_h'\right\|_{0,e}\Big\}, \tag{5.15c}$$

Proof. The proof of (5.15a) follows from (5.10), (5.11), estimates $\|\boldsymbol{\varphi}\|_{0,4;e} \leq c\|\boldsymbol{\varphi}\|_{1/2,00;\Sigma}$, $\|\boldsymbol{\varphi}_h\|_{0,4;e} \leq c\|\boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}$, and the fact that $\|\boldsymbol{\varphi}\|_{1/2,00;\Sigma}$ and $\|\boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}$ are bounded by data. In turn, estimates (5.15b) and (5.15c) follow from slight modifications of [32, Lemma 21]. □

We end this section by observing that the required efficiency of the *a posteriori* error estimator Θ (cf. (5.7)) is a direct consequence of Lemmas 5.3–5.6. In particular, the term $h_e\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2$ (analogously for $\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2$) appearing in Lemmas 5.4 and 5.5 can be bound as in (5.13) and $h_e\,\|\lambda-\lambda_h\|_{0,e}^2$ is bounded as follows:

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\lambda-\lambda_h\|_{0,e}^2 \;\leq\; h\,\|\lambda-\lambda_h\|_{0,\Sigma}^2 \;\leq\; C\,h\,\|\lambda-\lambda_h\|_{1/2,\Sigma}^2\,.$$

6 Numerical Results

This section illustrates the performance and accuracy of the fully-mixed finite element scheme (3.4) using the spaces defined in (4.1) and (4.2), along with the reliability and efficiency properties of the *a posteriori* error estimator Θ (cf. (5.3)) derived in Section 5. The implementation is based on a `FreeFEM` code [43]. In order to solve the nonlinear problem (3.4), given $(\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi})\in\mathbf{L}^4(\Omega_{\mathrm{S}})\times\mathbf{H}^{1/2}_{00}(\Sigma)$ we introduce the Gâteaux derivatives associated to $\mathbf{C}_{\boldsymbol{\phi}}$ and $\mathbf{C}_{\mathbf{w}_{\mathrm{S}}}$ (cf. (2.19)), i.e.,

$$\mathcal{D}\mathbf{C}(\boldsymbol{\phi})(\underline{\boldsymbol{\varphi}},\underline{\boldsymbol{\psi}}) := \rho\langle\boldsymbol{\phi}\cdot\mathbf{n},\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\rangle_\Sigma + \rho\langle\boldsymbol{\varphi}\cdot\mathbf{n},\boldsymbol{\phi}\cdot\boldsymbol{\psi}\rangle_\Sigma$$

and

$$\mathcal{D}\mathbf{C}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{u}},\underline{\boldsymbol{\tau}}) := \frac{\rho}{2\,\nu}((\mathbf{w}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}}+(\mathbf{u}_{\mathrm{S}}\otimes\mathbf{w}_{\mathrm{S}})^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}}\,,$$

for all $\underline{\boldsymbol{\varphi}}\in\mathbf{Y},\underline{\mathbf{u}}\in\mathbb{Q}$ and $(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}})\in\mathbb{H}$, and the functional

$$\mathbf{F}(\mathbf{w}_{\mathrm{S}},\boldsymbol{\phi})(\underline{\boldsymbol{\tau}},\underline{\boldsymbol{\psi}}) \;:=\; (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \frac{\rho}{2\,\nu}((\mathbf{w}_{\mathrm{S}}\otimes\mathbf{w}_{\mathrm{S}})^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + \rho\,\langle\boldsymbol{\phi}\cdot\mathbf{n},\boldsymbol{\phi}\cdot\boldsymbol{\psi}\rangle_\Sigma\,.$$

In this way, we propose the Newton-type strategy: Given $(\mathbf{u}^0_{\mathrm{S},h},\boldsymbol{\varphi}^0_h)\in\mathbf{L}^{\mathrm{S}}_h\times\boldsymbol{\Lambda}^{\mathrm{S}}_h$, for $m\geq 1$, find $((\underline{\boldsymbol{\sigma}}^m_h,\underline{\boldsymbol{\varphi}}^m_h),\underline{\mathbf{u}}^m_h)\in\mathbb{H}_h\times\mathbb{Q}_h$, such that

$$\begin{aligned}
&A((\underline{\boldsymbol{\sigma}}^m_h,\underline{\boldsymbol{\varphi}}^m_h),(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h)) + \mathcal{D}\mathbf{C}(\boldsymbol{\varphi}^{m-1}_h)(\underline{\boldsymbol{\varphi}}^m_h,\underline{\boldsymbol{\psi}}_h)\\
&\qquad + B((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{u}}^m_h) + \mathcal{D}\mathbf{C}(\mathbf{u}^{m-1}_{\mathrm{S},h})(\underline{\mathbf{u}}^m_h,\underline{\boldsymbol{\tau}}_h) \;=\; \mathbf{F}(\mathbf{u}^{m-1}_{\mathrm{S},h},\boldsymbol{\varphi}^{m-1}_h)(\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h)\,,\\
&B((\underline{\boldsymbol{\sigma}}^m_h,\underline{\boldsymbol{\varphi}}^m_h),\underline{\mathbf{v}}_h) \;=\; -(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S},h})_{\mathrm{S}}\,,
\end{aligned}\tag{6.1}$$

for all $((\underline{\boldsymbol{\tau}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{v}}_h)\in\mathbb{H}_h\times\mathbb{Q}_h$.

The iterative method is stopped when the relative error between two consecutive iterations of the complete coefficient vector, namely coeff^m and coeff^{m+1}, is sufficiently small, that is

$$\frac{\|\mathbf{coeff}^{m+1}\;-\;\mathbf{coeff}^m\|_{\mathtt{DoF}}}{\|\mathbf{coeff}^{m+1}\|_{\mathtt{DoF}}}\;\leq\;\mathtt{tol},$$

where $\|\cdot\|_{\mathtt{DoF}}$ stands for the usual Euclidean norm in $\mathbb{R}^{\mathtt{DoF}}$, with `DoF` denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{H}_h$ and $\mathbb{Q}_h$, and tol is a fixed tolerance chosen as $\mathsf{tol} = 1\mathrm{E}-06$.

The global error and the effectivity index associated to the global estimator Θ (cf. (5.3)) are denoted, respectively, by

$$\mathsf{e}(\vec{\mathbf{t}}) := \mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}) + \mathsf{e}(\mathbf{u}_{\mathrm{S}}) + \mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}}) + \mathsf{e}(\mathbf{u}_{\mathrm{D}}) + \mathsf{e}(p_{\mathrm{D}}) + \mathsf{e}(\boldsymbol{\varphi}) + \mathsf{e}(\lambda) \quad \text{and} \quad \mathsf{eff}(\Theta) := \frac{\mathsf{e}(\vec{\mathbf{t}})}{\Theta},$$

where

$$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}) := \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div}_{4/3};\Omega_{\mathrm{S}}}, \quad \mathsf{e}(\mathbf{u}_{\mathrm{S}}) := \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,4;\Omega_{\mathrm{S}}}, \quad \mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}}) := \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}},$$

$$\mathsf{e}(\mathbf{u}_{\mathrm{D}}) := \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}, \quad \mathsf{e}(p_{\mathrm{D}}) := \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}},$$

$$\mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}, \quad \mathsf{e}(\lambda) := \|\lambda - \lambda_h\|_{1/2,\Sigma}.$$

Notice that, for ease of computation, the interface norm $\|\lambda-\lambda_h\|_{1/2,\Sigma}$ will be replaced by $\|\lambda-\lambda_h\|_{(0,1),\Sigma}$ with

$$\|\xi\|_{(0,1),\Sigma} := \|\xi\|_{0,\Sigma}^{1/2}\,\|\xi\|_{1,\Sigma}^{1/2} \quad \forall\, \xi \in \mathrm{H}^1(\Sigma),$$

owing to the fact that $\mathrm{H}^{1/2}(\Sigma)$ is the interpolation space with index $1/2$ between $\mathrm{H}^1(\Sigma)$ and $\mathrm{L}^2(\Sigma)$. Similarly, the interface norm $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}$ will be replaced by $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{(0,1),\Sigma}$. We emphasize that the fluid pressure and its corresponding error can be computed via the postprocessing formula presented in (2.7). However, for simplicity, in the examples below, we only present plots based on the following formula:

$$p_{\mathrm{S},h} := -\frac{1}{n}\Big\{\mathrm{tr}(\boldsymbol{\sigma}_{\mathrm{S},h}) + \rho\,\mathrm{tr}(\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})\Big\} - \ell_h \quad \text{in} \quad \Omega_{\mathrm{S}}.$$

Moreover, using the fact that $\mathtt{DoF}^{-1/n} \cong h$, with $n = 2$ and $h = \max\big\{h_{\mathrm{S}}, h_{\mathrm{D}}, \widehat{h}_\Sigma\big\}$, the respective experimental rates of convergence are computed as

$$\mathsf{r}(\diamond) := -2\,\frac{\log(\mathsf{e}(\diamond)/\mathsf{e}'(\diamond))}{\log(\mathtt{DoF}/\mathtt{DoF}')} \quad \text{for each } \diamond \in \Big\{\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\gamma}_{\mathrm{S}}, p_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, p_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda, \vec{\mathbf{t}}\Big\},$$

where $\mathtt{DoF}$ and $\mathtt{DoF}'$ denote the total degrees of freedom associated to two consecutive triangulations with errors e and e', respectively. In turn, we take $\widehat{h}_\Sigma$ as two times h_Σ, which comes from the restriction on the mesh sizes $h_\Sigma \le C_0\,\widehat{h}_\Sigma$ when considering the constant $C_0 = 1/2$.

The examples considered in this section are described below. In all of them, for simplicity, we take $(\mathbf{u}_{\mathrm{S},h}^0, \boldsymbol{\varphi}_h^0) = (\mathbf{0},\mathbf{0})$ and impose the conditions $(\mathrm{tr}(\boldsymbol{\sigma}_{\mathrm{S},h}),1)_{\mathrm{S}} = 0$ and $(p_{\mathrm{D},h},1)_{\mathrm{D}} = 0$ using a penalization strategy. In turn, in the first two examples we consider the model parameters $\nu = 1$, $\rho = 1$, $\omega = 1$ and the tensor $\mathbf{K}$ is taken as the identity matrix $\mathbb{I}$, which satisfy (2.3).

Example 1 is used to corroborate the reliability and efficiency of the *a posteriori* error estimator Θ, whereas Examples 2 and 3 are utilized to illustrate the behavior of the associated adaptive algorithm in 2D domains with and without manufactured solution, respectively, which applies the following procedure from [50]:

(1) Start with a coarse mesh $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$.

(2) Solve the Newton iterative method (6.1) for the current mesh $\mathcal{T}_h$.

(3) Compute the local indicator Θ_T for each $T \in \mathcal{T}_h := \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, where

$$\Theta_T := \Big\{\Theta_{\mathrm{S},T}^2 + \Theta_{\mathrm{D},T}^2\Big\}^{1/2} + \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}(\boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,4/3;T}, \qquad \text{(cf. (5.4), (5.5))}$$

(4) Check the stopping criterion and decide whether to finish or go to next step.

(5) Use the automatic meshing algorithm adaptmesh from [44, Section 9.1.9] to refine each $T' \in \mathcal{T}_h$ satisfying:

$$\Theta_{T'} \geq C_{\mathsf{adm}} \frac{1}{\# T} \sum_{T \in \mathcal{T}_h} \Theta_T, \quad \text{for some } C_{\mathsf{adm}} \in (0,1), \tag{6.2}$$

where $\# T$ denotes the number of triangles of the mesh $\mathcal{T}_h$.

(6) Define resulting meshes as current meshes $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$, and go to step (2).

In particular, in Examples 2 and 3 below we take $C_{\mathsf{adm}} = 0.72$ (cf. (6.2)).

Example 1: Accuracy assessment in a 2D tombstone-shaped domain

In the first example, we focus on the accuracy of the fully-mixed method (6.1) (cf. (3.4)). We consider a semi-disk-shaped fluid domain coupled with a porous unit square, i.e.,

$$\Omega_{\mathrm{S}} := \Big\{(x_1,x_2): \quad x_1^2 + (x_2 - 0.5)^2 \, < \, 0.5^2, \; x_2 > 0.5\Big\} \quad \text{and} \quad \Omega_{\mathrm{D}} := (-0.5, 0.5)^2 \,,$$

with interface $\Sigma := (-0.5, 0.5) \times \{0.5\}$. We choose the data $\mathbf{f}_{\mathrm{S}}$ and $\mathbf{f}_{\mathrm{D}}$ such that a manufactured solution in the tombstone-shaped domain $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ is given by

$$\mathbf{u}_{\mathrm{S}}(x_1,x_2) := \begin{pmatrix} \cos(\pi x_1)\,\sin(\pi x_2) \\ -\sin(\pi x_1)\,\cos(\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \quad \mathbf{u}_{\mathrm{D}}(x_1,x_2) := \begin{pmatrix} \pi\,\exp(x_1)\,\sin(\pi x_2) \\ \exp(x_1)\,\cos(\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{D}},$$

$$p_\star(x_1,x_2) := \sin(\pi x_1)\,\sin(\pi x_2) \quad \text{in} \quad \Omega_\star\,, \quad \star \in \{\mathrm{S}, \mathrm{D}\}\,.$$

Notice that this solution satisfies $\mathbf{u}_{\mathrm{S}} \cdot \mathbf{n} = \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n}$ on Σ. However, the second transmission condition in (2.4) (cf. (2.10)) is not satisfied, and both the Dirichlet boundary condition for the Navier–Stokes velocity on Γ_{S} and the Neumann boundary condition for the Darcy velocity on Γ_{D} are non-homogeneous. This introduces additional contributions that are included in the right-hand side of the resulting system and in the *a posteriori* error estimator.

The errors and associated rates of convergence are reported in Table 1, which align with the theoretical bounds established in Theorem 4.1 for the finite element spaces described in Section 4. Additionally, we compute the global *a posteriori* error indicator Θ (cf. (5.3)) and assess its reliability and efficiency using the effectivity index. Note that the estimator remains consistently bounded. The domain configuration, along with some components of the numerical solution, is displayed in Figure 2. These results were computed using the AFW+Raviart–Thomas-based discretization (cf. (4.1)–(4.2)) with $53,511$ triangular elements (corresponding to $227,206$ DoF). We observe that the continuity of the normal trace of the velocities on Σ is preserved, as the second components of $\mathbf{u}_{\mathrm{S}}$ and $\mathbf{u}_{\mathrm{D}}$ coincide on Σ, as expected. It can also be seen that the pressure is continuous throughout the domain and maintains its sinusoidal behavior.

Example 2: Adaptivity in a 2D horseshoe-shaped domain

The second example is aimed at testing the features of adaptive mesh refinement based on the *a posteriori* error estimator Θ (cf. (5.3)). We consider the 2D horseshoe-shaped domain $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$,

DoF	it	h_S	$e(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$e(\boldsymbol{\gamma}_S)$	$r(\boldsymbol{\gamma}_S)$	$\widehat{h}_\Sigma$	$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$
263	3	0.330	2.3E-00	–	1.8E-01	–	2.8E-01	–	1/2	2.5E-01	–
961	4	0.191	1.2E-00	1.02	9.3E-02	1.00	1.7E-01	0.76	1/4	9.7E-02	1.48
3,629	4	0.091	5.5E-01	1.15	4.5E-02	1.10	8.2E-02	1.11	1/8	3.3E-02	1.62
14,280	4	0.049	2.7E-01	1.05	2.2E-02	1.02	4.2E-02	0.99	1/16	1.2E-02	1.52
57,479	4	0.024	1.3E-01	1.03	1.1E-02	1.01	2.0E-02	1.03	1/32	4.1E-03	1.50
227,206	4	0.013	6.5E-02	1.01	5.6E-03	1.00	1.0E-02	1.00	1/64	1.5E-03	1.51

h_D	$e(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$e(p_D)$	$r(p_D)$	$\widehat{h}_\Sigma$	$e(\lambda)$	$r(\lambda)$	$e(\vec{\mathbf{t}})$	$r(\vec{\mathbf{t}})$	Θ	$\mathsf{eff}(\Theta)$
0.373	6.6E-01	–	1.2E-01	–	1/2	3.0E-01	–	2.4E-00	–	7.1E-00	0.339
0.190	3.0E-01	1.20	5.9E-02	1.12	1/4	9.6E-02	1.76	1.2E-00	1.04	3.5E-00	0.354
0.095	1.5E-01	1.10	3.0E-02	1.02	1/8	3.3E-02	1.61	5.8E-01	1.15	1.7E-00	0.345
0.054	7.4E-02	0.98	1.5E-02	1.00	1/16	1.2E-02	1.52	2.8E-01	1.04	8.4E-01	0.337
0.025	3.7E-02	1.00	7.5E-03	1.00	1/32	4.1E-03	1.50	1.4E-01	1.03	4.1E-01	0.334
0.014	1.8E-02	1.01	3.7E-03	1.02	1/64	1.4E-03	1.51	6.9E-02	1.01	2.1E-01	0.333

Table 1: [Example 1] Number of degrees of freedom, Newton iteration count, meshsizes, errors, rates of convergence, global estimator, and effectivity index for the AFW+Raviart–Thomas-based discretization of the coupled Navier–Stokes/Darcy problem.

where $\Omega_D := (-1,1)\times(-0.5,0), \Sigma := (-1,1)\times\{0\}$, and $\Omega_S := (-1,1)\times(0,1.25)\setminus\widetilde{\Omega}_S$ with $\widetilde{\Omega}_S := (-0.75,0.75)\times(0.25,1.25)$. The data $\mathbf{f}_S$ and $\mathbf{f}_D$ are chosen so that the exact solution is given by

$$\mathbf{u}_S := \begin{pmatrix} \dfrac{(x_2-0.26)}{r_1(x_1,x_2)} + \dfrac{(x_2-0.26)}{r_2(x_1,x_2)} \\ -\dfrac{(x_1+0.74)}{r_1(x_1,x_2)} - \dfrac{(x_1-0.74)}{r_2(x_1,x_2)} \end{pmatrix}, \quad \mathbf{u}_D := \begin{pmatrix} \sin(2\,\pi\,x_1)\,\exp(x_2) \\ \sin(2\,\pi\,x_2)\,\exp(x_1) \end{pmatrix},$$

$$p_\star := x_2\sin(\pi x_1) \quad \text{in} \quad \Omega_\star, \qquad \text{with } \star\in\{\mathrm{S},\mathrm{D}\},$$

where

$$r_1(x_1,x_2) := \sqrt{(x_1+0.74)^2+(x_2-0.26)^2} \quad \text{and} \quad r_2(x_1,x_2) := \sqrt{(x_1-0.74)^2+(x_2-0.26)^2}.$$

Tables 2 and 3 along with Figure 3, summarize the convergence history of the method applied to a sequence of quasi-uniform and adaptively refined triangulations of the domain. Suboptimal rates are observed in the first case, whereas adaptive refinement based on the *a posteriori* error indicator Θ yields optimal convergence and stable effectivity indexes. Notice how the adaptive algorithms improve the method's efficiency by providing high-quality solutions at a lower computational cost. For instance, it is possible to achieve a better result (in terms of $\mathsf{e}(\vec{\mathbf{t}})$) with approximately only 2.5% of the degrees of freedom compared to the last quasi-uniform mesh for the fully-mixed scheme. Furthermore, Figure 4 shows the domain configuration in the initial mesh, the second component of the velocity, and the pressure field over the entire domain, computed using the adaptive AFW+Raviart–Thomas-based scheme (via Θ) with $662,948$ degrees of freedom and $78,623$ triangles. We observe that the second component of the velocity exhibits high gradients near the vertices $(-0.75,0.25)$ and $(0.75,0.25)$. Examples of some adapted meshes are shown in Figure 5, where a clear clustering of elements near the vertices in Ω_S of the 2D horseshoe-shaped domain is observed, as expected.

DoF	it	h_{S}	$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r(\boldsymbol{\gamma}_{\mathrm{S}})$	$\widehat{h}_{\Sigma}$	$\mathsf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$
1,701	4	0.188	4.7E+01	–	2.3E-01	–	1.9E-00	–	1/4	3.3E-01	–
6,914	4	0.100	3.7E+01	0.34	1.6E-01	0.48	1.2E-00	0.69	1/8	1.0E-01	1.65
26,067	4	0.050	2.6E+01	0.53	1.1E-01	0.60	7.2E-01	0.73	1/16	4.2E-02	1.38
104,256	4	0.026	1.5E+01	0.76	6.0E-02	0.89	4.0E-01	0.85	1/32	1.5E-02	1.47
410,453	4	0.014	8.1E-00	0.94	3.1E-02	0.96	2.1E-01	0.95	1/64	6.7E-03	1.18
1,644,689	4	0.007	4.2E-00	0.95	1.5E-02	1.04	1.1E-01	0.92	1/128	2.5E-03	1.44

h_{D}	$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathsf{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\widehat{h}_{\Sigma}$	$\mathsf{e}(\lambda)$	$r(\lambda)$	$\mathsf{e}(\vec{\mathbf{t}})$	$r(\vec{\mathbf{t}})$	Θ	$\mathsf{eff}(\Theta)$
0.200	1.3E-00	–	4.3E-02	–	1/4	6.8E-02	–	4.7E+01	–	5.6E+01	0.847
0.095	6.2E-01	1.04	1.4E-02	1.65	1/8	1.2E-02	2.53	3.7E+01	0.34	4.3E+01	0.861
0.049	3.2E-01	1.02	6.8E-03	1.06	1/16	3.2E-03	1.94	2.6E+01	0.53	3.1E+01	0.838
0.026	1.6E-01	0.99	3.4E-03	1.00	1/32	1.2E-03	1.43	1.5E+01	0.76	1.9E+01	0.821
0.013	7.9E-02	1.01	1.7E-03	1.01	1/64	4.3E-04	1.45	8.1E-00	0.94	9.9E-00	0.819
0.007	4.0E-02	1.00	8.5E-04	1.00	1/128	1.3E-04	1.73	4.2E-00	0.95	5.1E-00	0.817

Table 2: [Example 2] Number of degrees of freedom, Newton iteration count, meshsizes, errors, rates of convergence, global estimator, and effectivity index for the AFW+Raviart–Thomas-based discretization with quasi-uniform refinement for the coupled Navier–Stokes/Darcy problem.

DoF	it	$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathsf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$
1,701	4	4.7E+01	–	2.3E-01	–	1.9E-00	–	3.3E-01	–
3,326	3	3.2E+01	1.19	1.4E-01	1.57	9.3E-01	2.12	2.4E-01	0.92
5,637	4	1.5E+01	2.71	1.0E-01	1.16	4.2E-01	3.01	2.3E-01	0.20
9,928	4	7.6E-00	2.51	7.6E-02	0.97	2.4E-01	2.04	1.4E-01	1.68
20,117	4	4.9E-00	1.24	5.2E-02	1.10	1.6E-01	1.05	8.9E-02	1.36
40,759	4	3.5E-00	0.95	3.6E-02	1.04	1.1E-01	1.04	3.9E-02	2.30
79,674	4	2.5E-00	1.01	2.5E-02	1.10	8.1E-02	0.98	2.3E-02	1.57
161,769	4	1.8E-00	0.94	1.7E-02	1.03	5.7E-02	1.02	1.3E-02	1.74
323,197	4	1.3E-00	0.97	1.2E-02	1.07	4.1E-02	0.95	6.9E-03	1.73
662,948	4	9.1E-01	0.96	8.1E-03	1.06	2.8E-02	1.02	3.8E-03	1.69

$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathsf{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathsf{e}(\lambda)$	$r(\lambda)$	$\mathsf{e}(\vec{\mathbf{t}})$	$r(\vec{\mathbf{t}})$	Θ	$\mathsf{eff}(\Theta)$
1.29E-00	–	4.3E-02	–	6.8E-02	–	4.7E+01	–	5.6E+01	0.847
1.33E-00	–	2.8E-02	1.26	8.1E-02	–	3.2E+01	1.19	3.7E+01	0.866
1.32E-00	0.02	2.7E-02	0.15	9.0E-02	–	1.6E+01	2.70	1.9E+01	0.823
1.1E-00	0.49	3.2E-02	–	1.1E-01	–	7.7E-00	2.48	1.0E+01	0.768
8.4E-01	0.88	2.4E-02	0.85	7.6E-02	0.95	5.0E-00	1.23	6.6E-00	0.754
6.0E-01	0.96	1.7E-02	1.01	6.2E-02	0.57	3.6E-00	0.95	4.7E-00	0.754
4.4E-01	0.92	1.2E-02	1.03	3.0E-02	2.20	2.5E-00	1.01	3.4E-00	0.752
3.0E-01	1.08	7.4E-03	1.31	1.9E-02	1.29	1.8E-00	0.95	2.4E-00	0.754
2.3E-01	0.80	6.0E-03	0.60	1.4E-02	0.77	1.3E-00	0.96	1.7E-00	0.752
1.6E-01	1.06	3.8E-03	1.30	7.4E-03	1.85	9.2E-01	0.96	1.2E-00	0.750

Table 3: [Example 2] Number of degrees of freedom, Newton iteration count, errors, rates of convergence, global estimator, and effectivity index for the AFW+Raviart–Thomas-based discretization with adaptive refinement via Θ for the coupled Navier–Stokes/Darcy problem.

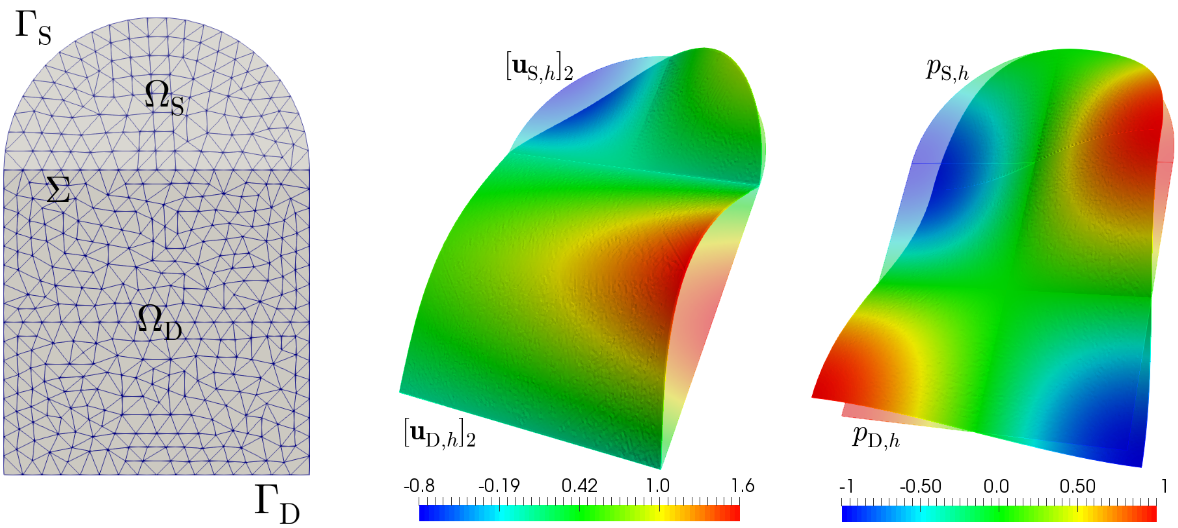


Figure 2: [Example 1] Domain configuration, second velocity component and pressure field in the whole domain.

Example 3: Flow through a rectangular domain with a porous filter

Inspired by [49, Test case 1 in Section 4.2] and [18, Example 2 in Section 5], we finally focus on studying the behavior of the Navier–Stokes/Darcy model for fluid flow through a rectangular domain with a square porous filter. The domain is defined by the rectangle $\Omega = \Omega_S \cup \Sigma \cup \Omega_D$, where $\Omega_D := (0.4, 0.6) \times (0, 0.2)$, $\Omega_S := (0, 2.6) \times (0, 0.3) \setminus \Omega_D$, and $\Sigma := \{0.4\} \times (0, 0.2) \cup (0.4, 0.6) \times \{0.2\} \cup \{0.6\} \times (0, 0.2)$, with boundaries $\Gamma_S := \Gamma_S^{\rm in} \cup \Gamma_S^{\rm top} \cup \Gamma_S^{\rm bottom} \cup \Gamma_S^{\rm out}$ and $\Gamma_D := (0.4, 0.6) \times \{0\}$, respectively, as detailed in the top plot of Figure 6. The model parameters are takes as

$$\nu = 10^{-6}\,, \quad \rho = 10^{-2}\,, \quad \mathbf{K} = [0.505, -0.495; -0.495, 0.505] \times 10^{-4}\,, \quad \text{and} \quad \omega = \nu^{-1}\sqrt{(\mathbf{K}\mathbf{t})\cdot\mathbf{t}}\,,$$

where the permeability tensor $\mathbf{K}$ is obtained as in [49, eq. (33)]:

$$\mathbf{K} = \mathbf{R}(\alpha)\begin{pmatrix} \frac{1}{100}\kappa & 0 \\ 0 & \kappa \end{pmatrix}\mathbf{R}^{-1}(\alpha)\,, \quad \text{with} \quad \mathbf{R}(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

with $\alpha = 45^\circ$ and $\kappa = 10^{-4}$, taking into account the effect of material anisotropy on the flow. In turn, the right-hand side data $\mathbf{f}_S$ and $\mathbf{f}_D$ are chosen as zero, and the boundary conditions are

$$\mathbf{u}_S = \begin{cases} (-x_2\,(x_2 - 0.3), 0)^{\rm t} & \text{on} \quad \Gamma_S^{\rm in} \cup \Gamma_S^{\rm out}\,, \\ (0,0)^{\rm t} & \text{on} \quad \Gamma_S^{\rm top} \cup \Gamma_S^{\rm bottom}\,, \end{cases} \quad \text{and} \quad \mathbf{u}_D \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_D\,.$$

In particular, the first row of boundary equations corresponds to the inflow/outflow driven by a parabolic fluid velocity from the left to the right boundary of the fluid region. In Table 4, we present the experimental rate of convergence of the global estimator Θ (cf. (5.3)) for six adapted meshes. Although the effectivity index cannot be computed for this example, the optimal convergence rate of Θ illustrates the reliability and efficiency of the estimator, as well as the accuracy of the obtained approximation in handling small viscosity and permeability. In Figure 6, we display the initial mesh, the computed magnitude of the velocities and pressures, which were built using the AFW+Raviart–Thomas-based scheme on a mesh with 27,377 triangle elements (actually representing 246,649 DoF)

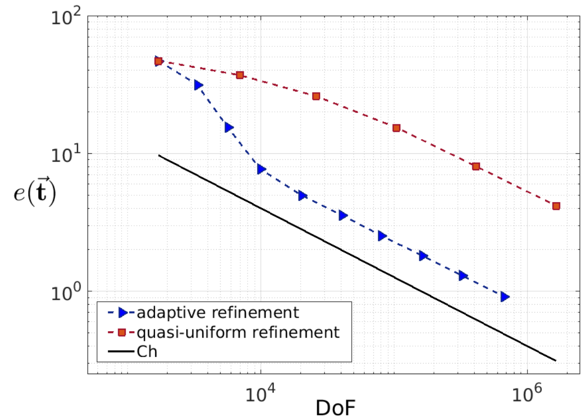


Figure 3: [Example 2] Log-log plot of $\mathsf{e}(\vec{\mathbf{t}})$ vs. `DoF` for quasi-uniform/adaptive refinements.

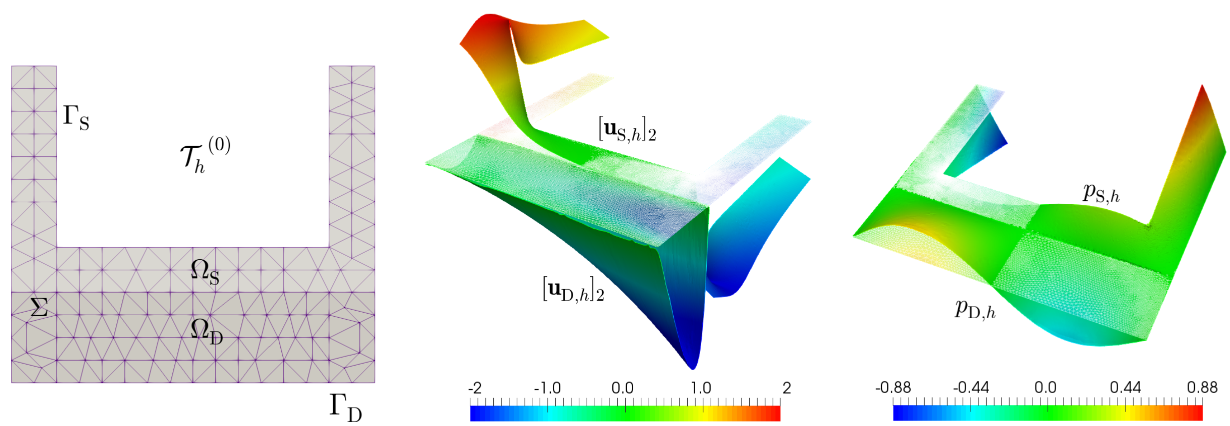


Figure 4: [Example 2] Initial mesh, second velocity component and pressure field in the whole domain.

obtained via Θ. We observe high velocity in the open channel above the filter and a vortex behind the obstacle. A sharp pressure gradient is evident in the region above the filter and within the filter, where the permeability anisotropy affects both the pressure and velocity fields. This example demonstrates the fully-mixed method's ability to produce an oscillation-free solution under challenging physical conditions, including low viscosity and permeability. Additionally, snapshots of some adapted meshes generated using Θ are shown in Figure 7. We observe appropriate refinement in regions with higher velocity and pressure. This suggests that the indicator Θ effectively addresses challenging model parameters and localizes the areas where the solutions are more significant, efficiently utilizing computational resources where they are most needed.

7 Concluding remarks

In this work, we have introduced and analyzed a novel Banach-space-based pseudostress-velocity-vorticity formulation for the coupled Navier–Stokes/Darcy system. The proposed approach treats the free-flow and porous-medium regions in a unified mixed finite element framework, incorporating

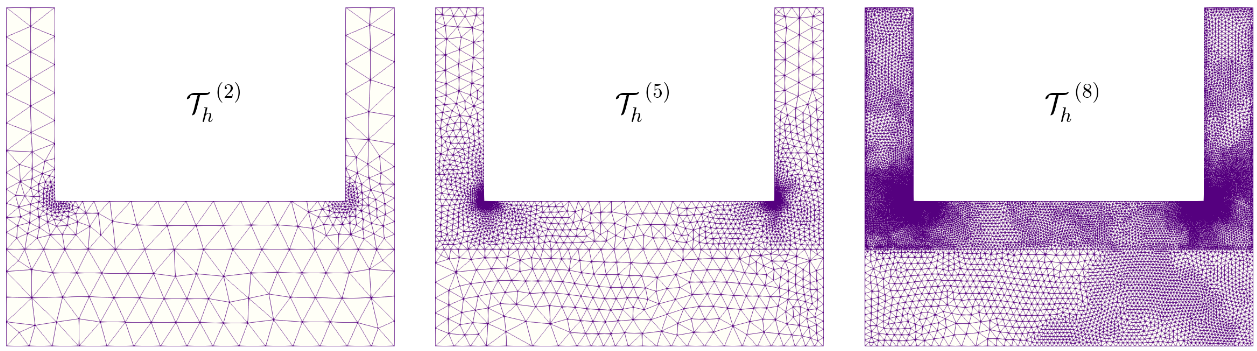


Figure 5: [Example 2] Three snapshots of adapted meshes according to the indicator Θ.

DoF	6491	11373	29885	62847	128904	246649
it	6	8	10	10	10	10
Θ	2.3E-00	1.3E-00	7.2E-01	4.9E-01	3.4E-01	2.5E-01
$r(\Theta)$	–	1.96	1.26	1.02	1.01	0.95

Table 4: [Example 3] Number of degrees of freedom, Newton iteration count, global estimator, and experimental rate of convergence of the global estimator.

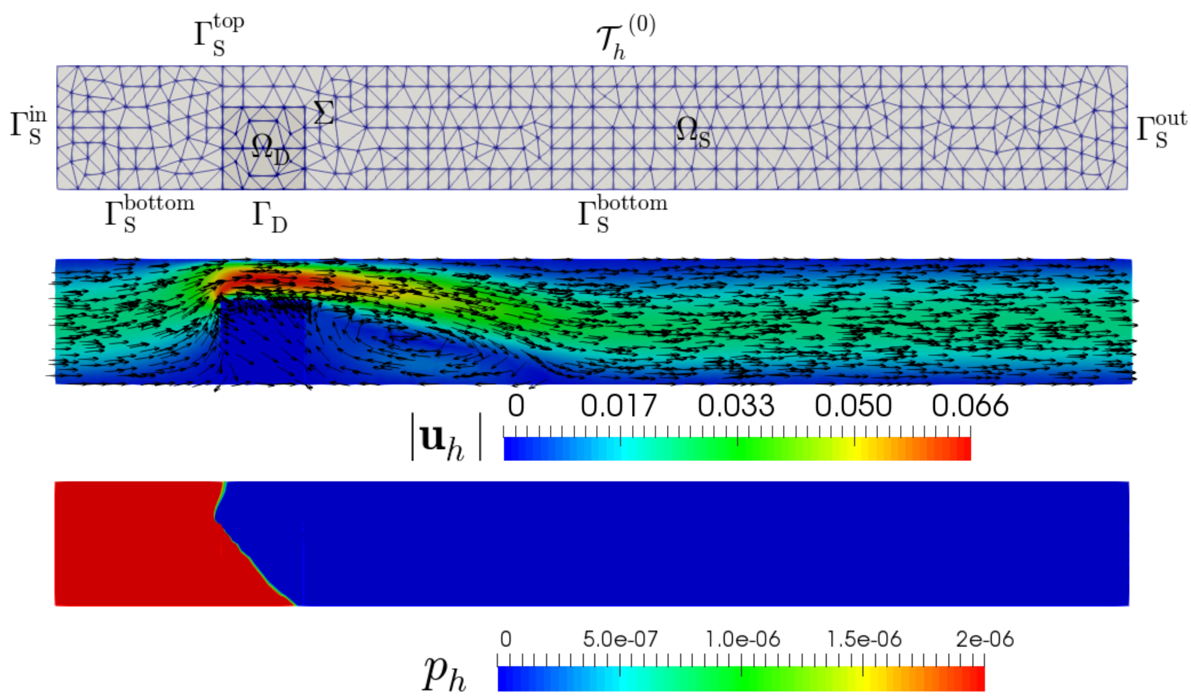


Figure 6: [Example 3] Initial mesh, computed velocities $\mathbf{u}_h = (\mathbf{u}_{S,h}, \mathbf{u}_{D,h})$ (arrows not scaled) and their magnitudes (color), and pressures $p_h = (p_{S,h}, p_{D,h})$ (color).

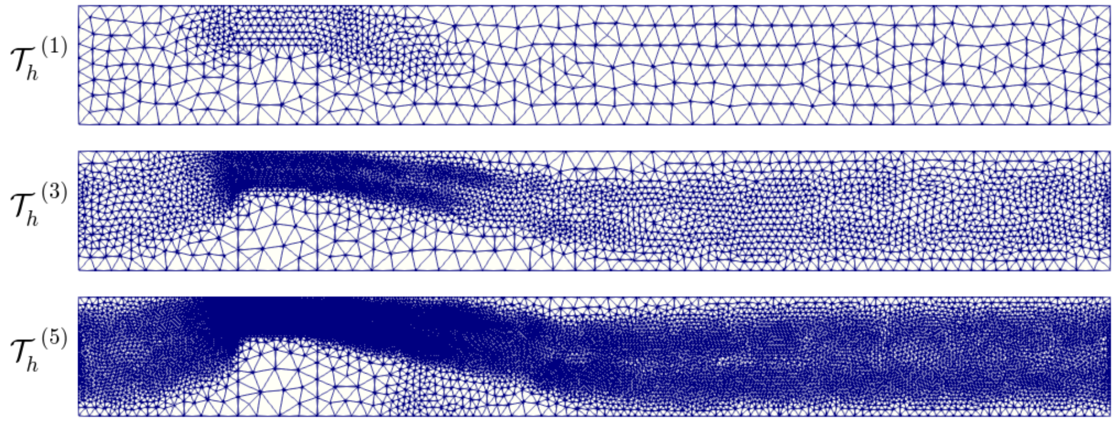


Figure 7: [Example 3] Three snapshots of adapted meshes according to the indicator Θ.

pseudostress, velocity, vorticity, and interface traces as primary unknowns for the Navier–Stokes region, while handling velocity, pressure, and interface traces for the Darcy region. This leads to a double saddle-point structure perturbed by the convective term of the Navier–Stokes equations. Well-posedness of both the continuous and discrete formulations is established under small data assumptions, leveraging fixed-point arguments, the Banach–Nečas–Babuška theorem, and suitable inf-sup conditions.

At the discrete level, we identified specific finite element subspaces, namely Arnold–Falk–Winther elements for the pseudostress, vorticity, and velocity in the free-flow region, and Raviart–Thomas elements for the velocity and discontinuous polynomials for the pressure, in the porous-medium region, which satisfy the required stability conditions. Optimal-order *a priori* error estimates are derived, confirming the theoretical convergence rates. Moreover, a fully computable residual-based *a posteriori* error estimator is proposed for the numerical scheme. Its reliability and efficiency are proven rigorously, following results similar to those in the literature, which are based on global inf-sup conditions, Helmholtz decompositions, inverse inequalities, and bubble-function arguments. Numerical experiments illustrate the accuracy, stability, and robustness of the method, confirming that the proposed estimator effectively guides adaptive mesh refinement and recovers optimal convergence rates even in the presence of solution singularities.

References

[1] D.N. Arnold, F. Brezzi, and J. Douglas Jr., *PEERS: A new mixed finite element method for plane elasticity.* Japan J. Appl. Math. 1 (1984), no. 2, pp. 347–367.

[2] D.N. Arnold, R.S. Falk, and R. Winther, *Mixed finite element methods for linear elasticity with weakly imposed symmetry.* Math. Comp. 76 (2007), pp. 1699–1723.

[3] D.N. Arnold, R.S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications.* Acta Numer. (2006), pp. 1–155.

[4] I. Babuška and G.N. Gatica, *A residual–based a posteriori error estimator for the Stokes–Darcy coupled problem.* SIAM J. Numer. Anal. 48 (2010), no. 2, pp. 498–523.

[5] L. Badea, M. Discacciati, and A. Quarteroni, *Numerical analysis of the Navier–Stokes/Darcy coupling.* Numer. Math. 115 (2010), no. 2, pp. 195–227.

[6] T.P. Barrios, G.N. Gatica, M. González, and N. Heuer, *A residual–based a posteriori error estimator for an augmented mixed finite element method in linear elasticity.* ESAIM Math. Model. Numer. Anal. 40 (2006), no. 5, pp. 843–869.

[7] G. Beavers and D. Joseph, *Boundary conditions at a naturally impermeable wall.* J. Fluid Mech. 30 (1967), pp. 197–207.

[8] I. Bermúdez, G.N. Gatica, and J.P. Silva, *A new Banach spaces–based mixed finite element method for the coupled Navier–Stokes and Darcy equations.* J. Comput. Appl. Math. 474 (2026), Paper No. 116982.

[9] D. Boffi, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications.* Springer Series in Computational Mathematics, 44. Springer–Verlag, New York, 2013.

[10] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods.* Springer Series in Computational Mathematics, 15. Springer–Verlag, New York, 1991.

[11] J. Camaño, S. Caucao, R. Oyarzúa, and S. Villa-Fuentes, *A posteriori error analysis of a momentum conservative mixed FEM for the stationary Navier–Stokes problem.* Appl. Numer. Math. 176 (2022), pp. 134–158.

[12] J. Camaño, C. García, and R. Oyarzúa, *Analysis of a momentum conservative mixed FEM for the stationary Navier–Stokes problem.* Numer. Methods Partial Differential Equations 37 (2021), no. 5, pp. 2895–2923.

[13] J. Camaño, G.N. Gatica, R. Oyarzúa, R. Ruiz-Baier, and P. Venegas, *New fully-mixed finite element methods for the Stokes–Darcy coupling.* Comput. Methods Appl. Mech. Engrg. 295 (2015), pp. 362–395.

[14] J. Camaño, C. Muñoz, and R. Oyarzúa, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces.* Electron. Trans. Numer. Anal. 48 (2018), pp. 114–130.

[15] J. Camaño, R. Oyarzúa, R. Ruiz-Baier, and G. Tierra, *Error analysis of an augmented mixed method for the Navier–Stokes problem with mixed boundary conditions.* IMA J. Numer. Anal. 38 (2018), no. 3, pp. 1452–1484.

[16] M. Cai, M. Mu, and J. Xu, *Numerical solution to a mixed Navier–Stokes/Darcy model by the two-grid approach.* SIAM J. Numer. Anal. 47 (2009), no. 5, pp. 3325–3338.

[17] C. Carstensen, *A posteriori error estimate for the mixed finite element method.* Math. Comp. 66 (1997), no. 218, pp. 465–476.

[18] S. Caucao, A. Dalal, T. Li, and I. Yotov, *Mixed finite element methods for the Navier–Stokes–Biot model.* In: I. Lirkov, S. Margenov (eds.), Large-Scale Scientific Computations (LSSC 2023). Lecture Notes in Computer Science, vol. 13952, Springer, Cham, 2024, pp. 19–31.

[19] S. Caucao, G.N. Gatica, and F. Sandoval, *A fully-mixed finite element method for the coupling of the Navier–Stokes and Darcy–Forchheimer equations.* Numer. Methods Partial Differential Equations 37 (2021), no. 3, pp. 2550–2587.

[20] S. Caucao, R. Oyarzúa, and S. Villa-Fuentes, *A posteriori error analysis of a momentum and thermal energy conservative mixed FEM for the Boussinesq equations.* Calcolo 59 (2022), no. 4, Paper No. 45.

[21] S. Caucao and P. Zúñiga, *A posteriori error analysis of a mixed FEM for the coupled Brinkman–Forchheimer/Darcy problem.* arXiv:2411.19695 (2024).

[22] P. Chidyagwai and B. Rivière, *On the solution of the coupled Navier–Stokes and Darcy equations.* Comput. Methods Appl. Mech. Engrg. 198 (2009), no. 47-48, pp. 3806–3820.

[23] E. Colmenares, G.N. Gatica, and S. Moraga, *A Banach spaces–based analysis of a new fully-mixed finite element method for the Boussinesq problem.* ESAIM Math. Model. Numer. Anal. 54 (2020), no. 5, pp. 1525–1568.

[24] M. Discacciati, E. Miglio, and A. Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows.* Appl. Numer. Math. 43 (2002), no. 1-2, pp. 57–74.

[25] M. Discacciati and R. Oyarzúa, *A conforming mixed finite element method for the Navier–Stokes/Darcy coupled problem.* Numer. Math. 135 (2017), no. 2, pp. 571–606.

[26] M. Discacciati and A. Quarteroni, *Navier–Stokes/Darcy coupling: modeling, analysis, and numerical approximation.* Rev. Mat. Complut. 22 (2009), no. 2, pp. 315–426.

[27] A. Ern and J.-L. Guermond, *Theory and Practice of Finite Elements.* Applied Mathematical Sciences, 159. Springer–Verlag, New York, 2004.

[28] J. Galvis and M. Sarkis, *Non matching mortar discretization analysis for the coupling Stokes–Darcy equations.* Electron. Trans. Numer. Anal. 26 (2007), pp. 350–384.

[29] G.N. Gatica, *A note on the efficiency of residual–based a posteriori error estimators for some mixed finite element methods.* Electron. Trans. Numer. Anal. 17 (2004), pp. 218–233.

[30] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications.* SpringerBriefs in Mathematics. Springer, Cham, 2014.

[31] G.N. Gatica, G.C. Hsiao, and S. Meddahi, *A coupled mixed finite element method for the interaction problem between an electromagnetic field and an elastic body.* SIAM J. Numer. Anal. 48 (2010), no. 4, pp. 1338–1368.

[32] G.N. Gatica, G.C. Hsiao, and S. Meddahi, *A residual–based a posteriori error estimator for a two-dimensional fluid–solid interaction problem.* Numer. Math. 114 (2009), no. 1, pp. 63–106.

[33] G.N. Gatica, A. Márquez, and S. Meddahi, *Analysis of the coupling of primal and dual-mixed finite element methods for a two-dimensional fluid–solid interaction problem.* SIAM J. Numer. Anal. 45 (2007), no. 5, pp. 2072–2097.

[34] G.N. Gatica, A. Márquez, and M.A. Sánchez, *Analysis of a velocity–pressure–pseudostress formulation for the stationary Stokes equations.* Comput. Methods Appl. Mech. Engrg. 199 (2010), no. 17-20, pp. 1064–1079.

[35] G.N. Gatica, A. Márquez, R. Oyarzúa, and R. Rebolledo, *Analysis of an augmented fully-mixed approach for the coupling of quasi-Newtonian fluids and porous media.* Comput. Methods Appl. Mech. Engrg. 270 (2014), pp. 76–112.

[36] G.N. Gatica, S. Meddahi, and R. Oyarzúa, *A conforming mixed finite element method for the coupling of fluid flow with porous media flow.* IMA J. Numer. Anal. 29 (2009), no. 1, pp. 86–108.

[37] G.N. Gatica, R. Oyarzúa, and F. Sayas, *Analysis of fully-mixed finite element methods for the Stokes–Darcy coupled problem.* Math. Comp. 80 (2011), no. 276, pp. 1911–1948.

[38] G.N. Gatica, R. Oyarzúa, and F. Sayas, *A residual–based a posteriori error estimator for a fully-mixed formulation of the Stokes–Darcy coupled problem.* Comput. Methods Appl. Mech. Engrg. 200 (2011), no. 21-22, pp. 1877–1891.

[39] G.N. Gatica, R. Oyarzúa, and N. Valenzuela, *A five-field augmented fully-mixed finite element method for the Navier–Stokes/Darcy coupled problem.* Comput. Math. Appl. 80 (2020), no. 8, pp. 1944–1963.

[40] G.N. Gatica and F. Sequeira, *Analysis of the HDG method for the Stokes–Darcy coupling.* Numer. Methods Partial Differential Equations 33 (2017), no. 3, pp. 885–917.

[41] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier–Stokes Equations. Theory and Algorithms.* Springer Series in Computational Mathematics, 5. Springer–Verlag, Berlin, 1986.

[42] V. Girault and B. Riviére, *DG approximation of coupled Navier–Stokes and Darcy equations by Beaver–Joseph–Saffman interface condition.* SIAM J. Numer. Anal. 47 (2009), no. 3, pp. 2052–2089.

[43] F. Hecht, *New development in FreeFem++.* J. Numer. Math. 20 (2012), pp. 251–265.

[44] F. Hecht, *FreeFem++.* Third edition, Version 3.58-1. Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris, 2018. Available at `http://www.freefem.org/ff++`.

[45] W. Jäger and A. Mikelić, *On the interface boundary condition of Beavers, Joseph and Saffman.* SIAM J. Appl. Math. 60 (2000), no. 4, pp. 1111–1127.

[46] A. Márquez, S. Meddahi, and F.J. Sayas, *Strong coupling of finite element methods for the Stokes–Darcy problem.* IMA J. Numer. Anal. 35 (2015), pp. 969–988.

[47] A. Masud, *A stabilized mixed finite element method for Darcy–Stokes flow.* Internat. J. Numer. Methods Fluids 54 (2007), no. 6-8, pp. 665–681.

[48] P.G. Saffman, *On the boundary condition at the interface of a porous medium.* Stud. Appl. Math. 50 (1971), pp. 93–101.

[49] M. Schneider, K. Weishaupt, D. Gläser, W.M. Boon, and R. Helmig, *Coupling staggered-grid and MPFA finite volume methods for free flow/porous-medium flow problems.* J. Comput. Phys. 401 (2020), Paper No. 109012.

[50] R. Verfürth, *A Review of A-Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques.* Wiley–Teubner, Chichester, 1996.

[51] L. Zuo and Y. Hou, *Numerical analysis for the mixed Navier–Stokes and Darcy problem with the Beavers–Joseph interface condition.* Numer. Methods Partial Differential Equations 31 (2015), no. 4, pp. 1009–1030.

Centro de Investigación en Ingeniería Matemática (CI^2MA)

PRE-PUBLICACIONES 2025

2025-04 Alonso Bustos, Sergio Caucao, Gabriel N. Gatica, Benjamín Venegas: *New fully mixed finite element methods for the coupled convective Brinkman-Forchheimer and nonlinear transport equations*

2025-05 Fahim Aslam, Zayd Hajjej, Jianghao Hao, Mauricio Sepúlveda: *Global Existence and Asymptotic Profile of an Infinite Memory Logarithmic Wave Equation with Fractional Derivative and Strong Damping*

2025-06 Jessika Camaño, Ricardo Oyarzúa, Katherine Rojo: *A momentum and mass conservative pseudostress-based mixed finite element method for the Stokes problem*

2025-07 Aníbal Coronel, Fernando Huancas, Mauricio Sepúlveda: *Theoretical and numerical approaches of reaction term identification in a SIS reaction diffusion system*

2025-08 Isaac Bermudez, Gabriel N. Gatica, Juan P. Silva: *A new Banach spaces-based mixed finite element method for the coupled Navier–Stokes and Darcy equations*

2025-09 Sergio Caucao, Gabriel N. Gatica, Saulo Medrado, Yuri D. Sobral: *A posteriori error analysis of mixed finite element methods for a regularized mu(I)-rheology model of granular materials*

2025-10 Veronica Anaya, Gerardo Chowell, Felipe Jara, Mauricio Sepúlveda: *Optimal Control of Inter-Population Disease Spread via Reaction–Diffusion Models*

2025-11 Ramiro Acevedo, Rommel Bustinza, Christian Gómez: *A Transient Eddy Current Problem via Potential Formulation with Current Excitation*

2025-12 Abraham J. Arenas, Juan Barajas-Calonge, Gilberto González-Parra, Luis M. Villada: *A second-order nonstandard finite difference scheme for eco-epidemiological predator-prey models*

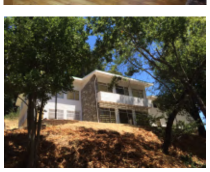
2025-13 Alonso Bustos, Sergio Caucao, Gabriel N. Gatica: *Mixed-primal and fully-mixed formulations for the convection-diffusion-reaction system based upon Brinkman–Forchheimer equations*

2025-14 Gabriel N. Gatica, Zeinab Gharibi, Ricardo Oyarzúa: *Banach spaces-based fully mixed finite element methods for the n-dimensional Boussinesq problem with temperature-dependent parameters*

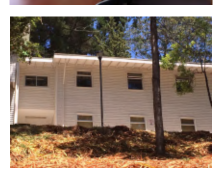
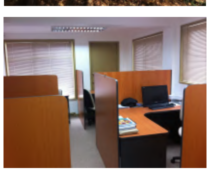
2025-15 Sergio Caucao, Ricardo Oyarzúa, Segundo Villa-Fuentes: *A priori and a posteriori error analyses of a fully-mixed finite element method for the coupled Navier–Stokes/Darcy problem*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: Director, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, Tel.: 41-2661324, o bien, visitar la página web del centro: http://www.ci2ma.udec.cl



Centro de Investigación en
Ingeniería Matemática (CI²MA)
Universidad de Concepción

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl