UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en
Ingeniería Matemática (CI^{2}MA)

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PREPRINT 2025-19

SERIE DE PRE-PUBLICACIONES

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Abstract

In this work we propose and analyze the well-posedness a new class of macroscopic vehicular traffic model described by a scalar nonlocal conservation law that simultaneously incorporates both upstream and downstream effects in the flow dynamics. Unlike nonlocal models previously described in the literature, which only account for downstream density averages (look-ahead behavior), the proposed model introduces an additional term depending on an upstream average (look-behind), allowing for a more realistic representation of anticipatory driver behavior under high-density conditions. Inspired by the multiplicative flux proposed in [I. Karafyllis, D. Theodosis, and M. Papageorgiou. Analysis and control of a non-local pde traffic flow model. International Journal of Control, 95(3):660–678, 2022], our model generalizes and adapts such ideas to an entropy weak solution framework, allowing for the presence of discontinuities and shock waves. The considered flux takes the form $\rho\, g(\rho)\, W(\hat{R}_\delta)\, V(R_\eta)$, where the nonlocal terms $\hat{R}_\delta$ and R_η represent backward- and forward-looking spatial averages of the density, respectively, and the functions W and V encode the drivers' responses to these observations. The main novelty of this work lies in establishing the existence and uniqueness theory for entropy weak solutions, together with a rigorous proof of Lipschitz continuous dependence of solutions not only on the initial data, but also on the kernel functions, under reasonable structural assumptions on the flux components. The proofs are achieved through the design of a conservative numerical scheme that preserves key structural properties of the continuous model, such as maximum principle, mass conservation, **BV** estimates, and $\mathbf{L^1}$-stability. Finally, we present numerical experiments that illustrate the behavior of solutions and the qualitative impact of nonlocal terms on traffic dynamics.

Keywords: Macroscopic traffic flow, nonlocal flux, upstream-downstream traffic model, nudging effect, finite volume method.

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1. Introduction

1.1. Scope

We are interested in proving the existence and uniqueness of entropy weak solutions for nonlocal conservation laws of the following type

$$\partial_t \rho(t,x) + \partial_x \Big(\rho(t,x) g(\rho(t,x)) W(\hat{R}_\delta(t,x)) V(R_\eta(t,x))\Big) = 0, \quad (t,x) \in (\mathbb{R}^+, \mathbb{R}) \quad (1)$$

$$\rho(0,x) = \rho_0(x), \quad (2)$$

with initial condition $\rho_0 \in \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}])$ and where $\hat{R}_\delta$ and R_η correspond to nonlocal terms defined as

$$\hat{R}_\delta(t,x) := (\rho * \hat{\omega}_\delta)(t,x) = \int_{x-\delta}^{x} \rho(t,y) \hat{\omega}_\delta(x-y) \mathrm{d}y, \quad (3)$$

$$R_\eta(t,x) := (\rho * \omega_\eta)(t,x) = \int_{x}^{x+\eta} \rho(t,y) \omega_\eta(y-x) \mathrm{d}y. \quad (4)$$

When ρ represents the density of cars, Eq (1) is a generalization of the nonlocal LWR traffic model [3, 9, 10, 14, 18, 20], where $V(R_\eta)$ corresponds to the mean speed downstream and the term $W(\hat{R}_\delta)$ corresponds to the mean speed upstream, $\hat{\omega}_\delta(\cdot)$, $\omega_\eta(\cdot)$ are kernel functions detailed later. The presence of W introduces an additional behavioral mechanism whereby drivers also take into account the density of vehicles behind them. This upstream dependence, often referred to as a *nudging effect* [19], has been recently introduced in the literature to enhance the realism and stability of macroscopic traffic models [13]. The vehicle nudging behavior suggests that vehicles in the traffic flow may induce a 'pushing effect' on their preceding vehicles. In other words, while the traditional vehicle-following behavior results in look-ahead interaction, the nudging behavior may result in look-behind interaction: the combination of the two effects would thus result in bidirectional inter-vehicle interactions [17].

1.2. Related work

Most of the nonlocal traffic flow models existing in the literature [2, 3, 9, 10, 14] account exclusively on downstream information, ignoring the possible impact of upstream traffic on drivers' behavior. Recent studies have shown that this type of influence is especially relevant in dense congestion scenarios. In particular, Karafyllis et al. [13] demonstrated that the introduction of upstream terms can stabilize traffic flow on ring roads, proposing control mechanisms based on specific feedback design problems. In addition, in that work, the authors provided conditions that guarantee the existence and uniqueness of classical solutions for their model on a ring road. On the other hand, in [12] the authors identify a threshold condition for shock formation for traffic flow models with Arrhenius look-ahead-behind dynamics with concave-convex flux of the form $f(u, \bar{u}, \tilde{u}) = u(1-u)^\gamma e^{-\bar{u}+\tilde{u}}$, with $\gamma \geq 1$ and $e^{-\bar{u}}$, $e^{\tilde{u}}$ called the Arrhenius-type slow-down and nudging factors, respectively; $\bar{u}(t,x) = \int_x^{x+\gamma_a} K_a u(t,y) \mathrm{d}y$ and $\tilde{u} = \int_{x-\gamma_b}^{x} K_b u(t,y) \mathrm{d}y$, where γ_a and γ_b are the horizon ahead and behind of drivers, respectively, and K_a, K_b represent constant proportional to positive interaction strengths. In the same work, to perform the analysis, the authors consider $\gamma_a = \gamma_b = K_a = K_b =$

1 and $\gamma = 2$. Likewise, in [16] the authors propose a traffic flow model with look-ahead relaxation and look-behind intensification by inserting look-ahead intensification dynamics to the flux. Finite-time shock formation conditions with various types of interaction potentials are identified in the proposed model.

Although the present work and [13] share the general structure of a nonlocal macroscopic model with a multiplicative flux depending on both upstream and downstream density averages, there are substantial differences in their objectives, theoretical framework, and scope of results. In [13], the analysis is carried out on a periodic domain (ring road) and focuses on sufficiently regular classical solutions ($W^{2,\infty}$), with the main goal of establishing local exponential stability conditions around equilibrium states through Lyapunov techniques and control theory. In that setting, the upstream term is primarily interpreted as a feedback (nudging) mechanism for the design of control strategies aimed at mitigating congestion waves. In contrast, the present work addresses entropy weak solutions in $\mathbf{L}^\infty \cap \mathbf{BV}$, allowing for the presence of discontinuities and shock waves, and establishes results on existence, uniqueness, and Lipschitz continuous dependence of solutions with respect to both the initial data and the kernel functions, without assuming proximity to an equilibrium state. Furthermore, while in [13] the numerical experiments are mainly used to illustrate stabilization scenarios under active control, in the present work conservative numerical schemes are developed to preserve structural properties of the continuous model and are employed to investigate the qualitative influence of bidirectional upstream–downstream interactions on the formation and mitigation of traffic waves in more general settings.

1.3. Outline of the paper

This work is organized as follows: In Section 2 we present the necessary assumptions on the parameters of the studied problem as well as the main results. In Section 3 we introduce the numerical scheme and derive some of its important properties such as the maximum principle, $\mathbf{BV}$ and $\mathbf{L^1}$-Lipschitz continuity in time estimates. These imply the convergence of the proposed scheme, which in turn covers the existence part of the well-posedness of the governing equation. Afterwards, Lipschitz continuous dependence of solutions to the problem on initial data, and kernels ω_η, $\hat{\omega}_\delta$ is proved in Section 4 using Kružkov's doubling of variables technique. Finally, in Section 5 we present some numerical tests, analyzing the $\mathbf{L^1}$ error of approximate solutions of the problem studied. In Appendix we collects the derivation of some estimates necessary throughout the paper.

2. Main results

Throughout the paper, we will denote $\mathcal{I}(r,s) := [\min\{r,s\}, \max\{r,s\}]$, for any $r, s \in \mathbb{R}$, $\|\cdot\|_{\mathbf{L^1}(\mathbb{R})} = \|\cdot\|_{\mathbf{L^1}}$ and also $\|\cdot\|_{\mathbf{L^\infty}(\mathbb{R})} := \|\cdot\|$. In addition, we will require the following assumptions to hold:

Assumptions 2.1. *The nonlocal conservation law* (1) *is studied under the following assumptions:*

i) $V \in \mathbf{C^2}([0, \rho_{\max}]; \mathbb{R}^+)$, with $0 \le V(\rho) \le V_{\max}$, $V'(\rho) \le 0$, $\rho \in [0, \rho_{\max}]$;

ii) $W \in \mathbf{C^2}([0, \rho_{\max}]; \mathbb{R}^+)$, with $1 \le W(\rho) \le W_{\max}$, $W'(\rho) \ge 0$, $\rho \in [0, \rho_{\max}]$;

iii) $g \in \mathbf{C^1}([0, \rho_{\max}]; \mathbb{R}^+)$, with $g'(\rho) \le 0$, for $\rho \in [0, \rho_{\max}]$ and $g(\rho_{\max}) = 0$;

iv) $\omega_\eta \in \mathbf{C^2}([0,\eta];\mathbb{R}^+)$ with $\omega_\eta'(x) \le 0$, $\int_0^\eta \omega_\eta(x)\mathrm{d}x = 1$, $\forall \eta > 0$.

v) $\hat{\omega}_\delta \in \mathbf{C^2}([0,\delta];\mathbb{R}^+)$ with $\hat{\omega}_\delta'(x) \le 0$, $\int_0^\delta \hat{\omega}_\delta(x)\mathrm{d}x = 1$, $\forall \delta > 0$.

The solutions of problem (1)-(2) are intended in the following sense.

Definition 1. *Let $\rho_0 \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,\rho_{\max}])$. A function $\rho \in \mathbf{C}([0,T];\mathbf{L^1}(\mathbb{R};[0,\rho_{\max}]))$, with $\rho(t,\cdot) \in \mathbf{BV}(\mathbb{R};[0,\rho_{\max}])$ for $t \in [0,T]$ is an entropy weak solution to Eq.* (1) *with initial datum ρ_0 if and only if for all $\varphi \in \mathbf{C^1_c}([0,T[\times\mathbb{R};\mathbb{R}_+)$ and $k \in \mathbb{R}$*

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \Big(|\rho - k| \partial_t \varphi + \operatorname{sgn}(\rho - k)\big(\rho g(\rho) - k g(k)\big) W(\hat{R}_\delta) V(R_\eta) \partial_x \varphi \\ - \operatorname{sgn}(\rho - k) k g(k) \partial_x (W(\hat{R}_\delta) V(R_\eta)) \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \\ + \int_{\mathbb{R}} \big|\rho_0(x) - k\big| \varphi(0,x) \,\mathrm{d}x \ge 0.\end{aligned}$$

The following theorem, which states the well-posedness of the model (1)-(2), is the main result of this paper.

Theorem 1. *Let $\rho_0 \in \mathbf{BV}(\mathbb{R};[0,\rho_{\max}])$ and Assumptions 2.1 hold, then for all $T > 0$ there exists a unique entropy weak solution $\rho \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV})([0,T] \times \mathbb{R};[0,\rho_{\max}]))$ of* (1)-(2) *in the sense of Definition 1.*

The proof of Theorem 1 is standard, existence is based on a construction of a converging sequence of approximate solutions defined by means of a numerical scheme based on the finite volume method, and uniqueness is proved by means of Lipschitz continuous dependence of weak entropy solutions, not only on initial data (Theorem 2), but also on the kernel functions ω_η and $\hat{\omega}_\delta$ (Theorem 3).

3. Existence of solutions for model (1)-(2)

In this section, we construct approximate solution of the model (1)-(2), derive uniform bounds on these, and provide a discrete entropy weak inequality. These ingredients allow us, by means of the Helly's Compactness Theorem, to guarantee convergence and existence of entropy weak solutions of (1)-(2).

3.1. Numerical discretization

We consider a uniform space mesh with cell sizes $\Delta x > 0$. For any $j \in \mathbb{Z}$, let $x_{j+1/2} = (j+1/2)\Delta x$ be the cell interfaces and $x_j = j\Delta x$ the center of the cell $C_j :=]x_{j-1/2}, x_{j+1/2}]$. Next, we take a time step $\Delta t > 0$ subject to a Courant-Friedrichs-Levy (CFL) condition, which will be specified later. For a fixed time horizon $T > 0$, we set $N_T \in \mathbb{N}$ such that $N_T \Delta t \le T < (N_T + 1)\Delta t$ and we define the time mesh as $t^n = n\Delta t$, for $n = 0, \dots, N_T$. Finally, we set $\lambda := \Delta t / \Delta x$.

We approximate the initial datum (2) as

$$\rho_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)\mathrm{d}x, \qquad j \in \mathbb{Z},$$

and we construct a finite volume approximate solution for Eq (1) as a piecewise constant function defined as

$$\rho^{\Delta}(t,x) = \rho_j^n, \qquad (t,x) \in [t^n, t^{n+1}[\times C_j.$$

In order to approximate the nonlocal terms (3) and (4) in the flux function of Eq (1), if $\hat{\omega}_\delta$ and ω_η satisfy Assumption 2.1-(iv, v), first we define

$$\hat{R}_\delta(t,x) := [\rho^{\Delta} * \hat{\omega}_\delta](t,x), \quad R_\eta(t,x) := [\rho^{\Delta} * \omega_\eta](t,x),$$

and next we choose $N_\eta := \lfloor \eta/\Delta x \rfloor \in \mathbb{N}$, $N_\delta := \lfloor \delta/\Delta x \rfloor \in \mathbb{N}$, and use a first order quadrature formula to approach these nonlocal terms in the cell interfaces $x_{j+\frac{1}{2}}$, for all $j \in \mathbb{Z}$,

$$R^n_{\eta,\, j+1/2} := \Delta x \sum_{k=0}^{N_\eta - 1} \omega_\eta^k \rho^n_{j+k+1}, \quad \text{where} \quad R_\eta(t^n, x_{j+\frac{1}{2}}) = R^n_{\eta,\, j+\frac{1}{2}} + \mathcal{O}(\Delta x) \tag{5}$$

$$\hat{R}^n_{\delta,\, j+1/2} := \Delta x \sum_{k=0}^{N_\delta - 1} \hat{\omega}_\delta^k \rho^n_{j-k-1}, \quad \text{where} \quad \hat{R}_\delta(t^n, x_{j+\frac{1}{2}}) = \hat{R}^n_{\delta,\, j+\frac{1}{2}} + \mathcal{O}(\Delta x), \tag{6}$$

where we define

$$\omega_\eta^k := \frac{1}{\Delta x}\int_{k\Delta x}^{(k+1)\Delta x} \omega_\eta(y)\mathrm{d}y, \quad k = 0,1,\dots,N_\eta - 1,$$

$$\hat{\omega}_\delta^k := \frac{1}{\Delta x}\int_{k\Delta x}^{(k+1)\Delta x} \hat{\omega}_\delta(y)\mathrm{d}y, \quad k = 0,1,\dots,N_\delta - 1.$$

In this way, we can define the following Hilliges-Weidlich (HW)-type numerical scheme [4, 5, 11]

$$\rho_j^{n+1} = \rho_j^n - \lambda\left(F^n_{j+1/2} - F^n_{j-1/2}\right), \qquad j \in \mathbb{Z}, \tag{7}$$

where the numerical flux $F^n_{j+1/2}$ is defined as

$$F^n_{j+1/2} = \rho^n_j g^n_{j+1} W^n_{j+1/2} V^n_{j+1/2}, \tag{8}$$

with $g^n_{j+1} = g(\rho^n_{j+1})$ and the nonlocal terms denoted by $W^n_{j+1/2} := W(\hat{R}^n_{\delta,\, j+1/2})$ and $V^n_{j+1/2} := V(R^n_{\eta,\, j+1/2})$.

Now, we are going to prove that the sequence of approximate solutions $\rho^{\Delta}(t,x)$ satisfies the assumptions of Helly's compactness Theorem.

3.2. Uniform bounds

Lemma 3.1 (Maximum principle)**.** *If Assumptions 2.1 and the following CFL condition hold,*

$$\Delta t \leq \frac{\Delta x}{\|W\|\left(\|g\|\|V\| + \rho_{\max}\left(\|g'\|\|V\| + \Delta x \omega_\eta^0 \|g\|\|V'\|\right)\right)}, \tag{9}$$

then if $\rho_0(x) \in [0, \rho_{\max}]$, for $x \in \mathbb{R}$, the approximate solutions satisfy

$$0 \leq \rho_j^n \leq \rho_{\max}, \quad \text{for all } j \in \mathbb{Z} \text{ and } n = 1, \dots, N_T. \tag{10}$$

Proof. The proof is done by induction and follows the same idea of [4]. We assume that $0 \le \rho_j^n \le \rho_{\max}$, for all $j \in \mathbb{Z}$ and $n = 1, \ldots, N_T$, and we prove $0 \le \rho_j^{n+1} \le \rho_{\max}$. Indeed,

$$\begin{aligned}
\rho_j^{n+1} &= \rho_j^n - \lambda \left[\rho_j^n g_{j+1}^n W_{j+1/2}^n V_{j+1/2}^n - \rho_{j-1}^n g_j^n W_{j-1/2}^n V_{j-1/2}^n\right] \\
&\le \rho_j^n + \lambda \rho_{\max} g_j^n W_{j-1/2}^n V_{j-1/2}^n \\
&\le \rho_j^n + \lambda \rho_{\max} W_{\max} g_j^n V_{j-1/2}^n,
\end{aligned}$$

now, we denote $G(\rho_j^n, \rho_{j+1}^n, \ldots, \rho_{j+N_\eta-1}^n) = \rho_{\max} W_{\max} g_j^n V_{j-1/2}^n$ and note that because of Assumptions 2.1 *iii*) we have

$$G(\rho_{\max}, \rho_{j+1}^n, \ldots, \rho_{j+N_\eta-1}^n) = \rho_{\max} W_{\max} g(\rho_{\max}) V_{j-1/2}^n = 0$$

and also by means of Assumptions 2.1 *i*), G is a non-increasing function with respect to each of its variable, indeed

$$\begin{aligned}
\partial_1 G &= \rho_{\max} W_{\max} \left(g_j' V_{j-1/2}^n + g_j V'(R_{\eta,j-1/2}^n) \partial_1 R_{\eta,j-1/2}^n \right) \\
&= \rho_{\max} W_{\max} \left(g_j' V_{j-1/2}^n + g_j V'(R_{\eta,j-1/2}^n) \Delta x \omega_\eta^0 \right) \le 0,
\end{aligned}$$

and for $i = 2, \ldots, N_\eta - 1$

$$\partial_i G = \rho_{\max} W_{\max} g_j^n V'(R_{\eta,j-1/2}^n) \Delta x \omega_\eta^i \le 0.$$

Thus, taking into account these facts we get

$$\begin{aligned}
\rho_j^{n+1} &\le \rho_j^n + \lambda \left(G(\rho_j^n, \rho_{j+1}^n, \ldots, \rho_{j+N-1}^n) - G(\rho_{\max}, \rho_{j+1}^n, \ldots, \rho_{j+N-1}^n) \right) \\
&= \rho_j^n - \lambda \partial_1 G(\nu_j^n)(\rho_{\max} - \rho_j^n) \\
&= \left(1 + \lambda \partial_1 G(\nu_j^n)\right) \rho_j^n - \lambda \partial_1 G(\nu_j^n) \rho_{\max} \\
&\le \rho_{\max},
\end{aligned}$$

for $\nu_j^n \in \mathcal{I}((\rho_j^n, \rho_{j+1}^n, \ldots \rho_{j+N-1}^n)), (\rho_{\max}, \rho_{j+1}^n, \ldots \rho_{j+N-1}^n))$. Observe that due to the CFL condition (9)

$$1 + \lambda \partial_1 G(\nu_j^n) \ge 0.$$

On the other hand, for computing the lower bound in Eq (10), we consider again the numerical scheme Eq (7) and Assumption 2.1 *iii*), so for all $j \in \mathbb{Z}$, $n = 1, \ldots, N_T$, we obtain the following estimate

$$\begin{aligned}
\rho_j^{n+1} &\ge \rho_j^n - \lambda \rho_j^n g_{j+1}^n W_{j+1/2}^n V_{j+1/2}^n \\
&= \rho_j^n + \lambda \left(\rho_j^n g(\rho_{\max}) W_{j+1/2}^n V_{j+1/2}^n - \rho_j^n g_{j+1}^n W_{j+1/2}^n V_{j+1/2}^n \right) \\
&= \rho_j^n + \lambda \rho_j^n W_{j+1/2}^n V_{j+1/2}^n g'(\zeta) \left(\rho_{\max} - \rho_{j+1}^n\right) \\
&\ge \left(1 + \lambda W_{j+1/2}^n V_{j+1/2}^n g'(\zeta) \rho_{\max}\right) \rho_j^n \\
&\ge 0.
\end{aligned}$$

In the previous estimates, we take $\zeta \in \mathcal{I}(\rho_{j+1}^n, \rho_{\max})$ which is obtained by means of the Mean Value theorem (used in the third line) and we have used the CFL condition (9) in the second to last line. □

Lemma 3.2 ($\mathbf{L^1}$-bound)**.** *Consider* $\rho_0 \in \mathbf{L^1}(\mathbb{R}; [0, \rho_{\max}])$*. If Assumptions 2.1 and CFL condition* (9) *hold, then the approximate solutions satisfy the following property*

$$\|\rho_\Delta(t, \cdot)\|_{\mathbf{L^1}} = \|\rho_0\|_{\mathbf{L^1}}, \quad \text{for all } t > 0. \tag{11}$$

Proof. By induction, assume that (11) hold for $t^n = n\Delta t$, then because of the positivity property proved in Lemma 3.1 and the conservative form of the numerical scheme (7), we have the following estimates for the $\mathbf{L^1}$ norm of the approximate solutions

$$\begin{aligned}
\|\rho_\Delta(t^{n+1}, \cdot)\|_{\mathbf{L^1}} &= \|\rho^{n+1}\|_{\mathbf{L^1}} \\
&= \Delta x \sum_{j\in\mathbb{Z}} \rho_j^{n+1} \\
&= \Delta x \sum_{j\in\mathbb{Z}} \left(\rho_j^n - \lambda \left(\rho_j^n g_{j+1}^n W_{j+1/2}^n V_{j+1/2}^n - \rho_{j-1}^n g_j^n W_{j-1/2}^n V_{j-1/2}^n\right)\right) \\
&= \Delta x \sum_{j\in\mathbb{Z}} \rho_j^n - \Delta t \sum_{j\in\mathbb{Z}} \rho_j^n g_{j+1}^n W_{j+1/2}^n V_{j+1/2}^n + \Delta t \sum_{j\in\mathbb{Z}} \rho_{j-1}^n g_j^n W_{j-1/2}^n V_{j-1/2}^n \\
&= \Delta x \sum_{j\in\mathbb{Z}} \rho_j^n - \Delta t \sum_{j\in\mathbb{Z}} \rho_j^n g_{j+1}^n W_{j+1/2}^n V_{j+1/2}^n + \Delta t \sum_{j\in\mathbb{Z}} \rho_j^n g_{j+1}^n W_{j+1/2}^n V_{j+1/2}^n \\
&= \Delta x \sum_{j\in\mathbb{Z}} \rho_j^n,
\end{aligned}$$

now by means of an iterative argument, we get the desired result (11). □

Lemma 3.3 (**BV** estimate in space)**.** *Let Assumptions 2.1 and CFL* (9) *hold, and consider* $\rho_0 \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0, \rho_{\max}])$*, then the numerical solutions constructed by means of the numerical scheme* (7) *satisfy the following estimate, for all* $n = 0, \ldots, N_T$*,*

$$\sum_{j\in\mathbb{Z}} |\rho_{j+1}^n - \rho_j^n| \leq e^{T\mathcal{H}_1} \left(\sum_{j\in\mathbb{Z}} |\rho_{j+1}^0 - \rho_{j+1}^0| + \frac{\mathcal{H}_2}{\mathcal{H}_1}(e^{T\mathcal{H}_1} - 1)\right) \tag{12}$$

with $\mathcal{H}_1$*,* $\mathcal{H}_2$ *positive constants depending on model parameters, which are defined as below,*

$$\begin{aligned}
\mathcal{H}_1 &= \left(\|W\|\|V'\|\mathcal{K}_1 + \|V\|\|W'\|\mathcal{K}_2\right)\left(\|g\| + \rho_{\max}\|g'\|\right) + \|g\|\rho_{\max}\|V'\|\|W\| \max\{\omega_\eta^0, \hat{\omega}_\delta^0\}, \\
\mathcal{H}_2 &= \|g\| \Big[2\|W\|\|V''\|\mathcal{K}_1^2 + 2\|V\|\|W''\|\mathcal{K}_2^2 + \|V'\|\left(\|W'\|\mathcal{K}_1\mathcal{K}_2 + \|W\|\mathcal{K}_3\right) \\
&\quad + \|W'\|\|V'\|\mathcal{K}_1\mathcal{K}_2 + \|V\|\mathcal{K}_4\Big] \|\rho^n\|_{\mathbf{L^1}},
\end{aligned}$$

and $\mathcal{K}_1$*,* $\mathcal{K}_2$*,* $\mathcal{K}_3$*,* $\mathcal{K}_4$*, computed in the Appendix section, defined as follows*

$$\begin{aligned}
\mathcal{K}_1 &= \|\omega_\eta'\|\|\rho^n\|_{\mathbf{L^1}} + 2\rho_{\max}\omega_\eta^0, \\
\mathcal{K}_2 &= \|\hat{\omega}_\delta'\|\|\rho^n\|_{\mathbf{L^1}} + 2\rho_{\max}\hat{\omega}_\delta^0, \\
\mathcal{K}_3 &= \|\omega_\eta''\|\|\rho\|_{\mathbf{L^1}} + 2\rho_{\max}\|\omega_\eta'\|, \\
\mathcal{K}_4 &= \|\hat{\omega}_\delta''\|\|\rho\|_{\mathbf{L^1}} + 2\rho_{\max}\|\hat{\omega}_\delta'\|.
\end{aligned}$$

Proof. Let us first recall the following inequalities, which will be useful for getting the desired result (12); those estimates are computed in the Appendix (Section 7),

$$
\begin{aligned}
|R^n_{\eta,j+1/2} - R^n_{\eta,j-1/2}| &\leq \Delta x \mathcal{K}_1, \\
|\tilde{R}^n_{j+1} - \tilde{R}^n_j| &\leq 2\Delta x \mathcal{K}_1, \\
|\hat{R}^n_{\delta,j+1/2} - \hat{R}^n_{\delta,j-1/2}| &\leq \Delta x \mathcal{K}_2, \\
|\bar{R}^n_{j+1} - \bar{R}^n_j| &\leq 2\Delta x \mathcal{K}_2, \\
|R^n_{\eta,j+3/2} - 2R^n_{\eta,j+1/2} + R^n_{\eta,j-1/2}| &\leq (\Delta x)^2 \mathcal{K}_3 + \Delta x \omega^0_\eta |\rho^n_{j+1} - \rho^n_j|, \\
|\hat{R}^n_{\delta,j+3/2} - 2\hat{R}^n_{\delta,j+1/2} + \hat{R}^n_{\delta,j-1/2}| &\leq (\Delta x)^2 \mathcal{K}_4 + \Delta x \hat{\omega}^0_\delta |\rho^n_{j+1} - \rho^n_j|.
\end{aligned}
\tag{13}
$$

In addition, we have the following estimates, which can be proven by following the proof of [8, Lemma 3.1] and using the fact that the total integral of kernel functions is unitary, as established in Assumption 2.1 $iv) - v)$,

$$
\sum_{j\in\mathbb{Z}} |R_{\eta,j+1/2} - R_{\eta,j-1/2}| \leq \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|, \tag{14}
$$

$$
\sum_{j\in\mathbb{Z}} |\hat{R}_{\delta,j+1/2} - \hat{R}_{\delta,j-1/2}| \leq \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|. \tag{15}
$$

The proof of Lemma 3.3 follows the ideas in [4, Lemma 4.3]. Based on the numerical scheme (7), we can write ρ^{n+1}_{j+1} as

$$
\rho^{n+1}_{j+1} = \rho^n_{j+1} - \lambda \left[\rho^n_{j+1} g^n_{j+2} W^n_{j+3/2} V^n_{j+3/2} - \rho^n_j g^n_{j+1} W^n_{j+1/2} V^n_{j+1/2} \right], \tag{16}
$$

and now computing the difference $\rho^{n+1}_{j+1} - \rho^{n+1}_j$, we obtain

$$
\rho^{n+1}_{j+1} - \rho^{n+1}_j = \mathcal{A}^n_j - \lambda \mathcal{B}^n_j
$$

where

$$
\begin{aligned}
\mathcal{A}^n_j &:= \rho^n_{j+1} - \rho^n_j - \lambda \Big[\rho^n_{j+1} g^n_{j+2} W^n_{j+3/2} V^n_{j+3/2} - \rho^n_j g^n_{j+1} W^n_{j+1/2} V^n_{j+1/2} - \rho^n_j g^n_{j+1} W^n_{j+3/2} V^n_{j+3/2} \\
&\quad + \rho^n_{j-1} g^n_j W^n_{j+1/2} V^n_{j+1/2} \Big], \\
\mathcal{B}^n_j &:= \rho^n_j g^n_{j+1} W^n_{j+3/2} V^n_{j+3/2} - \rho^n_j g^n_{j+1} W^n_{j+1/2} V^n_{j+1/2} + \rho^n_{j-1} g^n_j W^n_{j-1/2} V^n_{j-1/2} - \rho^n_{j-1} g^n_j W^n_{j+1/2} V^n_{j+1/2}.
\end{aligned}
$$

Consider first the term $\mathcal{A}^n_j$, adding and subtracting the terms $\rho^n_{j+1} g^n_{j+1} V^n_{j+3/2} W^n_{j+3/2}$ and $\rho^n_j g^n_j V^n_{j+1/2} W^n_{j+1/2}$, and doing straightforward computations on $\mathcal{A}^n_j$, allows us to rewrite this term as follows

$$
\begin{aligned}
\mathcal{A}^n_j &= (\rho^n_{j+1} - \rho^n_j) - \lambda \Big(\rho^n_{j+1} g'(\xi^n_{j+3/2}) V^n_{j+3/2} W^n_{j+3/2} (\rho^n_{j+2} - \rho^n_{j+1}) + g^n_{j+1} V^n_{j+3/2} W^n_{j+3/2} (\rho^n_{j+1} - \rho^n_j) \\
&\quad - \rho^n_j g'(\xi^n_{j+1/2}) V^n_{j+1/2} W^n_{j+1/2} (\rho^n_{j+1} - \rho^n_j) - g^n_j V^n_{j+1/2} W^n_{j+1/2} (\rho^n_j - \rho^n_{j-1}) \Big) \\
&= \left(1 - \lambda \left(g^n_{j+1} W^n_{j+3/2} V^n_{j+3/2} - \rho^n_j g'(\xi^n_{j+1/2}) W^n_{j+1/2} V^n_{j+1/2} \right) \right) (\rho^n_{j+1} - \rho^n_j) \\
&\quad - \lambda \rho^n_{j+1} g'(\xi^n_{j+\frac{3}{2}}) W^n_{j+\frac{3}{2}} V^n_{j+\frac{3}{2}} \left(\rho^n_{j+2} - \rho^n_{j+1} \right) + \lambda g^n_j W^n_{j+1/2} V^n_{j+1/2} \left(\rho^n_j - \rho^n_{j-1} \right),
\end{aligned}
\tag{17}
$$

where $\xi^n_{j+1/2} \in \mathcal{I}(\rho^n_j, \rho^n_{j+1})$ and $\xi^n_{j+3/2} \in \mathcal{I}(\rho^n_{j+1}, \rho^n_{j+2})$. Note that all terms on the right hand side in (17) are positive, in particular the first term is positive because of the CFL condition (9), and in this way if we take absolute value and sum over all $j \in \mathbb{Z}$ in (17), we get the following estimate

$$\sum_{j\in\mathbb{Z}} |\mathcal{A}^n_j| \leq \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|. \tag{18}$$

On the other hand, doing some computations on $\mathcal{B}^n_j$, we can rewrite this term as follows

$$\begin{aligned}
\mathcal{B}^n_j \;=\; & \rho^n_j g'(\xi^n_{j+1/2}) \Big(W^n_{j+3/2} V'(\tilde{R}^n_{j+1}) \Big(R^n_{\eta,j+3/2} - R^n_{\eta,j+1/2} \Big) \\
& + V^n_{j+1/2} W'(\bar{R}^n_{j+1}) \Big(\hat{R}^n_{\delta,j+3/2} - \hat{R}^n_{\delta,j+1/2} \Big) \Big) (\rho^n_{j+1} - \rho^n_j) \\
& + g^n_j \Big(W^n_{j+3/2} V'(\tilde{R}^n_{j+1}) \Big(R^n_{\eta,j+3/2} - R^n_{\eta,j+1/2} \Big) \\
& + V^n_{j+1/2} W'(\bar{R}^n_{j+1}) \Big(\hat{R}^n_{\delta,j+3/2} - \hat{R}^n_{\delta,j+1/2} \Big) \Big) (\rho^n_j - \rho^n_{j-1}) \\
& + \rho^n_{j-1} g^n_j \Big[W^n_{j+3/2} V''(\tilde{\tilde{R}}^n_{j+1/2}) \Big(R^n_{\eta,j+3/2} - R^n_{\eta,j+1/2} \Big) \Big(\tilde{R}^n_{j+1} - \tilde{R}^n_j \Big) \\
& + V^n_{j+1/2} W''(\bar{\bar{R}}^n_{j+1/2}) \Big(\hat{R}^n_{\delta,j+3/2} - \hat{R}^n_{\delta,j+1/2} \Big) (\bar{R}^n_{j+1} - \bar{R}^n_j) \\
& + V'(\tilde{R}^n_j) \Big(W'(\bar{R}^n_{j+1}) \Big(\hat{R}^n_{\delta,j+3/2} - \hat{R}^n_{\delta,j+1/2} \Big) \Big(R^n_{\eta,j+3/2} - R^n_{\eta,j+1/2} \Big) \\
& + W^n_{j+1/2} \Big(R^n_{\eta,j+3/2} - 2R^n_{\eta,j+1/2} + R^n_{\eta,j-1/2} \Big) \Big) \\
& + W'(\bar{R}^n_j) \Big(V'(\tilde{R}^n_j) \Big(R^n_{\eta,j+1/2} - R^n_{j-1/2} \Big) \Big(\hat{R}^n_{\delta,j+3/2} - \hat{R}^n_{\delta,j+1/2} \Big) \\
& + V^n_{j-1/2} \Big(\hat{R}^n_{\delta,j+3/2} - 2\hat{R}^n_{\delta,j+1/2} + \hat{R}^n_{\delta,j-1/2} \Big) \Big) \Big],
\end{aligned}$$

where $\tilde{R}^n_j \in \mathcal{I}(R^n_{\eta,j-1/2}, R^n_{\eta,j+1/2})$, $\bar{R}^n_j \in \mathcal{I}(\hat{R}^n_{\delta,j-1/2}, \hat{R}^n_{\delta,j+1/2})$, $\tilde{\tilde{R}}^n_{j+1/2} \in \mathcal{I}(\tilde{R}^n_j, \tilde{R}^n_{j+1})$ and $\bar{\bar{R}}^n_{j+1/2} \in \mathcal{I}(\bar{R}^n_j, \bar{R}^n_{j+1})$. Now we take absolute value on both sides of the previous equality, then we replacing the inequalities (13), (14), (15) and summing over all $j \in \mathbb{Z}$ we get

$$\begin{aligned}
\lambda \sum_{j\in\mathbb{Z}} \left|\mathcal{B}^n_j\right| \;\leq\; & \Delta t \rho_{\max} \|g'\| \left(\|W\| \|V'\| \mathcal{K}_1 + \|V\| \|W'\| \mathcal{K}_2 \right) \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j| \\
& + \Delta t \|g\| \left(\|W\| \|V'\| \mathcal{K}_1 + \|V\| \|W'\| \mathcal{K}_2 \right) \sum_{j\in\mathbb{Z}} |\rho^n_j - \rho^n_{j-1}| \\
& \Delta t \|g\| \Big[2\|W\| \|V''\| \mathcal{K}_1^2 + 2\|V\| \|W''\| \mathcal{K}_2^2 \\
& + \|V'\| \left(\|W'\| \mathcal{K}_1 \mathcal{K}_2 + \|W\| \mathcal{K}_3 \right) + \|W'\| \|V'\| \mathcal{K}_1 \mathcal{K}_2 + \|V\| \mathcal{K}_4 \Big] \Delta x \sum_{j\in\mathbb{Z}} |\rho^n_{j-1}| \\
& + \Delta t \rho_{\max} \|g\| \|V'\| \|W\| \max\{\omega^0_\eta, \hat{\omega}^0_\delta\} \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j| \\
\leq\; & \Delta t \mathcal{H}_1 \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j| + \Delta t \mathcal{H}_2.
\end{aligned} \tag{19}$$

Therefore, collecting the terms in (18) and (19) we obtain the following estimates

$$\sum_{j\in\mathbb{Z}} |\rho_{j+1}^{n+1} - \rho_j^{n+1}| \leq (1 + \Delta t \mathcal{H}_1) \sum_{j\in\mathbb{Z}} |\rho_{j+1}^n - \rho_j^n| + \Delta t \mathcal{H}_2.$$

A standard iterative procedure provides the desired result (12). □

Lemma 3.4 (**BV** estimate in space and time)**.** *Let $\rho_0 \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0, \rho_{\max}])$, Assumptions 2.1 and the CFL condition (9) hold, then for all $T > 0$, the numerical solutions constructed by means of the numerical scheme (7) satisfy the following estimate, for all $n = 1, \ldots, N_T$,*

$$\sum_{n=0}^{N_T-1} \sum_{j\in\mathbb{Z}} \Delta t |\rho_{j+1}^n - \rho_j^n| + (T - N_T \Delta t) \sum_{j\in\mathbb{Z}} |\rho_{j+1}^{N_T} - \rho_j^{N_T}| + \sum_{n=0}^{N_T-1} \sum_{j\in\mathbb{Z}} \Delta x |\rho_{j+1}^n - \rho_j^n| \leq T \mathcal{Q}(T), \quad (20)$$

where

$$\begin{aligned}
\mathcal{Q}(T) &= \mathrm{TV}(\rho) + \mathcal{M}(T), \\
\mathcal{M}(T) &= \left(\rho_{\max} \|g'\| + \|g\|\right) \|V\| \|W\| e^{T\mathcal{H}_1} \left(\sum_{j\in\mathbb{Z}} |\rho_{j+1}^0 - \rho_{j+1}^0| + \frac{\mathcal{H}_2}{\mathcal{H}_1} (e^T - 1) \right) \\
&\quad + \Delta t \|g\| \left(\|V\| \|W'\| \mathcal{K}_2 + \|V'\| \|W\| \mathcal{K}_1 \right) \|\rho^n\|_{\mathbf{L^1}}.
\end{aligned}$$

Proof. By means of (12) we obtain

$$\sum_{n=0}^{N_T-1} \sum_{j\in\mathbb{Z}} \Delta t |\rho_{j+1}^n - \rho_j^n| + (T - N_T \Delta t) \sum_{j\in\mathbb{Z}} |\rho_{j+1}^{N_T} - \rho_j^{N_T}| \leq T e^{T\mathcal{H}_1} \left(\sum_{j\in\mathbb{Z}} |\rho_{j+1}^0 - \rho_{j+1}^0| + \frac{\mathcal{H}_2}{\mathcal{H}_1} (e^{T\mathcal{H}_1} - 1) \right). \quad (21)$$

On the other hand, by means of the scheme (7), adding and subtracting appropriate terms, we have

$$\begin{aligned}
|\rho_j^{n+1} - \rho_j^n| &= \Big| \lambda \rho_j^n g'(\xi_{j+1/2}^n) W_{j+1/2}^n V_{j+1/2}^n (\rho_{j+1}^n - \rho_j^n) + \lambda \rho_j^n g_j^n V_{j+1/2}^n \left(W_{j+1/2}^n - W_{j-1/2}^n \right) \\
&\quad + \lambda \rho_j^n g_j^n W_{j-1/2}^n \left(V_{j+1/2}^n - V_{j-1/2}^n \right) + \lambda g_j^n W_{j-1/2}^n V_{j-1/2}^n \left(\rho_j^n - \rho_{j-1}^n \right) \Big| \\
&\leq \lambda \rho_{\max} \|g'\| \|W\| \|V\| |\rho_{j+1}^n - \rho_j^n| + \lambda \rho_j^n \|g\| \|V\| \|W'\| \left| \hat{R}_{\delta,j+1/2}^n - \hat{R}_{\delta,j-1/2}^n \right| \\
&\quad + \lambda \rho_j^n \|g\| \|W\| \|V'\| \left| R_{\eta,j+1/2}^n - R_{\eta,j-1/2}^n \right| + \lambda \|g\| \|W\| \|V\| \left| \rho_j^n - \rho_{j-1}^n \right| \\
&\leq \lambda \rho_{\max} \|g'\| \|W\| \|V\| |\rho_{j+1}^n - \rho_j^n| + \lambda \|g\| \|V\| \|W'\| \mathcal{K}_2 \Delta x \rho_j^n \\
&\quad + \lambda \|g\| \|W\| \|V'\| \mathcal{K}_1 \Delta x \rho_j^n + \lambda \|g\| \|W\| \|V\| \left| \rho_j^n - \rho_{j-1}^n \right|,
\end{aligned}$$

now multiplying by Δx and summing over all $j \in \mathbb{Z}$ in the previous inequality, we get

$$\begin{aligned}
\Delta x \sum_{j\in\mathbb{Z}} \left| \rho_j^{n+1} - \rho_j^n \right| &\leq \Delta t \left(\rho_{\max} \|g'\| + \|g\| \right) \|V\| \|W\| \sum_{j\in\mathbb{Z}} \left| \rho_{j+1}^n - \rho_j^n \right| \\
&\quad + \Delta t \|g\| \left(\|V\| \|W'\| \mathcal{K}_2 + \|V'\| \|W\| \mathcal{K}_1 \right) \|\rho^n\|_{\mathbf{L^1}} \\
&\leq \Delta t \left(\rho_{\max} \|g'\| + \|g\| \right) \|V\| \|W\| e^{T\mathcal{H}_1} \left(\sum_{j\in\mathbb{Z}} |\rho_{j+1}^0 - \rho_j^0| + \frac{\mathcal{H}_2}{\mathcal{H}_1} (e^{T\mathcal{H}_1} - 1) \right) \\
&\quad + \Delta t \|g\| \left(\|V\| \|W'\| \mathcal{K}_2 + \|V'\| \|W\| \mathcal{K}_1 \right) \|\rho^n\|_{\mathbf{L^1}} \\
&= \Delta t \mathcal{M}(T). \qquad (22)
\end{aligned}$$

Next, summing over $n \in \{0, \ldots, N_T - 1\}$ we get

$$\sum_{n=0}^{N_T-1} \sum_{j\in\mathbb{Z}} \Delta x \left|\rho_j^{n+1} - \rho_j^n\right| \leq N_T \Delta t \mathcal{M}(T). \tag{23}$$

Finally, collecting (21) and (23) obtain the desired estimate (20). □

3.3. Discrete entropy inequalities

Next we derive a discrete entropy inequality for the approximate solutions obtained by means of the scheme (7). This discrete entropy inequality is used in order to prove that the limit of numerical solutions is indeed a weak entropy solution in the sense of Definition (1). First, we denote

$$G_{j+1/2}^{n,k}(a,b) \quad := \quad \mathcal{F}_{j+1/2}^n(a \vee k, b \vee k) - \mathcal{F}_{j+1/2}^n(a \wedge k, b \wedge k),$$

where $a \vee b := \max\{a,b\}$, $a \wedge b := \min\{a,b\}$ and $\mathcal{F}_{j+1/2}^n(a,b) = ag(b)W_{j+1/2}^n V_{j+1/2}^n$.

Proposition 3.1. *Let Assumptions 2.1 and the CFL condition* (9) *hold, then the approximate solution computed by means of* (7) *satisfies the discrete entropy inequality*

$$\begin{aligned}&\left|\rho_j^{n+1} - k\right| - \left|\rho_j^n - k\right| + \lambda\left(G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j-1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n)\right)\\ &+\lambda \mathrm{sgn}(\rho_j^{n+1} - k)kg(k)\left(W_{j+1/2}^n V_{j+1/2}^n - W_{j-1/2}^n V_{j-1/2}^n\right) \leq 0,\end{aligned} \tag{24}$$

for all $j \in \mathbb{Z}$, $n = 0, \ldots, N_T - 1$ and $k \in \mathbb{R}$.

Proof. The proof follows [1, 6] with the appropriate change in the numerical flux. We define

$$H_j^n(u,p,z) = p - \lambda\left(\mathcal{F}_{j+1/2}^n(p,z) - \mathcal{F}_{j-1/2}^n(u,p)\right), \tag{25}$$

Observe that H_j^n is monotone non-decreasing with with respect to each variable, indeed we have

$$\begin{aligned}\frac{\partial H_j^n}{\partial u} &= \lambda g(p) W_{j-1/2}^n V_{j-1/2}^n \geq 0,\\ \frac{\partial H_j^n}{\partial p} &= 1 - \lambda\left(g(z)W_{j+1/2}^n V_{j+1/2}^n - ug'(p)W_{j-1/2}^n V_{j-1/2}^n\right) \geq 0, \text{ by the CFL condition (9)},\\ \frac{\partial H_j^n}{\partial z} &= -\lambda pg'(z) W_{j+1/2}^n V_{j+1/2}^n \geq 0.\end{aligned}$$

Observe that we can write the numerical scheme (7) as $\rho_j^{n+1} = H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n)$ and note also that for $k \in \mathbb{R}$, $H_j^n(k,k,k) = k - \lambda kg(k)\left(W_{j+1/2}^n V_{j+1/2}^n - W_{j-1/2}^n V_{j-1/2}^n\right)$, and furthermore, we have the identity

$$\begin{aligned}&H_j^n(\rho_{j-1}^n \vee k, \rho_j^n \vee k, \rho_{j+1}^n \vee k) - H_j^n(\rho_{j-1}^n \wedge k, \rho_j^n \wedge k, \rho_{j+1}^n \wedge k)\\ &= \left|\rho_j^n - k\right| - \lambda\left[G_{j+1/2}^{n,k}(\rho_j, \rho_{j+1}) - G_{j-1/2}^{n,k}(\rho_{j-1}, \rho_j)\right],\end{aligned} \tag{26}$$

and on the other hand, by monotonicity of the H_j^n map we get

$$\begin{aligned}
&H_j^n(\rho_{j-1}^n \vee k, \rho_j^n \vee k, \rho_{j+1}^n \vee k) - H_j^n(\rho_{j-1}^n \wedge k, \rho_j^n \wedge k, \rho_{j+1}^n \wedge k) \\
&\geq H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) \vee H_j^n(k,k,k) - H_j^n(\rho_{j-1}, \rho_j, \rho_{j+1}) \wedge H_j^n(k,k,k) \\
&= \left|H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - H_j^n(k,k,k)\right| \\
&= \mathrm{sgn}(H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - H_j^n(k,k,k)) \times \left(H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - H_j^n(k,k,k)\right) \\
&= \mathrm{sgn}(H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - k + \lambda k g(k)(W_{j+1/2}^n V_{j+1/2}^n - W_{j-1/2}^n V_{j-1/2}^n)) \times \\
&\quad \times \left(H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - k + \lambda k g(k)(W_{j+1/2}^n V_{j+1/2}^n - W_{j-1/2}^n V_{j-1/2}^n)\right) \\
&\geq \mathrm{sgn}(H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - k) \times \Bigg(H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - k \\
&\qquad\qquad + \lambda k g(k)(W_{j+1/2}^n V_{j+1/2}^n - W_{j-1/2}^n V_{j-1/2}^n)\Bigg) \\
&= \left|H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - k\right| \\
&\quad + \lambda\, \mathrm{sgn}(H_j^n(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - k) k g(k)(W_{j+1/2}^n V_{j+1/2}^n - W_{j-1/2}^n V_{j-1/2}^n) \\
&= \left|\rho_j^{n+1} - k\right| + \lambda\, \mathrm{sgn}(\rho_j^{n+1} - k) k g(k)(W_{j+1/2}^n V_{j+1/2}^n - W_{j-1/2}^n V_{j-1/2}^n), \qquad (27)
\end{aligned}$$

so, by means of the equations (26) and (27) we get the desired result (24). □

4. $\mathbf{L^1}$-stability of the entropy weak solutions

The following theorem states the $\mathbf{L^1}$-Lipschitz continuous dependence of solutions to (1) on the initial data.

Theorem 2. *Let Assumptions* (2.1) *hold and let ρ, σ be two entropy solutions to Eq* (1) *with initial data ρ_0, σ_0, respectively, then for any $T > 0$ there holds*

$$\|\rho(T,\cdot) - \sigma(T,\cdot)\|_{\mathbf{L^1}} \leq \mathrm{e}^{\mathcal{C}T} \|\rho_0 - \sigma_0\|_{\mathbf{L^1}}, \qquad (28)$$

where $\mathcal{C}$ is a positive constant depending on the model parameters.

Proof. The proof of Theorem 2 is quite standard and is adapted from [6, Theorem 2.1] with the appropriate changes in the flux function. □

We want to highlight the fact that if we increase the regularity of the parameters in Assumptions 2.1, as in [7]:

Assumptions 4.1.

i') $V \in (\mathbf{C^2} \cap \mathbf{W}^{2,\infty})(\mathbb{R};\mathbb{R})$, with $0 \leq V(\rho) \leq V_{\max}$, $V'(\rho) \leq 0$, $\rho \in \mathbb{R}$;

ii') $W \in (\mathbf{C^2} \cap \mathbf{W}^{2,\infty})(\mathbb{R};\mathbb{R})$, with $1 \leq W(\rho) \leq W_{\max}$, $W'(\rho) \geq 0$, $\rho \in \mathbb{R}$;

iv') $\omega_\eta \in (\mathbf{C^2} \cap \mathbf{W}^{1,1} \cap \mathbf{W}^{2,\infty})(\mathbb{R};\mathbb{R})$ with $\omega_\eta'(x) \leq 0$, $\displaystyle\int_{\mathbb{R}} \omega_\eta(x)\mathrm{d}x = 1$, $\forall \eta > 0$.

v') $\hat{\omega}_\delta \in (\mathbf{C^2} \cap \mathbf{W}^{1,1} \cap \mathbf{W}^{2,\infty})(\mathbb{R};\mathbb{R})$ with $\hat{\omega}_\delta'(x) \leq 0$, $\displaystyle\int_{\mathbb{R}} \hat{\omega}_\delta(x)\mathrm{d}x = 1$, $\forall \delta > 0$, we get the following result.

Theorem 3. *Let Assumptions (4.1) hold and let ρ, σ be two entropy solutions to (1) with initial data ρ_0, σ_0 respectively, and kernels ω_η^1, ω_η^2, $\hat{\omega}_\delta^1$, $\hat{\omega}_\delta^2$, then for any $T > 0$ there holds*

$$\|\rho(T,\cdot) - \sigma(T,\cdot)\|_{\mathbf{L^1}} \leq \left(\|\rho_0 - \sigma_0\|_{\mathbf{L^1}} + \mathcal{C}_{10}(T)\left(\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}}\right)\right) \mathrm{e}^{\mathcal{C}_9 T}, \quad (29)$$

with $\mathcal{C}_9$ and $\mathcal{C}_{10}(T)$ defined in Eq (33), Eq (34), respectively.

Proof. Let $\rho(t,x)$ and $\sigma(t,x)$ be a weak entropy solutions of the following equations

$$\begin{cases} \rho(t,x) + \partial_x(f(\rho(t,x))U(t,x))) = 0 \\ \rho(0,x) = \rho_0(x), \end{cases} \qquad \begin{cases} \partial_t\sigma(t,x) + \partial_x(f(\sigma(t,x))\tilde{U}(t,x))) = 0 \\ \sigma(0,x) = \sigma_0(x), \end{cases}$$

respectively, where we have denoted

$$f(\rho) \;:=\; \rho g(\rho), \qquad U(t,x) = W(t,x)V(t,x), \qquad \tilde{U}(t,x) = \tilde{W}(t,x)\tilde{V}(t,x),$$

with $W(t,x) := W((\rho * \hat{\omega}_\delta^1)(t,x))$, $V(t,x) := V((\rho * \omega_\eta^1)(t,x))$, $\tilde{W}(t,x) := W((\sigma * \hat{\omega}_\delta^2)(t,x))$ and $\tilde{V}(t,x) := V((\sigma * \omega_\eta^2)(t,x))$. First, we use the classical doubling techniques introduced by Kruzkov [15] to obtain the following inequality

$$\|\rho(T,\cdot) - \sigma(T,\cdot)\|_{\mathbf{L^1}} \leq \|\rho_0(x) - \sigma_0(x)\|_{\mathbf{L^1}} + \int_0^T \int_{\mathbb{R}} \mathcal{S}_1(t,x)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}} S_2(t,x)\mathrm{d}x\mathrm{d}t, \quad (30)$$

where we have denoted $\mathcal{S}_1(t,x) := |\mathcal{U}|\,|\partial_x\rho(t,x)||f'(\rho(t,x))|$, $\mathcal{S}_2(t,x) := |\partial_x\mathcal{U}|\,|f(\rho(t,x))|$ and $\mathcal{U} := \tilde{U}(t,x) - U(t,x)$. Next, we have the following estimates derived in Appendix 7:

$$\int_0^T \int_{\mathbb{R}} \mathcal{S}_1(t,x)\mathrm{d}x\mathrm{d}t \leq \mathcal{C}_1 \int_0^T \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}}\mathrm{d}t + \mathcal{C}_2(T)\left(\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}}\right) (31)$$

$$\int_0^T \int_{\mathbb{R}} \mathcal{S}_2(t,x)\mathrm{d}x\mathrm{d}t \leq \mathcal{C}_8 \int_0^T \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}}\mathrm{d}t + \mathcal{C}_6(T)\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \mathcal{C}_7(T)\|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}} (32)$$

where $\mathcal{C}_1$, $\mathcal{C}_8$ are positive constants depending on the parameters of the model (1) and $\mathcal{C}_2(T)$, $\mathcal{C}_6(T)$, $\mathcal{C}_7(T)$ also depend on T. These terms are computed in the appendix section. Now, if we replace (31) and (32) in (30) we get,

$$\begin{aligned} \|\rho(T,\cdot) - \sigma(T,\cdot)\|_{\mathbf{L^1}} \;\leq\; & \|\rho_0(x) - \sigma_0(x)\|_{\mathbf{L^1}} + \mathcal{C}_{10}(T)\left(\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}}\right) \\ & + \mathcal{C}_9 \int_0^T \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}}\mathrm{d}t, \end{aligned}$$

where we are denoting

$$\mathcal{C}_9 \;:=\; \mathcal{C}_1 + \mathcal{C}_8 \quad (33)$$

$$\mathcal{C}_{10}(T) \;:=\; \max\{\mathcal{C}_2(T) + \mathcal{C}_6(T), \mathcal{C}_2(T) + \mathcal{C}_7(T)\} \quad (34)$$

Finally, by applying Gronwall's lemma we get (11) and for $\rho_0(x) = \sigma_0(x)$, the uniqueness of entropy solutions. □

Proof of Theorem 1

Existence of solutions to the problem (1)-(2) follow from the results in Section 3. Indeed, Lemma 3.1 and Lemma 3.4, allow us to apply Helly's compactness theorem, by which we can guarantee the existence of a subsequence of approximate solutions ρ^{Δ} that converges in $\mathbf{L^1}$ to a function $\rho \in \mathbf{L}^{\infty}([0,T]\times\mathbb{R};\mathbb{R}^{+})$. Following a Lax-Wendroff-type argument, we can show that the limit function ρ is a weak entropy solution of (1)-(2) in the sense of Definition 1. Uniqueness is ensured by the Lipschitz continuous dependence of solutions to Eq(1) on initial data (Theorem 2) and also on kernel functions ω_η, $\hat{\omega}_\delta$ (Theorem 3), see Section 4.

5. Numerical Examples

In the following tests, we solve the nonlocal problem (1) numerically for $x \in [0,10]$ within a time horizon $T > 0$ and periodic boundary conditions, using the numerical scheme (7)-(8) with a fixed discretization of $M = 800$ cells of length $\Delta x = 1/200$ and a time step Δt satisfying the CFL condition (9). We consider the function $g(\rho) = (\rho_{\max} - \rho)$, with maximum speed $V_{\max} = 1$ and maximum density $\rho_{\max} = 1$. The other parameters are specified in each subsection.

In order to observe the "look-behind-ahead effect", we refer to Model 1 for (1) with a non-constant functions V and W, and to Model 2 to the "look-ahead" model with $W(\rho) = 1$.

5.1. Example 1

Inspired by the numerical examples in [13], we consider the speed functions

$$V(\rho) = V_{\max}(1-\rho), \quad V_{\max} = 1, \qquad W(\rho) = (k+1)\frac{V(-\rho)}{k+V(-\rho)}, \quad k = \frac{1}{2}, \tag{35}$$

and the kernel functions and supports given by

$$\omega_\eta(x) = \frac{1}{\eta}, \quad \eta = 1, \qquad \hat{\omega}_\delta(x) = \frac{\delta - x}{\delta^2} \quad \delta > 0,$$

and initial condition

$$\rho_0(x) = 0.8\chi_{[1,3]}(x) + 0.5\chi_{[5,8]}(x), \quad x \in [0,10].$$

In Figure 1 (a-c), we display the numerical approximation at different simulation times, $T = 3.15$, $T = 9.75$ and $T = 18$. In each plot, we compare $\rho_\Delta(\cdot, T)$ for Model 1 with $\delta = 0.1$, $\delta = 0.5$ and $\delta = 1$, with respect to Model 2. We can observe that the density ρ moves faster in the "look-behind-ahead" model than in the "look-ahead" model as δ increases. In Figure 1 (d), we show the $TV(\rho^{\Delta}(t,\cdot))$ for $0 \le t \le 18$, where we can observe that the total variation is smaller and decays faster for Model 1.

5.2. Example 2: Limit $\delta, \eta \to 0$

In this test we consider $\omega_\eta(x)$ and $\hat{\omega}_\delta(x)$ as in Example 1 with $\eta = 2\delta$, and initial condition

$$\rho_0(x) = 0.8\chi_{[3,6]}(x) + 0.1, \quad x \in [0,10].$$

We investigate numerically the limit $\delta \searrow 0$. In Figure 2, we display the numerical approximations corresponding to $\delta = 0.5$, $\delta = 0.1$ and $\delta = 0.05$, compared to the numerical solution of the local problem with flux function $f(\rho) = \rho g(\rho) V(\rho) W(\rho)$.

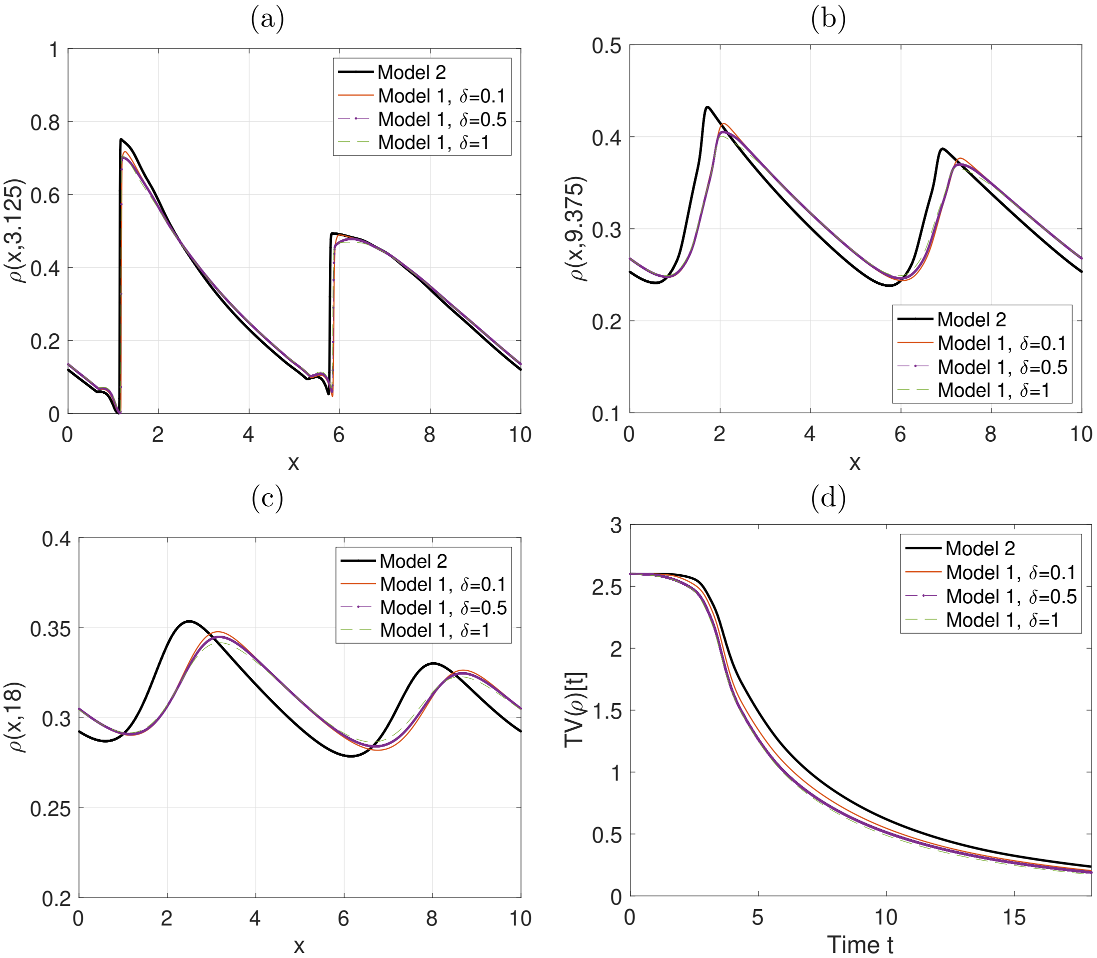

Figure 1: Example 1: Comparison of the "look-behind-ahead" model with $\hat{\omega}(x) = \frac{\delta - x}{\delta^2}$ and $\delta = 1$, $\delta = 0.5$ and $\delta = 0.1$ with respect to the "look-ahead" model at different simulation times, and their total variation decay.

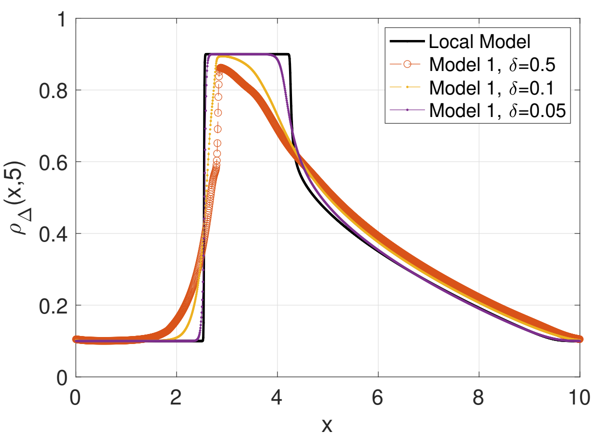

Figure 2: Example 2: Limit $\delta, \eta \to 0$ in Model 1.

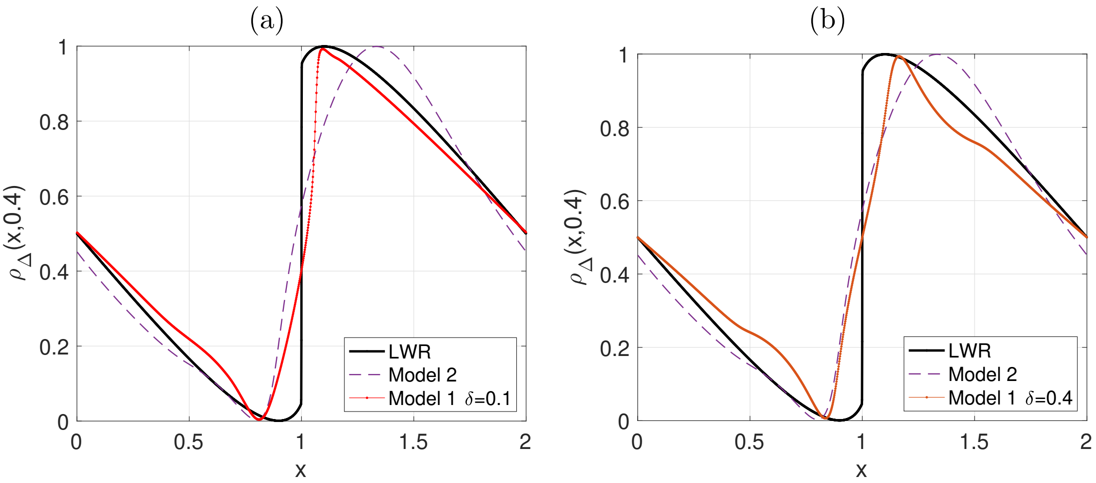

Figure 3: Example 3: Comparison Arrhenius look-ahead-behind model with respect to LWR traffic model.

5.3. Example 3: Arrhenius look-ahead-behind model

In this numerical test we consider the Arrhenius look-ahead-behind model studied in [12] where

$$g(\rho) = (1-\rho), \quad V(\rho) = e^{-\rho}, \quad W(\rho) = e^{\rho},$$

and nonlocal terms

$$(\rho * \omega_{\gamma_a})(t,x) = \frac{1}{\gamma_a}\int_x^{x+\gamma_a} K_a \rho(t,z)\mathrm{d}z, \qquad (\rho * \hat{\omega}_{\gamma_b})(t,x) = \frac{1}{\gamma_b}\int_{x-\gamma_b}^{x} K_b \rho(t,z)\mathrm{d}z, \tag{36}$$

where γ_a and γ_b are positive constant proportional to the look-ahead and behind distances, respectively, and K_a and K_b represent constants proportional to positive interaction strengths. Observe that functions g, V and W satisfies Assumptions 2.1 $(i-iii)$ for $\rho \in [0,1]$, however for $K_a \neq 1$ and $K_b \neq 1$, kernel functions does not satisfies Assumptions 2.1 $(iv-v)$. The main intention in this Example is to use the numerical scheme (7)-(8) to observe the effect of blow-up in ρ_x at finite time. According with [12], the higher values of K_b increase the blow-up, whereas the higher values of γ_a and γ_b suppress the blow-up.

6. Conclusions

This work has introduced and analyzed a macroscopic traffic flow model described by a nonlocal scalar conservation law whose flux combines, in a multiplicative manner, both downstream and upstream averaged densities. The proposed formulation is based on the nonlocal model of bidirectional flux introduced in [13], but extends it by establishing a well-posedness theory for entropy weak solutions and by allowing for discontinuities and shock waves. This bidirectional structure enables the simultaneous representation of the influence of traffic conditions ahead and behind each driver, thus offering a more complete and realistic description of vehicular dynamics than unidirectional nonlocal models previously studied in the literature.

From a theoretical perspective, we have proved the existence and uniqueness of entropy weak solutions, along with Lipschitz continuous dependence of the solutions on both the initial data and the convolution kernel functions. These results ensure the robustness of the model with respect to perturbations in the data and parameters, and broaden the analytical framework beyond the smooth solution setting considered in [13].

On the numerical side, we have developed a conservative finite volume scheme that preserves key structural properties of the continuous model, such as mass conservation, positivity, and $\mathbf{L^1}$-stability. Numerical experiments show that the inclusion of the upstream and downstream nonlocal terms qualitatively affects traffic dynamics, providing a more realistic framework for analyzing phenomena such as shock formation and propagation.

Overall, the results presented in this paper provide a solid mathematical and computational framework for the study of nonlocal conservation laws with bidirectional interactions, with clear potential for applications in the design and assessment of traffic control strategies in realistic settings.

Acknowledgments

LMV is supported by Centro de Modelamiento Matemático (CMM), project FB210005 of BASAL funds for Centers of Excellence.

7. Appendix: Technical estimates

Estimates in Lemma 3.3

For the discretized downstream convolution term we have the following estimates,

$$\begin{aligned}
\left|R^n_{\eta,j+1/2} - R^n_{\eta,j-1/2}\right| &= \left|\Delta x \sum_{k=0}^{N-1} \omega^k_\eta \rho^n_{j+k+1} - \Delta x \sum_{k=0}^{N-1} \omega^k_\eta \rho^n_{j+k}\right| \\
&= \left|\Delta x \left(\sum_{k=1}^{N} \omega^{k-1}_\eta \rho^n_{j+k} - \sum_{k=0}^{N-1} \omega^k_\eta \rho^n_{j+k}\right)\right| \\
&= \left|\Delta x \left(\sum_{k=0}^{N-1} \left(\omega^{k-1}_\eta - \omega^k_\eta\right) \rho^n_{j+k} + \omega^{N-1}_\eta \rho^n_{j+N} - \omega^0_\eta \rho^n_j\right)\right| \\
&\leq \Delta x \|\omega'_\eta\| \|\rho\|_{\mathbf{L^1}} + 2\Delta x \rho_{\max} \omega^0_\eta \\
&= \Delta x \mathcal{K}_1,
\end{aligned}$$

where we denote $\mathcal{K}_1 = \|\omega'_\eta\| \|\rho^n\|_{\mathbf{L^1}} + 2\rho_{\max}\omega^0_\eta$. Now for $0 \leq \theta$, $\mu \leq 1$, we have

$$\begin{aligned}
\left|\tilde{R}^n_{j+1} - \tilde{R}^n_j\right| &= \left|\theta R^n_{\eta,j+3/2} + (1-\theta) R^n_{\eta,j+1/2} - \mu R^n_{\eta,j+1/2} - (1-\mu) R^n_{\eta,j-1/2}\right| \\
&= \left|\theta(R^n_{\eta,j+3/2} - R^n_{\eta,j+1/2}) + (R^n_{\eta,j+1/2} - R^n_{\eta,j-1/2}) - \mu(R^n_{\eta,j+1/2} - R^n_{\eta,j-1/2})\right| \\
&= \left|\theta(R^n_{\eta,j+3/2} - R^n_{\eta,j+1/2}) + (1-\mu)(R^n_{\eta,j+1/2} - R^n_{\eta,j-1/2})\right| \\
&\leq |R^n_{\eta,j+3/2} - R^n_{\eta,j+1/2}| + |R^n_{\eta,j+1/2} - R^n_{\eta,j-1/2}| \\
&\leq 2\Delta x \|\omega'_\eta\| \|\rho\|_{\mathbf{L^1}} + 4\Delta x \rho_{\max} \omega^0_\eta \\
&= 2\Delta x \mathcal{K}_1,
\end{aligned}$$

also, we compute the following estimates

$$\begin{aligned}\left|\hat{R}^n_{\delta,j+1/2}-\hat{R}^n_{\delta,j-1/2}\right| &\leq \Delta x\|\omega_\delta'\|\|\rho\|_{\mathbf{L^1}}+2\Delta x\rho_{\max}\hat{\omega}^0_\delta\\ &= \Delta x\mathcal{K}_2,\end{aligned}$$

and for $0\leq\alpha,\ \beta\leq 1$ we have

$$\begin{aligned}\left|\bar{R}^n_{j+1}-\bar{R}^n_{j}\right| &\leq 2\Delta x\|\omega_\delta'\|\|\rho\|_{\mathbf{L^1}}+4\Delta x\rho_{\max}\hat{\omega}^0_\delta\\ &= 2\Delta x\mathcal{K}_2.\end{aligned}$$

where $\mathcal{K}_2=\|\omega_\delta'\|\|\rho^n\|_{\mathbf{L^1}}+2\rho_{\max}\hat{\omega}^0_\delta$. On the other hand, we have the following estimates

$$\begin{aligned}&\left|R^n_{\eta,j+3/2}-2R^n_{\eta,j+1/2}+R^n_{\eta,j-1/2}\right|\\
&=\left|\Delta x\left(\sum_{k=0}^{N-1}\omega_\eta^k\rho^n_{j+k+2}-2\sum_{k=0}^{N-1}\omega_\eta^k\rho^n_{j+k+1}+\sum_{k=0}^{N-1}\omega_\eta^k\rho^n_{j+k}\right)\right|\\
&=\left|\Delta x\left(\sum_{k=1}^{N}\omega_\eta^{k-1}\rho^n_{j+k+1}-2\sum_{k=0}^{N-1}\omega_\eta^k\rho^n_{j+k+1}+\sum_{k=-1}^{N-2}\omega_\eta^{k+1}\rho^n_{j+k+1}\right)\right|\\
&=\left|\Delta x\left(\sum_{k=0}^{N-1}\left(\omega_\eta^{k-1}-2\omega_\eta^k+\omega_\eta^{k+1}\right)\rho^n_{j+k+1}+\omega_\eta^{N-1}\rho^n_{j+N+1}-2\omega^0_\eta\rho^n_{j+1}+\omega^0_\eta\rho^n_j+\omega^1_\eta\rho^n_{j+1}-\omega_\eta^N\rho^n_{j+N}\right)\right|\\
&=\left|\Delta x\left((\Delta x)^2\sum_{k=0}^{N-1}\frac{\omega_\eta^{k-1}-2\omega_\eta^k+\omega_\eta^{k+1}}{(\Delta x)^2}\rho^n_{j+k+1}\right.\right.\\
&\qquad\left.\left.+\omega^0_\eta(\rho^n_j-\rho^n_{j+1})+\Delta x\rho^n_{j+1}\frac{\omega^1_\eta-\omega^0_\eta}{\Delta x}+\omega_\eta^{N-1}\rho^n_{j+N+1}-\underbrace{\omega_\eta^N}_{=0}\rho^n_{j+N}\right)\right|\\
&=\left|\Delta x\left((\Delta x)^2\sum_{k=0}^{N-1}\frac{\omega_\eta^{k-1}-2\omega_\eta^k+\omega_\eta^{k+1}}{(\Delta x)^2}\rho^n_{j+k+1}\right.\right.\\
&\qquad\left.\left.+\omega^0_\eta(\rho^n_j-\rho^n_{j+1})+\Delta x\rho^n_{j+1}\frac{\omega^1-\omega^0_\eta}{\Delta x}+\Delta x\left(\frac{\omega_\eta^{N-1}-\omega_\eta^N}{\Delta x}\right)\rho^n_{j+N+1}\right)\right|\\
&\leq(\Delta x)^2\|\omega_\eta''\|\|\rho\|_{\mathbf{L^1}}+\Delta x\omega^0_\eta|\rho^n_{j+1}-\rho^n_j|+2(\Delta x)^2\rho_{\max}\|\omega_\eta'\|\\
&=(\Delta x)^2\mathcal{K}_3+\Delta x\omega^0_\eta|\rho^n_{j+1}-\rho^n_j|.\end{aligned}$$

In a similar way, we get the following estimates for the discretized upstream convolution term

$$\begin{aligned}\left|\tilde{R}^n_{j+3/2}-2\tilde{R}^n_{j+1/2}+\tilde{R}^n_{j-1/2}\right| &\leq (\Delta x)^2\|\omega_\delta''\|\|\rho\|_{\mathbf{L^1}}+\Delta x\hat{\omega}^0_\delta|\rho^n_{j+1}-\rho^n_j|+2(\Delta x)^2\rho_{\max}\|\omega_\delta'\|\\ &= (\Delta x)^2\mathcal{K}_4+\Delta x\hat{\omega}^0_\delta|\rho^n_{j+1}-\rho^n_j|.\end{aligned}$$

Estimates in Theorem 3.

In this part of the work, we derive upper bounds for $\int_0^T \int_{\mathbb{R}} \mathcal{S}_1(t,x)\mathrm{d}x\mathrm{d}t$ and $\int_0^T \int_{\mathbb{R}} \mathcal{S}_2(t,x)\mathrm{d}x\mathrm{d}t$ in Eq. (30). First, we compute $|\mathcal{U}|$ in $\mathcal{S}_1$ as follows,

$$\begin{aligned} |\mathcal{U}| &= |\tilde{W}\tilde{V} - WV| \\ &= |\tilde{W}(\tilde{V} - V) + (\tilde{W} - W)V| \\ &\leq \|\tilde{W}\||\tilde{V} - V| + \|V\||\tilde{W} - W| \\ &= \|W\||\tilde{V} - V| + \|V\||\tilde{W} - W| \end{aligned} \tag{37}$$

Now we compute each term on the right-hand side of inequality (37).

$$\begin{aligned} |\tilde{V} - V| &= \left|V(\sigma * \omega_\eta^2) - V(\rho * \omega_\eta^1)\right| \\ &= \left|V(\sigma * \omega_\eta^2) - V(\rho * \omega_\eta^2) + V(\rho * \omega_\eta^2) - V(\rho * \omega_\eta^1)\right| \\ &= \left|V'(\xi_1)\left((\sigma - \rho) * \omega_\eta^2\right) + V'(\xi_2)\left(\rho * \left(\omega_\eta^2 - \omega_\eta^1\right)\right)\right| \\ &\leq \|V'\|\left(\|\omega_\eta^2\|\|\rho - \sigma\|_{\mathbf{L^1}} + \|\rho\|\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{L^1}}\right), \end{aligned} \tag{38}$$

where $\xi_1 \in \mathcal{I}(\sigma * \omega_\eta^2, \rho * \omega_\eta^2)$ and $\xi_2 \in \mathcal{I}(\rho * \omega_\eta^2, \rho * \omega_\eta^1)$. In a similar way, we get

$$\left|\tilde{W} - W\right| \leq \|W'\|\left(\|\hat{\omega}_\delta^2\|\|\rho - \sigma\|_{\mathbf{L^1}} + \|\rho\|\|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{L^1}}\right). \tag{39}$$

Thus, replacing (38) and (39) in (37) we obtain the following bound for $|\mathcal{U}|$,

$$\begin{aligned} |\mathcal{U}| &\leq \left(\|W\|\|V'\|\|\omega_\eta^2\| + \|V\|\|W'\|\|\hat{\omega}_\delta^2\|\right)\|\rho - \sigma\|_{\mathbf{L^1}} + \|\rho\|\left(\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{L^1}} + \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{L^1}}\right) \\ &\leq \left(\|W\|\|V'\|\|\omega_\eta^2\| + \|V\|\|W'\|\|\hat{\omega}_\delta^2\|\right)\|\rho - \sigma\|_{\mathbf{L^1}} + \|\rho\|\left(\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}}\right), \end{aligned}$$

then, with this estimate, we can compute a bound for (30),

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \mathcal{S}_1(t,x)\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}} \left(|\mathcal{U}||\partial_x\rho||f'(\rho(t,x))|\right)\mathrm{d}x\mathrm{d}t \\ &\leq \left(\|W\|\|V'\|\|\omega_\eta^2\| + \|V\|\|W'\|\|\hat{\omega}_\delta^2\|\right)\int_0^T \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}} \int_{\mathbb{R}} |\partial_x\rho(t,x)||f'(\rho(t,x))|\mathrm{d}x\mathrm{d}t \\ &\quad + T\|\rho\|\left(\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}}\right)\int_{\mathbb{R}} |\partial_x\rho(t,x)||f'(\rho(t,x))|\mathrm{d}x\mathrm{d}t \\ &\leq \left(\|W\|\|V'\|\|\omega_\eta^2\| + \|V\|\|W'\|\|\hat{\omega}_\delta^2\|\right)\|\rho\|_{\mathbf{BV}} \sup_{t\in[0,T]} \|f'(\rho(t,\cdot))\| \int_0^T \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}} \\ &\quad + T\left(\|\rho\|\|\rho\|_{\mathbf{BV}} \sup_{t\in[0,T]} \|f'(\rho(t,\cdot))\|\right)\left(\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}}\right) \\ &\leq \mathcal{C}_1 \int_0^T \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}} + \mathcal{C}_2(T)\left(\|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}}\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{C}_1 &= \leq \left(\|W\|\|V'\|\|\omega_\eta^2\| + \|V\|\|W'\|\|\hat{\omega}_\delta^2\|\right)\|\rho\|_{\mathbf{BV}} \sup_{t\in[0,T]} \|f'(\rho(t,\cdot))\| \\ \mathcal{C}_2(T) &= T\left(\|\rho\|\|\rho\|_{\mathbf{BV}} \sup_{t\in[0,T]} \|f'(\rho(t,\cdot))\|\right). \end{aligned}$$

Next, we bound $\partial_x|\mathcal{U}|$ in $\mathcal{S}_2(t,x)$,

$$\begin{aligned}\partial_x|\mathcal{U}| &= |\partial_x U(t,x) - \partial_x\vartheta(t,x)| \\ &= |\partial_x(\tilde{W}\tilde{V}) - \partial_x(WV)| \\ &\leq |\partial_x\tilde{W}||\tilde{V}-V| + \|V\||\partial_x\tilde{W}-\partial_x W| + \|\tilde{W}\||\partial_x\tilde{V}-\partial_x V| + |\partial_x V||\tilde{W}-W|, \end{aligned} \tag{40}$$

and bounded every term on the right-hand side of inequality (40). For first term,

$$\begin{aligned}|\partial_x\tilde{W}| &= |\partial_x W(\sigma*\hat{\omega}_\delta^2)| \\ &= |W'(\sigma*\hat{\omega}_\delta^2)(\sigma*\partial_x\hat{\omega}_\delta^2)| \\ &\leq \|W'\|\|\sigma\|_{\mathbf{L^1}}\|\partial_x\hat{\omega}_\delta^2\|. \end{aligned} \tag{41}$$

Now, for the second term on the right hand side of (40), we have the following estimates,

$$\begin{aligned}&|\partial_x\tilde{W}-\partial_x W| \\ &= \left|\partial_x W(\sigma*\hat{\omega}_\delta^2) - \partial_x W(\rho*\hat{\omega}_\delta^1)\right| \\ &= \left|W'(\sigma*\hat{\omega}_\delta^2)(\sigma*\partial_x\hat{\omega}_\delta^2) - W'(\rho*\hat{\omega}_\delta^1)(\rho*\partial_x\hat{\omega}_\delta^1)\right| \\ &= \left|\left(W'(\sigma*\hat{\omega}_\delta^2) - W'(\rho*\hat{\omega}_\delta^1)\right)(\sigma*\partial_x\hat{\omega}_\delta^2) + W'(\rho*\hat{\omega}_\delta^1)\left(\sigma*\partial_x\hat{\omega}_\delta^2 - \rho*\partial_x\hat{\omega}_\delta^1\right)\right| \\ &= \left|W''(\xi_3)(\sigma*\hat{\omega}_\delta^2 - \rho*\hat{\omega}_\delta^1)(\sigma*\partial_x\hat{\omega}_\delta^2) + W'(\rho*\hat{\omega}_\delta^1)(\sigma*\partial_x\hat{\omega}_\delta^2 - \rho*\partial_x\hat{\omega}_\delta^1)\right| \\ &= \Big|W''(\xi_3)\left((\sigma-\rho)*\hat{\omega}_\delta^2 + \rho*(\hat{\omega}_\delta^2-\hat{\omega}_\delta^1)\right)(\sigma*\partial_x\hat{\omega}_\delta^2) \\ &\qquad + W'(\rho*\hat{\omega}_\delta^1)\left((\sigma-\rho)*\partial_x\hat{\omega}_\delta^2\right) + \rho*(\partial_x\hat{\omega}_\delta^2 - \partial_x\hat{\omega}_\delta^1)\Big| \\ &\leq \|W''\|\|\sigma\|_{\mathbf{L^1}}\|\partial_x\hat{\omega}_\delta^2\|\left(\|\hat{\omega}_\delta^2\|\|\rho-\sigma\|_{\mathbf{L^1}} + \|\rho\|\|\hat{\omega}_\delta^2-\hat{\omega}_\delta^1\|_{\mathbf{L^1}}\right) \\ &\qquad + \|W'\|\left(\|\partial_x\hat{\omega}_\delta^2\|\|\rho-\sigma\|_{\mathbf{L^1}} + \|\rho\|\|\partial_x\hat{\omega}_\delta^2 - \partial_x\hat{\omega}_\delta^1\|_{\mathbf{L^1}}\right), \end{aligned} \tag{42}$$

where $\xi_3 \in \mathcal{I}(\sigma*\hat{\omega}_\delta^2, \rho*\hat{\omega}_\delta^1)$. Following an analogous process, we can bound the third term in (40), getting the following estimate,

$$\begin{aligned}|\partial_x\tilde{V}-\partial_x V| &\leq \|V''\|\|\sigma\|_{\mathbf{L^1}}\|\partial_x\omega_\eta^2\|\left(\|\omega_\eta^2\|\|\rho-\sigma\|_{\mathbf{L^1}} + \|\rho\|\|\omega_\eta^2-\omega_\eta^1\|_{\mathbf{L^1}}\right) \\ &\qquad + \|V'\|\left(\|\partial_x\omega_\eta^2\|\|\rho-\sigma\|_{\mathbf{L^1}} + \|\rho\|\|\partial_x\omega_\eta^2-\partial_x\omega_\eta^1\|_{\mathbf{L^1}}\right), \end{aligned} \tag{43}$$

and for the last term in (40) we have,

$$\begin{aligned}|\partial_x V| &= |\partial_x V(\rho*\omega_\eta^1)| \\ &= |V'(\rho*\omega_\eta^1)(\rho*\omega_\eta^1)| \\ &\leq \|V'\|\|\rho\|_{\mathbf{L^1}}\|\partial_x\omega_\eta^1\|. \end{aligned} \tag{44}$$

Now, replacing (38), (39), (41), (42), (43) and (44) in (40) and doing some straightforward computations, we get an upper bound for $|\partial_x \mathcal{U}|$ as follows,

$$\begin{aligned}
|\partial_x \mathcal{U}| \quad &\leq \quad \Bigg[\max\{\|\rho\|_{\mathbf{L^1}}, \|\sigma\|_{\mathbf{L^1}}\} \Bigg(\max\{\|\partial_x \hat{\omega}_\delta^2\|, \|\partial_x \omega_\eta^2\|\} \max\{\|\hat{\omega}_\delta^2\|, \|\omega_\eta^2\|\} (\|W'\| \|V'\| \\
&\quad + \|V\| \|W''\| + \|W\| \|V''\|) + \|V'\| \|W'\| \|\partial_x \omega_\eta^1\| \Bigg) \\
&\quad + (\|V\| \|W'\| + \|W\| \|V'\|) \max\{\|\partial_x \hat{\omega}_\delta^2\|, \|\partial_x \omega_\eta^2\|\} \Bigg] \|\rho - \sigma\|_{\mathbf{L^1}} \\
&\quad + \|\rho\| \Bigg[\Bigg(\|\sigma\|_{\mathbf{L^1}} \max\{\|\partial_x \hat{\omega}_\delta^2\|, \|\partial_x \omega_\eta^2\|\} (\|W'\| \|V'\| + \|W\| \|V''\|) + \|W\| \|V'\| \Bigg) \|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} \\
&\quad + \Bigg(\max\{\|\sigma\|_{\mathbf{L^1}} \|\partial_x \hat{\omega}_\delta^2\|, \|\rho\|_{\mathbf{L^1}} \|\partial_x \omega_\eta^1\|\} (\|V\| \|W''\| + \|V'\| \|W'\|) \\
&\quad + \|V\| \|W'\| \Bigg) \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}} \Bigg] \\
&= \quad \mathcal{C}_3 \|\rho - \sigma\|_{\mathbf{L^1}} + \mathcal{C}_4 \|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \mathcal{C}_5 \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}}, \qquad (45)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C}_3 \quad &= \quad \max\{\|\rho\|_{\mathbf{L^1}}, \|\sigma\|_{\mathbf{L^1}}\} \Bigg(\max\{\|\partial_x \hat{\omega}_\delta^2\|, \|\partial_x \omega_\eta^2\|\} \max\{\|\hat{\omega}_\delta^2\|, \|\omega_\eta^2\|\} (\|W'\| \|V'\| \\
&\quad + \|V\| \|W''\| + \|W\| \|V''\|) + \|V'\| \|W'\| \|\partial_x \omega_\eta^1\| \Bigg) \\
&\quad + (\|V\| \|W'\| + \|W\| \|V'\|) \max\{\|\partial_x \hat{\omega}_\delta^2\|, \|\partial_x \omega_\eta^2\|\}, \\
\mathcal{C}_4 \quad &= \quad \|\rho\| \Bigg(\|\sigma\|_{\mathbf{L^1}} \max\{\|\partial_x \hat{\omega}_\delta^2\|, \|\partial_x \omega_\eta^2\|\} (\|W'\| \|V'\| + \|W\| \|V''\|) + \|W\| \|V'\| \Bigg) \\
\mathcal{C}_5 \quad &= \quad \|\rho\| \Bigg(\max\{\|\sigma\|_{\mathbf{L^1}} \|\partial_x \hat{\omega}_\delta^2\|, \|\rho\|_{\mathbf{L^1}} \|\partial_x \omega_\eta^1\|\} (\|V\| \|W''\| + \|V'\| \|W'\|) + \|V\| \|W'\| \Bigg).
\end{aligned}$$

Finally,

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \mathcal{S}_2(t,x)\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}} |\partial_x \mathcal{U}(t,x)||f(\rho(t,x))|\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^T \int_{\mathbb{R}} \Big(\mathcal{C}_3 \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}} + \mathcal{C}_4 \|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \mathcal{C}_5 \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}} \Big) |f(\rho(t,x))|\mathrm{d}x\,\mathrm{d}t \\
&\leq \mathcal{C}_3 \int_{\mathbb{R}} |f(\rho(t,x))|\mathrm{d}x \int_0^T \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}}\mathrm{d}t + T\mathcal{C}_4 \int_{\mathbb{R}} |f(\rho(t,x))| \|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}}\,\mathrm{d}x \\
&\qquad + T\mathcal{C}_5 \int_{\mathbb{R}} |f(\rho(t,x))| \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}}\,\mathrm{d}x \\
&\leq \sup_{t\in[0,T]} \|f(\rho(t,\cdot))\|_{\mathbf{L^1}} \Big(\mathcal{C}_3 \int_0^T \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}}\,\mathrm{d}t \\
&\qquad + T\mathcal{C}_4 \|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + T\mathcal{C}_5 \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}} \Big) \\
&= \mathcal{C}_8 \int_0^T \|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L^1}}\mathrm{d}t + \mathcal{C}_6(T) \|\omega_\eta^2 - \omega_\eta^1\|_{\mathbf{W^{1,1}}} + \mathcal{C}_7(T) \|\hat{\omega}_\delta^2 - \hat{\omega}_\delta^1\|_{\mathbf{W^{1,1}}},
\end{aligned}
$$

where we denote $\mathcal{C}_6(T) := T\mathcal{C}_4 \sup_{t\in[0,T]} \|f(\rho(t,\cdot))\|_{\mathbf{L^1}}$, $\mathcal{C}_7(T) := T\mathcal{C}_5 \sup_{t\in[0,T]} \|f(\rho(t,\cdot))\|_{\mathbf{L^1}}$ and $\mathcal{C}_8 := \mathcal{C}_3 \sup_{t\in[0,T]} \|f(\rho(t,\cdot))\|_{\mathbf{L^1}}$.

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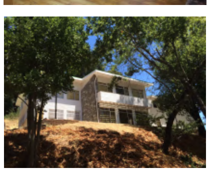

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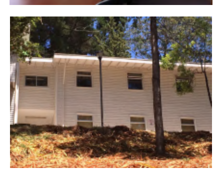

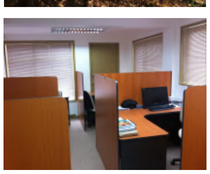

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