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Numerical Analysis of a three-field formulation for a reverse osmosis model

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Abstract

In this work, we study a coupled Navier–Stokes/convectiondiffusion model arising in the description of reverse osmosis membrane processes. The model accounts for the incompressible viscous flow of a saline solution and the transport of solute concentration, including appropriate interface conditions that capture the membrane permeability and rejection effects. We propose and analyze a variational formulation of the stationary problem, establishing existence, uniqueness, and positivity of solutions under suitable assumptions on the data. To approximate the problem numerically, we introduce a conforming finite element discretization and prove discrete stability estimates, as well as the well-posedness of the resulting nonlinear system. Furthermore, we develop a fixed-point strategy to handle the nonlinear coupling and provide error estimates for the discrete scheme. Numerical experiments are presented to confirm the theoretical results and to illustrate the performance of the method in relevant test cases for desalination applications.

Key words: reverse osmosis; desalination; Navier–Stokes equations; convection–diffusion equation; finite element method; variational formulation; well-posedness; positivity of concentration; numerical analysis; error estimates

Mathematics Subject Classifications (1991): 65N15, 65N30, 74F10, 74S05

1 Introduction

Reverse osmosis (RO) is one of the most widely used technologies for seawater desalination and freshwater production, owing to its efficiency, modularity, and ability to deliver high-quality permeate. The operation of RO membranes involves complex coupled phenomena, including incompressible fluid flow through narrow channels, solute transport driven by advection and diffusion, and nonlinear permeation across semipermeable boundaries. The accurate numerical modeling of such processes is crucial not only for understanding the fundamental mechanisms governing membrane performance, but also for guiding the optimization of RO modules and the design of next-generation desalination systems.

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Recent years have seen a growing interest in the numerical modeling of reverse osmosis (RO) processes, which are currently the dominant technology in large-scale desalination plants worldwide, accounting for nearly 70% of the installed capacity due to their lower energy consumption compared with thermal methods such as multi-stage flash [18, 13]. Most mathematical models are based on the Navier–Stokes and convection–diffusion equations although in some situations Brinkman-type equations have also been employed. From the numerical point of view, the majority of studies have focused on single-channel simulations, mainly because of the high computational cost of treating multiple channels. Classical approaches have relied on finite differences or finite volumes, whereas finite element simulations have often been carried out with commercial packages without a rigorous mathematical framework.

A significant advance was achieved in [6], where the authors proposed a coupled Navier–Stokes/transport model with nonlinear interface conditions, formulated a mixed variational problem with Lagrange multipliers to capture concentration traces, and proved well-posedness using Banachs fixed-point theorem together with the Banach–Nečas–Babuška theory. On the discrete level, they developed a stable mixed finite element scheme based on Raviart–Thomas spaces for auxiliary flux variables and piecewise polynomial approximations for velocity, pressure, and concentration. They also derived optimal *a priori* error estimates, which were confirmed by numerical experiments, demonstrating the robustness and accuracy of the method in realistic RO configurations.

In parallel, more recent work has focused on incorporating osmotic effects into finite element models. In [8], the authors developed a finite element formulation to investigate concentration polarization and osmotic phenomena in RO membranes. Their model couples the incompressible Navier–Stokes equations with an advection–diffusion equation for the salt concentration and integrates nonlinear interface conditions to represent the membrane transport. This approach captures both hydrodynamic and osmotic contributions to the transmembrane flux, allowing for detailed analysis of concentration polarization under realistic operating conditions. The numerical experiments reported in [8] illustrate how membrane selectivity and operational parameters affect solute buildup and flux decline, highlighting the importance of rigorous mathematical modeling in predicting the performance of RO modules.

Mathematical modeling of RO processes typically leads to coupled systems of partial differential equations (PDEs) of Navier–Stokes type for the hydrodynamics, coupled with convection–diffusion equations for the solute concentration. These equations are further linked by nonlinear boundary conditions at the membrane interfaces, which reflect the selective transport of water and salt. The resulting models pose significant analytical and computational challenges due to their nonlinearity, the interaction of boundary conditions with the interior flow, and the need to preserve key physical properties such as mass conservation and non-negativity of the solute concentration.

In this work we study a three-field formulation of the RO model studied in [6, 8], where the unknowns are the velocity, pressure, and solute concentration. At the continuous level, we derive the weak formulation of the coupled NavierStokesconvectiondiffusion system and establish its well-posedness by means of fixed-point arguments. A key feature of our analysis is the proof of positivity of the concentration under suitable conditions on the data, ensuring the physical consistency of the model. Furthermore, we introduce an equivalent reduced formulation that allows us to simplify the study of existence and uniqueness of solutions.

Building upon the continuous theory, we then propose and analyze a conforming finite element method for the numerical approximation of the model. The scheme employs standard inf-sup stable velocitypressure elements combined with continuous piecewise linear approximations for the concentration. We prove that the discrete problem inherits the stability properties of the continuous one, and we derive error estimates that guarantee optimal rates of convergence. Finally, we present a series of numerical experiments that confirm the theoretical findings and illustrate the performance of the proposed method in realistic test configurations.

1.1 Preliminaries

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with polyhedral boundary Γ. Throughout the paper we use standard notation for Sobolev spaces $\mathrm{W}^{m,p}(\Omega)$, where $p \in [1, \infty]$ and $m \geq 0$, and for the Lebesgue spaces $\mathrm{L}^p(\Omega) := \mathrm{W}^{0,p}(\Omega)$, equipped with the norms $\|\cdot\|_{\mathrm{W}^{m,p}(\Omega)}$ and $\|\cdot\|_{\mathrm{L}^p(\Omega)}$, respectively.

When $p = 2$, we write $\mathrm{H}^m(\Omega)$ instead of $\mathrm{W}^{m,2}(\Omega)$, and for simplicity, for $\mathrm{H}^m(\Omega)$, $\mathbf{H}^m(\Omega) := [\mathrm{H}^m(\Omega)]^2$, and $\mathbb{H}^m(\Omega) := [\mathrm{H}^m(\Omega)]^{2\times 2}$, the corresponding norms and seminorms will be denoted by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$.

For any vector fields $\mathbf{v} = (v_1, v_2)^t$ and $\mathbf{w} = (w_1, w_2)^t$, we define the gradient, divergence, and tensor product operators as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,2}, \qquad \operatorname{div} \mathbf{v} := \sum_{j=1}^{2} \frac{\partial v_j}{\partial x_j}, \qquad \mathbf{v} \otimes \mathbf{w} := \big(v_i w_j\big)_{i,j=1,2}.$$

Similarly, for any tensor fields $\mathbf{S} = (S_{ij})_{i,j=1,2}$ and $\mathbf{R} = (R_{ij})_{i,j=1,2}$, their tensor inner product is given by

$$\mathbf{S} : \mathbf{R} := \sum_{i,j=1}^{2} S_{ij} R_{ij},$$

where the superscript ^{t} denotes transposition.

Next, we let $\gamma_0 : \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Gamma)$ be the well-known trace operator, satisfying

$$\|\varphi\|_{0,\Gamma} \leq C_\Gamma \|\varphi\|_{1,\Omega} \quad \forall \varphi \in \mathrm{H}^1(\Omega), \tag{1.1}$$

with $C_\Gamma > 0$ and define the trace space of $\mathrm{H}^1(\Omega)$ as

$$\mathrm{H}^{1/2}(\Gamma) := \gamma_0(\mathrm{H}^1(\Omega)), \tag{1.2}$$

endowed with the norm

$$\|\psi\|_{1/2,\Gamma} := \inf \left\{ \|w\|_{1,\Omega} : w \in \mathrm{H}^1(\Omega),\ \gamma_0(w) = \psi \right\}. \tag{1.3}$$

Alternatively, as stated in [20, Remark 4.2.3], an equivalent norm for the space $\mathrm{H}^{1/2}(\Gamma)$ is given by

$$|||u|||_{1/2,\Gamma} = \left(\int_\Gamma |u|^2 + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x - y|^2} \right)^{1/2}. \tag{1.4}$$

In the sequel, the dual space of $\mathrm{H}^{1/2}(\Gamma)$ is denoted by $\mathrm{H}^{-1/2}(\Gamma)$ and we employ $\langle \cdot, \cdot \rangle_\Gamma$ to denote the duality pairing on $\mathrm{H}^{-1/2}(\Gamma) \times \mathrm{H}^{1/2}(\Gamma)$, which coincides with the $\mathrm{L}^2(\Gamma)$-inner product when applied to functions in $\mathrm{L}^2(\Gamma)$. The vector-valued counterparts are denoted by $\mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma)$, corresponding to $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$, respectively.

It is well known that

$$\mathrm{H}^1(\Omega) = \mathrm{H}^1_0(\Omega) \oplus \left[\mathrm{H}^1_0(\Omega)\right]^{\perp},$$

where $\mathrm{H}^1_0(\Omega)$ is the kernel of γ_0, and $\left[\mathrm{H}^1_0(\Omega)\right]^{\perp}$ denotes its orthogonal complement (see, e.g., [17, Theorem 1.3-1]). Then, defining the linear, bounded, bijective operator $\tilde{\gamma}_0 := \gamma_0\big|_{\left[\mathrm{H}^1_0(\Omega)\right]^{\perp}}$, it follows that

$$\|\tilde{\gamma}_0^{-1}(\psi)\|_{1,\Omega} = \|\psi\|_{1/2,\Gamma} \quad \forall \psi \in \mathrm{H}^{1/2}(\Gamma). \tag{1.5}$$

In what follows, we use a vector-valued version of $\tilde{\gamma}_0$, denoted by $\tilde{\boldsymbol{\gamma}}_0$, which is defined componentwise by $\tilde{\gamma}_0$. Furthermore, by the Sobolev embedding $\mathrm{H}^{1/2}(\Gamma) \hookrightarrow \mathrm{L}^4(\Gamma)$, we have

$$\|\psi\|_{\mathrm{L}^4(\Gamma)} \leq \widehat{C}_\Gamma \|\psi\|_{1/2,\Gamma} \quad \forall \psi \in \mathrm{H}^{1/2}(\Gamma), \tag{1.6}$$

with $\widehat{C}_\Gamma > 0$.

On the other hand, for $0 < \gamma \leq 1$ and following [15, 20], we define the Hölder boundary space on Γ by

$$C^{0,\gamma}(\Gamma) := \left\{ g \in C^0(\Gamma) : |g(x) - g(y)| \leq M|x-y|^\gamma,\, M > 0 \right\},$$

endowed with the norm

$$\|u\|_{C^{0,\gamma}(\Gamma)} := \left(\|u\|^2_{C^0(\Gamma)} + |u|^2_{C^{0,\gamma}(\Gamma)} \right)^{1/2}, \tag{1.7}$$

where

$$\|u\|_{C^0(\Gamma)} := \sup_{x \in \Gamma} |u(x)| \quad \text{and} \quad |g|_{C^{0,\gamma}(\Gamma)} := \sup_{\substack{x,y \in \Gamma \\ x \neq y}} \frac{|g(x) - g(y)|}{|x-y|^\gamma}.$$

The following technical result will be employed in the forthcoming analysis.

Lemma 1.1 *Let Ω be a Lipschitz domain with boundary $\Gamma = \partial\Omega$ and let $f \in \mathrm{H}^{1/2}(\Gamma)$, $g \in C^{0,\gamma}(\Gamma)$ with $0 < \gamma \leq 1$. Then $fg \in \mathrm{H}^{1/2}(\Gamma)$, and there exists $c_1 > 0$, such that*

$$\|fg\|_{1/2,\Gamma} \leq c_1 \|g\|_{C^{0,\gamma}(\Gamma)} \|f\|_{1/2,\Gamma}. \tag{1.8}$$

Proof. Given $f \in \mathrm{H}^{1/2}(\Gamma)$, $g \in C^{0,\gamma}(\Gamma)$ with $0 < \gamma \leq 1$, we start by noticing that from the definition of the norms $|||\cdot|||_{1/2,\Gamma}$ and $\|\cdot\|_{C^{0,\gamma}(\Gamma)}$ (see (1.4) and (1.7)), it is clear that

$$\int_\Gamma |f|^2 |g|^2 \leq \|g\|^2_{C^0(\Gamma)} \, |||f|||^2_{1/2,\Gamma} \leq \|g\|^2_{C^{0,\gamma}(\Gamma)} \, |||f|||^2_{1/2,\Gamma}. \tag{1.9}$$

In turn, since $f(x)g(x) - f(y)g(y) = (f(x) - f(y))g(x) + (g(x) - g(y))f(y)$ for all $x, y \in \Gamma$, employing $ab \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$, we easily obtain:

$$\int_\Gamma \int_\Gamma \frac{|f(x)g(x) - f(y)g(y)|^2}{|x-y|^2} \leq 2\|g\|^2_{C^0(\Gamma)} \, |||f|||^2_{1/2,\Gamma} + 2|g|^2_{C^{0,\gamma}(\Gamma)} |\Gamma| \, |||f|||^2_{1/2,\Gamma}. \tag{1.10}$$

From the inequalities (1.9) and (1.10), we readily obtain:

$$|||fg|||^2_{1/2,\Gamma} \leq c \|g\|_{C^{0,\gamma}(\Gamma)} \, |||f|||^2_{1/2,\Gamma} < +\infty, \tag{1.11}$$

with $c > 0$, which implies that $fg \in \mathrm{H}^{1/2}(\Gamma)$. In addition, since $\|\cdot\|_{1/2,\Gamma}$ and $|||\cdot|||_{1/2,\Gamma}$ are equivalent, (1.8) follows straightforwardly from (1.11). □

We now recall some definitions and technical results on extension operators (see, e.g., [12, 16]). To that end, we let $\tilde{\Gamma} \subseteq \Gamma$ be a proper subset of Γ (i.e., $\tilde{\Gamma} \neq \Gamma$), denote by $\tilde{\Gamma}^c$ its complement in Γ and let

$$E_{0,\tilde{\Gamma}} : \mathrm{H}^{1/2}(\tilde{\Gamma}) \to \mathrm{L}^2(\Gamma)$$

be the extension operator given by

$$E_{0,\tilde{\Gamma}}(\xi) := \begin{cases} \xi & \text{on } \tilde{\Gamma}, \\ 0 & \text{on } \tilde{\Gamma}^c, \end{cases} \qquad \forall\, \xi \in \mathrm{H}^{1/2}(\tilde{\Gamma}).$$

We then introduce the space

$$\mathrm{H}^{1/2}_{00}(\tilde{\Gamma}) := \big\{ \xi \in \mathrm{H}^{1/2}(\tilde{\Gamma}) : E_{0,\tilde{\Gamma}}(\xi) \in \mathrm{H}^{1/2}(\Gamma) \big\},$$

endowed with the norm

$$\|\xi\|_{1/2,00,\tilde{\Gamma}} := \big\| E_{0,\tilde{\Gamma}}(\xi) \big\|_{1/2,\Gamma}, \tag{1.12}$$

where $\|\cdot\|_{1/2,\Gamma}$ is the norm defined in (1.3).

We also define the extension operator

$$E_{\tilde{\Gamma}} : \mathrm{H}^{1/2}(\tilde{\Gamma}) \longrightarrow \mathrm{H}^{1/2}(\Gamma)\,, \quad \xi \longmapsto E_{\tilde{\Gamma}}(\xi) := z|_{\Gamma},$$

where $z \in \mathrm{H}^1(\Omega)$ is the unique solution of the boundary value problem

$$-\Delta z = 0 \text{ in } \Omega, \qquad z = \xi \text{ on } \tilde{\Gamma}, \qquad \nabla z \cdot \mathbf{n} = 0 \text{ on } \tilde{\Gamma}^c.$$

This operator satisfies

$$\|E_{\tilde{\Gamma}}(\xi)\|_{1/2,\Gamma} \leq C\, \|\xi\|_{1/2,\tilde{\Gamma}}, \tag{1.13}$$

where $C > 0$ is a constant independent of ξ.

In addition, we denote let

$$\mathrm{H}^1_{\tilde{\Gamma}}(\Omega) := \big\{ v \in \mathrm{H}^1(\Omega) : v|_{\tilde{\Gamma}} = 0 \big\} \quad \text{and} \quad \mathbf{H}^1_{\tilde{\Gamma}}(\Omega) := \big[\mathrm{H}^1_{\tilde{\Gamma}}(\Omega)\big]^2.$$

Throughout the paper, the norm $\|\cdot\|$ without subscripts denotes the natural norm of an element or operator in the corresponding product functional space. Moreover, the notations $a \lesssim b$ and $a \cong b$ means $a \leq C\, b$ and $a = C\, b$, respectively, for a constant $C > 0$ independent of the mesh size and physical parameters. The symbol $\mathbf{0}$ stands for a generic null vector, and, when no ambiguity arises, $|\cdot|$ denotes the absolute value in $\mathbb{R}$ and the Euclidean norm in $\mathbb{R}^2$ or $\mathbb{R}^{2\times 2}$. Finally, for any scalar function ϕ, we define its positive and negative parts by

$$\phi^+ := \tfrac{1}{2}\big(\phi + |\phi|\big), \qquad \phi^- := \tfrac{1}{2}\big(\phi - |\phi|\big), \tag{1.14}$$

so that

$$\phi = \phi^+ + \phi^-. \tag{1.15}$$

1.2 Model problem

To introduce the model problem, we consider the rectangular domain $\Omega := (0,a) \times (0,b) \subseteq \mathbb{R}^2$, where $a, b > 0$ are fixed constants, and denote by Γ its boundary. We partition Γ into the disjoint subsets Γ_{in}, Γ_{out}, Γ_{m^+}, $\Gamma_{m^-} \subseteq \Gamma$, such that

$$\Gamma = \overline{\Gamma}_{\text{in}} \cup \overline{\Gamma}_{m^+} \cup \overline{\Gamma}_{m^-} \cup \overline{\Gamma}_{\text{out}},$$

with

$$\Gamma_i \cap \Gamma_j = \emptyset \qquad \text{for all } i \neq j \text{ in } \{\text{in}, \text{out}, m^+, m^-\}.$$

In addition, we assume

$$\overline{\Gamma}_{m^+} \cap \overline{\Gamma}_{m^-} = \emptyset \quad \text{and} \quad \overline{\Gamma}_{\text{in}} \cap \overline{\Gamma}_{\text{out}} = \emptyset,$$

and we set $\Gamma_{\text{m}} := \Gamma_{m^+} \cup \Gamma_{m^-}$. Along the boundary Γ, we denote by $\mathbf{n} := (n_1, n_2)^t$ the unit outer normal vector field and by $\mathbf{t} := (-n_2, n_1)^t$ the associated counterclockwise tangential vector field, as illustrated in Figure 1.1.

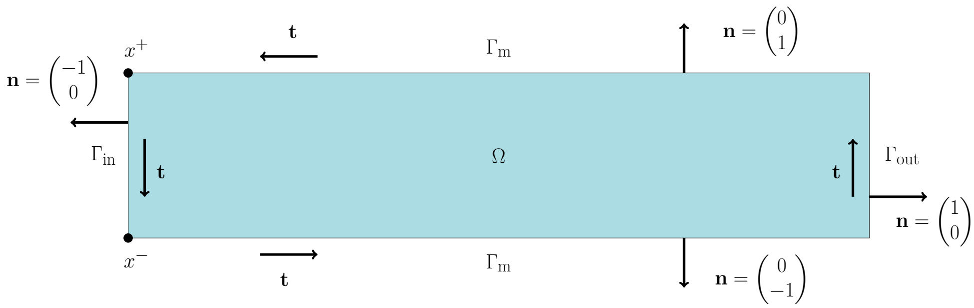

Figure 1.1: Sketch of the computational domain.

In this work we are concerned with approximating the solution of the stationary Navier–Stokes system

$$\begin{aligned} -\mathbf{div}\big(2\nu\nabla\mathbf{u} - p\mathrm{I}\big) + \rho\,\mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \Omega, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega,\\ \mathbf{u} = \mathbf{u}_{\text{in}} \quad \text{on} \quad \Gamma_{\text{in}}, \quad \mathbf{u}\cdot\mathbf{n} = A\big(\Delta P - iRT\varphi\big), \quad \mathbf{u}\cdot\mathbf{t} = 0 \quad \text{on} \quad \Gamma_{\text{m}},\\ \big(2\nu\nabla\mathbf{u} - p\mathrm{I}\big)\mathbf{n} = \frac{\rho}{2}(\mathbf{u}\cdot\mathbf{n})^{-}\mathbf{u} \quad \text{on} \quad \Gamma_{\text{out}}, \end{aligned} \tag{1.16}$$

coupled with the convection-diffusion equation

$$\begin{aligned} -\kappa\Delta\varphi + \mathbf{u}\cdot\nabla\varphi = 0 \quad \text{in} \quad \Omega, \qquad \varphi = \varphi_{\text{in}} \quad \text{on} \quad \Gamma_{\text{in}},\\ \big(\varphi\mathbf{u} - \kappa\nabla\varphi\big)\cdot\mathbf{n} = B\varphi \quad \text{on} \quad \Gamma_{\text{m}}, \quad \varphi \geq 0 \quad \text{on} \quad \Gamma_{\text{m}},\\ \kappa\nabla\varphi\cdot\mathbf{n} = \frac{1}{2}(\mathbf{u}\cdot\mathbf{n})^{-}\varphi \quad \text{on} \quad \Gamma_{\text{out}}, \end{aligned} \tag{1.17}$$

where I denotes the identity matrix in $\mathbb{R}^{2\times 2}$.

In the formulation above, the unknowns are the velocity $\mathbf{u}$, the pressure p, and the salt concentration φ of the fluid occupying the domain Ω. The given data are the fluid viscosity $\nu > 0$, the permeate diffusivity of salt in water $\kappa > 0$, the inlet velocity $\mathbf{u}_{\text{in}} = (u_{1,\text{in}}, u_{2,\text{in}})^t \in \mathbf{H}^{1/2}(\Gamma_{\text{in}})$, and the inlet salt concentration $\varphi_{\text{in}} \in \mathbb{R}^+$. The physical parameters A, i, R, T, ΔP, and B are assumed to be positive constants; their specific values are reported in Table 5.2. For convenience, we introduce the shorthand

$$a_1 := A\Delta P, \qquad a_2 := AiRT, \qquad a_3 := B, \qquad \mathbf{a} := (a_1, a_2, a_3).$$

We also set

$$\{x^-\} := \overline{\Gamma}_{\text{in}} \cap \overline{\Gamma}_{m^-}, \qquad \{x^+\} := \overline{\Gamma}_{\text{in}} \cap \overline{\Gamma}_{m^+}.$$

In what follows we assume that the inlet velocity $\mathbf{u}_{\text{in}} \in \mathbf{H}^{1/2}(\Gamma_{\text{in}})$ satisfies the compatibility conditions

$$\lim_{x\to x^{\pm}} \mathbf{u}_{\text{in}}(x)\cdot\mathbf{t} = \pm\big(a_1 - a_2\varphi_{\text{in}}\big), \qquad x \in \Gamma_{\text{in}}, \tag{1.18}$$

and

$$u_{1,\text{in}} \in \mathrm{H}^{1/2}_{00}(\Gamma_{\text{in}}). \tag{1.19}$$

In particular, if

$$u_{2,\text{in}}(x,y) = \frac{a_1 - a_2\varphi_{\text{in}}}{b}\,(2y-b), \qquad \forall (x,y) \in \Gamma_{\text{in}},$$

then $\mathbf{u}_{\text{in}}$ clearly satisfies (1.18) for any $u_{1,\text{in}} \in \mathrm{H}^{1/2}_{00}(\Gamma_{\text{in}})$. The reasoning behind the conditions (1.18) and (1.19) will become clear in the next section, where we introduce the associated variational formulation.

2 Continuous weak formulation and its well-posedness

Based on the model equations introduced above, we now derive the continuous weak formulation of the problem and analyze its well-posedness.

2.1 Weak formulation

To derive the variational formulation associated with the coupled system (1.16)-(1.17), we begin by multiplying the first equation of (1.16) by a test function $\mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\text{out}}}(\Omega)$, where $\Gamma^c_{\text{out}} := \Gamma_{\text{in}} \cup \Gamma_{\text{m}}$, integrating by parts, and using the last boundary condition in (1.16) to obtain

$$2\nu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} - \frac{\rho}{2}\int_{\Gamma_{\text{out}}} (\mathbf{u}\cdot\mathbf{n})^-\mathbf{u}\cdot\mathbf{v} + \rho\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \int_\Omega p\,\mathrm{div}\,\mathbf{v} = 0 \qquad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\text{out}}}(\Omega). \tag{2.1}$$

In turn, the incompressibility constraint, given by the second equation of (1.16), is imposed weakly as

$$\int_\Omega q\,\mathrm{div}\,\mathbf{u} = 0 \qquad \forall q \in \mathrm{L}^2(\Omega). \tag{2.2}$$

Finally, multiplying the first equation of (1.17) by a test function $\psi \in \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega)$, integrating by parts, and using the boundary conditions on Γ_{m} and Γ_{out} in (1.17), we arrive at

$$\int_\Omega \kappa\nabla\varphi\cdot\nabla\psi + a_3\int_{\Gamma_{\text{m}}} \varphi\psi + \int_\Omega (\mathbf{u}\cdot\nabla\varphi)\psi - \int_{\Gamma_{\text{m}}} \varphi\psi(\mathbf{u}\cdot\mathbf{n}) - \frac{1}{2}\int_{\Gamma_{\text{out}}} (\mathbf{u}\cdot\mathbf{n})^-\varphi\psi = 0 \qquad \forall \psi \in \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega). \tag{2.3}$$

According to the above derivations, the variational formulation of the coupled system (1.16)(1.17) reads: Find $(\mathbf{u}, p, \varphi) \in \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1(\Omega)$ such that

$$\mathbf{u} = \tilde{\mathbf{u}}_{\text{in}}(\varphi) \quad \text{on } \Gamma^c_{\text{out}}, \qquad \varphi = \varphi_{\text{in}} \quad \text{on } \Gamma_{\text{in}},$$

and

$$
\begin{aligned}
a_F(\mathbf{u},\mathbf{v}) + O_F(\mathbf{u};\mathbf{u},\mathbf{v}) + b_F(\mathbf{v},p) &= 0 \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\mathrm{out}}}(\Omega),\\
b_F(\mathbf{u},q) &= 0 \quad \forall\, q \in \mathrm{L}^2(\Omega),\\
a_C(\varphi,\psi) + O_C(\mathbf{u},\varphi,\psi) &= 0 \quad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega),
\end{aligned} \tag{2.4}
$$

where a_F, b_F and a_C are given by

$$
a_F(\mathbf{u},\mathbf{v}) := 2\nu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}, \quad b_F(\mathbf{v},q) := -\int_\Omega q \,\mathrm{div}\,\mathbf{v}, \quad a_C(\varphi,\psi) := \int_\Omega \kappa\nabla\varphi\cdot\nabla\psi + a_3 \int_{\Gamma_{\mathrm{m}}} \varphi\psi,
$$

whereas the nonlinear forms are defined by

$$
\begin{aligned}
O_F(\mathbf{w};\mathbf{u},\mathbf{v}) &:= \rho \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \frac{\rho}{2}\int_{\Gamma_{\mathrm{out}}} (\mathbf{w}\cdot\mathbf{n})^-\mathbf{u}\cdot\mathbf{v},\\
O_C(\mathbf{w},\varphi,\psi) &:= \int_\Omega (\mathbf{w}\cdot\nabla\varphi)\psi - \int_{\Gamma_{\mathrm{m}}} \varphi\psi(\mathbf{w}\cdot\mathbf{n}) - \frac{1}{2}\int_{\Gamma_{\mathrm{out}}} (\mathbf{w}\cdot\mathbf{n})^-\varphi\psi.
\end{aligned}
$$

Above, for any $\phi \in \mathrm{H}^1(\Omega)$ satisfying $\phi|_{\Gamma_{\mathrm{in}}} = \varphi_{\mathrm{in}}$, the concentration-dependent boundary function $\tilde{\mathbf{u}}_{\mathrm{in}}(\phi)$ is defined by

$$
\tilde{\mathbf{u}}_{\mathrm{in}}(\phi) := \begin{cases} \mathbf{u}_{\mathrm{in}} & \text{on } \Gamma_{\mathrm{in}},\\ \big(a_1 - a_2\phi^+\big)\mathbf{n} & \text{on } \Gamma_{\mathrm{m}}. \end{cases} \tag{2.5}
$$

Notice that, thanks to (1.18), (1.19), and [19, Theorem 1.5.2.3], we have $\tilde{\mathbf{u}}_{\mathrm{in}}(\phi) \in \mathbf{H}^{1/2}(\Gamma^c_{\mathrm{out}})$, which justifies the introduction of conditions (1.18) and (1.19).

2.2 Well-posedness

We now turn to the analysis of the variational formulation (2.4). Our first step is to establish the stability properties of the forms involved.

2.2.1 Stability Properties

We begin by observing that, after simple computations, it can be proved that the bilinear forms a_F, a_C and b are bounded:

$$
\begin{aligned}
|a_F(\mathbf{u},\mathbf{v})| &\lesssim \nu\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega},\ \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\mathrm{out}}}(\Omega),\\
|a_C(\varphi,\psi)| &\lesssim (\kappa + a_3)\|\varphi\|_{1,\Omega}\|\psi\|_{1,\Omega},\ \forall \varphi,\psi \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega),\\
|b_F(\mathbf{v},q)| &\lesssim \|q\|_{0,\Omega}\|\mathbf{v}\|_{1,\Omega}\ \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\mathrm{out}}}(\Omega), \forall q \in \mathrm{L}^2(\Omega),
\end{aligned} \tag{2.6}
$$

Recalling the continuous Sobolev embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$ holds (see [25, Theorem 1.3.4]), we obtain

$$
\|\mathbf{v}\|_{4,\Omega} \lesssim \|\mathbf{v}\|_{1,\Omega} \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.7}
$$

which together with (1.6), imply

$$
|O_F(\mathbf{w};\mathbf{u},\mathbf{v})| \lesssim \rho\|\mathbf{w}\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}, \quad |O_C(\mathbf{w},\varphi,\psi)| \lesssim \|\mathbf{w}\|_{1,\Omega}\|\varphi\|_{1,\Omega}\|\psi\|_{1,\Omega}. \tag{2.8}
$$

In addition, from the well-known Poincaré inequality, we deduce that

$$a_F(\mathbf{v},\mathbf{v}) \gtrsim \nu\|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\mathrm{out}}}(\Omega), \tag{2.9}$$

$$a_C(\psi,\psi) \gtrsim \kappa\|\psi\|_{1,\Omega}^2 + a_3\|\psi\|_{0,\Gamma_{\mathrm{m}}}^2 \gtrsim \kappa\|\psi\|_{1,\Omega}^2 \quad \forall \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega). \tag{2.10}$$

Now, we proceed to establish the corresponding inf-sup condition for b_F, which is crucial for ensuring the well-posedness and stability of problem (2.4).

Lemma 2.1 *There exists $\beta > 0$, such that*

$$\sup_{\substack{\mathbf{v}\in\mathbf{H}^1_{\Gamma^c_{\mathrm{out}}}(\Omega)\\ \mathbf{v}\neq\theta}} \frac{|b_F(\mathbf{v},q)|}{\|\mathbf{v}\|_{1,\Omega}} \gtrsim \|q\|_{0,\Omega}, \qquad \forall q \in \mathrm{L}^2(\Omega). \tag{2.11}$$

Proof. The result follows from the surjectivity of the operator div : $\mathbf{H}^1_0(\Omega) \to \mathrm{L}^2_0(\Omega) := \{q \in \mathrm{L}^2(\Omega) : \int_\Omega q = 0\}$, the orthogonal decomposition $\mathrm{L}^2(\Omega) = \mathrm{L}^2_0(\Omega) \oplus \mathbb{R}$ and the fact that $\mathbf{H}^1_0(\Omega) \subseteq \mathbf{H}^1_{\Gamma^c_{\mathrm{out}}}(\Omega)$. We omit further details. □

Finally, by applying integration by parts, we obtain the following identities, which will be used later:

$$\begin{aligned} O_F(\mathbf{w};\mathbf{v},\mathbf{v}) &= \frac{\rho}{2}\left(-\int_\Omega |\mathbf{v}|^2 \mathrm{div}\,(\mathbf{w}) + \int_\Gamma (\mathbf{w}\cdot\mathbf{n})|\mathbf{v}|^2\right) - \frac{\rho}{2}\int_{\Gamma_{\mathrm{out}}} (\mathbf{w}\cdot\mathbf{n})^-|\mathbf{v}|^2, \\ O_C(\mathbf{w};\psi,\psi) &= -\frac{1}{2}\int_\Omega \psi^2 \mathrm{div}\,(\mathbf{w}) + \frac{1}{2}\int_\Gamma \psi^2(\mathbf{w}\cdot\mathbf{n}) - \int_{\Gamma_{\mathrm{m}}} (\mathbf{w}\cdot\mathbf{n})\psi^2 - \frac{1}{2}\int_{\Gamma_{\mathrm{out}}} (\mathbf{w}\cdot\mathbf{n})^-\psi^2, \end{aligned} \tag{2.12}$$

for all $\mathbf{w},\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\psi \in \mathrm{H}^1(\Omega)$.

2.2.2 Positivity of the concentration

Before proving the well-posedness of (2.4) we first establish that if $(\mathbf{u},p,\varphi)$ is a solution of (2.4), then the concentration φ is positive in Ω under suitable assumptions on the data. This result can be deduced from the following lemma.

Lemma 2.2 *Let $\varphi_{\mathrm{in}}, a_1, a_3 \in \mathbb{R}^+$, such that $a_3 - \frac{1}{2}a_1 \geq 0$. Then, for any $(\mathbf{w},\phi) \in \mathbf{H}^1(\Omega)\times\mathrm{H}^1(\Omega)$ such that $\phi|_{\Gamma_{\mathrm{in}}} = \varphi_{\mathrm{in}}$, $\phi \geq 0$ on Γ_{m}, $\mathbf{w} = \tilde{\mathbf{u}}_{\mathrm{in}}(\phi)$ on Γ^c_{out}, and $\mathrm{div}\,\mathbf{w} = 0$, the solution $\varphi \in \mathrm{H}^1(\Omega)$ to the problem: Find $\varphi \in \mathrm{H}^1(\Omega)$ such that $\varphi = \varphi_{\mathrm{in}}$ on Γ_{in} and*

$$a_C(\varphi,\psi) + O_C(\mathbf{w},\varphi,\psi) = 0 \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega), \tag{2.13}$$

satisfies

$$\varphi \geq 0 \ \text{a.e. in } \Omega, \ \text{and } \varphi \geq 0 \quad \text{a.e. on } \Gamma_{\mathrm{m}}. \tag{2.14}$$

Proof. We start by choosing any sequence $G_n \in (0,\varphi_{\mathrm{in}})$ such that $\lim_{n\to\infty} G_n = 0$ (for instance, $G_n = \frac{\varphi_{\mathrm{in}}}{n}$). Clearly $\varphi_{\mathrm{in}} - G_n \geq 0$, $\forall\, n \in \mathbb{N}$, and since $\varphi|_{\Gamma_{\mathrm{in}}} = \varphi_{\mathrm{in}}$, it follows that $\varphi - G_n \geq 0$ on Γ_{in}, $\forall\, n \in \mathbb{N}$, which implies $(\varphi - G_n)^- = 0$ on Γ_{in}, $\forall\, n \in \mathbb{N}$.

Now, since $(\varphi - G_n) \in \mathrm{H}^1(\Omega)$, $\forall\, n \in \mathbb{N}$, similarly to [1, Lemma 5.2.24], one can show that $(\varphi - G_n)^- \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$, $\forall\, n \in \mathbb{N}$. Then, using the fact that $\mathrm{div}\,\mathbf{w} = 0$, integrating by parts we can deduce that

$$\int_\Omega (\mathbf{w}\cdot\nabla\varphi)(\varphi - G_n)^- = \frac{1}{2}\int_{\Gamma_{\mathrm{m}}\cup\Gamma_{\mathrm{out}}} (\mathbf{w}\cdot\mathbf{n})\left|(\varphi - G_n)^-\right|^2. \tag{2.15}$$

Then, from (2.13) with $\psi = (\varphi - G_n)^-$, employing (2.15) and using the fact that $\xi^-\xi = (\xi^-)^2$, for any function ξ, we obtain:

$$\begin{aligned} 0 = \; & a_C(\varphi, (\varphi - G_n)^-) + O_C(\mathbf{w}, \varphi, (\varphi - G_n)^-) \\ = \; & \kappa \int_\Omega |\nabla(\varphi - G_n)^-|^2 + a_3 \int_{\Gamma_\mathrm{m}} \varphi(\varphi - G_n)^- + \frac{1}{2}\int_{\Gamma_\mathrm{m}} (\mathbf{w}\cdot\mathbf{n})|(\varphi - G_n)^-|^2 \\ & + \frac{1}{2}\int_{\Gamma_\mathrm{out}} (\mathbf{w}\cdot\mathbf{n})|(\varphi - G_n)^-|^2 - \frac{1}{2}\int_{\Gamma_\mathrm{out}} (\mathbf{w}\cdot\mathbf{n})^-\varphi(\varphi - G_n)^- \\ & - \int_{\Gamma_\mathrm{m}} (\mathbf{w}\cdot\mathbf{n})\varphi(\varphi - G_n)^-, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Then, by (2.10) and the Dominated Convergence Theorem (see, e.g., [3, Lemma 3.31]), we can pass to the limit as $n \to \infty$ and obtain that

$$\begin{aligned} \kappa\|\varphi^-\|_{1,\Omega}^2 + a_3 \int_{\Gamma_\mathrm{m}} \varphi\varphi^- + \frac{1}{2}\int_{\Gamma_\mathrm{m}} (\mathbf{w}\cdot\mathbf{n})|\varphi^-|^2 + \frac{1}{2}\int_{\Gamma_\mathrm{out}} (\mathbf{w}\cdot\mathbf{n})|\varphi^-|^2 \\ - \frac{1}{2}\int_{\Gamma_\mathrm{out}} (\mathbf{w}\cdot\mathbf{n})^-\varphi\varphi^- - \int_{\Gamma_\mathrm{m}} (\mathbf{w}\cdot\mathbf{n})\varphi\varphi^- \lesssim 0, \end{aligned}$$

which implies

$$\kappa\|\varphi^-\|_{1,\Omega}^2 + a_3 \int_{\Gamma_\mathrm{m}} \varphi\varphi^- - \frac{1}{2}\int_{\Gamma_\mathrm{m}} (\mathbf{w}\cdot\mathbf{n})|\varphi^-|^2 + \frac{1}{2}\int_{\Gamma_\mathrm{out}} (\mathbf{w}\cdot\mathbf{n})^+|\varphi^-|^2 \lesssim 0.$$

In this way, recalling that $\mathbf{w}\cdot\mathbf{n} = a_1 - a_2\phi^+$ on Γ_m, from the latter we obtain:

$$\kappa\|\varphi^-\|_{1,\Omega}^2 + \left(a_3 - \frac{a_1}{2}\right)\int_{\Gamma_\mathrm{m}} |\varphi^-|^2 + \frac{a_2}{2}\int_{\Gamma_\mathrm{m}} \phi^+|\varphi^-|^2 + \frac{1}{2}\int_{\Gamma_\mathrm{out}} (\mathbf{w}\cdot\mathbf{n})^+|\varphi^-|^2 \lesssim 0.$$

which clearly implies the result. □

We end this section by observing that, as previously noted, if $(\mathbf{u}, p, \varphi)$ is a solution of (2.4), then $(\mathbf{w}, \phi) := (\mathbf{u}, \varphi)$ clearly satisfies the hypotheses of Lemma 2.2, from which it follows that the concentration φ satisfies (2.14), provided that $a_3 - \frac{1}{2}a_1 \geq 0$.

2.2.3 An equivalent reduced problem

To simplify the well-posedness analysis of problem (2.4), we now introduce an equivalent reduced formulation. For this purpose, we first define a suitable lifting operator that extends the Dirichlet datum $\tilde{\mathbf{u}}_\mathrm{in}(\varphi) \in \mathbf{H}^{1/2}(\Gamma_\mathrm{out}^c)$. The construction of such an operator, together with the derivation of its key properties, is established in the following lemma.

Lemma 2.3 *Let* $\varphi_\mathrm{in} \in \mathbb{R}^+$, $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ *and* $\mathbf{u}_\mathrm{in} \in \mathbf{H}^{1/2}(\Gamma_\mathrm{in})$ *satisfying* (1.18) *and* (1.19). *Then, for each* $\phi_0 \in \mathrm{H}^1_{\Gamma_\mathrm{in}}(\Omega)$ *such that* $\phi_0 + \varphi_{in} \geq 0$ *on* Γ_in, *there exists* $\mathbf{u}_1(\phi_0) \in \mathbf{H}^1(\Omega)$ *satisfying the identities*

$$\mathbf{u}_1(\phi_0)\big|_{\Gamma_\mathrm{out}^c} = \tilde{\mathbf{u}}_\mathrm{in}(\phi_0 + \varphi_\mathrm{in}), \quad \text{and} \quad \mathrm{div}\,(\mathbf{u}_1(\phi_0)) = 0 \text{ in } \Omega, \tag{2.16}$$

where $\tilde{\mathbf{u}}_\mathrm{in}$ *is defined in* (2.5). *In addition, there holds*

$$\|\mathbf{u}_1(\phi_0)\|_{1,\Omega} \lesssim \mathbf{C}(\mathbf{a}, \mathbf{u}_{in}, \varphi_\mathrm{in}) + a_2\|\phi_0\|_{1,\Omega}, \qquad \forall\, \phi_0 \in \mathrm{H}^1_{\Gamma_\mathrm{in}}(\Omega), \tag{2.17}$$

and

$$\|\mathbf{u}_1(\phi_0) - \mathbf{u}_1(\tilde{\phi}_0)\|_{1,\Omega} \lesssim a_2 \|\phi_0 - \tilde{\phi}_0\|_{1,\Omega}, \qquad \forall\, \phi_0, \tilde{\phi}_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega), \tag{2.18}$$

where

$$\mathbf{C}(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}) := (\|u_{1,in}\|_{1/2,00,\Gamma_{\rm in}} + \|\tilde{u}_2(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in})\|_{1/2,\Gamma^c_{\rm out}}), \tag{2.19}$$

with $\tilde{u}_2(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}) \in \mathrm{H}^{1/2}(\Gamma^c_{\rm out})$ *in* (2.17) *being the extension of* $u_{2,in}$ *given by:*

$$\tilde{u}_2(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}) = \begin{cases} u_{2,in} \quad on \quad \Gamma_{\rm in} \\ a_1 - a_2\varphi_{\rm in} \quad on \quad \Gamma_{m^+} \\ -(a_1 - a_2\varphi_{\rm in}) \quad on \quad \Gamma_{m^-}. \end{cases} \tag{2.20}$$

Proof. Given $\phi_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$ such that $\phi_0 + \varphi_{in} \geq 0$ on $\Gamma_{\rm in}$, we start by defining $\mathbf{z}_1(\phi_0) \in \mathbf{H}^1(\Omega)$ given by

$$\mathbf{z}_1(\phi_0) := \tilde{\boldsymbol{\gamma}}_{\mathbf{0}}^{-1}(\boldsymbol{E}_{\Gamma^c_{\rm out}}(\tilde{\mathbf{u}}_{\rm in}(\phi_0 + \varphi_{\rm in}))), \tag{2.21}$$

and $\mathbf{z}_0(\phi_0) \in \mathbf{H}^1_{\Gamma^c_{\rm out}}(\Omega)$ being the first component of the solution of the Stokes problem: Find $(\mathbf{z}_0(\phi_0), r(\phi_0)) \in \mathbf{H}^1_{\Gamma^c_{\rm out}}(\Omega) \times \mathrm{L}^2(\Omega)$, such that

$$\begin{aligned} \int_\Omega \nabla \mathbf{z}_0(\phi_0) : \nabla \mathbf{v} + \int_\Omega r(\phi_0)\, \mathrm{div}\, \mathbf{v} &= -\int_\Omega \nabla \mathbf{z}_1(\phi_0) : \nabla \mathbf{v}, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\rm out}}(\Omega), \\ \int_\Omega q\, \mathrm{div}\, \mathbf{z}_0(\phi_0) &= -\int_\Omega q\, \mathrm{div}\, \mathbf{z}_1(\phi_0), \qquad \forall\, q \in \mathrm{L}^2(\Omega). \end{aligned} \tag{2.22}$$

Then we simply define

$$\mathbf{u}_1(\phi_0) := \mathbf{z}_0(\phi_0) + \mathbf{z}_1(\phi_0), \tag{2.23}$$

and observe, from (2.21), the fact that $\mathbf{z}_0(\phi_0)\big|_{\Gamma^c_{\rm out}} = 0$, and the second equation of (2.22), that $\mathbf{u}_1(\phi_0)$ satisfies (2.16).

Now, to deduce (2.17) we first recall that the solution of (2.22) satisfies

$$\|\mathbf{z}_0(\phi_0)\|_{1,\Omega} + \|r(\phi_0)\|_{0,\Omega} \lesssim \|\tilde{\mathbf{u}}_{\rm in}(\phi_0 + \varphi_{\rm in})\|_{1/2,\Gamma^c_{\rm out}}. \tag{2.24}$$

Then, from the latter, (1.5) and (1.13), it follows that

$$\|\mathbf{u}_1(\phi_0)\|_{1,\Omega} \lesssim \|\tilde{\mathbf{u}}_{\rm in}(\phi_0 + \varphi_{\rm in})\|_{1/2,\Gamma^c_{\rm out}}. \tag{2.25}$$

Next, we let $g \in C^{0,\gamma}(\Gamma)$ be such that $g = 1$ on Γ_{m^+} and $g = -1$ on Γ_{m^-} and define

$$\mathbf{s}(\phi_0) := \big(E_{0,\Gamma_{\rm in}}(u_{1,\rm in}), \quad E_{\Gamma^c_{\rm out}}(\tilde{u}_2(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in})) - a_2\gamma_0(\phi_0) g\big)^t \in \mathbf{H}^{1/2}(\Gamma), \tag{2.26}$$

after simple computations it is not difficult to see that

$$\mathbf{s}(\phi_0)|_{\Gamma_{\rm in}} = \mathbf{u}_{\rm in} \quad \text{on} \quad \Gamma_{\rm in} \quad \text{and} \quad \mathbf{s}(\phi_0)|_{\Gamma_m} = \big(a_1 - a_2(\phi_0 + \varphi_{\rm in})^+\big)\, \mathbf{n} \quad \text{on} \quad \Gamma_m, \tag{2.27}$$

that is $\mathbf{s}(\phi_0) = \tilde{\mathbf{u}}_{\rm in}(\phi_0 + \varphi_{\rm in})$ on $\Gamma^c_{\rm out}$. In this way, from (1.1), (1.8), (1.12), (1.13), and (2.25), we deduce that

$$\|\mathbf{u}_1(\phi_0)\|_{1,\Omega} \lesssim \|\tilde{\mathbf{u}}_{\rm in}(\phi_0 + \varphi_{\rm in})\|_{1/2,\Gamma^c_{\rm out}} \cong \|\mathbf{s}(\phi_0)\|_{1/2,\Gamma^c_{\rm out}} \lesssim \mathbf{C}(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}) + a_2\|\phi_0\|_{1,\Omega}, \tag{2.28}$$

thus, (2.17) holds. Finally, we let ϕ_0, $\tilde{\phi}_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$, and apply again Lemma 1.1 together with the inequality in (1.1) to conclude that

$$\|\mathbf{s}(\phi_0) - \mathbf{s}(\tilde{\phi}_0)\|_{1/2,\Gamma} \leq a_2\|\gamma_0(\phi_0 - \tilde{\phi}_0) g\|_{1/2,\Gamma} \leq a_2 c_2 \|\phi_0 - \tilde{\phi}_0\|_{1,\Omega},$$

for some constant $c_2 > 0$, which, due to (2.22), implies (2.18). □

Remark 2.4 *Notice that if* $\phi_0 \in \mathrm{H}^{1+\delta}(\Omega) \cap \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$*, with* $\delta > 0$*, then the solution of* (2.22) *satisfies* $\mathbf{z}_0(\phi_0) \in \mathbf{H}^{1+\delta}(\Omega) \cap \mathbf{H}^1_{\Gamma^c_{\rm out}}(\Omega)$ *and* $r(\phi_0) \in \mathrm{H}^{\delta}(\Omega)$ *(see [4, 26]). This regularity result will be used in Section 4.*

Now, we let $\mathbf{V}$ be the kernel of the bilinear form b_F, that is

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\rm out}}(\Omega) : b_F(\mathbf{v}, q) = 0 \quad \forall q \in \mathrm{L}^2(\Omega)\} = \{\mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\rm out}}(\Omega) : \mathrm{div}\, \mathbf{v} = 0\}. \tag{2.29}$$

Then the aforementioned equivalent reduced problem associated to (2.4) reads: Find $(\mathbf{u}_0, \varphi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$, such that

$$\begin{array}{lcll} a_F(\mathbf{u}_0 + \mathbf{u}_1(\varphi_0), \mathbf{v}) \,+\, O_F(\mathbf{u}_0 + \mathbf{u}_1(\varphi_0); \mathbf{u}_0 + \mathbf{u}_1(\varphi_0), \mathbf{v}) & = & 0 & \forall\, \mathbf{v} \in \mathbf{V}, \\ a_C(\varphi_0 + \varphi_{\rm in}, \psi) + O_C(\mathbf{u}_0 + \mathbf{u}_1(\varphi_0), \varphi_0 + \varphi_{\rm in}, \psi) & = & 0 & \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega), \end{array} \tag{2.30}$$

where $\mathbf{u}_1^{\varphi_0}$ is the lifting introduced in Lemma 2.3 with $\phi_0 = \varphi_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$.

The following Lemma establishes the equivalence between problems (2.4) and (2.30).

Lemma 2.5 *If* $(\mathbf{u}, p, \varphi) \in \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1(\Omega)$*, with* $\mathbf{u} = \tilde{\mathbf{u}}_{\rm in}(\varphi)$ *on* $\Gamma^c_{\rm out}$ *and* $\varphi = \varphi_{\rm in}$ *on* $\Gamma_{\rm in}$*, is a solution of* (2.4)*, then* $(\mathbf{u}_0, \varphi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$*, with* $\mathbf{u}_0 = \mathbf{u} - \mathbf{u}_1(\varphi) \in \mathbf{V}$ *and* $\varphi_0 = \varphi - \varphi_{\rm in} \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$*, is a solution of* (2.30)*. Conversely, if* $(\mathbf{u}_0, \varphi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$ *is a solution of* (2.30)*, then there exists* $p \in \mathrm{L}^2(\Omega)$ *such that* $(\mathbf{u}_0 + \mathbf{u}_1(\varphi_0), p, \varphi_0 + \varphi_{\rm in}) \in \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *is a solution of* (2.4)*.*

Proof. The proof follows from the definition of the lifting $\mathbf{u}_1(\varphi_0)$ in Lemma 2.3 and the inf-sup condition (2.1). We omit further details and refer the reader to [24, Lemma 2.1] for a similar result. □

From this point forward, our attention will be directed towards establishing the well-posedness of the problem (2.30) by means of a suitable fixed-point strategy. To that end, we first introduce an associated fixed-point operator.

2.2.4 The fixed-point operator

Here, we follow a similar approach to that in [2] and describe the fixed-point strategy that will be used to prove the well-posedness of (2.30). We start by defining the following auxiliary operator $\mathcal{L} : \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega) \to \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$ given by

$$\mathcal{L}(\mathbf{w}_0, \phi_0) := \varphi_0,$$

with $\varphi_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$ being the unique solution (to be confirmed below) of the linearized problem: Find $\varphi_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$, such that

$$a_C(\varphi_0 + \varphi_{\rm in}, \psi) + O_C(\mathbf{w}_0 + \mathbf{u}_1(\phi_0), \varphi_0 + \varphi_{\rm in}, \psi) = 0 \quad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega), \tag{2.31}$$

where $\mathbf{u}_1(\phi_0) \in \mathbf{H}^1(\Omega)$ is the lifting defined in Lemma 2.3.

In addition we let $\mathcal{S} : \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega) \to \mathbf{V}$ be the operator given by

$$\mathcal{S}(\mathbf{w}_0, \phi_0) := \mathbf{u}_0, \tag{2.32}$$

with $\mathbf{u}_0 \in \mathbf{V}$ being the unique solution (to be confirmed below) of problem: Find $\mathbf{u}_0 \in \mathbf{V}$ such that

$$a_F(\mathbf{u}_0 + \mathbf{u}_1(\phi_0), \mathbf{v}) + O_F(\mathbf{w}_0 + \mathbf{u}_1(\phi_0); \mathbf{u}_0 + \mathbf{u}_1(\phi_0), \mathbf{v}) \quad = \quad 0 \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\mathrm{out}}}(\Omega). \tag{2.33}$$

Thus, we let $\mathcal{J} : \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega) \to \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$ be the operator defined by

$$\mathcal{J}(\mathbf{w}_0, \phi_0) := (\mathcal{S}(\mathbf{w}_0, \phi_0), \mathcal{L}(\mathcal{S}(\mathbf{w}_0, \phi_0), \phi_0)) \quad \forall (\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega), \tag{2.34}$$

and realize that analyzing the well-posedness of problem (2.30) is equivalent study the unique solvability of the fixed point problem : Find $(\mathbf{u}_0, \varphi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$, such that

$$\mathcal{J}(\mathbf{u}_0, \varphi_0) = (\mathbf{u}_0, \varphi_0). \tag{2.35}$$

According to the above, in what follows we focus on analyzing the fixed-point problem (2.35). Before doing that we first establish the well-definiteness of operator $\mathcal{J}$. This is addressed in the following subsection.

2.2.5 Well-definiteness of $\mathcal{J}$

Since the operator $\mathcal{J}$ is defined in terms of $\mathcal{L}$ and $\mathcal{S}$ (cf. (2.34)), to prove that $\mathcal{J}$ is well defined it suffices to show that $\mathcal{L}$ and $\mathcal{S}$ are themselves well defined, which is equivalent to establishing the well-posedness of problems (2.31) and (2.33), respectively. We begin with the analysis of problem (2.31).

Lemma 2.6 *Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3_+$ satisfying $a_3 - \frac{1}{2}a_1 \geq 0$ and let $(\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$ with $\phi_0 + \varphi_{\mathrm{in}} \geq 0$ on Γ_{m}. There exists a unique $\varphi_0 := \mathcal{L}(\mathbf{w}_0, \phi_0) \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$ solution to (2.31), satisfying*

$$\varphi_0 + \varphi_{\mathrm{in}} \geq 0 \ \text{on } \Gamma_{\mathrm{m}}. \tag{2.36}$$

In addition, the following estimate holds:

$$\|\mathcal{L}(\mathbf{w}_0, \phi_0)\|_{1,\Omega} \lesssim \kappa^{-1}\varphi_{\mathrm{in}}\,(a_3 + a_2\varphi_{\mathrm{in}} + a_1 + a_2\|\phi_0\|_{1,\Omega} + \|\mathbf{w}_0\|_{1,\Omega} + \|\mathbf{u}_1(\phi_0)\|_{1,\Omega})\,. \tag{2.37}$$

Proof. We begin by noting that, given $(\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$, problem (2.31) can be reformulated as: find $\varphi_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$, such that

$$a_C(\varphi_0, \psi) + O_C(\mathbf{w}_0 + \mathbf{u}_1(\phi_0), \varphi_0, \psi) = G_{\mathbf{w}_0,\phi_0}(\psi) \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega), \tag{2.38}$$

where the linear functional G on the right-hand side is given by

$$G_{\mathbf{w}_0,\phi_0}(\psi) := -a_C(\varphi_{\mathrm{in}}, \psi) - O_C(\mathbf{w}_0 + \mathbf{u}_1(\phi_0), \varphi_{\mathrm{in}}, \psi). \tag{2.39}$$

Thus, to prove the well-posedness of (2.38), it suffices, by the Lax–Milgram lemma, to show that the bilinear form on the left-hand side is elliptic on $\mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$ and $G_{\mathbf{w}_0,\phi_0}$ is a bounded functional. Indeed, using (2.16), the second identity in (2.12), and the fact that $\mathbf{u}_1(\phi_0) = a_1 - a_2(\phi_0 + \varphi_{\mathrm{in}})^+$ on Γ_{m} and $\mathbf{w}_0 = \mathbf{0}$ on Γ^c_{out} we obtain

$$\begin{aligned} O_C(\mathbf{w}_0 + \mathbf{u}_1(\phi_0), \psi, \psi) \quad &= -\frac{1}{2}\int_{\Gamma_{\mathrm{m}}} (\mathbf{u}_1(\phi_0)\cdot\mathbf{n})\psi^2 + \frac{1}{2}\int_{\Gamma_{\mathrm{out}}} ((\mathbf{w}_0 + \mathbf{u}_1(\phi_0))\cdot\mathbf{n})^+\psi^2, \\ &\geq -\frac{1}{2}\int_{\Gamma_{\mathrm{m}}} a_1\psi^2 + \frac{a_2}{2}\int_{\Gamma_{\mathrm{m}}} (\phi_0 + \varphi_{\mathrm{in}})^+\psi^2, \\ &\geq -\frac{a_1}{2}\int_{\Gamma_{\mathrm{m}}} \psi^2, \end{aligned} \tag{2.40}$$

for all $\psi \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$. Then, from the definition of a_C, (2.40), (2.10) and the fact that $a_3 - \frac{1}{2}a_1 \geq 0$, we conclude that

$$\begin{aligned} a_C(\psi,\psi) + O_C(\mathbf{w}_0 + \mathbf{u}_1(\phi_0), \psi, \psi) &\geq \kappa |\psi|^2_{1,\Omega} + \left(a_3 - \frac{a_1}{2}\right) \|\psi\|^2_{0,\Gamma_{\mathrm{m}}} \\ &\gtrsim \kappa \|\psi\|^2_{1,\Omega} \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega). \end{aligned} \tag{2.41}$$

Now, for the boundedness of $G_{\mathbf{w}_0,\phi_0}$ we use again that $\mathbf{u}_1(\phi_0) = a_1 - a_2(\phi_0 + \varphi_{\mathrm{in}})^+$ on Γ_{m} and $\mathbf{w}_0 = \mathbf{0}$ on Γ^c_{out}, to obtain

$$\begin{aligned} G_{\mathbf{w}_0,\phi_0}(\psi) &= -a_3\varphi_{\mathrm{in}} \int_{\Gamma_{\mathrm{m}}} \psi + \varphi_{\mathrm{in}} \int_{\Gamma_{\mathrm{m}}} \psi(a_1 - a_2(\phi_0 + \varphi_{\mathrm{in}})^+) + \frac{\varphi_{\mathrm{in}}}{2} \int_{\Gamma_{\mathrm{out}}} \psi((\mathbf{w}_0 + \mathbf{u}_1(\phi_0)) \cdot \mathbf{n})^- \\ &= -(a_3\varphi_{\mathrm{in}} + a_2\varphi^2_{\mathrm{in}}) \int_{\Gamma_{\mathrm{m}}} \psi + \varphi_{\mathrm{in}} \int_{\Gamma_{\mathrm{m}}} \psi(a_1 - a_2\phi_0) + \frac{\varphi_{\mathrm{in}}}{2} \int_{\Gamma_{\mathrm{out}}} \psi((\mathbf{w}_0 + \mathbf{u}_1(\phi_0)) \cdot \mathbf{n})^-. \end{aligned} \tag{2.42}$$

The latter combined with (1.1) imply

$$|G_{\mathbf{w}_0,\phi_0}(\psi)| \lesssim \varphi_{\mathrm{in}} \left(a_3 + a_2\varphi_{\mathrm{in}} + a_1 + a_2\|\phi_0\|_{1,\Omega} + \|\mathbf{w}_0\|_{1,\Omega} + \|\mathbf{u}_1(\phi_0)\|_{1,\Omega}\right) \|\psi\|_{1,\Omega}. \tag{2.43}$$

In this way, from (2.41), (2.43) and the Lax–Milgram Lemma we readily deduce the well-posedness of problem (2.38) and estimate (2.37). Finally, (2.36) follows from Lemma 2.2. □

Now we turn to prove the well-definiteness of $\mathcal{S}$.

Lemma 2.7 *Given* $(\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$*, there exists a unique* $\mathbf{u}_0 := \mathcal{S}(\mathbf{w}_0, \phi_0) \in \mathbf{V}$ *solution to problem* (2.33)*, satisfying*

$$\|\mathcal{S}(\mathbf{w}_0, \phi_0)\|_{1,\Omega} \lesssim \nu^{-1} \left(\nu + \rho\|\mathbf{w}_0\|_{1,\Omega} + \rho\|\mathbf{u}_1(\phi_0)\|_{1,\Omega}\right) \|\mathbf{u}_1(\phi_0)\|_{1,\Omega}. \tag{2.44}$$

Proof. Given $(\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$ similary to the proof of Lemma 2.6, we begin by noting that problem (2.33) can be equivalently rewritten as: Find $\mathbf{u}_0 \in \mathbf{V}$, such that

$$a_F(\mathbf{u}_0, \mathbf{v}) + O_F(\mathbf{w}_0 + \mathbf{u}_1(\phi_0); \mathbf{u}_0, \mathbf{v}) \quad = \quad F_{\mathbf{w}_0,\phi_0}(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}, \tag{2.45}$$

where the linear functional $F_{\mathbf{w}_0,\phi_0} : \mathbf{V} \to \mathbb{R}$ is defined as

$$\begin{aligned} F_{\mathbf{w}_0,\phi_0}(\mathbf{v}) :=& - a_F(\mathbf{u}_1(\phi_0), \mathbf{v}) - O_F(\mathbf{w}_0 + \mathbf{u}_1(\phi_0); \mathbf{u}_1(\phi_0), \mathbf{v}) \\ =& - 2\nu \int_\Omega \nabla \mathbf{u}_1(\phi_0) : \nabla \mathbf{v} - \rho \int_\Omega (\nabla \mathbf{u}_1(\phi_0))(\mathbf{w}_0 + \mathbf{u}_1(\phi_0)) \cdot \mathbf{v} \\ &+ \frac{\rho}{2} \int_{\Gamma_{\mathrm{out}}} ((\mathbf{w}_0 + \mathbf{u}_1(\phi_0)) \cdot \mathbf{n})^- \mathbf{u}_1(\phi_0) \cdot \mathbf{v}, \end{aligned} \tag{2.46}$$

for all $\mathbf{v} \in \mathbf{V}$.

Now, using the second identity in (2.16), the first identity in (2.12), we deduce that

$$\begin{aligned} O_F(\mathbf{w}_0 + \mathbf{u}_1(\phi_0); \mathbf{v}, \mathbf{v}) &= \frac{\rho}{2} \int_{\Gamma_{\mathrm{out}}} ((\mathbf{w}_0 + \mathbf{u}_1(\phi_0)) \cdot \mathbf{n}) |\mathbf{v}|^2 - \frac{\rho}{2} \int_{\Gamma_{\mathrm{out}}} ((\mathbf{w}_0 + \mathbf{u}_1(\phi_0)) \cdot \mathbf{n})^- |\mathbf{v}|^2 \\ &= \frac{\rho}{2} \int_{\Gamma_{\mathrm{out}}} ((\mathbf{w}_0 + \mathbf{u}_1(\phi_0)) \cdot \mathbf{n})^+ |\mathbf{v}|^2 \geq 0 \quad \forall\, \mathbf{v} \in \mathbf{V}. \end{aligned} \tag{2.47}$$

Hence, combining (2.47) and (2.9), we get

$$a_F(\mathbf{v},\mathbf{v}) + O_F(\mathbf{w}_0 + \mathbf{u}_1(\phi_0);\mathbf{v},\mathbf{v}) \gtrsim c_F\|\mathbf{v}\|_{1,\Omega}^2 \quad \forall\,\mathbf{v}\in\mathbf{V}. \tag{2.48}$$

On the other hand, from (1.1), (1.6), (2.7) and (2.46), we readily obtain

$$\begin{aligned}|F_{\mathbf{w}_0,\phi_0}(\mathbf{v})| &\lesssim \nu\|\mathbf{u}_1(\phi_0)\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega} + \rho\|\mathbf{w}_0+\mathbf{u}_1(\phi_0)\|_{1,\Omega}\|\mathbf{u}_1(\phi_0)\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\\ &\quad + \rho\|\mathbf{w}_0+\mathbf{u}_1(\phi_0)\|_{0,\Gamma_{\mathrm{out}}}\|\mathbf{u}_1(\phi_0)\|_{\mathbf{L}^4(\Gamma_{\mathrm{out}})}\|\mathbf{v}\|_{\mathbf{L}^4(\Gamma_{\mathrm{out}})}\\ &\lesssim (\nu + \rho\|\mathbf{w}_0\|_{1,\Omega} + \rho\|\mathbf{u}_1(\phi_0)\|_{1,\Omega})\,\|\mathbf{u}_1(\phi_0)\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega},\end{aligned} \tag{2.49}$$

for all $\mathbf{v}\in\mathbf{V}$. In this way, from (2.48), (2.49) and the Lax-Milgram Lemma (see, e.g., [14, Lemma 2.2]) we deduce the well-posedness of problem (2.45) and estimate (2.44). □

Having analyzed the well-definiteness of $\mathcal{L}$ and $\mathcal{S}$, now we are in position of establishing that $\mathcal{J}$ is well defined.

Lemma 2.8 *Let $\mathbf{a}=(a_1,a_2,a_3)\in\mathbb{R}^3_+$ satisfying $a_3-\frac{1}{2}a_1\ge 0$ and let $(\mathbf{w}_0,\phi_0)\in\mathbf{V}\times\mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$, with $\phi_0+\varphi_{\mathrm{in}}\ge 0$ on Γ_{m}. There exists a unique $(\mathbf{u}_0,\varphi_0)\in\mathbf{V}\times\mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$, such that*

$$\mathcal{J}(\mathbf{w}_0,\phi_0) = (\mathbf{u}_0,\varphi_0) \quad \textit{and} \quad \varphi_0+\varphi_{\mathrm{in}}\ge 0 \ \textit{on}\ \Gamma_{\mathrm{m}}. \tag{2.50}$$

In addition, the following estimate holds

$$\begin{aligned}\|\mathcal{J}(\mathbf{w}_0,\phi_0)\| &\lesssim \mathbf{B}_1(\mathrm{data}) + \mathbf{B}_2(\mathrm{data})\|\phi_0\|_{1,\Omega} + \mathbf{B}_3(\mathrm{data})\|\mathbf{w}_0\|_{1,\Omega}\\ &\quad + \mathbf{B}_4(\mathrm{data})\|\mathbf{w}_0\|_{1,\Omega}\|\phi_0\|_{1,\Omega} + \mathbf{B}_5(\mathrm{data})\|\phi_0\|^2_{1,\Omega},\end{aligned} \tag{2.51}$$

where

$$\begin{aligned}\mathbf{B}_1(\mathrm{data}) &:= \kappa^{-1}\varphi_{\mathrm{in}}(a_1+a_2\varphi_{\mathrm{in}}+a_3) + (1+\kappa^{-1}\varphi_{\mathrm{in}})\mathbf{C}(\mathbf{a},\mathbf{u}_{in},\varphi_{\mathrm{in}}),\\ &\quad +\nu^{-1}\rho(1+\kappa^{-1}\varphi_{\mathrm{in}})\mathbf{C}^2(\mathbf{a},\mathbf{u}_{in},\varphi_{\mathrm{in}})\\ \mathbf{B}_2(\mathrm{data}) &:= (1+\kappa^{-1}\varphi_{\mathrm{in}})a_2,\\ \mathbf{B}_3(\mathrm{data}) &:= \nu^{-1}\rho(1+\kappa^{-1}\varphi_{\mathrm{in}})\mathbf{C}(\mathbf{a},\mathbf{u}_{in},\varphi_{\mathrm{in}}),\\ \mathbf{B}_4(\mathrm{data}) &:= \nu^{-1}\rho(1+\kappa^{-1}\varphi_{\mathrm{in}})a_2,\\ \mathbf{B}_5(\mathrm{data}) &:= \nu^{-1}\rho(1+\kappa^{-1}\varphi_{\mathrm{in}})a_2^2.\end{aligned} \tag{2.52}$$

Proof. Recalling the definition of $\mathcal{J}$ (cf. (2.34)), the proof of well-definiteness of $\mathcal{J}$ is a direct consequence of Lemmas 2.6 and 2.7. In addition, after simple computations, from (2.37) and (2.44), it follows that

$$\begin{aligned}\|\mathcal{J}(\mathbf{w}_0,\phi_0)\| \lesssim\ & \kappa^{-1}\varphi_{\mathrm{in}}(a_1+a_2\varphi_{\mathrm{in}}+a_3) + \kappa^{-1}a_2\varphi_{\mathrm{in}}\|\phi_0\|_{1,\Omega} + (1+\kappa^{-1}\varphi_{\mathrm{in}})\|\mathbf{u}_1(\phi_0)\|_{1,\Omega}\\ &+\nu^{-1}\rho(1+\kappa^{-1}\varphi_{\mathrm{in}})\|\mathbf{w}_0\|_{1,\Omega}\|\mathbf{u}_1(\phi_0)\|_{1,\Omega} + \nu^{-1}\rho(1+\kappa^{-1}\varphi_{\mathrm{in}})\|\mathbf{u}_1(\phi_0)\|^2_{1,\Omega}.\end{aligned}$$

This inequality and (2.17) imply (2.51), which concludes the proof. □

2.2.6 Well-posedness of the continuous problem

Having proved the well-definiteness of operator $\mathcal{J}$, now we turn our attention to proving that $\mathcal{J}$ admits a fixed point. To that end, given $\lambda > 0$, we first introduce the following non-empty, closed subset of $\mathbf{H}^1_{\Gamma^c_{\text{out}}}(\Omega) \times \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega)$:

$$\mathbf{W}_\lambda := \left\{ (\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega) : \phi_0 + \varphi_{\text{in}} \geq 0 \text{ on } \Gamma_{\text{m}},\ \|(\mathbf{w}_0, \phi_0)\| \leq \lambda \right\}. \tag{2.53}$$

Under suitable assumptions on the data, it can be proved that $\mathcal{J}(\mathbf{W}_\lambda) \subseteq \mathbf{W}_\lambda$. This is establishes in the following result.

Lemma 2.9 *Let* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3_+$ *satisfying* $a_3 - \frac{1}{2}a_1 \geq 0$ *and let* $(\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega)$, *with* $\phi_0 + \varphi_{\text{in}} \geq 0$ *on* Γ_{m}. *For given* $\lambda > 0$, *assume further that*

$$\mathbf{B}_1(\text{data}) + (\mathbf{B}_2(\text{data}) + \mathbf{B}_3(\text{data}))\lambda + (\mathbf{B}_4(\text{data}) + \mathbf{B}_5(\text{data}))\lambda^2 \leq \lambda, \tag{2.54}$$

with $\mathbf{B}_i$, $i = 1, ..., 5$ *defined in* (2.52). *Then there holds* $\mathcal{J}(\mathbf{W}_\lambda) \subseteq \mathbf{W}_\lambda$.

Proof. The follows straightforwardly from (2.51) and assumption (2.54). We omit further details. □

We now turn to the proof of a Lipschitz continuity property for $\mathcal{J}$. As a first step, we establish the following intermediate result, which shows that $\mathcal{L}$ is Lipschitz continuous.

Lemma 2.10 *Let* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3_+$ *satisfying* $a_3 - \frac{1}{2}a_1 \geq 0$ *and let* $(\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega)$, *with* $\phi_0 + \varphi_{\text{in}} \geq 0$ *on* Γ_{m}. *For given* $\lambda > 0$, *assume further that* (2.54) *holds. Then, for all* $(\mathbf{w}_0, \phi_0), (\tilde{\mathbf{w}}_0, \tilde{\phi}_0) \in \mathbf{W}_\lambda$, *there holds*

$$\|\mathcal{L}(\mathbf{w}_0, \phi_0) - \mathcal{L}(\tilde{\mathbf{w}}_0, \tilde{\phi}_0)\| \lesssim C_{\mathcal{L}} \|(\mathbf{w}_0, \phi_0) - (\tilde{\mathbf{w}}_0, \tilde{\phi}_0)\|, \tag{2.55}$$

where $C_{\mathcal{L}} > 0$ *is given by:*

$$C_{\mathcal{L}} = k^{-1}(\lambda + \varphi_{\text{in}})(1 + a_2). \tag{2.56}$$

Proof. Given $(\mathbf{w}_0, \phi_0)$, $(\tilde{\mathbf{w}}_0, \tilde{\phi}_0) \in \mathbf{W}_\lambda$, we let $\varphi_0 = \mathcal{L}(\mathbf{w}_0, \phi_0)$ and $\tilde{\varphi}_0 = \mathcal{L}(\tilde{\mathbf{w}}_0, \tilde{\phi}_0)$. According the definition of $\mathcal{L}$ (cf. (2.31)), it follows that

$$a_C(\varphi_0, \psi) + O_C(\mathbf{w}_0 + \mathbf{u}_1(\phi_0), \varphi_0, \psi) = G_{\mathbf{w}_0, \phi_0}(\psi) \quad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega), \tag{2.57}$$

and

$$a_C(\tilde{\varphi}_0, \psi) + O_C(\tilde{\mathbf{w}}_0 + \mathbf{u}_1(\tilde{\phi}_0), \tilde{\varphi}_0, \psi) = G_{\tilde{\mathbf{w}}_0, \tilde{\phi}_0}(\psi) \quad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega), \tag{2.58}$$

with the functionals $G_{\mathbf{w}_0, \phi_0}$ and $G_{\tilde{\mathbf{w}}_0, \tilde{\phi}_0}$ defined as in (2.39). In addition, from Lemma 2.9 we know that

$$\|\varphi_0\|_{1,\Omega} \leq \lambda \quad \text{and} \quad \|\tilde{\varphi}_0\|_{1,\Omega} \leq \lambda. \tag{2.59}$$

Then, subtracting equations (2.57) and (2.58), we have that

$$a_C(\varphi_0 - \tilde{\varphi}_0, \psi) + O_C(\mathbf{w}_0 + \mathbf{u}_1(\phi_0), \varphi_0, \psi) - O_C(\tilde{\mathbf{w}}_0 + \mathbf{u}_1(\tilde{\phi}_0), \tilde{\varphi}_0, \psi) = G_{\mathbf{w}_0, \phi_0}(\psi) - G_{\tilde{\mathbf{w}}_0, \tilde{\phi}_0}(\psi),$$

and adding and subtracting suitable terms, from the latter we arrive at

$$\begin{aligned} a_C(\varphi_0 - \tilde{\varphi}_0, \psi) + O_C(\mathbf{w}_0 + \mathbf{u}_1(\phi_0), \varphi_0 - \tilde{\varphi}_0, \psi) &= -\, O_C(\mathbf{w}_0 - \tilde{\mathbf{w}}_0 + \mathbf{u}_1(\phi_0) - \mathbf{u}_1(\tilde{\phi}_0), \tilde{\varphi}_0, \psi) \\ &\quad + G_{\mathbf{w}_0, \phi_0}(\psi) - G_{\tilde{\mathbf{w}}_0, \tilde{\phi}_0}(\psi) \quad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega), \end{aligned} \tag{2.60}$$

where

$$
\begin{aligned}
G_{\mathbf{w}_0,\phi_0}(\psi) - G_{\tilde{\mathbf{w}}_0,\tilde{\phi}_0}(\psi) =& \frac{\varphi_{\rm in}}{2}\int_{\Gamma_{\rm out}} \psi([(\mathbf{w}_0+\mathbf{u}_1(\phi_0))\cdot\mathbf{n}]^- - [(\tilde{\mathbf{w}}_0+\mathbf{u}_1(\tilde{\phi}_0))\cdot\mathbf{n}]^-) \\
&- \varphi_{\rm in} a_2 \int_{\Gamma_{\rm m}} \psi(\phi_0-\tilde{\phi}_0) \quad \forall\,\psi\in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega).
\end{aligned}
\tag{2.61}
$$

Then in particular for $\psi = \varphi_0 - \tilde{\varphi}_0$ in (2.60), from (1.1), the second estimate in (2.8) and the Cauchy Schwarz inequality, we deduce that

$$
\begin{aligned}
\kappa\|\varphi_0-\tilde{\varphi}_0\|^2_{1,\Omega} \lesssim & \|\mathbf{w}_0-\tilde{\mathbf{w}}_0+\mathbf{u}_1(\phi_0)-\mathbf{u}_1(\tilde{\phi}_0)\|_{1,\Omega}\|\tilde{\varphi}_0\|_{1,\Omega}\|\varphi_0-\tilde{\varphi}_0\|_{1,\Omega} \\
&+ \varphi_{\rm in}\|\varphi_0-\tilde{\varphi}_0\|_{1,\Omega}\|\mathbf{w}_0-\tilde{\mathbf{w}}_0+\mathbf{u}_1(\phi_0)-\mathbf{u}_1(\tilde{\phi}_0)\|_{1,\Omega} \\
&+ \varphi_{\rm in} a_2\|\varphi_0-\tilde{\varphi}_0\|_{1,\Omega}\|\phi_0-\tilde{\phi}_0\|_{1,\Omega},
\end{aligned}
$$

which together with (2.18) and (2.59), yield

$$
\|\varphi_0-\tilde{\varphi}_0\|_{1,\Omega} \lesssim \kappa^{-1}(\lambda+\varphi_{\rm in})\|\mathbf{w}_0-\tilde{\mathbf{w}}_0\|_{1,\Omega} + k^{-1}a_2(\lambda+\varphi_{\rm in})\|\phi_0-\tilde{\phi}_0\|_{1,\Omega},
$$

which implies the result. □

Now, we provide a Lipschitz continuity result for $\mathcal{S}$.

Lemma 2.11 *Let $\mathbf{a}=(a_1,a_2,a_3)\in\mathbb{R}^3_+$ satisfying $a_3-\frac{1}{2}a_1\geq 0$ and let $(\mathbf{w}_0,\phi_0)\in\mathbf{V}\times\mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$, with $\phi_0+\varphi_{\rm in}\geq 0$ on $\Gamma_{\rm m}$. For given $\lambda>0$, assume further that (2.54) holds. Then, for all $(\mathbf{w}_0,\phi_0),(\tilde{\mathbf{w}}_0,\tilde{\phi}_0)\in\mathbf{W}_\lambda$, there holds*

$$
\|\mathcal{S}(\mathbf{w}_0,\phi_0)-\mathcal{S}(\tilde{\mathbf{w}}_0,\tilde{\phi}_0)\| \lesssim C_{\mathcal{S}}\|(\mathbf{w}_0,\phi_0)-(\tilde{\mathbf{w}}_0,\tilde{\phi}_0)\|, \tag{2.62}
$$

where $C_{\mathcal{S}}>0$ is given by

$$
C_{\mathcal{S}} = \nu^{-1}a_2(\nu+\rho(\mathbf{C}(\mathbf{a},\mathbf{u}_{in},\varphi_{\rm in})+(a_2+1)\lambda)). \tag{2.63}
$$

Proof. Given $(\mathbf{w}_0,\phi_0)$, $(\tilde{\mathbf{w}}_0,\tilde{\phi}_0)\in\mathbf{W}_\lambda$, we set $\mathbf{u}_0:=\mathcal{S}(\mathbf{w}_0,\phi_0)$ and $\tilde{\mathbf{u}}_0:=\mathcal{S}(\tilde{\mathbf{w}}_0,\tilde{\phi}_0)$, which satisfy

$$
a_F(\mathbf{u}_0,\mathbf{v}) + O_F\big(\mathbf{w}_0+\mathbf{u}_1(\phi_0);\,\mathbf{u}_0,\mathbf{v}\big) = F_{\mathbf{w}_0,\phi_0}(\mathbf{v}) \qquad \forall\,\mathbf{v}\in\mathbf{V}, \tag{2.64}
$$

and

$$
a_F(\tilde{\mathbf{u}}_0,\mathbf{v}) + O_F\big(\tilde{\mathbf{w}}_0+\mathbf{u}_1(\tilde{\phi}_0);\,\tilde{\mathbf{u}}_0,\mathbf{v}\big) = F_{\tilde{\mathbf{w}}_0,\tilde{\phi}_0}(\mathbf{v}) \qquad \forall\,\mathbf{v}\in\mathbf{V}, \tag{2.65}
$$

where the functionals on the right-hand sides are defined as in (2.46). According to Lemma 2.9 we recall that

$$
\|\mathbf{u}_0\|_{1,\Omega}\leq\frac{\lambda}{2} \quad\text{and}\quad \|\tilde{\mathbf{u}}_0\|_{1,\Omega}\leq\frac{\lambda}{2}. \tag{2.66}
$$

Similarly to the proof of Lemma 2.10 we subtract the equations (2.45) and (2.65), and add and subtract suitable terms, to deduce that

$$
a_F(\mathbf{u}_0-\tilde{\mathbf{u}}_0,\mathbf{v}) + O_F(\mathbf{w}_0+\mathbf{u}_1(\phi_0);\mathbf{u}_0-\tilde{\mathbf{u}}_0,\mathbf{v}) = \tilde{F}_{\mathbf{w}_0,\phi_0}(\mathbf{v}), \quad \forall\,\mathbf{v}\in\mathbf{V}, \tag{2.67}
$$

where

$$
\begin{aligned}
\tilde{F}_{\mathbf{w}_0,\phi_0}(\mathbf{v}) :=& - a_F(\mathbf{u}_1(\phi_0)-\mathbf{u}_1(\tilde{\phi}_0),\mathbf{v}) - O_F(\mathbf{w}_0+\mathbf{u}_1(\phi_0);\mathbf{u}_1(\phi_0)-\mathbf{u}_1(\tilde{\phi}_0),\mathbf{v}) \\
&- O_F(\mathbf{w}_0-\tilde{\mathbf{w}}_0+\mathbf{u}_1(\phi_0)-\mathbf{u}_1(\tilde{\phi}_0);\tilde{\mathbf{u}}_0+\mathbf{u}_1(\tilde{\phi}_0),\mathbf{v}).
\end{aligned}
\tag{2.68}
$$

In particular, taking $\mathbf{v} = \mathbf{u}_0 - \tilde{\mathbf{u}}_0$ above, and applying the first estimates in (2.6) and (2.8), we deduce that

$$\begin{aligned}\nu \|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{1,\Omega}^2 \lesssim\; & \nu \|\mathbf{u}_1(\phi_0) - \mathbf{u}_1(\tilde{\phi}_0)\|_{1,\Omega}\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{1,\Omega} \\ & + \rho\|\mathbf{w}_0 + \mathbf{u}_1(\phi_0)\|_{1,\Omega}\|\mathbf{u}_1(\phi_0) - \mathbf{u}_1(\tilde{\phi}_0)\|_{1,\Omega}\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{1,\Omega} \\ & + \rho\|\mathbf{w}_0 - \tilde{\mathbf{w}}_0 + \mathbf{u}_1(\phi_0) - \mathbf{u}_1(\tilde{\phi}_0)\|_{1,\Omega}\|\tilde{\mathbf{u}}_0 + \mathbf{u}_1(\tilde{\phi}_0)\|_{1,\Omega}\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{1,\Omega},\end{aligned} \tag{2.69}$$

which together with (2.17), (2.18) and (2.66), imply

$$\begin{aligned}\nu\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{1,\Omega} \lesssim\; & a_2(\nu + \rho(\mathbf{C}(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}) + (a_2+1)\lambda))\|\phi_0 - \tilde{\phi}_0\|_{1\Omega} \\ & + \rho(\mathbf{C}(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}) + (a_2+1)\lambda)\,\|\mathbf{w}_0 - \tilde{\mathbf{w}}_0\|_{1\Omega}\,,\end{aligned} \tag{2.70}$$

which readily implies the result. □

Now we are in position of establishing the Lipschitz continuity of $\mathcal{J}$.

Lemma 2.12 *Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3_+$ satisfying $a_3 - \frac{1}{2}a_1 \geq 0$ and let $(\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$, with $\phi_0 + \varphi_{\rm in} \geq 0$ on $\Gamma_{\rm m}$. For given $\lambda > 0$, assume further that (2.54) holds. Then, for each $(\mathbf{w}_0, \phi_0), (\tilde{\mathbf{w}}_0, \tilde{\phi}_0) \in \mathbf{W}_\lambda$ there holds*

$$\|\mathcal{J}(\mathbf{w}_0, \phi_0) - \mathcal{J}(\tilde{\mathbf{w}}_0, \tilde{\phi}_0)\| \lesssim C_{\mathcal{J}}\|(\mathbf{w}_0, \phi_0) - (\tilde{\mathbf{w}}_0, \tilde{\phi}_0)\|, \tag{2.71}$$

where $C_{\mathcal{J}}$ is given by

$$C_{\mathcal{J}} := ((1 + C_{\mathcal{L}})C_{\mathcal{S}} + C_{\mathcal{L}}), \tag{2.72}$$

with $C_{\mathcal{L}}$ and $C_{\mathcal{S}}$ being the positive constants defined in (2.56) and (2.63), respectively.

Proof. Recalling the definition of (2.34), from (2.55) and (2.62), we deduce that for each $(\mathbf{w}_0, \phi_0)$, $(\tilde{\mathbf{w}}_0, \tilde{\phi}_0)$, there holds

$$\begin{aligned}\|\mathcal{J}(\mathbf{w}_0, \phi_0) - \mathcal{J}(\tilde{\mathbf{w}}_0, \tilde{\phi}_0)\| & \lesssim (1 + C_{\mathcal{L}})\|\mathcal{S}(\mathbf{w}_0, \phi_0) - \mathcal{S}(\tilde{\mathbf{w}}_0, \tilde{\phi}_0)\|_{1,\Omega} + C_{\mathcal{L}}\|\phi_0 - \tilde{\phi}_0\|_{1,\Omega}, \\ & \lesssim (1 + C_{\mathcal{L}})C_{\mathcal{S}}\|\mathbf{w}_0 - \tilde{\mathbf{w}}_0\|_{1,\Omega} + ((1 + C_{\mathcal{L}})C_{\mathcal{S}} + C_{\mathcal{L}})\|\phi_0 - \tilde{\phi}_0\|_{1,\Omega},\end{aligned} \tag{2.73}$$

which implies the result. □

Finally, we introduce the main result of this section, namely, the well-posedness of problem (2.30).

Theorem 2.13 *Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3_+$ satisfying $a_3 - \frac{1}{2}a_1 \geq 0$ and let $(\mathbf{w}_0, \phi_0) \in \mathbf{V} \times \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$, with $\phi_0 + \varphi_{\rm in} \geq 0$ on $\Gamma_{\rm m}$. For given $\lambda > 0$, assume further that (2.54) holds. In addition, assume that*

$$C_{\mathcal{J}} = ((1 + C_{\mathcal{L}})C_{\mathcal{S}} + C_{\mathcal{L}}) < 1, \tag{2.74}$$

where $C_{\mathcal{L}}$ and $C_{\mathcal{S}}$ are the constants defined in (2.56) and (2.63), respectively. Then, there exists a unique $(\mathbf{u}_0, \varphi_0) \in \mathbf{W}_\lambda$, such that $(\mathbf{u}_0, \varphi_0) = \mathcal{J}(\mathbf{u}_0, \varphi_0)$. Equivalently, there exists a unique $(\mathbf{u}, p, \varphi) = (\mathbf{u}_0 + \mathbf{u}_1(\varphi_0), p, \varphi_0 + \varphi_{\rm in}) \in \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1(\Omega)$ solution to (2.4), with $(\mathbf{u}_0, \varphi_0) \in \mathbf{W}_\lambda$ and

$$\|p\|_{0,\Omega} \lesssim \nu\lambda(1 + a_2) + \lambda^2(1 + a_2^2) + \nu\mathbf{C}(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}) + \mathbf{C}^2(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}). \tag{2.75}$$

Proof. Now, for (2.75) we make use of the inf-sup condition (2.11), the first equation of (2.4) and the first estimates in (2.6)and (2.8), to deduce that

$$\|p\|_{0,\Omega} \lesssim \nu\|\mathbf{u}_0 + \mathbf{u}_1(\varphi_0)\|_{1,\Omega} + \|\mathbf{u}_0 + \mathbf{u}_1(\varphi_0)\|_{1,\Omega}^2,$$

which together with (2.17) and the fact that $\|\mathbf{u}_0\|_{1,\Omega} \leq \lambda$, imply (2.75). □

3 Conforming finite element approach

In this section, we introduce and analyze a conforming finite element scheme for approximating the solution of problem (2.4). We begin by presenting the Galerkin scheme and reviewing the discrete stability properties of the associated bilinear forms. As will be shown in the following subsections, the analysis of the discrete scheme closely parallels that of problem (2.30), which was used to study the continuous problem (2.4).

3.1 Preliminaries

We begin by taking arbitrary piece-wise polynomial finite element subspaces

$$\mathrm{H}_h, \Psi_h \subseteq \mathrm{H}^1(\Omega), \quad \mathbf{H}_h := [\mathrm{H}_h]^2, \quad \mathrm{Q}_h \subseteq \mathrm{L}^2(\Omega). \tag{3.1}$$

We also define the finite element subspaces $\mathbf{H}_{0,h}$ and $\Psi_{0,h}$ given by

$$\mathbf{H}_{0,h} := \{\mathbf{v}_h \in \mathbf{H}_h : \mathbf{v}_h = 0 \text{ on } \Gamma_{\mathrm{out}}^c\}, \quad \Psi_{0,h} := \{\psi_h \in \Psi_h : \psi_h = 0 \text{ on } \Gamma_{\mathrm{in}}\}, \tag{3.2}$$

which clearly satisfy $\mathbf{H}_{0,h} \subseteq \mathbf{H}^1_{\Gamma^c_{\mathrm{out}}}(\Omega)$ and $\Psi_{0,h} \subseteq \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$.

Above, h stands for the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) of diameter h_K, that is $h := \max\{h_K : K \in \mathcal{T}_h\}$.

For the subsequent analysis, from now on we assume that the pair $(\mathbf{H}_h, \mathrm{Q}_h)$ is inf-sup stable, namely: There exists a constant $\widehat{\beta}_F > 0$, independent of h, such that

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \cap \mathbf{H}^1_0(\Omega) \\ \mathbf{v}_h \neq 0}} \frac{b_F(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \widehat{\beta}_F \|q_h\|_{0,\Omega} \quad \forall q_h \in \mathrm{Q}_h \cap \mathrm{L}^2_0(\Omega). \tag{3.3}$$

In addition, we let $\mathrm{P}_1(K)$ be the space of polynomials on K of degree less than or equal to 1, and assume $\mathcal{P}^1_h \subseteq \mathrm{H}_h$, where

$$\mathcal{P}^1_h := \left\{\psi_h \in C(\bar{\Omega}) : \psi_h|_K \in \mathrm{P}_1(K), \quad \forall K \in \mathcal{T}_h\right\}.$$

In turn, for any $\tilde{\Gamma} \subseteq \Gamma$, and $g \in \mathrm{H}^{1/2}(\tilde{\Gamma})$ we define,

$$\mathcal{I}^{\mathrm{SZ}}_{h,\tilde{\Gamma}}(g) := [(\mathcal{I}^{\mathrm{SZ}}_h \circ \tilde{\gamma_0}^{-1} \circ E_{\tilde{\Gamma}})(g)]|_{\tilde{\Gamma}}, \tag{3.4}$$

where $\mathcal{I}^{\mathrm{SZ}}_h : \mathrm{H}^l(\Omega) \to \mathrm{H}_h$ denotes the Scott-Zhang interpolant (see, e.g., [27]), satisfying

$$\|\mathcal{I}^{\mathrm{SZ}}_h(\phi) - \phi\|_{l,\Omega} \leq C h^{l-1} \|\phi\|_{l,\Omega} \quad \forall \phi \in \mathrm{H}^l(\Omega), \quad 1 \leq l \leq 2. \tag{3.5}$$

In what follows, we also employ the vector versions of $\mathcal{I}^{\mathrm{SZ}}_h$ and $\mathcal{I}^{\mathrm{SZ}}_{h,\tilde{\Gamma}}$, denoted by $\boldsymbol{\mathcal{I}}^{\mathrm{SZ}}_h$ and $\boldsymbol{\mathcal{I}}^{\mathrm{SZ}}_{h,\tilde{\Gamma}}$, respectively.

Using (3.5) is possible to obtain an approximation property on $\tilde{\Gamma}$. This result is established in the following Lemma.

Lemma 3.1 *For any function $\phi \in \mathrm{H}^l(\Omega)$ we have that*

$$\|\mathcal{I}^{\mathrm{SZ}}_{h,\tilde{\Gamma}}(\phi|_{\tilde{\Gamma}}) - \phi|_{\tilde{\Gamma}}\|_{0,\tilde{\Gamma}} \lesssim h^{l-1} \|\phi\|_{l,\Omega}, \quad 1 \leq l \leq 2. \tag{3.6}$$

Proof. Given $l \in \{1,2\}$, we notice that $[\tilde{\gamma_0}^{-1}(E_{\tilde{\Gamma}}(\phi|_{\tilde{\Gamma}}))]|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}}$, and apply (1.1) and (3.5), to obtain

$$\begin{aligned} \|\mathcal{I}^{\mathrm{SZ}}_{h,\tilde{\Gamma}}(\phi|_{\tilde{\Gamma}}) - \phi|_{\tilde{\Gamma}}\|_{0,\tilde{\Gamma}} &\le \|\mathcal{I}^{\mathrm{SZ}}_{h}(\tilde{\gamma_0}^{-1}(E_{\tilde{\Gamma}}(\phi))) - \tilde{\gamma_0}^{-1}(E_{\tilde{\Gamma}}(\phi))\|_{0,\Gamma}, \\ &\lesssim \|\mathcal{I}^{\mathrm{SZ}}_{h}(\tilde{\gamma_0}^{-1}(E_{\tilde{\Gamma}}(\phi)) - \tilde{\gamma_0}^{-1}(E_{\tilde{\Gamma}}(\phi))\|_{l,\Omega}, \\ &\lesssim h^{l-1}\|\tilde{\gamma_0}^{-1}(E_{\tilde{\Gamma}}(\phi))\|_{l,\Omega}, \\ &\lesssim h^{l-1}\|\phi\|_{l,\Omega}, \end{aligned} \tag{3.7}$$

which concludes the proof. □

As observed in (3.3), the pair $(\mathbf{H}_h, \mathrm{Q}_h)$ is selected as a stable finite element pair for the Stokes problem. Nevertheless, it is well known that this choice does not, in general, guarantee divergence-free velocity fields. Consequently, at the discrete level, the second equation in the first line of (1.16) may not be necessarily satisfied. In view of this, we will consider discrete versions of the convective terms O_F and O_C, denoted respectively by O_F^h and O_C^h, both of which are linear in the last two components. More precisely, in what follows we consider the well-known skew-symmetric forms (see, e.g., [28]), given by

$$\begin{aligned} O_F^h(\mathbf{w}_h;\mathbf{u}_h,\mathbf{v}_h) &= \rho\int_\Omega [(\mathbf{w}_h\cdot\nabla)\,\mathbf{u}_h]\cdot\mathbf{v}_h + \frac{\rho}{2}\int_\Omega (\operatorname{div}\mathbf{w}_h)\,\mathbf{u}_h\cdot\mathbf{v}_h \\ &\quad -\frac{\rho}{2}\int_{\Gamma_{\mathrm{out}}}(\mathbf{w}_h\cdot\mathbf{n})^-\,\mathbf{u}_h\cdot\mathbf{v}_h \\ O_C^h(\mathbf{w}_h;\varphi_h,\psi_h) &= \int_\Omega (\mathbf{w}_h\cdot\nabla\varphi_h)\,\psi_h + \frac{1}{2}\int_\Omega (\operatorname{div}\mathbf{w}_h)\,\varphi_h\psi_h \\ &\quad -\frac{1}{2}\int_{\Gamma_{\mathrm{out}}}(\mathbf{w}_h\cdot\mathbf{n})^-\,\varphi_h\psi_h - \int_{\Gamma_{\mathrm{m}}}(\mathbf{w}_h\cdot\mathbf{n})\,\varphi_h\psi_h. \end{aligned}$$

We observe that, given $\mathbf{w}_h \in \mathbf{H}_h$, by applying integration by parts, the following properties for O_F^h and O_C^h can be readily derived:

$$O_F^h(\mathbf{w}_h;\mathbf{v}_h,\mathbf{v}_h) \ge 0, \quad \forall \mathbf{v}_h \in \mathbf{H}_{0,h}, \tag{3.8}$$

$$O_C^h(\mathbf{w}_h;\psi_h,\psi_h) \ge -\frac{1}{2}\int_{\Gamma_{\mathrm{m}}}(\mathbf{w}_h\cdot\mathbf{n})\psi_h^2, \quad \forall \psi_h \in \Psi_{0,h}. \tag{3.9}$$

Furthermore, we note that the discrete forms O_F^h and O_C^h are consistent in the following sense: Given $\mathbf{w} \in \mathbf{V}$ (cf. (2.29)), there hold

$$O_F^h(\mathbf{w};\mathbf{u},\mathbf{v}) = O_F(\mathbf{w};\mathbf{u},\mathbf{v}) \quad \text{and} \quad O_C^h(\mathbf{w};\varphi,\psi) = O_C(\mathbf{w};\varphi,\psi), \tag{3.10}$$

for all $\mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\varphi,\psi \in \mathrm{H}^1(\Omega)$.

Then, the Galerkin scheme associated with (2.4) reads: Find $(\mathbf{u}_h, p_h, \varphi_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ such that $\mathbf{u}_h = \tilde{\mathbf{u}}_{\mathrm{in},h}(\varphi_h)$ on Γ^c_{out}, $\varphi_h = \varphi_{\mathrm{in}}$ on Γ_{in}, and

$$\begin{aligned} a_F(\mathbf{u}_h,\mathbf{v}_h) + O_F^h(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) + b_F(\mathbf{v}_h,p_h) &= 0 \quad \forall \mathbf{v}_h \in \mathbf{H}_{0,h}, \\ b_F(\mathbf{u}_h,q_h) &= 0 \quad \forall q_h \in \mathrm{Q}_h, \\ a_C(\varphi_h,\psi_h) + O_C^h(\mathbf{u}_h,\varphi_h,\psi_h) &= 0 \quad \forall \psi_h \in \Psi_{0,h}, \end{aligned} \tag{3.11}$$

where the bilinear forms a_F, b_F, and a_C are defined as in (2.4), and for any $\phi \in \mathrm{H}^1_{\Gamma_{\mathrm{in}}}(\Omega)$ such that $\phi|_{\Gamma_{\mathrm{in}}} = \varphi_{\mathrm{in}}$, and the function $\tilde{\mathbf{u}}_{\mathrm{in},h}(\phi)$ is defined by

$$\tilde{\mathbf{u}}_{\mathrm{in},h}(\phi) := \boldsymbol{\mathcal{I}}^{\mathrm{SZ}}_{h,\Gamma^c_{\mathrm{out}}}(\tilde{\mathbf{u}}_{\mathrm{in}}(\phi)) = \begin{cases} \boldsymbol{\mathcal{I}}^{\mathrm{SZ}}_{h,\Gamma_{\mathrm{in}}}(\mathbf{u}_{\mathrm{in}}) & \text{on} \quad \Gamma_{\mathrm{in}} \\ (a_1 - a_2\mathcal{I}^{\mathrm{SZ}}_{h,\Gamma_{\mathrm{m}}}(\phi^+))\mathbf{n} & \text{on} \quad \Gamma_{\mathrm{m}} \end{cases}, \tag{3.12}$$

where $\tilde{\mathbf{u}}_{\rm in}$ is defined in (2.5).

We notice that, since in the definition of $\tilde{\mathbf{u}}_{\rm in}$ appears the positive part of ϕ_h, namely ϕ^+, $\tilde{\mathbf{u}}_{\rm in}$ does not belong to the trace space of $\mathbf{H}_h$, even if $\phi \in \Psi_h$. This forces us to interpolate $\tilde{\mathbf{u}}_{\rm in}(\phi)$ to define $\tilde{\mathbf{u}}_{{\rm in},h}$.

3.2 Analysis of the discrete problem

We now address the unique solvability of (3.11) by adapting the arguments of the continuous setting to the discrete framework. As a first step, we establish the discrete counterparts of the stability estimates introduced in Section 2.2.1.

3.2.1 Stability properties

First we observe that under assumption (3.1), the bilinear forms a_F, a_C, and b_F are bounded with the same constants as in the continuous case:

$$
\begin{aligned}
|a_F(\mathbf{u}_h,\mathbf{v}_h)| &\lesssim \nu\|\mathbf{u}_h\|_{1,\Omega}\|\mathbf{v}_h\|_{1,\Omega}, \quad \forall \mathbf{u}_h,\mathbf{v}_h \in \mathbf{H}_{0,h},\\
|a_C(\varphi_h,\psi_h)| &\lesssim (\kappa + a_3)\|\varphi_h\|_{1,\Omega}\|\psi_h\|_{1,\Omega}, \quad \forall \varphi_h,\psi_h \in \Psi_{0,h},\\
|b_F(\mathbf{v}_h,q_h)| &\lesssim \|q_h\|_{0,\Omega}\|\mathbf{v}_h\|_{1,\Omega}, \quad \forall \mathbf{v}_h \in \mathbf{H}_{0,h},\ \forall q_h \in \mathrm{Q}_h,
\end{aligned}
\tag{3.13}
$$

As in the continuous setting, we also have the following ellipticity properties of the bilinear forms a_F and a_C:

$$
a_F(\mathbf{v}_h,\mathbf{v}_h) \gtrsim \nu\|\mathbf{v}_h\|_{1,\Omega}^2, \quad \forall \mathbf{v}_h \in \mathbf{H}_{0,h}. \tag{3.14}
$$

$$
a_C(\psi_h,\psi_h) \gtrsim \kappa\|\psi_h\|_{1,\Omega}^2 + a_3\|\psi_h\|_{0,\Gamma_{\rm m}}^2 \gtrsim \kappa\|\psi_h\|_{1,\Omega}^2, \quad \forall \psi_h \in \Psi_{0,h}. \tag{3.15}
$$

Moreover, it is straightforward to verify that, for each $\mathbf{w}_h \in \mathbf{H}_{0,h}$, the bilinear forms $O_F^h(\mathbf{w}_h;\cdot,\cdot)$ and $O_C^h(\mathbf{w}_h;\cdot,\cdot)$ are bounded. More precisely, the following estimates hold:

$$
|O_F^h(\mathbf{w}_h;\mathbf{u}_h,\mathbf{v}_h)| \lesssim \rho\|\mathbf{w}_h\|_{1,\Omega}\|\mathbf{u}_h\|_{1,\Omega}\|\mathbf{v}_h\|_{1,\Omega}, \quad \forall\, \mathbf{w}_h,\mathbf{u}_h,\mathbf{v}_h \in \mathbf{H}_{0,h}, \tag{3.16}
$$

$$
|O_C^h(\mathbf{w}_h;\varphi_h,\psi_h)| \lesssim \|\mathbf{w}_h\|_{1,\Omega}\|\varphi_h\|_{1,\Omega}\|\psi_h\|_{1,\Omega}, \quad \forall\, \mathbf{w}_h \in \mathbf{H}_{0,h},\ \forall\, \varphi_h,\psi_h \in \Psi_{0,h}. \tag{3.17}
$$

Finally, Now we present the discrete counterpart of (2.11).

Lemma 3.2 *Let* $(\mathbf{H}_h,\mathrm{Q}_h)$ *be the pair satisfying* (3.3)*. Then the following estimate holds*

$$
\sup_{\substack{\mathbf{v}_h\in\mathbf{H}_{0,h}\\ \mathbf{v}_h\neq 0}} \frac{\displaystyle\int_\Omega q_h \mathrm{div}\,\mathbf{v}_h}{\|\mathbf{v}_h\|_{1,\Omega}} \gtrsim \|q_h\|_{0,\Omega} \quad \forall q_h \in \mathrm{Q}_h. \tag{3.18}
$$

Proof. The result follows from the orthogonal decomposition $\mathrm{Q}_h = (\mathrm{Q}_h \cap \mathrm{L}_0^2(\Omega)) \oplus \mathbb{R}$, estimate (3.3) and the fact that $\mathbf{H}_h \cap \mathbf{H}_0^1(\Omega) \subseteq \mathbf{H}_{0,h}$. We omit further details. □

3.2.2 Equivalent reduced discrete problem

Similarly to the continuous case, we prove well-posedness of problem (3.11) by means of a reduced equivalent version of problem (3.11). To that end we first introduce a suitable lifting for $\tilde{\mathbf{u}}_{{\rm in},h}(\varphi_h)$. This result is stated in the following lemma.

Lemma 3.3 *Let* $\varphi_{\rm in} \in \mathbb{R}^+$ *and* $\mathbf{u}_{\rm in} \in \mathbf{H}^{1/2}(\Gamma_{\rm in})$ *satisfying* (1.18) *and* (1.19). *Then, for each* $\phi_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$, *there exists* $\mathbf{u}_{1,h}(\phi_0) \in \mathbf{H}_h$ *satisfying*

$$\mathbf{u}_{1,h}(\phi_0)\big|_{\Gamma^c_{\rm out}} = \tilde{\mathbf{u}}_{{\rm in},h}(\phi_0 + \varphi_{\rm in}) \ \textit{and} \ \int_\Omega q_h \operatorname{div} \mathbf{u}_{1,h}(\phi_0) = 0 \quad \forall\, q_h \in \mathrm{Q}_h, \tag{3.19}$$

where $\tilde{\mathbf{u}}_{{\rm in},h}(\phi_{0,h} + \varphi_{\rm in})$ *is given as in* (3.12). *In addition, the following estimate holds*

$$\begin{aligned} &\|\mathbf{u}_{1,h}(\phi_0)\|_{1,\Omega} \lesssim \mathbf{C}(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}) + a_2\|\phi_0\|_{1,\Omega} \quad \forall\, \phi_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega), \\ &\|\mathbf{u}_{1,h}(\phi_0) - \mathbf{u}_{1,h}(\tilde{\phi}_0)\|_{1,\Omega} \lesssim a_2\|\phi_0 - \tilde{\phi}_0\|_{1,\Omega} \quad \forall\, \phi_0, \tilde{\phi}_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega), \end{aligned} \tag{3.20}$$

with $\mathbf{C}(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in})$ *defined in* (2.19).

Proof. Given $\phi_0 \in \mathrm{H}^1_{\Gamma_{\rm in}}(\Omega)$, we define $\mathbf{u}_{1,h}(\phi_0)$ by

$$\mathbf{u}_{1,h}(\phi_0) := \mathbf{z}_{0,h}(\phi_0) + \mathbf{z}_{1,h}(\phi_0), \tag{3.21}$$

where $\mathbf{z}_{1,h}(\phi_0) \in \mathbf{H}_h$ is defined as

$$\mathbf{z}_{1,h}(\phi_0) := \boldsymbol{\mathcal{I}}^{\rm SZ}_h(\mathbf{z}_1(\phi_0)) \in \mathbf{H}_h, \tag{3.22}$$

with $\mathbf{z}_1(\phi_0)$ given in (2.21), and $\mathbf{z}_{0,h}(\phi_0) \in \mathbf{H}_{0,h}$ is the first component of the unique solution of the discrete Stokes problem: Find $(\mathbf{z}_{0,h}(\phi_0), r_h(\phi_0)) \in \mathbf{H}_{0,h} \times \mathrm{Q}_h$, such that

$$\begin{aligned} \int_\Omega \nabla \mathbf{z}_{0,h}(\phi_0) : \nabla \mathbf{v_h} + \int_\Omega r_h(\phi_0) \operatorname{div} \mathbf{v_h} &= -\int_\Omega \nabla \mathbf{z}_{1,h}(\phi_0) : \nabla \mathbf{v_h}, \quad \forall\, \mathbf{v}_h \in \mathbf{H}_{0,h}, \\ \int_\Omega s_h \operatorname{div} \mathbf{z}_{0,h}(\phi_0) &= -\int_\Omega s_h \operatorname{div} \mathbf{z}_{1,h}(\phi_0), \qquad \forall\, s_h \in \mathrm{Q}_h. \end{aligned} \tag{3.23}$$

From (2.21), (3.4), (3.12), (3.21) and (3.22), we deduce that

$$\mathbf{u}_{1,h}(\phi_0)\big|_{\Gamma^c_{\rm out}} = \boldsymbol{\mathcal{I}}^{\rm SZ}_h(\mathbf{z}_1(\phi_0))\big|_{\Gamma^c_{\rm out}} = \boldsymbol{\mathcal{I}}^{\rm SZ}_{h,\Gamma^c_{\rm out}}(\tilde{\mathbf{u}}_{\rm in}(\phi_0 + \varphi_{\rm in})) = \tilde{\mathbf{u}}_{{\rm in},h}(\phi_0 + \varphi_{\rm in}),$$

which establishes the first identity in (3.19). In turn, the second identity in (3.19) can be easily deduced from the second equation of (3.23).

Now, for (3.20) we first recall that the inf-sup stability property of the pair $(\mathbf{H}_{0,h}, \mathrm{Q}_h \cap \mathrm{L}^2_0(\Omega))$ and the Babuška–Brezzi theory (see [14]) ensure that the solution of (3.23) satisfies

$$\|\mathbf{z}_{0,h}(\phi_0)\|_{1,\Omega} + \|r_h(\phi_0)\|_{0,\Omega} \lesssim \|\mathbf{z}_{1,h}(\phi_0)\|_{1,\Omega}.$$

From this estimate, (3.21), estimate (3.1) with $l = 1$ and the triangle inequality, we deduce that

$$\|\mathbf{u}_{1,h}(\phi_0)\|_{1,\Omega} \lesssim \|\mathbf{z}_{1,h}(\phi_0)\|_{1,\Omega} = \|\boldsymbol{\mathcal{I}}^{\rm SZ}_h(\mathbf{z}_1(\phi_0))\|_{1,\Omega} \lesssim \|\mathbf{z}_1(\phi_0)\|_{1,\Omega}.$$

In this way, combining the latter with the definition of $\mathbf{z}_1(\phi_0)$ in (2.21), and proceeding analogously to (2.28), we obtain

$$\|\mathbf{u}_{1,h}(\phi_0)\|_{1,\Omega} \lesssim \|\mathbf{z}_1(\phi_0)\|_{1,\Omega} \lesssim \|\tilde{\mathbf{u}}_{\rm in}(\phi_0 + \varphi_{\rm in})\|_{1/2,\Gamma^c_{\rm out}} \lesssim \mathbf{C}(\mathbf{a}, \mathbf{u}_{in}, \varphi_{\rm in}) + a_2\|\phi_0\|_{1,\Omega}.$$

We end the proof by noticing that the proof of the second estimate in (3.20) follows analogously as in Lemma 2.3. □

Now we let

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}_{0,h} : b_F(\mathbf{v}_h, q_h) = 0 \text{ for all } q_h \in \mathrm{Q}_h\}, \tag{3.24}$$

and observe that $\mathbf{u}_{1,h}(\varphi_{0,h}) \in \mathbf{V}_h$ for all $\phi_0 \in \Psi_{0,h}$. Then, according to the above definitions, now we introduce the following reduced version of problem (3.11): Find $(\mathbf{u}_{0,h}, \varphi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}$, such that

$$\begin{aligned} a_F(\mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\varphi_{0,h}), \mathbf{v}_h) + O_F^h(\mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\varphi_{0,h}); \mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\varphi_{0,h}), \mathbf{v}_h) &= 0 \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ a_C(\varphi_{0,h} + \varphi_{\mathrm{in}}, \psi_h) + O_C^h(\mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\varphi_{0,h}); \varphi_{0,h} + \varphi_{\mathrm{in}}, \psi_h) &= 0 \quad \forall\, \psi_h \in \Psi_{0,h}, \end{aligned} \tag{3.25}$$

where $\mathbf{u}_{1,h}(\varphi_{0,h})$ is the discrete lifting introduced in Lemma 3.3 with $\phi_0 = \varphi_{0,h} \in \Psi_{0,h}$, and $\mathbf{V}_h$ given by:

The following Lemma establishes the equivalence between problems (3.11) and (3.25).

Lemma 3.4 *If $(\mathbf{u}_h, p_h, \varphi_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ with $\mathbf{u}_h = \tilde{\mathbf{u}}_{\mathrm{in}}(\varphi_h)$ on Γ^c_{out}, $\varphi_h = \varphi_{\mathrm{in}}$ on Γ_{in}, is a solution of (3.11), then $(\mathbf{u}_{0,h}, \varphi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}$, with $\mathbf{u}_{0,h} = \mathbf{u}_h - \mathbf{u}_{1,h}(\varphi_h) \in \mathbf{V}_h$ and $\varphi_{0,h} = \varphi_h - \varphi_{\mathrm{in}} \in \Psi_{0,h}$ is a solution of (3.25). Conversely, if $(\mathbf{u}_{0,h}, \varphi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}$ is a solution of (3.25), then there exists $p_h \in \mathrm{Q}_h$, such that $(\mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\varphi_{0,h}), p_h, \varphi_{0,h} + \varphi_{\mathrm{in}}) \in \mathbf{H}_h \times Q_h \times \Psi_h$ is a solution of (3.11).*

Proof. The proof follows from the definition of the lifting $\mathbf{u}_{1,h}(\varphi_{0,h})$ in Lemma 3.3, the inf-sup condition (3.18) and the definition of $\mathbf{V}_h$. We omit further details and refer the reader to [24, Lemma 2.1] for a similar result. □

From this point forward, our attention will be directed towards establishing the well-posedness of the problem (3.25). To that end, and analogously to the continuous case, now we introduce an equivalent fixed-point problem associated to (3.25).

3.2.3 The discrete fixed-point operator

To prove the well-posedness of problem (3.25), we proceed analogously to the continuous case by using a fixed-point approach. We start by defining the auxiliary operator $\mathcal{L}_h : \mathbf{V}_h \times \Psi_{0,h} \to \Psi_{0,h}$ given by

$$\mathcal{L}_h(\mathbf{w}_{0,h}, \phi_{0,h}) := \varphi_{0,h}, \tag{3.26}$$

with $\varphi_{0,h} \in \Psi_{0,h}$ being the unique solution (to be confirmed below) of the linearized problem: Find $\varphi_{0,h} \in \Psi_{0,h}$, such that

$$a_C(\varphi_{0,h} + \varphi_{\mathrm{in}}, \psi_h) + O_C^h(\mathbf{w}_{0,h} + \mathbf{u}_{1,h}(\phi_{0,h}), \varphi_{0,h} + \varphi_{\mathrm{in}}, \psi_h) = 0 \quad \forall\, \psi_h \in \Psi_{0,h}. \tag{3.27}$$

In addition we let $\mathcal{S}_h : \mathbf{V}_h \times \Psi_{0,h} \to \mathbf{V}_h$, the operator given by

$$\mathcal{S}_h(\mathbf{w}_{0,h}, \phi_{0,h}) := \mathbf{u}_{0,h}, \tag{3.28}$$

with $\mathbf{u}_{0,h} \in \mathbf{V}_h$ being the unique solution (to be confirmed below) of problem: Find $\mathbf{u}_{0,h} \in \mathbf{V}_h$ such that

$$a_F(\mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\phi_{0,h}), \mathbf{v}_h) + O_F^h(\mathbf{w}_{0,h} + \mathbf{u}_{1,h}(\phi_{0,h}); \mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\phi_{0,h}), \mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h. \tag{3.29}$$

Thus, we let $\mathcal{J}_h : \mathbf{V}_h \times \Psi_{0,h} \to \mathbf{V}_h \times \Psi_{0,h}$ the operator given by

$$\mathcal{J}_h(\mathbf{w}_{0,h}, \phi_{0,h}) := (\mathcal{S}_h(\mathbf{w}_{0,h}, \phi_{0,h}), \mathcal{L}_h(\mathcal{S}_h(\mathbf{w}_{0,h}, \phi_{0,h}), \phi_{0,h})) \quad \forall (\mathbf{w}_{0,h}, \phi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}, \tag{3.30}$$

and realize that solving problem (3.25) is equivalent to seeking a unique solution of the fixed point problem: Find $(\mathbf{u}_{0,h}, \varphi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}$, such that

$$\mathcal{J}_h(\mathbf{u}_{0,h}, \varphi_{0,h}) = (\mathbf{u}_{0,h}, \varphi_{0,h}). \tag{3.31}$$

According to the above, in what follows we focus on proving that (3.31), has a unique solution. Before doing that, as in the continuous case we first prove that $\mathcal{J}_h$ is well defined.

3.2.4 Welldefiniteness of $\mathcal{J}_h$

Since $\mathcal{J}_h$ is defined in terms of operators $\mathcal{L}_h$ and $\mathcal{S}_h$, which in turn are associated to the uncoupled problems (3.27) and (3.29), respectively, the well-definiteness of $\mathcal{J}_h$ reduces, as in the continuous case, to proving the well-posedness of (3.27) and (3.29). We begin with the analysis of the discrete problem (3.27).

Lemma 3.5 *Let $\mathbf{a} \in \mathbb{R}^3_+$ satisfying $a_3 - \frac{1}{2}a_1 \geq 0$ and $(\mathbf{w}_{0,h}, \phi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}$ be such that*

$$a_2 \|\phi_{0,h} + \varphi_{\rm in}\|_{1,\Omega} \lesssim \kappa. \tag{3.32}$$

Then there exists a unique $\varphi_{0,h} := \mathcal{L}_h(\mathbf{w}_{0,h}, \phi_{0,h}) \in \Psi_{0,h}$ solution of (3.27). In addition, the following estimate holds

$$\|\mathcal{L}_h(\mathbf{w}_{0,h}, \phi_{0,h})\|_{1,\Omega} \lesssim \kappa^{-1} \varphi_{\rm in} \left(a_3 + a_1 + \kappa + \|\mathbf{w}_{0,h}\|_{1,\Omega} + \|\mathbf{u}_{1,h}(\phi_{0,h})\|_{1,\Omega} \right). \tag{3.33}$$

Proof. We begin by noting that, given $(\mathbf{w}_{0,h}, \phi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}$, problem (3.27) can be reformulated as: Find $\varphi_{0,h} \in \Psi_{0,h}$, such that

$$a_C(\varphi_{0,h}, \psi_h) + O_C^h(\mathbf{w}_{0,h} + \mathbf{u}_{1,h}(\phi_{0,h}), \varphi_{0,h}, \psi_h) = G_{\mathbf{w}_{0,h},\phi_{0,h}}(\psi_h) \qquad \forall\, \psi_h \in \Psi_{0,h}, \tag{3.34}$$

where the linear functional $G_{\mathbf{w}_{0,h},\phi_{0,h}}$ on the right-hand side is given by

$$G_{\mathbf{w}_{0,h},\phi_{0,h}}(\psi_h) := -a_C(\varphi_{\rm in}, \psi_h) - O_C^h(\mathbf{w}_{0,h} + \mathbf{u}_{1,h}(\phi_{0,h}), \varphi_{\rm in}, \psi_h). \tag{3.35}$$

Thus, to prove the well-posedness of (3.34), it suffices, by the Lax–Milgram Lemma, to show that the bilinear form on the left-hand side and the functional on the left-hand side are coercive and bounded, respectively, on $\Psi_{0,h}$.

First, using (3.9), (3.12) and the first identity in (3.19) we obtain the following inequality:

$$\begin{aligned} O_C^h(\mathbf{w}_{0,h} + \mathbf{u}_{1,h}(\phi_{0,h}), \psi_h, \psi_h) &\geq -\frac{1}{2}\int_{\Gamma_{\rm m}} (a_1 - a_2 \mathcal{I}^{\rm SZ}_{h,\Gamma_{\rm m}}((\phi_{0,h} + \varphi_{\rm in})^+))\psi_h^2 \\ = -\frac{a_1}{2}\int_{\Gamma_{\rm m}} \psi_h^2 + \frac{a_2}{2}\int_{\Gamma_{\rm m}} (\phi_{0,h} + \varphi_{\rm in})^+ \psi_h^2 &+ \frac{a_2}{2}\int_{\Gamma_{\rm m}} (\mathcal{I}^{\rm SZ}_{h,\Gamma_{\rm m}}((\phi_{0,h} + \varphi_{\rm in})^+) - (\phi_{0,h} + \varphi_{\rm in})^+)\psi_h^2, \end{aligned} \tag{3.36}$$

for all $\psi_h \in \Psi_{0,h}$. In turn, using Lemma 3.1 with $l = 1$, we deduce that

$$\frac{a_2}{2}\widehat{C}_\Gamma^2 \|\mathcal{I}^{\rm SZ}_{h,\Gamma_{\rm m}}((\phi_{0,h} + \varphi_{\rm in})^+) - (\phi_{0,h} + \varphi_{\rm in})^+\|_{0,\Gamma_{\rm m}} \lesssim \frac{a_2}{2}\|\phi_{0,h} + \varphi_{\rm in}\|_{1,\Omega}. \tag{3.37}$$

Then, from (1.1), (1.6), (3.15), (3.36), (3.37), and the Cauchy–Schwarz inequality, we conclude that

$$\begin{aligned} a_C(\psi_h,\psi_h) + O_C^h(\mathbf{w}_{0,h}+\mathbf{u}_{1,h}(\phi_{0,h}),\psi_h,\psi_h) \\ &\gtrsim \kappa\|\psi_h\|_{1,\Omega}^2 + (a_3 - \frac{1}{2}a_1)\|\psi_h\|_{0,\Gamma_\mathrm{m}}^2 + \frac{a_2}{2}\int_{\Gamma_\mathrm{m}} (\phi_{0,h}+\varphi_\mathrm{in})^+\psi_h^2 \\ &\quad - \frac{a_2}{2}\|\mathcal{I}_{h,\Gamma_\mathrm{m}}^{\mathrm{SZ}}((\phi_{0,h}+\varphi_\mathrm{in})^+) - (\phi_{0,h}+\varphi_\mathrm{in})^+\|_{0,\Gamma_\mathrm{m}}\|\psi_h\|_{\mathrm{L}^4(\Gamma_\mathrm{m})}^2 \\ &\gtrsim \kappa\|\psi_h\|_{1,\Omega}^2 - \frac{a_2}{2}C\|\phi_{0,h}+\varphi_\mathrm{in}\|_{1,\Omega}\|\psi_h\|_{1,\Omega}^2. \end{aligned} \tag{3.38}$$

This estimate and (3.32) imply

$$a_C(\psi_h,\psi_h) + O_C^h(\mathbf{w}_{0,h}+\mathbf{u}_{1,h}(\phi_{0,h}),\psi_h,\psi_h) \gtrsim \kappa\|\psi_h\|_{1,\Omega}^2, \qquad \forall\, \psi_h \in \Psi_{0,h}. \tag{3.39}$$

On the other hand, from (3.12) and the first identity in (3.19), we obtain

$$\begin{aligned} G_{\mathbf{w}_{0,h},\phi_{0,h}}(\psi_h) &= -a_3\varphi_\mathrm{in}\int_{\Gamma_\mathrm{m}}\psi_h + \varphi_\mathrm{in}\int_{\Gamma_\mathrm{m}}\psi_h(a_1 - a_2\mathcal{I}_{h,\Gamma_\mathrm{m}}^{\mathrm{SZ}}((\phi_{0,h}+\varphi_\mathrm{in})^+)) \\ &\quad + \frac{\varphi_\mathrm{in}}{2}\int_{\Gamma_\mathrm{out}}\psi_h((\mathbf{w}_{0,h}+\mathbf{u}_{1,h}(\phi_{0,h}))\cdot\mathbf{n})^-. \end{aligned}$$

The latter, together with (1.1), the first identity in (1.14), (3.5) and [27, Corollary 4.1], imply

$$|G_{\mathbf{w}_{0,h},\phi_{0,h}}(\psi_h)| \le \hat{C}_1\varphi_\mathrm{in}\left(a_3 + a_1 + \kappa + \|\mathbf{w}_{0,h}\|_{1,\Omega} + \|\mathbf{u}_1(\phi_{0,h})\|_{1,\Omega}\right)\|\psi_h\|_{1,\Omega}. \tag{3.40}$$

In this way, from (3.39), (3.40) and the Lax-Milgram Lemma, we obtain the well-posedness of problem (3.34) and estimate (3.33). □

In analogy with the continuous setting, we next examine the solvability of the discrete problem (3.29). For the sake of simplicity, we omit further technical details, as the arguments follow closely those already employed in the continuous case.

Lemma 3.6 *Let $(\mathbf{w}_{0,h},\phi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}$. Then, there exists a unique solution $\mathbf{u}_{0,h} \in \mathbf{V}_h$ to problem* (3.29)*, given by*

$$\mathbf{u}_{0,h} := \mathcal{S}_h(\mathbf{w}_{0,h},\phi_{0,h}),$$

and satisfying the estimate

$$\|\mathcal{S}_h(\mathbf{w}_{0,h},\phi_{0,h})\|_{1,\Omega} \le \hat{C}_2\nu^{-1}\left(\nu + \rho\|\mathbf{w}_{0,h}\|_{1,\Omega} + \rho\|\mathbf{u}_{1,h}(\phi_{0,h})\|_{1,\Omega}\right)\|\mathbf{u}_{1,h}(\phi_{0,h})\|_{1,\Omega}, \tag{3.41}$$

where $\hat{C}_2 > 0$ is a constant independent of the data.

Proof. As in the continuous case, problem (3.29) can be reformulated as follows: Find $\mathbf{u}_{0,h} \in \mathbf{V}_h$ such that

$$a_F(\mathbf{u}_{0,h},\mathbf{v}_h) + O_F^h(\mathbf{w}_{0,h}+\mathbf{u}_{1,h}(\phi_{0,h});\mathbf{u}_{0,h},\mathbf{v}_h) = F_{\mathbf{w}_{0,h},\phi_{0,h}}(\mathbf{v}_h) \quad \forall\,\mathbf{v}_h \in \mathbf{V}_h, \tag{3.42}$$

where the linear functional $F_{\mathbf{w}_{0,h},\phi_{0,h}}$ on the right-hand side is defined by

$$F_{\mathbf{w}_{0,h},\phi_{0,h}}(\mathbf{v}_h) := -a_F(\mathbf{u}_{1,h}(\phi_{0,h}),\mathbf{v}_h) - O_F^h(\mathbf{w}_{0,h}+\mathbf{u}_{1,h}(\phi_{0,h});\mathbf{u}_{1,h}(\phi_{0,h}),\mathbf{v}_h). \tag{3.43}$$

Then, following the same arguments as in Lemma 2.7, we obtain (3.41). □

We now establish the solvability of (3.25) by analyzing the equivalent fixed-point equation (3.31). To this end, we verify the conditions of Brouwer's fixed-point theorem, stated as follows (see, e.g., [10, Theorem 9.9-2]).

Theorem 3.7 *Let W be a compact and convex subset of a finite-dimensional Banach space X, and let $T : W \to W$ be continuous. Then T admits at least one fixed point.*

Accordingly, for $\lambda_d > 0$, we consider the closed ball in $\mathbf{V}_h \times \Psi_{0,h}$, defined by :

$$\mathbf{W}_h(\lambda_d) := \{(\mathbf{w}_{0,h}, \phi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h} : \|(\mathbf{w}_{0,h}, \phi_{0,h})\| \leq \lambda_d\}\,. \tag{3.44}$$

The following result provides the discrete analogue of Lemma (**??**), showing that, under appropriate assumptions on the data, $\mathcal{J}_h(\mathbf{W}_h(\lambda_d)) \subseteq \mathbf{W}_h(\lambda_d)$.

Lemma 3.8 *Let $\mathbf{C}_{\text{in}}$ be defined as in (2.19) and $\mathbf{a} \in \mathbb{R}^3_+$ satisfying $a_3 - \frac{1}{2}a_1 \geq 0$. Given $\lambda_d > 0$, and $\mathbf{W}_h(\lambda_d)$ defined as in (3.44). We assume that the data satisfy the following conditions:*

$$a_2 C\lambda_d + a_2\varphi_{\text{in}}C|\Omega|^{1/2} \leq C_P\kappa, \tag{3.45}$$

$$C_1\varphi_{\text{in}}\kappa^{-1}\widehat{\mathbf{R}}_{\mathcal{L}} \max\{1, \lambda_d\} \leq \frac{\lambda_d}{2}, \tag{3.46}$$

$$C_2(\mathbf{R}_{\mathcal{S}} + a_2\nu^{-1} + a_2\rho\nu^{-1} + \rho\nu^{-1}\mathbf{C}_{\text{in}} + \rho\nu^{-1}a_2^2)\max\{1, \lambda_d, \lambda_d^2\} \leq \frac{\lambda_d}{2}, \tag{3.47}$$

where $C_1, C_2 > 0$ are constants independent of the data. Then it follows that

$$\mathcal{J}_h(\mathbf{W}_h(\lambda_d)) \subseteq \mathbf{W}_h(\lambda_d). \tag{3.48}$$

Proof. Given $(\mathbf{w}_{0,h}, \phi_{0,h}) \in \mathbf{W}_h(\lambda_d)$, by the definition (3.44) and the triangle inequality, we obtain

$$a_2 C\|\phi_{0,h} + \varphi_{\text{in}}\|_{1,\Omega} \leq a_2 C\lambda_d + a_2 C\varphi_{\text{in}}|\Omega|^{1/2},$$

which, in view of (3.45), ensures that condition (3.32) is satisfied. This implies that there exists a unique

$$\varphi_{0,h} := \mathcal{L}_h(\mathbf{w}_{0,h}, \phi_{0,h}) \in \Psi_{0,h}$$

solving (3.27). Moreover, from (3.46), inequality (3.20), the continuous dependence estimate (3.33), together with (3.45) and the triangle inequality, we deduce that there exists a constant $C_1 > 0$, independent of $(\mathbf{w}_{0,h}, \phi_{0,h})$, such that

$$\begin{aligned}
\|\mathcal{L}_h(\mathbf{w}_{0,h}, \phi_{0,h})\|_{1,\Omega} &\leq \widehat{C}_1\varphi_{\text{in}}\kappa^{-1}\big(\widehat{\mathbf{R}}_{\mathcal{L}} + a_2\|\phi_{0,h}\|_{1,\Omega} + \|\mathbf{w}_{0,h}\|_{1,\Omega}\big) && (3.49)\\
&\leq C_1\varphi_{\text{in}}\kappa^{-1}\big(\widehat{\mathbf{R}}_{\mathcal{L}} + a_2\lambda_d + \lambda_d\big) && (3.50)\\
&\leq C_1\varphi_{\text{in}}\kappa^{-1}\big(\widehat{\mathbf{R}}_{\mathcal{L}} + a_2 + 1\big)\max\{1, \lambda_d\} && (3.51)\\
&\leq \tfrac{\lambda_d}{2}, && (3.52)
\end{aligned}$$

where

$$\widehat{\mathbf{R}}_{\mathcal{L}} := a_3 + a_1 + \kappa + \mathbf{C}_{\text{in}}. \tag{3.53}$$

Next, from (3.46), (3.20), and the continuous dependence estimate (3.41), there exists a constant $C_2 > 0$, also independent of $(\mathbf{w}_{0,h}, \phi_{0,h})$, and h such that

$$\begin{aligned}
\|\mathcal{S}_h(\mathbf{w}_{0,h}, \phi_{0,h})\|_{1,\Omega} &\leq C_2\Big(\mathbf{R}_{\mathcal{S}} + a_2\nu^{-1}\|\phi_{0,h}\|_{1,\Omega} + a_2\rho\nu^{-1}\|\mathbf{w}_{0,h}\|_{1,\Omega}\|\phi_{0,h}\|_{1,\Omega}\\
&\qquad + \rho\nu^{-1}\mathbf{C}_{\text{in}}\|\mathbf{w}_{0,h}\|_{1,\Omega} + \rho\nu^{-1}a_2^2\|\phi_{0,h}\|^2_{1,\Omega}\Big)\\
&\leq C_2\big(\mathbf{R}_{\mathcal{S}} + a_2\nu^{-1}\lambda_d + a_2\rho\nu^{-1}\lambda_d^2 + \rho\nu^{-1}\mathbf{C}_{\text{in}}\lambda_d + \rho\nu^{-1}a_2^2\lambda_d^2\big)\\
&\leq C_2\big(\mathbf{R}_{\mathcal{S}} + a_2\nu^{-1} + a_2\rho\nu^{-1} + \rho\nu^{-1}\mathbf{C}_{\text{in}} + \rho\nu^{-1}a_2^2\big)\max\{1, \lambda_d, \lambda_d^2\}\\
&\leq \tfrac{\lambda_d}{2},
\end{aligned} \tag{3.54}$$

where

$$\mathbf{R}_{\mathcal{S}} := \mathbf{C}_{\text{in}}(1 + \rho\nu^{-1}\mathbf{C}_{\text{in}}). \tag{3.55}$$

Finally, from the definition of the operator $\mathcal{J}_h$ (cf. (3.30)), together with (3.49), (3.51), (3.54), and (3.47), we obtain

$$\begin{aligned}\|\mathcal{J}_h(\mathbf{w}_{0,h}, \phi_{0,h})\|_{1,\Omega} &\leq \|\mathcal{S}_h(\mathbf{w}_{0,h}, \phi_{0,h})\|_{1,\Omega} + \|\mathcal{L}_h(\mathcal{S}_h(\mathbf{w}_{0,h}, \phi_{0,h}), \phi_{0,h})\|_{1,\Omega} \\ &\leq \tfrac{\lambda_d}{2} + C_1\varphi_{\text{in}}\kappa^{-1}\big(\widehat{\mathbf{R}}_{\mathcal{L}} + a_2\|\phi_{0,h}\|_{1,\Omega} + \|\mathcal{S}_h(\mathbf{w}_{0,h}, \phi_{0,h})\|_{1,\Omega}\big) \\ &\leq \tfrac{\lambda_d}{2} + C_1\varphi_{\text{in}}\kappa^{-1}\big(\widehat{\mathbf{R}}_{\mathcal{L}} + a_2\lambda_d + \tfrac{\lambda_d}{2}\big) \\ &\leq \tfrac{\lambda_d}{2} + C_1\varphi_{\text{in}}\kappa^{-1}\big(\widehat{\mathbf{R}}_{\mathcal{L}} + a_2 + 1\big)\max\{1, \lambda_d\} \\ &\leq \tfrac{\lambda_d}{2} + \tfrac{\lambda_d}{2} = \lambda_d,\end{aligned}$$

which concludes the proof of (3.48). □

Next, we present the discrete analogues of Lemmas 2.10 and 2.11. Since their proofs are either analogous or closely related to the continuous case, they are omitted. We only remark that Lemma 3.9 is established almost verbatim from Lemma 2.10, while Lemma 3.10 follows by similar arguments; hence, we refrain from giving further details.

Lemma 3.9 *Let $\lambda_d > 0$, and $\mathbf{W}_h(\lambda_d)$ as in (3.44). Assume that $\mathbf{a} \in \mathbb{R}^3_+$ satisfies $a_3 - \frac{1}{2}a_1 \geq 0$. Then $\mathcal{L}_h$ is a Lipschitz operator. That is, for all $(\mathbf{w}_{0,h}, \phi_{0,h}), (\tilde{\mathbf{w}}_{0,h}, \tilde{\phi}_{0,h}) \in \mathbf{W}_h(\lambda_d)$ it satisfies*

$$\|\mathcal{L}_h(\mathbf{w}_{0,h}, \phi_{0,h}) - \mathcal{L}_h(\tilde{\mathbf{w}}_{0,h}, \tilde{\phi}_{0,h})\| \leq \widehat{c}_1 C_{\mathcal{L}}\|(\mathbf{w}_{0,h}, \phi_{0,h}) - (\tilde{\mathbf{w}}_{0,h}, \tilde{\phi}_{0,h})\|, \tag{3.56}$$

where $C^2_{\mathcal{L}}$ is given by:

$$C_{\mathcal{L}} = \kappa^{-1}(\lambda_d + \varphi_{\text{in}} + a_2\varphi_{\text{in}}), \tag{3.57}$$

and $\widehat{c}_1$ being a positive constant with does not depend on data.

Lemma 3.10 *Let $\lambda_d > 0$, and $\mathbf{W}_h(\lambda_d)$ as in (3.44). Then $\mathcal{S}_h$ is a Lipschitz operator. That is, for all $(\mathbf{w}_{0,h}, \phi_{0,h}), (\tilde{\mathbf{w}}_{0,h}, \tilde{\phi}_{0,h}) \in \mathbf{W}_h(\lambda_d)$, it satisfies*

$$\|\mathcal{S}_h(\mathbf{w}_0, \phi_0) - \mathcal{S}_h(\tilde{\mathbf{w}}_{0,h}, \tilde{\phi}_{0,h})\| \leq \widehat{c}_2 C_{\mathcal{S}}\|(\mathbf{w}_{0,h}, \phi_{0,h}) - (\tilde{\mathbf{w}}_{0,h}, \tilde{\phi}_{0,h})\|, \tag{3.58}$$

where $C_{\mathcal{S}}$ is given by

$$C_{\mathcal{S}} = a_2 + \nu^{-1}\rho\,(2a_2 + 1)\,(\mathbf{C}_{\text{in}} + \lambda_d + a_2\lambda_d) \tag{3.59}$$

with $\widehat{c}_2$ being a positive constant which does not depend on the data.

As a direct consequence of the preceding lemmas, we now establish the continuity of the operator $\mathcal{J}_h$.

Lemma 3.11 *Let $\lambda_d > 0$, and $\mathbf{W}_h(\lambda_d)$ as in (3.44). Assume that $\mathbf{a} \in \mathbb{R}^3_+$ satisfies $a_3 - \frac{1}{2}a_1 \geq 0$. Then $\mathcal{J}_h$ is a Lipschitz operator. That is, for all $(\mathbf{w}_{0,h}, \phi_{0,h}), (\tilde{\mathbf{w}}_{0,h}, \tilde{\phi}_{0,h}) \in \mathbf{W}_h(\lambda_d)$, it satisfies*

$$\|\mathcal{J}_h(\mathbf{w}_{0,h}, \phi_{0,h}) - \mathcal{J}_h(\tilde{\mathbf{w}}_{0,h}, \tilde{\phi}_{0,h})\| \leq \widehat{c}_3 C_{\mathcal{J}}\|(\mathbf{w}_{0,h}, \phi_{0,h}) - (\tilde{\mathbf{w}}_{0,h}, \tilde{\phi}_{0,h})\|, \tag{3.60}$$

where $C^2_{\mathcal{J}}$ is given by

$$C^2_{\mathcal{J}} = (1 + C^2_{\mathcal{L}})C_{\mathcal{S}} + C^2_{\mathcal{L}} + (1 + C^2_{\mathcal{L}})C^2_{\mathcal{S}} \tag{3.61}$$

with $\widehat{c}_3$ being a positive constant which does not depend on the data $C_{\mathcal{L}}$ and $C_{\mathcal{S}}$ depending on the data are defined in (3.57) and (3.59) respectively.

We are now in a position to establish the existence of a fixed point of the operator $\mathcal{J}_h$. Under suitable assumptions on the data, this follows from Brouwer's fixed-point theorem (cf. 3.7), and is formalized in the following result.

Theorem 3.12 *Let $\lambda_d > 0$, and let $\mathbf{W}_h(\lambda_d)$ be defined as in (3.44). Assume that $\mathbf{a} \in \mathbb{R}^3_+$ satisfies $a_3 - \frac{1}{2}a_1 \geq 0$. Furthermore, suppose that the data satisfy*

$$C_1 \varphi_{\text{in}} \kappa^{-1} (\widehat{\mathbf{R}}_{\mathcal{L}} + a_2 + 1) \max\{1, \lambda_d\} \leq \frac{\lambda_d}{2}, \tag{3.62}$$

$$C_2 (\mathbf{R}_{\mathcal{S}} + a_2 \nu^{-1} + a_2 \rho \nu^{-1} + \rho \nu^{-1} \mathbf{C}_{\text{in}} + \rho \nu^{-1} a_2^2) \max\{1, \lambda_d, \lambda_d^2\} \leq \frac{\lambda_d}{2}, \tag{3.63}$$

with $C_1, C_2 > 0$ being constants with does not depend on data. Then, problem (3.25) has a unique solution $(\mathbf{u}_{0,h}, \varphi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}$, with $(\mathbf{u}_{0,h}, \varphi_{0,h}) \in \mathbf{W}_h(\lambda_d)$. Moreover, there hold

$$\|\varphi_{0,h}\|^2_{1,\Omega} \leq C_1 \varphi_{\text{in}} \kappa^{-1} (\widehat{\mathbf{R}}_{\mathcal{L}} + a_2 + 1) \max\{1, \lambda_d\},$$
$$\|\mathbf{u}_{0,h}\|^2_{1,\Omega} \leq C_2 (\mathbf{R}_{\mathcal{S}} + a_2 \nu^{-1} + a_2 \rho \nu^{-1} + \rho \nu^{-1} \mathbf{C}_{\text{in}} + \rho \nu^{-1} a_2^2) \max\{1, \lambda_d, \lambda_d^2\}.$$

Furthermore, by imposing stronger conditions on the data to guarantee that $\mathcal{J}_h$ is a contraction mapping, we derive the ensuing existence and uniqueness theorem for problem (3.25).

Theorem 3.13 *Let $\lambda_d > 0$, and let $\mathbf{W}_h(\lambda_d)$ be as in (3.44). In addition to the assumptions of Theorem 3.12, suppose that the data satisfy*

$$\widehat{c}_3^2 (1 + C_{\mathcal{L}}^2) C_{\mathcal{S}} + C_{\mathcal{L}}^2 + (1 + C_{\mathcal{L}}^2) C_{\mathcal{S}}^2 < 1. \tag{3.64}$$

Then, problem (3.25) admits a unique solution $(\mathbf{u}_{0,h}, \varphi_{0,h}) \in \mathbf{V}_h \times \Psi_{0,h}$, with $(\mathbf{u}_{0,h}, \varphi_{0,h}) \in \mathbf{W}_h(\lambda_d)$. Moreover, the same a priori estimates as in Theorem 3.12 hold.

Proof. The result follows directly from Theorem 3.12, together with assumption (3.64) and the Banach fixed-point theorem. □

4 A priori error analysis

In this section we derive an a priori error estimate for our Galerkin scheme with arbitrary finite element subspaces satisfying the hypotheses stated in Section 3.1. More precisely, given $(\mathbf{u}, p, \varphi) \in \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1(\Omega)$ and $(\mathbf{u}_h, p_h, \varphi_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ solutions of the continuous and discrete problems (2.4) and (3.11), respectively, we are interested in obtaining an upper bound for

$$\|(\mathbf{u}, p, \varphi) - (\mathbf{u}_h, p_h, \varphi_h)\|.$$

For this purpose, in what follows we set $(\mathbf{u}, p, \varphi) = (\mathbf{u}_0 + \mathbf{u}_1(\varphi_0), p, \varphi_0 + \varphi_{\text{in}})$, with $(\mathbf{u}_0, \varphi_0, p) \in \mathbf{H}^1_{\Gamma^c_{\text{out}}}(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega)$ satisfying

$$\begin{array}{rcll} \widehat{a}_F(\mathbf{u}_0, \mathbf{v}) + b_F(\mathbf{v}, p) & = & F_{\mathbf{u}_0, \varphi_0}(\mathbf{v}), & \forall\, \mathbf{v} \in \mathbf{H}^1_{\Gamma^c_{\text{out}}}(\Omega), \\ b_F(\mathbf{u}_0, q) & = & H_{\varphi_0}(q), & \forall\, q \in \mathrm{L}^2(\Omega), \\ \widehat{a}_C(\varphi_0, \psi) & = & G_{\mathbf{u}_0, \varphi_0}(\psi), & \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\text{in}}}(\Omega), \end{array} \tag{4.1}$$

with

$$\widehat{a}_F(\mathbf{w},\mathbf{v}) := a_F(\mathbf{w},\mathbf{v}) \,+\, O_F(\mathbf{u}_0+\mathbf{u}_1(\varphi_0);\mathbf{w},\mathbf{v}), \qquad H_{\varphi_0}(q) = -b_F(\mathbf{u}_1(\varphi_0),q), \\ \widehat{a}_C(\phi,\psi) := a_C(\phi,\psi) + O_C(\mathbf{u}_0+\mathbf{u}_1(\varphi_0),\phi,\psi), \tag{4.2}$$

where $F_{\mathbf{u}_0,\varphi_0}$ and $G_{\mathbf{u}_0,\varphi_0}$ are defined in (2.46) and (2.39) respectively.

Similarly, we set $(\mathbf{u}_h, p_h, \varphi_h) = (\mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\varphi_{0,h}), p_h, \varphi_{0,h} + \varphi_{\text{in}})$, with $(\mathbf{u}_{0,h}, p_h, \varphi_{0,h}) \in \mathbf{H}_{0,h} \times Q_h \times \Psi_{0,h}$ satisfying

$$\begin{array}{lcll} \widehat{a}^h_F(\mathbf{u}_{0,h},\mathbf{v}_h) \,+\, b_F(\mathbf{v}_h,p) & = & F_{\mathbf{u}_{0,h},\varphi_{0,h}}(\mathbf{v}_h), & \forall\,\mathbf{v}_h \in \mathbf{H}_{0,h}, \\ b_F(\mathbf{u}_{0,h},q_h) & = & H_{\varphi_{0,h}}(q_h), & \forall\,q_h \in Q_h, \\ \widehat{a}^h_C(\varphi_{0,h},\psi_h) & = & G_{\mathbf{u}_{0,h},\varphi_{0,h}}(\psi_h), & \forall\,\psi_h \in \Psi_{0,h}, \end{array} \tag{4.3}$$

with

$$\widehat{a}^h_F(\mathbf{w}_h,\mathbf{v}_h) := a_C(\mathbf{w}_h,\mathbf{v}_h) + O^h_C(\mathbf{u}_{0,h}+\mathbf{u}_{1,h}(\varphi_{0,h}),\mathbf{w}_h,\mathbf{v}_h), \quad H_{\varphi_{0,h}}(q_h) = -b_F(\mathbf{u}_{1,h}(\varphi_{0,h}),q_h), \\ \widehat{a}^h_C(\phi_h,\psi_h) := a_C(\phi_h,\psi_h) + O_C(\mathbf{u}_0+\mathbf{u}_1(\varphi_0),\phi_h,\psi_h), \tag{4.4}$$

where $F_{\mathbf{u}_{0,h},\varphi_{0,h}}$ and $G_{\mathbf{u}_{0,h},\varphi_{0,h}}$ are defined as in (3.43) and (3.35), respectively.

Since the first two equations in (4.1) and (4.3) share the same structure as the Strang-type estimate in Lemma 4.2, and the last equations in (4.1) and (4.3) correspond to the structure in Lemma 4.1, for the error analysis of our problem we recall two abstract results from [9, Theorem 26.1] and [9, Theorem 11.2], which will be employed in the subsequent analysis.

The first is the standard Strang Lemma for elliptic variational problems, which will be directly applied to the last equation in (4.1) together with the last equation in (4.3). The second is a generalized Strang-type estimate for saddle-point problems, where the continuous and discrete schemes differ only in the functionals involved, as is the case for the first two equations in (4.1) and (4.3).

We are now in a position to introduce the aforementioned abstract results, which will play a central role in the subsequent error analysis.

Lemma 4.1 *Let* H *be a Hilbert space,* $F \in \mathrm{H}'$*, and* $a : \mathrm{H}\times\mathrm{H} \to \mathbb{R}$ *a bounded and* H*-elliptic bilinear form. In addition, let* $\{\mathrm{H}_h\}_{h>0}$ *be a sequence of finite dimensional subspaces of* H*, and for each* $h>0$ *consider a bounded bilinear form* $a_h : \mathrm{H}_h\times\mathrm{H}_h \to \mathbb{R}$ *and a functional* $F_h \in \mathrm{H}'_h$*. Assume that the family* $\{a_h\}_{h>0}$ *is uniformly elliptic, that is, there exists a constant* $\tilde{\alpha} > 0$*, independent of* h*, such that*

$$a_h(v_h,v_h) \geq \tilde{\alpha}\|v_h\|^2_{\mathrm{H}} \quad \forall v_h \in \mathrm{H}_h,\ \forall h>0.$$

In turn, let $u \in \mathrm{H}$ *and* $u_h \in \mathrm{H}_h$ *be such that*

$$a(u,v) = F(v) \quad \forall v \in \mathrm{H}, \quad \textit{and} \quad a_h(u_h,v_h) = F_h(v_h) \quad \forall v_h \in \mathrm{H}_h.$$

Then, for each $h>0$ *there holds*

$$\|u-u_h\|_{\mathrm{H}} \leq C_{ST}\Bigg\{ \sup_{\substack{w_h\in\mathrm{H}_h\\ w_h\neq 0}} \frac{|F(w_h)-F_h(w_h)|}{\|w_h\|_{\mathrm{H}}} \\ + \inf_{\substack{v_h\in\mathrm{H}_h\\ v_h\neq 0}} \left(\|u-v_h\|_{\mathrm{H}} + \sup_{\substack{w_h\in\mathrm{H}_h\\ w_h\neq 0}} \frac{|a(v_h,w_h)-a_h(v_h,w_h)|}{\|w_h\|_{\mathrm{H}}} \right) \Bigg\}, \tag{4.5}$$

where $C_{ST} := \tilde{\alpha}^{-1}\max\{1, \|a\|\}$.

Lemma 4.2 *Let* H *and* Q *be Hilbert spaces,* $F \in \mathrm{H}'$, $G \in Q'$, *and let* $a : \mathrm{H} \times \mathrm{H} \to \mathbb{R}$ *and* $b : H \times Q \to \mathbb{R}$ *be bounded bilinear forms satisfying the hypotheses of the Babuka-Brezzi theory. Furthermore, let* $\{\mathrm{H}_h\}_{h>0}$ *and* $\{\mathrm{Q}_h\}_{h>0}$ *be sequences of finite dimensional subspaces of* H *and* Q, *respectively, and for each* $h > 0$ *consider functionals* $F_h \in \mathrm{H}_h'$ *and* $G_h \in Q_h'$. *In addition, assume that* a_h *and* b *satisfy the hypotheses of the discrete Babuka-Brezzi theory uniformly on* H_h *and* Q_h, *that is, there exist positive constants* $\bar{M}$, $\bar{\alpha}$ *and* $\bar{\beta}$, *independent of* h, *such that, denoting by* V_h *the discrete kernel of* b, *there holds*

$$a_h(\varphi_h, \psi_h) \leq \|a_h\|\|\varphi_h\|_{\mathrm{H}}\|\psi_h\|_{\mathrm{H}} \qquad \forall \varphi_h, \psi_h \in \mathrm{H}_h, \tag{4.6}$$

and

$$a_h(v_h, v_h) \geq \alpha_h\|v_h\|_{\mathrm{H}}^2 \qquad \forall v_h \in V_h \quad \text{and} \quad \sup_{\substack{\psi_h \in H_h \\ \psi_h \neq 0}} \frac{b(\psi_h, \xi_h)}{\|\psi_h\|_H} \;\geq\; \bar{\beta}\,\|\xi_h\|_Q \qquad \forall \xi_h \in \mathrm{Q}_h. \tag{4.7}$$

In turn, let $(\varphi, \omega) \in \mathrm{H} \times Q$ *and* $(\varphi_h, \omega_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, *such that*

$$\begin{aligned} a(\varphi, \psi) + b(\psi, \omega) &= F(\psi) \qquad && \forall \psi \in \mathrm{H}, \\ b(\varphi, \xi) &= G(\xi) && \forall \xi \in Q, \end{aligned}$$

and

$$\begin{aligned} a_h(\varphi_h, \psi_h) + b(\psi_h, \omega_h) &= F_h(\psi_h) \qquad && \forall \psi_h \in \mathrm{H}_h, \\ b(\varphi_h, \xi_h) &= G_h(\xi_h) && \forall \xi_h \in \mathrm{Q}_h. \end{aligned}$$

Then, for each $h > 0$ *there holds*

$$\begin{aligned} \|\varphi - \varphi_h\|_{\mathrm{H}} + \|\omega - \omega_h\|_Q \leq \bar{C}_{ST}\Bigg\{ &\inf_{\psi_h \in \mathrm{H}_h}\left(\|\varphi - \psi_h\|_{\mathrm{H}} + \sup_{\eta_h \in \mathrm{H}_h} \frac{a(\psi_h, \eta_h) - a_h(\psi_h, \eta_h)}{\|\eta_h\|_{\mathrm{H}}}\right) \\ &+ \inf_{\mu_h \in \mathrm{Q}_h} \|\omega - \mu_h\|_Q \quad + \sup_{\eta_h \in \mathrm{H}_h} \frac{|F(\eta_h) - F_h(\eta_h)|}{\|\eta_h\|_{\mathrm{H}}} \\ &+ \sup_{v_h \in \mathrm{Q}_h} \frac{|G(v_h) - G_h(v_h)|}{\|v_h\|_Q}\Bigg\} \end{aligned} \tag{4.8}$$

where $\bar{C}_{ST}$ is a positive constant depending only on $\|a\|$, $\|b\|$, $\|a_h\|$, $\bar{\alpha}$ and $\bar{\beta}$.

Lemma 4.3 *Let* $(\mathbf{u}, p, \varphi) \in \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *and* $(\mathbf{u}_h, p_h, \varphi_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ *be solutions of the continuous and discrete problems* (2.4) *and* (3.11), *respectively. Assume that*

$$c_1(\mathbf{C}_{\text{in}} + (a_2 + \rho)\lambda_d) \leq \frac{1}{2}. \tag{4.9}$$

Then, the following estimate holds:

$$\begin{aligned} \|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq c_1 &\inf_{\mathbf{w}_h \in \mathbf{H}_{0,h}} \|\mathbf{u}_0 - \mathbf{w}_h\|_{1,\Omega} + c_1 \inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \\ &+ c_2(\mathbf{C}_{\text{in}} + (a_2 + \rho)\lambda_d + \lambda + a_2(\lambda + \lambda_d) + \nu)\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega}, \end{aligned} \tag{4.10}$$

where c_1 *and* c_2 *are positive constants independent of* h.

Proof. From the definitions of $\widehat{a}_F$ and $\widehat{a}_F^h$ in (4.2) and (4.4), together with the inequalities in (2.6) and (2.8), we conclude that the bilinear forms $\widehat{a}_F$ and $\widehat{a}_F^h$ are both bounded and elliptic, with the same ellipticity constant given in (2.48). Moreover, $F_{\mathbf{u}_0,\varphi_0}$ and $F_{\mathbf{u}_{0,h},\varphi_{0,h}}$ are bounded linear functionals. Therefore, a direct application of Lemma (4.2) to the context (4.1) and (4.3) yields

$$
\begin{aligned}
\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \le C \Bigg\{ & \inf_{\mathbf{w}_h \in \mathbf{H}_{0,h}} \left(\|\mathbf{u}_0 - \mathbf{w}\|_{1,\Omega} + \sup_{\mathbf{v}_h \in \mathbf{H}_{0,h}} \frac{\widehat{a}_F(\mathbf{w}_h, \mathbf{v}_h) - \widehat{a}_F^h(\mathbf{w}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathrm{H}}} \right) \\
& + \inf_{\mu_h \in \mathrm{Q}_h} \|p - \mu_h\|_{0,\Omega} \quad + \sup_{\mathbf{v}_h \in \mathbf{H}_{0,h}} \frac{|F_{\mathbf{u}_0,\varphi_0}(\mathbf{v}_h) - F_{\mathbf{u}_{0,h},\varphi_{0,h}}(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{1,\Omega}} \\
& + \sup_{q_h \in \mathrm{Q}_h} \frac{|H_{\varphi_0}(q_h) - H_{\varphi_{0,h}}(q_h)|}{\|q_h\|_{0,\Omega}} \Bigg\}. \qquad (4.11)
\end{aligned}
$$

We now proceed to estimate each term appearing on the right-hand side of the foregoing inequality.

From the definitions of the functionals H_{φ_0} and $H_{\varphi_{0,h}}$ in (4.2) and (4.4), respectively, we readily deduce that

$$
\sup_{\substack{q_h \in Q_h \\ q_h \neq \theta}} \frac{|H_{\varphi_0}(q_h) - H_{\varphi_{0,h}}(q_h)|}{\|q_h\|_{0,\Omega}} = \sup_{\substack{q_h \in Q_h \\ q_h \neq \theta}} \frac{|b_F(\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h}), q_h)|}{\|q_h\|_{0,\Omega}} \lesssim \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega}. \qquad (4.12)
$$

On the other hand, from the definitions of the functionals $F_{\mathbf{u}_0,\varphi_0}$ and $F_{\mathbf{u}_{0,h},\varphi_{0,h}}$ in (2.46) and (3.43), we readily deduce that

$$
\begin{aligned}
|F_{\mathbf{u}_0,\varphi_0}(\mathbf{v}_h) - F_{\mathbf{u}_{0,h},\varphi_{0,h}}(\mathbf{v}_h)| \le & \, |a_F(\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h}), \mathbf{v}_h)| \\
& + |O_F(\mathbf{u}_0 + \mathbf{u}_1(\varphi_0); \mathbf{u}_1(\varphi_0), \mathbf{v}_h) - O_F^h(\mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\varphi_{0,h}); \mathbf{u}_{1,h}(\varphi_{0,h}), \mathbf{v}_h)|.
\end{aligned} \qquad (4.13)
$$

In what follows, we will bound (4.13). We start by estimating the first term in (4.13). To this end, we observe that, by applying the first inequality in (3.13), it follows that

$$
|a_F(\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h}), \mathbf{v}_h)| \lesssim \nu \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}. \qquad (4.14)
$$

Now, to bound the second term on the right-hand side of (4.13), we add and subtract suitable terms, we apply (2.17), Theorem 2.13 and (3.20) to deduce that

$$
\begin{aligned}
|O_F(\mathbf{u}_0 + \mathbf{u}_1(\varphi_0); \mathbf{u}_1(\varphi_0), \mathbf{v}_h) - & O_F^h(\mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\varphi_{0,h}); \mathbf{u}_{1,h}(\varphi_{0,h}), \mathbf{v}_h)| \\
& \le |O_F^h(\mathbf{u}_0 + \mathbf{u}_1(\varphi_0); \mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h}), \mathbf{v}_h)| \\
& \quad + |O_F^h(\mathbf{u}_0 - \mathbf{u}_{0,h} + \mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h}); \mathbf{u}_{1,h}(\varphi_{0,h}), \mathbf{v}_h)| \\
& \lesssim (\lambda + \mathbf{C}_{\mathrm{in}} + a_2 \lambda) \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega} \\
& \quad + (\mathbf{C}_{\mathrm{in}} + a_2 \lambda_d) \|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega} \\
& \quad + (\mathbf{C}_{\mathrm{in}} + a_2 \lambda_d) \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\| \|\mathbf{v}_h\|_{1,\Omega} \\
& \lesssim (\lambda + \mathbf{C}_{\mathrm{in}} + a_2(\lambda + \lambda_d)) \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega} \\
& \quad + (\mathbf{C}_{\mathrm{in}} + a_2 \lambda_d) \|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}.
\end{aligned} \qquad (4.15)
$$

Consequently, combining (4.13), (4.14), (4.15), and taking the supremum over $\mathbf{H}_{0,h}$, it follows that

$$\sup_{\substack{\mathbf{v}_h\in\mathbf{H}_{0,h}\\ \mathbf{v}_h\neq\theta}} \frac{|F_{\mathbf{u}_0,\varphi_0}(\mathbf{v}_h) - F_{\mathbf{u}_{0,h},\varphi_{0,h}}(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{1,\Omega}} \lesssim (\lambda + \mathbf{C}_{\text{in}} + a_2(\lambda+\lambda_d) + \nu)\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega} + (\mathbf{C}_{\text{in}} + a_2\lambda_d)\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega}. \tag{4.16}$$

Now, from the definitions of $\widehat{a}_F$ and $\widehat{a}_F^h$ in (4.2) and (4.4), together with (3.16), we also obtain that

$$\begin{aligned} |\widehat{a}_F(\mathbf{w}_h,\mathbf{v}_h) - \widehat{a}_F^h(\mathbf{w}_h,\mathbf{v}_h)| =& |O_F^h(\mathbf{u}_0 - \mathbf{u}_{0,h} + \mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h}); \mathbf{w}_h, \mathbf{v}_h)| \\ \lesssim& \rho\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega}\|\mathbf{w}_h\|_{1,\Omega}\|\mathbf{v}_h\|_{1,\Omega} \\ &+ \rho\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega}\|\mathbf{w}_h\|_{1,\Omega}\|\mathbf{v}_h\|_{1,\Omega}, \end{aligned} \tag{4.17}$$

which implies that

$$\sup_{\substack{\mathbf{v}_h\in\mathbf{H}_{0,h}\\ \mathbf{v}_h\neq\theta}} \frac{|\widehat{a}_F(\mathbf{w}_h,\mathbf{v}_h) - \widehat{a}_F^h(\mathbf{w}_h,\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{1,\Omega}} \lesssim \rho(\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} + \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega})\|\mathbf{w}_h\|_{1,\Omega}. \tag{4.18}$$

Then, from the hypothesis (4.9), (4.11), the inequalities (4.12), (4.16) and (4.18), we obtain that

$$\begin{aligned} \frac{1}{2}\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} +& \|p - p_h\|_{0,\Omega} \\ \leq&\ \widehat{c}_1 \inf_{\mathbf{w}_h\in\mathbf{H}_{0,h}} \|\mathbf{u}_0 - \mathbf{w}_h\|_{1,\Omega} + \widehat{c}_1 \inf_{q_h\in Q_h} \|p - p_h\|_{1,\Omega} \\ &+ \widehat{c}_1(\mathbf{C}_{\text{in}} + (a_2+\rho)\lambda_d + \lambda + a_2(\lambda+\lambda_d) + \nu)\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega}, \end{aligned} \tag{4.19}$$

which in turn implies the inequality in (4.10). □

Lemma 4.4 *Let* $(\mathbf{u}, p, \varphi) \in \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *and* $(\mathbf{u}_h, p_h, \varphi_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ *be solutions of the continuous and discrete problems* (2.4) *and* (3.11)*, respectively. Assume that the data satisfy* (4.9)*. Then, the following estimate holds:*

$$\begin{aligned} \|\varphi_0 - \varphi_{0,h}\|_{1,\Omega} \leq&\ \widehat{c}_3 \inf_{\phi_h\in\Psi_{0,h}} \|\varphi_0 - \phi_h\|_{1,\Omega} + \widehat{c}_3(\varphi_{\text{in}} + \lambda_d) \inf_{\mathbf{w}_h\in\mathbf{H}_{0,h}} \|\mathbf{u}_0 - \mathbf{w}_h\|_{1,\Omega} \\ &+ \widehat{c}_3(\varphi_{\text{in}} + \lambda_d) \inf_{q_h\in\mathrm{Q}_h} \|p - p_h\|_{0,\Omega} \\ &+ \widehat{c}_3(\varphi_{\text{in}} + \lambda_d)(\mathbf{C}_{\text{in}} + (a_2+\rho)\lambda_d + \lambda + a_2(\lambda+\lambda_d) + \nu + 1)\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega}. \end{aligned} \tag{4.20}$$

Proof. From the definitions of $\widehat{a}_C$ and $\widehat{a}_C^h$ in (4.2) and (4.4), together with the inequalities in (2.6) and (2.8), we conclude that the bilinear forms $\widehat{a}_F$ and $\widehat{a}_F^h$ are both bounded and elliptic, with the same ellipticity constant α given in (2.41). Moreover, $G_{\mathbf{u}_0,\varphi_0}$ and $G_{\mathbf{u}_{0,h},\varphi_{0,h}}$ are bounded linear functionals. Therefore, a direct application of Lemma (4.1) to the context (4.1) and (4.3) yields

$$\|\varphi_0 - \varphi_{0,h}\|_{\mathrm{H}} \leq C_{ST}\Bigg\{ \sup_{\substack{\psi_h\in \mathrm{H}_{0,h}\\ \psi_h\neq 0}} \frac{|G_{\mathbf{u}_0,\varphi_0}(\psi_h) - G_{\mathbf{u}_{0,h},\varphi_{0,h}}(\psi_h)|}{\|\psi_h\|_{1,\Omega}} + \inf_{\substack{\phi_h\in \mathrm{H}_h\\ \phi_h\neq 0}} \left(\|\varphi_0 - \phi_h\|_{\mathrm{H}} + \sup_{\substack{\psi_h\in \mathrm{H}_h\\ \psi_h\neq 0}} \frac{|\widehat{a}_C(\phi_h,\psi_h) - \widehat{a}_C^h(\phi_h,\psi_h)|}{\|\psi_h\|_{\mathrm{H}}} \right) \Bigg\}, \tag{4.21}$$

where $C_{ST} := \tilde{\alpha}^{-1}\max\{1, \|\widehat{a}_C\|\}$. We now proceed to estimate each term appearing on the right-hand side of the foregoing inequality.

From the definitions of the functionals $G_{\mathbf{u}_0,\varphi_0}$ and $G_{\mathbf{u}_{0,h},\varphi_{0,h}}$ in (2.39) and (3.35), together with (3.17), we readily deduce that

$$\begin{aligned} |G_{\mathbf{u}_0,\varphi_0}(\psi_h) - G_{\mathbf{u}_{0,h},\varphi_{0,h}}(\psi_h)| =& |O_C^h(\mathbf{u}_0 - \mathbf{u}_{0,h} + \mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h}); \varphi_{\mathrm{in}}, \psi_h)| \\ \lesssim & \varphi_{\mathrm{in}}(\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} + \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega})\|\psi_h\|_{1,\Omega}, \end{aligned}$$

which implies that

$$\sup_{\substack{\psi_h\in \Psi_{0,h}\\ \psi_h\neq\theta}} \frac{|G_{\mathbf{u}_0,\varphi_0}(\psi_h) - G_{\mathbf{u}_{0,h},\varphi_{0,h}}(\psi_h)|}{\|\psi_h\|_{1,\Omega}} \lesssim \varphi_{\mathrm{in}}\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} + \varphi_{\mathrm{in}}\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega}. \tag{4.22}$$

Now, from the definitions of $\widehat{a}_C$ and $\widehat{a}_C^h$ in (4.2) and (4.4), together with (3.17)

we also have that

$$\begin{aligned} |\widehat{a}_C(\phi_h,\psi_h) - \widehat{a}_C^h(\phi_h,\psi_h)| =& |O_C^h(\mathbf{u}_0 - \mathbf{u}_{0,h} + \mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h}); \phi_h, \psi_h)| \\ \lesssim & (\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} + \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega})\|\phi_h\|_{1,\Omega}\|\psi_h\|_{1,\Omega}, \end{aligned} \tag{4.23}$$

which implies that

$$\sup_{\substack{\psi_h\in \Psi_{0,h}\\ \psi_h\neq\theta}} \frac{|\widehat{a}_C(\phi_h,\psi_h) - \widehat{a}_C^h(\phi_h,\psi_h)|}{\|\psi_h\|_{1,\Omega}} \lesssim (\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} + \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega})\|\phi_h\|_{1,\Omega}.$$

This later, inequality (4.21) and (4.22) implies

$$\begin{aligned} \|\varphi_0 - \varphi_{0,h}\|_{1,\Omega} \leq \widehat{c}_3 \inf_{\phi_h\in\Psi_{0,h}} \|\varphi_0 - \phi_h\|_{1,\Omega} + \widehat{c}_3(\varphi_{\mathrm{in}} + \lambda_d)\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} \\ + \widehat{c}_3(\varphi_{\mathrm{in}} + \lambda_d)\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega}, \end{aligned} \tag{4.24}$$

wich together with (4.10) and (4.21) implies (4.20). □

From now on we assume that the finite element subspaces introduced in (3.1) and (3.2) satisfies the following approximation properties: There exists $C > 0$, independent of h, such that for each $\mathbf{u} \in \mathbf{H}^{1+s}(\Omega)$, $\varphi \in \mathrm{H}^{1+s}(\Omega)$ and $p \in \mathrm{H}^s(\Omega)$ with $s > 0$.

(AP1) $$\inf_{\mathbf{w}_h\in\mathbf{H}_{0,h}} \|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega} \leq Ch^s\,\|\mathbf{u}\|_{s+1,\Omega}, \tag{4.25}$$

(AP2) $$\inf_{\phi_h\in\Psi_{0,h}} \|\varphi - \phi_h\|_{1,\Omega} \leq Ch^s\,\|\varphi\|_{s+1,\Omega}, \tag{4.26}$$

(AP3) $$\inf_{q_h\in Q_h} \|p - q_h\|_{k,\Omega} \leq Ch^s\,\|p\|_{s,\Omega}. \tag{4.27}$$

Lemma 4.5 *Let $\varphi_0 \in \mathrm{H}^{1+\delta}(\Omega)$, with $\delta > 0$ given in Remark 2.4, and let $\mathbf{z}_0(\varphi_0)$, $r(\varphi_0)$, $\mathbf{z}_1(\varphi_0)$, and $\mathbf{z}_{1,h}(\varphi_0)$ be defined in* (2.21), (2.22), (3.22), *and* (3.23), *respectively. Then the following estimate holds:*

$$\begin{aligned}\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega} \leq Ch^\delta \Big(&\|\mathbf{z}_0(\varphi_0)\|_{1+\delta,\Omega} + \|r(\varphi_0)\|_{\delta,\Omega} + \|\mathbf{z}_1(\varphi_0)\|_{1+\delta,\Omega}\Big) \\ &+ a_2 C\|\varphi_0 - \varphi_{0,h}\|_{1,\Omega}.\end{aligned} \tag{4.28}$$

Proof. We start by noticing that, from the triangle inequality it readly follow that

$$\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega} \leq \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_0)\|_{1,\Omega} + \|\mathbf{u}_{1,h}(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega}. \tag{4.29}$$

Then, we bound each term on the right-hand side of (4.29). For the first term, we recall that a regularity result for the Stokes problem (2.22) implies that $\mathbf{z}_0(\varphi_0) \in \mathbf{H}^{1+\delta}(\Omega)$ and $r(\varphi_0) \in \mathrm{H}^\delta(\Omega)$ (see Remark 2.4), and noticing that, according to the definition of $\mathbf{z}_1(\varphi_0)$ in (2.21) and the fact that $\varphi_0 \in \mathrm{H}^{1+\delta}(\Omega)$, we have that $\mathbf{z}_1(\varphi_0) \in \mathrm{H}^{1+\delta}(\Omega)$. In turn, using standard results on Sobolev space interpolation (see [7, Chapter 14]) and (3.5), we obtain the following approximation property of $\mathcal{I}_h^{\mathrm{SZ}}$.

$$\|\phi - \mathcal{I}_h^{\mathrm{SZ}}(\phi)\|_{1,\Omega} \leq Ch^\delta \|\phi\|_{1+\delta,\Omega}, \forall \phi \in \mathrm{H}^{1+\delta}(\Omega). \tag{4.30}$$

Then applying Lemma 4.2 to the context (2.22) and (3.23), the inequalities (4.25), (4.27), (4.30) and the definition of $\mathbf{z}_{1,h}(\varphi_0)$ in (3.22) we deduce that

$$\begin{aligned}\|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_0)\|_{1,\Omega} &\lesssim \inf_{\mathbf{w}_h \in \mathbf{H}_{0,h}} \|\mathbf{z}_0(\varphi_0) - \mathbf{w}_h\|_{1,\Omega} + \inf_{q_h \in \mathrm{Q}_h} \|r(\varphi_0) - q_h\|_{0,\Omega} \\ &\quad + \|\mathbf{z}_1(\varphi_0) - \mathbf{z}_{1,h}(\varphi_0)\|_{1,\Omega} \\ &= \inf_{\mathbf{w}_h \in \mathbf{H}_{0,h}} \|\mathbf{z}_0(\varphi_0) - \mathbf{w}_h\|_{1,\Omega} + \inf_{q_h \in \mathrm{Q}_h} \|r(\varphi_0) - q_h\|_{0,\Omega} \\ &\quad + \|\mathbf{z}_1(\varphi_0) - \boldsymbol{\mathcal{I}}_h^{\mathrm{SZ}}(\mathbf{z}_1(\varphi_0))\|_{1,\Omega} \\ &\leq Ch^\delta\big(\|\mathbf{z}_0(\varphi_0)\|_{1+\delta,\Omega} + \|r(\varphi_0)\|_{\delta,\Omega} + \|\mathbf{z}_1(\varphi_0)\|_{1+\delta,\Omega}\big).\end{aligned} \tag{4.31}$$

Now, for the second term in (4.29), we apply (**??**) to deduce that

$$\|\mathbf{u}_{1,h}(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega} \lesssim a_2 \|\varphi_0 - \varphi_{0,h}\|_{1,\Omega}. \tag{4.32}$$

Then from (4.29), (4.31) and (4.32), we deduce the desired estimate (4.28). □

Theorem 4.6 *Let $(\mathbf{u}, p, \varphi) \in \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1(\Omega)$ and $(\mathbf{u}_h, p_h, \varphi_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ be the solutions of the continuous and discrete problems* (2.4) *and* (3.11), *respectively. Suppose that*

$$\widehat{c}_4(\mathbf{C}_{\mathrm{in}} + (a_2 + \rho)\lambda_d + \lambda + a_2(\lambda + \lambda_d) + \nu + 1)(\varphi_{\mathrm{in}} + \lambda_d + 1)a_2 \leq \tfrac{1}{2}, \tag{4.33}$$

and also that

$$\mathbf{u}_0 \in \mathbf{H}^{1+\delta}(\Omega), \quad p \in \mathrm{H}^\delta(\Omega), \quad \varphi_0 \in \mathrm{H}^{1+\delta}(\Omega), \tag{4.34}$$

for some $0 < \delta < 1$. Then there exists a constant $C > 0$, independent of the mesh size h, such that

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} + \|\varphi - \varphi_h\|_{1,\Omega} \leq Ch^\delta \Big(&\|\mathbf{u}_0\|_{1+\delta,\Omega} + \|p\|_{\delta,\Omega} + \|\varphi_0\|_{1+\delta,\Omega} \\ &+ \|\mathbf{z}_0(\varphi_0)\|_{1+\delta,\Omega} + \|r(\varphi_0)\|_{\delta,\Omega} + \|\mathbf{z}_1(\varphi_0)\|_{1+\delta,\Omega}\Big).\end{aligned} \tag{4.35}$$

Proof. Applying the fact that $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1(\varphi_0)$, $\varphi = \varphi_0 + \varphi_{\text{in}}$, $\mathbf{u}_h = \mathbf{u}_{0,h} + \mathbf{u}_{1,h}(\varphi_{0,h})$ and $\varphi_h = \varphi_{0,h} + \varphi_{\text{in}}$, the triangle inequality yields

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \leq \|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} + \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega}, \tag{4.36}$$

which together with (4.10) implies that

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} &\leq \|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} + \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega} \\
&\lesssim \inf_{\mathbf{w}_h \in \mathbf{H}_{0,h}} \|\mathbf{u}_0 - \mathbf{w}_h\|_{1,\Omega} + \inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \\
&\quad \widehat{C}_1 \, \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega},
\end{aligned} \tag{4.37}$$

with

$$\widehat{C}_1 := (\mathbf{C}_{\text{in}} + (a_2 + \rho)\lambda_d + \lambda + a_2(\lambda + \lambda_d) + \nu + 1). \tag{4.38}$$

Then, from this latter, the fact that $\varphi - \varphi_h = \varphi_0 - \varphi_{0,h}$ and (4.20) we deduce that

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} + \|\varphi - \varphi_h\|_{1,\Omega} \\
&\qquad \lesssim \inf_{\mathbf{w}_h \in \mathbf{H}_{0,h}} \|\mathbf{u}_0 - \mathbf{w}_h\|_{1,\Omega} + \inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \\
&\qquad\quad + \widehat{C}_1(\varphi_{\text{in}} + \lambda_d + 1) \, \|\mathbf{u}_1(\varphi_0) - \mathbf{u}_{1,h}(\varphi_{0,h})\|_{1,\Omega} \\
&\qquad\quad + (\varphi_{\text{in}} + \lambda_d)\left(\inf_{\mathbf{w}_h \in \mathbf{H}_{0,h}} \|\mathbf{u}_0 - \mathbf{w}_h\|_{1,\Omega} + \inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \right) \\
&\qquad\quad + \widehat{c}_3 \inf_{\phi_h \in \Psi_{0,h}} \|\varphi_0 - \phi_h\|_{1,\Omega}.
\end{aligned} \tag{4.39}$$

Then from this latter, the hypothesis (4.33), (4.28) and (4.38) we deduce that

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} + \tfrac{1}{2}\|\varphi - \varphi_h\|_{1,\Omega} \\
&\quad \leq \widehat{C}_2\left(\inf_{\mathbf{w}_h \in \mathbf{H}_{0,h}} \|\mathbf{u}_0 - \mathbf{w}_h\|_{1,\Omega} + \inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \right) + \widehat{c}_3 \inf_{\phi_h \in \Psi_{0,h}} \|\varphi_0 - \phi_h\|_{1,\Omega} \\
&\qquad + \widehat{C}_1(\varphi_{\text{in}} + \lambda_d + 1) h^{\delta} \big(\|\mathbf{z}_0(\varphi_0)\|_{1+\delta,\Omega} + \|r(\varphi_0)\|_{\delta,\Omega} + \|\mathbf{z}_1(\varphi_0)\|_{1+\delta,\Omega} \big).
\end{aligned} \tag{4.40}$$

with

$$\widehat{C}_2 := \varphi_{\text{in}} + \lambda_d.$$

Finally, combining the estimate (4.40) with the standard interpolation properties of the finite element spaces (AP1), (AP2) and (AP3), we deduce (4.35). □

5 Computational results

In this section, we present a series of experiments to assess the accuracy and robustness of the proposed method in Section 3. The section is divided into two parts, each addressing a different scenario. In the first scenario, we investigate the convergence and stability of the scheme. To this end, unstructured meshes with uniform refinement are employed, and the asymptotic behavior is analyzed for $k \geq 1$. In the second scenario, we consider a typical reverse osmosis system represented on a membrane channel with fully developed parabolic flow and permeable walls.

Unstructured graded meshes with refinement towards the membrane are used to enhance the resolution of the behavior near $\Gamma_{\rm m}$.

The fixed point iterations are performed until the error of two consecutive vector of coefficients $\boldsymbol{U}^j$ and $\boldsymbol{U}^{j+1}$ satisfy the following criteria:

$$\frac{\|\boldsymbol{U}^{j+1} - \boldsymbol{U}^j\|_2}{\|\boldsymbol{U}^{j+1}\|_2} \leq 10^{-7},$$

where $\|\cdot\|_2$ is the usual euclidean 2-norm. All the experiments consider $\boldsymbol{u}_h^0 = \boldsymbol{0}, \varphi_h^0 = 0$ as initial guess.

5.0.1 Convergence and stability test

In order to study the convergence and stability, let us consider the unit square domain $\Omega := (0,1)^2$ whose boundary conditions are distributed in the same manner as the geometry sketch presented in Figure 1.1. We now introduce some additional notations that are useful trougouht this section. The errors are denoted by

$$\mathsf{e}(\boldsymbol{u}) : \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}, \qquad \|p - p_h\|_{0,\Omega}, \qquad \mathsf{e}(\varphi) := \|\varphi - \varphi_h\|_{1,\Omega}.$$

Given two consecutive mesh sizes h and h', the experimental convergence order is computed as

$$r(\cdot) := \frac{\log(\mathsf{e}(\cdot)) - \log(\mathsf{e}'(\cdot))}{\log(h) - \log(h')},$$

where e and e' are two consecutive errors associated with the mesh sizes h and h', respectively.

From the results presented in Table 5.1, we consider a manufacture solution

$$\mathbf{u} = \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ -\cos(\pi y)\sin(\pi x) \end{pmatrix}, \quad p = \cos(\pi x)\exp(y), \quad \text{and} \quad \varphi = (1-2y)(-\cos(\pi y)\sin(\pi x)) + 1,$$

and we observe that optimal convergence rates are achieved for the selected finite elements. More precisely, when using Taylor–Hood+$\mathbb{P}_2$ elements, we obtain a convergence rate of $\mathcal{O}(h^{k+1})$ for the velocity and concentration approximations, and a rate of $\mathcal{O}(h^k)$ for the pressure, with $k = 1, 2$. In the case of the MINI element+$\mathbb{P}_1$, the expected asymptotic rate is $\mathcal{O}(h)$ for all variables, with a superconvergent rate of $\mathcal{O}(h^{1.5})$ for the pressure. This particular behavior was already studied in [11]. For computational efficiency, the mini element is the perfect choice, with a tradeoff in precision. For efficiency vs. accuracy tradeoff, the Taylor–Hood+$\mathbb{P}_2$ configuration offers optimal convergence with moderate DoFs. If maximum accuracy is needed and computational resources permit then the higher order Taylor-Hood combination $\mathbb{P}_3 - \mathbb{P}_2 - \mathbb{P}_3$ is the best choice. In all experiments, the fixed-point iteration reached the prescribed tolerance of 10^{-7} within 13 iterations.

5.0.2 Study of a membrane channel

This experiment aims to test the scheme when we face a membrane channel unit whose length is defined by a subsection of the channel that allows a full parabolic flow development. The domain is defined as $\Omega := (0,a) \times (0,b)$, where $a = 20$mm and $b = 2$mm. Within this domain, we consider the existence of a circular spacer of radius $r = 0.4$mm located initially at the

Table 5.1: Example 5.0.1. Convergence history of the three-field scheme in the square domain using different finite element families.

h	DoFs	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	$e(\varphi)$	$r(\varphi)$	IT
				$\mathbb{P}_{1,b} - \mathbb{P}_1 - \mathbb{P}_1$				
0.0707	1241	2.3538e-01	1.00	5.4039e-02	1.50	1.6831e-01	0.99	13
0.0471	2761	1.5669e-01	1.00	2.9357e-02	1.50	1.1255e-01	1.00	13
0.0354	4881	1.1741e-01	1.00	1.9044e-02	1.50	8.4506e-02	1.00	13
0.0283	7601	9.3874e-02	1.00	1.3612e-02	1.50	6.7640e-02	1.00	13
				$\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_2$				
0.0707	1681	7.6142e-03	2.00	1.3670e-03	2.03	6.2578e-03	1.99	13
0.0471	3721	3.3897e-03	2.00	5.9921e-04	2.02	2.7950e-03	1.99	13
0.0354	6561	1.9080e-03	2.00	3.3535e-04	2.01	1.5757e-03	1.99	13
0.0283	10201	1.2216e-03	2.00	2.1411e-04	2.01	1.0097e-03	2.00	13
				$\mathbb{P}_3 - \mathbb{P}_2 - \mathbb{P}_3$				
0.0707	3721	1.0367e-04	3.01	3.4282e-05	3.12	1.0788e-04	3.00	13
0.0471	8281	3.0563e-05	3.01	9.6635e-06	3.09	3.2023e-05	3.00	13
0.0354	14641	1.2861e-05	3.01	3.9709e-06	3.07	1.3523e-05	3.00	13
0.0283	22801	6.5754e-06	3.00	2.0028e-06	3.04	6.9279e-06	3.00	13

channel center (x_H, y_H) =(10mm, 1.0mm), which generates an inner boundary denoted by $\Gamma_{\rm w}$. This boundary is impermeable, meaning that there is no fluid flow or salt penetration through it. More precisely, we impose the following boundary conditions:

$$\mathbf{u} = \mathbf{0}, \qquad (\varphi\mathbf{u} - \kappa\nabla\varphi)\cdot\mathbf{n} = 0, \quad \text{on } \Gamma_{\rm w}.$$

As stated at the beginning of this section, we consider graded meshes towards $\Gamma_{\rm m}$ in order to maximize the recovery of information at the membrane and to improve stability (see for example [21, 22]).

The physical parameters are in inspired in [23] and summarized in Table 5.2. In particular, we consider the inlet velocity

$$\mathbf{u}_{\rm in} := \begin{pmatrix} \left[u_0 - \dfrac{2x}{b}(a_1 - a_2\varphi_{\rm in})\right]\left[1.5(1-\lambda^2)\right]\left[1 - \dfrac{1}{420}(2 - 7\lambda^2 - 7\lambda^4)\right] \\ (a_1 - a_2\varphi_{\rm in})\left[0.5\lambda(3-\lambda^2) - \dfrac{1}{280}\lambda(2 - 3\lambda^2 + \lambda^6)\right] \end{pmatrix},$$

where $\lambda = 2y/b - 1$ and $u_0 \in \{30, 60, 120\}$ mm/s. This choice of velocity profile is such that we have a fully developed fluid at the entrance of the channel that satisfies the conditions of a permeable channel fluid, where the maximum permeability is concentrated at $\{x^-\}$ and $\{x^+\}$. In that sense, we observe that the compatibility conditions (1.18) and (1.19) are satisfied. This choice is inspired by Berman-type flows (see for example [5]).

Furthermore, we move the spacer by increments of 0.2mm towards the bottom membrane, until the spacer center is located at a height $y = 0.4$ mm from the bottom membrane, which allow us to study the effects on salt accumulation, water recovery, and the formation of recirculation zones along the membrane.

We begin by plotting in Figure 5.1 the velocity behavior along the lower membrane, located at $y = 0$. The permeate velocity reflects the amount of fluid passing through the membrane at each axial location. In particular, the presence of the circular hole perturbs the velocity field and acts as a local recirculation zone generator, thereby modifying the overall flow behavior. All curves exhibit a steady, slight decrease in permeate velocity along the channel. For the first configuration ($y_H = 1.0\,\text{mm}$), higher inlet velocities result in increased permeate velocities throughout the channel, which is consistent with the expected rise in transmembrane pressure. When the hole is located at $y_H = 0.8\,\text{mm}$, a mild drop in permeate velocity appears, especially pronounced in the central region of the domain (between 10–15 mm), and particularly for $u_0 = 120\,\text{mm/s}$. This suggests that the hole introduces a stronger flow perturbation at this height, likely due to interaction with the developing boundary layer or enhanced mixing. When the hole is further lowered to $y_H = 0.6\,\text{mm}$, we observe a local nonzero minimum in permeate velocity. This minimum likely corresponds to the axial location where the disruption of the flow is most significant, owing to the hole's close proximity to the membrane. The resulting disturbance may locally reduce the pressure drop across the membrane or introduce recirculation zones. Moreover, for the highest inlet velocity ($u_0 = 120\,\text{mm/s}$), these effects are more pronounced compared to the other inlet profiles, indicating a stronger sensitivity of the flow to geometric perturbations under higher convective forcing.

For the second set of figures, Figure 5.2, which illustrates the pressure distribution for each hole configuration, we observe that higher inlet velocities u_0 result in higher overall pressure levels, as expected. However, the pattern of pressure drop along the channel varies depending on the vertical position of the hole, indicating differences in flow resistance and the presence of localized recirculation regions. The base case, corresponding to the configuration with the hole furthest from the membrane, exhibits the highest pressure throughout the channel. This configuration likely provides the greatest filtration driving force. In contrast, when the hole is positioned closer to the membrane, the pressure gradient becomes smoother, potentially reducing the risk of membrane fouling due to less severe pressure fluctuations. Nevertheless, this smoother gradient may come at the cost of a lower permeate flow rate, as the driving force for filtration is diminished in the region near the membrane.

Finally, we compare the previous results with the concentration profiles along the bottom membrane of the channel, as shown in Figure 5.3. These profiles help us evaluate solute accumulation and potential fouling risks. When comparing with the corresponding velocity and pressure behaviors, we observe that the cases with hole positions at $y_H = 1.0\,\text{mm}$ and $y_H = 0.8\,\text{mm}$ are quite similar in terms of their influence just downstream of the spacer, exhibiting a slight concentration perturbation after the 10 mm mark. The configuration with the hole at the channel center provides the most favorable conditions for permeation due to higher transmembrane pressure, but it may also increase the risk of membrane fouling due to higher solute accumulation near the surface. In contrast, shifting the spacer 0.2 mm closer to the membrane (i.e., $y_H = 0.8\,\text{mm}$) yields moderate permeation with some lateral flow redistribution and a mild increase in solute concentration. The most critical case occurs when the hole is positioned even closer to the membrane, at $y_H = 0.6\,\text{mm}$, where we observe a tenfold increase in solute concentration near the spacer. This sharp rise, especially between 11 mm and 15 mm, suggests the presence of a recirculation zone that promotes solute accumulation while potentially reducing fouling by limiting direct membrane contact. However, this configuration might also result in underutilized membrane area, excessive flow bypass, or suboptimal separation performance.

We conclude this experiment by presenting in Figures 5.4–5.6 the velocity streamlines, pressure distribution, and solute concentration profiles near the spacer, under the condition of an inlet velocity profile of $u_0 = 120\,\text{mm/s}$. Two spacer configurations are considered: one located at the channel center ($y_H = 1.0\,\text{mm}$) and another placed 0.4 mm from the bottom membrane ($y_H = 0.6\,\text{mm}$). In Figure 5.4, we observe that when the spacer is centered, the velocity streamlines remain symmetric, ensuring uniform membrane utilization across both walls. Conversely, when the spacer is closer to the membrane, the flow is deflected upwards over the hole, resulting in reduced interaction with the bottom membrane and uneven velocity distribution. Figure 5.5 confirms this behavior, showing a lower pressure drop for the off-center case ($y_H = 0.6\,\text{mm}$). This reduction correlates with diminished membrane utilization and smaller permeate velocity, as previously discussed. Finally, the concentration fields shown in Figure 5.6 reveal a clear accumulation of solute downstream of the hole in the $y_H = 0.6\,\text{mm}$ case. A localized concentration peak appears between 11.5 mm and 16.5 mm from the channel inlet, with the top membrane underutilized downstream of the spacer. This configuration leads to a 22% increase in maximum concentration compared to the centered case. The thick, high-concentration region near the bottom membrane suggests a local fouling risk and potential mass transport limitations. In summary, the centered spacer configuration yields more uniform flow and pressure fields, promoting higher membrane utilization and maximum flux. This makes it a favorable option for short-term, high-efficiency operation, though it may require careful fouling monitoring on both membranes. In contrast, the off-center configuration exhibits flow bypass and localized concentration buildup due to recirculation, which may compromise long-term performance unless mitigated by additional design features such as baffles. This observation opens up a direction for future research and optimization studies.

Table 5.2: Test 5.0.2. Physical parameters considered for the membrane channel.

Parameter	Meaning	Approximate value	Units
A	Membrane water permeability	3.75×10^{-6}	mm s^{-1}
ΔP	Hydrostatic transmembrane pressure	5572.875	kg mm^{-1} s^{-2}
i	Number of ions from salt solvation	2	—
R	Ideal gas constant	8.314×10^6	kg m^2 s^{-2} mol^{-1} K^{-1}
T	System temperature	298.0	K^2
ν	Fluid dynamic viscosity	8.9×10^{-7}	mm^2 s^1 kg^{-1}
ρ	Fluid density	1027.2×10^{-9}	mm^2 s^{-1}
κ	Diffusivity of salt in water	1.611×10^{-3}	kg mm^{-1} s^{-1}
B	Membrane salt permeability	5.56×10^{-6}	
u_0	Inlet velocity profile	30 – 120	mm s^{-1}
φ_{in}	Inlet salt molar concentration	600×10^{-9}	mol mm^{-3}

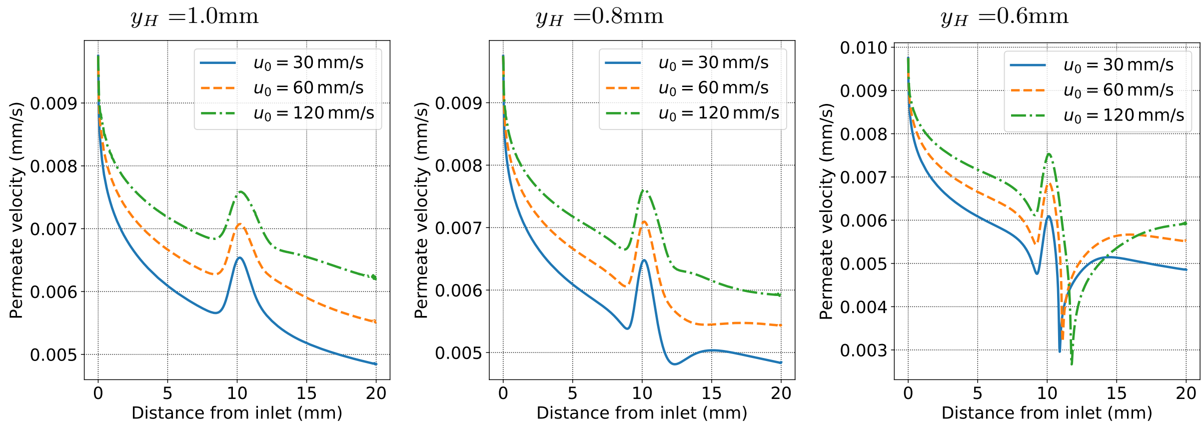

Figure 5.1: Example 5.0.2. Comparison of permeate velocity at $y = 0$mm for different inlet velocity profiles, with the spacer center located at different heights y_H from the bottom membrane.

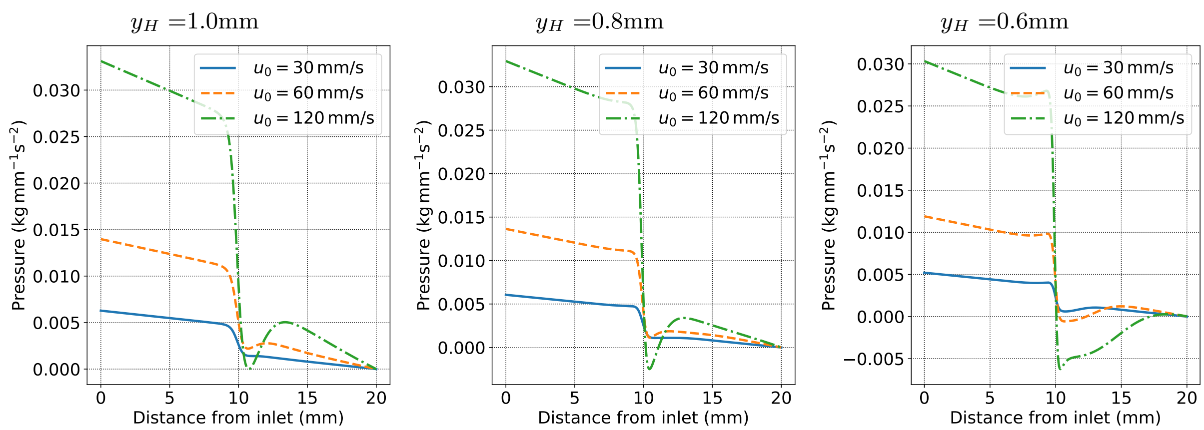

Figure 5.2: Example 5.0.2. Comparison of pressure drop at $y = 0$mm for different inlet velocity profiles, with the spacer center located at different heights y_H from the bottom membrane.

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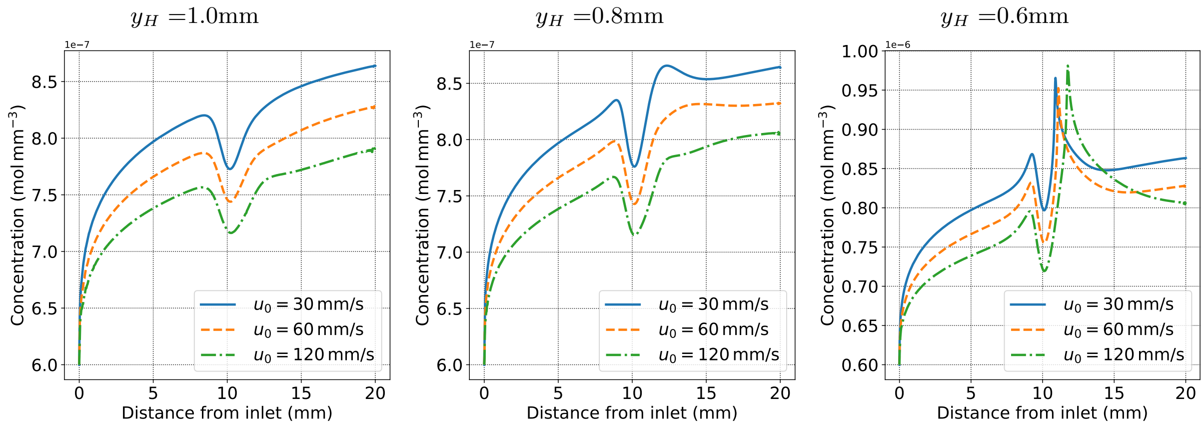

Figure 5.3: Example 5.0.2. Comparison of salt concentration at $y = 0$mm for different inlet velocity profiles, with the spacer center located at different heights y_H from the bottom membrane.

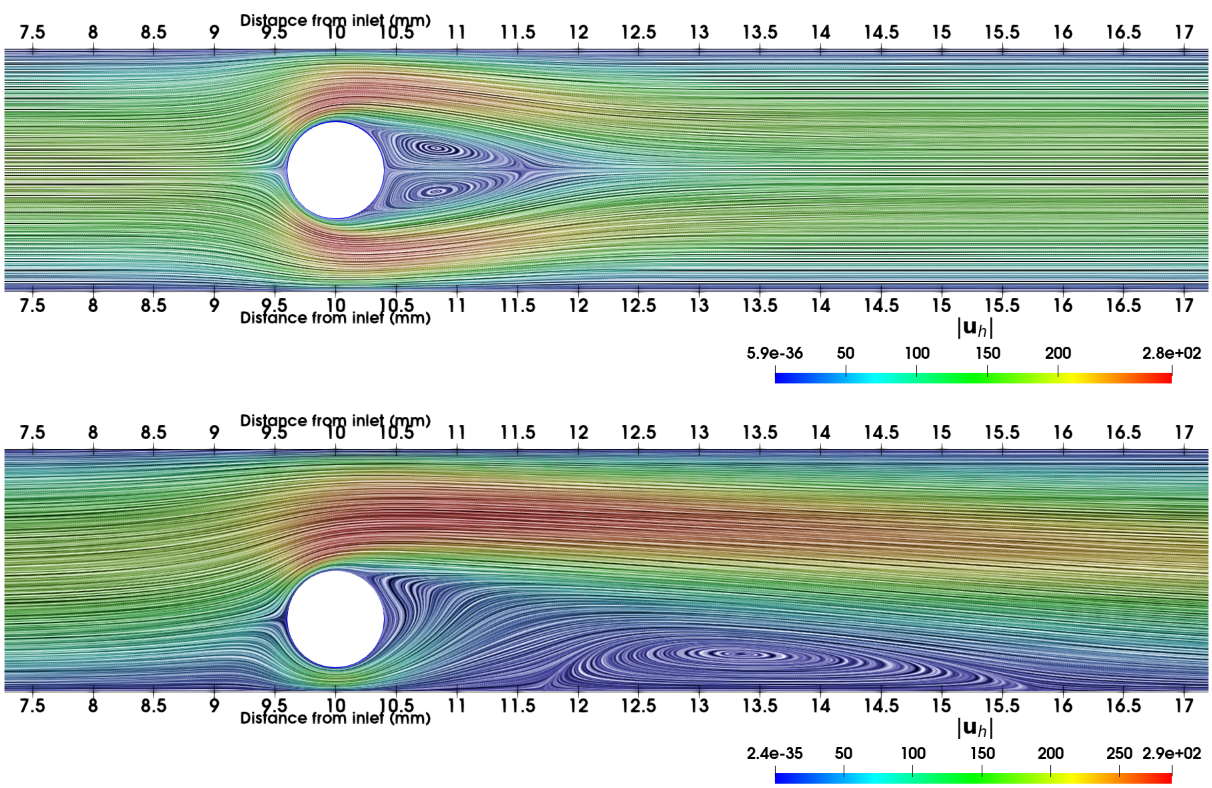

Figure 5.4: Example 5.0.2. Comparison of streamlines for an inlet velocity profile $u_0 = 120\,\mathrm{mm\,s^{-1}}$, with the circular spacer center located at $y = 1\,\mathrm{mm}$ (top) and $y = 0.6\,\mathrm{mm}$ (bottom) from the bottom membrane.

194–212.

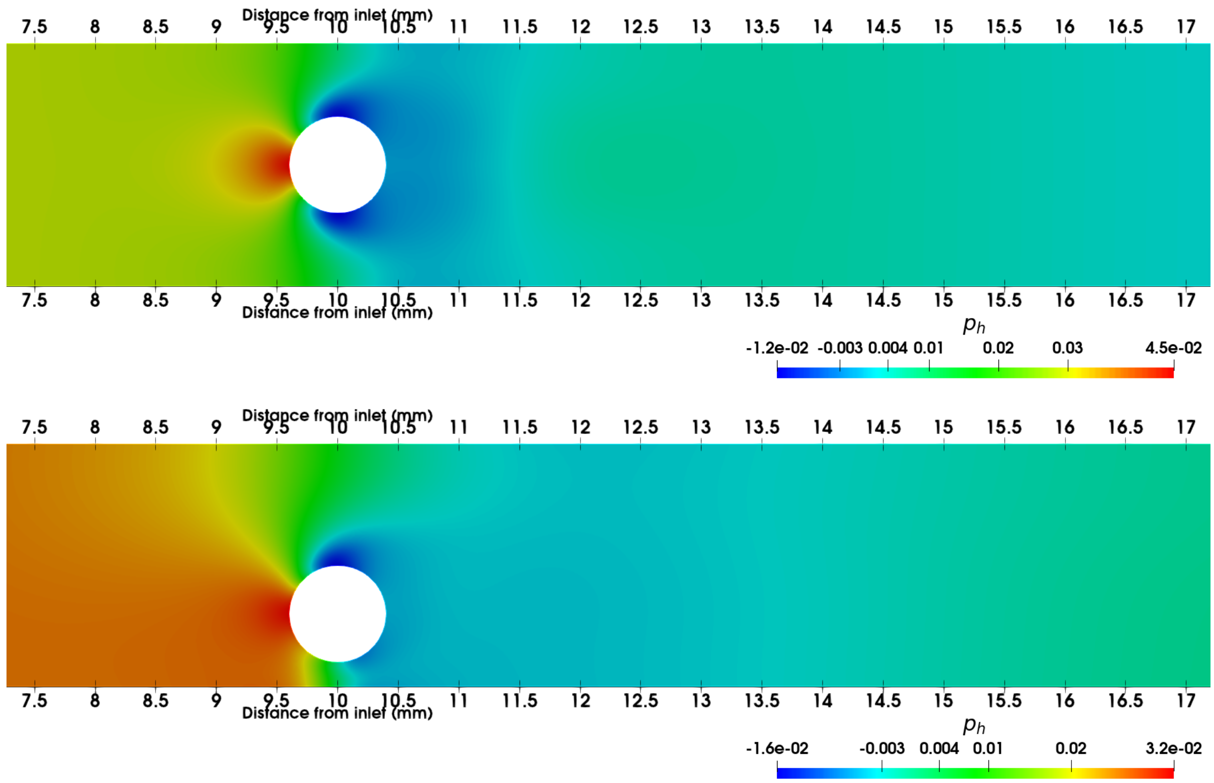

Figure 5.5: Example 5.0.2. Comparison of pressure profiles for an inlet velocity $u_0 = 120\,\mathrm{mm\,s^{-1}}$, with the circular spacer center located at $y = 1\,\mathrm{mm}$ (top) and $y = 0.6\,\mathrm{mm}$ (bottom) from the bottom membrane.

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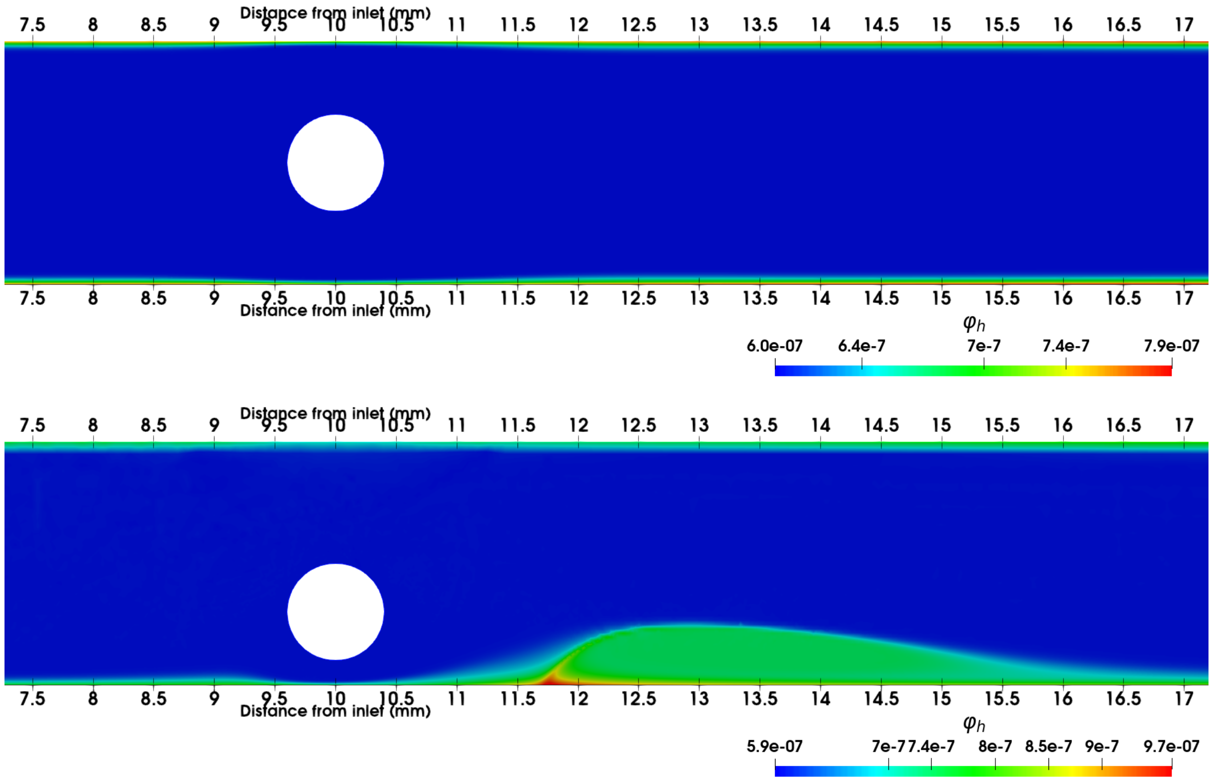

Figure 5.6: Example 5.0.2. Comparison of concentration profiles for an inlet velocity $u_0 = 120\,\mathrm{mm\,s^{-1}}$, with the circular spacer center located at $y = 1\,\mathrm{mm}$ (top) and $y = 0.6\,\mathrm{mm}$ (bottom) from the bottom membrane.

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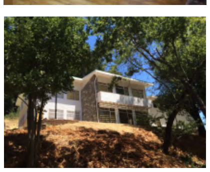

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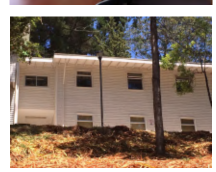
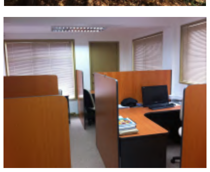

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