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A skew-symmetry-based mixed formulation for an Oseen-type Kelvin–Voigt–Brinkman–Forchheimer model*

Sergio Caucao[†] Gabriel N. Gatica[‡] Adrian Suarez[†] Ivan Yotov[§]

Abstract

We propose and analyze a new mixed formulation for an Oseen linearization of the Kelvin–Voigt–Brinkman–Forchheimer equations, which model viscoelastic flows at higher velocities in highly porous media. Besides the velocity field, our approach introduces the velocity gradient and the viscoelastic pseudostress tensors as auxiliary unknowns, thereby allowing the pressure to be eliminated from the system while still being recoverable through a simple postprocess. This leads to a three-field mixed variational formulation within a Banach space framework, where the aforementioned variables constitute the main unknowns, exploiting the skew-symmetric structure of one of the operators involved. Existence and uniqueness of a solution to the weak formulation are established, and the corresponding stability bounds are derived by employing classical results on nonlinear monotone operators. A semidiscrete continuous-in-time approximation is then introduced, based on Raviart–Thomas finite elements of degree $k \geq 0$ for the viscoelastic pseudostress tensor and discontinuous piecewise polynomials of degree k for the velocity and velocity gradient fields. Furthermore, by applying a backward Euler time discretization, a fully discrete finite element scheme is obtained. Well-posedness is established, and stability bounds together with the corresponding *a priori* error estimates are derived for both schemes. Several numerical experiments involving both manufactured and non-manufactured solutions are presented, confirming the theoretical convergence rates and illustrating the capability of the proposed method to handle challenging geometries with strong contrasts in physical parameters such as permeability and elasticity.

Key words: Kelvin–Voigt–Brinkman–Forchheimer equations, Oseen linearization, mixed finite element methods, skew-symmetric formulation, Banach spaces

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

High-speed fluid flows through porous media occur in many industrial applications, such as in environmental, chemical, and petroleum engineering. For instance, in groundwater remediation and oil and gas extraction, the flow may become fast near injection or production wells, or when the

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aquifer or reservoir is highly porous. Accurate modeling and simulation of these flows are essential in these fields to optimize processes, ensure safety, and minimize environmental impact. Mathematical models have been developed to address different aspects of such flows. The Forchheimer model [22] accounts for the nonlinearities inherent in high-velocity porous flow regimes. The Brinkman model [8] incorporates both viscous and permeability effects, allowing accurate simulations of fluid motion in various environments, including highly porous media. In turn, many applications of interest involve the flow of viscoelastic fluids through porous media, such as polymer and foam injection in oil and gas recovery, blood perfusion through biological tissues, and industrial filtration. The Kelvin–Voigt model [25] provides a fundamental framework for describing the viscoelastic behavior of fluids, as it captures both viscosity and elasticity. The Kelvin–Voigt–Brinkman–Forchheimer (KVBF) model [33], which generalizes and combines the advantages of the three models, is suitable for fast viscoelastic flows in highly porous media.

Regarding the literature, several works are devoted to the mathematical analysis of the KVBF equations (see, for instance, [33], [32], [29], [16], [4] and the references therein). In [33], the existence of a weak solution to the KVBF problem in its velocity-pressure formulation is proved using the Faedo–Galerkin method. Moreover, the existence, uniqueness, and stability of a stationary solution are established when the external force is time-independent and sufficiently small. Later, in [32], the KVBF model with continuous delay is analyzed. In particular, the authors show that, after establishing pullback-$\mathcal{D}$-type absorbing sets for the continuous solution process, the asymptotic compactness obtained through the decomposition method leads to the existence of pullback-$\mathcal{D}$ attractors. Furthermore, in [29], the existence and uniqueness of a strong solution to the KVBF equations are obtained by means of an approximation preserving the m-accretive property of the linear and nonlinear operators. The existence of an exponential attractor is also established, and the inviscid limit of the 3D KVBF equations towards the 3D Navier–Stokes–Voigt system and, subsequently, towards the simplified Bardina model is discussed. In [16], the authors propose a mixed variational formulation of the KVBF problem introducing vorticity as an additional variable, thus leading to a mixed formulation in terms of vorticity, velocity, and pressure. The existence and uniqueness of a weak solution are proved, and stability estimates are derived by means of a fixed-point strategy combined with the theory of monotone operators and Schauder's theorem. In addition, a semidiscrete-in-time approximation is introduced, employing stable Stokes elements for velocity and pressure, and continuous or discontinuous piecewise polynomial spaces for vorticity. Finally, a backward Euler scheme is used for time discretization, yielding a fully discrete system. More recently, in [4], the well-posedness of an initial-boundary value problem for the KVBF equations with memory and variable viscosity under non-homogeneous Dirichlet boundary conditions has been investigated. In that work, the authors proved the global-in-time existence and uniqueness of a strong solution by employing an operator-based technique relying on an abstract theorem on the local uniqueness of solutions in Banach spaces. This result broadens the existing analytical frameworks for the mathematical study of non-Newtonian fluid models in porous media.

On the other hand, several papers have been devoted to the design and analysis of numerical schemes for the simulation of the Brinkman–Forchheimer equations (see, for instance [27], [28], [26], [17],[15], [34], [3], and the references therein). In [27], the authors introduce and analyze a perturbed compressible system that serves as an approximation of the Brinkman–Forchheimer equations. They also develop a numerical method for this perturbed system based on a semi-implicit Euler scheme for time discretization and the use of lowest-order Raviart–Thomas elements for spatial discretization. In [28], a finite element method with pressure stabilization is developed. In [26], a time-discrete scheme is applied to a Brinkman–Forchheimer model with variable porosity to simulate the propagation of dissolution wormholes. In [17], a mixed formulation based on the pseudostress tensor and velocity is analyzed, with a fully discrete scheme and suboptimal error estimates. These estimates are improved in [15], where a three-field formulation including the velocity gradient is developed and analyzed. In [34], a Discontinuous Galerkin (DG) method is developed for a velocity-velocity gradient-pressure formulation of the unsteady Brinkman–Forchheimer problem. Well-posedness and error analyses are presented for both semidiscrete and fully discrete schemes. The method is robust with respect to the Brinkman parameter. More recently, in [3], a mixed variational formulation based on vorticity is analyzed, where the main unknowns are velocity, vorticity, and pressure, proving existence, uniqueness, stability, and optimal convergence rates for both semidiscrete and fully discrete schemes.

The aim of this work is to take a first step toward the development and analysis of a new mixed variational formulation for the KVBF model, based on the viscoelastic pseudostress and velocity gradient tensors. Motivated by [19, 18, 14, 15, 10] and in contrast with previous works on KVBF such as [16], we consider an Oseen-type linearization of the KVBF model and incorporate these tensors as additional unknowns alongside the velocity field. This approach offers several advantages, including direct and accurate approximations of the velocity gradient and the viscoelastic pseudostress tensor. Moreover, it yields optimal theoretical convergence rates without requiring small data assumptions or mesh quasi-uniformity, and provides an adequate postprocessing formula for the pressure. Another relevant contribution of this work is the generalization of the model studied in [15] by incorporating an Oseen-type linearized convective term and an additional time-derivative term, thereby extending the analysis to viscoelastic flows. We establish existence and uniqueness of a solution to the continuous weak formulation by employing techniques from [31] and [15], combined with the classical monotone operator theory in a Banach space setting and an abstract result from [14]. We also emphasize that our formulation exploits the skew-symmetric structure of a bilinear form involving the convective term, which allows us to avoid small data assumptions. Stability for the weak solution is established by means of an energy estimate. Regarding the semidiscrete continuous-in-time scheme, we employ Raviart–Thomas finite elements for the viscoelastic pseudostress tensor and piecewise discontinuous polynomial elements for both the velocity and the velocity gradient tensor. For time discretization, the backward Euler method is used. Furthermore, by adapting the tools employed for the analysis of the continuous problem, we prove well-posedness of the discrete schemes and derive the corresponding stability estimates. We also carry out an error analysis for both the semidiscrete continuous-in-time scheme and the fully discrete scheme, obtaining optimal convergence rates in space and time.

We have organized the content of this article as follows. In the remainder of this section, we introduce the standard notation and the necessary function spaces. In Section 2, we describe the model problem of interest and develop the variational velocity-velocity gradient-pseudostress formulation. In Section 3, we establish the well-posedness of the problem using an abstract result for parabolic problems together with the theory of monotone operators. Next, in Section 4, we present the semidiscrete continuous-in-time approximation, provide a particular family of stable finite elements, and obtain error estimates for the proposed method. Section 5 is devoted to the fully discrete approximation. The performance of the method is examined in Section 6 through several two- and three-dimensional numerical examples, both with and without manufactured solutions, confirming the expected convergence rates and illustrating its flexibility in handling spatially varying parameters within complex geometries. The paper ends with conclusions in Section 7.

In the remainder of this section we introduce some standard notation and needed functional spaces. Let $\Omega \subset \mathrm{R}^d$, $d \in \{2,3\}$, denote a domain with Lipschitz boundary Γ. For $\mathrm{s} \geq 0$ and $\mathrm{p} \in [1,+\infty]$, we denote by $\mathrm{L}^{\mathrm{p}}(\Omega)$ and $\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^{\mathrm{p}}(\Omega)}$ and $\|\cdot\|_{\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)}$, respectively. Note that $\mathrm{W}^{0,\mathrm{p}}(\Omega) = \mathrm{L}^{\mathrm{p}}(\Omega)$. If $\mathrm{p} = 2$, we write $\mathrm{H}^{\mathrm{s}}(\Omega)$ in place of $\mathrm{W}^{\mathrm{s},2}(\Omega)$, and denote the corresponding norm by $\|\cdot\|_{\mathrm{H}^{\mathrm{s}}(\Omega)}$. By $\mathbf{H}$ and $\mathbb{H}$ we will denote the corresponding vector and tensor counterparts of a generic scalar functional space H. The $\mathrm{L}^2(\Omega)$ inner product for scalar, vector, or tensor valued functions is denoted by $(\cdot,\cdot)_\Omega$. The $\mathrm{L}^2(\Gamma)$ inner product or duality pairing is denoted by $\langle\cdot,\cdot\rangle_\Gamma$. Moreover, given a separable Banach space V endowed with the norm $\|\cdot\|_{\mathrm{V}}$, and $p\in[1,+\infty]$, we let $\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})$ be the space of classes of functions $f:(0,T)\to\mathrm{V}$ that are Bochner measurable and such that $\|f\|_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})}<\infty$, with

$$\|f\|^{\mathrm{p}}_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})} := \int_0^T \|f(t)\|^{\mathrm{p}}_{\mathrm{V}}\,dt \quad \text{if } p\in[1,+\infty), \quad \text{and} \quad \|f\|_{\mathrm{L}^\infty(0,T;\mathrm{V})} := \operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_{\mathrm{V}} \quad \text{if } \mathrm{p}=+\infty\,.$$

Similarly, we let (cf. [21, Section 5.9.2])

$$\mathrm{W}^{1,\mathrm{p}}(0,T;\mathrm{V}) := \Big\{ f\in \mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V}) : \quad f'\in \mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})\Big\}\,, \tag{1.1}$$

with norm

$$\begin{aligned}\|f\|^{\mathrm{p}}_{\mathrm{W}^{1,\mathrm{p}}(0,T;\mathrm{V})} &:= \|f\|^{\mathrm{p}}_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})} + \|f'\|^{\mathrm{p}}_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})} \quad \text{if } \mathrm{p}\in[1,+\infty), \quad \text{and}\\ \|f\|_{\mathrm{W}^{1,\infty}(0,T;\mathrm{V})} &:= \|f\|_{\mathrm{L}^\infty(0,T;\mathrm{V})} + \|f'\|_{\mathrm{L}^\infty(0,T;\mathrm{V})} \quad \text{if } \mathrm{p}=+\infty\,.\end{aligned}$$

When $\mathrm{p}=2$ we simply use the notation $\mathrm{H}^1(0,T;\mathrm{V})$. Here, $f'\in\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})$ means that there exists $g\in\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})$ such that

$$\int_0^T F(f(t))\,\varphi'(t)\,dt \;=\; -\int_0^T F(g(t))\,\varphi(t)\,dt \qquad \forall\,(F,\varphi)\in V'\times C_0^\infty(0,T)\,.$$

In this way, $g:=f'$ is the distributional Bochner derivative of f and $\|f'\|_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})} := \|g\|_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})}$. In turn, for any vector fields $\mathbf{v}:=(v_i)_{i=1,d}$ and $\mathbf{w}:=(w_i)_{i=1,d}$, we define the gradient, divergence, and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,d}\,, \quad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^d \frac{\partial v_j}{\partial x_j}\,, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,d}\,.$$

Furthermore, for any tensor fields $\boldsymbol{\tau}=(\tau_{ij})_{i,j=1,d}$ and $\boldsymbol{\zeta}=(\zeta_{ij})_{i,j=1,d}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the tensor inner product, and the transpose, trace, and deviatoric operators, respectively, as

$$\boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^d \tau_{ij}\zeta_{ij}\,, \quad \boldsymbol{\tau}^{\mathtt{t}} = (\tau_{ji})_{i,j=1,d}, \quad \operatorname{tr}(\boldsymbol{\tau}) = \sum_{i=1}^d \boldsymbol{\tau}_{ii}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{d}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I}\,,$$

where $\mathbb{I}$ stands for the identity matrix in $\mathrm{R}^{d\times d}$. In what follows, when no confusion arises, $|\cdot|$ denotes the Euclidean norm in R^d or $\mathrm{R}^{d\times d}$. In addition, in the sequel we will make use of the well-known Hölder inequality given by

$$\int_\Omega |f\,g| \le \|f\|_{\mathrm{L}^{\mathrm{p}}(\Omega)}\,\|g\|_{\mathrm{L}^{\mathrm{q}}(\Omega)} \quad \forall\, f\in\mathrm{L}^{\mathrm{p}}(\Omega), \forall\, g\in\mathrm{L}^{\mathrm{q}}(\Omega), \quad \text{with} \quad \frac{1}{\mathrm{p}}+\frac{1}{\mathrm{q}}=1\,,$$

and Young's inequality, for $a,b\ge0$, and $\delta>0$,

$$a\,b \;\le\; \frac{\delta^{\mathrm{p}/2}}{\mathrm{p}}\,a^{\mathrm{p}} + \frac{1}{\mathrm{q}\,\delta^{\mathrm{q}/2}}\,b^{\mathrm{q}}\,. \tag{1.2}$$

Finally, we recall the continuous injection i_{p} of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{\mathrm{p}}(\Omega)$ for $\mathrm{p}\ge1$ if $d=2$ or $\mathrm{p}\in[1,6]$ if $d=3$. More precisely, we have the following inequality

$$\|w\|_{\mathrm{L}^{\mathrm{p}}(\Omega)} \;\le\; \|i_{\mathrm{p}}\|\,\|w\|_{\mathrm{H}^1(\Omega)} \quad \forall\, w\in\mathrm{H}^1(\Omega),$$

with $\|i_{\mathrm{p}}\|>0$ depending only on $|\Omega|$ and p (see [30, Theorem 1.3.4]). We will denote by $\mathbf{i}_{\mathrm{p}}$ the vector version of i_{p}. In particular, the injection $\mathbf{i}_4:\mathbf{H}^1(\Omega)\to\mathbf{L}^4(\Omega)$ is frequently used along the paper.

2 The model problem and its continuous formulation

In this section we introduce the model problem and derive the corresponding weak formulation.

2.1 The model problem

We are interested in an Oseen-type linearization of the Kelvin–Voigt–Brinkman–Forchheimer model (see, for instance, [33, 32, 29, 16]). More precisely, given the body force $\mathbf{f} : \Omega \times [0,T] \to \mathrm{R}^d$, the steady-state convective velocity field $\boldsymbol{\chi} : \Omega \to \mathrm{R}^d$, and a suitable initial datum $\mathbf{u}_0 : \Omega \to \mathrm{R}^d$, and denoting from now on $\partial_t := \frac{\partial}{\partial t}$, the aforementioned system of equations is given by

$$\begin{gathered}
\partial_t \mathbf{u} - \mathbf{div}\left(\kappa^2\,\partial_t \nabla\mathbf{u} + \nu\,\nabla\mathbf{u}\right) + (\nabla\mathbf{u})\boldsymbol{\chi} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega\times(0,T]\,,\\
\mathrm{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\times(0,T]\,,\\
\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma\times(0,T]\,,\quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \quad \Omega\,,\quad (p,1)_\Omega = 0 \quad \text{in} \quad (0,T]\,,
\end{gathered} \tag{2.1}$$

where the unknowns are the velocity field $\mathbf{u}$ and the scalar pressure p. In addition, the constant $\kappa > 0$ is a length scale parameter characterizing the elasticity of the fluid, $\nu > 0$ is the Brinkman coefficient (or the effective viscosity), $\mathtt{D} > 0$ is the Darcy coefficient, $\mathtt{F} > 0$ is the Forchheimer coefficient, and $\rho \in [3,4]$ is a given number. For simplicity, $\boldsymbol{\chi}$ is assumed to be time-independent and solenoidal, that is, $\mathrm{div}(\boldsymbol{\chi}) = 0$ in Ω. The extension to the case $\boldsymbol{\chi} = \mathbf{u}$ in $\Omega\times(0,T]$ in (2.1), corresponding to the original Kelvin–Voigt–Brinkman–Forchheimer model, will be addressed in future work.

Now, in order to derive our weak formulation, we first rewrite (2.1) as an equivalent first-order set of equations. To that end, unlike [16] and inspired by [19], [18], [14], and [10], we introduce the velocity gradient and viscoelastic pseudostress tensors as further unknowns, that is

$$\boldsymbol{\vartheta} := \nabla\mathbf{u}, \quad \boldsymbol{\sigma} := \kappa^2\,\partial_t\boldsymbol{\vartheta} + \nu\,\boldsymbol{\vartheta} - \frac{1}{2}\,(\mathbf{u}\otimes\boldsymbol{\chi}) - p\,\mathbb{I} \quad \text{in} \quad \Omega\times(0,T]\,. \tag{2.2}$$

In this way, applying the matrix trace to $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$, and utilizing the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in $\Omega\times(0,T]$, one arrives at $\mathrm{tr}(\boldsymbol{\vartheta}) = 0$ in $\Omega\times(0,T]$ and

$$p \;=\; -\frac{1}{d}\,\mathrm{tr}(\boldsymbol{\sigma}) - \frac{1}{2\,d}\,\mathrm{tr}(\mathbf{u}\otimes\boldsymbol{\chi}) \quad \text{in} \quad \Omega\times(0,T]\,. \tag{2.3}$$

In turn, using the solenoidal property of $\boldsymbol{\chi}$, we easily deduce that

$$\mathbf{div}(\mathbf{u}\otimes\boldsymbol{\chi}) \;=\; (\nabla\mathbf{u})\boldsymbol{\chi} + \mathrm{div}(\boldsymbol{\chi})\mathbf{u} \;=\; (\nabla\mathbf{u})\boldsymbol{\chi} \quad \text{in} \quad \Omega\times(0,T]. \tag{2.4}$$

Hence, applying deviatoric operator $^{\mathtt{d}}$ to the second equation of (2.2), and using (2.4) when taking the divergence of $\boldsymbol{\sigma}$, we find that the model problem (2.1) can be equivalently rewritten as the following system of equations with unknowns $\mathbf{u}$, $\boldsymbol{\vartheta}$, and $\boldsymbol{\sigma}$:

$$\begin{gathered}
\boldsymbol{\vartheta} \;=\; \nabla\mathbf{u}\,,\quad \boldsymbol{\sigma}^{\mathtt{d}} \;=\; \kappa^2\,\partial_t\boldsymbol{\vartheta} + \nu\,\boldsymbol{\vartheta} - \frac{1}{2}\,(\mathbf{u}\otimes\boldsymbol{\chi})^{\mathtt{d}} \quad \text{in} \quad \Omega\times(0,T]\,,\\
\partial_t\mathbf{u} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\mathbf{u} + \frac{1}{2}\,\boldsymbol{\vartheta}\boldsymbol{\chi} - \mathbf{div}(\boldsymbol{\sigma}) \;=\; \mathbf{f} \quad \text{in} \quad \Omega\times(0,T]\,,\\
\mathbf{u} \;=\; \mathbf{0} \quad \text{on} \quad \Gamma\times(0,T]\,,\quad \mathbf{u}(0) \;=\; \mathbf{u}_0 \quad \text{in} \quad \Omega\,,\quad (\mathrm{tr}(\boldsymbol{\sigma}) + \frac{1}{2}\mathrm{tr}(\mathbf{u}\otimes\boldsymbol{\chi}),1)_\Omega \;=\; 0 \quad \text{in} \quad (0,T]\,.
\end{gathered} \tag{2.5}$$

At this point, we stress that, as suggested by (2.3), the pressure p is eliminated from the formulation (2.5) and subsequently recovered in terms of $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\boldsymbol{\chi}$ by means of identity (2.3). This fact justifies the last equation in (2.5), which is equivalent to imposing $(p,1)_\Omega = 0$ in $(0,T]$. We also note that the present formulation involves the time derivative of the new unknown $\boldsymbol{\vartheta}$. Consequently, a compatible initial condition $\boldsymbol{\vartheta}(0) = \boldsymbol{\vartheta}_0$ in Ω is required, where $\boldsymbol{\vartheta}_0$ is consistent with system (2.5). Such an initial datum will be constructed in Lemma 3.7 (cf. Section 3.3).

2.2 The skew-symmetry-based mixed formulation

In this section we derive a three-field Banach mixed variational formulation for the system (2.5). To that end, we proceed as in [10, Section 3.1] (see also [19, 12, 18, 11, 14] for similar approaches) and extend the analysis derived there to our current unsteady regime. In fact, multiplying the first, second and third equations of (2.5) by suitable test functions $\boldsymbol{\tau}, \boldsymbol{\xi}$, and $\mathbf{v}$, respectively, integrating by parts, and using the Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\Gamma \times (0,T]$, we get

$$(\boldsymbol{\vartheta}, \boldsymbol{\tau})_\Omega + (\mathbf{u}, \mathbf{div}(\boldsymbol{\tau}))_\Omega = 0\,, \tag{2.6}$$

$$\kappa^2\, \partial_t(\boldsymbol{\vartheta}(t), \boldsymbol{\xi})_\Omega + \nu\,(\boldsymbol{\vartheta}, \boldsymbol{\xi})_\Omega - \frac{1}{2}\,((\mathbf{u}\otimes\boldsymbol{\chi})^{\mathtt{d}}, \boldsymbol{\xi})_\Omega - (\boldsymbol{\sigma}^{\mathtt{d}}, \boldsymbol{\xi})_\Omega = 0\,, \tag{2.7}$$

$$\partial_t(\mathbf{u}(t), \mathbf{v})_\Omega + \mathtt{D}\,(\mathbf{u}, \mathbf{v})_\Omega + \mathtt{F}\,(|\mathbf{u}|^{\rho-2}\mathbf{u}, \mathbf{v})_\Omega + \frac{1}{2}\,(\boldsymbol{\vartheta}, (\mathbf{v}\otimes\boldsymbol{\chi}))_\Omega - (\mathbf{div}(\boldsymbol{\sigma}), \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega\,, \tag{2.8}$$

for all $(\boldsymbol{\tau}, \boldsymbol{\xi}, \mathbf{v})$ in $\mathbb{X}\times\mathbb{Q}\times\mathbf{M}$, where $\mathbb{X}\,, \mathbb{Q}$ and $\mathbf{M}$ are spaces to be defined below. Notice that for the fourth term in (2.8) we have used the identity $(\boldsymbol{\vartheta}\boldsymbol{\chi})\cdot\mathbf{v} = \boldsymbol{\vartheta} : (\mathbf{v}\otimes\boldsymbol{\chi})$. In turn, we stress that the first terms of (2.7) and (2.8) are written in that way to emphasize that the time derivative only applies to the first component of the inner product $(\cdot,\cdot)_\Omega$.

We begin by noting that the first and second terms in (2.7) are well defined for $\boldsymbol{\vartheta}, \boldsymbol{\xi} \in \mathbb{L}^2(\Omega)$, but due to the incompressibility condition $\mathrm{div}(\mathbf{u}) = \mathrm{tr}(\boldsymbol{\vartheta}) = 0$, it makes sense to look for $\boldsymbol{\vartheta}$, and consequently for the test function $\boldsymbol{\xi}$ in

$$\mathbb{Q} \,:=\, \Big\{ \boldsymbol{\xi} \in \mathbb{L}^2(\Omega) : \quad \mathrm{tr}(\boldsymbol{\xi}) = 0 \quad \text{in} \quad \Omega \Big\}\,. \tag{2.9}$$

This implies that (2.7) can be rewritten equivalently as

$$\kappa^2\, \partial_t(\boldsymbol{\vartheta}(t), \boldsymbol{\xi})_\Omega + \nu\,(\boldsymbol{\vartheta}, \boldsymbol{\xi})_\Omega - \frac{1}{2}\,((\mathbf{u}\otimes\boldsymbol{\chi}), \boldsymbol{\xi})_\Omega - (\boldsymbol{\sigma}, \boldsymbol{\xi})_\Omega \,=\, 0 \quad \forall\, \boldsymbol{\xi}\in\mathbb{Q}\,. \tag{2.10}$$

In addition, we note that the first and fourth terms in (2.6) and (2.10) (or (2.7)), respectively, are well defined if $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$. In turn, assuming $\boldsymbol{\chi} \in \mathbf{L}^4(\Omega)$, applying the Hölder and Cauchy–Schwarz inequalities, and the Sobolev embedding of $\mathbf{L}^4(\Omega)$ into $\mathbf{L}^{2\,(\rho-2)}$, with $\rho \in [3,4]$, we find that the Forchheimer and Oseen convective terms, given by the third expressions in (2.8) and (2.10), and the fourth term in (2.8), can be bounded, respectively, as

$$\big|(|\mathbf{u}|^{\rho-2}\mathbf{u}, \mathbf{v})_\Omega\big| \,\leq\, |\Omega|^{(4-\rho)/4}\, \|\mathbf{u}\|^{\rho-1}_{\mathbf{L}^4(\Omega)}\, \|\mathbf{v}\|_{\mathbf{L}^4(\Omega)}\,, \tag{2.11}$$

$$|((\mathbf{u}\otimes\boldsymbol{\chi}), \boldsymbol{\xi})_\Omega| \,\leq\, \|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\, \|\boldsymbol{\chi}\|_{\mathbf{L}^4(\Omega)}\, \|\boldsymbol{\xi}\|_{\mathbb{L}^2(\Omega)}\,, \tag{2.12}$$

and

$$|(\boldsymbol{\vartheta}, (\mathbf{v}\otimes\boldsymbol{\chi}))_\Omega| \,\leq\, \|\boldsymbol{\vartheta}\|_{\mathbb{L}^2(\Omega)}\, \|\boldsymbol{\chi}\|_{\mathbf{L}^4(\Omega)}\, \|\mathbf{v}\|_{\mathbf{L}^4(\Omega)}\,, \tag{2.13}$$

which shows that they are well-defined for all $\mathbf{u}, \mathbf{v} \in \mathbf{L}^4(\Omega)$ and for all $\boldsymbol{\vartheta}, \boldsymbol{\xi} \in \mathbb{Q}$. Note that $\boldsymbol{\chi}$ could be assumed to be more regular so as to ensure that $\mathbf{u}, \mathbf{v} \in \mathbf{L}^\rho(\Omega)$, with $\rho \in [3,4]$, due to the natural norm induced by the Forchheimer term. However, in order to extend the analysis developed for the present approach to the case $\boldsymbol{\chi} = \mathbf{u}$, we have adopted this choice. Next, knowing the space to which $\mathbf{u}$ and $\mathbf{v}$

belong, the last terms in (2.6) and (2.8) force $\mathbf{div}(\boldsymbol{\tau})$ and $\mathbf{div}(\boldsymbol{\sigma})$, respectively, to live in $\mathbf{L}^{4/3}(\Omega)$. In turn, the fact that $\mathbf{L}^4(\Omega)$ is certainly contained in $\mathbf{L}^2(\Omega)$ guarantees that the first and second terms in (2.8) are clearly well defined. According to the above discussion, we introduce the Banach space

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^{4/3}(\Omega) \Big\},$$

equipped with the norm

$$\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div}_{4/3};\Omega)} := \|\boldsymbol{\tau}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\mathbf{L}^{4/3}(\Omega)},$$

and deduce that the equations (2.6), (2.8) and (2.10) are well defined if, besides setting $\mathbb{Q}$ as in (2.9), we choose

$$\mathbf{M} := \mathbf{L}^4(\Omega) \quad \text{and} \quad \mathbb{X} := \mathbb{H}(\mathbf{div}_{4/3};\Omega),$$

with their respective norms: $\|\cdot\|_{\mathbb{Q}} := \|\cdot\|_{\mathbb{L}^2(\Omega)}$, $\|\cdot\|_{\mathbf{M}} := \|\cdot\|_{\mathbf{L}^4(\Omega)}$, and $\|\cdot\|_{\mathbb{X}} := \|\cdot\|_{\mathbb{H}(\mathbf{div}_{4/3};\Omega)}$.

Now, for convenience of the subsequent analysis and similarly as in [11] (see also [23, 18, 14, 10]) we consider the decomposition

$$\mathbb{X} = \mathbb{X}_0 \oplus \mathrm{R}\,\mathbb{I},$$

where

$$\mathbb{X}_0 := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad (\mathrm{tr}(\boldsymbol{\tau}),1)_\Omega = 0 \Big\},$$

and $\mathrm{R}\,\mathbb{I}$ is its topological supplement with respect to $\mathbb{X}$. More precisely, each $\boldsymbol{\tau} \in \mathbb{X}$ can be decomposed uniquely as

$$\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}} + \tilde{c}\,\mathbb{I} \quad \text{with} \quad \tilde{\boldsymbol{\tau}} \in \mathbb{X}_0 \quad \text{and} \quad \tilde{c} := \frac{1}{d\,|\Omega|}(\mathrm{tr}(\boldsymbol{\tau}),1)_\Omega \in \mathrm{R}.$$

In particular, using from the last equation in (2.5) that $(\mathrm{tr}(\boldsymbol{\sigma}),1)_\Omega = -\dfrac{1}{2}(\mathrm{tr}(\mathbf{u}\otimes\boldsymbol{\chi}),1)_\Omega$, we obtain

$$\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} + \bar{c}\,\mathbb{I} \quad \text{with} \quad \bar{\boldsymbol{\sigma}} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad \bar{c} = -\frac{1}{2\,d\,|\Omega|}(\mathrm{tr}(\mathbf{u}\otimes\boldsymbol{\chi}),1)_\Omega. \tag{2.14}$$

In this way, knowing explicitly $\bar{c}$ in terms of $\mathbf{u}$ and $\boldsymbol{\chi}$, it remains to find the $\mathbb{X}_0$-component $\bar{\boldsymbol{\sigma}}$ of $\boldsymbol{\sigma}$ to fully determine it. In this regard, using the fact that $\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{div}(\bar{\boldsymbol{\sigma}})$ and $\boldsymbol{\sigma} : \boldsymbol{\xi} = \bar{\boldsymbol{\sigma}} : \boldsymbol{\xi}$, for all $\boldsymbol{\xi} \in \mathbb{Q}$, we deduce that (2.8) and (2.10) remain unchanged if $\boldsymbol{\sigma}$ is replaced there by $\bar{\boldsymbol{\sigma}}$, which, for simplicity, is still named $\boldsymbol{\sigma}$. In turn, since $\boldsymbol{\tau} := \mathbb{I}$ vanishes the left-hand side of (2.6), it follows that testing this equation against $\boldsymbol{\tau} \in \mathbb{X}$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{X}_0$. Next, in order to write the above formulation in a more suitable way for the analysis to be developed below, we now set the notations

$$\underline{\mathbf{u}} := (\mathbf{u},\boldsymbol{\vartheta}), \quad \underline{\mathbf{v}} := (\mathbf{v},\boldsymbol{\xi}) \in \mathbf{M}\times\mathbb{Q},$$

with corresponding norm given by

$$\|\underline{\mathbf{v}}\| := \|\mathbf{v}\|_{\mathbf{M}} + \|\boldsymbol{\xi}\|_{\mathbb{Q}} \quad \forall\,\underline{\mathbf{v}} \in \mathbf{M}\times\mathbb{Q}.$$

Hence, the weak formulation associated with the Oseen-type linearized Kelvin–Voigt–Brinkman–Forchheimer system (2.5) reads: Given $\mathbf{f} : [0,T] \to \mathbf{L}^2(\Omega)$, $\boldsymbol{\chi} \in \mathbf{M}$, and $(\mathbf{u}_0,\boldsymbol{\vartheta}_0) \in \mathbf{M}\times\mathbb{Q}$, find $(\underline{\mathbf{u}},\boldsymbol{\sigma}) : [0,T] \to (\mathbf{M}\times\mathbb{Q})\times\mathbb{X}_0$, such that $\underline{\mathbf{u}}(0) := (\mathbf{u}(0),\boldsymbol{\vartheta}(0)) = (\mathbf{u}_0,\boldsymbol{\vartheta}_0)$ and, for a.e. $t \in (0,T)$,

$$\begin{aligned}
\partial_t[\mathcal{E}(\underline{\mathbf{u}}(t)),\underline{\mathbf{v}}] + [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}(t)),\underline{\mathbf{v}}] + [\mathcal{B}^{\mathrm{t}}(\boldsymbol{\sigma}(t)),\underline{\mathbf{v}}] &= [F(t),\underline{\mathbf{v}}] \quad \forall\,\underline{\mathbf{v}} \in \mathbf{M}\times\mathbb{Q},\\
-[\mathcal{B}(\underline{\mathbf{u}}(t)),\boldsymbol{\tau}] &= 0 \qquad\qquad \forall\,\boldsymbol{\tau} \in \mathbb{X}_0,
\end{aligned} \tag{2.15}$$

where the operators $\mathcal{E}$, $\mathcal{A}(\boldsymbol{\chi}) : (\mathbf{M}\times\mathbb{Q}) \to (\mathbf{M}\times\mathbb{Q})'$, for the given $\boldsymbol{\chi}\in\mathbf{M}$, and $\mathcal{B} : (\mathbf{M}\times\mathbb{Q}) \to \mathbb{X}_0'$ are defined, respectively, as

$$[\mathcal{E}(\underline{\mathbf{u}}),\underline{\mathbf{v}}] := (\mathbf{u},\mathbf{v})_\Omega + \kappa^2(\boldsymbol{\vartheta},\boldsymbol{\xi})_\Omega\,, \tag{2.16}$$

$$[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}),\underline{\mathbf{v}}] := [\mathbf{a}(\underline{\mathbf{u}}),\underline{\mathbf{v}}] + [\mathbf{c}(\boldsymbol{\chi})(\underline{\mathbf{u}}),\underline{\mathbf{v}}]\,, \tag{2.17}$$

$$[\mathbf{a}(\underline{\mathbf{u}}),\underline{\mathbf{v}}] := \mathtt{D}\,(\mathbf{u},\mathbf{v})_\Omega + \mathtt{F}\,(|\mathbf{u}|^{\rho-2}\mathbf{u},\mathbf{v})_\Omega + \nu\,(\boldsymbol{\vartheta},\boldsymbol{\xi})_\Omega\,, \tag{2.18}$$

$$[\mathbf{c}(\boldsymbol{\chi})(\underline{\mathbf{u}}),\underline{\mathbf{v}}] := \frac{1}{2}\Big\{(\boldsymbol{\vartheta},(\mathbf{v}\otimes\boldsymbol{\chi}))_\Omega - (\boldsymbol{\xi},(\mathbf{u}\otimes\boldsymbol{\chi}))_\Omega\Big\}\,, \tag{2.19}$$

$$[\mathcal{B}(\underline{\mathbf{v}}),\boldsymbol{\tau}] := -\,(\mathbf{v},\mathbf{div}(\boldsymbol{\tau}))_\Omega - (\boldsymbol{\xi},\boldsymbol{\tau})_\Omega\,, \tag{2.20}$$

and F is the bounded linear functional given by

$$[F,\underline{\mathbf{v}}] := (\mathbf{f},\mathbf{v})_\Omega\,.$$

In all the terms above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators. In addition, we let $\mathcal{B}^{\mathtt{t}} : \mathbb{X}_0 \to \big(\mathbf{M}\times\mathbb{Q}\big)'$ be the operator defined by $[\mathcal{B}^{\mathtt{t}}(\boldsymbol{\tau}),\underline{\mathbf{v}}] = [\mathcal{B}(\underline{\mathbf{v}}),\boldsymbol{\tau}]$ for all $\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}$ and $\boldsymbol{\tau}\in\mathbb{X}_0$. Note that the adjoint operator $\mathcal{B}'\in\mathcal{L}\big(\mathbb{X}_0'',(\mathbf{M}\times\mathbb{Q})'\big)$ and $\mathcal{B}^{\mathtt{t}}$ are related by means of the identity $\mathcal{B}^{\mathtt{t}} = \mathcal{B}'\circ\mathcal{J}_0$, where $\mathcal{J}_0 : \mathbb{X}_0 \to \mathbb{X}_0''$ is the bijection characterizing the reflexivity of $\mathbb{X}_0$.

3 Well-posedness of the model

In this section, we establish the solvability of (2.15). To this end, we first collect some preliminary results that will be used in the forthcoming analysis.

3.1 Preliminary results

We begin by recalling a key result, which will be used to establish the existence of a solution to (2.15). In what follows, an operator Θ from a real vector space E to its algebraic dual $E^\star$ is symmetric and monotone if, respectively,

$$[\Theta(x),y] = [\Theta(y),x] \quad \forall\,x,y\in E\,, \quad \text{and} \quad [\Theta(x)-\Theta(y),x-y]\geq 0 \quad \forall\,x,y\in E\,.$$

In addition, let us denote by $R(\Theta)$ the range of Θ. We also recall that the dual of a seminormed space is the space of all linear functionals that are continuous with respect to the seminorm. The following result is a slight simplification of [31, Theorem IV.6.1(b)], which will be used to establish the existence of a solution to (2.15).

Theorem 3.1 *Let the linear, symmetric and monotone operator $\mathcal{N}$ be given from the real vector space E to its algebraic dual $E^\star$, and let E_b' be the Hilbert space which is the topological dual of the seminormed space $(E,|\cdot|_b)$, where*

$$|x|_b = [\mathcal{N}(x),x]^{1/2} \qquad \forall\,x\in E\,. \tag{3.1}$$

Let $\mathcal{M} : E \to E_b'$ be an operator with domain $\mathcal{D} = \Big\{x\in E \,:\, \mathcal{M}(x)\in E_b'\Big\}$. Assume that $\mathcal{M}$ is monotone and $R(\mathcal{N}+\mathcal{M}) = E_b'$. Then, for each $f\in \mathrm{W}^{1,1}(0,T;E_b')$ (cf. (1.1)) and for each $u_0\in\mathcal{D}$, there is a solution $u : [0,T]\to E$ of

$$\partial_t\big(\mathcal{N}(u(t))\big) + \mathcal{M}\big(u(t)\big) = f(t) \quad \textit{for a.e.} \quad 0<t<T\,, \tag{3.2}$$

with

$$\mathcal{N}(u) \in \mathrm{W}^{1,\infty}(0,T;E_b')\,, \quad u(t) \in \mathcal{D} \quad \text{for all } 0 \le t \le T\,, \quad \text{and} \quad \mathcal{N}(u(0)) = \mathcal{N}(u_0)\,.$$

For the proof of the range condition in Theorem 3.1, we will require the following abstract result [14, Theorem 3.1], which in turn, is a modification of [13, Theorem 3.1].

Theorem 3.2 *Let X_1, X_2 and Y be separable and reflexive Banach spaces, being X_1 and X_2 uniformly convex, and set $X := X_1 \times X_2$. Let $\mathcal{A} : X \to X'$ be a nonlinear operator, $\mathcal{B} \in \mathcal{L}(X, Y')$, and let V be the kernel of $\mathcal{B}$, that is,*

$$V := \Big\{ v = (v_1, v_2) \in X : \quad \mathcal{B}(v) = \mathbf{0} \Big\}\,.$$

Assume that

(i) *there exist constants $L_{\mathcal{A}} > 0$ and $\mathrm{p}_1, \mathrm{p}_2 \ge 2$, such that*

$$\|\mathcal{A}(u) - \mathcal{A}(v)\|_{X'} \,\le\, L_{\mathcal{A}} \sum_{i=1}^{2} \Big\{ \|u_i - v_i\|_{X_i} + \big(\|u_i\|_{X_i} + \|v_i\|_{X_i}\big)^{\mathrm{p}_i - 2} \|u_i - v_i\|_{X_i} \Big\}\,,$$

for all $u = (u_1, u_2), v = (v_1, v_2) \in X$.

(ii) *the family of operators $\big\{\mathcal{A}(\cdot + z) : V \to V' : \quad z \in X\big\}$ is uniformly strongly monotone, that is there exists $\gamma > 0$, such that*

$$[\mathcal{A}(u + z) - \mathcal{A}(v + z), u - v] \,\ge\, \gamma\, \|u - v\|_X^2\,,$$

for all $z \in X$, and for all $u, v \in V$, and

(iii) *there exists $\beta > 0$ such that*

$$\sup_{\mathbf{0} \ne v \in X} \frac{[\mathcal{B}(v), q]}{\|v\|_X} \,\ge\, \beta\, \|q\|_Y \quad \forall\, q \in Y\,.$$

Then, for each $(\mathcal{F}, \mathcal{G}) \in X' \times Y'$ there exists a unique $(u, p) \in X \times Y$ such that

$$\begin{aligned} [\mathcal{A}(u), v] + [\mathcal{B}(v), p] \quad &= \quad [\mathcal{F}, v] \quad \forall\, v \in X\,, \\ [\mathcal{B}(u), q] \quad &= \quad [\mathcal{G}, q] \quad \forall\, q \in Y\,. \end{aligned}$$

Next, we establish the stability properties of the operators involved in (2.15). We begin by noting that the operators $\mathcal{E}, \mathcal{B}$ and the functional F are linear. Furthermore, given $\boldsymbol{\chi} \in \mathbf{M}$, from (2.16), (2.17), (2.18), (2.19), and (2.20), and using (2.12)–(2.13), (2.11), and the Hölder and Cauchy–Schwarz inequalities, it follows that

$$|\big[\mathcal{B}(\underline{\mathbf{v}}), \boldsymbol{\tau}\big]| \,\le\, \|\underline{\mathbf{v}}\| \|\boldsymbol{\tau}\|_{\mathbb{X}} \quad \forall\, (\underline{\mathbf{v}}, \boldsymbol{\tau}) \in (\mathbf{M} \times \mathbb{Q}) \times \mathbb{X}_0\,, \tag{3.3}$$

$$|\big[F, \underline{\mathbf{v}}\big]| \,\le\, \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \,\le\, |\Omega|^{1/4} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\underline{\mathbf{v}}\| \quad \forall\, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q}\,, \tag{3.4}$$

$$|\big[\mathbf{c}(\boldsymbol{\chi})(\underline{\mathbf{u}}), \underline{\mathbf{v}}\big]| \,\le\, \|\boldsymbol{\chi}\|_{\mathbf{M}} \|\underline{\mathbf{u}}\| \|\underline{\mathbf{v}}\| \quad \forall\, \underline{\mathbf{u}}, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q}\,, \tag{3.5}$$

$$|\big[\mathbf{a}(\underline{\mathbf{u}}), \underline{\mathbf{v}}\big]| \,\le\, \mathtt{D}\, |\Omega|^{1/2}\, \|\mathbf{u}\|_{\mathbf{M}}\, \|\mathbf{v}\|_{\mathbf{M}} + \mathtt{F} |\Omega|^{(4-\rho)/4}\, \|\mathbf{u}\|_{\mathbf{M}}^{\rho-1}\, \|\mathbf{v}\|_{\mathbf{M}} + \nu \|\boldsymbol{\vartheta}\|_{\mathbb{Q}}\, \|\boldsymbol{\xi}\|_{\mathbb{Q}}\,, \tag{3.6}$$

and

$$
|[\mathcal{E}(\underline{\mathbf{u}}),\underline{\mathbf{v}}]| \,\le\, (|\Omega|^{1/2}+\kappa^2)\,\|\underline{\mathbf{u}}\|\,\|\underline{\mathbf{v}}\| \qquad \forall\,\underline{\mathbf{u}},\,\underline{\mathbf{v}}\in \mathbf{M}\times\mathbb{Q}\,, \tag{3.7}
$$

which implies that $\mathcal{B}$, F, $\mathbf{c}(\boldsymbol{\chi})$, $\mathbf{a}$ and $\mathcal{E}$ are bounded and continuous. In addition, there holds

$$
[\mathcal{E}(\underline{\mathbf{v}}),\underline{\mathbf{v}}] \,=\, \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\|\boldsymbol{\xi}\|^2_{\mathbb{Q}} \,\ge\, 0 \quad \forall\,\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\,, \tag{3.8}
$$

which says that $\mathcal{E}$, being linear, is monotone. On the other hand, given $\boldsymbol{\chi}\in\mathbf{M}$, it is readily seen, employing the Cauchy–Schwarz inequality, (3.5) and (3.6), that the nonlinear operator $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.17)) satisfies

$$
\begin{aligned}
|[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}),\underline{\mathbf{v}}]| \,&\le\, |[\mathbf{a}(\underline{\mathbf{u}}),\underline{\mathbf{v}}]| + |[\mathbf{c}(\boldsymbol{\chi})(\underline{\mathbf{u}}),\underline{\mathbf{v}}]| \\
&\le C_{\mathcal{A}}\Big\{\|\underline{\mathbf{u}}\| + \|\mathbf{u}\|^{\rho-1}_{\mathbf{M}} + \|\boldsymbol{\chi}\|_{\mathbf{M}}\|\underline{\mathbf{u}}\|\Big\}\|\underline{\mathbf{v}}\| \qquad \forall\,\underline{\mathbf{u}},\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\,,
\end{aligned} \tag{3.9}
$$

where $C_{\mathcal{A}} := \max\Big\{\mathtt{D}\,|\Omega|^{1/2}+\nu, \mathtt{F}\,|\Omega|^{(4-\rho)/4}, 1\Big\}$, which shows that $\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}})$ does belong to $\big(\mathbf{M}\times\mathbb{Q}\big)'$, thus proving that $\mathcal{A}(\boldsymbol{\chi})$ is well-defined. In addition, using similar arguments to (2.12) and (3.5), it is not difficult to see that the operator $\mathbf{c}(\boldsymbol{\chi})$ (cf. (2.19)) satisfies

$$
\begin{aligned}
|[\mathbf{c}(\boldsymbol{\chi})(\underline{\mathbf{u}}_1-\underline{\mathbf{u}}_2),\underline{\mathbf{v}}]| &\le \frac{1}{2}\Big\{\|\mathbf{v}\|_{\mathbf{M}}\|\boldsymbol{\chi}\|_{\mathbf{M}}\|\boldsymbol{\vartheta}_1-\boldsymbol{\vartheta}_2\|_{\mathbb{Q}} + \|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbf{M}}\|\boldsymbol{\chi}\|_{\mathbf{M}}\|\boldsymbol{\xi}\|_{\mathbb{Q}}\Big\} \\
&\le \|\boldsymbol{\chi}\|_{\mathbf{M}}\|\underline{\mathbf{u}}_1-\underline{\mathbf{u}}_2\|\|\underline{\mathbf{v}}\| \quad \forall\,\underline{\mathbf{u}}_i := (\mathbf{u}_i,\boldsymbol{\vartheta}_i)\,\big(i\in\{1,2\}\big),\,\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\,.
\end{aligned} \tag{3.10}
$$

In turn, observe from the definition of the operator $\mathbf{c}(\boldsymbol{\chi})$ (cf. (2.19)) that, for any $\boldsymbol{\chi}\in\mathbf{M}$, the following skew-symmetry property holds

$$
[\mathbf{c}(\boldsymbol{\chi})(\underline{\mathbf{v}}),\underline{\mathbf{v}}] \,=\, 0 \quad \forall\,\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\,. \tag{3.11}
$$

Finally, given the steady-state convective velocity $\boldsymbol{\chi}\in\mathbf{M}$ and recalling the definitions of the operators $\mathcal{E}$, $\mathcal{A}(\boldsymbol{\chi})$, and $\mathcal{B}$ (cf. (2.16), (2.17), (2.20)), we observe that problem (2.15) can be written in the form of (3.2) with

$$
\begin{gathered}
E \,:=\, \big(\mathbf{M}\times\mathbb{Q}\big)\times\mathbb{X}_0, \quad u \,:=\, \begin{pmatrix}\underline{\mathbf{u}}\\ \boldsymbol{\sigma}\end{pmatrix}, \quad \mathcal{N} \,:=\, \begin{pmatrix}\mathcal{E} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}, \\
\text{and} \qquad \mathcal{M} \,:=\, \begin{pmatrix}\mathcal{A}(\boldsymbol{\chi}) & \mathcal{B}^{\mathrm{t}}\\ -\mathcal{B} & \mathbf{0}\end{pmatrix}.
\end{gathered} \tag{3.12}
$$

Let E_b' be the Hilbert space that is the dual of E with the semi-norm induced by the operator $\mathcal{N}$ (cf. (3.1), (3.12), (2.16)), which thanks to the fact that $\kappa>0$, is given by

$$
|(\underline{\mathbf{v}},\boldsymbol{\tau})|_{\mathcal{N}} \,:=\, \big(\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\|\boldsymbol{\xi}\|^2_{\mathbb{L}^2(\Omega)}\big)^{1/2} \,\equiv\, \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\xi}\|_{\mathbb{L}^2(\Omega)} \quad \forall\,(\underline{\mathbf{v}},\boldsymbol{\tau})\in\big(\mathbf{M}\times\mathbb{Q}\big)\times\mathbb{X}_0\,.
$$

Then we define the spaces

$$
E_b' \,=\, \big(\mathbf{L}^2(\Omega)\times\mathbb{L}^2(\Omega)\big)\times\{\mathbf{0}\}\,, \quad \mathcal{D} \,:=\, \Big\{(\underline{\mathbf{u}},\boldsymbol{\sigma})\in E: \quad \mathcal{M}(\underline{\mathbf{u}},\boldsymbol{\sigma})\in E_b'\Big\}. \tag{3.13}
$$

In the next two sections, we verify the hypotheses of Theorem 3.1 and thereby establish the main result of this section, namely the well-posedness of (2.15).

3.2 Range condition

We begin with the verification of the range condition in Theorem 3.1. Let us consider the resolvent system associated with (2.15): Given $\boldsymbol{\chi} \in \mathbf{M}$, find $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) \in (\mathbf{M} \times \mathbb{Q}) \times \mathbb{X}_0$ such that

$$\begin{aligned} [(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{u}}), \underline{\mathbf{v}}] + [\mathcal{B}^{\mathrm{t}}(\boldsymbol{\sigma}), \underline{\mathbf{v}}] &= [\widehat{F}, \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q}\,, \\ [\mathcal{B}(\underline{\mathbf{u}}), \boldsymbol{\tau}] &= 0 \qquad\quad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0\,, \end{aligned} \tag{3.14}$$

where $\widehat{F} \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2(\Omega) \subset \mathbf{M}' \times \mathbb{Q}'$ is a functional given by $[\widehat{F}, \underline{\mathbf{v}}] := (\widehat{\mathbf{f}}_1, \mathbf{v})_\Omega + (\widehat{\mathbf{f}}_2, \boldsymbol{\xi})_\Omega$ for some $\widehat{\mathbf{f}}_1 \in \mathbf{L}^2(\Omega)$ and $\widehat{\mathbf{f}}_2 \in \mathbb{L}^2(\Omega)$. We will apply Theorem 3.2 to ensure solvability of (3.14). Indeed, we first observe that, thanks to the reflexivity, uniform convexity and separability of $\mathbf{L}^p(\Omega)$ for $p \in (1, +\infty)$, the spaces $\mathbf{M}$, $\mathbb{Q}$, and $\mathbf{X}_0$ are also reflexive, uniformly convex, and separable. Next, we establish a continuity property of the nonlinear operator $\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})$ that corresponds to hypothesis (i) of Theorem 3.2, with $\mathrm{p}_1 = \rho \in [3, 4]$ and $\mathrm{p}_2 = 2$.

Lemma 3.3 *Let $\rho \in [3, 4]$ and let $\boldsymbol{\chi} \in \mathbf{M}$ be given. Then there exists a constant $L_{\mathtt{KV}} > 0$, depending only on $\|\boldsymbol{\chi}\|_{\mathbf{M}}$, ν, $\mathtt{F}$, ρ, $\mathtt{D}$, κ and $|\Omega|$, such that*

$$\begin{aligned} &\|(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{u}}) - (\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{v}})\| \\ &\quad \leq L_{\mathtt{KV}} \left\{ \|\mathbf{u} - \mathbf{v}\|_{\mathbf{M}} + \big(\|\mathbf{u}\|_{\mathbf{M}} + \|\mathbf{v}\|_{\mathbf{M}}\big)^{\rho-2} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{M}} + \|\boldsymbol{\vartheta} - \boldsymbol{\xi}\|_{\mathbb{Q}} \right\}, \end{aligned} \tag{3.15}$$

for all $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\vartheta})$, $\underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\xi}) \in \mathbf{M} \times \mathbb{Q}$.

Proof. Given $\boldsymbol{\chi} \in \mathbf{M}$, let $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\vartheta})$, $\underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\xi}) \in \mathbf{M} \times \mathbb{Q}$. Then, according to the definitions of the operators $\mathcal{E}$ and $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.16)–(2.19)), and proceeding analogously to the boundedness estimates in (3.7) and (3.9), while using (3.10) together with the Sobolev embedding of $\mathbf{L}^4(\Omega)$ into $\mathbf{L}^\rho(\Omega)$, with embedding constant $|\Omega|^{(4-\rho)/(4\rho)}$, as well as the Hölder and Cauchy–Schwarz inequalities, we obtain that

$$\begin{aligned} &\|\big(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})\big)(\underline{\mathbf{u}}) - \big(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})\big)(\underline{\mathbf{v}})\| \,\leq\, (|\Omega|^{1/2} + \mathtt{D}\,|\Omega|^{1/2})\,\|\mathbf{u} - \mathbf{v}\|_{\mathbf{M}} \\ &\quad + \mathtt{F}\,|\Omega|^{(4-\rho)/(4\rho)}\,\||\mathbf{u}|^{\rho-2}\mathbf{u} - |\mathbf{v}|^{\rho-2}\mathbf{v}\|_{\mathbf{L}^\varrho(\Omega)} + (\kappa^2 + \nu)\,\|\boldsymbol{\vartheta} - \boldsymbol{\xi}\|_{\mathbb{Q}} + \|\boldsymbol{\chi}\|_{\mathbf{M}}\,\|\underline{\mathbf{u}} - \underline{\mathbf{v}}\|\,, \end{aligned} \tag{3.16}$$

with $\varrho \in [4/3, 3/2]$ and $1/\rho + 1/\varrho = 1$. In turn, applying [5, Lemma 2.1, eq. (2.1a)] to bound the second term on the right-hand side of (3.16), using again the Sobolev embedding of $\mathbf{L}^4(\Omega)$ into $\mathbf{L}^\rho(\Omega)$, and simple algebraic computations, we deduce that there exists $c_\rho > 0$, depending only on $|\Omega|$ and ρ, such that

$$\begin{aligned} \||\mathbf{u}|^{\rho-2}\mathbf{u} - |\mathbf{v}|^{\rho-2}\mathbf{v}\|_{\mathbf{L}^\varrho(\Omega)} &\leq c_\rho\,\big(\|\mathbf{u}\|_{\mathbf{L}^\rho(\Omega)} + \|\mathbf{v}\|_{\mathbf{L}^\rho(\Omega)}\big)^{\rho-2}\,\|\mathbf{u} - \mathbf{v}\|_{\mathbf{L}^\rho(\Omega)} \\ &\leq c_\rho\,|\Omega|^{(4-\rho)(\rho-1)/(4\rho)}\,(\|\mathbf{u}\|_{\mathbf{M}} + \|\mathbf{v}\|_{\mathbf{M}})^{\rho-2}\,\|\mathbf{u} - \mathbf{v}\|_{\mathbf{M}}\,. \end{aligned} \tag{3.17}$$

Then, replacing (3.17) into (3.16), and after simple computations, we obtain (3.15) with

$$L_{\mathtt{KV}} \,=\, \big(1 + \|\boldsymbol{\chi}\|_{\mathbf{M}}\big)\,\max\Big\{|\Omega|^{1/2} + \mathtt{D}\,|\Omega|^{1/2}, \mathtt{F}\,c_\rho\,|\Omega|^{(4-\rho)/4}, \kappa^2 + \nu, 1\Big\}\,,$$

which completes the proof. □

We continue our analysis by proving hypothesis (ii) of Theorem 3.2, namely, the strong monotonicity of $\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})$ (cf. (2.16)–(2.19)) for a given $\boldsymbol{\chi} \in \mathbf{M}$ on the kernel $\mathbf{K}$ of the operator $\mathcal{B}$ (cf. (2.20)). To this end, we first note that, proceeding similarly to [18, eq. (3.34)], $\mathbf{K}$ can be characterized as

$$\mathbf{K} = \Big\{\underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\xi}) \in \mathbf{M} \times \mathbb{Q} : \quad \nabla\mathbf{v} = \boldsymbol{\xi} \quad \text{and } \mathbf{v} \in \mathbf{H}^1_0(\Omega)\Big\}\,. \tag{3.18}$$

Lemma 3.4 *Let $\rho \in [3,4]$ and let $\boldsymbol{\chi} \in \mathbf{M}$ be given. Then the family of operators given by*

$$\Big\{ (\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\cdot + \underline{\mathbf{z}}) : \mathbf{K} \to \mathbf{K}' : \quad \underline{\mathbf{z}} \in \mathbf{M} \times \mathbb{Q} \Big\}$$

is uniformly strongly monotone, that is, there exists a constant $\gamma_{\mathtt{KV}} > 0$, depending only on $\mathtt{D}$, κ, ν, and $\|\mathbf{i}_4\|$, such that

$$\big[\big(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})\big)(\underline{\mathbf{u}} + \underline{\mathbf{z}}) - \big(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})\big)(\underline{\mathbf{v}} + \underline{\mathbf{z}}), \underline{\mathbf{u}} - \underline{\mathbf{v}} \big] \,\geq\, \gamma_{\mathtt{KV}} \, \|\underline{\mathbf{u}} - \underline{\mathbf{v}}\|^2 \,, \tag{3.19}$$

for all $\underline{\mathbf{z}} = (\mathbf{z}, \boldsymbol{\psi}) \in \mathbf{M} \times \mathbb{Q}$ and for all $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\vartheta})$, $\underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\xi}) \in \mathbf{K}$.

Proof. Let $\boldsymbol{\chi} \in \mathbf{M}$, $\underline{\mathbf{z}} = (\mathbf{z}, \boldsymbol{\psi}) \in \mathbf{M} \times \mathbb{Q}$ and $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\vartheta})$, $\underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\xi}) \in \mathbf{K}$. Then, according to the definition of the operators $\mathcal{E}$ and $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.16)–(2.19)), and using, thanks to the skew-symmetry property (3.11), that there holds $[\mathbf{c}(\boldsymbol{\chi})(\underline{\mathbf{u}} - \underline{\mathbf{v}}), \underline{\mathbf{u}} - \underline{\mathbf{v}}] = 0$, we first obtain

$$\begin{aligned} &\big[\big(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})\big)(\underline{\mathbf{u}} + \underline{\mathbf{z}}) - \big(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})\big)(\underline{\mathbf{v}} + \underline{\mathbf{z}}), \underline{\mathbf{u}} - \underline{\mathbf{v}} \big] \,=\, (1 + \mathtt{D}) \, \|\mathbf{u} - \mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} \\ &+ (\kappa^2 + \nu) \, \|\boldsymbol{\vartheta} - \boldsymbol{\xi}\|^2_{\mathbb{Q}} \,+\, \mathtt{F} \, (|\mathbf{u} + \mathbf{z}|^{\rho-2}(\mathbf{u} + \mathbf{z}) - |\mathbf{v} + \mathbf{z}|^{\rho-2}(\mathbf{v} + \mathbf{z}), \mathbf{u} - \mathbf{v})_\Omega \,. \end{aligned} \tag{3.20}$$

Next, noting from the definition of $\mathbf{K}$ (cf. (3.18)) that $\boldsymbol{\vartheta} - \boldsymbol{\xi} = \nabla(\mathbf{u} - \mathbf{v})$ in Ω and $\mathbf{u} - \mathbf{v} \in \mathbf{H}^1_0(\Omega)$, we can write

$$(\kappa^2 + \nu) \, \|\boldsymbol{\vartheta} - \boldsymbol{\xi}\|^2_{\mathbb{Q}} \,=\, \frac{(\kappa^2 + \nu)}{2} \, \|\boldsymbol{\vartheta} - \boldsymbol{\xi}\|^2_{\mathbb{Q}} \,+\, \frac{(\kappa^2 + \nu)}{2} \, |\mathbf{u} - \mathbf{v}|^2_{\mathbf{H}^1(\Omega)} \,. \tag{3.21}$$

In turn, employing [5, Lemma 2.1, eq. (2.1b)], we deduce that there exists $\widetilde{C} > 0$, depending only on $|\Omega|$, ρ, and $\mathtt{F}$, such that the last term in (3.20) is bounded below as

$$\mathtt{F} \, (|\mathbf{u} + \mathbf{z}|^{\rho-2}(\mathbf{u} + \mathbf{z}) - |\mathbf{v} + \mathbf{z}|^{\rho-2}(\mathbf{v} + \mathbf{z}), \mathbf{u} - \mathbf{v})_\Omega \,\geq\, \widetilde{C} \, \|\mathbf{u} - \mathbf{v}\|^\rho_{\mathbf{L}^\rho(\Omega)} \,\geq\, 0 \,. \tag{3.22}$$

Thus, replacing (3.21) and (3.22) back into (3.20), and using the continuous injection $\mathbf{i}_4$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{M} := \mathbf{L}^4(\Omega)$, we arrive at (3.19) with the constant

$$\gamma_{\mathtt{KV}} \,:=\, \min\left\{ 1 + \mathtt{D}, \frac{\kappa^2 + \nu}{2} \right\} \, \min\left\{ 1, \|\mathbf{i}_4\|^{-2} \right\} .$$

□

We end the verification of the hypotheses of Theorem 3.2, with the corresponding inf-sup condition for the operator $\mathcal{B}$.

Lemma 3.5 *There exists a constant $\beta > 0$ such that*

$$\sup_{\substack{\underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q} \\ \underline{\mathbf{v}} \neq \mathbf{0}}} \frac{[\mathcal{B}(\underline{\mathbf{v}}), \boldsymbol{\tau}]}{\|\underline{\mathbf{v}}\|} \,\geq\, \beta \, \|\boldsymbol{\tau}\|_{\mathbb{X}} \quad \forall \, \boldsymbol{\tau} \in \mathbb{X}_0 \,. \tag{3.23}$$

Proof. We refer the reader to [18, eq. (3.44), Lemma 3.3] for further details. □

Now, we are in a position of establishing the solvability of the resolvent system (3.14).

Lemma 3.6 *Given $\boldsymbol{\chi} \in \mathbf{M}$ and $(\widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_2) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$, there exists a unique $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \boldsymbol{\vartheta}), \boldsymbol{\sigma}) \in \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0$ solution of the resolvent system (3.14).*

Proof. Having established Lemmas 3.3, 3.4, and 3.5, the proof follows from a straightforward application of Theorem 3.2. □

3.3 Construction of compatible initial data

We now derive suitable initial data for (2.15), which are necessary for the corresponding application of Theorem 3.1.

Lemma 3.7 *Assume that* $\mathbf{u}_0 \in \mathbf{H}$*, where*

$$\mathbf{H} := \Big\{ \mathbf{v} \in \mathbf{H}^1_0(\Omega) : \quad \nabla \mathbf{v} \in \mathbb{L}^4(\Omega) \quad \textit{and} \quad \mathrm{div}(\mathbf{v}) = 0 \quad \textit{in} \quad \Omega \Big\} . \tag{3.24}$$

Define

$$\boldsymbol{\vartheta}_0 := \nabla \mathbf{u}_0 \,, \quad \boldsymbol{\sigma}_0 := \mathbf{0} \,, \quad \textit{and} \quad \underline{\mathbf{u}}_0 \,:=\, (\mathbf{u}_0, \boldsymbol{\vartheta}_0) \,. \tag{3.25}$$

Then, $(\underline{\mathbf{u}}_0, \boldsymbol{\sigma}_0) \in \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0$ *and, for a given* $\boldsymbol{\chi} \in \mathbf{M}$*,*

$$\left(\begin{array}{cc} \mathcal{A}(\boldsymbol{\chi}) & \mathcal{B}^{\mathrm{t}} \\ -\mathcal{B} & \mathbf{0} \end{array} \right) \left(\begin{array}{c} \underline{\mathbf{u}}_0 \\ \boldsymbol{\sigma}_0 \end{array} \right) \in \big(\mathbf{L}^2(\Omega) \times \mathbb{L}^2(\Omega) \big) \times \{\mathbf{0}\} \,. \tag{3.26}$$

Proof. Given $\mathbf{u}_0 \in \mathbf{H}$, the embedding $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ guarantees that $\mathbf{u}_0 \in \mathbf{M}$. In turn, for $\boldsymbol{\vartheta}_0$ as defined in (3.25), there holds $\mathrm{tr}(\boldsymbol{\vartheta}_0) = \mathrm{div}(\mathbf{u}_0) = 0$ in Ω, whence $\boldsymbol{\vartheta}_0 \in \mathbb{Q}$ and hence $(\mathbf{u}_0, \boldsymbol{\vartheta}_0) \in \mathbf{K}$ (cf. (3.18)). In addition, it trivially follows that $\boldsymbol{\sigma}_0 = \mathbf{0} \in \mathbb{X}_0$. Thus, given $\boldsymbol{\chi} \in \mathbf{M}$, it follows from the definition of $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.17)–(2.19)) and by integrating by parts the first identity in (3.25) that

$$\begin{aligned} [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_0), \underline{\mathbf{v}}] + [\mathcal{B}^{\mathrm{t}}(\boldsymbol{\sigma}_0), \underline{\mathbf{v}}] \quad &= \quad [F_0, \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q} \,, \\ -\, [\mathcal{B}(\underline{\mathbf{u}}_0), \boldsymbol{\tau}] \quad &= \quad 0 \qquad\quad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0 \,, \end{aligned} \tag{3.27}$$

where, $F_0 := \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_0)$ and

$$\begin{aligned} [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_0), \underline{\mathbf{v}}] \;&=\; (\mathtt{D}\, \mathbf{u}_0 + \mathtt{F}\, |\mathbf{u}_0|^{\rho-2} \mathbf{u}_0, \mathbf{v})_\Omega + \nu\, (\nabla \mathbf{u}_0, \boldsymbol{\xi})_\Omega \\ +\quad & \frac{1}{2} \Big\{ (\nabla \mathbf{u}_0, (\mathbf{v} \otimes \boldsymbol{\chi}))_\Omega - ((\mathbf{u}_0 \otimes \boldsymbol{\chi}), \boldsymbol{\xi})_\Omega \Big\} \,, \end{aligned} \tag{3.28}$$

so that applying the Cauchy–Schwarz and Hölder inequalities, we obtain

$$\begin{aligned} [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_0), \underline{\mathbf{v}}] \,\leq\, & C_0 \Big\{ \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \,+\, \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{L}^{2(\rho-1)}(\Omega)} \\ & + \|\boldsymbol{\chi}\|_{\mathbf{M}} \big(\|\nabla \mathbf{u}_0\|_{\mathbb{L}^4(\Omega)} \,+\, \|\mathbf{u}_0\|_{\mathbf{M}} \big) \Big\} \, \|(\mathbf{v}, \boldsymbol{\xi})\|_{\mathbf{L}^2(\Omega) \times \mathbb{L}^2(\Omega)} \,, \end{aligned} \tag{3.29}$$

for all $\underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q}$, with $2(\rho - 1) \in [4, 6]$ and $C_0 \,:=\, \max \Big\{ \mathtt{D}, \mathtt{F}, \nu, \dfrac{1}{2} \Big\}$. In this way, $F_0 := \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_0)$ can be uniquely identified with an element of $\mathbf{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$, and, from the second row of (3.27), its right-hand side is the null functional, thus proving (3.26). $\square$

We stress that the assumption on the initial condition $\mathbf{u}_0 \in \mathbf{H}$ is more restrictive than the one employed in [16, Lemma 3.8] for the analysis of the Kelvin–Voigt–Brinkman–Forchheimer problem, since it additionally requires $\nabla \mathbf{u}_0 \in \mathbb{L}^4(\Omega)$.

3.4 Main result

We now establish the well-posedness of problem (2.15).

Theorem 3.8 *For each compatible initial data* $(\underline{\mathbf{u}}_0, \boldsymbol{\sigma}_0) = ((\mathbf{u}_0, \boldsymbol{\vartheta}_0), \mathbf{0})$ *constructed in Lemma 3.7, and for each* $\mathbf{f} \in \mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega))$ *and* $\boldsymbol{\chi} \in \mathbf{M}$*, there exists a unique solution* $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \boldsymbol{\vartheta}), \boldsymbol{\sigma}) : [0,T] \to \big(\mathbf{M}\times\mathbb{Q}\big)\times\mathbb{X}_0$ *to* (2.15)*, such that* $(\mathbf{u}, \boldsymbol{\vartheta}) \in \mathrm{H}^1(0,T;\mathbf{M})\times\mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))$ *and* $(\mathbf{u}(0), \boldsymbol{\vartheta}(0)) = (\mathbf{u}_0, \boldsymbol{\vartheta}_0) = (\mathbf{u}_0, \nabla\mathbf{u}_0) \in \mathbf{K}$ *(cf.* (3.18)*).*

Proof. Similarly as in [16, Theorem 3.9], the proof reduces to show first that (2.15) fits the abstract context of Theorem 3.1, with the definitions (3.12) and (3.13). In fact, operator $\mathcal{N}$ is clearly linear, symmetric, and monotone (cf. (2.16), (3.8)). Moreover, according to the definition of $\mathbf{a}$ (cf. (2.18)) and the inequality (3.22), with $\mathbf{z} = \mathbf{0}$, it readily follows that

$$[\mathbf{a}(\underline{\mathbf{u}}) - \mathbf{a}(\underline{\mathbf{v}}), \underline{\mathbf{u}} - \underline{\mathbf{v}}] \,\geq\, \mathtt{D}\,\|\mathbf{u} - \mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \nu\,\|\boldsymbol{\vartheta} - \boldsymbol{\xi}\|^2_{\mathbb{L}^2(\Omega)}\,, \tag{3.30}$$

for all $\underline{\mathbf{u}} := (\mathbf{u}, \boldsymbol{\vartheta})$, $\underline{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\xi}) \in \mathbf{M}\times\mathbb{Q}$. Hence, given the time-independent convective velocity $\boldsymbol{\chi} \in \mathbf{M}$, we deduce from the definition of $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.17)) and the skew-symmetry property of $\mathbf{c}(\boldsymbol{\chi})$ (cf. (3.11)), that

$$\big[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{v}}), \underline{\mathbf{u}} - \underline{\mathbf{v}}\big] = [\mathbf{a}(\underline{\mathbf{u}}) - \mathbf{a}(\underline{\mathbf{v}}), \underline{\mathbf{u}} - \underline{\mathbf{v}}]\,, \tag{3.31}$$

which, together with (3.30), shows that $\mathcal{A}(\boldsymbol{\chi})$ is monotone and, consequently, that $\mathcal{M}$ (cf. (3.12)) is also monotone. In turn, we know from Lemma 3.6 that, given $\boldsymbol{\chi} \in \mathbf{M}$ and $((\widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_2), \mathbf{0}) \in E_b'$, there exists a unique $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) \in \big(\mathbf{M}\times\mathbb{Q}\big)\times\mathbb{X}_0$ such that $((\widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_2), \mathbf{0}) = (\mathcal{N} + \mathcal{M})(\underline{\mathbf{u}}, \boldsymbol{\sigma})$, thus proving that $R(\mathcal{N} + \mathcal{M}) = E_b'$. Finally, given $\mathbf{u}_0 \in \mathbf{H}$, a direct application of Lemma 3.7 ensures the existence of $\boldsymbol{\vartheta}_0 \in \mathbb{Q}$ and $\boldsymbol{\sigma}_0 = \mathbf{0} \in \mathbb{X}_0$ for any $\boldsymbol{\chi} \in \mathbf{M}$ such that $(\underline{\mathbf{u}}_0, \boldsymbol{\sigma}_0) = ((\mathbf{u}_0, \boldsymbol{\vartheta}_0), \mathbf{0}) \in \mathcal{D}$ (cf. (3.13), (3.26)). Therefore, applying Theorem 3.1 in our setting, we conclude that, given $\boldsymbol{\chi} \in \mathbf{M}$ and for each $\mathbf{f} \in \mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega))$, there exists a solution $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \boldsymbol{\vartheta}), \boldsymbol{\sigma})$ to problem (2.15), with $(\mathbf{u}, \boldsymbol{\vartheta}) \in \mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))\times\mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))$, satisfying $(\mathbf{u}(0), \boldsymbol{\vartheta}(0)) = (\mathbf{u}_0, \boldsymbol{\vartheta}_0) = (\mathbf{u}_0, \nabla\mathbf{u}_0) \in \mathbf{K}$. Next, we prove that $\mathbf{u} \in \mathrm{H}^1(0,T;\mathbf{M})$. Indeed, since the second row of (2.15) implies that $(\mathbf{u}, \boldsymbol{\vartheta}) : (0,T] \to \mathbf{K}$ and $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \hookrightarrow \mathbf{M}$ is a continuous embedding, we first observe that

$$\|\mathbf{u}\|^2_{\mathbf{M}} \,\leq\, \|\mathbf{i}_4\|^2\,\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} = \|\mathbf{i}_4\|^2\,\Big(\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\vartheta}\|^2_{\mathbb{L}^2(\Omega)}\Big). \tag{3.32}$$

In turn, knowing that $\partial_t\mathbf{u} \in \mathbf{L}^2(\Omega)$ and $\partial_t\boldsymbol{\vartheta} \in \mathbb{L}^2(\Omega)$, and noting that $\partial_t\boldsymbol{\vartheta} = \partial_t\nabla\mathbf{u} = \nabla(\partial_t\mathbf{u})$ in the distributional sense, we deduce that $\partial_t\mathbf{u} \in \mathbf{H}^1(\Omega)$, which yields in particular $\partial_t\mathbf{u} \in \mathbf{M} := \mathbf{L}^4(\Omega)$ with

$$\|\partial_t\,\mathbf{u}\|^2_{\mathbf{M}} \leq \|\mathbf{i}_4\|^2\,\|\partial_t\,\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} = \|\mathbf{i}_4\|^2\,\Big(\|\partial_t\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t\,\boldsymbol{\vartheta}\|^2_{\mathbb{L}^2(\Omega)}\Big)\,, \tag{3.33}$$

and thus $\partial_t\underline{\mathbf{u}} := (\partial_t\mathbf{u}, \partial_t\boldsymbol{\vartheta}) : (0,T] \to \mathbf{K}$. In this way, integrating (3.32) and (3.33) over $(0,T)$, we get

$$\|\mathbf{u}\|^2_{\mathrm{H}^1(0,T;\mathbf{M})} = \|\mathbf{u}\|^2_{\mathrm{L}^2(0,T;\mathbf{M})} + \|\partial_t\,\mathbf{u}\|^2_{\mathrm{L}^2(0,T;\mathbf{M})} \leq \|\mathbf{i}_4\|^2\Big(\|\mathbf{u}\|^2_{\mathrm{H}^1(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\vartheta}\|^2_{\mathrm{H}^1(0,T;\mathbb{L}^2(\Omega))}\Big)\,,$$

which, together with the regularity of the solution $(\mathbf{u}, \boldsymbol{\vartheta}) \in \mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))\times\mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))$ implies that $\mathbf{u} \in \mathrm{H}^1(0,T;\mathbf{M})$. We now prove the uniqueness of (2.15). To this end, let $\boldsymbol{\chi} \in \mathbf{M}$ be given, and let $(\underline{\mathbf{u}}_i, \boldsymbol{\sigma}_i) = ((\mathbf{u}_i, \boldsymbol{\vartheta}_i), \boldsymbol{\sigma}_i)$, with $i \in \{1,2\}$, be two solutions of (2.15) corresponding to the same data. We recall that the external time derivative in (2.15) acts only on the first component of $\mathcal{E}$, which, in particular, yields

$$\partial_t[\mathcal{E}(\underline{\mathbf{v}}(t)), \underline{\mathbf{v}}] \,=\, (\partial_t\mathbf{v}(t), \mathbf{v})_\Omega \,+\, \kappa^2\,(\partial_t\boldsymbol{\xi}(t), \boldsymbol{\xi})_\Omega \,=\, \frac{1}{2}\,\partial_t\Big(\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\|\boldsymbol{\xi}\|^2_{\mathbb{Q}}\Big)\,, \tag{3.34}$$

for all $\underline{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\xi}) : [0,T] \to \mathbf{M}\times\mathbb{Q}$. Thus, taking $\underline{\mathbf{v}} = \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2 \in \mathbf{M}\times\mathbb{Q}$ and $\boldsymbol{\tau} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \in \mathbb{X}_0$, subtracting the corresponding problems in (2.15), using (3.34) and the monotonicity property of $\mathcal{A}(\boldsymbol{\chi})$

(cf. (3.30)–(3.31)), we find

$$
\begin{aligned}
&\frac{1}{2}\,\partial_t\Big(\|\mathbf{u}_1-\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}+\kappa^2\|\boldsymbol{\vartheta}_1-\boldsymbol{\vartheta}_2\|^2_{\mathbb{Q}}\Big)\,+\,\mathtt{D}\,\|\mathbf{u}_1-\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}+\nu\,\|\boldsymbol{\vartheta}_1-\boldsymbol{\vartheta}_2\|^2_{\mathbb{Q}}\\
&\qquad\le\;\partial_t[\mathcal{E}(\underline{\mathbf{u}}_1-\underline{\mathbf{u}}_2),\underline{\mathbf{u}}_1-\underline{\mathbf{u}}_2]+\big[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_1)-\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_2),\underline{\mathbf{u}}_1-\underline{\mathbf{u}}_2\big]=0\,.
\end{aligned}
$$

Integrating the latter inequality in time from 0 to $t\in(0,T]$, and using that $\big(\mathbf{u}_1(0),\boldsymbol{\vartheta}_1(0)\big)=\big(\mathbf{u}_2(0),\boldsymbol{\vartheta}_2(0)\big)$, we deduce that

$$
\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_{\mathbf{L}^2(\Omega)}+\kappa^2\,\|\boldsymbol{\vartheta}_1(t)-\boldsymbol{\vartheta}_2(t)\|^2_{\mathbb{Q}}+2\int_0^t\Big(\mathtt{D}\,\|\mathbf{u}_1-\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}+\nu\,\|\boldsymbol{\vartheta}_1-\boldsymbol{\vartheta}_2\|^2_{\mathbb{Q}}\Big)ds\le 0\,. \tag{3.35}
$$

Therefore, it follows from (3.35) that $\mathbf{u}_1(t)=\mathbf{u}_2(t)$ and $\boldsymbol{\vartheta}_1(t)=\boldsymbol{\vartheta}_2(t)$ for all $t\in(0,T]$. Next, from the inf-sup condition of the operator $\mathcal{B}$ (cf. (3.23)) and the first equation of (2.15), we get

$$
\begin{aligned}
\beta\,\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|_{\mathbb{X}}\;&\le\;\sup_{\substack{\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\\ \underline{\mathbf{v}}\neq\mathbf{0}}}\frac{[\mathcal{B}^{\mathrm{t}}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2),\underline{\mathbf{v}}]}{\|\underline{\mathbf{v}}\|}\\
&=\;-\sup_{\substack{\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\\ \underline{\mathbf{v}}\neq\mathbf{0}}}\frac{\partial_t[\mathcal{E}(\underline{\mathbf{u}}_1-\underline{\mathbf{u}}_2),\underline{\mathbf{v}}]+[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_1)-\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_2),\underline{\mathbf{v}}]}{\|\underline{\mathbf{v}}\|}\;=\;0\,,
\end{aligned} \tag{3.36}
$$

which implies that $\boldsymbol{\sigma}_1(t)=\boldsymbol{\sigma}_2(t)$ for all $t\in(0,T]$, and therefore (2.15) has a unique solution. $\square$

We conclude this section with the corresponding stability bounds for the solution of (2.15).

Theorem 3.9 *Let $\rho\in[3,4]$. Assume that $\mathbf{f}\in\mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega))\cap\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))$, $\boldsymbol{\chi}\in\mathbf{M}$, and $\mathbf{u}_0\in\mathbf{H}$ satisfying (3.26). Then there exists a constant $C_{\mathtt{KV}\underline{\mathbf{u}}}>0$, depending only on $\nu,\mathtt{D}$, κ, and $\|\mathbf{i}_4\|$, such that*

$$
\begin{aligned}
&\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}+\|\boldsymbol{\vartheta}\|_{\mathrm{L}^\infty(0,T;\mathbb{Q})}+\|\mathbf{u}\|_{\mathrm{L}^2(0,T;\mathbf{M})}+\|\boldsymbol{\vartheta}\|_{\mathrm{L}^2(0,T;\mathbb{Q})}\\
&\qquad\le\;C_{\mathtt{KV}\underline{\mathbf{u}}}\Big(\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))}+\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)}\Big)\,.
\end{aligned} \tag{3.37}
$$

In addition, there exists a constant $C_{\mathtt{KV}\boldsymbol{\sigma}}>0$, depending only on $\|\boldsymbol{\chi}\|_{\mathbf{M}}$, $|\Omega|$, ν, $\mathtt{D}$, $\mathtt{F}$, ρ, β, κ and $\|\mathbf{i}_4\|$, such that

$$
\|\boldsymbol{\sigma}\|_{\mathrm{L}^2(0,T;\mathbb{X}_0)}\le C_{\mathtt{KV}\boldsymbol{\sigma}}\left\{\sum_{j\in\{2,\rho\}}\Big(\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))}+\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)}\Big)^{j-1}+\|\mathbf{u}_0\|^{\rho/2}_{\mathbf{L}^\rho(\Omega)}\right\}. \tag{3.38}
$$

Proof. We first derive the stability bound (3.37). For this purpose, we proceed as for the derivation of (3.35), testing (2.15) against $(\underline{\mathbf{v}},\boldsymbol{\tau})=(\underline{\mathbf{u}},\boldsymbol{\sigma})$, invoking the definition of $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.17)) along with the skew-symmetry property (3.11), the identity (3.34), and then applying the Cauchy–Schwarz and Young inequalities, to find that

$$
\begin{aligned}
&\partial_t\Big(\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}\,+\,\kappa^2\,\|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\Big)\,+\,2\,\mathtt{D}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}\,+\,2\,\mathtt{F}\,\|\mathbf{u}\|^\rho_{\mathbf{L}^\rho(\Omega)}\,+\,2\,\nu\,\|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\\
&\qquad=\;2\,(\mathbf{f},\mathbf{u})_\Omega\;\le\;2\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\;\le\;\frac{1}{\mathtt{D}}\,\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}\,+\,\mathtt{D}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}\,.
\end{aligned} \tag{3.39}
$$

Simplifying $\mathtt{D}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}$ at the end of (3.39), dropping $2\,\mathtt{F}\,\|\mathbf{u}\|^\rho_{\mathbf{L}^\rho(\Omega)}$, and then integrating the remaining terms from 0 to $t\in(0,T]$, we get

$$
\begin{aligned}
&\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}\,+\,\kappa^2\,\|\boldsymbol{\vartheta}(t)\|^2_{\mathbb{Q}}\,+\,\int_0^t\Big(\mathtt{D}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}\,+\,2\,\nu\|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\Big)ds\\
&\qquad\le\;\|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)}\,+\,\kappa^2\,\|\boldsymbol{\vartheta}(0)\|^2_{\mathbb{Q}}\,+\,\frac{1}{\mathtt{D}}\int_0^t\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}ds\,.
\end{aligned} \tag{3.40}
$$

Next, recalling from the second row of (2.15) that $(\mathbf{u},\boldsymbol{\vartheta}) : (0,T] \to \mathbf{K}$ (cf. (3.18)), we infer that the first two terms in (3.40) are bounded from below as follows:

$$\begin{aligned} \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \,+\, \kappa^2\,\|\boldsymbol{\vartheta}(t)\|^2_{\mathbb{Q}} \,\geq\, \min\left\{1,\frac{\kappa^2}{2}\right\}\left(\|\mathbf{u}(t)\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\vartheta}(t)\|^2_{\mathbb{Q}}\right) \\ \geq\, \gamma_\kappa\left(\|\mathbf{u}(t)\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}(t)\|^2_{\mathbb{Q}}\right), \end{aligned} \tag{3.41}$$

with $\gamma_\kappa \,:=\, \min\left\{1,\frac{\kappa^2}{2}\right\}\,\min\left\{1,\|\mathbf{i}_4\|^{-2}\right\}$. Similarly, the terms in the time integral on the left-hand side of (3.40) can be bounded as

$$\mathtt{D}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \,+\, 2\nu\,\|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}} \,\geq\, \gamma_{\mathtt{D}\nu}\left(\|\mathbf{u}\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\right), \tag{3.42}$$

with $\gamma_{\mathtt{D}\nu} := \min\{\mathtt{D},\nu\}\,\min\left\{1,\|\mathbf{i}_4\|^{-2}\right\}$. Thus, replacing (3.41) and (3.42) back into (3.40), we get

$$\begin{aligned} \gamma_\kappa\left(\|\mathbf{u}(t)\|^2_{\mathbf{M}} \,+\, \|\boldsymbol{\vartheta}(t)\|^2_{\mathbb{Q}}\right) \,+\, \gamma_{\mathtt{D}\nu}\int_0^t\left(\|\mathbf{u}\|^2_{\mathbf{M}} \,+\, \|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\right)ds \\ \leq\, \frac{1}{\mathtt{D}}\int_0^t \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}ds \,+\, \|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} \,+\, \kappa^2\,\|\boldsymbol{\vartheta}(0)\|^2_{\mathbb{Q}}, \end{aligned} \tag{3.43}$$

which, together with the fact that $(\mathbf{u}(0),\boldsymbol{\vartheta}(0)) = (\mathbf{u}_0,\boldsymbol{\vartheta}_0) = (\mathbf{u}_0,\nabla\mathbf{u}_0)$, yields (3.37) with $C_{\mathtt{KV}\underline{\mathbf{u}}} > 0$ depending only on $\nu, \mathtt{D}$, κ and $\|\mathbf{i}_4\|$. We now employ the inf-sup condition satisfied by $\mathcal{B}$ (cf. (3.23)) to deduce (3.38). Indeed, given $\boldsymbol{\chi}\in\mathbf{M}$, using the first equation of (2.15), and the stability bounds of F, $\mathcal{E}$, and $\mathcal{A}(\boldsymbol{\chi})$ (cf. (3.4), (3.7), and (3.9)), we first observe that

$$\begin{aligned} \beta\,\|\boldsymbol{\sigma}\|_{\mathbb{X}} \,\leq\, \sup_{\substack{\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\\ \underline{\mathbf{v}}\neq 0}} \frac{[F,\underline{\mathbf{v}}] - [\partial_t\mathcal{E}(\underline{\mathbf{u}}),\underline{\mathbf{v}}] - [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}),\underline{\mathbf{v}}]}{\|\underline{\mathbf{v}}\|} \\ \leq\, \widehat{C}_1\left\{\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\partial_t\underline{\mathbf{u}}\| + \|\mathbf{u}\|^{\rho-1}_{\mathbf{M}} + \big(1+\|\boldsymbol{\chi}\|_{\mathbf{M}}\big)\big(\|\mathbf{u}\|_{\mathbf{M}} + \|\boldsymbol{\vartheta}\|_{\mathbb{Q}}\big)\right\}, \end{aligned} \tag{3.44}$$

with a positive constant $\widehat{C}_1$ depending on $|\Omega|$, κ, and $C_{\mathcal{A}}$ (cf. (3.9)). Then, taking square in (3.44), integrating from 0 to $t\in(0,T]$, and rearranging the resulting terms, we deduce that there exists a positive constant C_1, depending only on $\widehat{C}_1$ and β, such that

$$\begin{aligned} \int_0^t \|\boldsymbol{\sigma}\|^2_{\mathbb{X}}ds \,\leq\, C_1\left\{\int_0^t\left(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \big(1+\|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big)\big(\|\mathbf{u}\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\big)\right)ds \right. \\ \left. +\,\int_0^t\left(\|\mathbf{u}\|^{2(\rho-1)}_{\mathbf{M}} + \|\partial_t\underline{\mathbf{u}}\|^2\right)ds\right\}. \end{aligned} \tag{3.45}$$

In turn, to bound the last term in (3.45), we differentiate in time the second equation of (2.15) and test it with $(\underline{\mathbf{v}},\boldsymbol{\tau}) = (\partial_t\underline{\mathbf{u}},\boldsymbol{\sigma})$, where $\partial_t\underline{\mathbf{u}} := (\partial_t\mathbf{u},\partial_t\boldsymbol{\vartheta}) : (0,T]\to\mathbf{K}$ (cf. (3.33)). Using arguments similar to those leading to (3.41), together with (3.5) and the Cauchy–Schwarz and Young inequalities, we obtain

$$\begin{aligned} \frac{1}{2}\,\partial_t\Big(\mathtt{D}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\|\mathbf{u}\|^{\rho}_{\mathbf{L}^{\rho}(\Omega)} + \nu\|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\Big) + \gamma_\kappa\|\partial_t\underline{\mathbf{u}}\|^2 \\ \leq\, \Big(|\Omega|^{1/4}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\chi}\|_{\mathbf{M}}\big(\|\mathbf{u}\|_{\mathbf{M}} + \|\boldsymbol{\vartheta}\|_{\mathbb{Q}}\big)\Big)\|\partial_t\underline{\mathbf{u}}\| \\ \leq\, C_2\left(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big(\|\mathbf{u}\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\big)\right) + \frac{\gamma_\kappa}{2}\|\partial_t\underline{\mathbf{u}}\|^2, \end{aligned} \tag{3.46}$$

with γ_κ as in (3.41) and $C_2 > 0$ depending only on $|\Omega|, \|\mathbf{i}_4\|$, and κ. Thus, integrating (3.46) from 0 to $t \in (0, T]$, we obtain

$$
\begin{aligned}
&\mathtt{D}\,\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\,\|\mathbf{u}(t)\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\vartheta}(t)\|^2_{\mathbb{Q}} + \gamma_\kappa \int_0^t \|\partial_t \underline{\mathbf{u}}\|^2 ds \\
&\quad\le \mathtt{D}\,\|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\|\mathbf{u}(0)\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\vartheta}(0)\|^2_{\mathbb{Q}} \\
&\quad+ 2\,C_2 \left\{ \int_0^t \Big(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big(\|\mathbf{u}\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\big)\Big)\, ds \right\}.
\end{aligned}
\tag{3.47}
$$

Thus, dropping the first three terms on the left-hand side of (3.47) and combining the resulting inequality with (3.45) yields

$$
\begin{aligned}
\int_0^t \|\boldsymbol{\sigma}\|^2_{\mathbb{X}} ds \,\le\, C_3 \bigg\{ &\int_0^t \Big(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \big(1 + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big)\big(\|\mathbf{u}\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\big)\Big) ds \\
&+ \|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}(0)\|^\rho_{\mathbf{L}^\rho(\Omega)} + \|\boldsymbol{\vartheta}(0)\|^2_{\mathbb{Q}} + \int_0^t \|\mathbf{u}\|^{2(\rho-1)}_{\mathbf{M}}\, ds \bigg\}.
\end{aligned}
\tag{3.48}
$$

with $C_3 > 0$ depending only on $\nu, \mathtt{D}, \mathtt{F}, \kappa$, and $\|\mathbf{i}_4\|$, so that, using again the fact that $(\mathbf{u}(0), \boldsymbol{\vartheta}(0)) = (\mathbf{u}_0, \boldsymbol{\vartheta}_0) = (\mathbf{u}_0, \nabla \mathbf{u}_0)$, recalling that $\boldsymbol{\chi}$ is time-independent and therefore $\|\boldsymbol{\chi}\|_{\mathbf{M}}$ can be taken outside the integral, and performing some algebraic manipulations, in particular noting that

$$
\int_0^t \|\mathbf{u}\|^{2(\rho-1)}_{\mathbf{M}}\, ds \,=\, \int_0^t \|\mathbf{u}\|^{2(\rho-2)}_{\mathbf{M}}\,\|\mathbf{u}\|^2_{\mathbf{M}}\, ds \,\le\, \|\mathbf{u}\|^{2(\rho-2)}_{\mathrm{L}^\infty(0,T;\mathbf{M})} \int_0^t \|\mathbf{u}\|^2_{\mathbf{M}}\, ds\,,
$$

we find that

$$
\begin{aligned}
\int_0^t \|\boldsymbol{\sigma}\|^2_{\mathbb{X}} ds \,\le\, C_2 \bigg\{ &\int_0^t \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}\, ds + \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^\rho_{\mathbf{L}^\rho(\Omega)} \\
&+ \big(1 + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big) \int_0^t \Big(\|\mathbf{u}\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}\|^2_{\mathbb{Q}}\Big)\, ds + \|\mathbf{u}\|^{2(\rho-2)}_{\mathrm{L}^\infty(0,T;\mathbf{M})} \int_0^t \|\mathbf{u}\|^2_{\mathbf{M}}\, ds \bigg\}.
\end{aligned}
\tag{3.49}
$$

Finally, by combining (3.49) with (3.37), we obtain (3.38), thus concluding the proof. $\square$

We find it important to remark that the analysis developed here can be extended to the problem (2.5) with a nonhomogeneous Dirichlet boundary condition $\mathbf{u} = \mathbf{u}_D$ on $\Gamma \times (0, T]$. To that end, (2.15) has to be rewritten as follows: Given $\mathbf{f} : [0, T] \to \mathbf{L}^2(\Omega)$, $\boldsymbol{\chi} \in \mathbf{M}$, $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{u}_0 \in \mathbf{M} \cap \mathbf{H}$ (cf. (3.24)), find $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) : [0, T] \to (\mathbf{M} \times \mathbb{Q}) \times \mathbb{X}_0$, such that $(\mathbf{u}(0), \boldsymbol{\vartheta}(0)) = (\mathbf{u}_0, \nabla \mathbf{u}_0)$ and, for a.e. $t \in (0, T)$,

$$
\begin{aligned}
\partial_t [\mathcal{E}(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] + [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] + [\mathcal{B}^{\mathrm{t}}(\boldsymbol{\sigma}(t)), \underline{\mathbf{v}}] \;&=\; [F(t), \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q}\,, \\
-\, [\mathcal{B}(\underline{\mathbf{u}}(t)), \boldsymbol{\tau}] \;&=\; \big[G, \boldsymbol{\tau}\big] \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0\,,
\end{aligned}
$$

where the functional $G \in \mathbb{X}_0'$ is given by $\big[G, \boldsymbol{\tau}\big] = \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle_\Gamma$ with $\langle \cdot, \cdot \rangle_\Gamma$ denoting the duality between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. We refer the reader to [11, Lemma 3.5] for the proof that $\boldsymbol{\tau}\mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma)$ for all $\boldsymbol{\tau} \in \mathbb{X}_0$. Then, we reformulate the problem as a parabolic problem for $(\mathbf{u}, \boldsymbol{\vartheta})$, and proceed as in [1, eq. (4.14), Section 4.1].

On the other hand, we end this section by noting in advance that (3.37) will be employed later on to handle the nonlinear terms associated with the operator $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.17)), which is necessary to obtain the corresponding error estimate.

4 Semidiscrete continuous-in-time approximation

In this section we introduce and analyze the semidiscrete continuous-in-time approximation of (2.15).

4.1 Discrete in space setting

Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω consisting of triangles K (when $d=2$) or tetrahedra K (when $d=3$) of diameter h_K, and define the mesh-size $h := \max\big\{h_K : \quad K \in \mathcal{T}_h\big\}$. In turn, given an integer $\ell \geq 0$ and a subset S of R^d, we denote by $\mathrm{P}_\ell(S)$ (resp. $\widetilde{\mathrm{P}}_\ell(S)$) the space of polynomials of total degree at most ℓ (resp. $=\ell$) defined on S. In addition, according to the convention in Section 1 we set $\mathbf{P}_\ell(S) := [\mathrm{P}_\ell(S)]^d$ and $\mathbb{P}_\ell(S) := [\mathrm{P}_\ell(S)]^{d\times d}$, and define the Raviart–Thomas space of order ℓ as

$$\mathbf{RT}_\ell(S) \,:=\, \mathbf{P}_\ell(S) \oplus \widetilde{\mathrm{P}}_\ell(S)\,\mathbf{x}\,,$$

where $\mathbf{x} := (x_1, ..., x_d)^{\mathrm{t}}$ is a generic vector of R^d. Next, given a fixed integer $k \geq 0$, we introduce the finite element subspaces

$$\begin{aligned}
\mathbf{M}_h &:= \Big\{\mathbf{v}_h \in \mathbf{M} : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,\\
\mathbb{Q}_h &:= \Big\{\boldsymbol{\xi}_h \in \mathbb{Q} : \quad \boldsymbol{\xi}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,\\
\mathbb{X}_h &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{X} : \quad \boldsymbol{\tau}_{h,i}|_K \in \mathbf{RT}_k(K) \quad \forall\, i \in \big\{1,\dots,d\big\}\,, \quad \forall\, K \in \mathcal{T}_h\Big\}\,,\\
\mathbb{X}_{0,h} &:= \mathbb{X}_h \,\cap\, \mathbb{X}_0\,,
\end{aligned} \tag{4.1}$$

where $\boldsymbol{\tau}_{h,i}$ denotes the ith-row of $\boldsymbol{\tau}_h$. Then, denoting from now on

$$\underline{\mathbf{u}}_h \,:=\, (\mathbf{u}_h, \boldsymbol{\vartheta}_h)\,, \quad \underline{\mathbf{v}}_h \,:=\, (\mathbf{v}_h, \boldsymbol{\xi}_h) \in \mathbf{M}_h \times \mathbb{Q}_h\,,$$

the semidiscrete continuous-in-time problem associated with (2.15) reads: Given $\mathbf{f} : [0,T] \to \mathbf{L}^2(\Omega)$, $\boldsymbol{\chi} \in \mathbf{M}$, and $(\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0}) \in \mathbf{M}_h \times \mathbb{Q}_h$, find $(\underline{\mathbf{u}}_h, \boldsymbol{\sigma}_h) : [0,T] \to (\mathbf{M}_h \times \mathbb{Q}_h) \times \mathbb{X}_{0,h}$, such that $(\mathbf{u}_h(0), \boldsymbol{\vartheta}_h(0)) = (\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0})$ and, for a.e. $t \in (0,T)$,

$$\begin{aligned}
\partial_t[\mathcal{E}(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] + [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] + [\mathcal{B}^{\mathrm{t}}(\boldsymbol{\sigma}_h(t)), \underline{\mathbf{v}}_h] \;&=\; [F(t), \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\,,\\
-\,[\mathcal{B}(\underline{\mathbf{u}}_h(t)), \boldsymbol{\tau}_h] \;&=\; 0 \qquad\qquad\; \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,.
\end{aligned} \tag{4.2}$$

The well-posedness of (4.2) will be established in the next section. For completeness, we note in advance that it follows analogously to its continuous counterpart given in Theorem 3.8. More precisely, we apply Theorem 3.1 to problem (4.2), which requires the solvability of a resolvent system associated with (4.2) and the construction of appropriate discrete initial data $(\underline{\mathbf{u}}_{h,0}, \boldsymbol{\sigma}_h) = ((\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0}), \boldsymbol{\sigma}_h) \in (\mathbf{M}_h \times \mathbb{Q}_h) \times \mathbb{X}_{h,0}$ that are compatible in the sense of Lemma 3.7. To that end, we begin by introducing $\mathbf{K}_h$, the discrete kernel of $\mathcal{B}$ (cf. (2.20)), that is,

$$\mathbf{K}_h \,:=\, \Big\{\underline{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\xi}_h) \in \mathbf{M}_h \times \mathbb{Q}_h : \quad (\mathbf{v}_h, \mathbf{div}(\boldsymbol{\tau}_h))_\Omega + (\boldsymbol{\xi}_h, \boldsymbol{\tau}_h)_\Omega = 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\Big\}\,. \tag{4.3}$$

We continue the discussion by recalling from [6, Section 4.3] (see also [18, Section 5]) two properties that will be needed in the sequel. The first one is the discrete inf-sup condition for $\mathcal{B}$, while the second is an auxiliary result which, given $\boldsymbol{\chi} \in \mathbf{M}$, will be used to establish the discrete strong monotonicity of $\mathcal{A}(\boldsymbol{\chi})$. For the corresponding proof, we refer the reader to [6, Lemma 4.2].

Lemma 4.1 *There exist positive constants* $\beta_{\mathtt{d}}$ *and* $C_{\mathtt{d}}$ *such that*

$$\sup_{\substack{\underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h \\ \underline{\mathbf{v}}_h \neq 0}} \frac{[\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\tau}_h]}{\|\underline{\mathbf{v}}_h\|} \,\geq\, \beta_{\mathtt{d}}\, \|\boldsymbol{\tau}_h\|_{\mathbb{X}} \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,, \tag{4.4}$$

and

$$\|\boldsymbol{\xi}_h\|_{\mathbb{Q}} \,\geq\, C_{\mathtt{d}}\, \|\mathbf{v}_h\|_{\mathbf{M}} \quad \forall\, \underline{\mathbf{v}}_h \,:=\, (\mathbf{v}_h, \boldsymbol{\xi}_h) \in \mathbf{K}_h\,. \tag{4.5}$$

4.2 Solvability Analysis

We first verify the range condition associated with (4.2) by applying the Theorem 3.2. More precisely, we now consider the resolvent system associated with (4.2), which is the discrete version of (3.14), that is: Given $\boldsymbol{\chi} \in \mathbf{M}$ and $(\widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_2) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$, find $(\underline{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \big(\mathbf{M}_h \times \mathbb{Q}_h\big) \times \mathbb{X}_{0,h}$ such that

$$\begin{aligned} [(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h] + [\mathcal{B}^{\mathrm{t}}(\boldsymbol{\sigma}_h), \underline{\mathbf{v}}_h] \;&=\; [\widehat{F}_h, \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\,, \\ [\mathcal{B}(\underline{\mathbf{u}}_h), \boldsymbol{\tau}_h] \;&=\; 0 \qquad\quad\;\; \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,, \end{aligned} \tag{4.6}$$

where $\widehat{F}_h \in \big(\mathbf{M}_h \times \mathbb{Q}_h\big)'$ is the functional defined as

$$[\widehat{F}_h, \underline{\mathbf{v}}_h] \,:=\, (\widehat{\mathbf{f}}_1, \mathbf{v}_h)_\Omega + (\widehat{\mathbf{f}}_2, \boldsymbol{\xi}_h)_\Omega \qquad \forall\, \underline{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\xi}_h) \in \mathbf{M}_h \times \mathbb{Q}_h\,. \tag{4.7}$$

The discrete versions of Lemmas 3.3 and 3.4, which state the strong monotonicity and continuity of $\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})$, are summarized as follows.

Lemma 4.2 *Let* $\rho \in [3,4]$ *and let* $\boldsymbol{\chi} \in \mathbf{M}$ *be given. Then the family of operators given by*

$$\Big\{ (\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\cdot + \underline{\mathbf{z}}_h) : \mathbf{K}_h \to \mathbf{K}_h' : \quad \underline{\mathbf{z}}_h \in \mathbf{M} \times \mathbb{Q} \Big\}$$

is uniformly strongly monotone, that is, there exists a constant $\gamma_{\mathtt{KVd}} > 0$*, depending only on* κ*,* ν*, and* $C_{\mathtt{d}}$ *(cf. (4.5)), such that*

$$\big[(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{u}}_h + \underline{\mathbf{z}}_h) - (\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{v}}_h + \underline{\mathbf{z}}_h),\; \underline{\mathbf{u}}_h - \underline{\mathbf{v}}_h \big] \,\geq\, \gamma_{\mathtt{KVd}} \|\underline{\mathbf{u}}_h - \underline{\mathbf{v}}_h\|^2\,, \tag{4.8}$$

for all $\underline{\mathbf{z}}_h = (\mathbf{z}_h, \boldsymbol{\psi}_h) \in \mathbf{M}_h \times \mathbb{Q}_h$ *and for all* $\underline{\mathbf{u}}_h = (\mathbf{u}_h, \boldsymbol{\vartheta}_h)$*,* $\underline{\mathbf{v}}_h = (\mathbf{v}_h, \boldsymbol{\xi}_h) \in \mathbf{K}_h$*. In addition, the operator* $\mathcal{E} + \mathcal{A}(\boldsymbol{\chi})$ *is continuous, exactly as stated in (3.15), and with the same constant* $L_{\mathtt{KV}}$*.*

Proof. Let $\boldsymbol{\chi} \in \mathbf{M}$, $\underline{\mathbf{z}}_h = (\mathbf{z}_h, \boldsymbol{\psi}_h) \in \mathbf{M}_h \times \mathbb{Q}_h$ and $\underline{\mathbf{u}}_h = (\mathbf{u}_h, \boldsymbol{\vartheta}_h)$, $\underline{\mathbf{v}}_h = (\mathbf{v}_h, \boldsymbol{\xi}_h) \in \mathbf{K}_h$. Then, using the same arguments that yield (3.20) (cf. the proof of Lemma 3.4) and (3.22), we readily obtain that

$$\big[(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{u}}_h + \underline{\mathbf{z}}_h) - (\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{v}}_h + \underline{\mathbf{z}}_h),\; \underline{\mathbf{u}}_h - \underline{\mathbf{v}}_h \big] \,\geq\, (\kappa^2 + \nu)\, \|\boldsymbol{\vartheta}_h - \boldsymbol{\xi}_h\|_{\mathbb{Q}}^2\,.$$

Thus, using (4.5) for $(\mathbf{u}_h - \mathbf{v}_h, \boldsymbol{\vartheta}_h - \boldsymbol{\xi}_h) \in \mathbf{K}_h$, we deduce

$$\big[(\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{u}}_h + \underline{\mathbf{z}}_h) - (\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}))(\underline{\mathbf{v}}_h + \underline{\mathbf{z}}_h),\; \underline{\mathbf{u}}_h - \underline{\mathbf{v}}_h \big] \,\geq\, \frac{\kappa^2 + \nu}{2} \Big(C_{\mathtt{d}}^2\, \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathbf{M}}^2 + \|\boldsymbol{\vartheta}_h - \boldsymbol{\xi}_h\|_{\mathbb{Q}}^2 \Big)\,,$$

which yields (4.8) with the constant $\gamma_{\mathtt{KVd}} \,:=\, \frac{\kappa^2+\nu}{2} \min\big\{ C_{\mathtt{d}}^2, 1 \big\}$. Furthermore, since no additional arguments or estimates beyond the inequalities (3.16)–(3.17) are required in this discrete setting, the continuity of $\mathcal{E} + \mathcal{A}(\boldsymbol{\chi}) : \big(\mathbf{M}_h \times \mathbb{Q}_h\big) \to \big(\mathbf{M}_h \times \mathbb{Q}_h\big)'$ follows directly from (3.15). □

The aforementioned well-posedness of the resolvent system (4.6), which follows straightforwardly from Theorem 3.2, Lemma 4.2, and the discrete inf-sup condition of $\mathcal{B}$ (cf. (4.4)), is now stated.

Lemma 4.3 *Given $\boldsymbol{\chi} \in \mathbf{M}$ and $(\widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_2) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$, there exists a unique $(\underline{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \big(\mathbf{M}_h \times \mathbb{Q}_h\big) \times \mathbb{X}_{0,h}$ solution of the resolvent system (4.6).*

We now provide appropriate initial data $(\underline{\mathbf{u}}_{h,0}, \boldsymbol{\sigma}_{h,0}) \in (\mathbf{M}_h \times \mathbb{Q}_h) \times \mathbb{X}_{0,h}$ for (4.2), which are taken as the Galerkin approximation of the solution $(\underline{\mathbf{u}}_0, \mathbf{0})$ of (3.27), that is, such that

$$
\begin{aligned}
[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] + [\mathcal{B}^{\mathrm{t}}(\boldsymbol{\sigma}_{h,0}), \underline{\mathbf{v}}_h] &= [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_0), \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h \,, \\
-\,[\mathcal{B}(\underline{\mathbf{u}}_{h,0}), \boldsymbol{\tau}_h] &= 0 \qquad\qquad\qquad\quad\; \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h} \,,
\end{aligned}
\tag{4.9}
$$

where $\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_0)$ is defined by (3.28). This choice is necessary to guarantee that the discrete initial datum is compatible in the sense of Lemma 3.7, which is needed for the application of Theorem 3.1. Notice that the well-posedness of problem (4.9) follows from similar arguments to the proof of Lemma 4.3. More precisely, proceeding as in the proof of (4.8) we are able to deduce for all $\underline{\mathbf{z}}_h = (\mathbf{z}_h, \boldsymbol{\psi}_h) \in \mathbf{M}_h \times \mathbb{Q}_h$ and for all $\underline{\mathbf{u}}_h = (\mathbf{u}_h, \boldsymbol{\vartheta}_h)$, $\underline{\mathbf{v}}_h = (\mathbf{v}_h, \boldsymbol{\xi}_h) \in \mathbf{K}_h$ the strong monotonicity property:

$$
\big[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h + \underline{\mathbf{z}}_h) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{v}}_h + \underline{\mathbf{z}}_h), \underline{\mathbf{u}}_h - \underline{\mathbf{v}}_h\big] \,\geq\, \gamma_{\mathtt{KVO}}\, \|\underline{\mathbf{u}}_h - \underline{\mathbf{v}}_h\|^2 \,,
\tag{4.10}
$$

with $\gamma_{\mathtt{KVO}} := \frac{\nu}{2} \min\{C_{\mathtt{d}}^2, 1\}$. In turn, following the steps of the proof of (3.15), we deduce that for all $\underline{\mathbf{u}}_h = (\mathbf{u}_h, \boldsymbol{\vartheta}_h)$ and $\underline{\mathbf{v}}_h = (\mathbf{v}_h, \boldsymbol{\xi}_h) \in \mathbf{M}_h \times \mathbb{Q}_h$, the following continuity property holds:

$$
\begin{aligned}
&\|\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{v}}_h)\| \\
&\leq\, L_{\mathtt{KVO}} \Big\{ \|\mathbf{u}_h - \mathbf{v}_h\| + \big(\|\mathbf{u}_h\|_{\mathbf{M}} + \|\mathbf{v}_h\|_{\mathbf{M}}\big)^{\rho-2} \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h - \boldsymbol{\xi}_h\|_{\mathbb{Q}} \Big\} \,,
\end{aligned}
\tag{4.11}
$$

with

$$
L_{\mathtt{KVO}} \,=\, \big(1 + \|\boldsymbol{\chi}\|_{\mathbf{M}}\big) \max\Big\{ \mathtt{D}\, |\Omega|^{1/2}, \mathtt{F}\, c_\rho\, |\Omega|^{(4-\rho)/4}, \nu, 1 \Big\} .
$$

Thus, the properties (4.10) and (4.11) of the operator $\mathcal{A}(\boldsymbol{\chi})$, in conjunction with the discrete inf-sup condition of $\mathcal{B}$ (cf. (4.4)) and Theorem 3.2, imply the existence and uniqueness of a solution $(\underline{\mathbf{u}}_{h,0}, \boldsymbol{\sigma}_{h,0}) \in (\mathbf{M}_h \times \mathbb{Q}_h) \times \mathbb{X}_{0,h}$ to (4.9). In addition, taking $(\underline{\mathbf{v}}_h, \boldsymbol{\tau}_h) = (\underline{\mathbf{u}}_h, \boldsymbol{\sigma}_h)$ in (4.9), we deduce from the definition of the operator $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.17)), the skew-symmetric property of $\mathbf{c}(\boldsymbol{\chi})$ (cf. (3.11)), and the continuity bound of $\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_0)$ (cf. (3.29)) that, there exists a constant $\widetilde{C}_0 > 0$, depending only on $\nu, \mathtt{D}$, and $\mathtt{F}$, and hence independent of h, such that

$$
\begin{aligned}
&\|\mathbf{u}_{h,0}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_{h,0}\|^\rho_{\mathbf{L}^\rho(\Omega)} + \|\boldsymbol{\vartheta}_{h,0}\|^2_{\mathbb{Q}} \\
&\quad \leq\, \widetilde{C}_0 \Big\{ \|\mathbf{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{\chi}\|^2_{\mathbf{M}} \big(\|\nabla \mathbf{u}_0\|^2_{\mathbb{L}^4(\Omega)} \,+\, \|\mathbf{u}_0\|^2_{\mathbf{M}}\big) + \|\mathbf{u}_0\|^{2(\rho-1)}_{\mathbf{L}^{2(\rho-1)}(\Omega)} \Big\} .
\end{aligned}
\tag{4.12}
$$

We are now in a position to establish the semi-discrete continuous in time analogue of Theorems 3.8 and 3.9.

Theorem 4.4 *For each compatible initial data $(\underline{\mathbf{u}}_{h,0}, \boldsymbol{\sigma}_{h,0}) = ((\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0}), \boldsymbol{\sigma}_{h,0})$ satisfying (4.9), and for each $\mathbf{f} \in \mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega))$ and $\boldsymbol{\chi} \in \mathbf{M}$, there exists a unique solution $(\underline{\mathbf{u}}_h, \boldsymbol{\sigma}_h) = ((\mathbf{u}_h, \boldsymbol{\vartheta}_h), \boldsymbol{\sigma}_h) : [0,T] \to \big(\mathbf{M}_h \times \mathbb{Q}_h\big) \times \mathbb{X}_{0,h}$ to (4.2), such that $(\mathbf{u}_h, \boldsymbol{\vartheta}_h) \in \mathrm{H}^1(0,T;\mathbf{M}_h) \times \mathrm{W}^{1,\infty}(0,T;\mathbb{Q}_h)$ and $(\mathbf{u}_h(0), \boldsymbol{\vartheta}_h(0)) = (\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0}) \in \mathbf{K}_h$ (cf. (4.3)). Moreover, assuming $\mathbf{f} \in \mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))$, there exists a constant $\widehat{C}_{\mathtt{KV}\underline{\mathbf{u}}} > 0$, depending only on $\|\boldsymbol{\chi}\|_{\mathbf{M}}$, $C_{\mathtt{d}}, \nu, \mathtt{D}, \mathtt{F}$, and κ, such that*

$$
\begin{aligned}
&\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{M})} + \|\boldsymbol{\vartheta}_h\|_{\mathrm{L}^\infty(0,T;\mathbb{Q})} + \|\mathbf{u}_h\|_{\mathrm{L}^2(0,T;\mathbf{M})} + \|\boldsymbol{\vartheta}_h\|_{\mathrm{L}^2(0,T;\mathbb{Q})} \\
&\leq\, \widehat{C}_{\mathtt{KV}\underline{\mathbf{u}}} \Big(\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\nabla \mathbf{u}_0\|_{\mathbb{L}^4(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{M}} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{L}^{2(\rho-1)}(\Omega)} \Big) .
\end{aligned}
\tag{4.13}
$$

In addition, there exists a constant $\widehat{C}_{\mathtt{KV}\boldsymbol{\sigma}} > 0$, depending only on $\|\boldsymbol{\chi}\|_{\mathbf{M}}$, $|\Omega|$, $C_{\mathtt{d}}$, ν, $\mathtt{D}$, $\mathtt{F}$, κ, and $\beta_{\mathtt{d}}$ such that

$$\begin{aligned} \|\boldsymbol{\sigma}_h\|_{\mathrm{L}^2(0,T;\mathbb{X})} \,\leq\, \widehat{C}_{\mathtt{KV}\boldsymbol{\sigma}} \sum_{j\in\{2,\rho\}} \Big\{ &\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \\ &+ \|\nabla\mathbf{u}_0\|_{\mathbb{L}^4(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{M}} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{L}^{2(\rho-1)}(\Omega)} \Big\}^{j-1}. \end{aligned} \tag{4.14}$$

Proof. Similarly as the proof of Theorem 3.8, we first show that (4.2) fits the abstract framework of Theorem 3.1 with the discrete counterparts of the definitions (3.12) and (3.13). Indeed, being the present operator $\mathcal{N}$ the discrete counterpart of the one defined in (3.12), it is clearly linear, symmetric, and monotone (cf. (3.7)–(3.8)). In addition, since $\mathcal{A}(\boldsymbol{\chi})$ is monotone for each $\boldsymbol{\chi}\in\mathbf{M}$ (cf. (3.30)–(3.31)), it follows that the present operator $\mathcal{M}$ is certainly monotone. In turn, we know from Lemma 4.3 that, given $\boldsymbol{\chi}\in\mathbf{M}$ and $(\widehat{\mathbf{f}}_1,\widehat{\mathbf{f}}_2)\in\mathbf{L}^2(\Omega)\times\mathbb{L}^2(\Omega)$, there exists a unique $(\underline{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\in\big(\mathbf{M}_h\times\mathbb{Q}_h\big)\times\mathbb{X}_{0,h}$ such that $(\mathcal{N}+\mathcal{M})(\underline{\mathbf{u}}_h,\boldsymbol{\sigma}_h) = (\widehat{F}_h|_{\mathbf{M}_h\times\mathbb{Q}_h},\mathbf{0})$ (cf. (4.7)), which constitutes the required discrete range condition. In this way, considering as initial data $(\underline{\mathbf{u}}_{h,0},\boldsymbol{\sigma}_{h,0}) = ((\mathbf{u}_{h,0},\boldsymbol{\vartheta}_{h,0}),\boldsymbol{\sigma}_{h,0})\in\big(\mathbf{M}_h\times\mathbb{Q}_h\big)\times\mathbb{X}_{0,h}$, which satisfies (4.9), we can apply Theorem 3.1 to our setting and conclude the existence of a solution $(\underline{\mathbf{u}}_h,\boldsymbol{\sigma}_h) = ((\mathbf{u}_h,\boldsymbol{\vartheta}_h),\boldsymbol{\sigma}_h)$ to problem (4.2) with $(\mathbf{u}_h,\boldsymbol{\vartheta}_h)\in\mathrm{W}^{1,\infty}(0,T;\mathbf{M}_h)\times\mathrm{W}^{1,\infty}(0,T;\mathbb{Q}_h)$, and satisfying $(\mathbf{u}_h(0),\boldsymbol{\vartheta}_h(0)) = (\mathbf{u}_{h,0},\boldsymbol{\vartheta}_{h,0})\in\mathbf{K}_h$ (cf. second row of (4.9)). In addition, observing that $\mathbf{M}_h$ is finite dimensional and that $\mathbf{u}_h\in\mathrm{W}^{1,\infty}(0,T;\mathbf{M}_h)$, we readily obtain that $\mathbf{u}_h\in\mathrm{H}^1(0,T;\mathbf{M}_h)$. The proof of uniqueness follows straightforwardly from its continuous counterpart by mimicking the estimates (3.34)–(3.36). We now derive the stability bounds (4.13) and (4.14) by proceeding analogously to the proof of Theorem 3.9. In fact, given $\boldsymbol{\chi}\in\mathbf{M}$, and testing equation (4.2) against $(\underline{\mathbf{v}}_h,\boldsymbol{\tau}_h) = (\underline{\mathbf{u}}_h,\boldsymbol{\sigma}_h)$, we readily obtain the discrete analogue of (3.40), namely

$$\begin{aligned} &\|\mathbf{u}_h(t)\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\,\|\boldsymbol{\vartheta}_h(t)\|^2_{\mathbb{Q}} + \int_0^t \Big(\mathtt{D}\,\|\mathbf{u}_h\|^2_{\mathbf{L}^2(\Omega)} + 2\,\nu\,\|\boldsymbol{\vartheta}_h\|^2_{\mathbb{Q}}\Big)ds \\ &\qquad \leq\, \|\mathbf{u}_h(0)\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\,\|\boldsymbol{\vartheta}_h(0)\|^2_{\mathbb{Q}} + \frac{1}{\mathtt{D}}\int_0^t \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}ds\,. \end{aligned} \tag{4.15}$$

Next, noting from the second row of (4.2) that $(\mathbf{u}_h,\boldsymbol{\vartheta}_h):(0,T]\to\mathbf{K}_h$ (cf. (4.3)), and using (4.5) together with some algebraic manipulations, we deduce from (4.15) the discrete analogue of (3.43), that is,

$$\begin{aligned} &\gamma_{\kappa\mathtt{d}}\Big(\|\mathbf{u}_h(t)\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h(t)\|^2_{\mathbb{Q}}\Big) + \gamma_{\mathtt{D}\nu\mathtt{d}}\int_0^t \Big(\|\mathbf{u}_h\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h\|^2_{\mathbb{Q}}\Big)ds \\ &\qquad \leq\, \frac{1}{\mathtt{D}}\int_0^t \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}ds + \|\mathbf{u}_h(0)\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\,\|\boldsymbol{\vartheta}_h(0)\|^2_{\mathbb{Q}}\,, \end{aligned} \tag{4.16}$$

with $\gamma_{\kappa\mathtt{d}} := \frac{\kappa^2}{2}\min\big\{C^2_{\mathtt{d}},1\big\}$ and $\gamma_{\mathtt{D}\nu\mathtt{d}} := \nu\,\min\big\{C^2_{\mathtt{d}},1\big\}$. Thus, (4.16) together with the fact that $(\mathbf{u}_h(0),\boldsymbol{\vartheta}_h(0)) = (\mathbf{u}_{h,0},\boldsymbol{\vartheta}_{h,0})$ and (4.12), yield (4.13). Finally, we derive (4.14), which follows closely the arguments of its continuous counterpart (3.38). We provide the main steps for completeness. In fact, using the discrete inf-sup condition of $\mathcal{B}$ (cf. (4.4)), the first equation of (4.2), the stability bounds of F, $\mathcal{E}$, and $\mathcal{A}(\boldsymbol{\chi})$ (cf. (3.4), (3.7), (3.9)), and integrating from 0 to $t\in(0,T]$, along with some algebraic manipulations, we deduce the discrete analogue of (3.45), that is

$$\begin{aligned} \int_0^t \|\boldsymbol{\sigma}_h\|^2_{\mathbb{X}}ds \,\leq\, \widehat{C}_1 \Big\{ &\int_0^t \Big(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \big(1+\|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big)\big(\|\mathbf{u}_h\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h\|^2_{\mathbb{Q}}\big)\Big)ds \\ &+ \int_0^t \Big(\|\mathbf{u}_h\|^{2(\rho-1)}_{\mathbf{M}} + \|\partial_t\underline{\mathbf{u}}_h\|^2\Big)\,ds\Big\}. \end{aligned} \tag{4.17}$$

with $\widehat{C}_1 > 0$ depending on $|\Omega|, \kappa, \mathtt{D}, \mathtt{F}, \nu$, and $\beta_{\mathtt{d}}$. In addition, in order to bound the last term in (4.17), we first observe that $\partial_t \underline{\mathbf{v}}_h : (0,T] \to \mathbf{K}_h$ for each $\underline{\mathbf{v}}_h(t) \in \mathbf{K}_h$ (cf. (4.3)). In fact, given $\underline{\mathbf{v}}_h : (0,T] \to \mathbf{K}_h$, simple algebraic computations lead to

$$\big[\mathcal{B}(\partial_t \underline{\mathbf{v}}_h), \boldsymbol{\tau}_h\big] = \partial_t\Big(\big[\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\tau}_h\big]\Big) - \big[\mathcal{B}(\underline{\mathbf{v}}_h), \partial_t \boldsymbol{\tau}_h\big] = 0\,, \tag{4.18}$$

where, the latter is obtained by using the fact that $\partial_t \boldsymbol{\tau}_h = 0$. It follows that $\partial_t \underline{\mathbf{u}}_h \in \mathbf{K}_h$ since $\underline{\mathbf{u}}_h \in \mathbf{K}_h$. Next, we differentiate in time the second equation of (4.2), choose $(\underline{\mathbf{v}}_h, \boldsymbol{\tau}_h) = (\partial_t \underline{\mathbf{u}}_h, \boldsymbol{\sigma}_h)$, and use (3.5) and (4.5), together with the Cauchy–Schwarz and Young inequalities, to arrive at the discrete version of (3.46), that is

$$\begin{aligned}
&\frac{1}{2}\partial_t\Big(\mathtt{D}\|\mathbf{u}_h\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\mathtt{F}}{\rho}\|\mathbf{u}_h\|^{\rho}_{\mathbf{L}^\rho(\Omega)} + \nu\|\boldsymbol{\vartheta}_h\|^2_{\mathbb{Q}}\Big) + \gamma_{\kappa\mathtt{d}}\|\partial_t \underline{\mathbf{u}}_h\|^2 \\
&\qquad \le \widehat{C}_2\Big(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big(\|\mathbf{u}_h\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h\|^2_{\mathbb{Q}}\big)\Big) + \frac{\gamma_{\kappa\mathtt{d}}}{2}\|\partial_t \underline{\mathbf{u}}_h\|^2\,,
\end{aligned} \tag{4.19}$$

with $\gamma_{\kappa\mathtt{d}}$ as in (4.16) and $\widehat{C}_2 > 0$ depending only on $|\Omega|, C_{\mathtt{d}}$, and κ. Thus, integrating (4.19) from 0 to $t \in (0,T]$, we deduce a simplified discrete version of (3.47), namely

$$\begin{aligned}
\gamma_{\kappa\mathtt{d}} \int_0^t \|\partial_t \underline{\mathbf{u}}_h\|^2 ds \,&\le\, \mathtt{D}\,\|\mathbf{u}_h(0)\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\|\mathbf{u}_h(0)\|^{\rho}_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\vartheta}_h(0)\|^2_{\mathbb{Q}} \\
&+ 2\,\widehat{C}_2\left\{\int_0^t \Big(\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big(\|\mathbf{u}_h\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h\|^2_{\mathbb{Q}}\big)\Big)\, ds\right\}.
\end{aligned} \tag{4.20}$$

Thus, replacing back (4.20) into (4.17), proceeding as in its continuous counterparts (3.48)–(3.49), and using again the fact that $(\mathbf{u}_h(0), \boldsymbol{\vartheta}_h(0)) = (\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0})$, yields

$$\begin{aligned}
\int_0^t \|\boldsymbol{\sigma}_h\|^2_{\mathbb{X}} ds \,&\le\, \widehat{C}_3\left\{\int_0^t \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}\, ds + \|\mathbf{u}_{h,0}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_{h,0}\|^{\rho}_{\mathbf{L}^\rho(\Omega)} + \|\boldsymbol{\vartheta}_{h,0}\|^2_{\mathbb{Q}}\right. \\
&\left. + \big(1 + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big)\int_0^t \Big(\|\mathbf{u}_h\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h\|^2_{\mathbb{Q}}\Big)\, ds + \|\mathbf{u}_h\|^{2(\rho-2)}_{\mathrm{L}^\infty(0,T;\mathbf{M})}\int_0^t \|\mathbf{u}_h\|^2_{\mathbf{M}}\, ds\right\}.
\end{aligned} \tag{4.21}$$

with $\widehat{C}_3 > 0$ depending on $|\Omega|, C_{\mathtt{d}}, \kappa, \mathtt{D}, \mathtt{F}, \nu$, and $\beta_{\mathtt{d}}$. Thus, employing the estimates (4.12) and (4.13) to bound the right-hand side of (4.21), we conclude (4.14). □

4.3 Error analysis

In this section we derive suitable error estimates for the semidiscrete scheme (4.2). To this end, we first recall that the discrete inf-sup condition $\mathcal{B}$ (cf. (4.4)), along with a classical result on mixed methods (see, e.g., [23, eq. (2.89) in Theorem 2.6]), ensure the existence of a constant $C > 0$, independent of h, such that

$$\inf_{\underline{\mathbf{v}}_h \in \mathbf{K}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\| \,\le\, C \inf_{\underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|\,. \tag{4.22}$$

4.3.1 Approximation properties

In order to obtain the theoretical rates of convergence for the discrete scheme (4.2), we state next the approximation properties of the finite element subspaces $\mathbf{M}_h, \mathbb{Q}_h$, and $\mathbb{X}_h$ (cf. (4.1)), which can be found in [7], [20], [23], and [12, Section 3.1] (see also [18, Section 5]).

($\mathbf{AP}_h^{\mathbf{u}}$) There exists a positive constant C, independent of h, such that for each $s \in [0, k+1]$ and for each $\mathbf{v} \in \mathbf{W}^{s,4}(\Omega)$, there holds

$$\inf_{\mathbf{v}_h \in \mathbf{M}_h} \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{M}} \,\leq\, C\, h^s \,\|\mathbf{v}\|_{\mathbf{W}^{s,4}(\Omega)} \,.$$

($\mathbf{AP}_h^{\boldsymbol{\vartheta}}$) There exists a positive constant C, independent of h, such that for each $s \in [0, k+1]$ and for each $\boldsymbol{\xi} \in \mathbb{H}^s(\Omega) \cap \mathbb{Q}$, there holds

$$\inf_{\boldsymbol{\xi}_h \in \mathbb{Q}_h} \|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{\mathbb{Q}} \,\leq\, C\, h^s \,\|\boldsymbol{\xi}\|_{\mathbb{H}^s(\Omega)} \,.$$

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) There exists a positive constant C, independent of h, such that for each $s \in (0, k+1]$ and for each $\boldsymbol{\tau} \in \mathbb{H}^s(\Omega) \cap \mathbb{X}_0$, with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{s,4/3}(\Omega)$, there holds

$$\inf_{\boldsymbol{\tau}_h \in \mathbb{X}_{0,h}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbb{X}} \,\leq\, C\, h^s \left\{ \|\boldsymbol{\tau}\|_{\mathbb{H}^s(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\mathbf{W}^{s,4/3}(\Omega)} \right\}.$$

Owing to (4.22), ($\mathbf{AP}_h^{\mathbf{u}}$), ($\mathbf{AP}_h^{\boldsymbol{\vartheta}}$), and ($\mathbf{AP}_h^{\boldsymbol{\sigma}}$), it follows that, under extra regularity assumptions on the exact solution, there exist constants $C(\underline{\mathbf{u}})$, $C(\partial_t \underline{\mathbf{u}})$, and $C(\boldsymbol{\sigma})$, depending on $\mathbf{u}$, $\boldsymbol{\vartheta}$, and $\boldsymbol{\sigma}$, respectively, such that

$$\begin{gathered} \inf_{\underline{\mathbf{v}}_h \in \mathbf{K}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\| \,\leq\, C(\underline{\mathbf{u}})\, h^s, \qquad \inf_{\underline{\mathbf{v}}_h \in \mathbf{K}_h} \|\partial_t \underline{\mathbf{u}} - \underline{\mathbf{v}}_h\| \,\leq\, C(\partial_t \underline{\mathbf{u}})\, h^s, \\ \text{and} \quad \inf_{\boldsymbol{\tau}_h \in \mathbb{X}_{0,h}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbb{X}} \,\leq\, C(\boldsymbol{\sigma})\, h^s \,. \end{gathered} \tag{4.23}$$

4.3.2 Preliminary computations

In this section we perform several preliminary computations aiming to derive later on the a priori error estimates and obtain the associated rates of convergence. In this regard, and in order to simplify the subsequent analysis, we first write $\mathbf{e}_{\underline{\mathbf{u}}} = (\mathbf{e}_{\mathbf{u}}, \mathbf{e}_{\boldsymbol{\vartheta}}) = (\mathbf{u} - \mathbf{u}_h, \boldsymbol{\vartheta} - \boldsymbol{\vartheta}_h)$, and $\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$. In this way, given arbitrary $\widehat{\underline{\mathbf{v}}}_h := (\widehat{\mathbf{v}}_h, \widehat{\boldsymbol{\xi}}_h) : (0,T] \to \mathbf{K}_h$ (cf. (4.3)) and $\widehat{\boldsymbol{\tau}}_h : (0,T] \to \mathbb{X}_{0,h}$, we decompose the errors into

$$\mathbf{e}_{\underline{\mathbf{u}}} \,=\, \boldsymbol{\delta}_{\underline{\mathbf{u}}} \,+\, \boldsymbol{\eta}_{\underline{\mathbf{u}}} \quad \text{and} \quad \mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\delta}_{\boldsymbol{\sigma}} + \boldsymbol{\eta}_{\boldsymbol{\sigma}} \,, \tag{4.24}$$

with

$$\begin{aligned} \boldsymbol{\delta}_{\underline{\mathbf{u}}} \,&=\, (\boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\delta}_{\boldsymbol{\vartheta}}) \,=\, \underline{\mathbf{u}} - \widehat{\underline{\mathbf{v}}}_h \,:=\, (\mathbf{u} - \widehat{\mathbf{v}}_h, \boldsymbol{\vartheta} - \widehat{\boldsymbol{\xi}}_h)\,, \qquad \boldsymbol{\delta}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \widehat{\boldsymbol{\tau}}_h\,, \\ \boldsymbol{\eta}_{\underline{\mathbf{u}}} \,&=\, (\boldsymbol{\eta}_{\mathbf{u}}, \boldsymbol{\eta}_{\boldsymbol{\vartheta}}) \,=\, \widehat{\underline{\mathbf{v}}}_h - \underline{\mathbf{u}}_h \,:=\, (\widehat{\mathbf{v}}_h - \mathbf{u}_h, \widehat{\boldsymbol{\xi}}_h - \boldsymbol{\vartheta}_h)\,, \qquad \boldsymbol{\eta}_{\boldsymbol{\sigma}} = \widehat{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h\,. \end{aligned} \tag{4.25}$$

Hence, by subtracting the continuous and discrete problems (2.15) and (4.2), respectively, we obtain the following error system

$$\begin{aligned} \partial_t[\mathcal{E}(\mathbf{e}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] + [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \mathbf{e}_{\boldsymbol{\sigma}}] \;&=\; 0 \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\,, \\ [\mathcal{B}(\mathbf{e}_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h] \;&=\; 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,. \end{aligned} \tag{4.26}$$

In what follows we proceed as in [15, Theorem 4.6]. Indeed, taking $\underline{\mathbf{v}}_h = \boldsymbol{\eta}_{\underline{\mathbf{u}}} = (\boldsymbol{\eta}_{\mathbf{u}}, \boldsymbol{\eta}_{\boldsymbol{\vartheta}}) : (0,T] \to \mathbf{K}_h$ (cf. (4.3)) in the first equation of (4.26), and using that $\boldsymbol{\eta}_{\underline{\mathbf{u}}} \in \mathbf{K}_h$, which yields $[\mathcal{B}(\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\eta}_{\boldsymbol{\sigma}}] = 0$, we get

$$\partial_t[\mathcal{E}(\mathbf{e}_{\underline{\mathbf{u}}}), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] + [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] + [\mathcal{B}(\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\delta}_{\boldsymbol{\sigma}}] \,=\, 0\,. \tag{4.27}$$

In turn, according to the definition of $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.17), (2.18), (2.19)), adding and subtracting the term $\mathcal{A}(\boldsymbol{\chi})(\widehat{\underline{\mathbf{v}}}_h)$, we have

$$\begin{aligned}[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] \,&=\, [\mathcal{A}(\boldsymbol{\chi})(\widehat{\underline{\mathbf{v}}}_h) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] \\ &\quad + [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}) - \mathcal{A}(\boldsymbol{\chi})(\widehat{\underline{\mathbf{v}}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}]\,,\end{aligned} \tag{4.28}$$

whereas, invoking the definition of $\mathcal{E}$ (cf. (2.16)) and (4.25), we can write,

$$\partial_t[\mathcal{E}(\mathbf{e}_{\underline{\mathbf{u}}}), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] \,=\, \frac{1}{2}\partial_t\Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} \,+\, \kappa^2\,\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}}\Big) \,+\, (\partial_t\boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{u}})_\Omega \,+\, \kappa^2\,(\partial_t\boldsymbol{\delta}_{\boldsymbol{\vartheta}}, \boldsymbol{\eta}_{\boldsymbol{\vartheta}})_\Omega\,. \tag{4.29}$$

In addition, bearing in mind the definition of $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.17)), recalling from (4.25) that $\boldsymbol{\eta}_{\underline{\mathbf{u}}} = \widehat{\underline{\mathbf{v}}}_h - \underline{\mathbf{u}}_h$, and employing (3.22) and the skew-symmetry property of $\mathbf{c}$ (cf. (3.11)), we obtain

$$[\mathcal{A}(\boldsymbol{\chi})(\widehat{\underline{\mathbf{v}}}_h) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] \,\geq\, \mathtt{D}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \nu\,\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}}\,. \tag{4.30}$$

In this way, replacing (4.28) and (4.29) back into (4.27), and then using (4.30), we arrive at

$$\begin{aligned}&\frac{1}{2}\partial_t\Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}}\Big) + \mathtt{D}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} \,+\, \nu\,\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}} \\ &\quad \leq\, -\,(\partial_t\boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{u}})_\Omega \,-\, \kappa^2\,(\partial_t\boldsymbol{\delta}_{\boldsymbol{\vartheta}}, \boldsymbol{\eta}_{\boldsymbol{\vartheta}})_\Omega \,-\, [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}) - \mathcal{A}(\boldsymbol{\chi})(\widehat{\underline{\mathbf{v}}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}] \,-\, [\mathcal{B}(\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\delta}_{\boldsymbol{\sigma}}]\,.\end{aligned} \tag{4.31}$$

Next, using the Cauchy–Schwarz and Hölder inequalities, invoking the definition of $\mathcal{A}(\boldsymbol{\chi})$ (cf. (2.17)) along with (3.17), and then employing the continuity property of $\mathcal{B}$ (cf. (3.3)), the terms on the right-hand side of (4.31) can be bounded as follows

$$\big|(\partial_t\boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{u}})_\Omega \,+\, \kappa^2(\partial_t\boldsymbol{\delta}_{\boldsymbol{\vartheta}}, \boldsymbol{\eta}_{\boldsymbol{\vartheta}})_\Omega\big| \,\leq\, |\Omega|^{1/2}\,\|\partial_t\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{M}} \,+\, \kappa^2\,\|\partial_t\boldsymbol{\delta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}}\,\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}}\,, \tag{4.32}$$

$$\begin{aligned}\big|[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}) - \mathcal{A}(\boldsymbol{\chi})(\widehat{\underline{\mathbf{v}}}_h), \boldsymbol{\eta}_{\underline{\mathbf{u}}}]\big| \,&\leq\, \mathtt{D}\,|\Omega|^{1/2}\,\|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{M}} \,+\, \nu\,\|\boldsymbol{\delta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}}\,\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}} \\ &\quad + \mathtt{F}\,c_\rho\,\big(\|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}} \,+\, 2\,\|\mathbf{u}\|_{\mathbf{M}}\big)^{\rho-2}\,\|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{M}} + \|\boldsymbol{\chi}\|_{\mathbf{M}}\,\|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|\,\|\boldsymbol{\eta}_{\underline{\mathbf{u}}}\|\,,\end{aligned} \tag{4.33}$$

and

$$\big|[\mathcal{B}(\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\delta}_{\boldsymbol{\sigma}}]\big| \,\leq\, \big(\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{M}} \,+\, \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}}\big)\,\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}\,. \tag{4.34}$$

Replacing (4.32), (4.33), and (4.34) back into (4.31) and performing some algebraic manipulations, we find that

$$\begin{aligned}&\partial_t\Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} \,+\, \kappa^2\,\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}}\Big) \,+\, \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} \,+\, \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}} \\ &\quad \leq\, c_1\,\Big\{\Big(\|\partial_t\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{\rho-1}_{\mathbf{M}} + \|\mathbf{u}\|^{\rho-2}_{\mathbf{M}}\,\|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}} + \|\boldsymbol{\chi}\|_{\mathbf{M}}\,\|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\| + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}\Big)\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{M}} \\ &\quad + \Big(\|\partial_t\boldsymbol{\delta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}} + \|\boldsymbol{\delta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}} + \|\boldsymbol{\chi}\|_{\mathbf{M}}\,\|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\| \,+\, \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}\Big)\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}}\Big\}\,,\end{aligned} \tag{4.35}$$

where c_1 is a positive constant depending on $|\Omega|$, $\mathtt{D}$, κ, $\mathtt{F}$, ρ, and ν. Integrating (4.35) from 0 to $t \in (0,T]$, using the fact that $\boldsymbol{\eta}_{\underline{\mathbf{u}}} = (\boldsymbol{\eta}_{\mathbf{u}}, \boldsymbol{\eta}_{\boldsymbol{\vartheta}}) \in \mathbf{K}_h$, which yields $C_{\mathtt{d}}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{M}} \,\leq\, \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}}$ (cf. (4.5)), applying Young inequality (cf. (1.2)), and recalling that $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ is bounded by data (cf. (3.37)), we deduce that

$$\begin{aligned}&\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{M}} + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}(t)\|^2_{\mathbb{Q}} + \int_0^t \Big(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}}\Big)ds \\ &\quad \leq c_2\,\Big\{\int_0^t \Big(\|\partial_t\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\rho-1)}_{\mathbf{M}}\Big)ds + \|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}(0)\|^2_{\mathbb{Q}}\Big\}\,,\end{aligned} \tag{4.36}$$

with a positive constant c_2 depending on $|\Omega|$, $\mathtt{D}$, κ, $\mathtt{F}$, ρ, ν, $C_{\mathtt{d}}$, $\|\mathbf{i}_4\|$, and data. On the other hand, in order to estimate $\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathrm{L}^2(0,T;\mathbb{X})}$, we observe from the discrete inf-sup condition of $\mathcal{B}$ (cf. (4.4)), the first equation of (4.26), and the continuity bounds of $\mathcal{B}$, $\mathcal{E}$, and $\mathcal{A}(\boldsymbol{\chi})$ (cf. (3.3), (3.7), (3.15)), that there holds

$$\begin{aligned}
\beta_{\mathtt{d}}\|\boldsymbol{\eta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} &\leq \sup_{\substack{\underline{\mathbf{v}}_h\in\mathbf{M}_h\times\mathbb{Q}_h\\ \underline{\mathbf{v}}_h\neq 0}} \frac{-\Big([\partial_t\mathcal{E}(\mathbf{e}_{\underline{\mathbf{u}}}),\underline{\mathbf{v}}_h] + [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h),\underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h),\boldsymbol{\delta}_{\boldsymbol{\sigma}}]\Big)}{\|\underline{\mathbf{v}}_h\|} \\
&\leq\ c_3\left\{\Big(1+\|\mathbf{u}\|_{\mathbf{M}}^{\rho-2} + \|\mathbf{u}_h\|_{\mathbf{M}}^{\rho-2} + \|\boldsymbol{\chi}\|_{\mathbf{M}}\Big)\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}}\right. \\
&\quad\left. +\ \|\partial_t\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}} + \|\partial_t\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}} + \Big(1+\|\boldsymbol{\chi}\|_{\mathbf{M}}\Big)\|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}\right\},
\end{aligned}$$

with a positive constant c_3 depending on $|\Omega|$, $\mathtt{D}$, $\mathtt{F}$, ν, ρ, and κ. Then, taking square in the above inequality, and integrating from 0 to $t\in(0,T]$, we obtain

$$\begin{aligned}
\int_0^t \|\boldsymbol{\eta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}^2 \ \leq\ c_4&\left\{\int_0^t\Big(\|\partial_t\mathbf{e}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}^2\Big)ds + \Big(1+\|\boldsymbol{\chi}\|_{\mathbf{M}}^2\Big)\int_0^t\|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}}^2 ds\right. \\
&\left. +\ \Big(1+\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}^{2(\rho-2)} + \|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}^{2(\rho-2)} + \|\boldsymbol{\chi}\|_{\mathbf{M}}^2\Big)\int_0^t\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}}^2 ds\right\},
\end{aligned}$$

where c_4 is a positive constant depending on $|\Omega|$, $\mathtt{D}$, $\mathtt{F}$, ν, ρ, κ, and $\beta_{\mathtt{d}}$. In turn, using the fact that $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ and $\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ are bounded by data (cf. (3.37), (4.13)), and employing (4.36), we conclude from the foregoing inequality that

$$\begin{aligned}
\int_0^t\|\boldsymbol{\eta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}^2 \ \leq\ c_5&\left\{\int_0^t\Big(\|\partial_t\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}^2 + \|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}}^{2(\rho-1)}\Big)ds\right. \\
&\left. +\ \int_0^t\|\partial_t\boldsymbol{\eta}_{\underline{\mathbf{u}}}\|^2\,ds + \|\boldsymbol{\eta}_{\mathbf{u}}(0)\|_{\mathbf{L}^2(\Omega)}^2 + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}(0)\|_{\mathbb{Q}}^2\right\},
\end{aligned} \tag{4.37}$$

with a positive constant c_5 depending on $|\Omega|$, $\mathtt{D}$, κ, $\mathtt{F}$, ρ, ν, $C_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, $\|\mathbf{i}_4\|$, and data.

4.3.3 A priori error estimates and rates of convergence

We now are in position to establish the main result of this section, namely, the theoretical rate of convergences of the discrete scheme (4.2). Note that optimal rates are obtained for all the unknowns.

Theorem 4.5 *Let* $(\underline{\mathbf{u}},\boldsymbol{\sigma}) : [0,T]\to(\mathbf{M}\times\mathbb{Q})\times\mathbb{X}_0$ *and* $(\underline{\mathbf{u}}_h,\boldsymbol{\sigma}_h) : [0,T]\to(\mathbf{M}_h\times\mathbb{Q}_h)\times\mathbb{X}_{0,h}$, *with* $(\mathbf{u},\boldsymbol{\vartheta})\in\mathrm{H}^1(0,T;\mathbf{M})\times\mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))$ *and* $(\mathbf{u}_h,\boldsymbol{\vartheta}_h)\in\mathrm{H}^1(0,T;\mathbf{M}_h)\times\mathrm{W}^{1,\infty}(0,T;\mathbb{Q}_h)$, *be the unique solutions of the continuous and semidiscrete problems* (2.15) *and* (4.2), *respectively. Assume further that there exits* $s\in(0,k+1]$, *such that* $\mathbf{u}(t)\in\mathbf{W}^{s,4}(\Omega),\boldsymbol{\vartheta}(t)\in\mathbb{H}^s(\Omega),\boldsymbol{\sigma}(t)\in\mathbb{H}^s(\Omega)$, *and* $\mathbf{div}(\boldsymbol{\sigma})(t)\in\mathbf{W}^{s,4/3}(\Omega)$, *for each* $t\in[0,T]$. *Then, there exists* $C(\underline{\mathbf{u}},\boldsymbol{\sigma})>0$ *depending only on* $C(\underline{\mathbf{u}})$, $C(\partial_t\underline{\mathbf{u}})$, $C(\boldsymbol{\sigma})$, $|\Omega|$, $\mathtt{D}$, $\mathtt{F}$, ρ, ν, κ, $\beta_{\mathtt{d}}$, $C_{\mathtt{d}}$, $\|\mathbf{i}_4\|$, *and data, such that*

$$\begin{aligned}
&\|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})} + \|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\mathrm{L}^\infty(0,T;\mathbb{Q})} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{M})} + \|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\mathrm{L}^2(0,T;\mathbb{Q})} \\
&\qquad + \|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathrm{L}^2(0,T;\mathbb{X})} \ \leq\ C(\underline{\mathbf{u}},\boldsymbol{\sigma})\Big(h^s + h^{s\,(\rho-1)}\Big).
\end{aligned} \tag{4.38}$$

Proof. We begin by bounding the term $\|\partial_t \boldsymbol{\eta}_{\underline{\mathbf{u}}}\|$ in (4.37). For this purpose, we choose $\underline{\mathbf{v}}_h = \partial \boldsymbol{\eta}_{\underline{\mathbf{u}}} := (\partial_t \boldsymbol{\eta}_{\mathbf{u}}, \partial_t \boldsymbol{\eta}_{\boldsymbol{\vartheta}}) \in \mathbf{K}_h$ (cf. (4.18)) in the first equation of (4.26), which, using that $[\mathcal{B}(\partial_t \boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\eta}_{\boldsymbol{\sigma}}] = 0$, yields

$$\begin{aligned}\frac{1}{2}\,\partial_t\Big(\mathtt{D}\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \nu\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}}\Big) + \|\partial_t\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\,\|\partial_t\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}} \,=\, -(\partial_t\boldsymbol{\delta}_{\mathbf{u}}, \partial_t\boldsymbol{\eta}_{\mathbf{u}})_\Omega \\ -\,\kappa^2\,(\partial_t\boldsymbol{\delta}_{\boldsymbol{\vartheta}}, \partial_t\boldsymbol{\eta}_{\boldsymbol{\vartheta}})_\Omega - \mathtt{D}\,(\boldsymbol{\delta}_{\mathbf{u}}, \partial_t\boldsymbol{\eta}_{\mathbf{u}})_\Omega - \nu\,(\boldsymbol{\delta}_{\boldsymbol{\vartheta}}, \partial_t\boldsymbol{\eta}_{\boldsymbol{\vartheta}})_\Omega - [\mathcal{B}(\partial_t\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\delta}_{\boldsymbol{\sigma}}] \\ -\,\mathtt{F}\,(|\mathbf{u}|^{\rho-2}\mathbf{u} - |\mathbf{u}_h|^{\rho-2}\mathbf{u}_h, \partial_t\boldsymbol{\eta}_{\mathbf{u}})_\Omega - [\mathbf{c}(\boldsymbol{\chi})(\mathbf{u}-\mathbf{u}_h), \partial_t\boldsymbol{\eta}_{\underline{\mathbf{u}}}]\,.\end{aligned}$$

In turn, by applying the Cauchy–Schwarz and Hölder inequalities, together with (3.10), using the estimate (4.5), and arguing in a similar way as in (4.19), we deduce that

$$\begin{aligned}&\frac{1}{2}\partial_t\Big(\mathtt{D}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \nu\,\|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}}\Big) + \|\partial_t\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \gamma_{\kappa\mathtt{d}}\,\|\partial_t\boldsymbol{\eta}_{\underline{\mathbf{u}}}\|^2 \,\le\, c_6\,\Big(\|\partial_t\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}} + \|\partial_t\boldsymbol{\delta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}} \\ &\quad + \|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}} + \|\boldsymbol{\delta}_{\boldsymbol{\vartheta}}\|_{\mathbb{Q}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \big(\|\boldsymbol{\chi}\|_{\mathbf{M}} + \|\mathbf{u}\|^{\rho-2}_{\mathbf{M}} + \|\mathbf{u}_h\|^{\rho-2}_{\mathbf{M}}\big)\,\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{M}}\Big)\|\partial_t\boldsymbol{\eta}_{\underline{\mathbf{u}}}\|\,,\end{aligned} \tag{4.39}$$

with $\gamma_{\kappa\mathtt{d}}$ as in (4.16) and a positive constant c_6 depending on $|\Omega|$, κ, $\mathtt{D}$, ν, $\mathtt{F}$, c_ρ, and ρ. Thus, integrating (4.39) from 0 to $t \in (0,T]$, and employing again the estimate (4.5), the Young inequality, and the fact that $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ and $\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ are bounded by data (cf. (4.13), (3.37)), we arrive at

$$\begin{aligned}&\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{M}} + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}(t)\|^2_{\mathbb{Q}} + \int_0^t \|\partial_t\boldsymbol{\eta}_{\underline{\mathbf{u}}}\|^2 ds \\ &\quad\le\, c_7\left\{\int_0^t \Big(\|\partial_t\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}} + \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{M}}\Big)\,ds \,+\, \|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}(0)\|^2_{\mathbb{Q}}\right\},\end{aligned} \tag{4.40}$$

with a positive constant c_7 depending on $|\Omega|, \kappa, \mathtt{D}, \nu, \mathtt{F}, \rho, C_{\mathtt{d}}, \|\mathbf{i}_4\|$, and data. Then, combining estimates (4.36), (4.37), and (4.40), and performing some algebraic manipulations, we deduce that

$$\begin{aligned}&\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{M}} + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}(t)\|^2_{\mathbb{Q}} + \int_0^t \Big(\|\partial_t\boldsymbol{\eta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\eta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}} + \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}}\Big)ds \\ &\quad\le\, c_8\left\{\int_0^t \Big(\|\partial_t\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\rho-1)}_{\mathbf{M}}\Big)ds + \|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}(0)\|^2_{\mathbb{Q}}\right\},\end{aligned} \tag{4.41}$$

with a positive constant c_8 depending on $|\Omega|, \kappa, \mathtt{D}, \nu, \mathtt{F}, \rho, C_{\mathtt{d}}, \beta_{\mathtt{d}}$, $\|\mathbf{i}_4\|$, and data. Finally, in order to bound the last two terms in (4.41), we deduce from the first row in (4.9), the error equation

$$[\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_0) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] - [\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\sigma}_{h,0}] \,=\, 0 \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\,. \tag{4.42}$$

Similarly to (4.25), we now denote $\boldsymbol{\delta}_{\underline{\mathbf{u}}_0} = (\boldsymbol{\delta}_{\mathbf{u}_0}, \boldsymbol{\delta}_{\boldsymbol{\vartheta}_0}) = (\mathbf{u}_0 - \widehat{\mathbf{v}}_h(0), \boldsymbol{\vartheta}_0 - \widehat{\boldsymbol{\xi}}_h(0))$ with arbitrary $(\widehat{\mathbf{v}}_h(0), \widehat{\boldsymbol{\xi}}_h(0)) \in \mathbf{K}_h \subset \mathbf{M}_h \times \mathbb{Q}_h$. Then, testing (4.42) with $\underline{\mathbf{v}}_h = (\boldsymbol{\eta}_{\mathbf{u}}(0), \boldsymbol{\eta}_{\boldsymbol{\vartheta}}(0)) \in \mathbf{K}_h$ and proceeding as in (4.35), recalling from Theorems 3.8 and 4.4 that $(\mathbf{u}(0), \boldsymbol{\vartheta}(0)) = (\mathbf{u}_0, \boldsymbol{\vartheta}_0)$ and $(\mathbf{u}_h(0), \boldsymbol{\vartheta}_h(0)) = (\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0})$, respectively, and using that $[\mathcal{B}(\boldsymbol{\eta}_{\mathbf{u}}(0), \boldsymbol{\eta}_{\boldsymbol{\vartheta}}(0)), \boldsymbol{\sigma}_{h,0}] = 0$, we get

$$\|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{M}} + \|\boldsymbol{\eta}_{\boldsymbol{\vartheta}}(0)\|^2_{\mathbb{Q}} \,\le\, c_9\,\Big(\|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\|^2 \,+\, \|\boldsymbol{\delta}_{\mathbf{u}_0}\|^{2(\rho-1)}_{\mathbf{M}}\Big)\,, \tag{4.43}$$

where c_9 is a positive constant depending on $|\Omega|$, $\mathtt{D}$, $\mathtt{F}$, $C_{\mathtt{d}}$, ρ, ν, $\|\mathbf{u}_0\|_{\mathbf{M}}$ and $\|\boldsymbol{\chi}\|_{\mathbf{M}}$. Thus, combining (4.41) with (4.43), and using the error decomposition (4.24), it follows that

$$\|\mathbf{e}_{\mathbf{u}}(t)\|^2_{\mathbf{M}} \,+\, \|\mathbf{e}_{\boldsymbol{\vartheta}}(t)\|^2_{\mathbb{Q}} \,+\, \int_0^t \Big(\|\mathbf{e}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\mathbf{e}_{\boldsymbol{\vartheta}}\|^2_{\mathbb{Q}} + \|\mathbf{e}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}}\Big)ds \,\le\, c_{10}\,\Psi(\underline{\mathbf{u}}, \boldsymbol{\sigma})\,, \tag{4.44}$$

with

$$\Psi(\underline{\mathbf{u}},\boldsymbol{\sigma}) \,:=\, \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}(t)\|^2 \,+\, \int_0^t \Big(\|\partial_t \boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}^2 + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^{2(\rho-1)}\Big) ds$$
$$+\, \|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\|^2 \,+\, \|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\|^{2(\rho-1)}\,,$$

where c_{10} denotes a positive constant depending on $|\Omega|$, κ, $\mathtt{D}$, ν, $\mathtt{F}$, ρ, $C_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, $\|\mathbf{i}_4\|$, and data. Finally, using the fact that $\widehat{\underline{\mathbf{v}}}_h \in \mathbf{K}_h$ and $\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_{0,h}$ are arbitrary, taking infimum in (4.44) over the corresponding discrete subespaces $\mathbf{K}_h$ and $\mathbb{X}_{0,h}$, and applying the estimates provided in (4.23), we derive (4.38) and conclude the proof. □

We end this section by observing that, based on (4.41), (4.38) can be expanded to include a bound on $\|\partial_t \mathbf{e}_{\underline{\mathbf{u}}}\|_{\mathrm{L}^2(0,T;\mathbf{M}\times\mathbb{Q})}$ and $\|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))}$.

5 Fully discrete approximation

In this section we introduce and analyze a fully discrete approximation of (4.2). To this end, and in order to employ the backward Euler method in the time discretization, we now consider $N \in \mathrm{N}$, let $\Delta t := \frac{T}{N}$ be the time step, and set $t_n := n\Delta t$, $n \in \big\{0,1,\dots,N\big\}$. In addition, we define the first order (backward) discrete time derivative $d_t u^n := (\Delta t)^{-1}(u^n - u^{n-1})$, where $u^n := u(t_n)$. Then the fully discrete method reads: given $\boldsymbol{\chi} \in \mathbf{M}$ and $(\underline{\mathbf{u}}_h^0, \boldsymbol{\sigma}_h^0) = ((\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0}), \boldsymbol{\sigma}_{h,0})$ satisfying (4.9), find $(\underline{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n) \in (\mathbf{M}_h \times \mathbb{Q}_h) \times \mathbb{X}_{0,h}$, $n \in \big\{1,\dots,N\big\}$, such that

$$\begin{aligned} d_t[\mathcal{E}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] + [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] + [\mathcal{B}^{\mathrm{t}}(\boldsymbol{\sigma}_h^n), \underline{\mathbf{v}}_h] \;&=\; [F^n, \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\,, \\ -\,[\mathcal{B}(\underline{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] \;&=\; 0 \qquad\quad\;\; \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,, \end{aligned} \tag{5.1}$$

where $[F^n, \underline{\mathbf{v}}_h] := (\mathbf{f}^n, \mathbf{v}_h)_\Omega$ and $\mathbf{f}^n := \mathbf{f}(t_n)$. In what follows, given a separable Banach space V endowed with the norm $\|\cdot\|_V$, we make use of the following discrete in time norms

$$\|u\|^2_{\ell^2(0,T;V)} := \Delta t \sum_{n=1}^N \|u^n\|_V^2 \quad \text{and} \quad \|u\|_{\ell^\infty(0,T;V)} := \max_{0\le n\le N} \|u^n\|_V. \tag{5.2}$$

We also recall the well-known identity

$$(d_t u_h^n, u_h^n)_\Omega \;=\; \frac{1}{2} d_t \|u_h^n\|^2_{\mathrm{L}^2(\Omega)} \,+\, \frac{1}{2}\Delta t\, \|d_t u_h^n\|^2_{\mathrm{L}^2(\Omega)}\,, \tag{5.3}$$

which follows from the definition of the discrete time derivative $d_t u_h^n$ and the polarization identity $(a-b,a) \;=\; \dfrac{1}{2}(|a|^2 - |b|^2 + |a-b|^2)$, applied to $a = u_h^n$ and $b = u_h^{n-1}$.

Next, we state the main result for the fully discrete scheme (5.1).

Theorem 5.1 *For each $(\underline{\mathbf{u}}_h^0, \boldsymbol{\sigma}_h^0) := ((\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0}), \boldsymbol{\sigma}_{h,0})$ satisfying problem (4.9) and for each $\mathbf{f}^n \in \mathbf{L}^2(\Omega)$, $n \in \big\{1,\dots,N\big\}$, and $\boldsymbol{\chi} \in \mathbf{M}$, there exists a unique solution $(\underline{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n) := ((\mathbf{u}_h^n, \boldsymbol{\vartheta}_h^n), \boldsymbol{\sigma}_h^n) \in (\mathbf{M}_h \times \mathbb{Q}_h) \times \mathbb{X}_{0,h}$ to (5.1), with $n \in \big\{1,\dots,N\big\}$. Moreover, there exists a positive constant $\widetilde{C}_{\mathtt{KV}\underline{\mathbf{u}}}$, depending only on $\|\boldsymbol{\chi}\|_{\mathbf{M}}$, $C_{\mathtt{d}}$, ν, $\mathtt{D}$, $\mathtt{F}$, and κ, such that*

$$\begin{aligned} &\|\mathbf{u}_h\|_{\ell^\infty(0,T;\mathbf{M})} + \|\boldsymbol{\vartheta}_h\|_{\ell^\infty(0,T;\mathbb{Q})} + \|\mathbf{u}_h\|_{\ell^2(0,T;\mathbf{M})} + \|\boldsymbol{\vartheta}_h\|_{\ell^2(0,T;\mathbb{Q})} \\ &\quad\le\, \widetilde{C}_{\mathtt{KV}\underline{\mathbf{u}}} \Big\{ \|\mathbf{f}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\nabla \mathbf{u}_0\|_{\mathbb{L}^4(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{M}} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{L}^{2(\rho-1)}(\Omega)} \Big\}\,, \end{aligned} \tag{5.4}$$

and there exists another positive constant $\widetilde{C}_{\mathtt{KV}\boldsymbol{\sigma}}$, depending only on $\|\boldsymbol{\chi}\|_{\mathbf{M}}$, $|\Omega|$, $C_{\mathtt{d}}$, ν, $\mathtt{D}$, $\mathtt{F}$, ρ, κ, and $\beta_{\mathtt{d}}$, such that

$$\begin{aligned} \|\boldsymbol{\sigma}_h\|_{\ell^2(0,T;\mathbb{X})} \,\leq\, \widetilde{C}_{\mathtt{KV}\boldsymbol{\sigma}} \sum_{j\in\{2,\rho\}} &\Big\{ \|\mathbf{f}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \\ &+ \|\nabla\mathbf{u}_0\|_{\mathbb{L}^4(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{M}} + \|\mathbf{u}_0\|^{\rho-1}_{\mathbf{L}^{2(\rho-1)}(\Omega)} \Big\}^{j-1}. \end{aligned} \tag{5.5}$$

Proof. The well-posedness of the fully discrete problem (5.1) at each time step t_n, $n\in 1,\dots,N$, can be demonstrated using arguments similar to those employed in the proof of Lemma 4.3. The derivation of estimates (5.4) and (5.5) proceeds in a similar manner to that presented in the proof of Theorem 4.4. Indeed, by choosing $(\underline{\mathbf{v}}_h,\boldsymbol{\tau}_h) = (\underline{\mathbf{u}}_h^n,\boldsymbol{\sigma}_h^n)$ in (5.1), and employing the identity (5.3), the definition of the operator $\mathcal{A}(\boldsymbol{\chi})$ (see (2.17)), the skew-symmetry property (3.11), as well as the classical Cauchy–Schwarz and Young inequalities (cf. (1.2)), we obtain

$$\begin{aligned} &\frac{1}{2}\,d_t\Big(\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} \,+\, \kappa^2\,\|\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\Big) \,+\, \frac{1}{2}\Delta t\Big(\|d_t\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} \,+\, \kappa^2\,\|d_t\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\Big) \\ &\quad + \mathtt{D}\,\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} \,+\, \mathtt{F}\,\|\mathbf{u}_h^n\|^\rho_{\mathbf{L}^\rho(\Omega)} \,+\, \nu\,\|\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}} \,\leq\, \frac{1}{4\,\mathtt{D}}\,\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} \,+\, \mathtt{D}\,\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)}\,. \end{aligned} \tag{5.6}$$

Thus, summing up over the time index $n\in\big\{1,\dots,m\big\}$, with $m\in\big\{1,\dots,N\big\}$, in (5.6), multiplying by Δt, and proceeding similarly to (4.15)–(4.16), we deduce the discrete version of (3.43), namely

$$\begin{aligned} &\gamma_{\kappa\mathtt{d}}\Big(\|\mathbf{u}_h^m\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h^m\|^2_{\mathbb{Q}}\Big) + \gamma_{\kappa\mathtt{d}}\,(\Delta t)^2\sum_{n=1}^m\Big(\|d_t\mathbf{u}_h^n\|^2_{\mathbf{M}} + \|d_t\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\Big) \\ &\quad + \gamma_{\mathtt{D}\nu\mathtt{d}}\,\Delta t\sum_{n=1}^m\Big(\|\mathbf{u}_h^n\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\Big) \leq \frac{\Delta t}{2\mathtt{D}}\sum_{n=1}^m\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_h^0\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2\,\|\boldsymbol{\vartheta}_h^0\|^2_{\mathbb{Q}}\,, \end{aligned} \tag{5.7}$$

with $\gamma_{\kappa\mathtt{d}}$ and $\gamma_{\mathtt{D}\nu\mathtt{d}}$ as in (4.16). Notice that, in order to simplify the stability bound, we have neglected the term $\|\mathbf{u}_h^n\|^\rho_{\mathbf{L}^\rho(\Omega)}$ on the left-hand side of (5.6). Thus, analogously to (4.16), and using the estimate (4.12) in (5.7), we deduce the stability bound (5.4). Unlike its continuous counterpart (3.37), however, the constant $\widetilde{C}_{\mathtt{KV}\underline{\mathbf{u}}}$ in (5.4) depends on $\mathtt{F}$ and $C_{\mathtt{d}}$ due to the use of (4.12) and (4.5). On the other hand, from the discrete inf-sup condition of $\mathcal{B}$ (cf. (4.4)) and the first equation of (5.1) related to $\underline{\mathbf{v}}_h$, we deduce the discrete version of (3.44), that is

$$\beta_{\mathtt{d}}\|\boldsymbol{\sigma}_h^n\|_{\mathbb{X}} \,\leq\, \widehat{c}\,\Big\{\|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)} \,+\, \|d_t\underline{\mathbf{u}}_h^n\| + \|\mathbf{u}_h^n\|^{\rho-1}_{\mathbf{M}} + \big(1+\|\boldsymbol{\chi}\|_{\mathbf{M}}\big)\big(\|\mathbf{u}_h^n\|_{\mathbf{M}} \,+\, \|\boldsymbol{\vartheta}_h^n\|_{\mathbb{Q}}\big)\Big\}, \tag{5.8}$$

with a positive constant $\widehat{c}$, depending on $|\Omega|$, κ, and $C_{\mathcal{A}}$ (cf. (3.9)). Then, squaring (5.8), summing up over the time index $n\in\big\{1,\dots,m\big\}$, with $m\in\big\{1,\dots,N\big\}$, and mutiplying by Δt, we deduce analogously to (3.45), that there exists $\widehat{c}_1>0$, depending on $|\Omega|$, κ, $\mathtt{D}$, $\mathtt{F}$, ν, and $\beta_{\mathtt{d}}$, such that

$$\begin{aligned} \Delta t\sum_{n=1}^m\|\boldsymbol{\sigma}_h^n\|^2_{\mathbb{X}} \,\leq\, \widehat{c}_1\,\Bigg\{ &\Delta t\sum_{n=1}^m\Big(\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \big(1+\|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big)\big(\|\mathbf{u}_h^n\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\big)\Big) \\ &+ \Delta t\sum_{n=1}^m\Big(\|\mathbf{u}_h^n\|^{2(\rho-1)}_{\mathbf{M}} + \|d_t\underline{\mathbf{u}}_h^n\|^2\Big)\Bigg\}. \end{aligned} \tag{5.9}$$

Next, in order to bound the last term in (5.9), we choose $(\underline{\mathbf{v}}_h,\boldsymbol{\tau}_h) := \big((d_t(\mathbf{u}_h^n), d_t(\boldsymbol{\vartheta}_h^n),\boldsymbol{\sigma}_h^n\big)$ in (5.1), apply some algebraic manipulation, use the identity (5.3), the estimate (4.5), and the Cauchy–Schwarz and Young inequalities, to obtain the discrete version of (3.46) (cf. (4.19))

$$\begin{aligned}&\frac{1}{2}\,d_t\Big(\mathtt{D}\,\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \nu\,\|\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\Big) \,+\, \mathtt{F}\,(|\mathbf{u}_h^n|^{\rho-2}\mathbf{u}_h^n, d_t\mathbf{u}_h^n)_\Omega + \frac{1}{2}\,\Delta t\Big(\mathtt{D}\,\|d_t\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \nu\,\|d_t\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\Big)\\ &\qquad + \gamma_{\kappa d}\,\|d_t\underline{\mathbf{u}}_h^n\|^2 \,\leq\, \widehat{c}_2\,\Big(\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big(\|\mathbf{u}_h^n\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\big)\Big) + \frac{\gamma_{\kappa d}}{2}\,\|d_t\underline{\mathbf{u}}_h^n\|^2\,,\end{aligned} \tag{5.10}$$

with $\gamma_{\kappa d}$ as in (4.19), and a positive constant $\widehat{c}_2$ depending on $|\Omega|$, $C_{\mathtt{d}}$, and κ. In turn, employing the Hölder and Young inequalities, we are able to deduce (cf. [15, eq. (5.13)]):

$$(|\mathbf{u}_h^n|^{\rho-2}\mathbf{u}_h^n, d_t\mathbf{u}_h^n)_\Omega \,\geq\, \frac{(\Delta t)^{-1}}{\rho}\,\Big(\|\mathbf{u}_h^n\|^\rho_{\mathbf{L}^\rho(\Omega)} - \|\mathbf{u}_h^{n-1}\|^\rho_{\mathbf{L}^\rho(\Omega)}\Big) \,=\, \frac{1}{\rho}\,d_t\|\mathbf{u}_h^n\|^\rho_{\mathbf{L}^\rho(\Omega)}\,. \tag{5.11}$$

Thus, combining (5.10) with (5.11), using Young's inequality, summing up over the time index $n \in \{1,\dots,m\}$, with $m \in \{1,\dots,N\}$, and multiplying by Δt, we obtain

$$\begin{aligned}&\mathtt{D}\,\|\mathbf{u}_h^m\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\|\mathbf{u}_h^m\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\vartheta}_h^m\|^2_{\mathbb{Q}} + \gamma_{\kappa\mathtt{d}}\,\Delta t\sum_{n=1}^m \|d_t\underline{\mathbf{u}}_h^n\|^2\\ &\quad\leq\, 2\,\widehat{c}_2\,\Delta t\sum_{n=1}^m\Big(\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big(\|\mathbf{u}_h^n\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\big)\Big)\\ &\quad\quad + \mathtt{D}\,\|\mathbf{u}_h^0\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathtt{F}}{\rho}\|\mathbf{u}_h^0\|^\rho_{\mathbf{L}^\rho(\Omega)} + \nu\,\|\boldsymbol{\vartheta}_h^0\|^2_{\mathbb{Q}}\,,\end{aligned} \tag{5.12}$$

with $\gamma_{\kappa\mathtt{d}}$ as in (4.16). Combining (5.12) with (5.9), and using the fact that $(\mathbf{u}_h^0, \boldsymbol{\vartheta}_h^0) = (\mathbf{u}_{h,0}, \boldsymbol{\vartheta}_{h,0})$, we deduce that

$$\begin{aligned}&\Delta t\sum_{n=1}^m \|\boldsymbol{\sigma}_h^n\|^2_{\mathbb{X}} \,\leq\, \widehat{c}_3\,\Bigg\{\Delta t\sum_{n=1}^m \|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_{h,0}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_{h,0}\|^\rho_{\mathbf{L}^\rho(\Omega)} + \|\boldsymbol{\vartheta}_{h,0}\|^2_{\mathbb{Q}}\\ &\quad + \big(1 + \|\boldsymbol{\chi}\|^2_{\mathbf{M}}\big)\,\Delta t\sum_{n=1}^m\Big(\|\mathbf{u}_h^n\|^2_{\mathbf{M}} + \|\boldsymbol{\vartheta}_h^n\|^2_{\mathbb{Q}}\Big) + \|\mathbf{u}_h\|^{2(\rho-2)}_{\ell^\infty(0,T;\mathbf{M})}\,\Delta t\sum_{n=1}^m \|\mathbf{u}_h^n\|^2_{\mathbf{M}}\Bigg\}\,,\end{aligned} \tag{5.13}$$

with $m \in \{1,\dots,N\}$ and $\widehat{c}_3$ a positive constant depending on $|\Omega|$, $C_{\mathtt{d}}$, κ, $\mathtt{D}$, $\mathtt{F}$, ν, ρ, and $\beta_{\mathtt{d}}$. Then, using (5.4) and (4.12) to bound the terms on the right-hand side of (5.13), we obtain (5.5). $\square$

Finally, we derive the rates of convergence for the fully discrete scheme (5.1). For this purpose, we subtract this latter from the continuous counterparts (2.15) at each time step $n \in \{1,\dots,N\}$, thus obtaining the following error system

$$\begin{aligned}d_t[\mathcal{E}(\mathbf{e}^n_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] + [\mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}^n) - \mathcal{A}(\boldsymbol{\chi})(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \mathbf{e}^n_{\boldsymbol{\sigma}}] &\;=\; [r_n(\underline{\mathbf{u}}), \underline{\mathbf{v}}_h] \quad \forall\,\underline{\mathbf{v}}_h \in \mathbf{M}_h\times\mathbb{Q}_h\,,\\ [\mathcal{B}(\mathbf{e}^n_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h] &\;=\; 0 \qquad\qquad\quad \forall\,\boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,,\end{aligned}$$

where

$$[r_n(\underline{\mathbf{u}}), \underline{\mathbf{v}}_h] \,:=\, (r_n(\mathbf{u}), \mathbf{v}_h)_\Omega \,+\, \kappa^2\,(r_n(\boldsymbol{\vartheta}), \boldsymbol{\xi}_h)_\Omega\,,$$

and r_n denotes the difference between the time derivative and its discrete analogue, that is

$$r_n(\underline{\mathbf{u}}) := (r_n(\mathbf{u}), r_n(\boldsymbol{\vartheta}))\,,$$

with

$$r_n(\mathbf{u}) \,=\, d_t\mathbf{u}^n - \partial_t\mathbf{u}(t_n) \quad\text{and}\quad r_n(\boldsymbol{\vartheta}) \,=\, d_t\boldsymbol{\vartheta}^n - \partial_t\boldsymbol{\vartheta}(t_n)\,.$$

In addition, we recall from [9, Lemma 4] that, if $\underline{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\vartheta}) \in \mathrm{H}^2(0,T;\mathbf{L}^2(\Omega)) \times \mathrm{H}^2(0,T;\mathbb{Q})$, then there exists a positive constant C, independent of Δt, such that

$$\Delta t \sum_{n=1}^{N} \|r_n(\underline{\mathbf{u}})\|^2 \,\leq\, C \left(\|\partial_{tt}\mathbf{u}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\partial_{tt}\boldsymbol{\vartheta}\|_{\mathrm{L}^2(0,T;\mathbb{Q})} \right) (\Delta t)^2 \,. \tag{5.14}$$

Therefore, the proof of the theoretical rate of convergence of the fully discrete scheme (5.1) follows the structure of the proof of Theorem 4.5, using discrete-in-time arguments, as in the proof of Theorem 5.1, and the estimate (5.14) (see [15, Theorem 5.4] for a similar approach). We summarize the above in the following result.

Theorem 5.2 *Let $s \in (0, k+1]$ such that the assumptions of Theorem 4.5 hold. Then, for the solution of the fully discrete problem (5.1) there exists a positive constant $\widehat{C}(\underline{\mathbf{u}}, \boldsymbol{\sigma})$, depending only on $C(\underline{\mathbf{u}}), C(\partial_t \underline{\mathbf{u}}), C(\partial_{tt}\underline{\mathbf{u}})$, $C(\boldsymbol{\sigma}), |\Omega|, \mathtt{D}, \mathtt{F}, \rho, \nu, \kappa, \beta_{\mathtt{d}}, C_{\mathtt{d}}, \|\mathbf{i}_4\|$, and data, such that*

$$\begin{aligned} \|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{M})} \,+\, \|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\ell^\infty(0,T;\mathbb{Q})} \,+\, \|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{M})} \,+\, \|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\ell^2(0,T;\mathbb{Q})} & \\ +\, \|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\ell^2(0,T;\mathbb{X})} \,\leq\, \widehat{C}(\underline{\mathbf{u}}, \boldsymbol{\sigma}) \left(h^s \,+\, h^{s\,(\rho-1)} \,+\, \Delta t \right). & \end{aligned}$$

6 Numerical results

In this section, we present three numerical results that illustrate the performance of the fully discrete method (5.1). The implementation is based on a `FreeFEM` code [24]. We use quasi-uniform triangulations and the finite element subespaces detailed in Section 4.1 (cf. (4.1)). In what follows, we refer to the corresponding sets of finite element subspaces generated by $k = 0$ and $k = 1$, as simply $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0$ and $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$, respectively. The nonlinearity is handled using a Newton–Raphson algorithm with a fixed tolerance of $\mathsf{tol} = 1\mathrm{E}-06$. The iterative process at each time step is terminated when the relative error between two consecutive iterates of the complete coefficient vector, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\mathtt{DoF}}}{\|\mathbf{coeff}^{m+1}\|_{\mathtt{DoF}}} \leq \mathsf{tol}\,,$$

where $\|\cdot\|_{\mathtt{DoF}}$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DoF}}$, with `DoF` denoting the total number of degrees of freedom defined by the finite element subspaces $\mathbf{M}_h, \mathbb{Q}_h$ and $\mathbb{X}_{0,h}$ (cf. (4.1)).

We stress that, according to the notation used for the fully discrete norm (5.2), and besides the unknowns $\mathbf{u}, \boldsymbol{\vartheta}$, and $\boldsymbol{\sigma}$, we are also able to compute the pressure error:

$$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} = \left\{ \Delta t \sum_{n=1}^{N} \|p^n - p_h^n\|^2_{\mathrm{L}^2(\Omega)} \right\}^{1/2},$$

where for each $n = 1, \ldots, N$, p_h^n stands for the post-processed pressure suggested by the identity (2.3) and (2.14), that is

$$p_h^n = -\frac{1}{d}\,\mathrm{tr}(\boldsymbol{\sigma}_h^n) - \frac{1}{2\,d}\,\mathrm{tr}(\mathbf{u}_h^n \otimes \boldsymbol{\chi}) - \bar{c}_h \quad \text{with} \quad \bar{c}_h = -\frac{1}{2\,d\,|\Omega|}\,(\mathrm{tr}(\mathbf{u}_h^n \otimes \boldsymbol{\chi}), 1)_\Omega\,. \tag{6.1}$$

The examples considered in this section are described next. In the first two examples, and for the sake of simplicity, we choose $\nu = 1$, $\mathtt{D} = 1$, $\mathtt{F} = 10$, and $\kappa = 1$. In addition, the condition

$(\mathrm{tr}(\boldsymbol{\sigma}_h^n), 1)_\Omega = 0$ is implemented using a scalar Lagrange multiplier (adding one row and one column to the matrix system that solves (5.1) for $\mathbf{u}_h^n$, $\boldsymbol{\vartheta}_h^n$ and $\boldsymbol{\sigma}_h^n$).

Examples 1 and 2 are used to corroborate the rate of convergence in two and three dimensional domains, respectively. The total simulation time for these examples is $T = 0.01$ and the time step is $\Delta t = 10^{-3}$. The time step is sufficiently small, so that time discretization error does not affect the convergence rates. On the other hand, Example 3 is used to analyze the method's behavior under various scenarios, considering different Darcy and Forchheimer coefficients, as well as varying values of the elasticity parameter κ. For these cases, the total simulation time and the time step are chosen as $T = 1$ and $\Delta t = 10^{-2}$, respectively.

Example 1: Two-dimensional smooth exact solution

In this test, we study the convergence of the space discretization using an analytical solution. The domain is the square $\Omega = (0,1)^2$. We consider $\rho = 3$ and choose the datum $\mathbf{f}$ so that the steady-state convective velocity $\boldsymbol{\chi}$ and the exact solution of (2.5) is given by the following smooth functions:

$$\boldsymbol{\chi} = \begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}, \quad \mathbf{u} = \sin(\pi t)\,\boldsymbol{\chi}, \quad \text{and} \quad p = \exp(t)\cos(\pi x)\sin\left(\frac{\pi y}{2}\right).$$

Notice that the given exact solution $\mathbf{u}$ is non-homogeneous on the boundary so that the right-hand side must be adjusted properly as described by the end of Section 3.4.

In Figure 6.1, we display the solution obtained with the mixed $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$ approximation, using a mesh with size $h = 0.0127$ and $39,146$ triangular elements, which results in $979,674$ `DoF`, at time $T = 0.01$. Note that this approach allows us to compute not only the original unknowns but also the pressure field through formula (6.1). Tables 6.1 and 6.2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the average number of Newton iterations. The results illustrate that the optimal spatial convergence rates $\mathcal{O}(h^{k+1})$ predicted by Theorem 5.2 (see also Theorem 4.5) are attained for $k = 0, 1$.

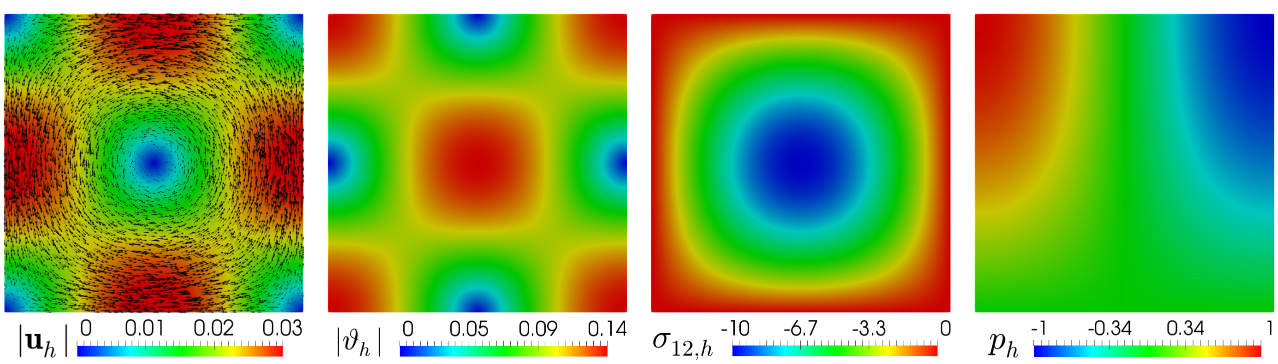

Figure 6.1: [Example 1]: Computed magnitude of the velocity and velocity gradient, viscoelastic pseudostress tensor component, and pressure field.

Example 2: Three-dimensional smooth exact solution

In the second example, we consider the cubic domain $\Omega = (0,1)^3$ and $\rho = 3.5$. Similarly to the first example, the right-hand side function $\mathbf{f}$ is computed from (2.5) using the following time-independent

DoF	h	$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{M})}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\ell^\infty(0,T;\mathbb{Q})}$ error	rate	$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{M})}$ error	rate
304	0.3726	7.78E-03	–	2.81E-02	–	4.83E-04	–
1,248	0.1964	3.41E-03	1.2873	1.38E-02	1.1096	2.12E-04	1.2873
4,896	0.0970	1.71E-03	0.9785	6.96E-03	0.9762	1.06E-04	0.9785
19,456	0.0478	8.26E-04	1.0288	3.53E-03	0.9573	5.13E-05	1.0288
77,648	0.0245	4.20E-04	1.0114	1.76E-03	1.0438	2.61E-05	1.0114
313,680	0.0127	2.08E-04	1.0768	8.70E-04	1.0792	1.29E-05	1.0769

$\|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\ell^2(0,T;\mathbb{Q})}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\ell^2(0,T;\mathbb{X})}$ error	rate	$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$ error	rate	iter
1.75E-03	–	1.13E+00	–	1.03E-01	–	2.1
8.59E-04	1.1096	4.89E-01	1.3013	4.43E-02	1.3197	2.1
4.32E-04	0.9762	2.44E-01	0.9853	2.12E-02	1.0452	2.1
2.19E-04	0.9573	1.19E-01	1.0129	1.02E-02	1.0381	2.1
1.09E-04	1.0438	6.02E-02	1.0228	5.29E-03	0.9750	2.1
5.40E-05	1.0792	2.98E-02	1.0760	2.55E-03	1.1170	2.1

Table 6.1: [Example 1] Number of degrees of freedom, mesh sizes, errors, rates of convergences, and average number of Newton iterations for the $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0$ approximation of the Oseen-type Kelvin–Voigt–Brinkman–Forchheimer model with $\rho = 3$, $\nu = 1$, $\mathtt{D} = 1$, $\mathtt{F} = 10$, and $\kappa = 1$.

convective velocity $\boldsymbol{\chi}$ and a manufactured solution:

$$\boldsymbol{\chi} = \begin{pmatrix} \sin(\pi x)\cos(\pi y)\cos(\pi z) \\ -2\cos(\pi x)\sin(\pi y)\cos(\pi z) \\ \cos(\pi x)\cos(\pi y)\sin(\pi z) \end{pmatrix}, \quad \mathbf{u} = \sin(\pi t)\,\boldsymbol{\chi}, \quad \text{and} \quad p = \exp(t)(x-0.5)^3\sin(y+z)\,.$$

The model is complemented with appropriate Dirichlet boundary condition and consistent initial data.

The numerical solutions at time $T = 0.01$ are shown in Figure 6.2. They were built using the fully-mixed $\mathbf{P}_0-\mathbb{P}_0-\mathbb{RT}_0$ approximation with mesh size $h = 0.0642$ and $118,098$ tetrahedral elements, which correspond to $2,020,788$ DoF. The convergence history for a set of quasi-uniform mesh refinements using $k = 0$ is shown in Table 6.3. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$ for all the unknowns, as it was proved by Theorem 5.2 (see also Theorem 4.5).

Example 3: Flow through porous media with fracture network

In the last example, inspired by [2, Section 5.2.5], we focus on flows through porous media with a fracture network. We consider the square domain $\Omega = (-1,1)^2$, containing an internal network of thin fractures (denoted by Ω_f) that intersect at sharp angles, as shown in the first plot of Figure 6.3. We study numerically the case where $\boldsymbol{\chi}$ is time-dependent and satisfies $\boldsymbol{\chi} = \mathbf{u}$ in (2.5), that is, the original Kelvin–Voigt–Brinkman–Forchheimer model posed on the entire domain $\Omega \times [0,T]$. For this problem, a suitable modification of Newton's method for (5.1), accounting for the nonlinearity of the convective term, is developed. The parameters are chosen as $\rho = 4$, $\nu = 1$, and $\kappa = 1$, but with different values of D and F inside and outside the fractures, namely:

$$\mathtt{D} = \begin{cases} 1 & \text{in} \quad \Omega_f \\ 1000 & \text{in} \quad \overline{\Omega}\setminus\Omega_f \end{cases} \quad \text{and} \quad \mathtt{F} = \begin{cases} 10 & \text{in} \quad \Omega_f \\ 1 & \text{in} \quad \overline{\Omega}\setminus\Omega_f \end{cases}. \tag{6.2}$$

DoF	h	$\|\mathbf{e_u}\|_{\ell^\infty(0,T;\mathbf{M})}$ error	rate	$\|\mathbf{e}_{\boldsymbol\vartheta}\|_{\ell^\infty(0,T;\mathbb{Q})}$ error	rate	$\|\mathbf{e_u}\|_{\ell^2(0,T;\mathbf{M})}$ error	rate
932	0.3726	8.38E-04	–	4.77E-03	–	5.20E-05	–
3,864	0.1964	2.10E-04	2.1629	8.93E-04	2.6158	1.30E-05	2.1629
15,228	0.0970	5.25E-05	1.9635	2.22E-04	1.9734	3.26E-06	1.9635
60,656	0.0478	1.36E-05	1.9133	5.26E-05	2.0341	8.41E-07	1.9131
242,362	0.0245	3.40E-06	2.0703	1.36E-05	2.0270	2.11E-07	2.0708
979,674	0.0127	8.53E-07	2.1180	3.59E-06	2.0417	5.23E-08	2.1363

$\|\mathbf{e}_{\boldsymbol\vartheta}\|_{\ell^2(0,T;\mathbb{Q})}$ error	rate	$\|\mathbf{e}_{\boldsymbol\sigma}\|_{\ell^2(0,T;\mathbb{X})}$ error	rate	$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$ error	rate	iter
2.96E-04	–	1.26E-01	–	1.22E-02	–	2.1
5.54E-05	2.6158	3.13E-02	2.1741	2.47E-03	2.4898	2.1
1.38E-05	1.9734	7.93E-03	1.9488	6.34E-04	1.9277	2.1
3.27E-06	2.0343	2.03E-03	1.9258	1.51E-04	2.0235	2.1
8.42E-07	2.0294	5.08E-04	2.0725	3.88E-05	2.0385	2.1
2.17E-07	2.0783	1.26E-04	2.1391	1.05E-05	1.9965	2.1

Table 6.2: [Example 1] Number of degrees of freedom, mesh sizes, errors, rates of convergences, and average number of Newton iterations for the $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$ approximation of the Oseen-type Kelvin–Voigt–Brinkman–Forchheimer model with $\rho = 3$, $\nu = 1$, $\mathtt{D} = 1$, $\mathtt{F} = 10$, and $\kappa = 1$.

In turn, the body force term is $\mathbf{f} = \mathbf{0}$, the initial condition is zero, and the boundaries conditions are

$$\boldsymbol\sigma\mathbf{n} = \begin{cases} (-0.5(y-1), 0) & \text{on} \quad \Gamma_{\text{left}}, \\ (0, -0.5(x-1)) & \text{on} \quad \Gamma_{\text{bottom}}, \end{cases} \qquad \boldsymbol\sigma\mathbf{n} = (0,0) \quad \text{on} \quad \Gamma_{\text{right}} \cup \Gamma_{\text{top}}, \tag{6.3}$$

which drives the flow in a diagonal direction from the left-bottom corner to the right-top corner of the square domain Ω.

In Figure 6.3 we display the computed magnitude of the velocity, the velocity gradient tensor, and the viscoelastic pseudostress tensor at times $T = 0.01$, 0.2, and 1. These results were obtained using the fully-mixed $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$ approximation on a mesh with $48,891$ triangular elements, which correspond to $1,222,690$ DoF. We note that the velocity in the fractures is higher that the velocity in the porous medium, due to smaller fractures thickness and the parameter setting (6.2). Also, the velocity is higher in branches of the network where the fluid enters from the left-bottom corner and decreases toward the right-top corner of the domain. In addition, we observe a sharp velocity gradient across the interfaces between the fractures and the porous medium. The viscoelastic pseudostress is consistent with the boundary conditions (6.3) and is more diffused since it includes the pressure field. This example illustrates the ability of the method to provide accurate resolution and numerically stable results for heterogeneous inclusions with high aspect ratio and complex geometry, as presented in the network of thin fractures. We further investigated the robustness of the method with respect to the elasticity parameter κ. In Figure 6.4, we show the computed magnitudes of the velocity, the velocity gradient tensor, and the viscoelastic pseudostress tensor for the parameters in (6.2), considering $\kappa \in \{3, 2, 1, 10^{-2}, 10^{-4}\}$. We observe that the elasticity parameter κ has a dissipative effect: as κ decreases, the velocity in the fractures increases and the velocity gradient across the fracture becomes more pronounced, while the viscoelastic pseudostress remains consistent throughout this variation. This study shows that the method produces stable and physically meaningful results

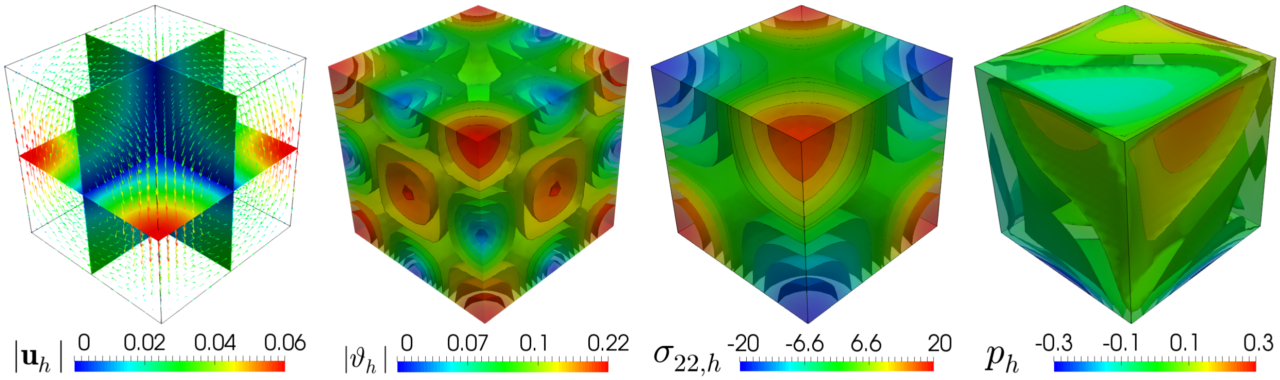

Figure 6.2: [Example 2]: Computed magnitudes of the velocity and velocity gradient, viscoelastic pseudostress tensor component, and pressure field.

over a wide range of parameters, such as $\mathtt{D}$, $\mathtt{F}$, and κ. Furthermore, we observe that the method is robust with respect to small values of κ, such as 10^{-4}. Notice that, in the limit case $\kappa = 0$, the classical convective Brinkman–Forchheimer model is recovered.

7 Conclusions

In this paper, we presented a new velocity-velocity gradient-pseudostress formulation for an Oseen-type linearization of the Kelvin–Voigt–Brinkman–Forchheimer (KVBF) system together with its mixed finite element approximation. The system models fast and unsteady viscoelastic flows in highly porous media. The formulation offers several advantages, among which we first highlight the fact that no small-data assumptions are required. In addition, it provides a well-posed setting for large data, achieving optimal convergence rates, and directly approximating two physical variables of interest, namely the velocity gradient and the viscoelastic pseudostress tensors, while also recovering the pressure field through a simple postprocessing step. We establish the well-posedness of the weak formulation, as well as stability and error estimates for the semidiscrete and fully discrete mixed finite element approximations. The numerical results confirm the effectiveness and stability of the method in highly demanding scenarios. In particular, the example involving an internal network of thin fractures demonstrates the ability of the formulation to accurately resolve heterogeneous media with large contrasts in the parameters $\mathtt{D}$ and $\mathtt{F}$, correctly reproducing the concentration and deviation of the flow along the fractures and its diagonal propagation induced by the boundary conditions. The simulations show significantly higher velocities inside the fractures than in the surrounding porous matrix, along with sharp velocity gradients across the interfaces. Moreover, the viscoelastic pseudostress remains consistent with the boundary conditions and exhibits a smoothed behavior due to the contribution of the pressure field. Finally, the robustness analysis with respect to the elasticity parameter κ shows that the method produces stable and physically meaningful results over a wide range of values, maintaining a consistent pseudostress pattern and accurately capturing the dissipative effect associated with decreasing κ. Overall, these experiments validate the numerical robustness of the proposed scheme and its capability to resolve viscoelastic flows in complex geometries with strong material discontinuities. In particular, the final numerical example suggests that the skew-symmetric approach can be extended to the original KVBF model and highlights the importance of mesh refinement around the fracture network, as illustrated in Figure 6.3. This observation motivates future work on establishing the well-posedness

DoF	h	$\|\mathbf{e_u}\|_{\ell^\infty(0,T;\mathbf{M})}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\ell^\infty(0,T;\mathbb{Q})}$ error	rate	$\|\mathbf{e_u}\|_{\ell^2(0,T;\mathbf{M})}$ error	rate
888	0.8660	1.80E-02	–	8.07E-02	–	1.12E-03	–
6,816	0.4330	9.45E-03	0.9280	4.34E-02	0.8963	5.86E-04	0.9280
53,376	0.2165	4.86E-03	0.9590	2.21E-02	0.9734	3.02E-04	0.9590
422,400	0.1083	2.45E-03	0.9894	1.11E-02	0.9933	1.52E-04	0.9894
2,020,788	0.0642	1.45E-03	0.9969	6.58E-03	0.9981	9.01E-05	0.9969

$\|\mathbf{e}_{\boldsymbol{\vartheta}}\|_{\ell^2(0,T;\mathbb{Q})}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\ell^2(0,T;\mathbb{X})}$ error	rate	$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$ error	rate	iter
5.01E-03	–	3.78E+00	–	1.51E-01	–	2.1
2.69E-03	0.8963	2.01E+00	0.9139	8.39E-02	0.8483	2.1
1.37E-03	0.9734	1.02E+00	0.9795	4.30E-02	0.9633	2.1
6.89E-04	0.9933	5.11E-01	0.9951	2.16E-02	0.9918	2.1
4.08E-04	0.9981	3.03E-01	0.9986	1.28E-02	0.9986	2.1

Table 6.3: [Example 2] Number of degrees of freedom, mesh sizes, errors, rates of convergences, and average number of Newton iterations for the $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0$ approximation of the Oseen-type Kelvin–Voigt–Brinkman–Forchheimer model with $\rho = 3.5$, $\nu = 1$, $\mathtt{D} = 1$, $\mathtt{F} = 10$, and $\kappa = 1$.

and error estimates for a skew-symmetry-based mixed formulation of the KVBF model, as well as developing an *a posteriori* error analysis for the proposed method, which could then guide adaptive mesh refinement and yield more accurate approximations for this class of problems.

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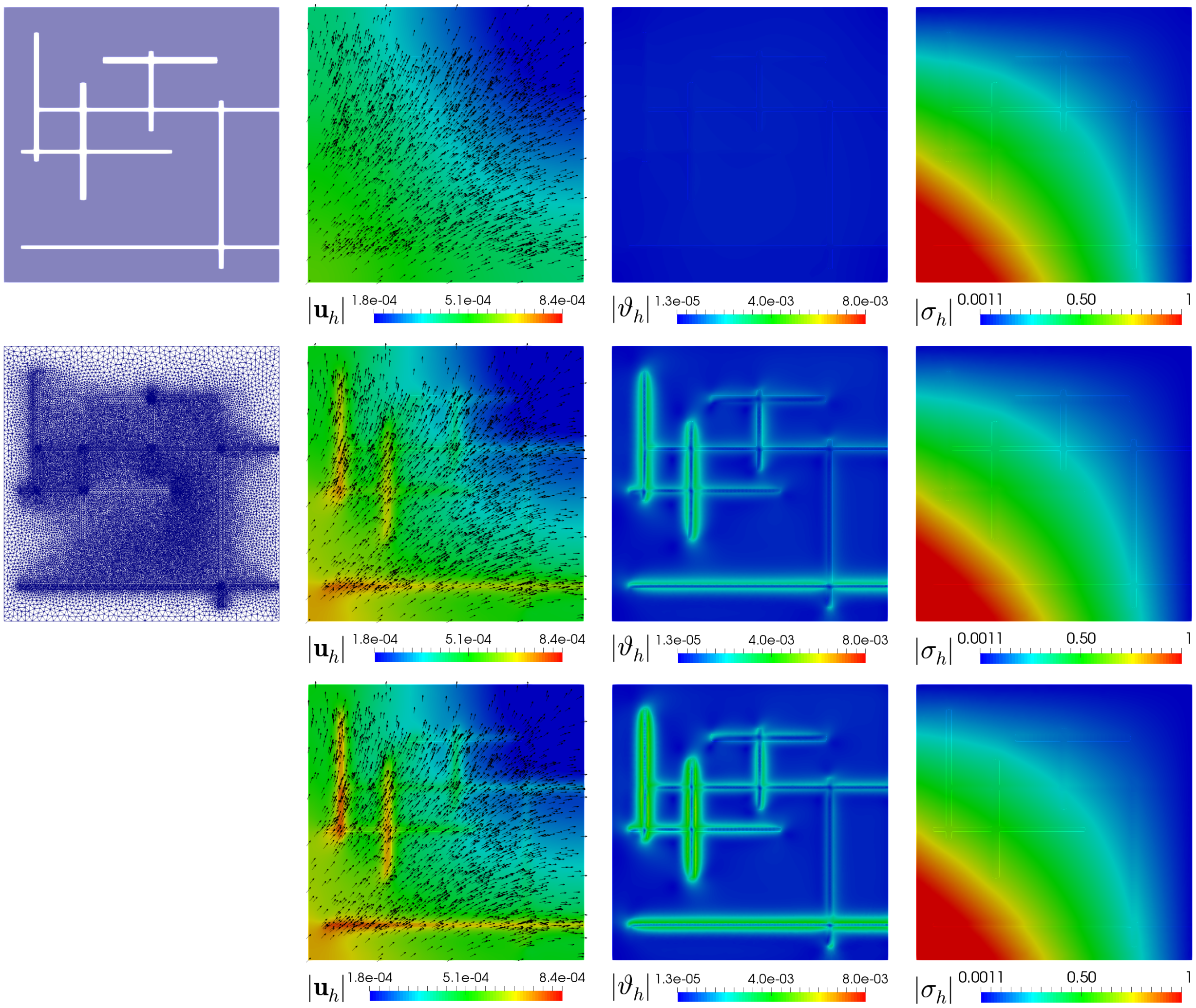

Figure 6.3: [Example 3]: Domain configuration and prescribed mesh (plots in first column), and computed magnitude of the velocity, velocity gradient tensor, and viscoelastic pseudostress tensor at time $T = 0.01$ (top plots), at time $T = 0.5$ (middle plots), and at time $T = 1$ (bottom plots).

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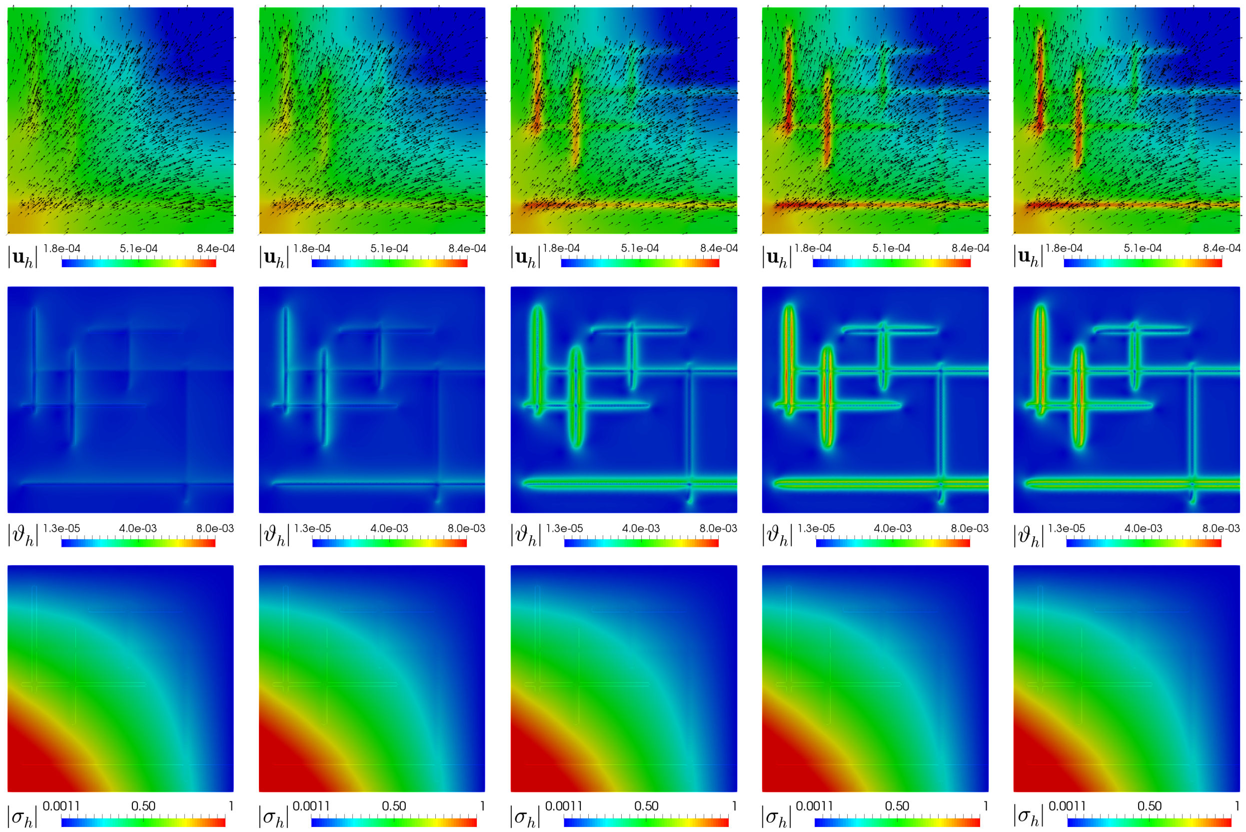

Figure 6.4: [Example 3]: Computed magnitude of the velocity, velocity gradient tensor, and viscoelastic pseudostress tensor at time $T = 1$, with $\rho = 4$, fracture setting $\mathtt{F} = 10$ and $\mathtt{D} = 1$, and porous media setting $\mathtt{F} = 1$ and $\mathtt{D} = 1000$, for $\kappa \in \{3, 2, 1, 10^{-2}, 10^{-4}\}$ (from left to right).

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