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A Mixed Finite Element Method Based on Pseudostress and Stream-Function for the Navier–Stokes Problem in 2D *

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Abstract

In this work, we propose and analyze a new pseudostress–stream function dual-mixed variational formulation for the stationary Navier–Stokes problem in two dimensions with nonhomogeneous Dirichlet boundary conditions. More precisely, after rewriting the Navier–Stokes system in terms of the pseudostress tensor and the velocity field, the latter is decomposed by means of a Helmholtz–Weyl decomposition in $\mathbf{L}^p(\Omega)$, which leads to a three-field formulation in which the pseudostress, the stream function, and a Lagrange multiplier associated with momentum conservation are the primary unknowns. The associated Galerkin scheme is defined using lowest-order Raviart–Thomas elements for the pseudostress tensor, continuous piecewise linear elements for the stream function, and Crouzeix–Raviart elements for the Lagrange multiplier. We prove the well-posedness of both the continuous and discrete formulations by means of the Banach–Nečas–Babuška theorem and Banach's fixed-point theorem, under a sufficiently small data assumption. In addition, we derive a priori error estimates and establish the optimal rate of convergence. Finally, numerical experiments are provided to validate the theoretical results and to demonstrate the efficiency and accuracy of the proposed method. The present contribution retains the main advantages of classical pseudostress-based formulations, such as the capability to recover additional variables of interest and to ensure momentum conservation, while introducing new features, including exactly divergence-free velocity approximations and a direct approximation of the stream function.

Key words: Navier–Stokes problem; mass and momentum conservation; Helmholtz–Weyl decomposition; Raviart–Thomas elements, stream-function, pseudostress

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1 Introduction

The numerical approximation of incompressible fluid flow remains a central topic in scientific computing, with applications ranging from engineering and geophysical flows to biological systems. In this context, the stationary Navier–Stokes equations constitute a fundamental model, describing the balance of momentum and mass conservation in viscous incompressible fluids.

A major challenge in the numerical treatment of this system lies in the incompressibility constraint, which couples the velocity and pressure fields and gives rise to the well-known inf–sup stability condition. Over the past decades, a wide variety of approaches have been developed to address this difficulty, including mixed finite element methods, stabilized formulations and discontinuous Galerkin (DG) schemes (see, e.g., [17, 18, 27, 28, 30, 32, 37] and the references therein).

Among these approaches, mixed formulations based on pseudostress variables have attracted considerable attention, as they allow for the direct approximation of physically relevant quantities while ensuring local conservation properties (see e.g. [8, 9, 10, 11, 12, 14, 16]). In particular, the introduction of the pseudostress tensor enables the reformulation of the momentum equation into a first-order system, leading to stable and momentum-conservative discretizations (see, e.g., [11, 12, 25]).

An alternative approach to enforcing mass conservation at the discrete level for the Navier–Stokes equations is provided by so-called stream function-based methods, which ensure the incompressibility constraint exactly by construction. More precisely, by expressing the velocity field as the *curl* of a scalar potential in two dimensions (or a vector potential in three dimensions), one obtains a fourth-order variational formulation in which the primary unknown is the corresponding stream function (see, e.g., [26, Sections I.5.2 and I.5.3]). In this framework, the velocity field is recovered by applying the *curl* operator to the stream function, thereby yielding an exactly divergence-free approximation. This approach has led to a variety of discretization strategies, including C^1 conforming finite elements, DG methods, and Virtual Element Methods (see, e.g., [1, 2, 4, 21, 36, 39] and the references therein).

In this work, we combine the pseudostress framework with the stream function approach to propose a new dual-mixed finite element method for the two-dimensional Navier–Stokes problem. The resulting scheme delivers an exactly divergence-free approximation of the velocity while retaining the main advantages of pseudostress formulations, such as momentum conservation and the capability to approximate several additional variables of interest. More precisely, starting from a reformulation of the pseudostress–velocity formulation introduced in [12], we proceed similarly to [14, 15, 34] and apply a Helmholtz–Weyl decomposition to the velocity field to derive a new three-field formulation in which the pseudostress, the stream function, and a Lagrange multiplier enforcing momentum conservation are the primary unknowns. This multiplier can be shown a posteriori to vanish identically. The analysis of both the continuous and discrete problems is carried out using the Banach–Nečas–Babuška theorem and Banach's fixed-point theorem, under the assumption of sufficiently small data. In addition, we derive the corresponding Céa estimate and prove the optimal convergence of the method, which is confirmed by numerical experiments.

We emphasize that, unlike previous works on stream function formulations, our approach avoids the resolution of a fourth-order system for the stream function. In addition, most stream function formulations available in the literature are restricted to the Stokes problem, and only a few extend the analysis to the Navier–Stokes equations. In contrast, the present approach allows us to treat the full Navier–Stokes problem within a mixed framework while preserving exact mass conservation and avoiding higher-order operators.

The remainder of this paper is organized as follows. In Section 2, we present the main aspects of the continuous problem. In particular, we introduce the pseudostress–stream function formulation and analyze its well-posedness by means of a fixed-point strategy combined with the classical Banach–Nečas–Babuška theorem and Banach's fixed-point theorem. Next, in Section 3, we introduce and analyze the associated Galerkin scheme, following closely the analysis developed for the continuous problem. We also establish the corresponding a priori error estimate and prove the optimal convergence of the method. Finally, in Section 4, we present several numerical experiments illustrating the good performance of the scheme.

2 The model problem and its mixed formulation

In this section we present the model problem and derive the variational formulation. We begin by introducing some notations and definitions.

2.1 Preliminaries

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with a polygonal connected boundary Γ, and let $\mathbf{n}$ be the outward unit normal vector on Γ. We will use standard notations for the Lebesgue spaces $\mathrm{L}^p(\Omega)$, with $p > 1$ and the Sobolev spaces $\mathrm{W}^{r,p}(\Omega)$ with $r \geq 0$, endowed with the norms and seminorms $\|\cdot\|_{0,p;\Omega}$, $\|\cdot\|_{r,p;\Omega}$, $|\cdot|_{r,p;\Omega}$, respectively. Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$ and if $p = 2$, we write $\mathrm{H}^r(\Omega)$ instead of $\mathrm{W}^{r,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms, denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{r,\Omega}$, respectively. We also write $|\cdot|_{r,\Omega}$ for the H^r-seminorm. On the other hand, $\mathrm{W}^{r,p}_0(\Omega)$ denotes the closure in $\mathrm{W}^{r,p}(\Omega)$ of all distribution with compact support in Ω which belongs to $\mathrm{W}^{r,p}(\Omega)$. In addition, $\mathrm{H}^{1/2}(\Gamma)$ denotes the trace space of $\mathrm{H}^1(\Omega)$, and $\mathrm{H}^{-1/2}(\Gamma)$ stands for its dual. The duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ is denoted by $\langle\cdot,\cdot\rangle_\Gamma$ and, for sufficiently smooth functions, it coincides with the $\mathrm{L}^2(\Gamma)$ inner product. Furthermore, we will denote by $\mathbf{S}$ and $\mathbb{S}$ the corresponding vectorial and tensorial counterparts of the generic scalar functional space S. Unless otherwise stated, given any pair $(\cdot,\cdot)$ in a product space $X \times Y$, we denote $\|(\cdot,\cdot)\| := \|\cdot\|_X + \|\cdot\|_Y$.

For convenience, for arbitrary scalar fields v and w, vector fields $\mathbf{v} = (v_i)_{i=1,2}$ and $\mathbf{w} = (w_i)_{i=1,2}$, and tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,2}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,2}$, we shall use the notation

$$(v,w)_\Omega := \int_\Omega vw, \quad (\mathbf{v},\mathbf{w})_\Omega := \int_\Omega \mathbf{v}\cdot\mathbf{w}, \quad \text{and} \quad (\boldsymbol{\tau},\boldsymbol{\zeta})_\Omega := \int_\Omega \boldsymbol{\tau} : \boldsymbol{\zeta},$$

and we also recall the Hölder inequality:

$$|(f,g)_\Omega| \leq \|f\|_{0,p;\Omega}\|g\|_{0,q;\Omega} \quad \forall\, f \in \mathrm{L}^p(\Omega),\ \forall\, g \in \mathrm{L}^q(\Omega), \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Now we recall from [38, Theorem 1.3.4] the following Sobolev embedding estimate:

$$\|w\|_{0,r,\Omega} \leq C(r)\|w\|_{1,\Omega} \quad \forall\, w \in \mathrm{H}^1(\Omega), \qquad \text{for all } r \geq 1, \tag{2.1}$$

where $C(r) > 0$ depends only on r and Ω. In addition, it is easy to see from [22, Lemma B.63] that the following Poincaré-type inequalities hold:

$$c_1(\Omega)\|w\|_{1,r;\Omega} \leq |w|_{1,r;\Omega}, \quad \forall\, w \in \mathrm{W}^{1,r}_0(\Omega), \qquad \text{for all } r \geq 1, \tag{2.2}$$

and

$$c_2(\Omega)\|w\|_{1,r;\Omega} \leq |w|_{1,r;\Omega}, \quad \forall\, w \in \widetilde{\mathrm{W}}^{1,r}(\Omega), \qquad \text{for all } r \geq 1, \tag{2.3}$$

where

$$\widetilde{\mathrm{W}}^{1,r}(\Omega) := \{\theta \in \mathrm{W}^{1,r}(\Omega) : (\theta, 1)_\Omega = 0\},$$

and $c_1(\Omega)$, $c_2(\Omega) > 0$ depend only on Ω.

For any two vector fields $\mathbf{v} = (v_i)_{i=1,2}$ and $\mathbf{w} = (w_i)_{i=1,2}$, we define the gradient, divergence, and tensor product operators, as follows

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,2}, \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^{2} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,2}.$$

For arbitrary tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,2}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,2}$, we also define $\mathbf{div}\,\boldsymbol{\tau}$ as the divergence operator div acting row-wise on $\boldsymbol{\tau}$. In addition, the transpose, trace, tensor inner product, and deviatoric part are given, respectively, by

$$\boldsymbol{\tau}^t := (\tau_{ji})_{j,i=1,2}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ denotes the identity tensor in $\mathbb{R}^{2\times 2}$. Moreover, for any scalar field θ, we define the operator curl by

$$\operatorname{curl} \theta := \left(\frac{\partial \theta}{\partial x_2}, -\frac{\partial \theta}{\partial x_1}\right)^t.$$

Next, given $p > 1$, we proceed as in [13] and introduce the Banach space

$$\mathbf{H}(\operatorname{div}_p; \Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ \operatorname{div} \mathbf{v} \in \mathrm{L}^p(\Omega)\},$$

endowed with the norm

$$\|\mathbf{v}\|_{\operatorname{div}_p;\Omega} := \left\{\|\mathbf{v}\|_{0,\Omega}^2 + \|\operatorname{div} \mathbf{v}\|_{0,p;\Omega}^2\right\}^{1/2}.$$

In the particular case $p = 2$, we omit the subscript and write $\mathbf{H}(\operatorname{div}; \Omega) = \mathbf{H}(\operatorname{div}_2; \Omega)$. We also set

$$\mathbf{H}(\operatorname{div}_0; \Omega) := \{\mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega) : \ \operatorname{div} \mathbf{v} = 0 \quad \text{in} \quad \Omega\}.$$

We further define the tensor-valued counterpart of $\mathbf{H}(\operatorname{div}_p; \Omega)$ by

$$\mathbb{H}(\mathbf{div}_p; \Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \ \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^p(\Omega)\},$$

and introduce

$$\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) := \{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega) : \ (\operatorname{tr}(\boldsymbol{\tau}), 1)_\Omega = 0\}.$$

Moreover, we recall from [12, Section 2.3] that the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3}; \Omega) = \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \oplus P_0(\Omega)\mathbb{I},$$

holds, where $P_0(\Omega)$ denotes the space of constant polynomials on Ω. In particular, on $\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ we recall from [12, Lemma 3.1] (see also [24, Lemma 2.3] for the Hilbert case) that

$$C(\Omega)\|\boldsymbol{\tau}\|_{0,\Omega}^2 \leq \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,4/3;\Omega}^2 \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega), \tag{2.4}$$

where $C(\Omega)$ depends only on Ω.

Finally, throughout the rest of the paper, we use $\mathbf{0}$ to denote a generic null vector (or tensor), and the symbols C and c, with or without subscripts, bars, tildes, or hats, to denote generic positive constants independent of the discretization parameters and of ν, not necessarily the same at each occurrence. The symbols $\lesssim$ and $\gtrsim$ are used to indicate inequalities that hold up to such constants.

2.2 Strong equations

We consider the stationary Navier–Stokes equations:

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \ & \mathbf{f} && \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} = \ & 0 && \text{in} \quad \Omega, \\ \mathbf{u} = \ & \mathbf{u}_D && \text{on} \quad \Gamma, \\ (p,1)_\Omega = \ & 0, && \end{aligned} \tag{2.5}$$

where the unknowns are the velocity field $\mathbf{u} := (u_1, u_2)^t$ and the pressure p of a fluid occupying the region Ω. The given data are $\nu > 0$ denoting the fluid viscosity, an external force $\mathbf{f}$ acting on Ω, and the boundary velocity $\mathbf{u}_D$ on Γ. Note that $\mathbf{u}_D$ must satisfy the compatibility condition

$$\langle \mathbf{u}_D\cdot\mathbf{n}, 1\rangle_\Gamma = 0, \tag{2.6}$$

which follows from the incompressibility constraint. Moreover, the last condition in (2.5) is imposed to ensure uniqueness of the pressure.

Now, in order to derive our mixed approach, we begin by introducing the following reformulation of the pseudostress tensor employed in [16]

$$\boldsymbol{\sigma} := \nabla\mathbf{u} - \nu^{-1}(\mathbf{u}\otimes\mathbf{u}) + \nu^{-1}c_{\mathbf{u}}\,\mathbb{I} - \nu^{-1}p\,\mathbb{I} \quad \text{in} \quad \Omega, \tag{2.7}$$

with $c_{\mathbf{u}} := \dfrac{1}{2|\Omega|}(\operatorname{tr}(\mathbf{u}\otimes\mathbf{u}),1)_\Omega \in \mathbb{R}$.

Then, noticing that owing to the incompressibility condition $\operatorname{tr}(\nabla\mathbf{u}) = \operatorname{div}\mathbf{u} = 0$ in Ω, the following identity holds:

$$p = -\frac{\nu}{2}\operatorname{tr}(\boldsymbol{\sigma}) - \frac{1}{2}\operatorname{tr}(\mathbf{u}\otimes\mathbf{u}) + c_{\mathbf{u}} \quad \text{in} \quad \Omega, \tag{2.8}$$

we can proceed analogously to [16] to rewrite equations (2.5) as the following first-order:

$$\begin{gathered} \boldsymbol{\sigma}^{\mathrm{d}} = \nabla\mathbf{u} - \nu^{-1}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega, \quad -\mathbf{div}\,\boldsymbol{\sigma} = \nu^{-1}\mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma, \quad (\operatorname{tr}(\boldsymbol{\sigma}),1)_\Omega = 0. \end{gathered} \tag{2.9}$$

Above, the unknowns are the velocity $\mathbf{u}$ and the tensor $\boldsymbol{\sigma}$, and the pressure can be easily recovered from (2.8). Notice that in view of (2.8), the condition ensuring uniqueness of the pressure is now imposed through the last equation of (2.9).

2.3 Variational formulation

We now proceed to derive the variational formulation associated with (2.9) that will allow us to obtain the desired momentum- and mass-conservative mixed scheme. We begin following [12, Section 2.3] to deduce from (2.9) the following variational problem: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ such that

$$\begin{aligned} (\boldsymbol{\sigma}^{\rm d}, \boldsymbol{\tau}^{\rm d})_\Omega + (\mathbf{div}\,\boldsymbol{\tau}, \mathbf{u})_\Omega + \frac{1}{\nu}(\mathbf{u}\otimes\mathbf{u}, \boldsymbol{\tau}^{\rm d})_\Omega &= \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma, \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \\ (\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{v})_\Omega &= -\frac{1}{\nu}(\mathbf{f}, \mathbf{v})_\Omega, \quad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega). \end{aligned} \tag{2.10}$$

We note that (2.10) does not coincide exactly with the variational formulation introduced in [12], since the tensor $\boldsymbol{\sigma}$ used here is a reformulation of the one considered in [12]. Nevertheless, the well-posedness of (2.10) follows straightforwardly from [12, Theorem 3.8]. More precisely, we have the following result, whose proof is omitted.

Theorem 2.1 *Let* $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$ *and* $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$*, such that*

$$C\left(\nu^{-1}\|\mathbf{u}_D\|_{1/2,\Gamma} + \nu^{-2}\|\mathbf{f}\|_{0,4/3;\Omega}\right) < 1,$$

where $C > 0$ *only depends on* Ω*. Then, there exists a unique* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ *solution to* (2.10)*. In addition, the following estimate holds:*

$$\|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3},\Omega} + \|\mathbf{u}\|_{0,4;\Omega} \lesssim \|\mathbf{u}_D\|_{1/2,\Gamma} + \nu^{-1}\|\mathbf{f}\|_{0,4/3;\Omega}.$$

Now, according to [35, Theorem 1.1 and Proposition 2.2], and recalling that $\Omega \subset \mathbb{R}^2$ is simply connected, which in particular implies that its first Betti number vanishes, there exists $\epsilon > 0$, depending only on Γ, such that, for every $p \in (4/3 - \epsilon,\, 4 + \epsilon)$, the following Helmholtz–Weyl decomposition holds and is stable:

$$\mathbf{L}^p(\Omega) = \operatorname{curl} \widetilde{\mathrm{W}}^{1,p}(\Omega) \oplus \nabla \mathrm{W}^{1,p}_0(\Omega). \tag{2.11}$$

More precisely, for each $\mathbf{z} \in \mathbf{L}^p(\Omega)$, there exist unique $\xi \in \widetilde{\mathrm{W}}^{1,p}(\Omega)$ and $\psi \in \mathrm{W}^{1,p}_0(\Omega)$ such that

$$\mathbf{z} = \operatorname{curl}\xi + \nabla\psi \quad \text{and} \quad |\xi|_{1,p;\Omega} + |\psi|_{1,p;\Omega} \lesssim \|\mathbf{z}\|_{0,p;\Omega}. \tag{2.12}$$

Then, if $\mathbf{u} \in \mathbf{L}^4(\Omega)$ is the velocity solution of (2.10), we employ (2.11) with $p = 4$, to deduce that there exists $\omega \in \widetilde{\mathrm{W}}^{1,4}(\Omega)$ and $\varphi \in \mathrm{W}^{1,4}_0(\Omega)$, such that

$$\mathbf{u} = \operatorname{curl}\omega + \nabla\varphi. \tag{2.13}$$

Based on this decomposition, we now propose the following equivalent variational formulation: Find $(\boldsymbol{\sigma}, (\omega, \varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}^{1,4}_0(\Omega))$, such that:

$$\begin{aligned} (\boldsymbol{\sigma}^{\rm d}, \boldsymbol{\tau}^{\rm d})_\Omega + (\mathbf{div}\,\boldsymbol{\tau}, \operatorname{curl}\omega + \nabla\varphi)_\Omega + \nu^{-1}(\operatorname{curl}\omega\otimes\operatorname{curl}\omega, \boldsymbol{\tau}^{\rm d})_\Omega &= \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma, \\ (\mathbf{div}\,\boldsymbol{\sigma}, \operatorname{curl}\theta + \nabla\psi)_\Omega &= -\nu^{-1}(\mathbf{f}, \operatorname{curl}\theta + \nabla\psi)_\Omega, \end{aligned} \tag{2.14}$$

for all $(\boldsymbol{\tau}, (\theta, \psi)) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}^{1,4}_0(\Omega))$.

Using (2.11), it can be proved that (2.14) is indeed equivalent to (2.10). To that end, we first introduce the following preliminary result.

Lemma 2.2 *Let $\epsilon > 0$ be the constant given in [35, Proposition 2.2], let $p \in (4/3-\epsilon,\, 4+\epsilon)$ be such that the decomposition (2.11) holds, and let $q \in \mathbb{R}$ be the conjugate exponent of p, that is, $1/p + 1/q = 1$. Then there exists a constant $\beta_0 > 0$ such that*

$$\sup_{0 \neq \psi \in \mathrm{W}_0^{1,p}(\Omega)} \frac{(\nabla\psi, \nabla\varphi)_\Omega}{|\psi|_{1,p;\Omega}} \geq \beta_0\, |\varphi|_{1,q;\Omega} \quad \forall \varphi \in \mathrm{W}_0^{1,q}(\Omega). \tag{2.15}$$

Proof. Given $\varphi \in \mathrm{W}_0^{1,q}(\Omega)$, we first observe that

$$(\left||\nabla\varphi|^{q-2}\nabla\varphi\right|^p, 1)_\Omega = (|\nabla\varphi|^q, 1)_\Omega < +\infty,$$

which implies that $|\nabla\varphi|^{q-2}\nabla\varphi \in \mathbf{L}^p(\Omega)$. Then, from (2.11), we deduce that there exist unique $\xi \in \widetilde{\mathrm{W}}^{1,p}(\Omega)$ and $\chi \in \mathrm{W}_0^{1,p}(\Omega)$, such that

$$|\nabla\varphi|^{q-2}\nabla\varphi = \operatorname{curl}\xi + \nabla\chi \tag{2.16}$$

and also, since the sum is direct and topological, the following estimate holds

$$|\xi|_{1,p;\Omega} + |\chi|_{1,p;\Omega} \lesssim \||\nabla\varphi|^{q-2}\nabla\varphi\|_{0,p;\Omega} = |\varphi|_{1,q;\Omega}^{q-1}. \tag{2.17}$$

Then, using (2.16), integration by parts and (2.17), we obtain

$$\begin{aligned}\sup_{0 \neq \psi \in \mathrm{W}_0^{1,p}(\Omega)} \frac{(\nabla\psi, \nabla\varphi)_\Omega}{|\psi|_{1,p;\Omega}} &\geq \frac{(\nabla\chi, \nabla\varphi)_\Omega}{|\chi|_{1,p;\Omega}} = \frac{((|\nabla\varphi|^{q-2}\nabla\varphi - \operatorname{curl}\xi), \nabla\varphi)_\Omega}{|\chi|_{1,p;\Omega}} \\ &\gtrsim \frac{(|\nabla\varphi|^{q-2}\nabla\varphi, \nabla\varphi)_\Omega - (\operatorname{curl}\xi, \nabla\varphi)_\Omega}{|\varphi|_{1,q;\Omega}^{q-1}} = \frac{|\varphi|_{1,q;\Omega}^q}{|\varphi|_{1,q;\Omega}^{q-1}},\end{aligned}$$

which completes the proof. □

We now prove the equivalence between problems (2.10) and (2.14).

Lemma 2.3 *If $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ is a solution to (2.10), then $\mathbf{u} = \operatorname{curl}\omega$ for some $\omega \in \widetilde{\mathrm{W}}^{1,4}(\Omega)$, and $(\boldsymbol{\sigma}, (\omega, 0))$ is a solution to (2.14). Conversely, if $(\boldsymbol{\sigma}, (\omega, \varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$ is a solution to (2.14), then $\varphi = 0$ in Ω and $(\boldsymbol{\sigma}, \mathbf{u}) = (\boldsymbol{\sigma}, \operatorname{curl}\omega)$ is a solution to (2.10).*

Proof. Let $(\boldsymbol{\sigma}, (\omega, \varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \left(\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega)\right)$ be a solution of (2.14), and let

$$\boldsymbol{\tau}_\phi := (\phi - |\Omega|^{-1}(\phi, 1)_\Omega)\,\mathbb{I} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \quad \text{with} \quad \phi \in \mathrm{W}_0^{1,4/3}(\Omega).$$

Observe that the compatibility condition (2.6), together with the fact that $\phi|_\Gamma = 0$, implies

$$\langle \boldsymbol{\tau}_\phi \mathbf{n}, \mathbf{u}_D \rangle_\Gamma = 0.$$

Therefore, taking $\boldsymbol{\tau} = \boldsymbol{\tau}_\phi$ in the first equation of (2.14), and using that $\boldsymbol{\tau}_\phi^{\mathrm{d}} = \mathbf{0}$ and $\mathbf{div}(\boldsymbol{\tau}_\phi) = \nabla\phi$, we obtain

$$(\nabla\phi, \nabla\varphi)_\Omega = 0. \tag{2.18}$$

Since $\phi \in \mathrm{W}_0^{1,4/3}(\Omega)$ is arbitrary, it follows from (2.18) and the inf–sup condition (2.15), with $p = 4/3$ and $q = 4$, that $|\varphi|_{1,4;\Omega} = 0$. Consequently, owing to the Poincaré inequality (2.2) with $r = 4$, we deduce that $\varphi = 0$ in Ω.

According to the above, setting $\mathbf{u} = \operatorname{curl}\omega \in \mathbf{L}^4(\Omega)$, we conclude that $(\boldsymbol{\sigma}, \mathbf{u})$ satisfies the first equation of (2.10). Moreover, by the decomposition (2.11), the second equation of (2.14) immediately yields the second equation of (2.10). Therefore, $(\boldsymbol{\sigma}, \mathbf{u}) = (\boldsymbol{\sigma}, \operatorname{curl}\omega)$ is a solution of (2.10).

Conversely, let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$ be the unique solution of (2.10), and let $\omega \in \widetilde{\mathrm{W}}^{1,4}(\Omega)$ and $\varphi \in \mathrm{W}_0^{1,4}(\Omega)$ be such that (2.13) holds. Then, the first equation of (2.10) can be equivalently rewritten as

$$(\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + (\mathbf{div}\,\boldsymbol{\tau}, \operatorname{curl}\omega + \nabla\varphi)_\Omega + \frac{1}{\nu}\big((\operatorname{curl}\omega + \nabla\varphi) \otimes (\operatorname{curl}\omega + \nabla\varphi), \boldsymbol{\tau}^{\mathrm{d}}\big)_\Omega = \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle_\Gamma, \quad (2.19)$$

for all $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$. But, proceeding exactly as above, that is, by taking $\boldsymbol{\tau}_\phi := (\phi - |\Omega|^{-1}(\phi, 1)_\Omega)\,\mathbb{I} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ in the first equation of (2.19), with arbitrary $\phi \in \mathrm{W}_0^{1,4/3}(\Omega)$, we deduce that $\varphi = 0$ in Ω, and consequently, $(\boldsymbol{\sigma}, (\omega, \varphi))$ satisfies the first equation of (2.14). Moreover, by applying again the decomposition (2.11), we infer that the second equation of (2.10) immediately yields the second equation of (2.14). Therefore, $(\boldsymbol{\sigma}, (\omega, \varphi))$ is a solution of (2.14), which completes the proof. □

Notice that, owing to the decomposition (2.11), the momentum conservation of the system is strongly imposed in $\mathbf{L}^{4/3}(\Omega)$ through the second equation of (2.14). Hence, the unknown φ acts as a Lagrange multiplier associated with momentum conservation and allows one to preserve the symmetry of the saddle-point structure. For this reason, this unknown is retained in the formulation, although it is shown to vanish identically. We also point out, in advance, that by approximating the solution of (2.14) with suitable finite element spaces, one obtains a direct approximation of the stream function ω, and by setting $\mathbf{u} = \operatorname{curl}\omega$, one immediately recovers an exactly divergence-free approximation of the velocity. Finally, we observe that Lemma 2.3, together with the well-posedness of problem (2.10), guarantees the existence of a solution to (2.14). It remains to establish uniqueness and the corresponding stability estimate, which will be addressed in the following section.

2.4 Well-posedness

In this section, similarly to [12], we apply a fixed-point strategy and combine the classical Banach–Nečas–Babuška theorem with the Banach fixed-point theorem to establish the well-posedness of (2.14) under a suitable smallness assumption on the data. To this end, we rewrite (2.14) as a perturbed saddle-point problem and introduce the associated fixed-point problem.

First, we let $\mathbf{a} : \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \to \mathbb{R}$, $\mathbf{b} : \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega)) \to \mathbb{R}$, $\mathbf{c} : \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \to \mathbb{R}$, $F : \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \to \mathbb{R}$ and $G : \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega) \to \mathbb{R}$, be the forms and functionals given by

$$\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := (\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega, \quad \mathbf{b}(\boldsymbol{\tau}, (\theta, \psi)) := (\mathbf{div}\,\boldsymbol{\tau}, \mathbf{curl}\,\theta + \nabla\psi)_\Omega, \quad (2.20)$$

$$\mathbf{c}(\widehat{\omega}; \omega, \boldsymbol{\tau}) := \nu^{-1}(\operatorname{curl}\widehat{\omega} \otimes \operatorname{curl}\omega, \boldsymbol{\tau}^{\mathrm{d}})_\Omega, \quad (2.21)$$

$$F(\boldsymbol{\tau}) := \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D \rangle_\Gamma \quad \text{and} \quad G(\theta, \psi) := -\nu^{-1}(\mathbf{f}, \operatorname{curl}\theta + \nabla\psi)_\Omega. \quad (2.22)$$

Then, it is clear that problem (2.14) can be rewritten as: Find $(\boldsymbol{\sigma},(\omega,\varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega)\times \mathrm{W}_0^{1,4}(\Omega))$, such that:

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},(\omega,\varphi)) \,+\, \mathbf{c}(\omega;\omega,\boldsymbol{\tau}) \;&= F(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \\ \mathbf{b}(\boldsymbol{\sigma},(\theta,\psi)) \;&= G(\theta,\psi) \quad \forall\,(\theta,\psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega)\times \mathrm{W}_0^{1,4}(\Omega). \end{aligned} \tag{2.23}$$

Now, define the operator:

$$\mathcal{J} : \widetilde{\mathrm{W}}^{1,4}(\Omega) \to \widetilde{\mathrm{W}}^{1,4}(\Omega), \quad \widehat{\omega} \to \mathcal{J}(\widehat{\omega}) = \omega, \tag{2.24}$$

where, given $\widehat{\omega} \in \widetilde{\mathrm{W}}^{1,4}(\Omega)$, ω is the second component of the solution to the linearized version of problem (2.23): Find $(\boldsymbol{\sigma},(\omega,\varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega)\times \mathrm{W}_0^{1,4}(\Omega))$, such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},(\omega,\varphi)) \,+\, \mathbf{c}(\widehat{\omega};\omega,\boldsymbol{\tau}) \;&= F(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \\ \mathbf{b}(\boldsymbol{\sigma},(\theta,\psi)) \;&= G(\theta,\psi) \quad \forall\,(\theta,\psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega)\times \mathrm{W}_0^{1,4}(\Omega). \end{aligned} \tag{2.25}$$

Hence, it is not difficult to see that $(\boldsymbol{\sigma},(\omega,\varphi)) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega)\times \mathrm{W}_0^{1,4}(\Omega))$ is a solution of (2.23) if and only if ω is a fixed point of $\mathcal{J}$, that is

$$\mathcal{J}(\omega) = \omega. \tag{2.26}$$

In what follows, we focus on proving that $\mathcal{J}$ admits a unique fixed point. Before proceeding with the solvability analysis, we first establish the well-definedness of the fixed point operator. We note in advance that, according to the definition of $\mathcal{J}$, it suffices to show that problem (2.25) is well-posed.

2.4.1 Well-definitiness of $\mathcal{J}$

We now apply the classical Banach–Nečas–Babuška theorem to prove that problem (2.25) is well-posed. To that end, we first establish the stability properties of the forms involved.

We begin by observing that straightforward calculations show that the forms $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are bounded. More precisely,

$$|\mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau})| \le \|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3};\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}, \qquad |\mathbf{b}(\boldsymbol{\tau},(\omega,\psi))| \le \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}\,\|(\omega,\psi)\|, \tag{2.27}$$

and

$$|\mathbf{c}(\widehat{\omega};\omega,\boldsymbol{\tau})| \le \frac{1}{\nu}\,\|\widehat{\omega}\|_{1,4;\Omega}\,\|\omega\|_{1,4;\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}. \tag{2.28}$$

In addition, it is straightforward to verify that G is bounded, and [12, Lemma 3.5] ensures that F is bounded as well. More precisely, the following estimates hold:

$$|G(\omega,\psi)| \le \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\,\|(\omega,\psi)\|, \qquad |F(\boldsymbol{\tau})| \le C_F\|\mathbf{u}_D\|_{1/2,\Gamma}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}, \tag{2.29}$$

where $C_F > 0$ is a constant depending on the constants in (2.1) and (2.4).

We now let $\mathbb{V}$ be the kernel of $\mathbf{b}$, that is

$$\mathbb{V} := \{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:\, \mathbf{b}(\boldsymbol{\tau},(\omega,\psi)) = 0, \quad \forall\,(\omega,\psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega)\times \mathrm{W}_0^{1,4}(\Omega)\},$$

using the decomposition (2.11) with $p = 4$, it is clear that the space $\mathbb{V}$ can be characterized as follows

$$\mathbb{V} = \{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:\, \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega\}.$$

Then, from (2.4), we easily see that $\mathbf{a}$ satisfies

$$\mathbf{a}(\boldsymbol{\tau},\boldsymbol{\tau}) \gtrsim \|\boldsymbol{\tau}\|^2_{\mathbf{div}_{4/3};\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{V}. \tag{2.30}$$

We now recall from [12, Lemma 3.3] that the following inf–sup condition holds:

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)} \frac{(\mathbf{div}\,\boldsymbol{\tau},\mathbf{z})_\Omega}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}} \gtrsim \|\mathbf{z}\|_{0,4;\Omega} \qquad \forall\, \mathbf{z} \in \mathbf{L}^4(\Omega). \tag{2.31}$$

Then, combining (2.3), (2.12), and (2.31), we readily deduce that the bilinear form $\mathbf{b}$ satisfies the inf–sup condition

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)} \frac{\mathbf{b}(\boldsymbol{\tau},(\theta,\psi))}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}} \gtrsim \|(\theta,\psi)\| \qquad \forall\,(\theta,\psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}^{1,4}_0(\Omega). \tag{2.32}$$

Finally, using the stability properties (2.27), (2.30) and (2.32), and applying [22, Proposition 2.36], it is not difficult to see that the bilinear form $\mathbf{A} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ defined by

$$\mathbf{A}((\boldsymbol{\sigma},\omega,\varphi),(\boldsymbol{\tau},\theta,\psi)) := \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},(\omega,\varphi)) + \mathbf{b}(\boldsymbol{\sigma},(\theta,\psi)), \tag{2.33}$$

with

$$\mathbb{X} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}^{1,4}_0(\Omega),$$

satisfies:

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},(\theta,\psi))\in\mathbb{X}} \frac{\mathbf{A}((\boldsymbol{\zeta},\vartheta,\phi),(\boldsymbol{\tau},\theta,\psi))}{\|(\boldsymbol{\tau},\theta,\psi)\|} \geq \gamma\|(\boldsymbol{\zeta},\vartheta,\phi)\| \quad \forall(\boldsymbol{\zeta},\vartheta,\phi) \in \mathbb{X}, \tag{2.34}$$

where γ is a positive constant independent of ν.

The following result establishes the well-definiteness of $\mathcal{J}$ under a suitable smallness assumption on the data.

Lemma 2.4 *Let* $\mathrm{K} \subseteq \widetilde{\mathrm{W}}^{1,4}(\Omega)$ *be the convex and bounded set given by*

$$\mathrm{K} := \left\{\widehat{\omega} \in \widetilde{\mathrm{W}}^{1,4}(\Omega) \,:\, \quad \|\widehat{\omega}\|_{1,4;\Omega} \leq \frac{2}{\gamma}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right)\right\}, \tag{2.35}$$

with γ *and* C_F *being the positive constant defined above in* (2.34) *and* (2.29)*, respectively, and assume that*

$$\frac{4}{\nu\gamma^2}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right) \leq 1. \tag{2.36}$$

Then, given $\widehat{\omega} \in \mathrm{K}$*, there exists a unique* $\omega \in \mathrm{K}$ *such that* $\mathcal{J}(\widehat{\omega}) = \omega$.

Proof. Given $\widehat{\omega} \in \mathrm{K}$, we begin by defining the bilinear form

$$\mathbf{A}^{\widehat{\omega}}((\boldsymbol{\sigma},\omega,\varphi),(\boldsymbol{\tau},\theta,\psi)) := \mathbf{A}((\boldsymbol{\sigma},\omega,\varphi),(\boldsymbol{\tau},\theta,\psi)) + \mathbf{c}(\widehat{\omega};\omega,\boldsymbol{\tau}), \tag{2.37}$$

where $\mathbf{A}$ and $\mathbf{c}$ are the forms defined in (2.33) and (2.21), respectively. Then, it is easy to see that problem (2.25) can be equivalently rewritten as: Find $(\boldsymbol{\sigma}, \omega, \varphi) \in \mathbb{X}$, such that

$$\mathbf{A}^{\widehat{\omega}}((\boldsymbol{\sigma}, \omega, \varphi), (\boldsymbol{\tau}, \theta, \psi)) = F(\boldsymbol{\tau}) + G(\theta, \psi) \quad \forall (\boldsymbol{\tau}, \theta, \psi) \in \mathbb{X}. \tag{2.38}$$

In this way, to prove the well-definiteness of $\mathcal{J}$, in what follows we equivalently prove that $\mathbf{A}^{\widehat{\omega}}$ satisfies the hypotheses of the Banach–Nečas–Babuška theorem [22, Theorem 2.6].

First, analogously to [12, Theorem 3.7], we combine the global inf–sup condition (2.34) with the boundedness of $\mathbf{c}$ (cf. (2.28)), to deduce that

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}, \theta, \psi) \in \mathbb{X}} \frac{\mathbf{A}^{\widehat{\omega}}((\boldsymbol{\zeta}, \vartheta, \phi), (\boldsymbol{\tau}, \theta, \psi))}{\|(\boldsymbol{\tau}, \theta, \psi)\|} \geq \Big(\gamma - \frac{1}{\nu}\|\widehat{\omega}\|_{1,4;\Omega}\Big)\|(\boldsymbol{\zeta}, \vartheta, \phi)\| \quad \forall (\boldsymbol{\zeta}, \vartheta, \phi) \in \mathbb{X}. \tag{2.39}$$

Therefore, from the definition of K (cf. (2.35)), assumption (2.36) and estimate (2.39), we obtain

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}, \theta, \psi) \in \mathbb{X}} \frac{\mathbf{A}^{\widehat{\omega}}((\boldsymbol{\zeta}, \vartheta, \phi), (\boldsymbol{\tau}, \theta, \psi))}{\|(\boldsymbol{\tau}, \theta, \psi)\|} \geq \frac{\gamma}{2}\|(\boldsymbol{\zeta}, \vartheta, \phi)\| \quad \forall (\boldsymbol{\zeta}, \vartheta, \phi) \in \mathbb{X}. \tag{2.40}$$

Similarly, noticing that $\mathbf{A}(\cdot,\cdot)$ is symmetric, we employ (2.40) and (2.28), together with (2.36), to obtain

$$\sup_{(\boldsymbol{\tau}, \theta, \psi) \in \mathbb{X}} \mathbf{A}^{\hat{\omega}}((\boldsymbol{\tau}, \theta, \psi), (\boldsymbol{\zeta}, \vartheta, \phi)) \geq \frac{\gamma}{2}\|(\boldsymbol{\zeta}, \vartheta, \phi)\| > 0 \quad \forall (\boldsymbol{\zeta}, \vartheta, \phi) \in \mathbb{X}\backslash\{\mathbf{0}\}. \tag{2.41}$$

In this way, from (2.40) and (2.41) we obtain that $\mathbf{A}^{\widehat{\omega}}(\cdot,\cdot)$ satisfies the hypotheses of the Banach–Nečas–Babuška theorem (cf. [22, Theorem 2.6]), which allows us to conclude the existence of a unique $(\boldsymbol{\sigma}, \omega, \varphi) \in \mathbb{X}$ solution to (2.25), or equivalently, the existence of a unique $\omega \in \widetilde{\mathrm{W}}^{1,4}(\Omega)$ such that $\mathcal{J}(\widehat{\omega}) = \omega$.

Finally, applying (2.40) with $(\boldsymbol{\zeta}, \vartheta, \phi) = (\boldsymbol{\sigma}, \omega, \varphi)$ and (2.38) we readily obtain that

$$\|\omega\|_{1,4;\Omega} \leq \|(\boldsymbol{\sigma}, \omega, \varphi)\| \leq \frac{2}{\gamma}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right), \tag{2.42}$$

which implies that ω belongs to K and concludes the proof. □

2.4.2 Existence and uniqueness of solution

We now establish the existence and uniqueness of a solution to problem (2.23). To this end, as anticipated above, we apply the Banach fixed-point theorem to show that the operator $\mathcal{J}$ admits a unique fixed point in K. This is stated in the following result.

Theorem 2.5 *Let $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$ and $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ be such that*

$$\frac{4}{\nu\gamma^2}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right) < 1. \tag{2.43}$$

Then, there exists a unique $(\boldsymbol{\sigma}, \omega, \varphi) \in \mathbb{X} = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega)$ solution to (2.23)*. Moreover, the solution satisfies*

$$\|(\boldsymbol{\sigma}, \omega, \varphi)\| \leq \frac{2}{\gamma}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right). \tag{2.44}$$

Proof. In what follows, we prove that $\mathcal{J}$ is a contraction mapping. To this end, let $\widehat{\omega}_1, \widehat{\omega}_2 \in \mathrm{K}$, and, in view of Lemma 2.4 and assumption (2.43), let $\omega_1, \omega_2 \in \mathrm{K}$ be such that

$$\mathcal{J}(\widehat{\omega}_1) = \omega_1 \quad \text{and} \quad \mathcal{J}(\widehat{\omega}_2) = \omega_2.$$

Then, from the definition of $\mathcal{J}$ (cf. (2.24)) and (2.25), it follows that there exist unique $(\boldsymbol{\sigma}_1, \varphi_1)$, $(\boldsymbol{\sigma}_2, \varphi_2) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathrm{W}_0^{1,4}(\Omega)$, such that for all $(\boldsymbol{\tau}, \theta, \psi) \in \mathbb{X}$, there hold

$$\mathbf{A}^{\widehat{\omega}_1}((\boldsymbol{\sigma}_1, \omega_1, \varphi_1), (\boldsymbol{\tau}, \theta, \psi)) = F(\boldsymbol{\tau}) + G(\theta, \psi)$$

and

$$\mathbf{A}^{\widehat{\omega}_2}((\boldsymbol{\sigma}_2, \omega_2, \varphi_2), (\boldsymbol{\tau}, \theta, \psi)) = F(\boldsymbol{\tau}) + G(\theta, \psi)$$

Subtracting the two equations, recalling the definition of $\mathbf{A}^{\widehat{\omega}}$ (cf. (2.37)), and adding and subtracting the term $\mathbf{c}(\widehat{\omega}_1; \omega_2, \boldsymbol{\tau})$, we easily arrive at

$$\mathbf{A}^{\widehat{\omega}_1}((\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \omega_1 - \omega_2, \varphi_1 - \varphi_2), (\boldsymbol{\tau}, \theta, \psi)) = -\,\mathbf{c}(\widehat{\omega}_1 - \widehat{\omega}_2; \omega_2, \boldsymbol{\tau}),$$

for all $(\boldsymbol{\tau}, \theta, \psi) \in \mathbb{X}$. Therefore, recalling that $\omega_1 \in \mathrm{K}$, from the above identity, together with (2.40) and the boundedness of $\mathbf{c}$ (cf. (2.28)), we obtain

$$\begin{aligned}\frac{\gamma}{2}\|\omega_1 - \omega_2\|_{1,4;\Omega} &\leq \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\theta,\psi)\in\mathbb{X}} \frac{\mathbf{A}^{\widehat{\omega}_1}((\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \omega_1 - \omega_2, \varphi_1 - \varphi_2), (\boldsymbol{\tau}, \theta, \psi))}{\|(\boldsymbol{\tau}, \theta, \psi)\|} \\ &= \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\theta,\psi)\in\mathbb{X}} \frac{-\,\mathbf{c}(\widehat{\omega}_1 - \widehat{\omega}_2; \omega_2, \boldsymbol{\tau})}{\|(\boldsymbol{\tau}, \theta, \psi)\|} \leq \frac{1}{\nu}\|\widehat{\omega}_1 - \widehat{\omega}_2\|_{1,4;\Omega}\|\omega_2\|_{1,4;\Omega},\end{aligned}$$

which, together with the fact that $\omega_2 \in \mathrm{K}$ (cf. (2.42)), implies that

$$\|\mathcal{J}(\widehat{\omega}_1) - \mathcal{J}(\widehat{\omega}_2)\|_{1,4;\Omega} = \|\omega_1 - \omega_2\|_{1,4;\Omega} \leq \frac{4}{\nu\gamma^2}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right)\|\widehat{\omega}_1 - \widehat{\omega}_2\|_{1,4;\Omega}.$$

The latter, together with assumption (2.43) and the Banach fixed-point theorem clearly implies the unique solvability of the fixed-point problem (2.26), which equivalently implies the well-posedness of problem (2.23).

Finally, analogously to the proof of Lemma 2.4, estimate (2.44) is obtained from (2.40) with $(\boldsymbol{\zeta}, \vartheta, \phi) = (\boldsymbol{\sigma}, \omega, \varphi)$ and $\widehat{\omega} = \omega$, which concludes the proof. □

3 Finite element discretization

In this section, we introduce and analyze a Galerkin discretization of the mixed formulation (2.23) that preserves the momentum conservation property of the system and provides an exactly divergence-free approximation of the velocity.

As we shall see next, the solvability of the discrete scheme can be established by employing arguments analogous to those used for the continuous problem. We conclude the section by deriving the corresponding a priori error estimate and the associated rates of convergence for the Galerkin approximation.

3.1 Discrete scheme

Let $\mathcal{T}_h$ be a regular family of triangulations of the polygonal domain $\overline{\Omega}$ into triangles $T \subset \mathbb{R}^2$ with diameter h_T, such that

$$\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T,$$

and define

$$h := \max\{h_T : \; T \in \mathcal{T}_h\}.$$

Given an integer $l \geq 0$ and a subset $S \subset \mathbb{R}^2$, we denote by $P_l(S)$ the space of polynomials on S of total degree at most l. Then, for each $T \in \mathcal{T}_h$, we define the local lowest-order Raviart–Thomas space (see, for instance, [7]) by

$$\mathbf{RT}_0(T) := [P_0(T)]^2 \oplus P_0(T)\mathbf{x},$$

where $\mathbf{x} := (x_1, x_2)^t$ denotes a generic vector in $\mathbb{R}^2$.

Next, let $\mathcal{E}_h$ denote the set of edges of $\mathcal{T}_h$, and let h_e be the diameter of each edge $e \in \mathcal{E}_h$. We define

$$\mathcal{E}_h(\Omega) := \big\{ e \in \mathcal{E}_h \; : \; e \subseteq \Omega \big\} \quad \text{and} \quad \mathcal{E}_h(\Gamma) := \big\{ e \in \mathcal{E}_h \; : \; e \subseteq \Gamma \big\}.$$

Moreover, for any piecewise functions v, we denote by $[\![\cdot]\!]$ the usual jump operator across interior edges, namely

$$[\![v]\!] := (v|_{T_+})|_e - (v|_{T_-})|_e \quad \text{with} \quad e = \partial T_+ \cap \partial T_-,$$

where T_+ and T_- are the two elements of $\mathcal{T}_h$ sharing the edge e.

With this notation, we introduce the classical Crouzeix–Raviart space (see, for instance, [19]) defined by

$$\begin{aligned} \mathrm{H}_h^{\varphi} := \quad & \Big\{ \psi_h \in \mathrm{L}^2(\Omega) : \; \psi_h|_T \in P_1(T) \; \forall\, T \in \mathcal{T}_h, \quad ([\![\psi_h]\!], 1)_e = 0 \; \forall\, e \in \mathcal{E}_h(\Omega), \\ & \quad \text{and} \quad (\psi_h, 1)_e = 0 \; \forall\, e \in \mathcal{E}_h(\Gamma) \Big\}. \end{aligned}$$

Then, defining the spaces

$$\begin{aligned} \mathbb{H}_h^{\boldsymbol{\sigma}} := \quad & \big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) \; : \; \mathbf{c}^t \boldsymbol{\tau}_h|_T \in \mathbf{RT}_0(T), \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \quad \forall\, T \in \mathcal{T}_h \big\}, \\ \mathrm{H}_h^{\omega} := \quad & \big\{ \theta_h \in \mathrm{C}(\overline{\Omega}) \; : \; \theta_h|_T \in \mathrm{P}_1(T), \quad \forall\, T \in \mathcal{T}_h \big\}, \end{aligned}$$

and their corresponding subspaces

$$\mathbb{H}_{h,0}^{\boldsymbol{\sigma}} := \mathbb{H}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad \mathrm{H}_{h,0}^{\omega} := \mathrm{H}_h^{\omega} \cap \widetilde{\mathrm{W}}^{1,4}(\Omega),$$

we propose the following Galerkin scheme to approximate the solution of problem (2.23): Find $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}} \times (\mathrm{H}_{h,0}^{\omega} \times \mathrm{H}_h^{\varphi})$, such that:

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}_h(\boldsymbol{\tau}_h, (\omega_h, \varphi_h)) + \mathbf{c}(\omega_h; \omega_h, \boldsymbol{\tau}_h) \quad &= F(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}}, \\ \mathbf{b}_h(\boldsymbol{\sigma}_h, (\theta_h, \psi_h)) \quad &= G_h(\theta_h, \psi_h) \quad \forall\, (\theta_h, \psi_h) \in \mathrm{H}_{h,0}^{\omega} \times \mathrm{H}_h^{\varphi}, \end{aligned} \tag{3.1}$$

where the forms $\mathbf{a}$ and $\mathbf{c}$, as well as the functional F, are defined in (2.20), (2.21) and (2.22), respectively, whereas the form $\mathbf{b}_h : \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times [\mathrm{W}_0^{1,4}(\Omega) + \mathrm{H}_h^{\varphi}]) \to \mathbb{R}$ and the functional $G_h : \widetilde{\mathrm{W}}^{1,4}(\Omega) \times [\mathrm{W}_0^{1,4}(\Omega) + \mathrm{H}_h^{\varphi}] \to \mathbb{R}$ are defined as follows:

$$\mathbf{b}_h(\boldsymbol{\tau}, (\omega_h, \psi_h)) := (\mathbf{div}\,\boldsymbol{\tau}, \mathbf{curl}\,\omega_h + \nabla_h \psi_h)_\Omega,$$

$$G_h(\omega_h, \psi_h) := -\frac{1}{\nu}(\mathbf{f}, \mathrm{curl}\,\omega_h + \nabla_h \psi_h)_\Omega,$$

with ∇_h being the discrete gradient for discontinuous functions, that is, $\nabla_h \psi_h|_T = \nabla(\psi_h|_T), \forall\, T \in \mathcal{T}_h$.

It is important to emphasize that the spaces $\mathrm{H}_{h,0}^{\omega}$ and H_h^{φ}, used to approximate the unknowns ω and φ, respectively, are chosen since they satisfy the key discrete Helmholtz decomposition (see [3, Theorem 4.1]):

$$[P_0(\mathcal{T}_h)]^2 = \mathbf{curl}\,\mathrm{H}_{h,0}^{\omega} \oplus \nabla_h \mathrm{H}_h^{\varphi}, \tag{3.2}$$

where $P_0(\mathcal{T}_h)$ denotes the space of piecewise constant functions. In particular, similarly to the analysis of the continuous formulation, the decomposition (3.2) will play a crucial role in establishing the well-posedness of (3.1) and in ensuring momentum conservation.

We also note that, since H_h^{φ} is not a subspace of $\mathrm{W}_0^{1,4}(\Omega)$, we equip H_h^{φ} with the following broken norm (see, for instance, [19]):

$$\|\psi_h\|_h = \left(\sum_{T \in \mathcal{T}_h} |\psi_h|_{1,4;T}^4 \right)^{1/4}.$$

Using this norm for H_h^{φ}, it is not difficult to see that the decomposition (3.2) is stable in the following sense: There exists $C > 0$, independent of h and ν, such that

$$|\theta_h|_{1,4;\Omega} + \|\phi_h\|_h \leq C\|\mathbf{z}_h\|_{0,4;\Omega}, \tag{3.3}$$

for all $\mathbf{z}_h \in [P_0(\mathcal{T}_h)]^2$, $\theta_h \in \mathrm{H}_{h,0}^{\omega}$ and $\phi_h \in \mathrm{H}_h^{\varphi}$, such that $\mathbf{z}_h = \mathrm{curl}\,\theta_h + \nabla_h \phi_h$.

3.2 Analysis of the Galerkin scheme

Analogously to the continuous case, we define the discrete counterpart of $\mathcal{J}$ (cf. (2.24)) by

$$\mathcal{J}_h : \mathrm{H}_{h,0}^{\omega} \to \mathrm{H}_{h,0}^{\omega}, \quad \widehat{\omega}_h \to \mathcal{J}_h(\widehat{\omega}_h) = \omega_h, \tag{3.4}$$

where, for a given $\widehat{\omega}_h \in \mathrm{H}_{h,0}^{\omega}$, the function ω_h denotes the second component of the solution to the following problem: find $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}} \times (\mathrm{H}_{h,0}^{\omega} \times \mathrm{H}_h^{\varphi})$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}_h(\boldsymbol{\tau}_h, (\omega_h, \varphi_h)) + \mathbf{c}(\widehat{\omega}_h; \omega_h, \boldsymbol{\tau}_h) &= F(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}}, \\ \mathbf{b}_h(\boldsymbol{\sigma}_h, (\theta_h, \psi_h)) &= G_h(\theta_h, \psi_h) && \forall\, (\theta_h, \psi_h) \in \mathrm{H}_{h,0}^{\omega} \times \mathrm{H}_h^{\varphi}. \end{aligned} \tag{3.5}$$

Consequently, the well-posedness of the discrete problem (3.1) reduces to identifying suitable hypotheses under which the fixed-point problem of finding $\omega_h \in \mathrm{H}_{h,0}^{\omega}$, such that

$$\mathcal{J}_h(\omega_h) = \omega_h, \tag{3.6}$$

admits a unique solution.

We first study the well-definedness of $\mathcal{J}_h$, which reduces to proving the well-posedness of (3.5). This is established next.

3.2.1 Well-definedness of $\mathcal{J}_h$

Analogously to Section 2.4.1, in what follows we apply the Banach–Nečas–Babuška theorem to establish the well-posedness of (3.5). To this end, we first observe that the boundedness of the forms $\mathbf{a}$ and $\mathbf{c}$, as well as of the functional F, is inherited from the corresponding continuous estimates (2.27), (2.28), and (2.29). In addition, the following bounds for the discrete form $\mathbf{b}_h$ and the functional G_h follow readily from local applications of Hölder's inequality:

$$|\mathbf{b}_h(\boldsymbol{\tau}_h,(\omega_h,\psi_h))| \leq \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}\,\|(\omega_h,\psi_h)\| \quad \text{and} \quad |G_h(\omega_h,\psi_h)| \leq \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\,\|(\omega_h,\psi_h)\|, \tag{3.7}$$

where, in this case, $\|(\omega_h,\psi_h)\|$ is defined by $\|(\omega_h,\psi_h)\| := \|\omega_h\|_{1,4;\Omega} + \|\psi_h\|_h$.

Now let $\mathbb{V}_h$ denote the discrete kernel of the bilinear form $\mathbf{b}_h$, that is,

$$\mathbb{V}_h := \left\{\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} : \ \mathbf{b}_h(\boldsymbol{\tau}_h,(\omega_h,\psi_h)) = 0, \quad \forall\,(\omega_h,\psi_h) \in \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h}\right\}.$$

In view of the decomposition (3.2) and the fact that $\mathbf{div}(\mathbb{H}^{\boldsymbol{\sigma}}_{h,0}) \subseteq [P_0(\mathcal{T}_h)]^2$, it is not difficult to see that $\mathbb{V}_h$ admits the characterization

$$\mathbb{V}_h = \{\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} : \ \mathbf{div}\,\boldsymbol{\tau}_h = \mathbf{0} \quad \text{in} \quad \Omega\}. \tag{3.8}$$

As a consequence, and arguing analogously to the continuous case (cf. (2.30)), we readily obtain from (3.8) that

$$\mathbf{a}(\boldsymbol{\tau}_h,\boldsymbol{\tau}_h) \gtrsim \|\boldsymbol{\tau}_h\|^2_{\mathbf{div}_{4/3};\Omega} \quad \forall\,\boldsymbol{\tau}_h \in \mathbb{V}_h. \tag{3.9}$$

Let us now recall from [12, Lemma 4.3] that the following estimate holds

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_{h,0}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_h,\mathbf{z}_h)_\Omega}{\|\boldsymbol{\tau}_h\|_{4/3,\mathbf{div},\Omega}} \gtrsim \|\mathbf{z}_h\|_{0,4;\Omega} \quad \forall\,\mathbf{z}_h \in [P_0(\mathcal{T}_h)]^2.$$

Using this estimate, along with (3.3) and the Poincaré inequality (2.3), it follows that

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_{h,0}} \frac{\mathbf{b}_h(\boldsymbol{\tau}_h,(\omega_h,\psi_h))}{\|\boldsymbol{\tau}_h\|_{4/3,\mathbf{div},\Omega}} \gtrsim \|(\omega_h,\psi_h)\| \qquad \forall\,(\omega_h,\psi_h) \in \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h}. \tag{3.10}$$

In this way, defining the bilinear form $\mathbf{A}_h : \mathbb{X}_h \times \mathbb{X}_h \to \mathbb{R}$ given by

$$\mathbf{A}_h((\boldsymbol{\sigma}_h,\omega_h,\varphi_h),(\boldsymbol{\tau}_h,\theta_h,\psi_h)) := \mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + \mathbf{b}_h(\boldsymbol{\tau}_h,(\omega_h,\varphi_h)) + \mathbf{b}_h(\boldsymbol{\sigma}_h,(\theta_h,\psi_h)), \tag{3.11}$$

with

$$\mathbb{X}_h := \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h},$$

from the stability properties (2.27), (3.7), (3.9) and (3.10), and applying [22, Proposition 2.36] it is not difficult to see that the bilinear form $\mathbf{A}_h$ satisfies:

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\theta_h,\psi_h)\in\mathbb{X}_h} \frac{\mathbf{A}_h((\boldsymbol{\zeta}_h,\vartheta_h,\phi_h),(\boldsymbol{\tau}_h,\theta_h,\psi_h))}{\|(\boldsymbol{\tau}_h,\theta_h,\psi_h)\|} \geq \widetilde{\gamma}\|(\boldsymbol{\zeta}_h,\vartheta_h,\phi_h)\|, \tag{3.12}$$

for all $(\boldsymbol{\zeta}_h,\vartheta_h,\phi_h) \in \mathbb{X}_h$, where $\widetilde{\gamma} > 0$ is independent of h and ν.

Now we are in position of establishing the well-definiteness of $\mathcal{J}_h$.

Lemma 3.1 *Let* $\mathrm{K}_h \subseteq \mathrm{H}^{\omega}_{h,0}$ *be the convex and bounded set given by*

$$\mathrm{K}_h := \left\{ \omega_h \in \mathrm{H}^{\omega}_{h,0} : \quad \|\omega_h\|_{1,4;\Omega} \le \frac{2}{\tilde{\gamma}} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right) \right\}, \tag{3.13}$$

with $\tilde{\gamma} > 0$ *being the constant introduced in the estimate* (3.12)*, and assume that*

$$\frac{4}{\nu \tilde{\gamma}^2} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right) \le 1. \tag{3.14}$$

Then, given $\widehat{\omega}_h \in \mathrm{K}_h$*, there exists a unique* $\omega_h \in \mathrm{K}_h$ *such that* $\mathcal{J}_h(\widehat{\omega}_h) = \omega_h$*.*

Proof. Analogously to the proof of Lemma 2.4, given $\widehat{\omega}_h \in \mathrm{K}_h$, we first introduce the bilinear form

$$\mathbf{A}_h^{\widehat{\omega}_h}\big((\boldsymbol{\sigma}_h, \omega_h, \varphi_h), (\boldsymbol{\tau}_h, \theta_h, \psi_h)\big) := \mathbf{A}_h\big((\boldsymbol{\sigma}_h, \omega_h, \varphi_h), (\boldsymbol{\tau}_h, \theta_h, \psi_h)\big) + \mathbf{c}(\widehat{\omega}_h; \omega_h, \boldsymbol{\tau}_h), \tag{3.15}$$

and then use (3.12) and (3.14) to deduce that $\mathbf{A}_h^{\widehat{\omega}_h}$ satisfies

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}_h, (\theta_h, \psi_h)) \in \mathbb{X}_h} \frac{\mathbf{A}_h^{\widehat{\omega}_h}\big((\boldsymbol{\zeta}_h, \vartheta_h, \phi_h), (\boldsymbol{\tau}_h, \theta_h, \psi_h)\big)}{\|(\boldsymbol{\tau}_h, \theta_h, \psi_h)\|} \ge \frac{\widetilde{\gamma}}{2} \|(\boldsymbol{\zeta}_h, \vartheta_h, \phi_h)\| \quad \forall (\boldsymbol{\zeta}_h, \vartheta_h, \phi_h) \in \mathbb{X}_h. \tag{3.16}$$

In this way, proceeding as in the continuous case, estimate (3.16) and the discrete Banach–Nečas–Babuška theorem [22, Theorem 2.22] yield the existence of a unique solution $(\boldsymbol{\sigma}_h, \omega_h, \varphi_h) \in \mathbb{X}_h := \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h$ to (3.5), with $\omega_h \in \mathrm{H}^{\omega}_{h,0}$ satisfying $\mathcal{J}_h(\widehat{\omega}_h) = \omega_h$.

Moreover, arguing analogously to (2.42), we obtain the stability estimate

$$\|\omega_h\|_{1,4;\Omega} \le \|(\boldsymbol{\sigma}_h, \omega_h, \varphi_h)\| \le \frac{2}{\tilde{\gamma}} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right), \tag{3.17}$$

which implies that $\omega_h \in \mathrm{K}_h$. $\square$

3.2.2 Existence and uniqueness of solution of the discrete problem

We no establish the main result of this section, namely, the existence and uniqueness of a solution to the fixed-point problem (3.6), or, equivalently, the well-posedness of (3.1). We note in advance that its proof follows arguments analogous to those used in the proof of Theorem 2.5, and therefore most details are omitted.

Theorem 3.2 *Let* $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$ *and* $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ *such that*

$$\frac{4}{\nu \widetilde{\gamma}^2} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right) < 1. \tag{3.18}$$

Then, the operator $\mathcal{J}_h$ *(cf.* (3.4)*) has a unique fixed-point* $\omega_h \in \mathrm{K}_h$*. Equivalently, the discrete problem* (3.1) *has a unique solution* $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times (\mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h)$*. Moreover, there hold*

$$\|(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h))\| \le \frac{2}{\widetilde{\gamma}} \left(C_F \|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu} \|\mathbf{f}\|_{0,4/3;\Omega} \right). \tag{3.19}$$

Proof. First we observe that, as for the continuous case, assumption (3.18) ensures the well-definiteness of the operator $\mathcal{J}_h$ (cf. Lemma 3.1). In turn, proceeding exactly as in the proof of Theorem 2.5 it is possible to obtain the following estimate

$$\|\mathcal{J}_h(\widehat{\omega}_{h,1}) - \mathcal{J}_h(\widehat{\omega}_{h,2})\|_{1,4;\Omega} \leq \frac{4}{\nu\widetilde{\gamma}^2}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right)\|\widehat{\omega}_{h,1} - \widehat{\omega}_{h,2}\|_{1,4;\Omega},$$

for all $\widehat{\omega}_{h,1}$, $\widehat{\omega}_{h,2} \in \mathrm{K}_h$. In this way, using estimate (3.18) we deduce that $\mathcal{J}_h$ is a contraction mapping on K_h, thus problem (3.6), or equivalently (3.1), is well-posed. Finally, estimate (3.19) is obtained analogously to (2.42). We omit further details. □

Remark 3.3 *Observe that, from the second equation in* (3.1), *we have*

$$\big(\mathbf{div}\,\boldsymbol{\sigma}_h + \nu^{-1}\mathbf{f}, \operatorname{curl}\theta_h + \nabla_h\psi_h\big)_\Omega = 0 \quad \forall\,(\theta_h,\psi_h) \in \mathrm{H}^\omega_{h,0} \times \mathrm{H}^\varphi_h.$$

Thanks to the discrete Helmholtz decomposition (3.2), *this implies*

$$\big(\mathbf{div}\,\boldsymbol{\sigma}_h + \nu^{-1}\mathbf{f}, \mathbf{z}_h\big)_\Omega = 0 \quad \forall\,\mathbf{z}_h \in [P_0(\mathcal{T}_h)]^2,$$

and therefore

$$\mathbf{div}\,\boldsymbol{\sigma}_h = -\nu^{-1}\mathbf{P}_h(\mathbf{f}).$$

Consequently, the method satisfies the discrete equilibrium equation exactly, and hence it is momentum-conservative whenever $\mathbf{f} \in [P_0(\mathcal{T}_h)]^2$.

If $\mathbf{f} \notin [P_0(\mathcal{T}_h)]^2$, *then, under the additional regularity assumption* $\mathbf{f} \in \mathbf{W}^{1,4/3}(\Omega)$, *we obtain*

$$\|\nu^{-1}\mathbf{f} - \mathbf{div}\boldsymbol{\sigma}_h\|_{0,4/3;\Omega} = \nu^{-1}\|\mathbf{f} - \mathbf{P}_h(\mathbf{f})\|_{0,4/3;\Omega} \leq C\nu^{-1}h|\mathbf{f}|_{1,4/3;\Omega},$$

where $\mathbf{P}_h$ *denotes the component-wise (vector-valued) version of the* $\mathrm{L}^2(\Omega)$*-projection onto* $P_0(\mathcal{T}_h)$, *denoted by* $\mathcal{P}_h$, *characterized by*

$$(\mathcal{P}_h(v) - v, z_h)_\Omega = 0 \quad \forall\, z_h \in P_0(\mathcal{T}_h),$$

and satisfying the following approximation estimate (see [22, Proposition 1.135]): for each $0 \leq m \leq 1$ *and each* $w \in \mathrm{W}^{m,r}(\Omega)$, *with* $1 \leq r \leq \infty$, *it holds that*

$$\|w - \mathcal{P}_h(w)\|_{0,r;\Omega} \leq Ch^m|w|_{m,r;\Omega}.$$

Moreover, recalling that

$$\operatorname{curl}\mathrm{H}^\omega_{h,0} = \{\mathbf{v}_h \in \mathbf{H}(\mathrm{div};\Omega) : \mathbf{v}_h|_T \in \mathbf{RT}_0(T), \quad \forall\,T \in \mathcal{T}_h\} \cap \mathbf{H}(\mathrm{div}_0;\Omega),$$

by setting

$$\mathbf{u}_h = \operatorname{curl}\omega_h \quad in \quad \Omega,$$

we obtain an exactly divergence-free approximation of the velocity $\mathbf{u}$, *and hence the method is mass-conservative.*

Finally, it is also worth noting that, although the analysis in this section has been carried out using $\mathbf{RT}_0$ *elements to approximate the pseudostress, the* $\mathbf{BDM}_1$ *element is also a viable alternative. For further details on the properties of* $\mathbf{BDM}_1$ *element, see, e.g., [6].*

3.3 Convergence

We now study the convergence of the Galerkin scheme (3.1). More precisely, we first derive the a priori error estimate for (3.1) and then, under an additional regularity assumption on the exact solution and using the approximation properties of the discrete spaces introduced in (3.1), we establish the corresponding theoretical rate of convergence. We begin by providing some preliminary results:

3.3.1 Preliminary results

We begin by noting that, since the gradient operator ∇ and the discrete gradient operator ∇_h coincide in $\mathrm{W}_0^{1,4}(\Omega)$, it is clear that the following identity holds

$$\mathbf{b}_h(\boldsymbol{\tau},(\theta,\psi)) = \mathbf{b}(\boldsymbol{\tau},(\theta,\psi)), \quad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega),\ \forall\,(\theta,\psi) \in \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega). \tag{3.20}$$

Thus, if $(\boldsymbol{\sigma},\omega,0) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega)$ is the unique solution of (2.23), then, from (3.20) and the identity $\mathbf{div}\,\boldsymbol{\sigma} = -\nu^{-1}\mathbf{f}$ in Ω, the exact solution satisfies

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}_h) + \mathbf{b}_h(\boldsymbol{\tau}_h,(\omega,0)) + \mathbf{c}(\omega;\omega,\boldsymbol{\tau}_h) &= F(\boldsymbol{\tau}_h), \qquad \forall\,\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0},\\ \mathbf{b}_h(\boldsymbol{\sigma},(\theta_h,\psi_h)) &= G_h(\theta_h,\psi_h), \quad \forall\,(\theta_h,\psi_h) \in \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h. \end{aligned}$$

which can be rewritten in terms of the bilinear form $\mathbf{A}_h$ (cf. (3.11)), as

$$\mathbf{A}_h\big((\boldsymbol{\sigma},\omega,0),(\boldsymbol{\tau}_h,\theta_h,\psi_h)\big) + \mathbf{c}(\omega;\omega,\boldsymbol{\tau}_h) = F(\boldsymbol{\tau}_h) + G_h(\theta_h,\psi_h), \tag{3.21}$$

for all $(\boldsymbol{\tau}_h,\theta_h,\psi_h) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h$.

Similarly, if $(\boldsymbol{\sigma}_h,(\omega_h,\varphi_h)) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \big(\mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h\big)$ is the unique solution of (3.1), then the corresponding discrete equations can be expressed in terms of $\mathbf{A}_h$ as

$$\mathbf{A}_h\big((\boldsymbol{\sigma}_h,\omega_h,\varphi_h),(\boldsymbol{\tau}_h,\theta_h,\psi_h)\big) + \mathbf{c}(\omega_h;\omega_h,\boldsymbol{\tau}_h) = F(\boldsymbol{\tau}_h) + G_h(\theta_h,\psi_h), \tag{3.22}$$

for all $(\boldsymbol{\tau}_h,\theta_h,\psi_h) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h$.

In this way, setting

$$\mathbf{e}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \quad \text{and} \quad \mathbf{e}_\omega := \omega - \omega_h\,,$$

from (3.21) and (3.22), and adding and subtracting appropriate terms, we deduce the following Galerkin orthogonality property

$$\mathbf{A}_h((\mathbf{e}_{\boldsymbol{\sigma}},\mathbf{e}_\omega,-\varphi_h),(\boldsymbol{\tau}_h,\theta_h,\psi_h)) + \mathbf{c}(\mathbf{e}_\omega;\omega,\boldsymbol{\tau}_h) + \mathbf{c}(\omega_h;\mathbf{e}_\omega,\boldsymbol{\tau}_h) = 0, \tag{3.23}$$

for all $(\boldsymbol{\tau}_h,\theta_h,\psi_h) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h$.

3.3.2 A priori estimate and theoretical rate of convergence

The following result provides the a priori error estimate for the Galerkin scheme (3.1).

Theorem 3.4 *Assume that*

$$\frac{8}{\nu\min\{\gamma,\tilde{\gamma}\}^2}\left(C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \frac{1}{\nu}\|\mathbf{f}\|_{0,4/3;\Omega}\right) \le 1, \tag{3.24}$$

with γ *and* $\widetilde{\gamma}$ *the constants introduced in* (2.34) *and* (3.12)*, respectively. Let* $(\boldsymbol{\sigma},(\omega,0)) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}_0^{1,4}(\Omega))$ *and* $(\boldsymbol{\sigma}_h,(\omega_h,\varphi_h)) \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}} \times (\mathrm{H}_{h,0}^{\omega} \times \mathrm{H}_h^{\varphi})$ *be the unique solutions of the continuous and discrete problems* (2.23) *and* (3.1)*, respectively. Then, there holds*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} + \|\omega-\omega_h\|_{1,4;\Omega} + \|\varphi_h\|_h \lesssim \inf_{(\boldsymbol{\tau}_h,\theta_h)\in\mathbb{H}_{h,0}^{\boldsymbol{\sigma}}\times\mathrm{H}_{h,0}^{\omega}} \left\{ \|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} + \|\omega-\theta_h\|_{1,4;\Omega} \right\}. \tag{3.25}$$

Proof. We begin by observing that, under assumption (3.24), both the continuous and discrete problems, (2.23) and (3.1), respectively, are well posed.

Now, let $\boldsymbol{\zeta}_h$ and ϑ_h be arbitrary elements in $\mathbb{H}_{h,0}^{\boldsymbol{\sigma}}$ and $\mathrm{H}_{h,0}^{\omega}$, respectively, and decompose the errors $\mathbf{e}_{\boldsymbol{\sigma}}$ and $\mathbf{e}_{\omega}$ as follows:

$$\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\xi}_{\boldsymbol{\sigma}} + \boldsymbol{\chi}_{\boldsymbol{\sigma}} \quad \text{and} \quad \mathbf{e}_{\omega} = \xi_{\omega} + \chi_{\omega}, \tag{3.26}$$

with

$$\boldsymbol{\xi}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{\zeta}_h, \quad \boldsymbol{\chi}_{\boldsymbol{\sigma}} := \boldsymbol{\zeta}_h - \boldsymbol{\sigma}_h, \quad \xi_{\omega} := \omega - \vartheta_h \quad \text{and} \quad \chi_{\omega} := \vartheta_h - \omega_h.$$

Then, from (3.23), and adding and subtracting suitable terms, we deduce

$$\begin{aligned} \mathbf{A}_h((\boldsymbol{\chi}_{\boldsymbol{\sigma}},\chi_{\omega},-\varphi_h),(\boldsymbol{\tau}_h,\theta_h,\psi_h)) + \mathbf{c}(\omega_h;\chi_{\omega},\boldsymbol{\tau}_h) &= -\mathbf{A}_h((\boldsymbol{\xi}_{\boldsymbol{\sigma}},\xi_{\omega},0),(\boldsymbol{\tau}_h,\theta_h,\psi_h)) \\ &\quad -\mathbf{c}(\chi_{\omega};\omega,\boldsymbol{\tau}_h) - \mathbf{c}(\omega_h;\xi_{\omega},\boldsymbol{\tau}_h) - \mathbf{c}(\xi_{\omega};\omega,\boldsymbol{\tau}_h). \end{aligned} \tag{3.27}$$

Therefore, recalling that the discrete solution satisfies (3.19), which implies that $\omega_h \in \mathrm{K}_h$ (cf. (3.13)), employing the inf-sup condition (3.12) on the left-hand side of (3.27), the boundedness of the forms $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{b}_h$ (cf. (2.27), (2.28) and (3.7)) on the right-hand side of (3.27), and proceeding as for (3.16), we obtain

$$\begin{aligned} \|\chi_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{4/3};\Omega} + \|\chi_{\omega}\|_{1,4;\Omega} + \|\varphi_h\|_h \leq \frac{2}{\widetilde{\gamma}}\Big(&2\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{4/3};\Omega} + \nu^{-1}\|\chi_{\omega}\|_{1,4;\Omega}\|\omega\|_{1,4;\Omega} \\ &+(1+\nu^{-1}\|\omega_h\|_{1,4;\Omega} + \nu^{-1}\|\omega\|_{1,4;\Omega})\|\xi_{\omega}\|_{1,4;\Omega}\Big). \end{aligned}$$

which together with (3.24) and the fact that ω and ω_h satisfy (2.44) and (3.19), respectively, implies

$$\|\chi_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{4/3};\Omega} + \left(1 - \frac{2}{\widetilde{\gamma}\nu}\|\omega\|_{1,4;\Omega}\right)\|\chi_{\omega}\|_{1,4;\Omega} + \|\varphi_h\|_h \lesssim \Big(\|\xi_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{4/3};\Omega} + \|\xi_{\omega}\|_{1,4;\Omega}\Big).$$

In view of the previous bound, together with hypothesis (3.24) and the stability estimate (2.44), we easily obtain

$$\|\chi_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{4/3};\Omega} + \|\chi_{\omega}\|_{1,4;\Omega} + \|\varphi_h\|_h \lesssim \|\xi_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{4/3};\Omega} + \|\xi_{\omega}\|_{1,4;\Omega}.$$

In this way, from the latter, (3.26), the triangle inequality and the fact that $\boldsymbol{\zeta}_h \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}}$ and $(\boldsymbol{\zeta}_h,\vartheta_h) \in \mathbb{H}_{h,0}^{\boldsymbol{\sigma}} \times \mathrm{H}_h^{\varphi}$ are arbitrary, we readily obtain (3.25), thus concluding the proof. □

Now we recall from [12, Section 4.2] and [22, Section 1.5] that the following approximation properties hold:

$$\inf_{\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\zeta}-\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \lesssim h\{\|\boldsymbol{\zeta}\|_{1,\Omega}+\|\mathbf{div}\,\boldsymbol{\zeta}\|_{1,4/3;\Omega}\} \quad \forall \boldsymbol{\zeta}\in\mathbb{H}^1(\Omega), \text{ s.t. } \mathbf{div}\,\boldsymbol{\zeta}\in\mathbf{W}^{1,4/3}(\Omega), \tag{3.28}$$

and

$$\inf_{\theta_h\in \mathrm{H}_h^{\omega}} \|\xi-\theta_h\|_{1,4;\Omega} \lesssim h\|\xi\|_{2,4;\Omega} \quad \forall \xi\in \mathrm{W}^{2,4}(\Omega). \tag{3.29}$$

The following theorem establishes the theoretical rate of convergence associated with the Galerkin scheme (3.1).

Theorem 3.5 *Let* $(\boldsymbol{\sigma},(\omega,0))\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times(\widetilde{\mathrm{W}}^{1,4}(\Omega)\times \mathrm{W}_0^{1,4}(\Omega))$ *and* $(\boldsymbol{\sigma}_h,(\omega_h,\varphi_h))\in\mathbb{H}_{h,0}^{\boldsymbol{\sigma}}\times(\mathrm{H}_{h,0}^{\omega}\times\mathrm{H}_h^{\varphi})$ *be the unique solutions of* (2.23) *and* (3.1)*, respectively, with* $\mathbf{f}$ *and* $\mathbf{u}_D$ *satisfying* (3.24)*, and assume that* $\boldsymbol{\sigma}\in\mathbb{H}^1(\Omega)$*,* $\mathbf{div}\,\boldsymbol{\sigma}\in\mathbf{W}^{1,4/3}(\Omega)$ *and* $\omega\in\mathrm{W}^{2,4}(\Omega)$*. Then, the following convergence estimate holds:*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}+\|\omega-\omega_h\|_{1,4;\Omega}+\|\varphi_h\|_h \lesssim h\,\{|\boldsymbol{\sigma}|_{1,\Omega}+|\mathbf{div}\boldsymbol{\sigma}|_{1,4/3;\Omega}+\|\omega\|_{2,4;\Omega}\}.$$

Proof. The result is a straightforward application of Theorem 3.4 and the approximation properties (3.28) and (3.29). We omit further details. □

3.4 Computing other variables of interest

As highlighted in [11] and [12], the pseudostress tensor provides valuable information about other quantities of interest in fluid mechanics. In particular, once the pseudostress tensor and the velocity are available, one can recover the pressure p, the velocity gradient $\mathbf{G}:=\nabla\mathbf{u}$, the vorticity $\boldsymbol{\gamma}:=\frac{1}{2}(\nabla\mathbf{u}-\nabla\mathbf{u}^t)$, and the stress tensor $\widetilde{\boldsymbol{\sigma}}:=\nu(\nabla\mathbf{u}+\nabla\mathbf{u}^t)-p\mathbb{I}$, all of them in terms of the variables mentioned above. Indeed, recalling that the velocity can be recovered from the stream function as $\mathbf{u}=\operatorname{curl}\omega$ (see Lemma 2.3) and using the definition of the pseudostress in (2.7), these quantities can be expressed in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$ as follows:

$$\begin{aligned}
p &= -\frac{1}{2}\left(\nu\,\mathrm{tr}\,(\boldsymbol{\sigma})+\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u})-\frac{1}{|\Omega|}\big(\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}),1\big)_\Omega\right),\\
\mathbf{G} &= \boldsymbol{\sigma}^{\mathrm{d}}+\frac{1}{\nu}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}},\\
\boldsymbol{\gamma} &= \frac{1}{2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}^t),\\
\widetilde{\boldsymbol{\sigma}} &= \nu(\boldsymbol{\sigma}^{\mathrm{d}}+\boldsymbol{\sigma}^t)-\frac{1}{2|\Omega|}\big(\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}),1\big)_\Omega\,\mathbb{I}+2(\mathbf{u}\otimes\mathbf{u})-\frac{1}{2}\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u})\mathbb{I}.
\end{aligned} \tag{3.30}$$

Motivated by the above, and setting the divergence-free Raviart–Thomas velocity $\mathbf{u}_h=\operatorname{curl}\omega_h$ (see Remark 3.3), we propose the following approximations of the aforementioned quantities:

$$\begin{aligned}
p_h &= -\frac{1}{2}\left(\nu\,\mathrm{tr}\,(\boldsymbol{\sigma}_h)+\mathrm{tr}\,(\mathbf{u}_h\otimes\mathbf{u}_h)-\frac{1}{|\Omega|}\big(\mathrm{tr}\,(\mathbf{u}_h\otimes\mathbf{u}_h),1\big)_\Omega\right),\\
\mathbf{G}_h &= \boldsymbol{\sigma}_h^{\mathrm{d}}+\frac{1}{\nu}(\mathbf{u}_h\otimes\mathbf{u}_h)^{\mathrm{d}},\\
\boldsymbol{\gamma}_h &= \frac{1}{2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^t),\\
\widetilde{\boldsymbol{\sigma}}_h &= \nu\big(\boldsymbol{\sigma}_h^{\mathrm{d}}+\boldsymbol{\sigma}_h^t\big)-\frac{1}{2|\Omega|}\big(\mathrm{tr}\,(\mathbf{u}_h\otimes\mathbf{u}_h),1\big)_\Omega\,\mathbb{I}+2(\mathbf{u}_h\otimes\mathbf{u}_h)-\mathrm{tr}\,(\mathbf{u}_h\otimes\mathbf{u}_h)\mathbb{I}.
\end{aligned}$$

Here, $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times (\mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h)$ denotes the solution of the discrete problem (3.1).

In this way, the present approach extends the results of [11] and [12] by introducing a new discrete scheme that yields an exactly divergence-free velocity approximation with optimal convergence. At the same time, it retains one of the main advantages of the approaches in [11] and [12], namely, the ability to compute accurate approximations of the additional variables in (3.30).

The following result, whose proof is a direct consequence of Theorem 3.5 and suitable algebraic manipulations, establishes the corresponding approximation result for this post-processing procedure.

Corollary 3.6 *Let* $(\boldsymbol{\sigma}, (\omega, 0)) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times (\widetilde{\mathrm{W}}^{1,4}(\Omega) \times \mathrm{W}^{1,4}_0(\Omega))$ *and* $(\boldsymbol{\sigma}_h, (\omega_h, \varphi_h)) \in \mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times (\mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h)$ *be the unique solutions of* (2.23) *and* (3.1)*, respectively, with* $\mathbf{f}$ *and* $\mathbf{u}_D$ *satisfying* (3.24)*, let* $\mathbf{u} = \mathrm{curl}\,\omega$ *and* $\mathbf{u}_h = \mathrm{curl}\,\omega_h$ *and assume that* $\boldsymbol{\sigma} \in \mathbb{H}^1(\Omega)$, $\mathbf{div}\boldsymbol{\sigma} \in \mathbf{W}^{1,4/3}(\Omega)$ *and* $\omega \in \mathrm{W}^{2,4}(\Omega)$*. Then, there holds*

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \|p - p_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|\mathbf{G} - \mathbf{G}_h\|_{0,\Omega} + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega} \\ \lesssim h\big\{|\boldsymbol{\sigma}|_{1,\Omega} + |\mathbf{div}\,\boldsymbol{\sigma}|_{1,4/3;\Omega} + \|\omega\|_{2,4;\Omega}\big\}.\end{aligned}$$

4 Numerical Results

In this section, we present four numerical examples illustrating the performance of our finite element scheme and confirming the theoretical results. We begin by mentioning that the numerical results that follow are obtained by imposing the conditions $(\mathrm{tr}\,(\boldsymbol{\sigma}_h), 1)_\Omega = 0$ and $(\omega_h, 1)_\Omega = 0$ through a penalty strategy using a scalar Lagrange multiplier (adding only one row and one column to the system for each unknown). Our implementation is based on *Freefem++* code (see [29]), in conjunction with the direct linear solver UMFPACK (see [20]). We apply a Newton method with a fixed tolerance $tol = 1E-08$, and the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, i.e.,

$$\frac{|\mathrm{coeff}^{m+1} - \mathrm{coeff}^{m}|}{|\mathrm{coeff}^{m+1}|} \leq tol,$$

where $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^{dof}$, with dof standing for the total number of degrees of freedom defining $\mathbb{H}^{\boldsymbol{\sigma}}_h$, H^{ω}_h, and H^{φ}_h.

We now introduce some additional notations. The individual errors are denoted by $\mathtt{e}(\star)$, and let $\mathtt{r}(\star)$ be the experimental rate of convergence given by

$$\mathtt{r}(\star) := \frac{\log(\mathtt{e}(\star)/\mathtt{e}'(\star))}{\log(h/h')}$$

for $\star \in \{\boldsymbol{\sigma}, \omega, \varphi, \mathbf{u}, p, \boldsymbol{\gamma}, \mathbf{G}, \widetilde{\boldsymbol{\sigma}}\}$, where h and h' denote two consecutive mesh sizes with their respective errors $\mathtt{e}$ and $\mathtt{e}'$. The post-processed variables $\mathbf{u}_h$, p_h, $\boldsymbol{\gamma}_h$, $\mathbf{G}_h$, and $\tilde{\boldsymbol{\sigma}}_h$ are approximated through the post-processing formulas presented in Section 3.4.

Example 1: Convergence against smooth exact solutions in a convex domain

In our first example we illustrate the accuracy of our method considering a manufactured exact solution defined on $\Omega = (0,1) \times (0,1)$. We consider the viscosity of the fluid $\nu = 1$, and the terms on the right-hand side are adjusted so that the exact solution is given by the functions:

$$\mathbf{u}(x,y) := \begin{pmatrix} \pi\exp(x)\cos(\pi y) \\ -\exp(x)\sin(\pi y) \end{pmatrix}, \qquad p(x,y) := x^3 + y^3 - \frac{1}{2},$$

We present in Table 4.1 the convergence history for a sequence of quasi-uniform mesh refinements when the finite element spaces $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h}$, described in Section 3.1, are used. It can be observed there that the rates of convergence are the ones expected from Theorem 3.5 and Corollary 3.6.

Errors and rates of convergence for the mixed $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h}$ approximations

h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\omega)$	$\mathtt{r}(\omega)$	$\mathtt{e}(\varphi)$	$\mathtt{r}(\varphi)$	Iter
0.373	213	11.1e+01	–	1.37e-00	–	8.15e-01	–	4
0.196	825	4.93e-00	1.203	6.16e-01	1.175	3.72e-01	1.158	4
0.098	3149	2.51e-00	1.012	3.02e-01	1.067	1.82e-01	1.069	4
0.048	12337	1.26e-00	1.011	1.50e-01	1.022	9.01e-02	1.028	4
0.028	48883	6.27e-01	1.009	7.56e-02	0.997	4.62e-02	0.971	4
0.014	196755	3.07e-01	1.025	3.65e-02	1.045	2.25e-02	1.036	3
0.007	775189	1.53e-01	1.015	1.86e-02	0.982	1.11e-02	1.029	3

Postprocessed variables

$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{G})$	$\mathtt{r}(\mathbf{G})$	$\mathtt{e}(\gamma)$	$\mathtt{r}(\gamma)$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$
1.43e-00	–	1.92e-00	–	5.57e-00	–	3.04e-00	–	9.95e-00	–
6.54e-01	1.155	8.07e-01	1.281	2.46e-00	1.205	1.61e-00	0.940	4.01e-00	1.341
3.27e-01	1.037	3.48e-01	1.257	1.22e-00	1.051	8.13e-01	1.016	1.97e-00	1.060
1.60e-01	1.042	1.55e-01	1.179	6.28e-01	0.972	4.34e-01	0.920	9.81e-01	1.024
8.17e-02	0.979	7.74e-02	1.014	3.11e-01	1.022	2.12e-01	1.037	4.91e-01	1.005
3.97e-02	1.036	3.65e-02	1.081	1.51e-01	1.039	1.04e-01	1.032	2.37e-01	1.046
1.99e-02	1.011	1.75e-02	1.067	7.59e-02	0.998	5.33e-02	0.970	1.17e-01	1.036

Table 4.1: Example 1: Mesh sizes, degrees of freedom, errors, rates of convergence, and number of iterations for the mixed $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h}$ approximations of the Navier–Stokes equations.

Example 2: Kovasznay flow: convergence and dependence on the viscosity parameter

The second example focuses on the performance of the iterative method as a function of the viscosity ν, considering the analytical solution $(\mathbf{u}, p)$ obtained by Kovasznay in [33]. For the domain $\Omega := (0,1) \times (0,1)$ and a prescribed value of ν, this solution is given by

$$\mathbf{u}(x,y) = \begin{pmatrix} 1 - e^{\lambda x}\cos(2\pi y) \\ \\ \dfrac{\lambda}{2\pi} e^{\lambda x}\sin(2\pi y) \end{pmatrix}, \quad p(x,y) = -\frac{1}{2}e^{2\lambda x} + \overline{p},$$

where

$$\lambda := \frac{-8\pi^2}{\nu^{-1} + \sqrt{\nu^{-2} + 16\pi^2}},$$

and the constant $\overline{p}$ is such that $(p,1)_\Omega = 0$. Note that for the Kovasznay flow, the external force vanishes, i.e., $\mathbf{f} = 0$ in Ω. Consequently, the tensor $\boldsymbol{\sigma}$, defined in (2.7), satisfies a conservative form of the momentum equation $\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{0}$ in Ω.

In Table 4.2, we report the performance of Newton's method as a function of the viscosity parameter ν, with $\nu \in \{1, 10^{-1}, 10^{-2}, 10^{-3}\}$, for a sequence of mesh sizes. The computations are carried out using the finite element spaces introduced in Section 3. As expected, we observe that the number of Newton iterations increases as the viscosity parameter ν decreases. Blank entries indicate that the iterative method requires more than 100 iterations to reach the stopping criterion. The table also includes the l^∞-norm of $\mathbf{div}\boldsymbol{\sigma}_h$ (reported for $\nu = 1$), for which values close to zero are expected, reflecting the momentum conservation property of the proposed method.

In Figure 4.1, we present the approximations of the magnitude of the discrete velocity $|\mathbf{u}_h|$, the stream function ω_h, and the pressure p_h, for different values of the viscosity parameter. In view of the analytical structure of the Kovasznay solution, the velocity field exhibits the characteristic behavior of the Kovasznay flow, where the exponential term $e^{\lambda x}$ governs the spatial variation along the horizontal direction. For relatively large viscosities (e.g., $\nu = 1$), the decay in x is mild, and the solution displays a smooth transition from a region of higher gradients near the left boundary to an almost uniform state in the rest of the domain. As ν decreases, the exponential term $e^{\lambda x}$ becomes nearly constant in x, and the solution progressively loses its dependence on the horizontal direction. In this regime, the variation near the inflow boundary $x = 0$ becomes less pronounced, and away from this region the flow is essentially independent of x. This results in vertical band patterns, where the solution varies predominantly in the y-direction. The same behavior is clearly reflected in the stream function ω_h.

Regarding the pressure, for $\nu = 1$ and $\nu = 10^{-1}$, vertical bands can also be identified, showing variations primarily in the horizontal direction and concentrated near the left boundary, while the pressure remains almost constant in the rest of the domain. As ν decreases, these variations become less noticeable, and the pressure appears nearly uniform throughout the domain, with only very small residual variations. This behavior is explained by an exponential decay whose intensity decreases as the viscosity is reduced.

Newton iterations and behavior of $\|\mathbf{div}\boldsymbol{\sigma}_h\|_{l^\infty}$ for different mesh sizes

ν	$h = 0.1964$	$h = 0.0978$	$h = 0.0481$	$h = 0.0279$	$h = 0.0142$	$h = 0.0073$
1	4	4	4	4	3	3
10^{-1}	5	5	4	4	4	4
10^{-2}	6	5	5	5	5	5
10^{-3}	–	–	–	6	6	6
$\|\mathbf{div}\boldsymbol{\sigma}_h\|_{l^\infty}$	4.263e-14	8.527e-14	2.842e-13	6.821e-13	2.046e-12	4.547e-12

Table 4.2: Example 2: Number of Newton iterations with respect to the viscosity parameter ν and mesh size h, together with the l^∞-norm of $\mathbf{div}\boldsymbol{\sigma}_h$ for $\nu = 1$, using the mixed $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_h$ discretization of the Navier–Stokes equations.

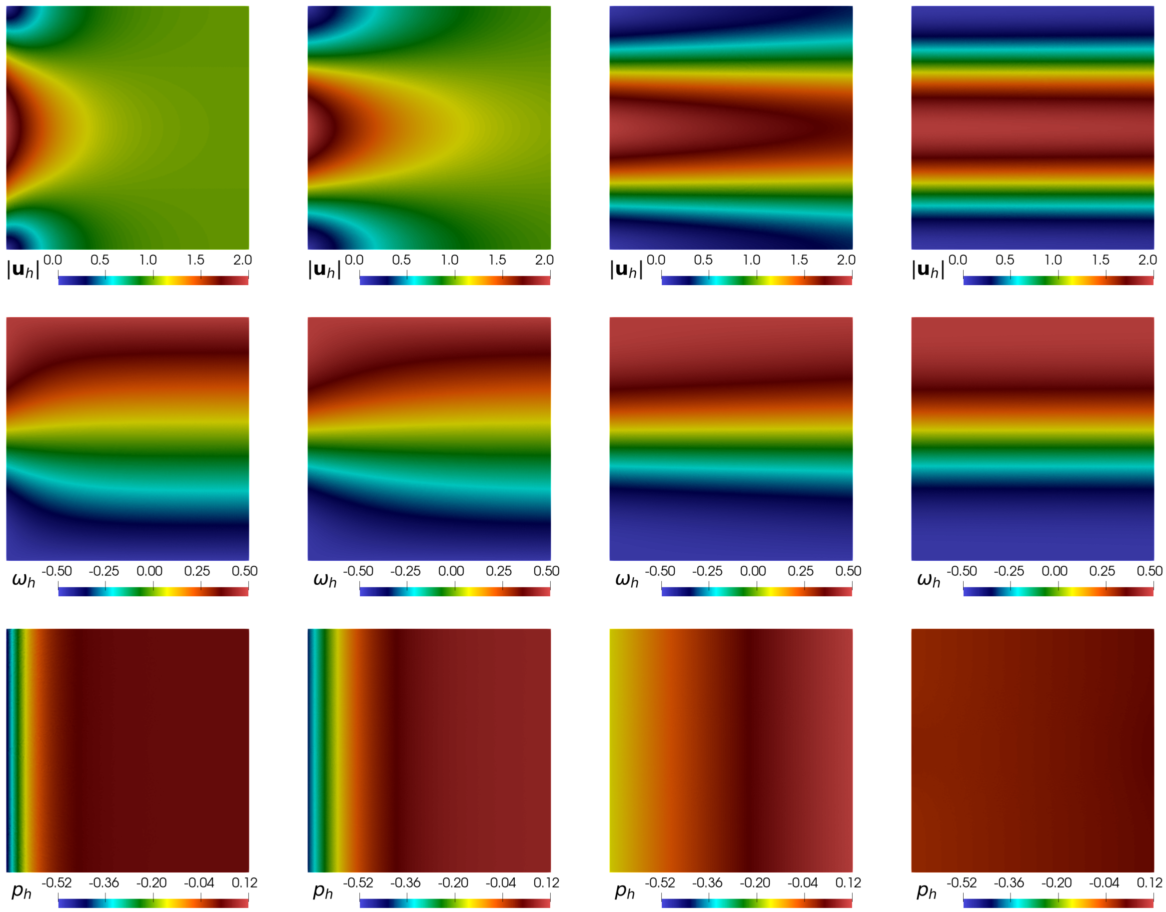

Figure 4.1: EXAMPLE 2: Magnitude of the discrete velocity $|\mathbf{u}_h|$ (top row), the stream function ω_h (middle row), and the pressure p_h (bottom row) for different values of the viscosity parameter $\nu = 1, 10^{-1}, 10^{-2}, 10^{-3}$.

Example 3: Driven cavity flow with regularization: streamlines and Reynolds number effects

In our third example, we study a regularized driven cavity problem. The domain is given by the unit square $\Omega = (0,1)^2$ and we consider a structured mesh with meshsize $h = 0.0143$. The data are given as in Reference [31, Example D.4], that is, a null body force $\mathbf{f} = \mathbf{0}$ and the prescribed velocity boundary

$$\mathbf{u}(x,1) = \begin{pmatrix} u_1(x) \\ 0 \end{pmatrix}$$

where

$$u_1(x) = \begin{cases} 1 - \dfrac{1}{4}\left(1 - \cos\left(\dfrac{x_1 - x}{x_1}\pi\right)\right)^2 & \text{for } x \in [0, x_1], \\ 1 & \text{for } x \in (x_1, 1 - x_1), \\ 1 - \dfrac{1}{4}\left(1 - \cos\left(\dfrac{x - (1 - x_1)}{x_1}\pi\right)\right)^2 & \text{for } x \in [1 - x_1, 1]. \end{cases}$$

In addition, all simulations in this example were carried out with $x_1 = 0.1$. In Figure 4.2, we display the velocity streamlines built using the $\mathbb{H}^{\boldsymbol{\sigma}}_{h,0} \times \mathrm{H}^{\omega}_{h,0} \times \mathrm{H}^{\varphi}_{h}$ scheme with 172,101 degrees of freedom (34,228 triangles), for Reynolds number $Re \in \{1, 10, 100, 1000\}$, with $Re = 1/\nu$. It can be seen that, as expected, and similarly to the results obtained in [31, Example D.4], that in the case of low Reynolds number $Re = 1$ there is a large vortex whose centre is close to the upper boundary. On the other hand, for increasing Reynolds numbers, the main vortex moves towards the centre of the cavity, while smaller counter-rotating vortices appear in both lower corners, with a stronger presence in the right one.

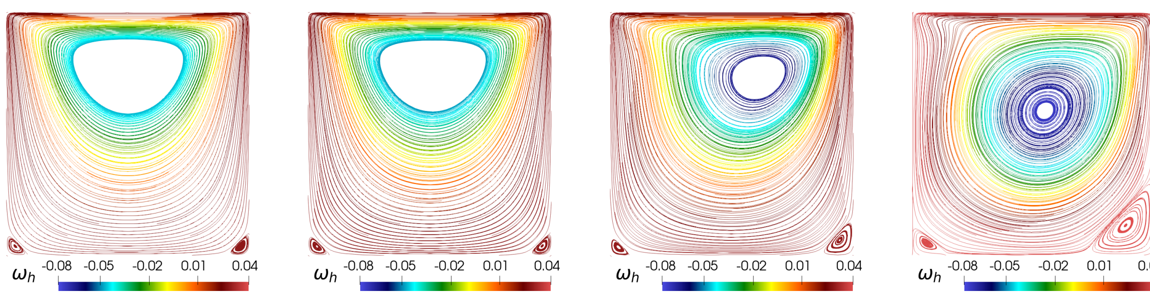

Figure 4.2: Example 3: Velocity streamlines for $\nu = 1, 10^{-1}, 10^{-2}, 10^{-3}$ with $h = 0.0143$.

Example 4: Backward-facing step flow

Finally, the fourth example examines mass loss in the standard backward-facing step flow test, similarly as in [5]. For this test, we consider a rectangular domain $\Omega = [0, 10] \times [0, 1]$ with a re-entrant corner at $(2, 0.5)$. The boundary Γ is partitioned into three segments: the inflow boundary Γ_{in}, the outflow boundary Γ_{out}, and the wall boundary Γ_{wall}, where $\Gamma_{wall} = \Gamma \setminus (\overline{\Gamma}_{in} \cup \overline{\Gamma}_{out})$ (see Figure 4.3). We consider $\nu = 1.0$ and $\mathbf{f}(x_1, x_2) = \mathbf{0}$ in Ω. As in [5], the

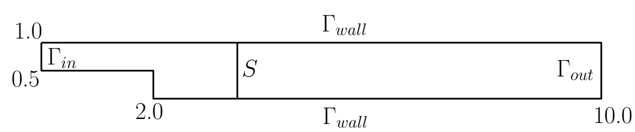

Figure 4.3: Example 4: Geometry for the backward-facing step flow test.

boundary conditions are prescribed as follows: a parabolic inflow profile $\mathbf{u}_D(x_1, x_2) = (8(x_2 -$

$0.5)(1 - x_2), 0)^t$ on Γ_{in}, a no-slip condition $\mathbf{u}_D(x_1, x_2) = \mathbf{0}$ on Γ_{wall}, and a parabolic outflow profile $\mathbf{u}_D(x_1, x_2) = x_2(1 - x_2)$ on Γ_{out}.

To evaluate mass conservation in the discrete solution, we measure the total mass flow across a sequence of vertical surfaces connecting the top and bottom boundaries of the computational domain. The line labeled "S" in Figure 4.3 illustrates a representative example of such a surface for the test problem.

Setting $\mathbf{u} = \text{curl } \omega$ in Ω, from the divergence theorem it follows that $\int_{\Gamma_{in}} \mathbf{u} \cdot \mathbf{n} = \int_S \mathbf{u} \cdot \mathbf{n}_S$, for any S connecting the top and bottom walls of the domain. Then, suggested by the above, in what follows, mass conservation in the discrete solution will be quantified by the percentage mass loss across the surface S, defined as

$$\%m_{loss} := 100 \left| \int_{\Gamma_{in}} \mathbf{u}_h \cdot \mathbf{n} - \int_S \mathbf{u}_h \cdot \mathbf{n}_S \right| / \left| \int_{\Gamma_{in}} \mathbf{u}_h \cdot \mathbf{n} \right|, \tag{4.1}$$

with $\mathbf{u}_h = \text{curl } \omega_h$ in Ω.

We compare the mass loss in the pseudostress-stream-function formulation against the standard discrete pseudostress-based scheme from [12]

We use a computational grid consisting of 189.350 triangles with a mesh size of $h = 0.0185$, leading to a total of $N = 951.151$ degrees of freedom for both schemes.

The results of our study are summarized in Figure 4.4, where we compare the mass losses obtained with the proposed pseudostress-stream-function formulation (solid line) with those obtained using the formulation from [12] (dashed line). The comparison is performed along 100 lines S uniformly distributed throughout Ω. The figure clearly demonstrates a significant improvement in mass conservation, as quantified by the percent mass loss formula (4.1). Specifically, with the pseudostress-stream-function formulation, the maximum mass loss remains below 0.1%, whereas with the scheme from [12] it exceeds 0.7%.

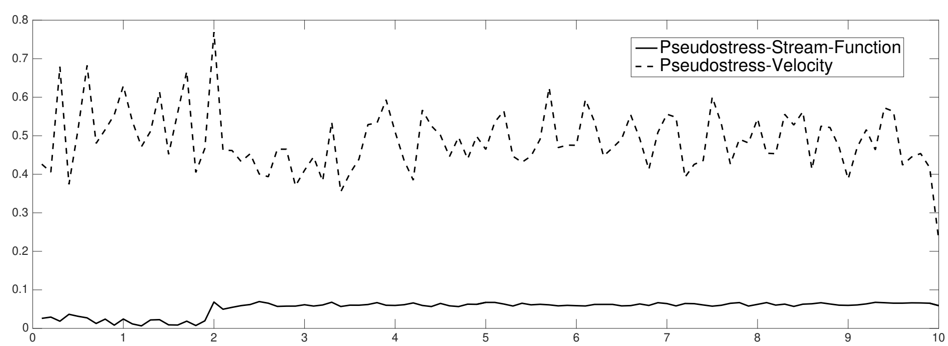

Figure 4.4: Example 3: Comparison of mass losses obtained with the formulation pseudostress-stream-function formulation (solid line) and those obtained using the formulation from [12] (dashed line).

5 Conclusions

In this work, we have proposed and analyzed a new pseudostress-stream-function dual-mixed finite element method for the stationary Navier–Stokes problem in two dimensions with nonhomogeneous boundary condition. The formulation is derived by combining a pseudostress-based approach with a Helmholtz–Weyl decomposition of the velocity field, leading to a three-field variational problem involving the pseudostress, the stream function, and a Lagrange multiplier associated with momentum conservation.

The resulting scheme provides an exactly divergence-free approximation of the velocity and preserves the momentum balance at the discrete level. In addition, it retains the main advantages of pseudostress-based formulations, including the possibility of recovering additional variables of interest directly from the computed solution.

We have established the well-posedness of both the continuous and the discrete formulations by means of the Banach–Nečas–Babuška theorem combined with a fixed-point argument, under a suitable small-data assumption. Furthermore, we have derived a priori error estimates and proved optimal convergence rates for the proposed Galerkin scheme.

The numerical results confirm the theoretical findings and illustrate the accuracy and robustness of the method. In particular, they show the effectiveness of the approach in delivering divergence-free velocity approximations together with accurate post-processed quantities.

Future research will address the extension of the present framework to the three-dimensional case and to more general configurations of Ω, including multiply connected domains. In particular, in the three-dimensional setting, such an extension is not straightforward. Indeed, the validity range of the Helmholtz–Weyl decomposition (2.11) is restricted to $p \in (3/2 - \epsilon, 3 + \epsilon)$ (see [23]), which differs from the two-dimensional case considered here. As a consequence, the functional setting underlying the analysis must be suitably modified, requiring an appropriate choice of Banach spaces for both the unknowns and the test functions.

6 Ethical Statement

The submitted work is original and is not published elsewhere in any form or language.

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