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A Banach spaces-based fully mixed finite element method for the thermo-electro-hydrodynamic Boussinesq problem*

Gabriel N. Gatica[†] Salim Meddahi[‡]
Kevin W. Pucha-Atán[§] Ricardo Ruiz-Baier[¶]

Abstract

In this work, we introduce and analyze a fully mixed finite element method for the stationary thermo–electro–hydrodynamic (TEHD) Boussinesq problem. The model consists of a nonlinear, strongly coupled system of equations for the velocity, pressure, temperature, and electric potential. We rewrite the governing equations as a first-order system by introducing additional variables. This reformulation enables a consistent and stable treatment of the coupling that arises from both convective effects and electric body forces. The resulting continuous formulation consists of three nonlinear saddle-point problems, which we decouple by introducing suitable linearizations of the corresponding variational equations. We analyze the resulting subproblems using the Babuška–Brezzi theory and the Lax–Milgram theorem in the Banach space setting. This analysis yields a unique admissible choice of Lebesgue exponents for the spaces of unknowns and test functions, and it also restricts the study to two dimensions. Under additional regularity for one of the subproblems and standard small-data conditions, we then apply Banach's fixed-point theorem to establish the unique solvability of the fully coupled problem. A discrete version of this strategy, combined with Brouwer's theorem, allows us to prove—under appropriate hypotheses on the discrete spaces—the existence of a solution to the associated Galerkin scheme. We derive the corresponding Céa estimate and introduce specific finite element subspaces that satisfy the required assumptions. Finally, we present numerical experiments to illustrate the performance of the method and to confirm the predicted convergence rates.

Key words: TEHD Boussinesq, abstract Babuška–Brezzi theory, fixed-point arguments, mixed finite element methods in Banach spaces, fully mixed formulation, a priori error analysis.

Mathematics Subject Classifications (2020): 65N30, 65N12, 65N15, 76D05, 76R10, 76W05.

1 Introduction

The interaction between fluid motion, temperature, and electric fields gives rise to coupled systems commonly described within the framework of thermo–electro–hydrodynamics. A standard modeling

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[†] CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: `ggatica@ci2ma.udec.cl`.

[‡] Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, Calvo Sotelo s/n, Oviedo, España, email: `salim@uniovi.es`.

[§] CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: `kpucha@udec.cl`.

[¶] School of Mathematics, Monash University, 9 Rainforest Walk, 3800, Victoria, Australia; and Universidad Adventista de Chile, Casilla 7-D Chillán, Chile, email: `ricardo.ruizbaier@monash.edu`.

approach consists of extending the classical Boussinesq equations to incorporate electric effects, leading to a nonlinear and strongly coupled system for the velocity, pressure, temperature, and electric potential. The coupling arises not only through the convective terms but also via the electric body forces, typically introduced through Gauss' law. These models are relevant in a wide range of applications. In engineering contexts, electrohydrodynamic mechanisms provide effective tools for controlling transport processes in thermal management, microfluidics, and energy-related systems, as well as in flows involving charged species and electrokinetic effects [18]. In addition, analogous models appear in geophysical settings, where the interaction between buoyancy-driven flows and electromagnetic forces plays a significant role. In particular, dielectrophoretic forces can be interpreted as an artificial gravity, enabling the experimental simulation of convection phenomena under controlled conditions [4, 17].

Despite these developments, the available literature on thermo–electro–hydrodynamic models is largely focused on numerical simulations, experimental studies, and linear stability analyses. Several works address their numerical approximation using finite volume methods [1, 19, 38], finite element methods [6, 24], and pseudo-spectral techniques [20]. Moreover, linear stability theory has been widely employed to characterize the onset of electro–thermo–convective instabilities (see, e.g., [21, 22, 23]). However, rigorous mathematical results concerning the numerical analysis of the fully coupled system remain scarce. Notable exceptions are the recent contributions [33, 34], where an $\mathbf{H}^1$-conformal formulation is analyzed. More recently, model reduction techniques based on proper orthogonal decomposition have been proposed to improve computational efficiency in [35].

The literature on the stationary Boussinesq problem has experienced significant development from the viewpoint of mixed variational formulations. In particular, the augmented mixed–primal formulation introduced in [13] for the case of constant viscosity was subsequently extended to a fully-mixed formulation in [15]. In this latter work, the introduction of the pseudo-stress tensor together with an additional auxiliary vector variable—combining the temperature, its gradient, and the velocity—yields improvements from both theoretical and computational perspectives. These ideas were further developed in [14], where augmented formulations with Galerkin-type schemes were proposed for both mixed–primal and fully-mixed settings. This framework has been extended to more strongly coupled variants. For instance, the case of temperature-dependent viscosity is studied in [2], where the introduction of the pseudo-stress and strain-rate tensors provides additional flexibility for the three-dimensional analysis, thereby overcoming previous limitations to two-dimensional settings (see also [11] for a related study of this more strongly coupled model). Another relevant extension is presented in [12], where the viscosity depends on both temperature and the concentration of a solute, leading to an Oberbeck–Boussinesq-type system. As in the previous cases, the introduction of suitable auxiliary variables allows for a consistent treatment of the coupling and the derivation of a fully-mixed formulation. Moreover, these approaches yield optimal a priori error estimates and have been complemented with reliable a posteriori error estimators, enabling the design of adaptive numerical strategies (see, e.g., [30]). The same methodology has been successfully applied to more complex multiphysics problems, such as double-diffusion systems and electrochemical flows, where additional variables associated with electric fields and ionic fluxes are incorporated into the formulation, as reported in [8, 9, 16].

The aforementioned developments highlight the flexibility of these formulations and their ability to accommodate increasingly complex couplings. This motivates the present work, where we extend the fully-mixed Banach space framework to a thermo–electro–hydrodynamic Boussinesq system by introducing additional variables that allow the model to be rewritten as a first-order system. More precisely, in addition to the primary variables (velocity, pressure, temperature, and electric potential), and following the ideas in [8, 15], we introduce the velocity gradient and an incomplete pseudo-stress tensor, which enables the elimination of the pressure as a primary unknown and its subsequent recovery through a post-processing procedure. We further incorporate the temperature gradient, the electric field, and the electric flux as independent variables. This setting provides a consistent treatment of the coupled problem, particularly with regard to the dielectrophoretic force term. The resulting fully-mixed formulation is derived in appropriate Banach spaces by testing the governing equations with suitable functions and applying standard integration by parts formulas. The system is then decoupled into subproblems, which are studied using the Babuška–Brezzi theory and the Lax–Milgram theorem, following the approach in [8, 31]. As a consequence, the continuous analysis can be carried out only for a particular choice of the Lebesgue exponents involved, as well as in the two-dimensional case. The aforementioned subproblems are then coupled through a fixed-point argument so that–under appropriate additional regularity assumptions on one of the solutions and smallness assumptions on the data–Banach's theorem yields the unique solvability of the fully coupled problem.

At the discrete level, we propose a general Galerkin scheme under abstract conditions and establish its well-posedness by arguments analogous to those used in the continuous setting, combined with Brouwer's theorem. Furthermore, we introduce specific finite element spaces satisfying the required hypotheses under a smallness assumption on the relative variation of the dielectric permittivity, together with a geometric restriction on the domain, thereby yielding concrete numerical schemes. Finally, numerical experiments are presented to illustrate the performance of the method and confirm the theoretical rates of convergence.

Outline. The remainder of the paper is organized as follows. In Section 2, we introduce the model equations and define the additional variables required to rewrite the original problem in a form suitable for the derivation of a fully-mixed formulation, which is carried out in Section 3. In Section 4, we establish the well-posedness of the continuous problem by means of a fixed-point strategy, combining smallness assumptions on the data with additional regularity properties on certain auxiliary linearized problems. This methodology is subsequently adapted to the analysis of the Galerkin scheme introduced in Section 5, where we provide abstract conditions on the discrete spaces and derive the corresponding a priori error estimates. In Section 6, we construct specific finite element spaces that satisfy these assumptions and establish their associated rates of convergence. This work concludes with Section 7, where numerical experiments validating the theoretical results are presented.

As a final introductory remark, we stress that, while we have already indicated that the main results of this paper ultimately hold only for specific Lebesgue exponents and in the two-dimensional case, we nevertheless begin by describing the model and developing its subsequent analysis on a domain of R^n, $n \in \{2, 3\}$, and employing generic exponents. This approach, which is maintained for as long as possible, allows the reader to clearly identify the points at which the limitations of the analysis arise.

Preliminaries. We end this section with some notation and useful formulae to be employed along the paper. Let Ω be a Lipschitz-continuous domain of R^n, $n \in \{2, 3\}$, whose outward unit normal at its boundary Γ is denoted $\boldsymbol{\nu}$. Then, the standard terminology is adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$, with $t \in [1, +\infty)$, and Sobolev spaces $\mathrm{W}^{\ell,t}(\Omega)$ and $\mathrm{W}^{\ell,t}_0(\Omega)$, with $\ell \geq 0$, whose corresponding norms and seminorms, are denoted by $\|\cdot\|_{0,t;\Omega}$, $\|\cdot\|_{\ell,t;\Omega}$, and $|\cdot|_{\ell,t;\Omega}$, respectively, irrespective of whether it refers to the scalar, vector, or tensorial version of them. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and that for $t = 2$ we just write $\mathrm{H}^\ell(\Omega)$ instead of $\mathrm{W}^{\ell,2}(\Omega)$, with its norm and seminorm denoted by $\|\cdot\|_{\ell,\Omega}$ and $|\cdot|_{\ell,\Omega}$, respectively. Now, letting t, $t' \in (1, +\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we let $\mathrm{W}^{1/t',t}(\Gamma)$ and $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the trace space of $\mathrm{W}^{1,t}(\Omega)$ and its dual, respectively, and denote by $\langle\cdot,\cdot\rangle$ the corresponding duality pairing between them. In particular, when $t = t' = 2$, we simply write $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ instead of $\mathrm{W}^{1/2,2}(\Gamma)$ and $\mathrm{W}^{-1/2,2}(\Gamma)$, respectively. In addition, given any generic scalar functional space S, we let $\mathbf{S}$ and $\mathbb{S}$ be its vector and tensorial counterparts. On the other hand, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \qquad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \qquad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n},$$

whereas, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,n}\,, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}\,, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I}\,,$$

where $\mathbb{I}$ stands for the identity tensor of $\mathbb{R} := \mathrm{R}^{n\times n}$. In turn, for each t, $j \in [1,+\infty)$ such that $t \geq j$, we introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega) \Big\}\,, \tag{1.1a}$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \Big\}\,, \tag{1.1b}$$

$$\mathbf{H}^t(\mathrm{div}_j;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^t(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^j(\Omega) \Big\}\,, \tag{1.1c}$$

which are endowed with the natural norms

$$\begin{aligned} \|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} &:= \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega)\,,\\ \|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} &:= \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,, \end{aligned}$$

and

$$\|\boldsymbol{\tau}\|_{t,\mathrm{div}_j;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,j;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_j;\Omega)\,.$$

Regarding these spaces, we recall that, proceeding as in [27, eq. (1.43), Section 1.3.4] (see also [11, Section 3.1]), one can prove that for each $t \in \begin{cases} (1,+\infty) & \text{if } n=2 \\ [6/5,+\infty) & \text{if } n=3 \end{cases}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_t;\Omega)\times \mathrm{H}^1(\Omega)\,, \tag{1.2}$$

and analogously

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \boldsymbol{v}\rangle = \int_\Omega \Big\{ \boldsymbol{\tau} : \nabla \boldsymbol{v} + \boldsymbol{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega)\times \mathbf{H}^1(\Omega)\,, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ denotes in (1.2) (resp. (1.3)) the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma)$) and $\mathrm{H}^{-1/2}(\Gamma)$ (resp. $\mathbf{H}^{-1/2}(\Gamma)$). Similarly, given t, $t' \in (1,+\infty)$ conjugate to each other, there also holds (cf. [25, Corollary B.57])

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\times \mathrm{W}^{1,t'}(\Omega)\,, \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$. Furthermore, letting $\mathrm{i}_{t'} : \mathrm{H}^1(\Omega) \to \mathrm{L}^{t'}(\Omega)$ and $\mathbf{i}_{t'} : \mathbf{H}^1(\Omega) \to \mathbf{L}^{t'}(\Omega)$ be the corresponding continuous injections, whose existences are guaranteed by the specified ranges for t, we easily deduce from (1.2) and (1.3), respectively, that

$$\|\boldsymbol{\tau}\cdot\boldsymbol{\nu}\|_{-1/2,\Gamma} \leq \max\big\{1, \|\mathrm{i}_{t'}\|\big\}\, \|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega),$$

and

$$\|\boldsymbol{\tau}\,\boldsymbol{\nu}\|_{-1/2,\Gamma} \leq \max\big\{1, \|\mathbf{i}_{t'}\|\big\}\, \|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,. \tag{1.5}$$

Similarly, applying [25, Lemma A.36], we deduce the existence of a positive constant $c_{t'}$, such that for each $\varphi \in \mathrm{W}^{1/t,t'}(\Gamma)$, there exists $v_\varphi \in \mathrm{W}^{1,t'}(\Omega)$ such that $v_\varphi|_\Gamma = \varphi$, and $\|v_\varphi\|_{1,t';\Omega} \leq c_{t'}\, \|\varphi\|_{1/t,t';\Gamma}$, and thus, it follows from (1.4) that

$$\|\boldsymbol{\tau}\cdot\boldsymbol{\nu}\|_{-1/t,t;\Gamma} \leq c_{t'}\, \|\boldsymbol{\tau}\|_{t,\mathrm{div}_t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\,. \tag{1.6}$$

2 Model problem

The thermal-electro-hydrodynamic (TEHD) Boussinesq problem models the dynamics of a dielectric and non-isothermal fluid occupying a Lipschitz-continuous domain Ω of R^n with boundary Γ, under the influence of a dielectrophoretic (DEP) force. More precisely, denoting by $\mathbf{u}$, $\mathtt{p}$, ϑ, and ϕ, the velocity, pressure, temperature, and internal electric potential of the fluid, respectively, and letting $\boldsymbol{\nu}$ be the outward unit normal vector at Γ, the corresponding equations are given by

$$\begin{array}{rcll}
-\mu\,\Delta\mathbf{u} + \big(\nabla\mathbf{u}\big)\mathbf{u} + \nabla\mathtt{p} \,-\, \delta_d\,|\nabla\phi|^2\,\nabla\vartheta & = & -\delta_t\,\vartheta\,\mathbf{g} + \mathbf{f} & \text{in} \quad \Omega\,, \\
\mathrm{div}(\mathbf{u}) & = & 0 & \text{in} \quad \Omega\,, \\
-\,\kappa\,\Delta\vartheta \,+\, \mathbf{u}\cdot\nabla\vartheta & = & f & \text{in} \quad \Omega\,, \\
\mathrm{div}\big(\eta(\vartheta)\,\nabla\phi\big) & = & g & \text{in} \quad \Omega\,, \\
\mathbf{u} = \mathbf{u}_D\,, \quad \vartheta = \vartheta_D\,, \quad \eta(\vartheta)\nabla\phi\cdot\boldsymbol{\nu} & = & 0 & \text{on} \quad \Gamma\,, \\
\displaystyle\int_\Omega \mathtt{p} \,=\, 0\,, \quad \int_\Omega \phi & = & 0, &
\end{array} \tag{2.1}$$

where $\mathbf{f}$, f and g are source functions, $|\nabla\phi|^2\,\nabla\vartheta$ is the DEP force term, $\mathbf{g}$ is the gravitational field, μ is the kinematic viscosity, δ_d is the DEP coefficient, δ_t is the thermal expansion coefficient, κ is the thermal diffusion coefficient, and η is the temperature dependent dielectric permittivity. In addition, regarding δ_d, there holds $\delta_d = (2\delta_f)^{-1}\,\eta_r\,\eta_0\,\delta_r$, where δ_f is the fluid density, η_r and η_0 are the dielectric constants, and $\delta_r = \eta'(\vartheta_r)$, with ϑ_r denoting the reference temperature. We also assume that η is bounded and Lipschitz-continuous, which means that there exist positive constants η_1, η_2, and L_η, such that

$$\begin{array}{l}
0 \,<\, \eta_1 \,\leq\, \eta(x) \,\leq\, \eta_2 \qquad \forall\, x \in \mathrm{R}\,, \quad \text{and} \\
|\eta(x) - \eta(z)| \,\leq\, L_\eta\,|x - z| \qquad \forall\, x,\, z \in \mathrm{R}\,.
\end{array} \tag{2.2}$$

In addition, the homogeneous Neumann boundary condition (2.1) imposes the compatibility condition $\displaystyle\int_\Omega g \,=\, 0$, while the last two equalities ensure solution uniqueness. Next, in order to derive a mixed formulation of (2.1) suitably handling the DEP force term, we introduce the auxiliary unknowns

$$\mathbf{t} \,:=\, \nabla\mathbf{u}\,, \quad \boldsymbol{\sigma} \,:=\, \mu\,\nabla\mathbf{u} \,-\, \frac{1}{2}\big(\mathbf{u}\otimes\mathbf{u}\big) \,-\, \mathtt{p}\,\mathbb{I}\,, \quad \boldsymbol{\chi} \,:=\, \kappa\,\nabla\vartheta\,, \quad \boldsymbol{\rho} \,:=\, \nabla\phi\,, \quad \text{and} \quad \boldsymbol{\xi} \,:=\, \eta(\vartheta)\,\boldsymbol{\rho}\,, \tag{2.3}$$

and thus we get, in particular,

$$\boldsymbol{\sigma} \,=\, \mu\,\mathbf{t} \,-\, \frac{1}{2}\big(\mathbf{u}\otimes\mathbf{u}\big) \,-\, \mathtt{p}\,\mathbb{I} \quad \text{and} \quad \nabla\vartheta \,=\, \kappa^{-1}\,\boldsymbol{\chi}\,. \tag{2.4}$$

In this way, using, thanks to the incompressibility condition, that $\mathbf{div}\big(\mathbf{u}\otimes\mathbf{u}\big) \,=\, \big(\nabla\mathbf{u}\big)\,\mathbf{u} \,=\, \mathbf{t}\,\mathbf{u}$, the first equation of (2.1) can be rewritten as

$$-\,\mathbf{div}(\boldsymbol{\sigma}) \,+\, \frac{1}{2}\,\mathbf{t}\,\mathbf{u} \,-\, \kappa^{-1}\,\delta_d\,|\boldsymbol{\rho}|^2\,\boldsymbol{\chi} \,=\, -\,\delta_t\,\vartheta\,\mathbf{g} \,+\, \mathbf{f}\,, \tag{2.5}$$

whereas the third and fourth ones become, respectively,

$$-\,\mathrm{div}(\boldsymbol{\chi}) \,+\, \kappa^{-1}\,\mathbf{u}\cdot\boldsymbol{\chi} \,=\, f \qquad \text{and} \qquad \mathrm{div}(\boldsymbol{\xi}) \,=\, g\,.$$

At this point we remark that $\boldsymbol{\sigma}$, as defined in (2.3), is named INCOMPLETE PSEUDOSTRESS TENSOR: its divergence yields just one half of the convective term $\big(\nabla\mathbf{u}\big)\,\mathbf{u}$. The remaining half is given precisely by the expression $\frac{1}{2}\,\mathbf{t}\,\mathbf{u}$ in (2.5). The convenience of proceeding in this way will become clear in the forthcoming sections. On the other hand, applying matrix trace and the deviatoric operator $^{\mathrm{d}}$ to $\boldsymbol{\sigma}$ in (2.4), and then using that $\mathrm{tr}(\mathbf{t}) = \mathrm{div}(\mathbf{u}) = 0$ and that $\mathbb{I}^{\mathrm{d}}$ is the null matrix, we deduce that

$$\mathtt{p} \,=\, -\frac{1}{n}\,\mathrm{tr}\Big(\boldsymbol{\sigma} + \frac{1}{2}\,(\mathbf{u}\otimes\mathbf{u})\Big)\,, \qquad \text{and} \qquad \boldsymbol{\sigma}^{\mathrm{d}} \,=\, \mu\,\mathbf{t} \,-\, \frac{1}{2}\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}}\,. \tag{2.6}$$

It follows that the pressure can be eliminated from the original system, and subsequently recovered in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$ (cf. (2.6)). Furthermore, the corresponding uniqueness condition (cf. (2.1)), in view of (2.6), can be stated as

$$\int_\Omega \mathrm{tr}\Big(\boldsymbol{\sigma} + \frac{1}{2}\,(\mathbf{u}\otimes\mathbf{u})\Big) \,=\, 0\,.$$

As a consequence of the previous discussion, we find that the original system (2.1) can be rewritten, equivalently, as: Find $\mathbf{t}$, $\mathbf{u}$, $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$, ϑ, $\boldsymbol{\rho}$, $\boldsymbol{\xi}$, and ϕ in suitable spaces to be specified later on, such that:

$$\begin{array}{rcll}
\mathbf{t} \,-\, \nabla\mathbf{u} & = & \mathbf{0} & \text{in} \quad \Omega \\
-\,\mathbf{div}(\boldsymbol{\sigma}) + \dfrac{1}{2}\,\mathbf{t}\,\mathbf{u} \,-\, \kappa^{-1}\,\delta_d\,|\boldsymbol{\rho}|^2\,\boldsymbol{\chi} & = & -\,\delta_t\,\vartheta\,\mathbf{g} \,+\, \mathbf{f} & \text{in} \quad \Omega\,, \\
\mu\,\mathbf{t} \,-\, \dfrac{1}{2}\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \,-\, \boldsymbol{\sigma}^{\mathrm{d}} & = & \mathbf{0} & \text{in} \quad \Omega\,, \\
\boldsymbol{\chi} \,-\, \kappa\,\nabla\vartheta & = & \mathbf{0} & \text{in} \quad \Omega\,, \\
-\,\mathrm{div}(\boldsymbol{\chi}) \,+\, \kappa^{-1}\,\mathbf{u}\cdot\boldsymbol{\chi} & = & f & \text{in} \quad \Omega\,, \\
\boldsymbol{\rho} \,-\, \nabla\phi & = & \mathbf{0} & \text{in} \quad \Omega\,, \\
\boldsymbol{\xi} \,-\, \eta(\vartheta)\,\boldsymbol{\rho} & = & \mathbf{0} & \text{in} \quad \Omega\,, \\
\mathrm{div}(\boldsymbol{\xi}) & = & g & \text{in} \quad \Omega\,, \\
\mathbf{u} = \mathbf{u}_D\,, \quad \vartheta = \vartheta_D\,, \quad \boldsymbol{\xi}\cdot\boldsymbol{\nu} & = & 0 & \text{on} \quad \Gamma\,, \\
\displaystyle\int_\Omega \mathrm{tr}\Big(\boldsymbol{\sigma} + \frac{1}{2}\,(\mathbf{u}\otimes\mathbf{u})\Big) = 0 \quad \text{and} \quad \int_\Omega \phi & = & 0. &
\end{array} \tag{2.7}$$

Note here that, under the boundedness assumption on the permittivity η (cf. (2.2)), the Neumann boundary condition for ϕ in the last row of (2.1) has been replaced, equivalently, by $\boldsymbol{\xi}\cdot\boldsymbol{\nu} = 0$ on Γ. In addition, due to the incompressibility of the fluid (second equation of (2.1)), $\mathbf{u}_D$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\nu} \,=\, 0\,. \tag{2.8}$$

3 The mixed variational formulation

Here we derive the variational formulation of (2.7) by splitting the analysis according to the subsets of equations forming the nonlinear coupled system, namely, fluid equations (first three rows), thermal equations (fourth and fifth rows), and electric potential equations (sixth, seventh, and eighth rows). The whole coupled variational formulation is summarized at the end.

3.1 The fluid equations

We begin by testing the third equation of (2.7) with $\mathbf{s}\in\mathbb{L}^2(\Omega)$, thus formally obtaining

$$\mu\int_\Omega \mathbf{t}:\mathbf{s} \,-\, \frac{1}{2}\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}}:\mathbf{s} \,-\, \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}}:\mathbf{s} \,=\, 0\,. \tag{3.1}$$

It readily follows that the first and third terms of (3.1) are well-defined if $\mathbf{t}$ and $\boldsymbol{\sigma}$ (and hence $\boldsymbol{\sigma}^{\mathrm{d}}$) belong to $\mathbb{L}^2(\Omega)$. In addition, since the trace of any deviatoric tensor is 0, it is also clear from the third equation of (2.7) that $\mathrm{tr}(\mathbf{t}) = 0$, whence we must look for this unknown in the space

$$\mathbb{L}^2_{\mathtt{tr}}(\Omega) := \Big\{ \mathbf{r} \in \mathbb{L}^2(\Omega) : \quad \mathrm{tr}(\mathbf{r}) = 0 \Big\} .$$

In turn, straightforward applications of Cauchy–Schwarz's inequality yield

$$\left| \int_\Omega \big(\mathbf{u} \otimes \mathbf{u} \big)^{\mathtt{d}} : \mathbf{s} \right| \,\le\, n^{1/2} \, \|\mathbf{u}\|^2_{0,4;\Omega} \, \|\mathbf{s}\|_{0,\Omega} \,, \tag{3.2}$$

showing that the second term of (3.1) makes sense if $\mathbf{u} \in \mathbf{L}^4(\Omega)$. Additionally, using that each $\mathbf{s} \in \mathbb{L}^2(\Omega)$ can be decomposed in a unique form as $\mathbf{s} = \mathbf{s}^{\mathtt{d}} + \frac{1}{n}\,\mathrm{tr}(\mathbf{s})\,\mathbb{I}$, with $\mathbf{s}^{\mathtt{d}} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega)$ and $\frac{1}{n}\,\mathrm{tr}(\mathbf{s})\,\mathbb{I}$, a $\mathrm{L}^2(\Omega)$ multiple of the identity tensor $\mathbb{I}$, we deduce that testing (3.1) against $\mathbb{L}^2(\Omega)$ is equivalent to doing it against $\mathbb{L}^2_{\mathtt{tr}}(\Omega)$, in which case this equation can be rewritten as

$$\mu \int_\Omega \mathbf{t} : \mathbf{s} \,-\, \frac{1}{2} \int_\Omega \big(\mathbf{u} \otimes \mathbf{u} \big) : \mathbf{s} \,-\, \int_\Omega \boldsymbol{\sigma} : \mathbf{s} \,=\, 0 \qquad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) \,. \tag{3.3}$$

Next, formally testing the second equation of (2.7) with a vector field $\mathbf{v}$, we get

$$- \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) + \frac{1}{2} \int_\Omega \mathbf{t}\,\mathbf{u} \cdot \mathbf{v} \,-\, \kappa^{-1}\,\delta_d \int_\Omega |\boldsymbol{\rho}|^2\,\boldsymbol{\chi} \cdot \mathbf{v} \,=\, -\,\delta_t \int_\Omega \vartheta\,\mathbf{g} \cdot \mathbf{v} \,+\, \int_\Omega \mathbf{f} \cdot \mathbf{v} \,, \tag{3.4}$$

from which, knowing already the spaces to which $\mathbf{t}$ and $\mathbf{u}$ must belong, and proceeding similarly as for (3.2), we notice that the second term on the left-hand side is well-defined if $\mathbf{v}$ belongs as well to $\mathbf{L}^4(\Omega)$. Moreover, this later fact leads the first term to make sense if $\mathbf{div}(\boldsymbol{\sigma})$ lies in $\mathbf{L}^{4/3}(\Omega)$, thus needing finally to look for $\boldsymbol{\sigma}$ in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ (cf. (1.1b)). Similarly, knowing the space from where $\mathbf{v}$ is taken, and thanks now to consecutive applications of Hölder's inequality, we find that

$$\left| \int_\Omega |\boldsymbol{\rho}|^2\,\boldsymbol{\chi} \cdot \mathbf{v} \right| \,\le\, \|\boldsymbol{\rho}\|^2_{0,8t/3;\Omega} \, \|\boldsymbol{\chi}\|_{0,4t'/3;\Omega} \, \|\mathbf{v}\|_{0,4;\Omega} \,, \tag{3.5}$$

where t, $t' \in (1,+\infty)$ are conjugate to each other. Hence, the third term on the left-hand side of (3.4) is well-defined if $\boldsymbol{\rho}$ and $\boldsymbol{\chi}$ are sought in $\mathbf{L}^j(\Omega)$ and $\mathbf{L}^r(\Omega)$, respectively, where

$$j \,=\, 8t/3\,, \quad \ell = \frac{j}{j-1}\ \text{(conjugate of } j)\,, \quad r \,=\, 4t'/3\,, \quad \text{and} \quad s = \frac{r}{r-1}\ \text{(conjugate of } r)\,. \tag{3.6}$$

The specific choice of t, and hence of t', j, r, and the respective conjugates ℓ and s, will be addressed later on, so that meanwhile we consider generic values for the indexes defined in (3.6). Now, regarding the right-hand side of (3.4), we postpone the discussion on its first term until we know where to seek the temperature ϑ, whereas its second one certainly requires $\mathbf{f}$ to belong to $\mathbf{L}^{4/3}(\Omega)$. Furthermore, the space to which $\mathbf{t}$ belongs along with the first equation of (2.7), imply that $\mathbf{u} \in \mathbf{H}^1(\Omega)$, whence applying the integration by parts formula (1.3) with $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and assuming that $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, we obtain

$$- \int_\Omega \boldsymbol{\tau} : \mathbf{t} \,-\, \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \,=\, -\,\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D \rangle \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)\,. \tag{3.7}$$

Moreover, according to the decomposition $\mathbb{H}(\mathbf{div}_{4/3};\Omega) \,=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,\oplus\, \mathrm{R}\,\mathbb{I}$, where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \,=\, 0 \Big\} ,$$

and thanks to (2.8), we realize that imposing (3.7) with $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it with $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. In turn, $\boldsymbol{\sigma}$ can be decomposed uniquely as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + d_0\,\mathbb{I}$, with $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $d_0 \in \mathrm{R}$, which, invoking the last equation of (2.7), is computed as

$$d_0 \,:=\, \frac{1}{n\,|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, -\frac{1}{2\,n\,|\Omega|} \int_\Omega \mathrm{tr}\big(\mathbf{u} \otimes \mathbf{u}\big)\,.$$

Besides, it is easy to see that (3.3) and (3.4) are unaltered if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$, so that from now on we re-denote $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Summarizing, adding (3.3) and (3.4), keeping (3.7) as it is, setting the spaces

$$\mathbf{H}_1 \,:=\, \mathbb{L}^2_{\mathbf{tr}}(\Omega)\,, \quad \mathbf{H}_2 \,:=\, \mathbf{L}^4(\Omega)\,, \quad \mathbf{H} \,:=\, \mathbf{H}_1 \times \mathbf{H}_2\,, \quad \mathbf{Q} \,:=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,$$

and introducing the notation

$$\vec{\mathbf{t}} := (\mathbf{t},\mathbf{u})\,, \quad \vec{\mathbf{s}} := (\mathbf{s},\mathbf{v})\,, \quad \vec{\mathbf{r}} := (\mathbf{r},\mathbf{w}) \in \mathbf{H}\,,$$

the variational formulation of the fluid equations can be written as: Find $(\vec{\mathbf{t}},\boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{A}_{\mathbf{u}}(\vec{\mathbf{t}},\vec{\mathbf{s}}) \,+\, \mathbf{B}(\vec{\mathbf{s}},\boldsymbol{\sigma}) \quad &= \quad \mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta}(\vec{\mathbf{s}}) \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,, \\ \mathbf{B}(\vec{\mathbf{t}},\boldsymbol{\tau}) \quad &= \quad \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,, \end{aligned} \tag{3.8}$$

where the bilinear forms $\mathbf{A}_{\mathbf{z}} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$, for each $\mathbf{z} \in \mathbf{H}_2$, and $\mathbf{B} : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$, are defined as

$$\mathbf{A}_{\mathbf{z}}(\vec{\mathbf{r}},\vec{\mathbf{s}}) \,:=\, \mu \int_\Omega \mathbf{r} : \mathbf{s} + \frac{1}{2}\int_\Omega \Big\{ \mathbf{r}\,\mathbf{z}\cdot\mathbf{v} - (\mathbf{w}\otimes\mathbf{z}) : \mathbf{s} \Big\} \qquad \forall\, \vec{\mathbf{r}},\, \vec{\mathbf{s}} \in \mathbf{H}\,, \quad \text{and} \tag{3.9a}$$

$$\mathbf{B}(\vec{\mathbf{s}},\boldsymbol{\tau}) \,:=\, -\int_\Omega \boldsymbol{\tau} : \mathbf{s} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\vec{\mathbf{s}},\boldsymbol{\tau}) \in \mathbf{H}\times\mathbf{Q}\,. \tag{3.9b}$$

In turn, the linear functionals $\mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta} : \mathbf{H} \to \mathrm{R}$, for each $(\boldsymbol{\rho},\boldsymbol{\chi}) \in \mathbf{L}^j(\Omega) \times \mathbf{L}^r(\Omega)$ and ϑ in a space to be specified later on, and $\mathbf{G} : \mathbf{Q} \to \mathrm{R}$, are given by

$$\begin{aligned} \mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta}(\vec{\mathbf{s}}) \,&:=\, \kappa^{-1}\,\delta_d \int_\Omega |\boldsymbol{\rho}|^2\,\boldsymbol{\chi}\cdot\mathbf{v} \,-\, \delta_t \int_\Omega \vartheta\,\mathbf{g}\cdot\mathbf{v} \,+\, \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\vec{\mathbf{s}} \in \mathbf{H}\,, \quad \text{and} \\ \mathbf{G}(\boldsymbol{\tau}) \,&:=\, -\,\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D\rangle \qquad \forall\,\boldsymbol{\tau} \in \mathbf{Q}\,. \end{aligned} \tag{3.10}$$

Note that (3.8) shows a saddle point structure with a nonlinear main operator induced by $\mathbf{A}_{\mathbf{u}}$. Later on, in Section 4.1, we explain how to linearize it so that the Babuška–Brezzi theory in Banach spaces (cf. [25, Theorem 2.34]) can be applied to the resulting formulation.

3.2 The thermal equations

Knowing already that $\boldsymbol{\chi}$ must belong to $\mathbf{L}^r(\Omega)$, the fourth equation of (2.7) suggests that ϑ may be originally sought in $\mathrm{W}^{1,r}(\Omega)$. Hence, applying the integration by parts formula (1.4), with $t = s$ and $t' = r$, to $\vartheta \in \mathrm{W}^{1,r}(\Omega)$ and $\boldsymbol{\eta} \in \mathbf{H}^s(\mathrm{div}_s;\Omega)$, and assuming that $\vartheta_D \in \mathrm{W}^{1/s,r}(\Gamma)$, we get

$$\int_\Omega \boldsymbol{\chi}\cdot\boldsymbol{\eta} \,+\, \kappa \int_\Omega \vartheta\,\mathrm{div}(\boldsymbol{\eta}) \,=\, \kappa\,\langle \boldsymbol{\eta}\cdot\boldsymbol{\nu}, \vartheta_D\rangle \qquad \forall\,\boldsymbol{\eta} \in \mathbf{H}^s(\mathrm{div}_s;\Omega)\,, \tag{3.11}$$

from which we notice that actually it suffices to look for ϑ in $\mathrm{L}^r(\Omega)$ as well. In turn, regarding the fifth equation of (2.7), we first look at its second term on the left-hand side. To this end, we consider a scalar test function ω, and proceed similarly to the derivation of (3.5), to obtain

$$\left| \int_\Omega \mathbf{u}\cdot\boldsymbol{\chi}\,\omega \right| \,\leq\, \|\mathbf{u}\|_{0,4;\Omega}\,\|\boldsymbol{\chi}\|_{0,r;\Omega}\,\|\omega\|_{0,4t/3;\Omega}\,, \tag{3.12}$$

with t, t', and r as in (3.6), which says that the left-hand side of (3.12) is well-defined if ω is taken in $\mathrm{L}^p(\Omega)$, where

$$p \,:=\, 4t/3 \quad \text{and} \quad q \,:=\, \frac{p}{p-1} \;\text{ denotes its conjugate}\,. \tag{3.13}$$

Testing the fifth equation of (2.7) against $\omega \in \mathrm{L}^p(\Omega)$, adding to $\boldsymbol{\chi}$ the condition that $\operatorname{div}(\boldsymbol{\chi}) \in \mathrm{L}^q(\Omega)$, and assuming that $f \in \mathrm{L}^q(\Omega)$, we deduce that

$$\kappa \int_\Omega \omega \operatorname{div}(\boldsymbol{\chi}) \,-\, \int_\Omega \mathbf{u}\cdot\boldsymbol{\chi}\,\omega \,=\, -\kappa \int_\Omega f\,\omega \qquad \forall\,\omega \in \mathrm{L}^p(\Omega)\,. \tag{3.14}$$

Hence, gathering (3.11) and (3.14), and introducing the spaces

$$\mathrm{X}_2 \,:=\, \mathbf{H}^r(\operatorname{div}_q;\Omega)\,, \quad \mathrm{M}_1 \,:=\, \mathrm{L}^r(\Omega)\,, \quad \mathrm{X}_1 \,:=\, \mathbf{H}^s(\operatorname{div}_s;\Omega)\,, \quad \mathrm{M}_2 \,:=\, \mathrm{L}^p(\Omega)\,, \tag{3.15}$$

the variational formulation of the thermal equations reduces to: Find $(\boldsymbol{\chi},\vartheta) \in \mathrm{X}_2 \times \mathrm{M}_1$ such that

$$\begin{aligned} a(\boldsymbol{\chi},\boldsymbol{\eta}) \,+\, b_1(\boldsymbol{\eta},\vartheta) \quad &= \quad \mathrm{F}(\boldsymbol{\eta}) \qquad \forall\,\boldsymbol{\eta} \in \mathrm{X}_1\,,\\ b_2(\boldsymbol{\chi},\omega) \,-\, \int_\Omega \mathbf{u}\cdot\boldsymbol{\chi}\,\omega \quad &= \quad \mathrm{G}(\omega) \qquad \forall\,\omega \in \mathrm{M}_2\,, \end{aligned} \tag{3.16}$$

where the bilinear forms $a : \mathrm{X}_2\times\mathrm{X}_1 \to \mathrm{R}$ and $b_i : \mathrm{X}_i\times\mathrm{M}_i \to \mathrm{R}$, $i \in \{1,2\}$, and the functionals $\mathrm{F} : \mathrm{X}_1 \to \mathrm{R}$ and $\mathrm{G} : \mathrm{M}_2 \to \mathrm{R}$, are defined as

$$a(\boldsymbol{\mu},\boldsymbol{\eta}) \,:=\, \int_\Omega \boldsymbol{\mu}\cdot\boldsymbol{\eta} \qquad \forall\,(\boldsymbol{\mu},\boldsymbol{\eta}) \in \mathrm{X}_2 \times X_1\,, \tag{3.17a}$$

$$b_i(\boldsymbol{\eta},\omega) \,:=\, \kappa \int_\Omega \omega \operatorname{div}(\boldsymbol{\eta}) \qquad \forall\,(\boldsymbol{\eta},\omega) \in \mathrm{X}_i \times \mathrm{M}_i\,, \tag{3.17b}$$

$$\mathrm{F}(\boldsymbol{\eta}) \,:=\, \kappa\,\langle \boldsymbol{\eta}\cdot\boldsymbol{\nu}, \vartheta_D\rangle \qquad \forall\,\boldsymbol{\eta} \in \mathrm{X}_1\,, \quad \text{and} \tag{3.17c}$$

$$\mathrm{G}(\omega) \,:=\, -\kappa \int_\Omega f\,\omega \qquad \forall\,\omega \in \mathrm{M}_2\,. \tag{3.17d}$$

Regarding the structure of (3.16), we stress that it represents a nonlinear perturbation (given by the term $-\displaystyle\int_\Omega \mathbf{u}\cdot\boldsymbol{\chi}\,\omega$) of a typical mixed formulation to which the generalized Babuška–Brezzi theory in Banach spaces (cf. [5, Theorem 2.1, Corollary 2.1]) could be applied. Due to this fact, (3.16) can be identified as a nonlinearly perturbed generalized saddle point problem.

We end this section by referring to the second term on the right-hand side of (3.10). Indeed, similarly to (3.12), it is easily seen that

$$\left|\int_\Omega \vartheta\,\mathbf{g}\cdot\mathbf{v}\right| \,\le\, \|\vartheta\|_{0,r;\Omega}\,\|\mathbf{g}\|_{0,p;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,, \tag{3.18}$$

which, being $\mathbf{g}$ a constant vector, confirms that the aforementioned term, and hence the functional $\mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta}$ (cf. (3.10)), is well-defined for $(\boldsymbol{\rho},\boldsymbol{\chi},\vartheta) \in \mathbf{L}^j(\Omega)\times\mathbf{L}^r(\Omega)\times\mathrm{L}^r(\Omega)$.

3.3 The electric potential equations

We first realize from the seventh equation of (2.7) and the boundedness property of η (cf. (2.2)) that it makes sense to look for $\boldsymbol{\xi}$ in the same spaces as $\boldsymbol{\rho}$, that is $\mathbf{L}^j(\Omega)$, so that, recalling from (3.6) that ℓ stands for the conjugate of j, the natural testing of the equation relating $\boldsymbol{\rho}$ and $\boldsymbol{\xi}$ reduces to

$$\int_\Omega \eta(\vartheta)\,\boldsymbol{\rho}\cdot\boldsymbol{\varrho} \,-\, \int_\Omega \boldsymbol{\xi}\cdot\boldsymbol{\varrho} \,=\, 0 \qquad \forall\,\boldsymbol{\varrho} \in \mathbf{L}^\ell(\Omega)\,. \tag{3.19}$$

In turn, similarly as observed at the beginning of Section 3.2, the sixth equation suggests that at first instance one could look for ϕ in $\mathrm{W}^{1,j}(\Omega)$. Thus, applying now (1.4), with $t = \ell$ and $t' = j$, to $\phi \in \mathrm{W}^{1,j}(\Omega)$ and $\boldsymbol{\lambda} \in \mathbf{H}^\ell(\operatorname{div}_\ell;\Omega)$, we find that

$$\int_\Omega \boldsymbol{\rho}\cdot\boldsymbol{\lambda} \,=\, \int_\Omega \nabla\phi\cdot\boldsymbol{\lambda} \,=\, -\int_\Omega \phi \operatorname{div}(\boldsymbol{\lambda}) \,+\, \langle \boldsymbol{\lambda}\cdot\boldsymbol{\nu}, \phi\rangle\,,$$

from which we notice, analogously to the respective consequence of (3.11), that actually it suffices to seek ϕ in $\mathrm{L}^j(\Omega)$. Moreover, assuming that $\boldsymbol{\lambda}\cdot\boldsymbol{\nu} = 0$ on Γ, the foregoing equation becomes

$$- \int_\Omega \boldsymbol{\rho}\cdot\boldsymbol{\lambda} - \int_\Omega \phi \, \mathrm{div}(\boldsymbol{\lambda}) \;=\; 0 \qquad \forall\, \boldsymbol{\lambda} \in \mathbf{H}_0^\ell(\mathrm{div}_\ell;\Omega)\,, \tag{3.20}$$

where, for each $t \in (1,+\infty)$, we set

$$\mathbf{H}_0^t(\mathrm{div}_t;\Omega) \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega) : \quad \boldsymbol{\tau}\cdot\boldsymbol{\nu} \;=\; 0 \quad \text{on} \quad \Gamma \Big\}\,. \tag{3.21}$$

We notice here that the second term on the left-hand side of (3.20) vanishes when ϕ is constant. Therefore, to ensure uniqueness, we incorporate the corresponding condition from (2.1) into the functional space. Thus, the internal electric potential unknown is sought from now on the space

$$\mathrm{L}_0^j(\Omega) \;:=\; \Big\{ \varphi \in \mathrm{L}^j(\Omega) : \quad \int_\Omega \varphi = 0 \Big\}\,.$$

Furthermore, regarding the eighth equation of (2.7), we impose $\mathrm{div}(\boldsymbol{\xi}) \in \mathrm{L}^j(\Omega)$, which certainly requires to assume that $g \in \mathrm{L}^j(\Omega)$, and hence this equation is tested as

$$\int_\Omega \varphi \, \mathrm{div}(\boldsymbol{\xi}) \;=\; \int_\Omega \varphi\, g \qquad \forall\, \varphi \in \mathrm{L}^\ell(\Omega)\,, \tag{3.22}$$

but thanks to the decomposition $\mathrm{L}^\ell(\Omega) \;=\; \mathrm{L}_0^\ell(\Omega) \oplus \mathrm{R}$ and the boundary condition satisfied by $\boldsymbol{\xi}$ (cf. ninth row of (2.7)), we conclude that $\boldsymbol{\xi}$ must be sought in $\mathbf{H}_0^j(\mathrm{div}_j;\Omega)$ (cf. (3.21)), and also that (3.22) is equivalent to be tested against $\varphi \in \mathrm{L}_0^\ell(\Omega)$, that is

$$\int_\Omega \varphi \, \mathrm{div}(\boldsymbol{\xi}) \;=\; \int_\Omega \varphi\, g \qquad \forall\, \varphi \in \mathrm{L}_0^\ell(\Omega)\,. \tag{3.23}$$

In this way, adding (3.19) and (3.20), taking the negative of (3.23), introducing the spaces

$$\begin{aligned} \mathbf{X}_2 \;&:=\; \mathbf{L}^j(\Omega) \times \mathbf{H}_0^j(\mathrm{div}_j;\Omega)\,, \qquad & \mathbf{Y}_1 \;&:=\; \mathrm{L}_0^j(\Omega)\,,\\ \mathbf{X}_1 \;&:=\; \mathbf{L}^\ell(\Omega) \times \mathbf{H}_0^\ell(\mathrm{div}_\ell;\Omega)\,, \qquad & \mathbf{Y}_2 \;&:=\; \mathrm{L}_0^\ell(\Omega)\,, \end{aligned}$$

and setting the notation

$$\vec{\boldsymbol{\rho}} \;:=\; (\boldsymbol{\rho},\boldsymbol{\xi}) \in \mathbf{X}_2\,, \quad \vec{\boldsymbol{\varrho}} \;:=\; (\boldsymbol{\varrho},\boldsymbol{\lambda}) \in \mathbf{X}_1\,, \quad \vec{\boldsymbol{\varkappa}} \;:=\; (\boldsymbol{\varkappa},\boldsymbol{\varsigma}) \in \mathbf{X}_i,\ i \in \big\{1,2\big\}\,,$$

we arrive at the following variational formulation of the electric potential equations: Find $\big(\vec{\boldsymbol{\rho}},\phi\big) \in \mathbf{X}_2\times\mathbf{Y}_1$ such that

$$\begin{aligned} \mathcal{A}_\vartheta(\vec{\boldsymbol{\rho}},\vec{\boldsymbol{\varrho}}) \,+\, \mathcal{B}_1\big(\vec{\boldsymbol{\varrho}},\phi\big) \quad &= \quad 0 \qquad && \forall\, \vec{\boldsymbol{\varrho}} \in \mathbf{X}_1\,,\\ \mathcal{B}_2\big(\vec{\boldsymbol{\rho}},\varphi\big) \quad &= \quad \mathcal{F}(\varphi) \qquad && \forall\, \varphi \in \mathbf{Y}_2\,, \end{aligned} \tag{3.24}$$

where the bilinear forms $\mathcal{A}_\theta : \mathbf{X}_2 \times \mathbf{X}_1 \to \mathrm{R}$, for each $\theta \in \mathrm{M}_1$ (cf. (3.15)), and $\mathcal{B}_i : \mathbf{X}_i \times \mathbf{Y}_i \to \mathrm{R}$, $i \in \big\{1,2\big\}$, and the functional $\mathcal{F} : \mathbf{Y}_2 \to \mathrm{R}$, are defined as

$$\mathcal{A}_\theta(\vec{\boldsymbol{\varkappa}},\vec{\boldsymbol{\varrho}}) \;:=\; \int_\Omega \eta(\theta)\, \boldsymbol{\varkappa}\cdot\boldsymbol{\varrho} \,-\, \int_\Omega \boldsymbol{\varrho}\cdot\boldsymbol{\varsigma} \,-\, \int_\Omega \boldsymbol{\varkappa}\cdot\boldsymbol{\lambda} \qquad \forall\, (\vec{\boldsymbol{\varkappa}},\vec{\boldsymbol{\varrho}}) \in \mathbf{X}_2 \times \mathbf{X}_1\,, \tag{3.25a}$$

$$\mathcal{B}_i(\vec{\boldsymbol{\varkappa}},\psi) \;:=\; -\int_\Omega \psi \, \mathrm{div}(\boldsymbol{\varsigma}) \qquad \forall\, (\vec{\boldsymbol{\varkappa}},\psi) \in \mathbf{X}_i \times \mathbf{Y}_i\,, \quad \text{and} \tag{3.25b}$$

$$\mathcal{F}(\varphi) \;:=\; -\int_\Omega \varphi\, g \qquad \forall\, \varphi \in \mathbf{Y}_2\,. \tag{3.25c}$$

Now, similarly as for (3.16), we highlight here that (3.24) constitutes another mixed formulation fitting the generalized Babuška–Brezzi theory in Banach spaces (cf. [5, Theorem 2.1, Corollary 2.1]). Moreover, the operator induced by $\mathcal{A}_\theta$ shows the same structure, so that we can call (3.24) a generalized twofold saddle point problem. Indeed, it suffices to see that, for each $\theta \in \mathrm{M}_1$, there holds

$$\mathcal{A}_\theta(\vec{\boldsymbol{\varkappa}}, \vec{\boldsymbol{\varrho}}) \,:=\, \mathbf{a}_\theta(\boldsymbol{\varkappa}, \boldsymbol{\varrho}) \,+\, \mathbf{b}_1(\boldsymbol{\varrho}, \boldsymbol{\varsigma}) \,+\, \mathbf{b}_2(\boldsymbol{\varkappa}, \boldsymbol{\lambda}) \qquad \forall\, (\vec{\boldsymbol{\varkappa}}, \vec{\boldsymbol{\varrho}}) \in \mathbf{X}_2 \times \mathbf{X}_1 \,,$$

where $\mathbf{a}_\theta : \mathbf{L}^j(\Omega) \times \mathbf{L}^\ell(\Omega) \to \mathrm{R}$, $\mathbf{b}_1 : \mathbf{L}^\ell(\Omega) \times \mathbf{H}_0^j(\mathrm{div}_j;\Omega) \to \mathrm{R}$, and $\mathbf{b}_2 : \mathbf{L}^j(\Omega) \times \mathbf{H}_0^\ell(\mathrm{div}_\ell;\Omega) \to \mathrm{R}$, are the bilinear forms defined as

$$\mathbf{a}_\theta(\boldsymbol{\varkappa}, \boldsymbol{\varrho}) \,:=\, \int_\Omega \eta(\theta)\, \boldsymbol{\varkappa} \cdot \boldsymbol{\varrho} \qquad \forall\, (\boldsymbol{\varkappa}, \boldsymbol{\varrho}) \in \mathbf{L}^j(\Omega) \times \mathbf{L}^\ell(\Omega) \,, \tag{3.26a}$$

$$\mathbf{b}_1(\boldsymbol{\varrho}, \boldsymbol{\varsigma}) \,:=\, -\int_\Omega \boldsymbol{\varrho} \cdot \boldsymbol{\varsigma} \qquad \forall\, (\boldsymbol{\varrho}, \boldsymbol{\varsigma}) \in \mathbf{L}^\ell(\Omega) \times \mathbf{H}_0^j(\mathrm{div}_j;\Omega) \,, \quad \text{and} \tag{3.26b}$$

$$\mathbf{b}_2(\boldsymbol{\varkappa}, \boldsymbol{\lambda}) \,:=\, -\int_\Omega \boldsymbol{\varkappa} \cdot \boldsymbol{\lambda} \qquad \forall\, (\boldsymbol{\varkappa}, \boldsymbol{\lambda}) \in \mathbf{L}^j(\Omega) \times \mathbf{H}_0^\ell(\mathrm{div}_\ell;\Omega) \,, \tag{3.26c}$$

so that the matrix structure of $\mathcal{A}_\theta$ becomes $\begin{pmatrix} \mathbf{a}_\theta & \mathbf{b}_1 \\ \mathbf{b}_2 & \mathbf{0} \end{pmatrix}$.

Gathering (3.8), (3.16), and (3.24), we arrive at the fully mixed variational formulation of (2.7): Find $(\vec{\mathbf{t}}, \boldsymbol{\sigma}) := \big((\mathbf{t}, \mathbf{u}), \boldsymbol{\sigma}\big) \in \mathbf{H} \times \mathbf{Q}$, $(\boldsymbol{\chi}, \vartheta) \in \mathrm{X}_2 \times \mathrm{M}_1$, and $\big(\vec{\boldsymbol{\rho}}, \phi\big) := \big((\boldsymbol{\rho}, \boldsymbol{\xi}), \phi\big) \in \mathbf{X}_2 \times \mathbf{Y}_1$, such that

$$\begin{array}{lcll}
\mathbf{A}_{\mathbf{u}}(\vec{\mathbf{t}}, \vec{\mathbf{s}}) \,+\, \mathbf{B}(\vec{\mathbf{s}}, \boldsymbol{\sigma}) & = & \mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta}(\vec{\mathbf{s}}) & \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,, \\
\mathbf{B}(\vec{\mathbf{t}}, \boldsymbol{\tau}) & = & \mathbf{G}(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,, \\
a(\boldsymbol{\chi}, \boldsymbol{\eta}) \,+\, b_1(\boldsymbol{\eta}, \vartheta) & = & \mathrm{F}(\boldsymbol{\eta}) & \forall\, \boldsymbol{\eta} \in \mathrm{X}_1\,, \\
b_2(\boldsymbol{\chi}, \omega) \,-\, \displaystyle\int_\Omega \mathbf{u} \cdot \boldsymbol{\chi}\, \omega & = & \mathrm{G}(\omega) & \forall\, \omega \in \mathrm{M}_2\,, \\
\mathcal{A}_\vartheta(\vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\varrho}}) \,+\, \mathcal{B}_1\big(\vec{\boldsymbol{\varrho}}, \phi\big) & = & 0 & \forall\, \vec{\boldsymbol{\varrho}} \in \mathbf{X}_1\,, \\
\mathcal{B}_2\big(\vec{\boldsymbol{\rho}}, \varphi\big) & = & \mathcal{F}(\varphi) & \forall\, \varphi \in \mathbf{Y}_2\,.
\end{array} \tag{3.27}$$

4 The continuous solvability analysis

In this section, we address the continuous well-posedness of (3.27) by means of a fixed-point strategy based on the introduction of suitable solution operators associated with each decoupled problem.

4.1 The fixed-point strategy

We begin by letting $\mathbf{S} : \mathbf{L}^4(\Omega) \times \mathbf{L}^j(\Omega) \times \mathrm{X}_2 \times \mathrm{M}_1 \to \mathbf{L}^4(\Omega)$ be the operator defined by

$$\mathbf{S}(\mathbf{z}, \boldsymbol{\varkappa}, \boldsymbol{\zeta}, \theta) \,:=\, \underset{\sim}{\mathbf{u}} \qquad \forall\, (\mathbf{z}, \boldsymbol{\varkappa}, \boldsymbol{\zeta}, \theta) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^j(\Omega) \times \mathrm{X}_2 \times \mathrm{M}_1 \,,$$

where $(\vec{\underset{\sim}{\mathbf{t}}}, \underset{\sim}{\boldsymbol{\sigma}}) \,=\, \big((\underset{\sim}{\mathbf{t}}, \underset{\sim}{\mathbf{u}}), \underset{\sim}{\boldsymbol{\sigma}}\big) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution of the problem arising from the first and second rows of (3.27) after replacing $\mathbf{A}_{\mathbf{u}}$ and $\mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta}$ by $\mathbf{A}_{\mathbf{z}}$ and $\mathbf{F}_{\boldsymbol{\varkappa},\boldsymbol{\zeta},\theta}$ respectively, that is

$$\begin{array}{lcll}
\mathbf{A}_{\mathbf{z}}(\vec{\underset{\sim}{\mathbf{t}}}, \vec{\mathbf{s}}) \,+\, \mathbf{B}(\vec{\mathbf{s}}, \underset{\sim}{\boldsymbol{\sigma}}) & = & \mathbf{F}_{\boldsymbol{\varkappa},\boldsymbol{\zeta},\theta}(\vec{\mathbf{s}}) & \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,, \\
\mathbf{B}(\vec{\underset{\sim}{\mathbf{t}}}, \boldsymbol{\tau}) & = & \mathbf{G}(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,,
\end{array} \tag{4.1}$$

Similarly, we let $\mathbf{T} : \mathbf{L}^4(\Omega) \to \mathrm{X}_2 \times \mathrm{M}_1$ be the operator given by

$$\mathbf{T}(\mathbf{z}) \,=\, (\mathbf{T}_1(\mathbf{z}), \mathbf{T}_2(\mathbf{z})) \,:=\, (\underset{\sim}{\boldsymbol{\chi}}, \underset{\sim}{\vartheta}) \qquad \forall\, \mathbf{z} \in \mathbf{L}^4(\Omega) \,, \tag{4.2}$$

where $(\underset{\sim}{\boldsymbol{\chi}}, \underset{\sim}{\vartheta}) \in \mathrm{X}_2 \times \mathrm{M}_1$ is the unique solution of the problem arising from the third and fourth rows of (3.27) after replacing $\mathbf{u}$ by $\mathbf{z}$ in the free term, that is

$$\begin{aligned} a(\underset{\sim}{\boldsymbol{\chi}}, \boldsymbol{\eta}) \,+\, b_1(\boldsymbol{\eta}, \underset{\sim}{\vartheta}) \quad &= \quad \mathrm{F}(\boldsymbol{\eta}) \qquad && \forall\, \boldsymbol{\eta} \in \mathrm{X}_1\,, \\ b_2(\underset{\sim}{\boldsymbol{\chi}}, \omega) \,-\, \int_\Omega \mathbf{z} \cdot \underset{\sim}{\boldsymbol{\chi}}\, \omega \quad &= \quad \mathrm{G}(\omega) \qquad && \forall\, \omega \in \mathrm{M}_2\,, \end{aligned} \tag{4.3}$$

Finally, we let $\boldsymbol{\mathcal{R}} : \mathrm{L}^r(\Omega) \to \mathbf{L}^j(\Omega)$ be the operator defined by

$$\boldsymbol{\mathcal{R}}(\theta) \,:=\, \underset{\sim}{\boldsymbol{\rho}} \qquad \forall\, \theta \in \mathrm{L}^r(\Omega)\,,$$

where $\big(\vec{\underset{\sim}{\boldsymbol{\rho}}}, \underset{\sim}{\phi}\big) = \big((\underset{\sim}{\boldsymbol{\rho}}, \underset{\sim}{\boldsymbol{\xi}}), \underset{\sim}{\phi}\big) \in \mathbf{X}_2 \times \mathbf{Y}_1$ is the unique solution of the problem arising from the last two rows of (3.27) after replacing $\mathcal{A}_\vartheta$ by $\mathcal{A}_\theta$, that is

$$\begin{aligned} \mathcal{A}_\theta(\vec{\underset{\sim}{\boldsymbol{\rho}}}, \vec{\boldsymbol{\varrho}}) \,+\, \mathcal{B}_1\big(\vec{\boldsymbol{\varrho}}, \underset{\sim}{\phi}\big) \quad &= \quad 0 \qquad && \forall\, \vec{\boldsymbol{\varrho}} \in \mathbf{X}_1\,, \\ \mathcal{B}_2\big(\vec{\underset{\sim}{\boldsymbol{\rho}}}, \varphi\big) \quad &= \quad \mathcal{F}(\varphi) \qquad && \forall\, \varphi \in \mathbf{Y}_2\,. \end{aligned} \tag{4.4}$$

In this way, introducing the operator $\boldsymbol{\Xi} : \mathbf{L}^4(\Omega) \to \mathbf{L}^4(\Omega)$ as

$$\boldsymbol{\Xi}(\mathbf{z}) \,:=\, \mathbf{S}(\mathbf{z}, \boldsymbol{\mathcal{R}}(\mathbf{T}_2(\mathbf{z})), \mathbf{T}_1(\mathbf{z}), \mathbf{T}_2(\mathbf{z})) \qquad \forall\, \mathbf{z} \in \mathbf{L}^4(\Omega), \tag{4.5}$$

we realize that solving (3.27) is equivalent to seeking $\mathbf{u} \in \mathbf{L}^4(\Omega)$ such that

$$\boldsymbol{\Xi}(\mathbf{u}) \,=\, \mathbf{u}\,.$$

In the next sections, we employ the Babuška–Brezzi theory in Banach spaces (cf. [5, Corollary 2.1, Theorem 2.1] and [25, Theorem 2.34] for the general and classical cases, respectively), and the Banach–Nečas–Babuška Theorem in Banach spaces as well (cf. [25, Theorem 2.6]), to establish the well-posedness of problems (4.1), (4.3), and (4.4).

4.2 Well-definedness of operator S

In what follows we apply the classical Babuška–Brezzi theory (cf. [25, Theorem 2.34]) to prove the well-posedness of (4.1). To this end, we first stress that, being its structure the same as the ones of the problems stated in [8, eq. (3.9)] and [11, eq. (3.23)], our analysis below is based on the results provided in those works. We begin with the boundedness properties of the bilinear forms and functionals involved. Indeed, successive applications of the Cauchy–Schwarz and Hölder inequalities, along with the estimate (1.5) for $t' = 4$, which makes use of the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, yield the existence of positive constants, given and denoted as

$$\begin{gathered} \|\mathbf{A}_\mathbf{z}\| \,=\, \mu + \frac{1}{2}\,\|\mathbf{z}\|_{0,4;\Omega}\,, \quad \|\mathbf{B}\| \,=\, 1, \quad \|\mathbf{G}\| \,=\, \max\big\{1, \|\mathbf{i}_4\|\big\}\ \|\mathbf{u}_D\|_{1/2,\Gamma}\,, \quad \text{and} \\ \|\mathbf{F}_{\boldsymbol{\varkappa},\zeta,\theta}\| \,=\, \kappa^{-1}\,\delta_d\,\|\boldsymbol{\varkappa}\|^2_{0,j;\Omega}\,\|\zeta\|_{0,r;\Omega} \,+\, \delta_t\,\|\theta\|_{0,r;\Omega}\,\|\mathbf{g}\|_{0,p;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\,, \end{gathered} \tag{4.6}$$

such that

$$\begin{aligned} |\mathbf{A}_\mathbf{z}(\vec{\mathbf{r}}, \vec{\mathbf{s}})| \quad &\le \quad \|\mathbf{A}_\mathbf{z}\|\,\|\vec{\mathbf{r}}\|_\mathbf{H}\,\|\vec{\mathbf{s}}\|_\mathbf{H} \qquad && \forall\, \vec{\mathbf{r}}, \vec{\mathbf{s}} \in \mathbf{H}\,, \\ |\mathbf{B}(\vec{\mathbf{s}}, \boldsymbol{\tau})| \quad &\le \quad \|\mathbf{B}\|\,\|\vec{\mathbf{s}}\|_\mathbf{H}\,\|\boldsymbol{\tau}\|_\mathbf{Q} \qquad && \forall\, (\vec{\mathbf{s}}, \boldsymbol{\tau}) \in \mathbf{H} \times \mathbf{Q}\,, \\ |\mathbf{G}(\boldsymbol{\tau})| \quad &\le \quad \|\mathbf{G}\|\,\|\boldsymbol{\tau}\|_\mathbf{Q} \qquad && \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,, \quad \text{and} \\ |\mathbf{F}_{\boldsymbol{\varkappa},\zeta,\theta}(\vec{\mathbf{s}})| \quad &\le \quad \|\mathbf{F}_{\boldsymbol{\varkappa},\zeta,\theta}\|\,\|\vec{\mathbf{s}}\|_\mathbf{H} \qquad && \forall\, \vec{\mathbf{s}} \in \mathbf{H}\,. \end{aligned} \tag{4.7}$$

Next, we aim to establish the inf-sup condition for $\mathbf{A}_{\mathbf{z}}$ required by [25, Theorem 2.34, eq. (2.28)]. Indeed, we let $\mathbf{K}$ be the kernel of the operator induced by the bilinear form $\mathbf{B}$, that is

$$\mathbf{K} \,:=\, \Big\{ \vec{\mathbf{s}} \,:=\, (\mathbf{s},\mathbf{v}) \in \mathbf{H} : \quad \mathbf{B}(\vec{\mathbf{s}},\boldsymbol{\tau}) = 0 \quad \forall\, \boldsymbol{\tau} \in \mathbf{Q} \Big\}, \tag{4.8}$$

which, making use of the definition of $\mathbf{B}$ (cf. (3.9b)), and following the same procedure as in [8, Section 3.2.1., eq. (3.12)], becomes

$$\mathbf{K} \,:=\, \Big\{ \vec{\mathbf{s}} \,:=\, (\mathbf{s},\mathbf{v}) \in \mathbf{H} : \quad \mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad \text{and} \quad \nabla\mathbf{v} = \mathbf{s} \Big\}.$$

Then, we have the following result.

Lemma 4.1. *There exists a positive constant* $\boldsymbol{\alpha}$ *such that for each* $\mathbf{z} \in \mathbf{L}^4(\Omega)$ *there holds*

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{K}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{z}}(\vec{\mathbf{r}},\vec{\mathbf{s}})}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geq\, \boldsymbol{\alpha}\,\|\vec{\mathbf{r}}\|_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{r}} \in \mathbf{K}. \tag{4.9}$$

Proof. Given $\mathbf{z} \in \mathbf{L}^4(\Omega)$, we find that for all $\vec{\mathbf{s}} = (\mathbf{s},\mathbf{v}) \in \mathbf{H}$ there holds the identity $\mathbf{s}\mathbf{z}\cdot\mathbf{v} = (\mathbf{v}\otimes\mathbf{z}) : \mathbf{s}$, which, along with the definition of $\mathbf{A}_{\mathbf{z}}$ (cf. (3.9a)), imply that

$$\mathbf{A}_{\mathbf{z}}(\vec{\mathbf{s}},\vec{\mathbf{s}}) \,=\, \mu \int_\Omega \mathbf{s} : \mathbf{s} \,=\, \mu\,\|\mathbf{s}\|_{0,\Omega}^2 \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{H}. \tag{4.10}$$

Then, letting c_P be the positive constant yielding the Friedrichs–Poincaré inequality, which says that $|\mathbf{w}|_{1,\Omega}^2 \,\geq\, c_P\,\|\mathbf{w}\|_{1,\Omega}^2$ for all $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$, and proceeding as in the derivation of [8, eq. (3.12)], we deduce from (4.10) that for any $\vec{\mathbf{s}} \in \mathbf{K}$ there holds

$$\mathbf{A}_{\mathbf{z}}(\vec{\mathbf{s}},\vec{\mathbf{s}}) \,=\, \mu\,\|\mathbf{s}\|_{0,\Omega}^2 \,=\, \frac{\mu}{2}\,\|\mathbf{s}\|_{0,\Omega}^2 + \frac{\mu}{2}\,|\mathbf{v}|_{1,\Omega}^2 \,\geq\, \frac{\mu}{2}\Big\{\|\mathbf{s}\|_{0,\Omega}^2 + \frac{c_P}{\|\mathbf{i}_4\|^2}\,\|\mathbf{v}\|_{0,4;\Omega}^2\Big\},$$

which yields the $\mathbf{K}$-ellipticity of $\mathbf{A}_{\mathbf{z}}$, that is

$$\mathbf{A}_{\mathbf{z}}(\vec{\mathbf{s}},\vec{\mathbf{s}}) \,\geq\, \boldsymbol{\alpha}\,\|\vec{\mathbf{s}}\|_{\mathbf{H}}^2 \qquad \forall\, \vec{\mathbf{s}} \in \mathbf{K}, \tag{4.11}$$

with $\boldsymbol{\alpha} \,:=\, \dfrac{\mu}{2}\min\Big\{1, \dfrac{c_P}{\|\mathbf{i}_4\|^2}\Big\}$. Finally, (4.11) readily leads to (4.9), thus completing the proof. □

We now address the inf-sup condition for $\mathbf{B}$ required by [25, Theorem 2.34, eq. (2.29)]. For this purpose we will use the fact that, for each $t \in \begin{cases} (1,+\infty) & \text{if } n = 2 \\ [6/5,+\infty) & \text{if } n = 3 \end{cases}$, there exists a positive constant C_t, depending only on Ω, such that

$$C_t\|\boldsymbol{\tau}\|_{0,\Omega}^2 \,\leq\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 \,+\, \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathrm{div}_t;\Omega). \tag{4.12}$$

Details on this inequality can be found in [8, Section 3.2.1]. Then, we have the following result.

Lemma 4.2. *There exists a positive constant* $\boldsymbol{\beta}$*, depending only on* $C_{4/3}$*, such that*

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{H}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathbf{B}(\vec{\mathbf{s}},\boldsymbol{\tau})}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geq\, \boldsymbol{\beta}\,\|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q} \,:=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega). \tag{4.13}$$

Proof. Given $\boldsymbol{\tau} \in \mathbf{Q}$, it is clear that $\boldsymbol{\tau}^{\mathrm{d}} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$. Letting $\vec{\mathbf{r}} := (-\boldsymbol{\tau}^{\mathrm{d}}, \mathbf{0}) \in \mathbf{H} := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{L}^4(\Omega)$, we get

$$\sup_{\substack{\vec{\mathbf{s}} \in \mathbf{H} \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{\mathbf{B}(\vec{\mathbf{s}}, \boldsymbol{\tau})}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geq\, \frac{\mathbf{B}(\vec{\mathbf{r}}, \boldsymbol{\tau})}{\|\vec{\mathbf{r}}\|_{\mathbf{H}}} \,=\, \frac{\displaystyle\int_\Omega \boldsymbol{\tau} : \boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}} \,=\, \frac{\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega}}{\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}} \,=\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}\,. \tag{4.14}$$

On the other hand, we also have that $\mathbf{w} := \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^{4/3}$, and therefore, defining

$$\mathbf{w}_4 \,:=\, \begin{cases} |\mathbf{w}|^{4/3-2}\mathbf{w} & \text{if} \quad \mathbf{w} \neq 0\,, \\ \mathbf{0} & \text{otherwise}\,, \end{cases}$$

it readily follows that $\mathbf{w}_4 \in \mathbf{L}^4(\Omega)$ and

$$\int_\Omega \mathbf{w}_4 \cdot \mathbf{w} \,=\, \|\mathbf{w}_4\|_{0,4;\Omega}\, \|\mathbf{w}\|_{0,4/3;\Omega}. \tag{4.15}$$

Setting now $\vec{\mathbf{r}} := (\mathbf{0}, -\mathbf{w}_4) \in \mathbf{H} = \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbf{L}^4(\Omega)$, and using (4.15) together with the definition of $\mathbf{w} \in \mathbf{L}^{4/3}(\Omega)$, we deduce

$$\sup_{\substack{\vec{\mathbf{s}} \in \mathbf{H} \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{\mathbf{B}(\vec{\mathbf{s}}, \boldsymbol{\tau})}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geq\, \frac{\mathbf{B}(\vec{\mathbf{r}}, \boldsymbol{\tau})}{\|\vec{\mathbf{r}}\|_{\mathbf{H}}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{w}_4 \cdot \mathbf{div}(\boldsymbol{\tau})}{\|\mathbf{w}_4\|_{0,4;\Omega}} \,=\, \frac{\|\mathbf{w}_4\|_{0,4;\Omega}\, \|\mathbf{w}\|_{0,4/3;\Omega}}{\|\mathbf{w}_4\|_{0,4;\Omega}} \,=\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}\,. \tag{4.16}$$

Finally, suitably combining (4.14) and (4.16), and employing (4.12) with $t = 4/3$, we are lead to (4.13) with a positive constant $\boldsymbol{\beta}$ depending only on $C_{4/3}$. $\square$

We are able now to establish the main result of this section.

Theorem 4.3. *For each* $(\mathbf{z}, \boldsymbol{\varkappa}, \boldsymbol{\zeta}, \theta) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^j(\Omega) \times \mathrm{X}_2 \times \mathrm{M}_1$ *there exists a unique* $(\vec{\underset{\sim}{\mathbf{t}}}, \underset{\sim}{\boldsymbol{\sigma}}) = \big((\underset{\sim}{\mathbf{t}}, \underset{\sim}{\mathbf{u}}), \underset{\sim}{\boldsymbol{\sigma}}\big) \in \mathbf{H} \times \mathbf{Q}$ *solution of problem* (4.1)*, and hence one can define* $\mathbf{S}(\mathbf{z}, \boldsymbol{\varkappa}, \boldsymbol{\zeta}, \theta) := \underset{\sim}{\mathbf{u}} \in \mathbf{L}^4(\Omega)$*. Moreover, there exists a positive constant* $C_{\mathbf{S}}$*, depending only on* $|\Omega|$*,* $\|\mathbf{i}_4\|$*,* μ*,* κ*,* δ_d*,* δ_t*,* $\boldsymbol{\alpha}$*, and* $\boldsymbol{\beta}$*, such that*

$$\begin{aligned} \|\mathbf{S}(\mathbf{z}, \boldsymbol{\varkappa}, \boldsymbol{\zeta}, \theta)\|_{0,4;\Omega} \,&=\, \|\underset{\sim}{\mathbf{u}}\|_{0,4;\Omega} \,\leq\, \|\vec{\underset{\sim}{\mathbf{t}}}\|_{\mathbf{H}} \,\leq\, C_{\mathbf{S}} \Big\{ \|\boldsymbol{\varkappa}\|^2_{0,j;\Omega}\, \|\boldsymbol{\zeta}\|_{0,r;\Omega} \\ &+ \|\mathbf{g}\|_{0,p;\Omega}\, \|\theta\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma} \left(1 + \|\mathbf{z}\|_{0,4;\Omega}\right) \Big\}\,. \end{aligned} \tag{4.17}$$

Proof. Given $(\mathbf{z}, \boldsymbol{\varkappa}, \boldsymbol{\zeta}, \theta) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^j(\Omega) \times \mathrm{X}_2 \times \mathrm{M}_1$, it is clear from Lemmas 4.1 and 4.2 that $\mathbf{A}_{\mathbf{z}}$ and $\mathbf{B}$ satisfy the hypotheses of [25, Theorem 2.34], which, along with the fact that $\mathbf{F}_{\boldsymbol{\varkappa},\boldsymbol{\zeta},\theta} \in \mathbf{H}'$ and $\mathbf{G} \in \mathbf{Q}'$ (cf. (4.6), (4.7)), imply the unique solvability of (4.1). Then, (4.17) follows from a straightforward application of the first a priori estimate in [25, Theorem 2.34, eq. (2.30)], and by employing the boundedness properties of $\mathbf{A}_{\mathbf{z}}$, $\mathbf{F}_{\boldsymbol{\varkappa},\boldsymbol{\zeta},\theta}$, and $\mathbf{G}$ provided in (4.6) and (4.7). $\square$

Next, we provide the a priori estimate for the component $\underset{\sim}{\boldsymbol{\sigma}}$ of the unique solution of (4.1), which will be used later on. Indeed, from the second inequality of [25, Theorem 2.34, eq. (2.30)] we deduce the existence of a positive constant $\widetilde{C}_{\mathbf{S}}$, depending only on $\|\mathbf{i}_4\|$, μ, κ, δ_d, δ_t, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$, such that

$$\|\underset{\sim}{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \leq \widetilde{C}_{\mathbf{S}} \big(1 + \|\mathbf{z}\|_{0,4;\Omega}\big) \Big\{ \|\boldsymbol{\varkappa}\|^2_{0,j;\Omega} \|\boldsymbol{\zeta}\|_{0,r;\Omega} + \|\mathbf{g}\|_{0,p;\Omega} \|\theta\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \big(1 + \|\mathbf{z}\|_{0,4;\Omega}\big) \Big\}\,. \tag{4.18}$$

4.3 Well-definedness of operator $\mathbf{T}$

We now apply the generalized Babuška–Brezzi theory (cf. [5, Theorem 2.1, Corollary 2.1]) and the Banach–Nečas–Babuška theorem (cf. [25, Theorem 2.6]) to establish the well-posedness of (4.3), equivalently, the well-definedness of $\mathbf{T}$. To this end, and since the corresponding structure is similar to that of [8, eq. (2.50)], we basically follow the same approach from [8, Section 3.2.3].

We begin by letting $\mathbb{A} : \big(\mathrm{X}_2 \times \mathrm{M}_1\big) \times \big(\mathrm{X}_1 \times \mathrm{M}_2\big) \to \mathrm{R}$ be the bounded bilinear form arising from (3.27) after adding the left-hand sides of its third and fourth rows, except the free term, that is

$$\begin{gathered}\mathbb{A}\big((\boldsymbol{\mu},\psi),(\boldsymbol{\eta},\omega)\big) \,:=\, a(\boldsymbol{\mu},\boldsymbol{\eta}) \,+\, b_1(\boldsymbol{\eta},\psi) + b_2(\boldsymbol{\mu},\omega)\\ \forall\,(\boldsymbol{\mu},\psi) \in \mathrm{X}_2 \times \mathrm{M}_1\,, \quad \forall\,(\boldsymbol{\eta},\omega) \in \mathrm{X}_1 \times \mathrm{M}_2\,.\end{gathered} \tag{4.19}$$

In addition, given $\mathbf{z} \in \mathbf{L}^4(\Omega)$, we let $\mathbb{A}_{\mathbf{z}} : \big(\mathrm{X}_2 \times \mathrm{M}_1\big) \times \big(\mathrm{X}_1 \times \mathrm{M}_2\big) \to \mathrm{R}$ be the bounded bilinear form given by all the terms on the left-hand sides of the third and fourth rows of (3.27), that is

$$\begin{gathered}\mathbb{A}_{\mathbf{z}}\big((\boldsymbol{\mu},\psi),(\boldsymbol{\eta},\omega)\big) \,:=\, \mathbb{A}\big((\boldsymbol{\mu},\psi),(\boldsymbol{\eta},\omega)\big) \,-\, \int_\Omega \mathbf{z}\cdot\boldsymbol{\mu}\,\omega\\ \forall\,(\boldsymbol{\mu},\psi) \in \mathrm{X}_2 \times \mathrm{M}_1\,, \quad \forall\,(\boldsymbol{\eta},\omega) \in \mathrm{X}_1 \times \mathrm{M}_2\,,\end{gathered} \tag{4.20}$$

and realize that (4.3) can be reformulated as: Find $(\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta}) \in \mathrm{X}_2 \times \mathrm{M}_1$ such that

$$\mathbb{A}_{\mathbf{z}}\big((\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta}),(\boldsymbol{\eta},\omega)\big) \,=\, \mathrm{F}(\boldsymbol{\eta}) + \mathrm{G}(\omega), \qquad \forall\,(\boldsymbol{\eta},\omega) \in \mathrm{X}_1 \times \mathrm{M}_2\,. \tag{4.21}$$

In order to conclude the well-posedness of (4.21), and hence of (4.3), we now aim to prove that, for a suitably chosen $\mathbf{z}$, $\mathbb{A}_{\mathbf{z}}$ satisfies the hypotheses of the Banach–Nečas–Babuška theorem, for which, in turn, it suffices to show previously that $\mathbb{A}$ induces an invertible operator. More precisely, due to the structure of $\mathbb{A}$ (cf. (4.19)), the latter is accomplished next by showing that the bilinear forms a, b_1, and b_2 satisfy the hypotheses of the generalized Babuška–Brezzi theory (cf. [5, Theorem 2.1, Corollary 2.1]). Regarding these goals, we stress that, while the respective analysis is basically the one developed in [8, Sections 2.3 and 3.2.3], for clearness we provide most details in what follows. Indeed, we first confirm the stability properties of all the forms involved by deriving, thanks to the Cauchy–Schwarz and Hölder inequalities, along with the estimate (1.6) for $t' = r$ (which yields the constant c_r that appears below), the existence of positive constants, given and denoted as

$$\|a\| \,=\, 1, \quad \|b_i\| \,=\, \kappa, \quad \|c\| \,=\, 1, \quad \|\mathrm{F}\| \,=\, \kappa\, c_r\, \|\vartheta_D\|_{1/s,r;\Gamma}\,, \quad \text{and} \quad \|\mathrm{G}\| \,=\, \kappa\, \|f\|_{0,q;\Omega}\,, \tag{4.22}$$

such that

$$\begin{array}{rcll} |a(\boldsymbol{\mu},\boldsymbol{\eta})| & \le & \|a\|\,\|\boldsymbol{\mu}\|_{\mathrm{X}_2}\,\|\boldsymbol{\eta}\|_{\mathrm{X}_1} & \forall\,(\boldsymbol{\mu},\boldsymbol{\eta}) \in \mathrm{X}_2 \times \mathrm{X}_1\,,\\ |b_i(\boldsymbol{\eta},\omega)| & \le & \|b_i\|\,\|\boldsymbol{\eta}\|_{X_i}\,\|\omega\|_{M_i} & \forall\,(\boldsymbol{\eta},\omega) \in \mathrm{X}_i \times \mathrm{M}_i\,, \quad i \in \{1,2\}\,,\\ \left|\displaystyle\int_\Omega \mathbf{z}\cdot\boldsymbol{\mu}\,\omega\right| & \le & \|c\|\,\|\mathbf{z}\|_{0,4;\Omega}\,\|\boldsymbol{\mu}\|_{\mathrm{X}_2}\,\|\omega\|_{\mathrm{M}_2} & \forall\,(\boldsymbol{\mu},\omega) \in \mathrm{X}_2 \times \mathrm{M}_2\,,\\ |\mathrm{F}(\boldsymbol{\eta})| & \le & \|\mathrm{F}\|\,\|\boldsymbol{\eta}\|_{\mathrm{X}_1} & \forall\,\boldsymbol{\eta} \in \mathrm{X}_1\,, \quad \text{and}\\ |\mathrm{G}(\omega)| & \le & \|\mathrm{G}\|\,\|\omega\|_{\mathrm{M}_2} & \forall\,\omega \in \mathrm{M}_2. \end{array} \tag{4.23}$$

It is clear from (4.23) that $\mathbb{A}$ is bounded with boundedness constant $\|\mathbb{A}\|$ depending on $\|a\|$, $\|b_1\|$, and $\|b_2\|$. In turn, $\mathbb{A}_{\mathbf{z}}$ is bounded as well with constant $\|\mathbb{A}_{\mathbf{z}}\|$ depending on $\|\mathbb{A}\|$, $\|c\|$, and $\|\mathbf{z}\|_{0,4;\Omega}$.

Furthermore, we now let K_i, $i \in \big\{1,2\big\}$, be the kernels of the operators induced by the bilinear forms b_i (cf. (3.17b)), which reduce, respectively, to

$$K_1 \,:=\, \Big\{\boldsymbol{\eta} \in \mathbf{H}^s(\mathrm{div}_s;\Omega) : \quad \mathrm{div}(\boldsymbol{\eta}) = 0 \quad \text{in} \quad \Omega\Big\}, \quad \text{and} \tag{4.24a}$$

$$K_2 \,:=\, \Big\{ \boldsymbol{\eta} \in \mathbf{H}^r(\mathrm{div}_q;\Omega) : \quad \mathrm{div}(\boldsymbol{\eta}) = 0 \quad \text{in} \quad \Omega \Big\} \,. \tag{4.24b}$$

For the inf-sup conditions to be satisfied by a in K_1 and K_2, we recall next [8, Lemma 3.3], whose proof makes use of the scalar version of [29, Theorem 3.2], which, in turn, refers to the $\mathbf{W}^{1,t}(\Omega)$-solvability of an equation in divergence form with an homogeneous Dirichlet boundary condition.

Lemma 4.4. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let t, $t' \in (1,+\infty)$ conjugate to each other with t (and hence t') lying in $\begin{cases} [4/3,4] & \text{if } n=2 \\ [3/2,3] & \text{if } n=3 \end{cases}$. Then, there exists a linear and bounded operator $D_t : \mathbf{L}^t(\Omega) \to \mathbf{L}^t(\Omega)$ such that*

$$\mathrm{div}(D_t(\mathbf{w})) \,=\, 0 \quad \mathit{in} \quad \Omega \qquad \forall\, \mathbf{w} \in \mathbf{L}^t(\Omega)\,. \tag{4.25}$$

Moreover, for each $\mathbf{z} \in \mathbf{L}^{t'}(\Omega)$ such that $\mathrm{div}(\mathbf{z}) \,=\, 0 \quad in \quad \Omega$, there holds

$$\int_\Omega \mathbf{z} \cdot D_t(\mathbf{w}) \,=\, \int_\Omega \mathbf{z} \cdot \mathbf{w} \qquad \forall\, \mathbf{w} \in \mathrm{L}^t(\Omega)\,. \tag{4.26}$$

A similar result to Lemma 4.4, whose proof employs the analogue of [29, Theorem 3.2], but for a more general elliptic operator in divergence form, and under Neumann boundary conditions, will be stated and utilized later on in Section 4.4.

To apply Lemma 4.4, we must first verify that the Lebesgue indices of the functions involved satisfy its hypotheses. Bearing in mind that $K_2 \subseteq L^r(\Omega)$ and $K_1 \subseteq L^s(\Omega)$, we require r (or equivalently its conjugate s) to lie in the prescribed ranges. From (3.6) we know that $r = 4t'/3$ with $t' > 1$, and we find that this condition is fulfilled when t' and its conjugate t are taken as indicated below

$$t' \in \begin{cases} (1,3] & \text{if } n=2 \\ [9/8,9/4] & \text{if } n=3 \end{cases}, \qquad t \in \begin{cases} [3/2,+\infty) & \text{if } n=2 \\ [9/5,9] & \text{if } n=3 \end{cases}, \tag{4.27}$$

which yields

$$r \,\in \begin{cases} (4/3,4] & \text{if } n=2 \\ [3/2,3] & \text{if } n=3 \end{cases}, \qquad s \,\in \begin{cases} [4/3,4) & \text{if } n=2 \\ [3/2,3] & \text{if } n=3 \end{cases}. \tag{4.28}$$

In addition, using from (3.13) that $p = 4t/3$, the respective ranges for p and its conjugate q become

$$p \in \begin{cases} [2,+\infty) & \text{if } n=2 \\ [12/5,12] & \text{if } n=3 \end{cases}, \qquad q \in \begin{cases} (1,2] & \text{if } n=2 \\ [12/11,12/7] & \text{if } n=3 \end{cases}. \tag{4.29}$$

Needless to say, once t (or its conjugate t') is fixed according to (4.27), the rest of the indexes, namely r, p and their respective conjugates s and q, are automatically fixed within the specified ranges.

We are now in position to show that a, b_1, and b_2 satisfy the hypotheses of [5, Theorem 2.1, Corollary 2.1]. We begin with those regarding a, which actually coincide with the inf-sup conditions proved in [8, Lemma 3.4].

Lemma 4.5. *There exists a positive constant α such that*

$$\sup_{\substack{\boldsymbol{\eta} \in K_1 \\ \boldsymbol{\eta} \neq 0}} \frac{a(\boldsymbol{\mu},\boldsymbol{\eta})}{\|\boldsymbol{\eta}\|_{\mathrm{X}_1}} \geq \alpha\, \|\boldsymbol{\mu}\|_{\mathrm{X}_2} \qquad \forall\, \boldsymbol{\mu} \in K_2\,, \quad \mathit{and} \tag{4.30a}$$

$$\sup_{\boldsymbol{\mu} \in K_2} a(\boldsymbol{\mu},\boldsymbol{\eta}) > 0 \qquad \forall\, \boldsymbol{\eta} \in K_1\,,\ \boldsymbol{\eta} \neq \mathbf{0}\,. \tag{4.30b}$$

Proof. Given $\boldsymbol{\mu} \in K_2$ (cf. (4.24b)), that is $\boldsymbol{\mu} \in \mathbf{H}^r(\mathrm{div}_q;\Omega)$, with $\mathrm{div}(\boldsymbol{\eta}) = 0$, we define

$$\boldsymbol{\mu}_s := \begin{cases} |\boldsymbol{\mu}|^{r-2}\,\boldsymbol{\mu} & \text{if } \boldsymbol{\mu} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \boldsymbol{\mu} = \mathbf{0} \end{cases}, \tag{4.31}$$

which, owing to duality arguments, it is easily seen to verify

$$\boldsymbol{\mu}_s \in \mathbf{L}^s(\Omega) \quad \text{and} \quad \int_\Omega \boldsymbol{\mu}\cdot\boldsymbol{\mu}_s = \|\boldsymbol{\mu}\|_{0,r;\Omega}^r = \|\boldsymbol{\mu}_s\|_{0,s;\Omega}^s = \|\boldsymbol{\mu}\|_{0,r;\Omega}\,\|\boldsymbol{\mu}_s\|_{0,s;\Omega}\,. \tag{4.32}$$

Then, bounding by below with $D_s(\boldsymbol{\mu}_s) \in \mathbf{L}^s(\Omega)$, which clearly belongs to K_1 due to (4.25), and using the identities from (4.26) and (4.32), along with the boundedness of D_s (cf. Lemma 4.4), we find that

$$\sup_{\substack{\boldsymbol{\eta}\in K_1\\ \boldsymbol{\eta}\neq 0}} \frac{a(\boldsymbol{\mu},\boldsymbol{\eta})}{\|\boldsymbol{\eta}\|_{\mathrm{X}_1}} \ge \frac{\displaystyle\int_\Omega \boldsymbol{\mu}\cdot D_s(\boldsymbol{\mu}_s)}{\|D_s(\boldsymbol{\mu}_s)\|_{0,s;\Omega}} = \frac{\displaystyle\int_\Omega \boldsymbol{\mu}\cdot\boldsymbol{\mu}_s}{\|D_s(\boldsymbol{\mu}_s)\|_{0,s;\Omega}} = \frac{\|\boldsymbol{\mu}\|_{0,r;\Omega}\,\|\boldsymbol{\mu}_s\|_{0,s;\Omega}}{\|D_s(\boldsymbol{\mu}_s)\|_{0,s;\Omega}} \ge \frac{1}{\|D_s\|}\,\|\boldsymbol{\mu}\|_{0,r;\Omega}\,,$$

which proves (4.30a) with $\alpha := \|D_s\|^{-1}$. Given $\boldsymbol{\eta} \in K_1 \setminus \{\mathbf{0}\}$ (cf. (4.24a)), we set, analogously as for $\boldsymbol{\mu}$, the function $\boldsymbol{\eta}_r := \begin{cases} |\boldsymbol{\eta}|^{s-2}\,\boldsymbol{\eta} & \text{if } \boldsymbol{\eta} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \boldsymbol{\eta} = \mathbf{0} \end{cases}$, which satisfies (4.32) with $\boldsymbol{\eta}$ and s instead of $\boldsymbol{\mu}$ and r, respectively. Noting again from (4.25) that $D_r(\boldsymbol{\eta}_r) \in K_2$, we proceed similarly as before to obtain

$$\sup_{\boldsymbol{\mu}\in K_2} a(\boldsymbol{\mu},\boldsymbol{\eta}) \ge a(D_r(\boldsymbol{\eta}_r),\boldsymbol{\eta}) = \int_\Omega \boldsymbol{\eta}\cdot D_r(\boldsymbol{\eta}_r) = \int_0 \boldsymbol{\eta}\cdot\boldsymbol{\eta}_r = \|\boldsymbol{\eta}\|_{0,s;\Omega}^s > 0\,,$$

which proves (4.30b) and concludes the proof. □

The continuous inf-sup conditions for the bilinear forms b_1 and b_2 are addressed next. For b_2 we use the Sobolev embedding (cf. [25, Corollary B.43], [39, Theorem 1.3.4]), which establishes, in particular, that $\mathbf{W}^{1,q}(\Omega)$ is embedded into $\mathbf{L}^r(\Omega)$ if either $1 \le q < n$ and $r \in [1, q^*]$, with $q^* := \frac{nq}{n-q}$, or $q = n$ and $r \in [1, +\infty)$. According to the expressions for r and q given by (3.6) and (3.13), we have $r = \frac{4}{3}\,t' = \frac{4t}{3(t-1)}$, $q = \frac{4t}{4t-3}$, and $q^* = \frac{4nt}{n(4t-3)-4t}$, so that, performing minor algebraic manipulations, we find that imposing the constraint $r \le q^*$ is equivalent to requiring $n \le 4$, which is certainly the case for $n \in \{2,3\}$. In this way, and since $q \le n$ (cf. (4.29)), there always holds the embedding of $\mathbf{W}^{1,q}(\Omega)$ into $\mathbf{L}^r(\Omega)$ for the ranges specified by (4.27) - (4.29).

The announced result regarding b_1 and b_2 is stated now.

Lemma 4.6. *For each $i \in \{1,2\}$ there exists a positive constant β_i such that*

$$\sup_{\substack{\boldsymbol{\eta}\in \mathrm{X}_i\\ \boldsymbol{\eta}\neq \mathbf{0}}} \frac{b_i(\boldsymbol{\eta},\omega)}{\|\boldsymbol{\eta}\|_{\mathrm{X}_i}} \ge \beta_i\,\|\omega\|_{\mathrm{M}_i} \qquad \forall\,\omega \in \mathrm{M}_i\,. \tag{4.33}$$

Proof. The proof for $i = 1$ follows similar arguments to those employed to prove Lemma 4.5. In fact, given $\omega \in \mathrm{M}_1 = \mathrm{L}^r(\Omega)$, we adopt the scalar version of (4.31) and define

$$\omega_s := \begin{cases} |\omega|^{r-2}\,\omega & \text{if } \omega \neq 0 \\ 0 & \text{if } \omega = 0 \end{cases}, \tag{4.34}$$

which is easily seen to verify the analogue of (4.32),that is

$$\omega_s \in \mathrm{L}^s(\Omega) \quad \text{and} \quad \int_\Omega \omega\,\omega_s = \|\omega\|_{0,r;\Omega}^r = \|\omega_s\|_{0,s;\Omega}^s = \|\omega\|_{0,r;\Omega}\,\|\omega_s\|_{0,s;\Omega}\,. \tag{4.35}$$

Then, a straightforward application of the scalar version of [29, Theorem 3.2] guarantees the existence of a unique $u \in \mathrm{W}^{1,s}(\Omega)$ such that $\operatorname{div}(\nabla u) = \omega_s$ in Ω, $u = 0$ on Γ, and $\|u\|_{1,s;\Omega} \leq C_s \, \|\omega_s\|_{0,s;\Omega}$, where C_s is a positive constant depending only on s. Next, defining $\widetilde{\boldsymbol{\eta}} := \nabla u \in \mathrm{L}^s(\Omega)$, it follows that $\operatorname{div}(\widetilde{\boldsymbol{\eta}}) = \omega_s$ in Ω, whence $\widetilde{\boldsymbol{\eta}} \in \mathrm{X}_1 = \mathbf{H}^s(\operatorname{div}_s;\Omega)$, and $\|\widetilde{\boldsymbol{\eta}}\|_{\mathrm{X}_1} = \|\widetilde{\boldsymbol{\eta}}\|_{s,\operatorname{div}_s;\Omega} \leq \big(1 + C_s\big) \, \|\omega_s\|_{0,s;\Omega}$. Thus, bounding by below with $\widetilde{\boldsymbol{\eta}}$, and employing the identity from (4.35), we obtain

$$\sup_{\substack{\boldsymbol{\eta} \in \mathrm{X}_1 \\ \boldsymbol{\eta} \neq \mathbf{0}}} \frac{b_1(\boldsymbol{\eta},\omega)}{\|\boldsymbol{\eta}\|_{\mathrm{X}_1}} \,\geq\, \frac{b_1(\widetilde{\boldsymbol{\eta}},\omega)}{\|\widetilde{\boldsymbol{\eta}}\|_{\mathrm{X}_1}} \,=\, \frac{\kappa \displaystyle\int_\Omega \omega\, \omega_s}{\|\widetilde{\boldsymbol{\eta}}\|_{s,\operatorname{div}_s;\Omega}} \,=\, \frac{\kappa\, \|\omega\|_{0,r;\Omega} \, \|\omega_s\|_{0,s;\Omega}}{\|\widetilde{\boldsymbol{\eta}}\|_{s,\operatorname{div}_s;\Omega}} \,,$$

which, using the a priori estimate for $\|\widetilde{\boldsymbol{\eta}}\|_{s,\operatorname{div}_s;\Omega}$, yields (4.33) for b_1 with $\beta_1 := \kappa \, \big(1 + C_s\big)^{-1}$. In turn, the proof for b_2 is basically the same of [8, Lemma 3.5], but since minor differences arise, and for sake of completeness, most of the required details are provided in what follows. Indeed, given now $\omega \in \mathrm{M}_2 = \mathrm{L}^p(\Omega)$, we define ω_q as in (4.34), but with p and q instead of r and s, respectively, which certainly satisfies the analogue of (4.35), that is

$$\omega_q \,\in\, \mathrm{L}^q(\Omega) \quad \text{and} \quad \int_\Omega \omega\, \omega_q \,=\, \|\omega\|_{0,p;\Omega}^p \,=\, \|\omega_q\|_{0,q;\Omega}^q \,=\, \|\omega\|_{0,p;\Omega} \, \|\omega_q\|_{0,q;\Omega} \,. \tag{4.36}$$

Now, let $\mathcal{O} \subset \mathrm{R}^n$ be a convex bounded domain containing $\bar{\Omega}$, and let $f \,:=\, \begin{cases} \omega_q & \text{in} \quad \Omega \,, \\ 0 & \text{in} \quad \mathcal{O} \setminus \Omega \,. \end{cases}$ Since $f \in \mathrm{L}^q(\mathcal{O})$, we know from [26, Corollary 1], valid for $q \in (1,2]$ (cf. (4.29)), that there exists a unique $z \in \mathrm{W}_0^{1,q}(\mathcal{O}) \cap \mathrm{W}^{2,q}(\mathcal{O})$ such that $\Delta z \,=\, f$ in $\mathcal{O}$, $z = 0$ on $\partial\mathcal{O}$, and for which there holds $\|z\|_{2,q;\mathcal{O}} \,\leq\, c_q \, \|f\|_{0,q;\mathcal{O}} \,=\, c_q \, \|\omega_q\|_{0,q;\Omega}$, with a positive constant c_q depending only on q and $\mathcal{O}$. Defining now $\widehat{\boldsymbol{\eta}} := \nabla z|_\Omega \,\in\, \mathbf{W}^{1,q}(\Omega)$, we can assert, according to the previously noticed embedding from $\mathbf{W}^{1,q}(\Omega)$ into $\mathbf{L}^r(\Omega)$, which we denote from now on $\mathbf{i}_{q,r}$, that $\widehat{\boldsymbol{\eta}} \in \mathbf{L}^r(\Omega)$, so that, invoking the aforementioned bound for $\|z\|_{2,q;\mathcal{O}}$, we obtain

$$\|\widehat{\boldsymbol{\eta}}\|_{0,r;\Omega} \,\leq\, \|\mathbf{i}_{q,r}\| \, \|\widehat{\boldsymbol{\eta}}\|_{1,q;\Omega} \,\leq\, \|\mathbf{i}_{q,r}\| \, \|z\|_{2,q;\Omega} \,\leq\, \|\mathbf{i}_{q,r}\| \, \|z\|_{2,q;\mathcal{O}} \,\leq\, \|\mathbf{i}_{q,r}\| \, c_q \, \|\omega_q\|_{0,q;\Omega} \,. \tag{4.37}$$

Moreover, there clearly holds $\operatorname{div}(\widehat{\boldsymbol{\eta}}) \,=\, \Delta z \,=\, f \,=\, w_q$ in Ω, and thus $\widehat{\boldsymbol{\eta}} \in \mathrm{X}_2 := \mathbf{H}^r(\operatorname{div}_q, \Omega)$, with

$$\|\widehat{\boldsymbol{\eta}}\|_{X_2} \,=\, \|\widehat{\boldsymbol{\eta}}\|_{0,r;\Omega} \,+\, \|\operatorname{div}(\widehat{\boldsymbol{\eta}})\|_{0,q;\Omega} \,=\, \|\widehat{\boldsymbol{\eta}}\|_{0,r;\Omega} \,+\, \|w_q\|_{0,q;\Omega} \,\leq\, \big(1 + \|\mathbf{i}_{q,r}\| \, c_q\big) \, \|\omega_q\|_{0,q;\Omega} \,, \tag{4.38}$$

where the estimate from (4.37) was employed in the last step yielding (4.38). Finally, bounding by below with $\widehat{\boldsymbol{\eta}}$, and using the identity from (4.36), we find that

$$\sup_{\substack{\boldsymbol{\eta} \in \mathrm{X}_2 \\ \boldsymbol{\eta} \neq \mathbf{0}}} \frac{b_2(\boldsymbol{\eta},\omega)}{\|\boldsymbol{\eta}\|_{\mathrm{X}_2}} \,\geq\, \frac{b_2(\widehat{\boldsymbol{\eta}},\omega)}{\|\widehat{\boldsymbol{\eta}}\|_{\mathrm{X}_2}} \,=\, \frac{\kappa \displaystyle\int_\Omega \omega\, \omega_q}{\|\widehat{\boldsymbol{\eta}}\|_{\mathrm{X}_2}} \,=\, \frac{\kappa\, \|\omega\|_{0,p;\Omega} \, \|\omega_q\|_{0,q;\Omega}}{\|\widehat{\boldsymbol{\eta}}\|_{\mathrm{X}_2}} \,,$$

which, employing (4.38), implies (4.33) for b_2 with $\beta_2 := \kappa \, \big(1 + \|\mathbf{i}_{q,r}\| \, c_q\big)^{-1}$. □

Bearing in mind the boundedness properties given by (4.23), and thanks to Lemmas 4.5 and 4.6, the hypotheses from [5, Theorem 2.1, Corollary 2.1] have been accomplished by the bilinear forms a, b_1, and b_2, and hence, in virtue of the corresponding a priori estimates provided in [5, Corollary 2.1, eqs. 2.15 and 2.16], we conclude the existence of a positive constant $\alpha_{\mathbf{T}}$, depending only on α, β_1, β_2, and $\|a\|$, such that the global inf-sup conditions for $\mathbb{A}$ hold, that is

$$\sup_{\substack{(\boldsymbol{\eta},\omega) \in \mathrm{X}_1 \times \mathrm{M}_2 \\ (\boldsymbol{\eta},\omega) \neq \mathbf{0}}} \frac{\mathbb{A}\big((\boldsymbol{\mu},\psi),(\boldsymbol{\eta},\omega)\big)}{\|(\boldsymbol{\eta},\omega)\|_{\mathrm{X}_1 \times \mathrm{M}_2}} \,\geq\, \alpha_{\mathbf{T}} \, \|(\boldsymbol{\mu},\psi)\|_{\mathrm{X}_2 \times \mathrm{M}_1} \qquad \forall\, (\boldsymbol{\mu},\psi) \in \mathrm{X}_2 \times \mathrm{M}_1 \,, \quad \text{and} \tag{4.39a}$$

$$\sup_{\substack{(\boldsymbol{\mu},\psi)\in \mathrm{X}_2\times\mathrm{M}_1\\(\boldsymbol{\mu},\psi)\neq\mathbf{0}}} \frac{\mathbb{A}\big((\boldsymbol{\mu},\psi),(\boldsymbol{\eta},\omega)\big)}{\|(\boldsymbol{\mu},\psi)\|_{\mathrm{X}_2\times\mathrm{M}_1}} \,\geq\, \alpha_{\mathbf{T}}\,\|(\boldsymbol{\eta},\omega)\|_{\mathrm{X}_1\times\mathrm{M}_2} \qquad \forall\,(\boldsymbol{\eta},\omega)\in \mathrm{X}_1\times\mathrm{M}_2\,. \tag{4.39b}$$

Next, according to the definition of $\mathbb{A}_{\mathbf{z}}$ (cf. (4.20)), for a given $\mathbf{z}\in\mathbf{L}^4(\Omega)$, and resorting to the bound of $\big|\int_\Omega \mathbf{z}\cdot\boldsymbol{\mu}\,\omega\big|$ provided by (4.22) and (4.23), we easily deduce from (4.39a) that

$$\sup_{\substack{(\boldsymbol{\eta},\omega)\in \mathrm{X}_1\times\mathrm{M}_2\\(\boldsymbol{\eta},\omega)\neq\mathbf{0}}} \frac{\mathbb{A}_{\mathbf{z}}\big((\boldsymbol{\mu},\psi),(\boldsymbol{\eta},\omega)\big)}{\|(\boldsymbol{\eta},\omega)\|_{\mathrm{X}_1\times\mathrm{M}_2}} \,\geq\, \big(\alpha_{\mathbf{T}}\,-\,\|\mathbf{z}\|_{0,4;\Omega}\big)\,\|(\boldsymbol{\mu},\psi)\|_{\mathrm{X}_2\times\mathrm{M}_1} \quad \forall\,(\boldsymbol{\mu},\psi)\in \mathrm{X}_2\times\mathrm{M}_1\,,$$

so that, under the following assumption

$$\|\mathbf{z}\|_{0,4;\Omega}\,\leq\,\frac{\alpha_{\mathbf{T}}}{2}\,, \tag{4.40}$$

we arrive at

$$\sup_{\substack{(\boldsymbol{\eta},\omega)\in \mathrm{X}_1\times\mathrm{M}_2\\(\boldsymbol{\eta},\omega)\neq\mathbf{0}}} \frac{\mathbb{A}_{\mathbf{z}}\big((\boldsymbol{\mu},\psi),(\boldsymbol{\eta},\omega)\big)}{\|(\boldsymbol{\eta},\omega)\|_{\mathrm{X}_1\times\mathrm{M}_2}} \,\geq\, \frac{\alpha_{\mathbf{T}}}{2}\,\|(\boldsymbol{\mu},\psi)\|_{\mathrm{X}_2\times\mathrm{M}_1} \quad \forall\,(\boldsymbol{\mu},\psi)\in \mathrm{X}_2\times\mathrm{M}_1\,. \tag{4.41}$$

Similarly, but starting from (4.39b) instead of (4.39a), and under the same assumption (4.40), we get

$$\sup_{\substack{(\boldsymbol{\mu},\psi)\in \mathrm{X}_2\times\mathrm{M}_1\\(\boldsymbol{\mu},\psi)\neq\mathbf{0}}} \frac{\mathbb{A}_{\mathbf{z}}\big((\boldsymbol{\mu},\psi),(\boldsymbol{\eta},\omega)\big)}{\|(\boldsymbol{\mu},\psi)\|_{\mathrm{X}_2\times\mathrm{M}_1}} \,\geq\, \frac{\alpha_{\mathbf{T}}}{2}\,\|(\boldsymbol{\eta},\omega)\|_{\mathrm{X}_1\times\mathrm{M}_2} \qquad \forall\,(\boldsymbol{\eta},\omega)\in \mathrm{X}_1\times\mathrm{M}_2\,. \tag{4.42}$$

Consequently, we are now in position to establish the well-definedness of operator $\mathbf{T}$.

Theorem 4.7. *For each $\mathbf{z}\in\mathbf{L}^4(\Omega)$ satisfying (4.40), there exists a unique $(\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta})\in \mathrm{X}_2\times\mathrm{M}_1$ solution of (4.21) (equivalently, (4.3)), and hence one can define $\mathbf{T}(\mathbf{z}) := (\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta})\in \mathrm{X}_2\times\mathrm{M}_1$. Moreover, there exists a positive constant $C_{\mathbf{T}}$, depending only on $\alpha_{\mathbf{T}}$, κ, and c_r, such that*

$$\|\mathbf{T}(\mathbf{z})\|_{\mathrm{X}_2\times\mathrm{M}_1}\,\leq\, C_{\mathbf{T}}\Big\{\|\vartheta_D\|_{1/s,r;\Gamma}\,+\,\|f\|_{0,q;\Omega}\Big\}\,. \tag{4.43}$$

Proof. Having proved (4.41) and (4.42), the existence of a unique solution $(\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta})\in \mathrm{X}_2\times\mathrm{M}_1$ of (4.21) follows from a straightforward application of the Banach–Nečas–Babuška theorem (cf. [25, Theorem 2.6]). In turn, the a priori estimate given by [25, eq. 2.5, Theorem 2.6], along with the boundedness properties of F and G (cf. (4.22) and (4.23)), yield (4.43) and end the proof. □

4.4 Well-definedness of operator $\boldsymbol{\mathcal{R}}$

In this section we apply once again the generalized Babuška–Brezzi theory (cf. [5, Theorem 2.1, Corollary 2.1, Section 2.1]), but now to establish the well-posedness of (4.4), equivalently, the well-definedness of $\boldsymbol{\mathcal{R}}$. In this regard, and due to the previously observed generalized twofold saddle point structure of (4.4), we stress that in this case the utilization of that theory is twice since, when applying it to the whole system (4.4), it is employed in turn, to derive the required inf-sup conditions for $\mathcal{A}_\theta$ with respect to the null spaces of the operators induced by $\mathcal{B}_1$ and $\mathcal{B}_2$.

We begin with the boundedness properties of all the forms involved. Indeed, given $\theta\in\mathrm{M}_1 := \mathrm{L}^r(\Omega)$, simple applications of Hölder's inequality, along with the upper bound for $\eta(\theta)$ (cf. (2.2)) in the particular case of $\mathbf{a}_\theta$, yield the existence of positive constants, given and denoted as

$$\begin{gathered}\|\mathbf{a}\|\,:=\,\eta_2\,,\quad \|\mathbf{b}_1\|\,:=\,1\,,\quad \|\mathbf{b}_2\|\,:=\,1\,,\quad \|\mathcal{A}\|\,:=\,\max\big\{2,1+\eta_2\big\}\,,\\ \|\mathcal{B}_1\|\,:=\,1\,,\quad \|\mathcal{B}_2\|\,:=\,1\,,\quad\text{and}\quad \|\mathcal{F}\|\,:=\,\|g\|_{0,j;\Omega}\,,\end{gathered} \tag{4.44}$$

such that

$$\begin{aligned}
|\mathbf{a}_\theta(\boldsymbol{\varkappa}, \boldsymbol{\varrho})| \,&\le\, \|\mathbf{a}\|\, \|\boldsymbol{\varkappa}\|_{0,j;\Omega}\, \|\boldsymbol{\varrho}\|_{0,\ell;\Omega} && \forall\, (\boldsymbol{\varkappa}, \boldsymbol{\varrho}) \in \mathbf{L}^j(\Omega) \times \mathbf{L}^\ell(\Omega)\,, \\
|\mathbf{b}_1(\boldsymbol{\varrho}, \boldsymbol{\varsigma})| \,&\le\, \|\mathbf{b}_1\|\, \|\boldsymbol{\varrho}\|_{0,\ell;\Omega}\, \|\boldsymbol{\varsigma}\|_{j,\mathrm{div}_j;\Omega} && \forall\, (\boldsymbol{\varrho}, \boldsymbol{\varsigma}) \in \mathbf{L}^\ell(\Omega) \times \mathbf{H}^j_0(\mathrm{div}_j;\Omega)\,, \\
|\mathbf{b}_2(\boldsymbol{\varkappa}, \boldsymbol{\lambda})| \,&\le\, \|\mathbf{b}_2\|\, \|\boldsymbol{\varkappa}\|_{0,j;\Omega}\, \|\boldsymbol{\lambda}\|_{\ell,\mathrm{div}_\ell;\Omega} && \forall\, (\boldsymbol{\varkappa}, \boldsymbol{\lambda}) \in \mathbf{L}^j(\Omega) \times \mathbf{H}^\ell_0(\mathrm{div}_\ell;\Omega)\,, \\
|\mathcal{A}_\theta(\vec{\boldsymbol{\varkappa}}, \vec{\boldsymbol{\varrho}})| \,&\le\, \|\mathcal{A}\|\, \|\vec{\boldsymbol{\varkappa}}\|_{\mathbf{X}_2}\, \|\vec{\boldsymbol{\varrho}}\|_{\mathbf{X}_1} && \forall\, (\vec{\boldsymbol{\varkappa}}, \vec{\boldsymbol{\varrho}}) \in \mathbf{X}_2 \times \mathbf{X}_1\,, \\
|\mathcal{B}_i(\vec{\boldsymbol{\varkappa}}, \psi)| \,&\le\, \|\mathcal{B}_i\|\, \|\vec{\boldsymbol{\varkappa}}\|_{\mathbf{X}_i}\, \|\psi\|_{\mathbf{Y}_i} && \forall\, (\vec{\boldsymbol{\varkappa}}, \psi) \in \mathbf{X}_i \times \mathbf{Y}_i\,, \quad i \in \big\{1, 2\big\}\,, \quad \text{and} \\
|\mathcal{F}(\varphi)| \,&\le\, \|\mathcal{F}\|\, \|\varphi\|_{0,\ell;\Omega} && \forall\, \varphi \in \mathbf{Y}_2\,.
\end{aligned} \tag{4.45}$$

Next, bearing in mind the definitions of $\mathcal{B}_1$ and $\mathcal{B}_2$ (cf. (3.25b)), we readily find that the null spaces induced by $\mathcal{B}_1$ and $\mathcal{B}_2$ become

$$\begin{aligned}
\mathcal{K}_1 \,&:=\, \Big\{ \vec{\boldsymbol{\varkappa}} := (\boldsymbol{\varkappa}, \boldsymbol{\varsigma}) \in \mathbf{X}_1 : \quad \mathcal{B}_1(\vec{\boldsymbol{\varkappa}}, \psi) \,=\, 0 \quad \forall\, \psi \in \mathbf{Y}_1 \Big\} \\
&=\, \Big\{ \vec{\boldsymbol{\varkappa}} := (\boldsymbol{\varkappa}, \boldsymbol{\varsigma}) \in \mathbf{X}_1 : \quad \mathrm{div}(\boldsymbol{\varsigma}) = 0 \quad \text{in} \quad \Omega \Big\}, \quad \text{and} \\
\mathcal{K}_2 \,&:=\, \Big\{ \vec{\boldsymbol{\varkappa}} := (\boldsymbol{\varkappa}, \boldsymbol{\varsigma}) \in \mathbf{X}_2 : \quad \mathcal{B}_2(\vec{\boldsymbol{\varkappa}}, \psi) \,=\, 0 \quad \forall\, \psi \in \mathbf{Y}_2 \Big\} \\
&=\, \Big\{ \vec{\boldsymbol{\varkappa}} := (\boldsymbol{\varkappa}, \boldsymbol{\varsigma}) \in \mathbf{X}_2 : \quad \mathrm{div}(\boldsymbol{\varsigma}) = 0 \quad \text{in} \quad \Omega \Big\},
\end{aligned} \tag{4.46}$$

so that, introducing the notation

$$\begin{aligned}
\widetilde{\mathbf{H}}^j_0(\mathrm{div}_j;\Omega) \,&:=\, \Big\{ \boldsymbol{\varsigma} \in \mathbf{H}^j_0(\mathrm{div}_j;\Omega) : \quad \mathrm{div}(\boldsymbol{\varsigma}) = 0 \quad \text{in} \quad \Omega \Big\}, \quad \text{and} \\
\widetilde{\mathbf{H}}^\ell_0(\mathrm{div}_\ell;\Omega) \,&:=\, \Big\{ \boldsymbol{\varsigma} \in \mathbf{H}^\ell_0(\mathrm{div}_\ell;\Omega) : \quad \mathrm{div}(\boldsymbol{\varsigma}) = 0 \quad \text{in} \quad \Omega \Big\},
\end{aligned} \tag{4.47}$$

they can be written equivalently as

$$\mathcal{K}_1 \,=\, \mathbf{L}^\ell(\Omega) \times \widetilde{\mathbf{H}}^\ell_0(\mathrm{div}_\ell;\Omega) \quad \text{and} \quad \mathcal{K}_2 \,=\, \mathbf{L}^j(\Omega) \times \widetilde{\mathbf{H}}^j_0(\mathrm{div}_j;\Omega)\,. \tag{4.48}$$

Then, as a first step in the verification of the hypotheses of [5, Theorem 2.1, Section 2.1] for (4.4), we aim to prove that the required inf-sup conditions for $\mathcal{A}_\theta$ on $\mathcal{K}_1$ and $\mathcal{K}_2$ (cf. [5, eqs. (2.8) and (2.9)]) are accomplished, which, in turn, reduces to establishing that $\mathbf{a}_\theta$, $\mathbf{b}_1$, and $\mathbf{b}_2$ satisfies the hypotheses of [5, Theorem 2.1, Section 2.1] in $\mathcal{K}_1 \times \mathcal{K}_2$. To this end, and particularly for the inf-sup conditions to be satisfied by $\mathbf{a}_\theta$, we need to identify the kernels of $\mathbf{b}_1$ and $\mathbf{b}_2$ when restricted to $\mathcal{K}_1 \times \mathcal{K}_2$, which we denote by V_1 and V_2, respectively. Thus, according to the definitions of these bilinear forms (cf. (3.26b), (3.26c)), and bearing in mind (4.48), we readily find the characterizations

$$\mathrm{V}_1 \,=\, \Big\{ \boldsymbol{\varrho} \in \mathbf{L}^\ell(\Omega) : \quad \int_\Omega \boldsymbol{\varrho} \cdot \boldsymbol{\varsigma} = 0 \quad \forall\, \boldsymbol{\varsigma} \in \widetilde{\mathbf{H}}^j_0(\mathrm{div}_j;\Omega) \Big\}, \quad \text{and}$$

$$\mathrm{V}_2 \,=\, \Big\{ \boldsymbol{\varkappa} \in \mathbf{L}^j(\Omega) : \quad \int_\Omega \boldsymbol{\varkappa} \cdot \boldsymbol{\lambda} = 0 \quad \forall\, \boldsymbol{\lambda} \in \widetilde{\mathbf{H}}^\ell_0(\mathrm{div}_\ell;\Omega) \Big\}.$$

Additionally, we need to invoke the following solvability result for the Neumann boundary value problem of a particular elliptic equation in divergence form.

Theorem 4.8. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2, 3\}$, and let $g \in \mathrm{L}^t(\Omega)$, $\mathbf{g} \in \mathbf{L}^t(\Omega)$, and $g_N \in \mathrm{W}^{-1/t,t}(\Gamma)$, with $t \in (1, +\infty)$, such that g and g_N satisfy the compatibility condition*

$$\int_\Omega g \,=\, \langle g_N, 1 \rangle\,,$$

where, as indicated in Section 1, $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$, with $t' \in (1, +\infty)$ being the conjugate of t. In turn, let $\xi : \Omega \to \mathrm{R}$ be a measurable function for which there *exist constants* ξ_1, ξ_2 *such that* $0 < \xi_1 \le \xi(x) \le \xi_2$ *for all* $x \in \Omega$. *Then, for each* $t \in \left\{ \begin{array}{ll} [4/3, 4] & \text{if } n = 2 \\ [3/2, 3] & \text{if } n = 3 \end{array} \right.$ *there exists a unique* $u \in \widetilde{\mathrm{W}}^{1,t}(\Omega) := \left\{ v \in \mathrm{W}^{1,t}(\Omega) : \quad \int_\Omega v = 0 \right\}$, *such that*

$$\operatorname{div}\left(\frac{1}{\xi}\,\nabla u - \mathbf{g}\right) = g \quad \text{in} \quad \Omega \quad \text{and} \quad \left(\frac{1}{\xi}\,\nabla u - \mathbf{g}\right) \cdot \boldsymbol{\nu} = g_N \quad \text{on} \quad \Gamma\,. \tag{4.49}$$

Moreover, there exists a positive constant C, *depending only on* n, t, ξ_1, ξ_2, *and* Ω, *such that*

$$\|u\|_{1,t;\Omega} \;\le\; C\left\{ \|g\|_{0,t;\Omega} + \|\mathbf{g}\|_{0,t;\Omega} + \|g_N\|_{-1/t,t;\Omega} \right\}.$$

Proof. It follows as a particular case of the more general result given by [32, Theorem 1.2]. □

We are now ready to provide the similar result to Lemma 4.4 that was announced in Section 4.3.

Lemma 4.9. *Let* Ω *be a bounded Lipschitz-continuous domain of* R^n, $n \in \{2, 3\}$, *and let* t, $t' \in (1, +\infty)$ *conjugate to each other with* t *(and hence* t'*) lying in* $\left\{ \begin{array}{ll} [4/3, 4] & \text{if } n = 2 \\ [3/2, 3] & \text{if } n = 3 \end{array} \right.$. *In addition, let* $\xi : \Omega \to \mathrm{R}$ *be as in Theorem 4.8. Then, there exists a linear and bounded operator* $D_{t,\xi} : \mathbf{L}^t(\Omega) \to \mathbf{L}^t(\Omega)$, *with* $\|D_{t,\xi}\|$ *depending only on* n, t, ξ_1, ξ_2, *and* Ω, *such that*

$$\operatorname{div}(D_{t,\xi}(\mathbf{w})) \;=\; 0 \quad \text{in} \quad \Omega \quad \text{and} \quad D_{t,\xi}(\mathbf{w}) \cdot \boldsymbol{\nu} \;=\; 0 \quad \text{on} \quad \Gamma \qquad \forall\, \mathbf{w} \in \mathbf{L}^t(\Omega)\,. \tag{4.50}$$

Moreover, for each $\mathbf{z} \in \mathbf{L}^{t'}(\Omega)$ *such that* $\operatorname{div}(\mathbf{z}) \;=\; 0 \quad \text{in} \quad \Omega$ *and* $\mathbf{z} \cdot \boldsymbol{\nu} \;=\; 0 \quad \text{on} \quad \Gamma$, *there holds*

$$\int_\Omega \xi\, \mathbf{z} \cdot D_{t,\xi}(\mathbf{w}) \;=\; \int_\Omega \xi\, \mathbf{z} \cdot \mathbf{w} \qquad \forall\, \mathbf{w} \in \mathrm{L}^t(\Omega). \tag{4.51}$$

Proof. Given $\mathbf{w} \in \mathbf{L}^t(\Omega)$, it suffices to define $D_{t,\xi}(\mathbf{w}) := \mathbf{w} - \frac{1}{\xi}\,\nabla u$, where $u \in \widetilde{\mathrm{W}}^{1,t}(\Omega)$ is the unique solution of (4.49) with $g = 0$, $\mathbf{g} = \mathbf{w}$, and $g_N = 0$. Further details are omitted. □

We plan to apply Lemma 4.9 to prove the inf-sup conditions for $\mathbf{a}_\theta$ on V_1 and V_2. This will require ℓ and j to lie in the ranges specified there. Knowing from (3.6) that $j = 8t/3$, simple algebraic computations show that $j \in \left\{ \begin{array}{ll} [4/3, 4] & \text{if } n = 2 \\ [3/2, 3] & \text{if } n = 3 \end{array} \right.$ if and only if $t \in \left\{ \begin{array}{ll} [1/2, 3/2] & \text{if } n = 2 \\ [9/16, 9/8] & \text{if } n = 3 \end{array} \right.$, which, intersected with the range for t given by (4.27), yields the only feasible choice $t = 3/2$ if $n = 2$, and no possible choice if $n = 3$. Hence, from now on we just consider $n = 2$ and $t = 3/2$, equivalently $t' = 3$, which, according to (3.6), (4.28), and (4.29), implies the following fixed Lebesgue indexes

$$j \;=\; 4\,, \quad \ell \;=\; 4/3\,, \quad r \;=\; 4\,, \quad s \;=\; 4/3\,, \quad p \;=\; 2\,, \quad \text{and} \quad q \;=\; 2\,. \tag{4.52}$$

Although (4.52) provides specific values, throughout the rest of the paper we will continue to use the generic indices j, ℓ, r, s, p, and q. This allows other choices for these indices to be tested later—for instance, to measure errors in numerical experiments. In Section 7 we suggest that (4.52) is not a necessary condition but rather a technical limitation of our analysis.

We are ready now to establish the required inf-sup conditions for $\mathbf{a}_\theta$.

Lemma 4.10. *There exists a positive constant* $\alpha_{\mathbf{a}}$ *such that for each* $\theta \in M_1 := \mathrm{L}^r(\Omega)$,

$$\sup_{\substack{\boldsymbol{\varrho} \in \mathrm{V}_1 \\ \boldsymbol{\varrho} \neq \mathbf{0}}} \frac{\mathbf{a}_\theta(\boldsymbol{\varkappa}, \boldsymbol{\varrho})}{\|\boldsymbol{\varrho}\|_{0,\ell;\Omega}} \;\ge\; \alpha_{\mathbf{a}}\, \|\boldsymbol{\varkappa}\|_{0,j;\Omega} \qquad \forall\, \boldsymbol{\varkappa} \in \mathrm{V}_2\,. \tag{4.53}$$

In addition, there holds

$$\sup_{\boldsymbol{\varkappa} \in \mathrm{V}_2} \mathbf{a}_\theta(\boldsymbol{\varkappa}, \boldsymbol{\varrho}) \;>\; 0 \qquad \forall\, \boldsymbol{\varrho} \in \mathrm{V}_1,\ \boldsymbol{\varrho} \neq \mathbf{0}\,. \tag{4.54}$$

Proof. Given $\theta \in \mathrm{M}_1$, we first consider $\boldsymbol{\varkappa} \in \mathrm{V}_2 \subseteq \mathbf{L}^j(\Omega)$, $\boldsymbol{\varkappa} \neq \mathbf{0}$, and define $\boldsymbol{\varkappa}_\ell$ as in (4.31), but with j and ℓ instead of r and s, respectively, which clearly satisfies (cf. (4.32))

$$\boldsymbol{\varkappa}_\ell \,\in\, \mathbf{L}^\ell(\Omega) \quad \text{and} \quad \int_\Omega \boldsymbol{\varkappa} \cdot \boldsymbol{\varkappa}_\ell \,=\, \|\boldsymbol{\varkappa}\|_{0,j;\Omega}\, \|\boldsymbol{\varkappa}_\ell\|_{0,\ell;\Omega}\,. \tag{4.55}$$

Then, we apply Lemma 4.9 with $t = \ell$ and $\xi := \big(\eta(\theta)\big)^{-1}$, and observe, thanks to (4.50), that $D_{\ell,\xi}(\boldsymbol{\varkappa}_\ell) \in \widetilde{\mathbf{H}}_0^\ell(\mathrm{div}_\ell;\Omega)$, which, according to the definition of V_2, yields

$$\int_\Omega \boldsymbol{\varkappa} \cdot D_{\ell,\xi}(\boldsymbol{\varkappa}_\ell) \,=\, 0\,. \tag{4.56}$$

In turn, letting $\widetilde{\boldsymbol{\varrho}} := \xi\,\big(\boldsymbol{\varkappa}_\ell - D_{\ell,\xi}(\boldsymbol{\varkappa}_\ell)\big) \in \mathbf{L}^\ell(\Omega)$, and employing (4.51), we easily deduce that for each $\boldsymbol{\varsigma} \in \widetilde{\mathbf{H}}_0^j(\mathrm{div}_j;\Omega)$ there holds

$$\int_\Omega \widetilde{\boldsymbol{\varrho}} \cdot \boldsymbol{\varsigma} = \int_\Omega \xi\, \boldsymbol{\varkappa}_\ell \cdot \boldsymbol{\varsigma} - \int_\Omega \xi\, \boldsymbol{\varsigma} \cdot D_{\ell,\xi}(\boldsymbol{\varkappa}_\ell) = \int_\Omega \xi\, \boldsymbol{\varkappa}_\ell \cdot \boldsymbol{\varsigma} - \int_\Omega \xi\, \boldsymbol{\varsigma} \cdot \boldsymbol{\varkappa}_\ell \,=\, 0\,,$$

which shows that $\widetilde{\boldsymbol{\varrho}} \in \mathrm{V}_1$. Hence, bearing in mind the definitions of $\mathbf{a}_\theta$ (cf. (3.26a)) and ξ, and using the identity from (4.55) along with (4.56), we find that

$$\mathbf{a}_\theta(\boldsymbol{\varkappa}, \widetilde{\boldsymbol{\varrho}}) \,=\, \int_\Omega \eta(\theta)\, \boldsymbol{\varkappa} \cdot \widetilde{\boldsymbol{\varrho}} \,=\, \int_\Omega \boldsymbol{\varkappa} \cdot \boldsymbol{\varkappa}_\ell \,-\, \int_\Omega \boldsymbol{\varkappa} \cdot D_{\ell,\xi}(\boldsymbol{\varkappa}_\ell) \,=\, \|\boldsymbol{\varkappa}\|_{0,j;\Omega}\, \|\boldsymbol{\varkappa}_\ell\|_{0,\ell;\Omega}\,, \tag{4.57}$$

which shows that $\widetilde{\boldsymbol{\varrho}} \neq \mathbf{0}$. Also, the lower bound of η (cf. (2.2)) and the boundedness of $D_{\ell,\xi}$ imply

$$\|\widetilde{\boldsymbol{\varrho}}\|_{0,\ell;\Omega} \,\leq\, \eta_1^{-1}\,\big(1 + \|D_{\ell,\xi}\|\big)\, \|\boldsymbol{\varkappa}_\ell\|_{0,\ell;\Omega}\,. \tag{4.58}$$

Bounding by below with $\widetilde{\boldsymbol{\varrho}}$, and invoking (4.57), we obtain

$$\sup_{\substack{\boldsymbol{\varrho} \in \mathrm{V}_1 \\ \boldsymbol{\varrho} \neq \mathbf{0}}} \frac{\mathbf{a}_\theta(\boldsymbol{\varkappa}, \boldsymbol{\varrho})}{\|\boldsymbol{\varrho}\|_{0,\ell;\Omega}} \,\geq\, \frac{\mathbf{a}_\theta(\boldsymbol{\varkappa}, \widetilde{\boldsymbol{\varrho}})}{\|\widetilde{\boldsymbol{\varrho}}\|_{0,\ell;\Omega}} \,=\, \frac{\|\boldsymbol{\varkappa}\|_{0,j;\Omega}\, \|\boldsymbol{\varkappa}_\ell\|_{0,\ell;\Omega}}{\|\widetilde{\boldsymbol{\varrho}}\|_{0,\ell;\Omega}}\,,$$

which, thanks to (4.58), implies (4.53) with $\alpha_{\mathbf{a}} := \eta_1\,\big(1 + \|D_{\ell,\xi}\|\big)^{-1}$. The proof of (4.54) proceeds in a similar manner. In fact, given $\boldsymbol{\varrho} \in \mathrm{V}_1 \subseteq \mathbf{L}^\ell(\Omega)$, $\boldsymbol{\varrho} \neq \mathbf{0}$, we define $\boldsymbol{\varrho}_j$ as in (4.31) again, but now with ℓ and j instead of r and s, and notice that (cf. (4.32))

$$\boldsymbol{\varrho}_j \in \mathbf{L}^j(\Omega) \quad \text{and} \quad \int_\Omega \boldsymbol{\varrho} \cdot \boldsymbol{\varrho}_j \,=\, \|\boldsymbol{\varrho}\|_{0,\ell;\Omega}^\ell\,.$$

Then, letting $\widetilde{\boldsymbol{\varkappa}} := \xi\,\big(\boldsymbol{\varrho}_j - D_{j,\xi}(\boldsymbol{\varrho}_j)\big)$, and proceeding similarly as before, we are able to show that

$$\sup_{\boldsymbol{\varkappa} \in \mathrm{V}_2} \mathbf{a}_\theta(\boldsymbol{\varkappa}, \boldsymbol{\varrho}) \,\geq\, \mathbf{a}_\theta(\widetilde{\boldsymbol{\varkappa}}, \boldsymbol{\varrho}) \,=\, \|\boldsymbol{\varrho}\|_{0,\ell;\Omega}^\ell \,>\, 0\,,$$

thus ending the proof. □

We address next the inf-sup conditions for $\mathbf{b}_i$, $i \in \{1,2\}$.

Lemma 4.11. *There exist positive constants* β_i, $i \in \{1,2\}$, *such that*

$$\sup_{\substack{\boldsymbol{\varrho} \in \mathbf{L}^\ell(\Omega) \\ \boldsymbol{\varrho} \neq \mathbf{0}}} \frac{\mathbf{b}_1(\boldsymbol{\varrho}, \boldsymbol{\varsigma})}{\|\boldsymbol{\varrho}\|_{0,\ell;\Omega}} \,\geq\, \beta_1\, \|\boldsymbol{\varsigma}\|_{j,\mathrm{div}_j;\Omega} \qquad \forall\, \boldsymbol{\varsigma} \in \widetilde{\mathbf{H}}_0^j(\mathrm{div}_j;\Omega)\,, \quad \textit{and} \tag{4.59a}$$

$$\sup_{\substack{\boldsymbol{\varkappa} \in \mathbf{L}^j(\Omega) \\ \boldsymbol{\varkappa} \neq \mathbf{0}}} \frac{\mathbf{b}_2(\boldsymbol{\varkappa}, \boldsymbol{\lambda})}{\|\boldsymbol{\varkappa}\|_{0,j;\Omega}} \,\geq\, \beta_2\, \|\boldsymbol{\lambda}\|_{\ell;\mathrm{div}_\ell;\Omega} \qquad \forall\, \boldsymbol{\lambda} \in \widetilde{\mathbf{H}}_0^\ell(\mathrm{div}_\ell;\Omega)\,. \tag{4.59b}$$

Proof. Given $\boldsymbol{\varsigma} \in \widetilde{\mathbf{H}}_0^j(\mathrm{div}_j;\Omega)$, we define $\boldsymbol{\varsigma}_\ell$ as in (4.31) once again, but with j and ℓ instead of r and s, so that there hold

$$\boldsymbol{\varsigma}_\ell \in \mathbf{L}^\ell(\Omega) \quad \text{and} \quad \int_\Omega \boldsymbol{\varsigma} \cdot \boldsymbol{\varsigma}_\ell \,=\, \|\boldsymbol{\varsigma}\|_{0,j;\Omega} \, \|\boldsymbol{\varsigma}_\ell\|_{0,\ell;\Omega} \,. \tag{4.60}$$

Then, invoking the definition of $\mathbf{b}_1$ (cf. (3.26b)), bounding by below with $\boldsymbol{\varsigma}_\ell$, and using the identity (4.60), we readily obtain

$$\sup_{\substack{\boldsymbol{\varrho} \in \mathbf{L}^\ell(\Omega) \\ \boldsymbol{\varrho} \neq \mathbf{0}}} \frac{\mathbf{b}_1(\boldsymbol{\varrho}, \boldsymbol{\varsigma})}{\|\boldsymbol{\varrho}\|_{0,\ell;\Omega}} \,\geq\, \frac{\mathbf{b}_1(-\boldsymbol{\varsigma}_\ell, \boldsymbol{\varsigma})}{\|\boldsymbol{\varsigma}_\ell\|_{0,\ell;\Omega}} \,=\, \frac{\displaystyle\int_\Omega \boldsymbol{\varsigma} \cdot \boldsymbol{\varsigma}_\ell}{\|\boldsymbol{\varsigma}_\ell\|_{0,\ell;\Omega}} \,=\, \|\boldsymbol{\varsigma}\|_{0,j;\Omega} \,=\, \|\boldsymbol{\varsigma}\|_{j,\mathrm{div}_j;\Omega} \,,$$

which proves (4.59a) with $\beta_1 = 1$. The proof of (4.59b) proceeds analogously by exchanging the roles of ℓ and j. Indeed, given $\boldsymbol{\lambda} \in \widetilde{\mathbf{H}}_0^\ell(\mathrm{div}_\ell;\Omega)$, we define $\boldsymbol{\lambda}_j \in \mathbf{L}^j(\Omega)$ as in (4.31), with ℓ and j instead of r and s, respectively, so that we finally obtain

$$\sup_{\substack{\boldsymbol{\varkappa} \in \mathbf{L}^j(\Omega) \\ \boldsymbol{\varkappa} \neq \mathbf{0}}} \frac{\mathbf{b}_2(\boldsymbol{\varkappa}, \boldsymbol{\lambda})}{\|\boldsymbol{\varkappa}\|_{0,j;\Omega}} \,\geq\, \frac{\mathbf{b}_2(-\boldsymbol{\lambda}_j, \boldsymbol{\lambda})}{\|\boldsymbol{\lambda}_j\|_{0,j;\Omega}} \,=\, \frac{\displaystyle\int_\Omega \boldsymbol{\lambda} \cdot \boldsymbol{\lambda}_j}{\|\boldsymbol{\lambda}_j\|_{0,j;\Omega}} \,=\, \|\boldsymbol{\lambda}\|_{0,\ell;\Omega} \,=\, \|\boldsymbol{\lambda}\|_{\ell,\mathrm{div}_\ell;\Omega} \,,$$

which proves (4.59b) with $\beta_2 = 1$. □

Having $\mathbf{a}_\theta$, $\mathbf{b}_1$, and $\mathbf{b}_2$ satisfied the hypotheses of [5, Theorem 2.1], we deduce equivalently, the bijectivity of the operator induced by $\mathcal{A}_\theta$ in $\mathcal{K}_1 \times \mathcal{K}_2$, which, thanks to the Banach–Nečas–Babuška Theorem (cf. [25, Theorem 2.6]), means, in turn, that there hold

$$\sup_{\substack{\vec{\boldsymbol{\varrho}} \in \mathcal{K}_1 \\ \vec{\boldsymbol{\varrho}} \neq \mathbf{0}}} \frac{\mathcal{A}_\theta(\vec{\boldsymbol{\varkappa}}, \vec{\boldsymbol{\varrho}})}{\|\vec{\boldsymbol{\varrho}}\|_{\mathbf{X}_1}} \,\geq\, \alpha_{\mathcal{A}} \, \|\vec{\boldsymbol{\varkappa}}\|_{\mathbf{X}_2} \qquad \forall\, \vec{\boldsymbol{\varkappa}} \in \mathcal{K}_2 \,, \quad \text{and} \tag{4.61a}$$

$$\sup_{\substack{\vec{\boldsymbol{\varkappa}} \in \mathcal{K}_2 \\ \vec{\boldsymbol{\varkappa}} \neq \mathbf{0}}} \mathcal{A}_\theta(\vec{\boldsymbol{\varkappa}}, \vec{\boldsymbol{\varrho}}) \,>\, 0 \qquad \forall\, \vec{\boldsymbol{\varrho}} \in \mathcal{K}_1 \,, \; \vec{\boldsymbol{\varrho}} \neq \mathbf{0} \,, \tag{4.61b}$$

where $\alpha_{\mathcal{A}}$ is a positive constant depending only on $\|\mathbf{a}\|$ (cf. (4.44)), $\alpha_{\mathbf{a}}$ (cf. Lemma 4.10), and β_1 and β_2 (cf. Lemma 4.11). The explicit form of $\alpha_{\mathcal{A}}$ in terms of the aforementioned constants can be derived from the a priori estimates provided by [5, Corollary 2.1, eqs. (2.15) and (2.16)].

We next address the inf-sup conditions for the bilinear forms $\mathcal{B}_i$, $i \in \{1, 2\}$.

Lemma 4.12. *There exist positive constants* $\beta_{\mathcal{B},i}$, $i \in \{1, 2\}$, *such that*

$$\sup_{\substack{\vec{\boldsymbol{\varkappa}} \in \mathbf{X}_i \\ \vec{\boldsymbol{\varkappa}} \neq \mathbf{0}}} \frac{\mathcal{B}_i(\vec{\boldsymbol{\varkappa}}, \psi)}{\|\vec{\boldsymbol{\varkappa}}\|_{\mathbf{X}_i}} \,\geq\, \beta_{\mathcal{B},i} \, \|\psi\|_{\mathbf{Y}_i} \qquad \forall\, \psi \in \mathbf{Y}_i \,. \tag{4.62}$$

Proof. Since the bilinear forms $\mathcal{B}_1$ and $\mathcal{B}_2$ share the same structure, though certainly with different spaces, the proofs of (4.62) are analogous, and hence we just proceed with $i = 2$. Given $\psi \in \mathbf{Y}_2 := \mathrm{L}_0^\ell(\Omega)$, we let ψ_j as in (4.34), with ℓ and j instead of r and s, respectively, so that (cf. (4.35))

$$\psi_j \in \mathbf{L}^j(\Omega) \quad \text{and} \quad \int_\Omega \psi \, \psi_j \,=\, \|\psi\|_{0,\ell;\Omega} \, \|\psi_j\|_{0,j;\Omega} \,.$$

Then, we define $\psi_{j,0} := \psi_j - \frac{1}{\Omega} \int_\Omega \psi_j$, which clearly belongs to $\mathrm{L}_0^j(\Omega)$, and satisfies

$$\|\psi_{j,0}\|_{0,j;\Omega} \,\leq\, c_j \, \|\psi_j\|_{0,j;\Omega} \,, \tag{4.63}$$

where c_j is a positive constant depending only on j and Ω. Hence, applying Theorem 4.8, with $\xi = 1$, $g = \psi_{j,0}$, $\mathbf{g} = \mathbf{0}$, and $g_N = 0$, we conclude that there exist a unique $z \in \widetilde{\mathrm{W}}^{1,j}(\Omega)$ and a positive constant C_j, depending only on n, j, and Ω, such that

$$\operatorname{div}(\nabla z) \,=\, \psi_{j,0} \quad \text{in} \quad \Omega\,, \quad \nabla z \cdot \boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma\,, \quad \text{and} \quad \|z\|_{1,j;\Omega} \,\le\, C_j\, \|\psi_{j,0}\|_{0,j;\Omega}\,. \tag{4.64}$$

Next, defining $\underset{\sim}{\boldsymbol{\varsigma}} := \nabla z \in \mathbf{L}^j(\Omega)$, it follows from (4.64) that $\operatorname{div}(\underset{\sim}{\boldsymbol{\varsigma}}) = \psi_{j,0}$ in Ω and $\underset{\sim}{\boldsymbol{\varsigma}} \cdot \boldsymbol{\nu} = 0$ on Γ, so that $\underset{\sim}{\boldsymbol{\varsigma}} \in \mathbf{H}^j_0(\operatorname{div}_j;\Omega)$, and, using (4.63), there holds

$$\|\underset{\sim}{\boldsymbol{\varsigma}}\|_{j,\operatorname{div}_j;\Omega} \,\le\, \big(1 + C_j\big)\, \|\psi_{j,0}\|_{0,j;\Omega} \,\le\, c_j\, \big(1 + C_j\big)\, \|\psi_j\|_{0,j;\Omega}\,. \tag{4.65}$$

Bounding by below with $\vec{\underset{\sim}{\boldsymbol{\varkappa}}} := (\mathbf{0}, -\underset{\sim}{\boldsymbol{\varsigma}}) \in \mathbf{X}_2$, bearing in mind the definition of $\mathcal{B}_2$ (cf. (3.25b)), and noting that the fact that $\psi \in \mathrm{L}^\ell_0(\Omega)$ yields $\int_\Omega \psi\,\psi_{j,0} = \int_\Omega \psi\,\psi_j$, we find that

$$\sup_{\substack{\vec{\boldsymbol{\varkappa}} \in \mathbf{X}_2\\ \vec{\boldsymbol{\varkappa}} \neq \mathbf{0}}} \frac{\mathcal{B}_2(\vec{\boldsymbol{\varkappa}}, \psi)}{\|\vec{\boldsymbol{\varkappa}}\|_{\mathbf{X}_2}} \,\ge\, \frac{\mathcal{B}_2(\vec{\underset{\sim}{\boldsymbol{\varkappa}}}, \psi)}{\|\vec{\underset{\sim}{\boldsymbol{\varkappa}}}\|_{\mathbf{X}_2}} \,=\, \frac{\displaystyle\int_\Omega \psi\,\psi_j}{\|\vec{\underset{\sim}{\boldsymbol{\varkappa}}}\|_{\mathbf{X}_2}} \,=\, \frac{\|\psi\|_{0,\ell;\Omega}\, \|\psi_j\|_{0,j;\Omega}}{\|\vec{\underset{\sim}{\boldsymbol{\varkappa}}}\|_{\mathbf{X}_2}}\,,$$

which, along with the bound for $\|\vec{\underset{\sim}{\boldsymbol{\varkappa}}}\|_{\mathbf{X}_2}$ provided by (4.65), yields (4.62) with $\beta_{\mathcal{B},2} := \big(c_j\,\big(1 + C_j\big)\big)^{-1}$. $\square$

We can establish now the well-definedness of the operator $\boldsymbol{\mathcal{R}}$.

Theorem 4.13. *For each $\theta \in \mathrm{L}^r(\Omega)$ there exists a unique $(\vec{\underset{\sim}{\boldsymbol{\rho}}}, \underset{\sim}{\phi}) = \big((\underset{\sim}{\boldsymbol{\rho}}, \underset{\sim}{\boldsymbol{\xi}}), \underset{\sim}{\phi}\big) \in \mathbf{X}_2 \times \mathbf{Y}_1$ solution of (4.4), and hence one can define $\boldsymbol{\mathcal{R}}(\theta) := \underset{\sim}{\boldsymbol{\rho}} \in \mathbf{L}^j(\Omega)$. Moreover, there exists a positive constant $C_{\boldsymbol{\mathcal{R}}}$, depending only on $\alpha_{\mathcal{A}}$, $\beta_{\mathcal{B},2}$, and $\|\mathcal{A}\|$ (cf. (4.44)), such that*

$$\|\boldsymbol{\mathcal{R}}(\theta)\|_{0,j;\Omega} \,=\, \|\underset{\sim}{\boldsymbol{\rho}}\|_{0,j;\Omega} \,\le\, \|\vec{\underset{\sim}{\boldsymbol{\rho}}}\|_{\mathbf{X}_2} \,\le\, C_{\boldsymbol{\mathcal{R}}}\, \|g\|_{0,j;\Omega}. \tag{4.66}$$

Proof. Thanks to (4.61a), (4.61b), and Lemma 4.12, the unique solvability of (4.4) follows from [5, Theorem 2.1]. In turn, noting that $\|\boldsymbol{\mathcal{R}}(\theta)\|_{0,j;\Omega} = \|\underset{\sim}{\boldsymbol{\rho}}\|_{0,j;\Omega} \le \|\vec{\underset{\sim}{\boldsymbol{\rho}}}\|_{\mathbf{X}_2}$, the a priori estimate given by [5, Corollary 2.1, eq. (2.16)], along with the boundedness of $\mathcal{F}$ (cf. (4.44), (4.45)), yield (4.66). $\square$

Similarly as in (4.18), we now establish the a priori estimate for the component $\underset{\sim}{\phi}$ of the unique solution of (4.4), which will also be utilized later on. In fact, applying [5, Corollary 2.1, eq. (2.15)], we deduce that there exists a positive constant $\widetilde{C}_{\boldsymbol{\mathcal{R}}}$, depending only on $\alpha_{\mathcal{A}}$, $\beta_{\mathcal{B},1}$, $\beta_{\mathcal{B},2}$, and $\|\mathcal{A}\|$, such that

$$\|\underset{\sim}{\phi}\|_{\mathbf{Y}_1} \,\le\, \widetilde{C}_{\boldsymbol{\mathcal{R}}}\, \|g\|_{0,j;\Omega}\,. \tag{4.67}$$

4.5 Solvability analysis of the fixed-point equation

Here we apply the classical Banach fixed-point theorem to prove that, under suitable assumptions on the data, the operator $\boldsymbol{\Xi}$ (cf. (4.5)) has a unique fixed point. We begin the analysis by establishing sufficient conditions ensuring a ball-mapping property for $\boldsymbol{\Xi}$. Bearing in mind the assumption on $\mathbf{z}$ from Theorem 4.7 (cf. (4.40)), we let $\delta := \dfrac{\alpha_{\mathbf{T}}}{2}$ and define the closed ball

$$W_\delta \,:=\, \Big\{ \mathbf{z} \in \mathbf{L}^4(\Omega) : \quad \|\mathbf{z}\|_{0,4;\Omega} \,\le\, \delta \Big\}\,. \tag{4.68}$$

Then, given $\mathbf{z} \in W_\delta$, we have from the definition of $\boldsymbol{\Xi}$ (cf. (4.5)) and the estimate for $\mathbf{S}$ (cf. (4.17)) that

$$\begin{aligned} \|\boldsymbol{\Xi}(\mathbf{z})\|_{0,4;\Omega} \,=\, \|\mathbf{S}\big(\mathbf{z}, \boldsymbol{\mathcal{R}}\big(\mathbf{T}_2(\mathbf{z})\big), \mathbf{T}_1(\mathbf{z}), \mathbf{T}_2(\mathbf{z})\big)\|_{0,4;\Omega} \,\le\, C_{\mathbf{S}} \Big\{ \|\boldsymbol{\mathcal{R}}(\mathbf{T}_2(\mathbf{z}))\|^2_{0,j;\Omega}\, \|\mathbf{T}_1(\mathbf{z})\|_{0,r;\Omega} \\ + \quad \|\mathbf{g}\|_{0,p;\Omega}\, \|\mathbf{T}_2(\mathbf{z})\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\, \big(1 + \|\mathbf{z}\|_{0,4;\Omega}\big) \Big\}\,, \end{aligned}$$

from which, using the corresponding estimates for $\mathbf{T}$ (cf. (4.43)) and $\boldsymbol{\mathcal{R}}$ (cf. (4.66)), along with (4.40), we deduce that there exists a positive constant $C(\delta)$, depending only on $C_{\mathbf{S}}$, $C_{\mathbf{T}}$, $C_{\boldsymbol{\mathcal{R}}}$, and δ, such that

$$\|\Xi(\mathbf{z})\|_{0,4;\Omega} \,\le\, C(\delta)\,\Big\{\big(\|g\|^2_{0,j;\Omega} + \|\mathbf{g}\|_{0,p;\Omega}\big)\,\big(\|\vartheta_D\|_{1/s,r;\Gamma} + \|f\|_{0,q;\Omega}\big) + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,.$$

In this way, we have proved the following result.

Lemma 4.14. *Assume that the data is sufficiently small so that*

$$C(\delta)\,\Big\{\big(\|g\|^2_{0,j;\Omega} \,+\, \|\mathbf{g}\|_{0,p;\Omega}\big)\,\big(\|\vartheta_D\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,q;\Omega}\big) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\} \,\le\, \delta\,. \tag{4.69}$$

Then, $\Xi(W_\delta) \,\subseteq\, W_\delta$.

We now aim to show that Ξ is a contraction, for which, according to its definition (cf. (4.5)), we first establish the continuity properties of $\mathbf{S}$, $\mathbf{T}$, and $\boldsymbol{\mathcal{R}}$. We begin with that of $\mathbf{S}$.

Lemma 4.15. *There exists a positive constant $L_{\mathbf{S}}$, depending only on n, κ, $C_{\mathbf{S}}$, $\boldsymbol{\alpha}$, δ_d and δ_t such that*

$$\begin{aligned}\|\mathbf{S}(\utilde{\mathbf{z}},\utilde{\boldsymbol{\varkappa}},\utilde{\boldsymbol{\zeta}},\utilde{\theta}) \,-\, \mathbf{S}(\underline{\mathbf{z}},\underline{\boldsymbol{\varkappa}},\underline{\boldsymbol{\zeta}},\underline{\theta})\|_{0,4;\Omega} \,\le\, L_{\mathbf{S}}\,\Big\{&\big(\|\utilde{\boldsymbol{\varkappa}}\|_{0,j;\Omega} \,+\, \|\underline{\boldsymbol{\varkappa}}\|_{0,j;\Omega}\big)\,\|\utilde{\boldsymbol{\zeta}}\|_{0,r;\Omega}\,\|\utilde{\boldsymbol{\varkappa}} - \underline{\boldsymbol{\varkappa}}\|_{0,j;\Omega}\\ &+\|\underline{\boldsymbol{\varkappa}}\|^2_{0,j;\Omega}\,\|\utilde{\boldsymbol{\zeta}} - \underline{\boldsymbol{\zeta}}\|_{0,r;\Omega} \,+\, \|\mathbf{g}\|_{0,p;\Omega}\,\|\utilde{\theta} - \underline{\theta}\|_{0,r;\Omega} \,+\, \mathcal{F}(\underline{\mathbf{z}},\underline{\boldsymbol{\varkappa}},\underline{\boldsymbol{\zeta}},\underline{\theta})\,\|\utilde{\mathbf{z}} - \underline{\mathbf{z}}\|_{0,4;\Omega}\Big\}\end{aligned} \tag{4.70}$$

for all $(\utilde{\mathbf{z}},\utilde{\boldsymbol{\varkappa}},\utilde{\boldsymbol{\zeta}},\utilde{\theta}), (\underline{\mathbf{z}},\underline{\boldsymbol{\varkappa}},\underline{\boldsymbol{\zeta}},\underline{\theta}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^j(\Omega) \times \mathrm{X}_2 \times \mathrm{M}_1$, *where*

$$\begin{aligned}\mathcal{F}(\underline{\mathbf{z}},\underline{\boldsymbol{\varkappa}},\underline{\boldsymbol{\zeta}},\underline{\theta}) \,:=\, C_{\mathbf{S}}\,\Big\{&\|\underline{\boldsymbol{\varkappa}}\|^2_{0,j;\Omega}\,\|\underline{\boldsymbol{\zeta}}\|_{0,r;\Omega} \,+\, \|\underline{\theta}\|_{0,r;\Omega}\,\|\mathbf{g}\|_{0,p;\Omega}\\ &+\|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\,\big(1 + \|\underline{\mathbf{z}}\|_{0,4;\Omega}\big)\Big\}\,.\end{aligned} \tag{4.71}$$

Proof. Given $(\utilde{\mathbf{z}},\utilde{\boldsymbol{\varkappa}},\utilde{\boldsymbol{\zeta}},\utilde{\theta})$, $(\underline{\mathbf{z}},\underline{\boldsymbol{\varkappa}},\underline{\boldsymbol{\zeta}},\underline{\theta}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^j(\Omega) \times \mathrm{X}_2 \times \mathrm{M}_1$, we let $(\utilde{\vec{\mathbf{t}}},\utilde{\boldsymbol{\sigma}}) \,=\, \big((\utilde{\mathbf{t}},\utilde{\mathbf{u}}),\utilde{\boldsymbol{\sigma}}\big) \in \mathbf{H} \times \mathbf{Q}$ and $(\underline{\vec{\mathbf{t}}},\underline{\boldsymbol{\sigma}}) \,=\, \big((\underline{\mathbf{t}},\underline{\mathbf{u}}),\underline{\boldsymbol{\sigma}}\big) \in \mathbf{H} \times \mathbf{Q}$ be the respective solutions of (4.1), so that $\mathbf{S}(\utilde{\mathbf{z}},\utilde{\boldsymbol{\varkappa}},\utilde{\boldsymbol{\zeta}},\utilde{\theta}) \,:=\, \utilde{\mathbf{u}}$ and $\mathbf{S}(\underline{\mathbf{z}},\underline{\boldsymbol{\varkappa}},\underline{\boldsymbol{\zeta}},\underline{\theta}) \,:=\, \underline{\mathbf{u}}$. Then, subtracting the corresponding equations defining (4.1), we obtain

$$\begin{array}{lcll}\mathbf{A}_{\utilde{\mathbf{z}}}(\utilde{\vec{\mathbf{t}}},\vec{\mathbf{s}}) \,-\, \mathbf{A}_{\underline{\mathbf{z}}}(\underline{\vec{\mathbf{t}}},\vec{\mathbf{s}}) \,+\, \mathbf{B}(\vec{\mathbf{s}},\utilde{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}) & = & \mathbf{F}_{\utilde{\boldsymbol{\varkappa}},\utilde{\boldsymbol{\zeta}},\utilde{\theta}}(\vec{\mathbf{s}}) \,-\, \mathbf{F}_{\underline{\boldsymbol{\varkappa}},\underline{\boldsymbol{\zeta}},\underline{\theta}}(\vec{\mathbf{s}}) & \forall\,\vec{\mathbf{s}} \in \mathbf{H}\,,\\ \mathbf{B}(\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}},\vec{\mathbf{s}}) & = & 0 & \forall\,\boldsymbol{\tau} \in \mathbf{Q}\,.\end{array} \tag{4.72}$$

It follows readily from the second equation of (4.72) that $\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}} \in \mathbf{K}$ (cf. (4.8)), so that taking $\vec{\mathbf{s}} := \utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}$ in the first one, and invoking the definitions of $\mathbf{F}_{\utilde{\boldsymbol{\varkappa}},\utilde{\boldsymbol{\zeta}},\utilde{\theta}}$ and $\mathbf{F}_{\underline{\boldsymbol{\varkappa}},\underline{\boldsymbol{\zeta}},\underline{\theta}}$ (cf. (3.10)), we deduce that

$$\mathbf{A}_{\utilde{\mathbf{z}}}(\utilde{\vec{\mathbf{t}}},\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}) - \mathbf{A}_{\underline{\mathbf{z}}}(\underline{\vec{\mathbf{t}}},\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}) \,=\, \kappa^{-1}\,\delta_d \int_\Omega \big(|\utilde{\boldsymbol{\varkappa}}|^2\,\utilde{\boldsymbol{\zeta}} - |\underline{\boldsymbol{\varkappa}}|^2\underline{\boldsymbol{\zeta}}\big)\cdot(\utilde{\mathbf{u}} - \underline{\mathbf{u}}) \,-\, \delta_t \int_\Omega (\utilde{\theta} - \underline{\theta})\,\mathbf{g}\cdot(\utilde{\mathbf{u}} - \underline{\mathbf{u}})\,. \tag{4.73}$$

In turn, applying the $\mathbf{K}$-ellipticity of $\mathbf{A}_{\utilde{\mathbf{z}}}$ (cf. (4.11)), employing the identity (4.73), and bearing in mind the definitions of $\mathbf{A}_{\utilde{\mathbf{z}}}$ and $\mathbf{A}_{\underline{\mathbf{z}}}$ (cf. (3.9a)), we find that

$$\begin{aligned}\boldsymbol{\alpha}\,\|\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}\|^2_{\mathbf{H}} \,\le\, \mathbf{A}_{\utilde{\mathbf{z}}}(\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}},\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}) \,&=\, \mathbf{A}_{\utilde{\mathbf{z}}}(\utilde{\vec{\mathbf{t}}},\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}) \,-\, \mathbf{A}_{\utilde{\mathbf{z}}}(\underline{\vec{\mathbf{t}}},\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}})\\ &=\, \mathbf{A}_{\utilde{\mathbf{z}}}(\utilde{\vec{\mathbf{t}}},\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}) \,-\, \mathbf{A}_{\underline{\mathbf{z}}}(\underline{\vec{\mathbf{t}}},\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}}) \,+\, \big(\mathbf{A}_{\underline{\mathbf{z}}} - \mathbf{A}_{\utilde{\mathbf{z}}}\big)(\underline{\vec{\mathbf{t}}},\utilde{\vec{\mathbf{t}}} - \underline{\vec{\mathbf{t}}})\\ &=\, \kappa^{-1}\,\delta_d \int_\Omega \big(|\utilde{\boldsymbol{\varkappa}}|^2\utilde{\boldsymbol{\zeta}} - |\underline{\boldsymbol{\varkappa}}|^2\underline{\boldsymbol{\zeta}}\big)\cdot(\utilde{\mathbf{u}} - \underline{\mathbf{u}}) \,-\, \delta_t \int_\Omega (\utilde{\theta} - \underline{\theta})\,\mathbf{g}\cdot(\utilde{\mathbf{u}} - \underline{\mathbf{u}})\\ &+\frac{1}{2}\int_\Omega \underline{\mathbf{t}}\,(\underline{\mathbf{z}} - \utilde{\mathbf{z}})\cdot(\utilde{\mathbf{u}} - \underline{\mathbf{u}}) \,-\, \frac{1}{2}\int_\Omega \underline{\mathbf{u}}\otimes(\underline{\mathbf{z}} - \utilde{\mathbf{z}}) : (\utilde{\mathbf{t}} - \underline{\mathbf{t}})\,.\end{aligned} \tag{4.74}$$

Then, adding and subtracting the term $|\underline{\boldsymbol{\varkappa}}|^2\,\underset{\sim}{\boldsymbol{\zeta}}$ and splitting the integral, we get

$$\left|\int_\Omega \big(|\underset{\sim}{\boldsymbol{\varkappa}}|^2\,\underset{\sim}{\boldsymbol{\zeta}} - |\underline{\boldsymbol{\varkappa}}|^2\underline{\boldsymbol{\zeta}}\big)\cdot(\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}})\right| \,\le\, \left|\int_\Omega \big(|\underset{\sim}{\boldsymbol{\varkappa}}|^2 - |\underline{\boldsymbol{\varkappa}}|^2\big)\,\underset{\sim}{\boldsymbol{\zeta}}\cdot(\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}})\right| + \left|\int_\Omega |\underline{\boldsymbol{\varkappa}}|^2\,(\underset{\sim}{\boldsymbol{\zeta}}-\underline{\boldsymbol{\zeta}})\cdot(\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}})\right|,$$

from which, factoring by difference of squares, using the reverse triangle inequality and applying Hölder's inequality successively in accordance with the definition of the indexes j and r (cf. (4.52)), we obtain

$$\begin{aligned}\left|\int_\Omega \big(|\underset{\sim}{\boldsymbol{\varkappa}}|^2\,\underset{\sim}{\boldsymbol{\zeta}} - |\underline{\boldsymbol{\varkappa}}|^2\underline{\boldsymbol{\zeta}}\big)\cdot(\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}})\right| \,\le\, \Big\{&\big(\|\underset{\sim}{\boldsymbol{\varkappa}}\|_{0,j;\Omega} + \|\underline{\boldsymbol{\varkappa}}\|_{0,j;\Omega}\big)\,\|\underset{\sim}{\boldsymbol{\varkappa}}-\underline{\boldsymbol{\varkappa}}\|_{0,j;\Omega}\,\|\underset{\sim}{\boldsymbol{\zeta}}\|_{0,r;\Omega}\\ &+\|\underline{\boldsymbol{\varkappa}}\|^2_{0,j;\Omega}\,\|\underset{\sim}{\boldsymbol{\zeta}}-\underline{\boldsymbol{\zeta}}\|_{0,r;\Omega}\Big\}\,\|\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}}\|_{0,4;\Omega}\,.\end{aligned} \tag{4.75}$$

Next, simple applications of the Cauchy–Schwarz inequality along with (3.18) yield

$$\begin{aligned}\left|\int_\Omega(\underset{\sim}{\theta}-\underline{\theta})\,\mathbf{g}\cdot(\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}})\right| \,&\le\, \|\underset{\sim}{\theta}-\underline{\theta}\|_{0,r;\Omega}\,\|\mathbf{g}\|_{0,p;\Omega}\,\|\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}}\|_{0,4;\Omega}\\ \left|\int_\Omega\underline{\mathbf{t}}\,(\underline{\mathbf{z}}-\underset{\sim}{\mathbf{z}})\cdot(\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}})\right| \,&\le\, \|\underline{\mathbf{t}}\|_{0;\Omega}\,\|\underset{\sim}{\mathbf{z}}-\underline{\mathbf{z}}\|_{0,4;\Omega}\,\|\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}}\|_{0,4;\Omega}\,,\quad\text{and}\\ \left|\int_\Omega\underline{\mathbf{u}}\otimes(\underline{\mathbf{z}}-\underset{\sim}{\mathbf{z}}):(\underset{\sim}{\mathbf{t}}-\underline{\mathbf{t}})\right| \,&\le\, \sqrt{n}\,\|\underline{\mathbf{u}}\|_{0,4;\Omega}\,\|\underset{\sim}{\mathbf{z}}-\underline{\mathbf{z}}\|_{0,4;\Omega}\,\|\underset{\sim}{\mathbf{t}}-\underline{\mathbf{t}}\|_{0,\Omega}\,.\end{aligned} \tag{4.76}$$

Employing the estimates (4.75) and (4.76) in (4.74), bounding $\|\underline{\mathbf{t}}\|_{0,\Omega}$ and $\|\underline{\mathbf{u}}\|_{0,4;\Omega}$ by $\|\underline{\vec{\mathbf{t}}}\|_{\mathbf{H}}$, as well as $\|\underset{\sim}{\mathbf{u}}-\underline{\mathbf{u}}\|_{0,4;\Omega}$ and $\|\underset{\sim}{\mathbf{t}}-\underline{\mathbf{t}}\|_{0,\Omega}$ by $\|\underset{\sim}{\vec{\mathbf{t}}}-\underline{\vec{\mathbf{t}}}\|_{\mathbf{H}}$, and then simplifying the latter, and using the corresponding estimation for $\|\underline{\vec{\mathbf{t}}}\|_{\mathbf{H}}$ (cf. (4.17)), we conclude the required inequality (4.70) with a positive constant $L_{\mathbf{S}}$ depending only on n, κ, $C_{\mathbf{S}}$, $\boldsymbol{\alpha}$, δ_d and δ_t, thus, ending the proof. □

The continuity of $\mathbf{T}$ is addressed next. For this, we remark that the following proof is essentially the same as the one for [8, Lemma 3.10], but we developed it for sake of completeness.

Lemma 4.16. *There exists a positive constant $L_{\mathbf{T}}$, depending only on $\alpha_{\mathbf{T}}$ and $C_{\mathbf{T}}$ such that*

$$\|\mathbf{T}(\underset{\sim}{\mathbf{z}})-\mathbf{T}(\underline{\mathbf{z}})\|_{\mathrm{X}_2\times\mathrm{M}_1} \,\le\, L_{\mathbf{T}}\Big\{\|\vartheta_D\|_{1/s,r;\Gamma} + \|f\|_{0,q;\Omega}\Big\}\,\|\underset{\sim}{\mathbf{z}}-\underline{\mathbf{z}}\|_{0,4;\Omega}, \tag{4.77}$$

for all $\underset{\sim}{\mathbf{z}}$, $\underline{\mathbf{z}}\in\mathbf{L}^4(\Omega)$ satisfying (4.40).

Proof. Given $\underset{\sim}{\mathbf{z}},\underline{\mathbf{z}}\in\mathbf{L}^4(\Omega)$, we let $\mathbf{T}(\underset{\sim}{\mathbf{z}}) := (\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta})\in\mathrm{X}_2\times\mathrm{M}_1$ and $\mathbf{T}(\underline{\mathbf{z}}) := (\underline{\boldsymbol{\chi}},\underline{\vartheta})\in\mathrm{X}_2\times\mathrm{M}_1$ be the respective solutions of (4.21), that is

$$\begin{aligned}\mathbb{A}_{\underset{\sim}{\mathbf{z}}}\big((\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta}),(\boldsymbol{\eta},\omega)\big) \,&=\, \mathrm{F}(\boldsymbol{\eta})+\mathrm{G}(\omega),\qquad\forall\,(\boldsymbol{\eta},\omega)\in\mathrm{X}_1\times\mathrm{M}_2\,,\quad\text{and}\\ \mathbb{A}_{\underline{\mathbf{z}}}\big((\underline{\boldsymbol{\chi}},\underline{\vartheta}),(\boldsymbol{\eta},\omega)\big) \,&=\, \mathrm{F}(\boldsymbol{\eta})+\mathrm{G}(\omega),\qquad\forall\,(\boldsymbol{\eta},\omega)\in\mathrm{X}_1\times\mathrm{M}_2\,.\end{aligned}$$

It follows from the foregoing identities and the definitions of $\mathbb{A}_{\underset{\sim}{\mathbf{z}}}$ and $\mathbb{A}_{\underline{\mathbf{z}}}$ (cf. (4.20)) that

$$\begin{aligned}\mathbb{A}_{\underset{\sim}{\mathbf{z}}}\big((\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta})-(\underline{\boldsymbol{\chi}},\underline{\vartheta}),(\boldsymbol{\eta},\omega)\big) \,&=\, \mathbb{A}_{\underset{\sim}{\mathbf{z}}}\big((\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta}),(\boldsymbol{\eta},\omega)\big) \,-\, \mathbb{A}_{\underset{\sim}{\mathbf{z}}}\big((\underline{\boldsymbol{\chi}},\underline{\vartheta}),(\boldsymbol{\eta},\omega)\big)\\ &=\, \mathbb{A}_{\underline{\mathbf{z}}}\big((\underline{\boldsymbol{\chi}},\underline{\vartheta}),(\boldsymbol{\eta},\omega)\big) \,-\, \mathbb{A}_{\underset{\sim}{\mathbf{z}}}\big((\underline{\boldsymbol{\chi}},\underline{\vartheta}),(\boldsymbol{\eta},\omega)\big) \,=\, \int_\Omega(\underset{\sim}{\mathbf{z}}-\underline{\mathbf{z}})\cdot\underline{\boldsymbol{\chi}}\,\omega\,,\end{aligned} \tag{4.78}$$

so that, applying the global inf-sup condition (4.41) to $\mathbb{A}_{\underset{\sim}{\mathbf{z}}}$ and $(\boldsymbol{\mu},\psi) := (\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta})-(\underline{\boldsymbol{\chi}},\underline{\vartheta})\in\mathrm{X}_2\times\mathrm{M}_1$, and using (4.78), we find that

$$\|(\underset{\sim}{\boldsymbol{\chi}},\underset{\sim}{\vartheta})-(\underline{\boldsymbol{\chi}},\underline{\vartheta})\|_{\mathrm{X}_2\times\mathrm{M}_1} \,\le\, \frac{2}{\boldsymbol{\alpha}_{\mathbf{T}}}\sup_{\substack{(\boldsymbol{\eta},\omega)\in\mathrm{X}_1\times\mathrm{M}_2\\(\boldsymbol{\eta},\omega)\neq\mathbf{0}}}\frac{\displaystyle\int_\Omega(\underset{\sim}{\mathbf{z}}-\underline{\mathbf{z}})\cdot\underline{\boldsymbol{\chi}}\,\omega}{\|(\boldsymbol{\eta},\omega)\|_{\mathrm{X}_1\times\mathrm{M}_2}}\,.$$

Thus, proceeding as for the derivation of (3.12), and bounding $\|\underline{\boldsymbol{\chi}}\|_{0,r;\Omega}$ by $\|(\underline{\boldsymbol{\chi}},\underline{\vartheta})\|_{\mathrm{X}_2\times\mathrm{M}_1}$, we get

$$\|(\utilde{\boldsymbol{\chi}},\utilde{\vartheta}) - (\underline{\boldsymbol{\chi}},\underline{\vartheta})\|_{\mathrm{X}_2\times\mathrm{M}_1} \,\le\, \frac{2}{\boldsymbol{\alpha}_{\mathbf{T}}}\|(\underline{\boldsymbol{\chi}},\underline{\vartheta})\|_{\mathrm{X}_2\times\mathrm{M}_1}\,\|\utilde{\mathbf{z}} - \underline{\mathbf{z}}\|_{0,4;\Omega}\,.$$

from which, along with the a priori estimate (4.43) for $\|(\underline{\boldsymbol{\chi}},\underline{\vartheta})\|_{\mathrm{X}_2\times\mathrm{M}_1} \,=\, \|\mathbf{T}(\underline{\mathbf{z}})\|_{\mathrm{X}_2\times\mathrm{M}_1}$, we arrive at (4.77) with $L_{\mathbf{T}} \,:=\, 2\,\boldsymbol{\alpha}_{\mathbf{T}}^{-1}\,C_{\mathbf{T}}$. □

In order to examine the continuity of the operator $\boldsymbol{\mathcal{R}}$, we require a continuous embedding $\mathbf{i}_\epsilon : \mathbf{W}^{\epsilon,j}(\Omega) \to \mathbf{C}^0(\bar{\Omega})$, for some $\epsilon > 0$, and further smoothness on the solution of (4.4). The former is guaranteed for $\epsilon > \frac{n}{j}$ by Rellich–Kondrachov Theorem (cf. [25, Theorem B.46]), whereas for the latter, given $g \in \mathrm{W}^{\epsilon,j}(\Omega)$, we assume the following regularity hypothesis.

(**RH**) for each $\theta \in \mathrm{L}^r(\Omega)$, the solution $(\utilde{\vec{\boldsymbol{\rho}}},\utilde{\phi}) \,:=\, \big((\utilde{\boldsymbol{\rho}},\utilde{\boldsymbol{\xi}}),\utilde{\phi}\big) \in \mathbf{X}_2\times\mathbf{Y}_1$ of problem (4.4) satisfies $\utilde{\boldsymbol{\rho}} \in \mathbf{W}^{\epsilon,j}(\Omega)$, $\utilde{\boldsymbol{\xi}} \in \mathbf{W}^{\epsilon,j}(\Omega)\cap\mathbf{H}_0^j(\mathrm{div}_j;\Omega)$, $\utilde{\phi} \in \mathrm{W}^{\epsilon,j}(\Omega)\cap\mathrm{L}_0^j(\Omega)$; and there exists a constant $C_\epsilon > 0$ such that

$$\|\utilde{\boldsymbol{\rho}}\|_{\epsilon,j;\Omega} \,+\, \|\utilde{\boldsymbol{\xi}}\|_{\epsilon,j;\Omega} \,+\, \|\utilde{\phi}\|_{\epsilon,j;\Omega} \,\le\, C_\epsilon\,\|g\|_{\epsilon,j;\Omega}\,. \tag{4.79}$$

We are now in a position to establish the following Lemma.

Lemma 4.17. *There exists a positive constant* $L_{\boldsymbol{\mathcal{R}}}$*, depending only on* L_η (cf. (2.2)), $\|\mathbf{i}_\epsilon\|, C_\epsilon$, $\alpha_{\mathcal{A}}$ (cf. (4.61a))*, such that*

$$\|\boldsymbol{\mathcal{R}}(\utilde{\theta}) - \boldsymbol{\mathcal{R}}(\underline{\theta})\|_{0,j;\Omega} \,\le\, L_{\boldsymbol{\mathcal{R}}}\,\|g\|_{\epsilon,j;\Omega}\,\|\utilde{\theta} - \underline{\theta}\|_{0,r;\Omega}, \tag{4.80}$$

for all $\utilde{\theta},\underline{\theta} \in \mathrm{L}^r(\Omega)$*.*

Proof. Given $\utilde{\theta}$, $\underline{\theta} \in \mathrm{L}^r(\Omega)$, we let $(\utilde{\vec{\boldsymbol{\rho}}},\utilde{\phi}) \,=\, \big((\utilde{\boldsymbol{\rho}},\utilde{\boldsymbol{\xi}}),\utilde{\phi}\big) \in \mathbf{X}_2\times\mathbf{Y}_1$ and $(\underline{\vec{\boldsymbol{\rho}}},\underline{\phi}) \,=\, \big((\underline{\boldsymbol{\rho}},\underline{\boldsymbol{\xi}}),\underline{\phi}\big) \in \mathbf{X}_2\times\mathbf{Y}_1$ be the respective solutions of (4.4), so that

$$\boldsymbol{\mathcal{R}}(\utilde{\theta}) \,:=\, \utilde{\boldsymbol{\rho}} \quad \text{and} \quad \boldsymbol{\mathcal{R}}(\underline{\theta}) \,:=\, \underline{\boldsymbol{\rho}}\,. \tag{4.81}$$

Thus, the subtraction of the equations forming (4.4) with $\utilde{\theta}$ and $\underline{\theta}$, leads to

$$\begin{aligned} \mathcal{A}_{\utilde{\theta}}(\utilde{\vec{\boldsymbol{\rho}}},\vec{\boldsymbol{\varrho}}) \,-\, \mathcal{A}_{\underline{\theta}}(\underline{\vec{\boldsymbol{\rho}}},\vec{\boldsymbol{\varrho}}) \,+\, \mathcal{B}_1(\vec{\boldsymbol{\varrho}},\utilde{\phi} - \underline{\phi}) \quad &= \quad 0 \qquad \forall\,\vec{\boldsymbol{\varrho}} \in \mathbf{X}_1\,,\\ \mathcal{B}_2\big(\utilde{\vec{\boldsymbol{\rho}}} - \underline{\vec{\boldsymbol{\rho}}},\varphi\big) \quad &= \quad 0 \qquad \forall\,\varphi \in \mathbf{Y}_2\,. \end{aligned} \tag{4.82}$$

The first equation of (4.82) ensures that $\mathcal{A}_{\utilde{\theta}}(\utilde{\vec{\boldsymbol{\rho}}},\vec{\boldsymbol{\varrho}}) \,=\, \mathcal{A}_{\underline{\theta}}(\underline{\vec{\boldsymbol{\rho}}},\vec{\boldsymbol{\varrho}})$ for all $\vec{\boldsymbol{\varrho}} \in \mathcal{K}_1$ (cf. (4.46)), from which, similarly as for (4.78) but now employing the definitions of $\mathcal{A}_{\utilde{\theta}}$ and $\mathcal{A}_{\underline{\theta}}$ (cf. (3.25a)) , we obtain

$$\begin{aligned} \mathcal{A}_{\utilde{\theta}}(\utilde{\vec{\boldsymbol{\rho}}} - \underline{\vec{\boldsymbol{\rho}}},\vec{\boldsymbol{\varrho}}) \,&=\, \mathcal{A}_{\utilde{\theta}}(\utilde{\vec{\boldsymbol{\rho}}},\vec{\boldsymbol{\varrho}}) - \mathcal{A}_{\utilde{\theta}}(\underline{\vec{\boldsymbol{\rho}}},\vec{\boldsymbol{\varrho}}) \,=\, \mathcal{A}_{\underline{\theta}}(\underline{\vec{\boldsymbol{\rho}}},\vec{\boldsymbol{\varrho}}) - \mathcal{A}_{\utilde{\theta}}(\underline{\vec{\boldsymbol{\rho}}},\vec{\boldsymbol{\varrho}})\\ &=\, \int_\Omega \big(\eta(\underline{\theta}) - \eta(\utilde{\theta})\big)\,\underline{\boldsymbol{\rho}}\cdot\boldsymbol{\varrho} \qquad \forall\,\vec{\boldsymbol{\varrho}} \in \mathcal{K}_1\,. \end{aligned} \tag{4.83}$$

Then, noting from the second equation of (4.82) that $\vec{\boldsymbol{\varkappa}} \,:=\, \utilde{\vec{\boldsymbol{\rho}}} - \underline{\vec{\boldsymbol{\rho}}} \in \mathcal{K}_2$ (cf. (4.46)), we apply the inf-sup condition for $\mathcal{A}_{\utilde{\theta}}$ (cf. (4.61a)) with $\vec{\boldsymbol{\varkappa}} \in \mathcal{K}_2$, along with (4.83), to get

$$\alpha_{\mathcal{A}}\|\utilde{\vec{\boldsymbol{\rho}}} - \underline{\vec{\boldsymbol{\rho}}}\|_{\mathbf{X}_2} \,\le\, \sup_{\substack{\vec{\boldsymbol{\varrho}}\in\mathcal{K}_1\\ \vec{\boldsymbol{\varrho}}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \big(\eta(\underline{\theta}) - \eta(\utilde{\theta})\big)\,\underline{\boldsymbol{\rho}}\cdot\boldsymbol{\varrho}}{\|\vec{\boldsymbol{\varrho}}\|_{\mathbf{X}_1}}\,,$$

which, employing the Lipschitz continuity property for η (cf. (2.2)), applying Hölder's inequality with j and ℓ, bounding $\|\boldsymbol{\varrho}\|_{0,\ell;\Omega}$ by $\|\vec{\boldsymbol{\varrho}}\|_{\mathbf{X}_1}$ and simplifying it, yields

$$\alpha_{\mathcal{A}}\|\vec{\underset{\sim}{\boldsymbol{\rho}}} - \vec{\underline{\boldsymbol{\rho}}}\|_{\mathbf{X}_2} \,\le\, L_\eta \big\|\, |\underset{\sim}{\theta} - \underline{\theta}|\, |\underline{\boldsymbol{\rho}}|\, \big\|_{0,j;\Omega} \,. \tag{4.84}$$

Finally, knowing from (**RH**) that $\underline{\boldsymbol{\rho}} \in \mathbf{W}^{\epsilon,j}(\Omega)$, and using the embedding $\mathbf{i}_\epsilon$, the fact that $j = r$ (cf. (4.52)), and the a priori estimate (4.79), we get

$$\begin{aligned}\big\|\, |\underset{\sim}{\theta} - \underline{\theta}|\, |\underline{\boldsymbol{\rho}}|\, \big\|_{0,j;\Omega} \,&\le\, \|\underset{\sim}{\theta} - \underline{\theta}\|_{0,r;\Omega}\, \|\underline{\boldsymbol{\rho}}\|_{0,\infty;\Omega} \,\le\, \|\underset{\sim}{\theta} - \underline{\theta}\|_{0,r;\Omega}\, \|\mathbf{i}_\epsilon\|\, \|\underline{\boldsymbol{\rho}}\|_{\epsilon,j;\Omega} \\ &\le\, \|\underset{\sim}{\theta} - \underline{\theta}\|_{0,r;\Omega}\, \|\mathbf{i}_\epsilon\|\, C_\epsilon\, \|g\|_{\epsilon,j;\Omega}\,,\end{aligned}$$

which, replaced back into (4.84), and invoking (4.81), yields (4.80) with $L_{\boldsymbol{\mathcal{R}}} := L_\eta\, \|\mathbf{i}_\epsilon\|\, C_\epsilon\, \alpha_{\mathcal{A}}^{-1}$. □

We now establish the Lipschitz continuity property for the fixed point operator $\boldsymbol{\Xi}$ in the closed ball W_δ, which follows by a straightforward application of Lemmas 4.15 - 4.17, and the regularity hypothesis (**RH**). Indeed, given $\underset{\sim}{\mathbf{z}}, \underline{\mathbf{z}} \in W_\delta$ (cf. (4.68)), we deduce from the definition of $\boldsymbol{\Xi}$ (cf. (4.5)) and the continuity property of $\mathbf{S}$ (cf. Lemma 4.15, (4.70)) that

$$\begin{aligned}\|\boldsymbol{\Xi}(\underset{\sim}{\mathbf{z}}) - \boldsymbol{\Xi}(\underline{\mathbf{z}})\|_{0,4;\Omega} \,\le\, L_{\mathbf{S}} \Big\{ &\Big(\|\boldsymbol{\mathcal{R}}(\mathbf{T}_2(\underset{\sim}{\mathbf{z}}))\|_{0,j;\Omega} \,+\, \|\boldsymbol{\mathcal{R}}(\mathbf{T}_2(\underline{\mathbf{z}}))\|_{0,j;\Omega}\Big)\, \|\mathbf{T}_1(\underset{\sim}{\mathbf{z}})\|_{0,r;\Omega} \\ &\times \|\boldsymbol{\mathcal{R}}(\mathbf{T}_2(\underset{\sim}{\mathbf{z}})) - \boldsymbol{\mathcal{R}}(\mathbf{T}_2(\underline{\mathbf{z}}))\|_{0,j;\Omega} \,+\, \|\boldsymbol{\mathcal{R}}(\mathbf{T}_2(\underline{\mathbf{z}}))\|^2_{0,j;\Omega}\, \|\mathbf{T}_1(\underset{\sim}{\mathbf{z}}) - \mathbf{T}_1(\underline{\mathbf{z}})\|_{0,r;\Omega} \\ &+ \|\mathbf{g}\|_{0,p;\Omega}\, \|\mathbf{T}_2(\underset{\sim}{\mathbf{z}}) - \mathbf{T}_2(\underline{\mathbf{z}})\|_{0,r;\Omega} \,+\, \mathcal{F}\big(\underline{\mathbf{z}}, \boldsymbol{\mathcal{R}}(\mathbf{T}_2(\underline{\mathbf{z}})), \mathbf{T}_1(\underline{\mathbf{z}}), \mathbf{T}_2(\underline{\mathbf{z}})\big)\, \|\underset{\sim}{\mathbf{z}} - \underline{\mathbf{z}}\|_{0,4;\Omega}\Big\}\,.\end{aligned} \tag{4.85}$$

Then, since $\mathbf{T}_2(\underset{\sim}{\mathbf{z}}), \mathbf{T}_2(\underline{\mathbf{z}}) \in \mathrm{M}_1 := \mathrm{L}^r(\Omega)$ (cf. (4.2)), the continuity of $\boldsymbol{\mathcal{R}}$ (cf. Lemma 4.17) implies that

$$\|\boldsymbol{\mathcal{R}}(\mathbf{T}_2(\underset{\sim}{\mathbf{z}})) - \boldsymbol{\mathcal{R}}(\mathbf{T}_2(\underline{\mathbf{z}}))\|_{0,j;\Omega} \,\le\, L_{\boldsymbol{\mathcal{R}}}\, \|g\|_{\epsilon,j;\Omega}\, \|\mathbf{T}_2(\underset{\sim}{\mathbf{z}}) - \mathbf{T}_2(\underline{\mathbf{z}})\|_{0,r;\Omega}\,. \tag{4.86}$$

On the other hand, the identity (4.71), the a priori estimates of $\mathbf{T}$ (cf. (4.43)) and $\boldsymbol{\mathcal{R}}$ (cf. (4.66)), and the fact that $\|\underline{\mathbf{z}}\|_{0,4;\Omega} \le \delta$, gives the existence of a positive constant $\underline{C}(\delta)$ depending on $C_{\mathbf{S}}$, $C_{\mathbf{T}}$, $C_{\boldsymbol{\mathcal{R}}}$, and δ, such that, denoting

$$\mathtt{D}(\vartheta_D, f) \,:=\, \Big\{\|\vartheta_D\|_{1/s,r;\Gamma} + \|f\|_{0,q;\Omega}\Big\}\,, \tag{4.87}$$

we find that

$$\mathcal{F}\big(\underline{\mathbf{z}}, \boldsymbol{\mathcal{R}}(\mathbf{T}_2(\underline{\mathbf{z}})), \mathbf{T}_1(\underline{\mathbf{z}}), \mathbf{T}_2(\underline{\mathbf{z}})\big) \,\le\, \underline{C}(\delta)\Big\{\big(\|g\|^2_{0,j;\Omega} \,+\, \|\mathbf{g}\|_{0,p;\Omega}\big)\, \mathtt{D}(\vartheta_D, f) + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,. \tag{4.88}$$

Substituting (4.86) back into (4.85), we then apply the continuity of $\mathbf{T}$ (cf. Lemma 4.16) along with the estimates of $\mathbf{T}$ (cf. (4.43)) and $\boldsymbol{\mathcal{R}}$ (cf. (4.66). In the resulting inequality, we use (4.88) and then factorize with respect to $\|\underset{\sim}{\mathbf{z}} - \underline{\mathbf{z}}\|_{0,4;\Omega}$, which leads to the existence of a positive constant $L_{\boldsymbol{\Xi}}$ depending on $L_{\mathbf{S}}$, $L_{\mathbf{T}}$, $L_{\boldsymbol{\mathcal{R}}}$, $C_{\mathbf{S}}$, $C_{\mathbf{T}}$, $C_{\boldsymbol{\mathcal{R}}}$ and $\underline{C}(\delta)$ such that

$$\begin{aligned}\|\boldsymbol{\Xi}(\underset{\sim}{\mathbf{z}}) - \boldsymbol{\Xi}(\underline{\mathbf{z}})\|_{0,4;\Omega} \,\le\, L_{\boldsymbol{\Xi}} \Big\{ &\Big(\|g\|_{0,j;\Omega}\, \|g\|_{\epsilon,j;\Omega}\, \mathtt{D}(\vartheta_D, f) \,+\, \|g\|^2_{0,j;\Omega} \,+\, \|\mathbf{g}\|_{0,p;\Omega}\Big)\, \mathtt{D}(\vartheta_D, f) \\ &+ \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\, \|\underset{\sim}{\mathbf{z}} - \underline{\mathbf{z}}\|_{0,4;\Omega}\,.\end{aligned} \tag{4.89}$$

Moreover, letting $\widehat{C}_\epsilon$ be the positive constant such that

$$\|g\|_{0,j;\Omega} \,\le\, \widehat{C}_\epsilon\, \|g\|_{\epsilon,j;\Omega},$$

and denoting $\widehat{L}_{\boldsymbol{\Xi}} = L_{\boldsymbol{\Xi}} \max\big\{\widehat{C}_\epsilon, 1\big\}$, the inequality (4.89) simplifies to

$$\begin{aligned}\|\boldsymbol{\Xi}(\underset{\sim}{\mathbf{z}}) - \boldsymbol{\Xi}(\underline{\mathbf{z}})\|_{0,4;\Omega} \,\le\, \widehat{L}_{\boldsymbol{\Xi}} \Big\{ &\Big(\|g\|^2_{\epsilon,j;\Omega}\, \big(\mathtt{D}(\vartheta_D, f) \,+\, 1\big) \,+\, \|\mathbf{g}\|_{0,p;\Omega}\Big)\, \mathtt{D}(\vartheta_D, f) \\ &+ \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\, \|\underset{\sim}{\mathbf{z}} - \underline{\mathbf{z}}\|_{0,4;\Omega} \qquad \forall\, \underset{\sim}{\mathbf{z}},\, \underline{\mathbf{z}} \in W_\delta\,.\end{aligned} \tag{4.90}$$

We now state the main result of this section.

Theorem 4.18. *Assume* (**RH**) *and that the data are sufficiently small so that there hold* (4.69) *and*

$$\widehat{L}_{\boldsymbol{\Xi}} \left\{ \Big(\|g\|^2_{\epsilon,j;\Omega} \left(\mathtt{D}(\vartheta_D, f) \,+\, 1 \right) \,+\, \|\mathbf{g}\|_{0,p;\Omega} \Big) \, \mathtt{D}(\vartheta_D, f) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma} \right\} \,<\, 1. \tag{4.91}$$

Then, the operator $\boldsymbol{\Xi}$ *has a unique fixed point* $\mathbf{u} \in W_\delta$*. Equivalently, the coupled system* (3.27) *has a unique solution* $(\vec{\mathbf{t}}, \boldsymbol{\sigma}) := \big((\mathbf{t}, \mathbf{u}), \boldsymbol{\sigma}\big) \in \mathbf{H} \times \mathbf{Q}$*,* $(\boldsymbol{\chi}, \vartheta) \in \mathrm{X}_2 \times \mathrm{M}_1$*, and* $\big(\vec{\boldsymbol{\rho}}, \phi\big) := \big((\boldsymbol{\rho}, \boldsymbol{\xi}), \phi\big) \in \mathbf{X}_2 \times \mathbf{Y}_1$*, with* $\mathbf{u} \in W_\delta$*. Moreover, there hold the following a priori estimates*

$$\begin{aligned}
&\|(\vec{\mathbf{t}}, \boldsymbol{\sigma})\|_{\mathbf{H}\times\mathbf{Q}} \le C_1 \Big\{ \big(\|g\|^2_{0,j;\Omega} + \|\mathbf{g}\|_{0,p;\Omega} \big) \big(\|\vartheta_D\|_{1/s,r;\Gamma} + \|f\|_{0,q;\Omega} \big) + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} \,, \\
&\|(\boldsymbol{\chi}, \vartheta)\|_{\mathrm{X}_2\times\mathrm{M}_1} \le C_{\mathbf{T}} \Big\{ \|\vartheta_D\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,q;\Omega} \Big\} \,, \qquad \textit{and} \\
&\|\big(\vec{\boldsymbol{\rho}}, \phi\big)\|_{\mathbf{X}_2\times\mathbf{Y}_1} \le C_2 \, \|g\|_{0,j;\Omega} \,,
\end{aligned}$$

where C_1 *is a positive constant depending on* $C_{\mathbf{S}}$*,* $\widetilde{C}_{\mathbf{S}}$*,* $C_{\mathbf{T}}$*,* $C_{\boldsymbol{\mathcal{R}}}$*, and* δ*; whereas* $C_2 \,:=\, C_{\boldsymbol{\mathcal{R}}} + \widetilde{C}_{\boldsymbol{\mathcal{R}}}$*.*

Proof. We recall from Lemma 4.14 that (4.69) guarantees that $\boldsymbol{\Xi}$ maps W_δ into itself. Since $\boldsymbol{\Xi}$ is Lipschitz continuous (cf. (4.90)), relation (4.91) shows that it is, in fact, a contraction. Therefore, from Banach's fixed-point Theorem, there exists a unique solution $\mathbf{u} \in W_\delta$ of (3.27). In turn, the a priori estimates follow easily from (4.17), (4.18), (4.43), (4.66), (4.67), and the bound $\|\mathbf{u}\|_{0,4;\Omega} \,\le\, \delta$. $\square$

5 The Galerkin scheme

In this section, we introduce a Galerkin scheme associated with the fully mixed formulation (3.27) and analyze its well-posedness using a discrete counterpart of the fixed-point approach developed in Section 4.1. We begin by presenting the scheme on generic finite-dimensional spaces and stating the assumptions required for solvability. An a priori error analysis is derived at the end of the section.

5.1 The discrete problem

We start by considering a regular family of triangulations $\big\{\mathcal{T}_h\big\}_{h>0}$ of $\overline{\Omega}$ made up of triangles K of diameter h_K, and set $h := \max\{\, h_K\colon\ K \in \mathcal{T}_h\}$. Then, we consider arbitrary finite element subspaces $\mathbb{H}^{\mathbf{t}}_h$, $\mathbf{H}^{\mathbf{u}}_h$, $\mathbb{H}^{\boldsymbol{\sigma}}_h$, $\mathrm{X}_{2,h}$, $\mathrm{M}_{1,h}$, $\mathrm{X}_{1,h}$, $\mathrm{M}_{2,h}$, $\mathbf{X}^{\boldsymbol{\rho}}_{2,h}$, $\mathbf{X}^{\boldsymbol{\xi}}_{2,h}$, $\mathbf{Y}_{1,h}$, $\mathbf{X}^{\boldsymbol{\varrho}}_{1,h}$, $\mathbf{X}^{\boldsymbol{\lambda}}_{2,h}$ and $\mathbf{Y}_{2,h}$ of the spaces $\mathbb{L}^2_{\mathtt{tr}}(\Omega)$, $\mathbf{L}^4(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, $\mathbf{H}^r(\mathrm{div}_q;\Omega)$, $\mathrm{L}^r(\Omega)$, $\mathbf{H}^s(\mathrm{div}_s;\Omega)$, $\mathrm{L}^p(\Omega)$, $\mathbf{L}^j(\Omega)$, $\mathbf{H}^j_0(\mathrm{div}_j;\Omega)$, $\mathrm{L}^j_0(\Omega)$, $\mathbf{L}^\ell(\Omega)$, $\mathbf{H}^\ell_0(\mathrm{div}_\ell;\Omega)$, and $\mathrm{L}^\ell_0(\Omega)$, respectively, each provided with the same norm of the spaces in which they are contained. Specific choices of them, verifying suitable conditions to be incorporated in the forthcoming analysis, will be provided later in Section 6. Now, defining the product spaces

$$\mathbf{H}_h := \mathbb{H}^{\mathbf{t}}_h \times \mathbf{H}^{\mathbf{u}}_h \,, \mathbf{Q}_h := \mathbb{H}^{\boldsymbol{\sigma}}_h \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,, \quad \mathbf{X}_{2,h} := \mathbf{X}^{\boldsymbol{\rho}}_{2,h} \times \mathbf{X}^{\boldsymbol{\xi}}_{2,h} \,, \mathbf{X}_{1,h} := \mathbf{X}^{\boldsymbol{\varrho}}_{1,h} \times \mathbf{X}^{\boldsymbol{\lambda}}_{1,h} \,,$$

and setting the notation

$$\vec{\mathbf{t}}_h := (\mathbf{t}_h, \mathbf{u}_h) \,, \quad \vec{\mathbf{s}}_h := (\mathbf{s}_h, \mathbf{v}_h) \in \mathbf{H}_h \,, \quad \vec{\boldsymbol{\rho}}_h := (\boldsymbol{\rho}_h, \boldsymbol{\xi}_h) \in \mathbf{X}_{2,h} \,, \quad \vec{\boldsymbol{\varrho}}_h := (\boldsymbol{\varrho}_h, \boldsymbol{\lambda}_h) \in \mathbf{X}_{1,h} \,,$$

the Galerkin scheme associated with (3.27) reads: Find $(\vec{\mathbf{t}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\boldsymbol{\chi}_h, \vartheta_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and $\big(\vec{\boldsymbol{\rho}}_h, \phi_h\big) \in \mathbf{X}_{2,h} \times \mathbf{Y}_{1,h}$, such that

$$\begin{aligned}
\mathbf{A}_{\mathbf{u}_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) \,+\, \mathbf{B}(\vec{\mathbf{s}}_h, \boldsymbol{\sigma}_h) &= \mathbf{F}_{\boldsymbol{\rho}_h, \boldsymbol{\chi}_h, \vartheta_h}(\vec{\mathbf{s}}_h) && \forall\, \vec{\mathbf{s}}_h \in \mathbf{H}_h \,, \\
\mathbf{B}(\vec{\mathbf{t}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h \,, \\
a(\boldsymbol{\chi}_h, \boldsymbol{\eta}_h) \,+\, b_1(\boldsymbol{\eta}_h, \vartheta_h) &= \mathrm{F}(\boldsymbol{\eta}_h) && \forall\, \boldsymbol{\eta}_h \in \mathrm{X}_{1,h} \,, \\
b_2(\boldsymbol{\chi}_h, \omega_h) \,-\, \int_\Omega \mathbf{u}_h \cdot \boldsymbol{\chi}_h \, \omega_h &= \mathrm{G}(\omega_h) && \forall\, \omega_h \in \mathrm{M}_{2,h} \,, \\
\mathcal{A}_{\vartheta_h}\big(\vec{\boldsymbol{\rho}}_h, \vec{\boldsymbol{\varrho}}_h\big) \,+\, \mathcal{B}_1\big(\vec{\boldsymbol{\varrho}}_h, \phi_h\big) &= 0 && \forall\, \vec{\boldsymbol{\varrho}}_h \in \mathbf{X}_{1,h} \,, \\
\mathcal{B}_2\big(\vec{\boldsymbol{\rho}}_h, \varphi_h\big) &= \mathcal{F}(\varphi_h) && \forall\, \varphi_h \in \mathbf{Y}_{2,h} \,.
\end{aligned} \tag{5.1}$$

5.2 Discrete fixed point strategy

To address the solvability of (5.1), here we adopt the discrete version of the fixed point strategy used in Section 4.1. We begin by letting $\mathbf{S}_h : \mathbf{H}^{\mathbf{u}}_h \times \mathbf{X}^{\boldsymbol{\rho}}_{2,h} \times \mathrm{X}_{2,h} \times \mathrm{M}_{1,h} \to \mathbf{H}^{\mathbf{u}}_h$ be the operator defined by

$$\mathbf{S}_h(\mathbf{z}_h, \boldsymbol{\varkappa}_h, \boldsymbol{\zeta}_h, \theta_h) \,:=\, \underset{\sim}{\mathbf{u}}_h \qquad \forall\,(\mathbf{z}_h, \boldsymbol{\varkappa}_h, \boldsymbol{\zeta}_h, \theta_h) \in \mathbf{H}^{\mathbf{u}}_h \times \mathbf{X}^{\boldsymbol{\rho}}_{2,h} \times \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}\,,$$

where $(\vec{\underset{\sim}{\mathbf{t}}}_h, \underset{\sim}{\boldsymbol{\sigma}}_h) = \big((\underset{\sim}{\mathbf{t}}_h, \underset{\sim}{\mathbf{u}}_h), \underset{\sim}{\boldsymbol{\sigma}}_h\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution of the discrete counterpart of the problem (4.1), which arises from the first two rows of (5.1) after replacing $\mathbf{A}_{\mathbf{u}_h}$ and $\mathbf{F}_{\boldsymbol{\rho}_h, \boldsymbol{\chi}_h, \vartheta_h}$ by $\mathbf{A}_{\mathbf{z}_h}$ and $\mathbf{F}_{\boldsymbol{\varkappa}_h, \boldsymbol{\zeta}_h, \theta_h}$ respectively, that is

$$\begin{array}{llll}
\mathbf{A}_{\mathbf{z}_h}(\vec{\underset{\sim}{\mathbf{t}}}_h, \vec{\mathbf{s}}_h) + \mathbf{B}(\vec{\mathbf{s}}_h, \underset{\sim}{\boldsymbol{\sigma}}_h) & = & \mathbf{F}_{\boldsymbol{\varkappa}_h, \boldsymbol{\zeta}_h, \theta_h}(\vec{\mathbf{s}}_h) & \qquad \forall\,\vec{\mathbf{s}}_h \in \mathbf{H}_h\,, \\
\mathbf{B}(\vec{\underset{\sim}{\mathbf{t}}}_h, \boldsymbol{\tau}_h) & = & \mathbf{G}(\boldsymbol{\tau}_h) & \qquad \forall\,\boldsymbol{\tau}_h \in \mathbf{Q}_h\,.
\end{array} \tag{5.2}$$

In turn, we let $\mathbf{T}_h : \mathbf{H}^{\mathbf{u}}_h \to \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ be the operator given by

$$\mathbf{T}_h(\mathbf{z}_h) \,=\, \big(\mathbf{T}_{1,h}(\mathbf{z}_h), \mathbf{T}_{2,h}(\mathbf{z}_h)\big) \,:=\, (\underset{\sim}{\boldsymbol{\chi}}_h, \underset{\sim}{\vartheta}_h) \qquad \forall\,\mathbf{z}_h \in \mathbf{H}^{\mathbf{u}}_h\,,$$

where $(\underset{\sim}{\boldsymbol{\chi}}_h, \underset{\sim}{\vartheta}_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ is the unique solution of the discrete counterpart of the problem (4.3), which arises from the third and fourth rows of (5.1) after replacing $\mathbf{u}_z$ by $\mathbf{z}_h$ in the free term, that is

$$\begin{array}{llll}
a(\underset{\sim}{\boldsymbol{\chi}}_h, \boldsymbol{\eta}_h) + b_1(\boldsymbol{\eta}_h, \underset{\sim}{\vartheta}_h) & = & \mathrm{F}(\boldsymbol{\eta}_h) & \qquad \forall\,\boldsymbol{\eta}_h \in \mathrm{X}_{1,h}\,, \\
b_2(\underset{\sim}{\boldsymbol{\chi}}_h, \omega_h) - \displaystyle\int_\Omega \mathbf{z}_h \cdot \underset{\sim}{\boldsymbol{\chi}}_h\,\omega_h & = & \mathrm{G}(\omega_h) & \qquad \forall\,\omega_h \in \mathrm{M}_{2,h}\,.
\end{array} \tag{5.3}$$

Similarly, we let $\boldsymbol{\mathcal{R}}_h : \mathrm{M}_{1,h} \to \mathbf{X}^{\boldsymbol{\rho}}_{2,h}$ be the operator defined by

$$\boldsymbol{\mathcal{R}}_h(\theta_h) \,:=\, \underset{\sim}{\boldsymbol{\rho}}_h \qquad \forall\,\theta_h \in \mathrm{M}_{1,h}\,,$$

where $(\vec{\underset{\sim}{\boldsymbol{\rho}}}_h, \underset{\sim}{\phi}_h) = \big((\underset{\sim}{\boldsymbol{\rho}}_h, \underset{\sim}{\boldsymbol{\xi}}_h), \underset{\sim}{\phi}_h\big) \in \mathbf{X}_{2,h} \times \mathbf{Y}_{1,h}$ is the unique solution of the discrete counterpart of the problem (4.4), which arises from the fifth and sixth rows of (5.1) after replacing $\mathcal{A}_{\vartheta_h}$ by $\mathcal{A}_{\theta_h}$, that is

$$\begin{array}{llll}
\mathcal{A}_{\theta_h}(\vec{\underset{\sim}{\boldsymbol{\rho}}}_h, \vec{\boldsymbol{\varrho}}_h) + \mathcal{B}_1(\vec{\boldsymbol{\varrho}}_h, \underset{\sim}{\phi}_h) & = & 0 & \qquad \forall\,\vec{\boldsymbol{\varrho}}_h \in \mathbf{X}_{1,h}\,, \\
\mathcal{B}_2(\vec{\underset{\sim}{\boldsymbol{\rho}}}_h, \varphi_h) & = & \mathcal{F}(\varphi_h) & \qquad \forall\,\varphi_h \in \mathbf{Y}_{2,h}\,.
\end{array} \tag{5.4}$$

Finally, we define the operator $\boldsymbol{\Xi}_h : \mathbf{H}^{\mathbf{u}}_h \to \mathbf{H}^{\mathbf{u}}_h$ as

$$\boldsymbol{\Xi}_h(\mathbf{z}_h) \,:=\, \mathbf{S}_h\big(\mathbf{z}_h, \boldsymbol{\mathcal{R}}_h(\mathbf{T}_{2,h}(\mathbf{z})), \mathbf{T}_{1,h}(\mathbf{z}), \mathbf{T}_{2,h}(\mathbf{z})\big) \qquad \forall\,\mathbf{z}_h \in \mathbf{H}^{\mathbf{u}}_h\,. \tag{5.5}$$

Solving (5.1) is equivalent to seeking a fixed point of $\boldsymbol{\Xi}_h$, that is: Find $\mathbf{u}_h \in \mathbf{H}^{\mathbf{u}}_h$ such that

$$\boldsymbol{\Xi}_h(\mathbf{u}_h) \,=\, \mathbf{u}_h\,. \tag{5.6}$$

In the next section, we prove the well-posedness of the uncoupled problems (5.2), (5.3) and (5.4), equivalently the well-definedness of $\mathbf{S}_h$, $\mathbf{T}_h$, $\boldsymbol{\mathcal{R}}_h$, and thus, that of $\boldsymbol{\Xi}_h$.

5.3 Well-definedness of $\mathbf{S}_h$, $\mathbf{T}_h$, $\boldsymbol{\mathcal{R}}_h$ and $\boldsymbol{\Xi}_h$

Similarly to Sections 4.2-4.4, we now examine the well-definedness of $\mathbf{S}_h$, $\mathbf{T}_h$ and $\boldsymbol{\mathcal{R}}_h$. In fact, these properties will be established as discrete counterparts of Theorems 4.3, 4.7 and 4.13 by applying the discrete versions of both classical and generalized Babuška–Brezzi theories (cf. [25, Proposition 2.42] and [5, Corollary 2.2]), and the Banach–Nečas–Babuška theorem (cf. [25, Theorem 2.22]). Hypotheses on the finite element subspaces will be introduced along the analysis.

We start with the well-definedness of $\mathbf{S}_h$, equivalently, the well-posedness of (5.2), by observing that the discrete kernel of the bilinear form $\mathbf{B}$ is

$$\mathbf{K}_h \,:=\, \Big\{ \vec{\mathbf{s}}_h \in \mathbf{H}_h : \quad \mathbf{B}(\vec{\mathbf{s}}_h, \boldsymbol{\tau}_h) = 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h \Big\}.$$

In this case, and differently from Section 4.2, we are not able to explicitly characterize $\mathbf{K}_h$ as we did for $\mathbf{K}$ (cf. (4.8)) since the distributional argument is not valid at the discrete level. Instead, we assume that

(**H.1**) there exists a positive constant $C_{\mathtt{d}}$, independent of h, such that

$$\|\mathbf{s}_h\|_{0,\Omega} \,\geq\, C_{\mathtt{d}} \|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\, \vec{\mathbf{s}}_h \,:=\, (\mathbf{s}_h, \mathbf{v}_h) \in \mathbf{K}_h\,, \quad \text{and}$$

(**H.2**) there exists a positive constant $\boldsymbol{\beta}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{s}}_h \in \mathbf{H}_h \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\mathbf{B}(\vec{\mathbf{s}}_h, \boldsymbol{\tau}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathbf{H}}} \,\geq\, \boldsymbol{\beta}_{\mathtt{d}}\, \|\boldsymbol{\tau}_h\|_{\mathbf{Q}} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h\,.$$

Now, similarly as in the proof of Lemma 4.1, but now using (**H.1**) instead of the Poincaré inequality and the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, we deduce that for each $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ there holds

$$\mathbf{A}_{\mathbf{z}_h}(\vec{\mathbf{s}}_h, \vec{\mathbf{s}}_h) \,=\, \mu\, \|\mathbf{s}_h\|^2_{0,\Omega} \,\geq\, \frac{\mu}{2}\, \|\mathbf{s}_h\|^2_{0,\Omega} \,+\, \frac{\mu}{2}\, C^2_{\mathtt{d}}\, \|\mathbf{v}_h\|^2_{0,4;\Omega} \qquad \forall\, \vec{\mathbf{s}}_h \,:=\, (\mathbf{s}_s, \mathbf{v}_h) \in \mathbf{K}_h\,, \tag{5.7}$$

thereby proving the $\mathbf{K}_h$-ellipticity of $\mathbf{A}_{\mathbf{z}_h}$ with constant $\boldsymbol{\alpha}_{\mathtt{d}} \,:=\, \dfrac{\mu}{2}\, \min\Big\{1, C^2_{\mathtt{d}}\Big\}$.

The well-definedness of $\mathbf{S}_h$ can be stated now.

Theorem 5.1. *For each* $(\mathbf{z}_h, \boldsymbol{\varkappa}_h, \boldsymbol{\zeta}_h, \theta_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{X}^{\boldsymbol{\rho}}_{2,h} \times \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ *there exists a unique* $(\vec{\underset{\sim}{\mathbf{t}}}_h, \underset{\sim}{\boldsymbol{\sigma}}_h) \,:=\, \big((\underset{\sim}{\mathbf{t}}_h, \underset{\sim}{\mathbf{u}}_h), \underset{\sim}{\boldsymbol{\sigma}}_h\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ *solution of problem* (5.2)*, and hence one can define* $\mathbf{S}_h(\mathbf{z}_h, \boldsymbol{\varkappa}_h, \boldsymbol{\zeta}_h, \theta_h) := \underset{\sim}{\mathbf{u}}_h \in \mathbf{H}_h^{\mathbf{u}}$*. Moreover, there exists a positive constant* $C_{\mathbf{S},\mathtt{d}}$*, depending only on* $\|\mathbf{i}_4\|$*,* μ*,* κ*,* δ_d*,* δ_t*,* $\boldsymbol{\alpha}_{\mathtt{d}}$*, and* $\boldsymbol{\beta}_{\mathtt{d}}$*, such that the following a priori estimate holds true*

$$\begin{aligned} \|\mathbf{S}_h(\mathbf{z}_h, \boldsymbol{\varkappa}_h, \boldsymbol{\zeta}_h, \theta_h)\|_{0,4;\Omega} \,\leq\, \|\underset{\sim}{\mathbf{u}}_h\|_{0,4;\Omega} \,\leq\, \|\vec{\underset{\sim}{\mathbf{t}}}_h\|_{\mathbf{H}} \,\leq\, C_{\mathbf{S},\mathtt{d}} \Big\{ \|\boldsymbol{\varkappa}_h\|^2_{0,j;\Omega}\, \|\boldsymbol{\zeta}_h\|_{0,r;\Omega} \\ + \|\mathbf{g}\|_{0,p;\Omega}\, \|\theta_h\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\, \big(1 + \|\mathbf{z}_h\|_{0,4;\Omega}\big) \Big\}. \end{aligned} \tag{5.8}$$

Proof. It is clear that the required discrete inf-sup condition for $\mathbf{A}_{\mathbf{z}_h}$ follows from (5.7), which, along with that of $\mathbf{B}$ (cf. (**H.2**)), and thanks to [25, Proposition 2.42], imply the existence of a unique solution to (5.2). Moreover, the a priori estimate (5.8) follows from [25, eq. (2.30)]. $\square$

Analogously to (4.18), and for subsequent use, the a priori estimate for the component $\underset{\sim}{\boldsymbol{\sigma}}_h$ of the unique solution to (5.2) is given by

$$\begin{aligned} \|\underset{\sim}{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} \,\leq\, \widetilde{C}_{\mathbf{S},\mathtt{d}}\, \big(1 + \|\mathbf{z}_h\|_{0,4;\Omega}\big) \Big\{ \|\boldsymbol{\varkappa}_h\|^2_{0,j;\Omega}\, \|\boldsymbol{\zeta}_h\|_{0,r;\Omega} \,+\, \|\mathbf{g}\|_{0,p;\Omega}\, \|\theta_h\|_{0,r;\Omega} \\ + \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\, \big(1 + \|\mathbf{z}_h\|_{0,4;\Omega}\big) \Big\}, \end{aligned} \tag{5.9}$$

where $\widetilde{C}_{\mathbf{S},\mathtt{d}}$ is a positive constant depending only on $\|\mathbf{i}_4\|$, μ, κ, δ_d, δ_t, $\boldsymbol{\alpha}_{\mathtt{d}}$, and $\boldsymbol{\beta}_{\mathtt{d}}$.

Regarding the well-definedness of $\mathbf{T}_h$, observe that the discrete kernels of b_1 and b_2 are

$$\begin{aligned} K_{1,h} \,&:=\, \Big\{ \boldsymbol{\eta}_h \in \mathrm{X}_{1,h} : \quad b_1(\boldsymbol{\eta}_h, \omega_h) = 0 \qquad \forall\, \omega_h \in M_{1,h} \Big\}, \quad \text{and} \\ K_{2,h} \,&:=\, \Big\{ \boldsymbol{\eta}_h \in \mathrm{X}_{2,h} : \quad b_2(\boldsymbol{\eta}_h, \omega_h) = 0 \qquad \forall\, \omega_h \in M_{2,h} \Big\} \end{aligned}$$

respectively, and introduce the following hypothesis:

(**H.3**) there exists a positive constant $\alpha_{\mathtt{d}}$, independent of h, such that

$$\begin{aligned}\sup_{\substack{\boldsymbol{\eta}_h\in K_{1,h}\\ \boldsymbol{\eta}_h\neq 0}} \frac{a(\boldsymbol{\mu}_h,\boldsymbol{\eta}_h)}{\|\boldsymbol{\eta}_h\|_{\mathrm{X}_1}} \geq \alpha_{\mathtt{d}}\,\|\boldsymbol{\mu}_h\|_{\mathrm{X}_2} \qquad \forall\,\boldsymbol{\mu}_h\in K_{2,h}\,, \quad \text{and}\\ \sup_{\boldsymbol{\mu}_h\in K_{2,h}} a(\boldsymbol{\mu}_h,\boldsymbol{\eta}_h) > 0 \qquad \forall\,\boldsymbol{\eta}_h\in K_{1,h}\,,\ \boldsymbol{\eta}_h\neq\mathbf{0}\,,\end{aligned}$$

(**H.4**) for each $i\in\big\{1,2\big\}$, there exists a positive constant $\beta_{i,\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\eta}_h\in \mathrm{X}_{i,h}\\ \boldsymbol{\eta}_h\neq \mathbf{0}}} \frac{b_i(\boldsymbol{\eta}_h,\omega_h)}{\|\boldsymbol{\eta}_h\|_{\mathrm{X}_i}} \,\geq\, \beta_{i,\mathtt{d}}\,\|\omega_h\|_{\mathrm{M}_i} \qquad \forall\,\omega_h\in \mathrm{M}_{i,h}\,.$$

Due to (**H.3**) and (**H.4**), an application of [5, Corollary 2.2] implies the discrete global inf-sup condition for $\mathbb{A}$ (cf. (4.19)) with a positive constant $\alpha_{\mathbf{T},\mathtt{d}}$ depending only on $\alpha_{\mathtt{d}}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$ and $\|a\|$ (cf. (4.22)). Consequently, the same property is transferred to $\mathbb{A}_{\mathbf{z}_h}$ (cf. (4.20)) for each $\mathbf{z}_h\in\mathbf{H}^{\mathbf{u}}_h$ that satisfies the discrete version of (4.40), that is

$$\|\mathbf{z}_h\|_{0,4;\Omega} \,\leq\, \frac{\alpha_{\mathbf{T},\mathtt{d}}}{2}\,. \tag{5.10}$$

Accordingly to the previous discussion, the well-definedness of $\mathbf{T}_h$ is stated as follows.

Theorem 5.2. *For each* $\mathbf{z}_h\in\mathbf{H}^{\mathbf{u}}_h$ *satisfying* (5.10), *there exists a unique* $(\underset{\sim}{\boldsymbol{\chi}}_h,\vartheta_h)\in\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}$ *solution of* (5.3), *and hence one can define* $\mathbf{T}_h(\mathbf{z}_h) := (\underset{\sim}{\boldsymbol{\chi}}_h,\vartheta_h)\in\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}$. *Moreover, there exists a positive constant* $C_{\mathbf{T},\mathtt{d}}$, *depending only on* $\alpha_{\mathbf{T},\mathtt{d}}$, κ, *and* c_r (*cf.* (4.22)), *such that*

$$\|\mathbf{T}_h(\mathbf{z}_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \,\leq\, C_{\mathbf{T},\mathtt{d}}\Big\{\|\vartheta_D\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,q;\Omega}\Big\}\,.$$

Proof. It is a simple application of [25, Theorem 2.22]. Further details are omitted. □

In the following, we deal with the well-definedness of $\boldsymbol{\mathcal{R}}_h$, equivalently, the well-posedness of (5.4), for which, the procedure developed in Section 4.4 will be adapted to the present discrete framework. In this manner, we first observe that the discrete kernels of $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$\begin{aligned}\mathcal{K}_{1,h} \,&:=\, \Big\{\vec{\boldsymbol{\varkappa}}_h\in\mathbf{X}_{1,h}:\quad \mathcal{B}_1(\vec{\boldsymbol{\varkappa}}_h,\psi_h) \,=\, 0 \quad \forall\,\psi_h\in\mathbf{Y}_{1,h}\Big\} \quad \text{and}\\ \mathcal{K}_{2,h} \,&:=\, \Big\{\vec{\boldsymbol{\varkappa}}_h\in\mathbf{X}_{2,h}:\quad \mathcal{B}_2(\vec{\boldsymbol{\varkappa}}_h,\psi_h) \,=\, 0 \quad \forall\,\psi_h\in\mathbf{Y}_{2,h}\Big\}\,,\end{aligned}$$

so that, under the hypotheses

(**H.5**) $\mathrm{div}(\mathbf{X}^{\boldsymbol{\lambda}}_{1,h})\subseteq\mathbf{Y}_{1,h}$ and $\mathrm{div}(\mathbf{X}^{\boldsymbol{\xi}}_{2,h})\subseteq\mathbf{Y}_{2,h}$,

along with (4.47), they can be written as

$$\mathcal{K}_{1,h} \,:=\, \mathbf{X}^{\boldsymbol{\varrho}}_{1,h}\times\big(\mathbf{X}^{\boldsymbol{\lambda}}_{1,h}\cap\widetilde{\mathbf{H}}^{\ell}_0(\mathrm{div}_\ell;\Omega)\big) \quad\text{and}\quad \mathcal{K}_{2,h} \,:=\, \mathbf{X}^{\boldsymbol{\rho}}_{2,h}\times\big(\mathbf{X}^{\boldsymbol{\xi}}_{2,h}\cap\widetilde{\mathbf{H}}^{j}_0(\mathrm{div}_j;\Omega)\big)\,.$$

Hence, the discrete kernels induced by $\mathbf{b}_1$ and $\mathbf{b}_2$ restricted to $\mathcal{K}_{1,h}\times\mathcal{K}_{2,h}$ are

$$\begin{aligned}\mathrm{V}_{1,h} \,&=\, \Big\{\boldsymbol{\varrho}_h\in\mathbf{X}^{\boldsymbol{\varrho}}_{1,h}:\quad \mathbf{b}_1(\boldsymbol{\varrho}_h,\boldsymbol{\varsigma}_h)=0 \quad \forall\,\boldsymbol{\varsigma}_h\in\mathbf{X}^{\boldsymbol{\xi}}_{2,h}\cap\widetilde{\mathbf{H}}^{j}_0(\mathrm{div}_j;\Omega)\Big\} \quad\text{and}\\ \mathrm{V}_{2,h} \,&=\, \Big\{\boldsymbol{\varkappa}_h\in\mathbf{X}^{\boldsymbol{\rho}}_{2,h}:\quad \mathbf{b}_2(\boldsymbol{\varkappa}_h,\boldsymbol{\lambda}_h)=0 \quad \forall\,\boldsymbol{\lambda}_h\in\mathbf{X}^{\boldsymbol{\lambda}}_{1,h}\cap\widetilde{\mathbf{H}}^{\ell}_0(\mathrm{div}_\ell;\Omega)\Big\}\,.\end{aligned} \tag{5.11}$$

Moreover, we introduce further hypotheses which constitute discrete counterparts of the inf-sup conditions from Lemmas 4.10-4.12. Specifically, we assume that

(**H.6**) there exists a positive constant $\alpha_{\mathbf{a},\mathtt{d}}$, independent of h, such that for each $\theta_h \in \mathrm{M}_{1,h}$ there hold

$$\sup_{\substack{\boldsymbol{\varrho}_h \in \mathrm{V}_{1,h} \\ \boldsymbol{\varrho}_h \neq \mathbf{0}}} \frac{\mathbf{a}_{\theta_h}(\boldsymbol{\varkappa}_h, \boldsymbol{\varrho}_h)}{\|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}} \geq \alpha_{\mathbf{a},\mathtt{d}} \, \|\boldsymbol{\varkappa}_h\|_{0,j;\Omega} \qquad \forall\, \boldsymbol{\varkappa}_h \in \mathrm{V}_{2,h}\,, \qquad \text{and} \tag{5.12a}$$

$$\sup_{\boldsymbol{\varkappa}_h \in \mathrm{V}_{2,h}} \mathbf{a}_{\theta_h}(\boldsymbol{\varkappa}_h, \boldsymbol{\varrho}_h) > 0 \qquad \forall\, \boldsymbol{\varrho}_h \in \mathrm{V}_{1,h},\ \boldsymbol{\varrho}_h \neq \mathbf{0}\,, \tag{5.12b}$$

(**H.7**) there exist positive constants $\beta_{i,\mathtt{d}}$, $i \in \{1,2\}$, such that

$$\sup_{\substack{\boldsymbol{\varrho}_h \in \mathbf{X}^{\boldsymbol{\varrho}}_{1,h} \\ \boldsymbol{\varrho}_h \neq \mathbf{0}}} \frac{\mathbf{b}_1(\boldsymbol{\varrho}_h, \boldsymbol{\varsigma}_h)}{\|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}} \geq \beta_{1,\mathtt{d}} \, \|\boldsymbol{\varsigma}_h\|_{j,\mathrm{div}_j;\Omega} \qquad \forall\, \boldsymbol{\varsigma}_h \in \mathbf{X}^{\boldsymbol{\xi}}_{2,h} \cap \widetilde{\mathbf{H}}^j_0(\mathrm{div}_j;\Omega)\,, \quad \text{and}$$

$$\sup_{\substack{\boldsymbol{\varkappa}_h \in \mathbf{X}^{\boldsymbol{\rho}}_{2,h} \\ \boldsymbol{\varkappa}_h \neq \mathbf{0}}} \frac{\mathbf{b}_2(\boldsymbol{\varkappa}_h, \boldsymbol{\lambda}_h)}{\|\boldsymbol{\varkappa}_h\|_{0,j;\Omega}} \geq \beta_{2,\mathtt{d}} \, \|\boldsymbol{\lambda}_h\|_{\ell;\mathrm{div}_\ell;\Omega} \qquad \forall\, \boldsymbol{\lambda}_h \in \mathbf{X}^{\boldsymbol{\lambda}}_{1,h} \cap \widetilde{\mathbf{H}}^\ell_0(\mathrm{div}_\ell;\Omega)\,; \quad \text{and}$$

(**H.8**) for each $i \in \{1,2\}$, there exists a positive constant $\beta_{\mathcal{B},i,\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\boldsymbol{\varkappa}}_h \in \mathbf{X}_{i,h} \\ \vec{\boldsymbol{\varkappa}}_h \neq \mathbf{0}}} \frac{\mathcal{B}_i(\vec{\boldsymbol{\varkappa}}_h, \psi_h)}{\|\vec{\boldsymbol{\varkappa}}_h\|_{\mathbf{X}_i}} \geq \beta_{\mathcal{B},i,\mathtt{d}} \, \|\psi_h\|_{\mathbf{Y}_i} \qquad \forall\, \psi_h \in \mathbf{Y}_{i,h}\,.$$

In this way, assumptions (**H.6**)-(**H.7**) yield the discrete analogues of (4.61a)-(4.61b) by arguing as in their continuous derivation, but now invoking [5, Corollary 2.2] and [25, Theorem 2.22]. These results correspond precisely to the discrete inf–sup conditions for $\mathcal{A}_{\theta_h}$ on $\mathcal{K}_{1,h} \times \mathcal{K}_{2,h}$, namely

$$\sup_{\substack{\vec{\boldsymbol{\varrho}}_h \in \mathcal{K}_{1,h} \\ \vec{\boldsymbol{\varrho}}_h \neq \mathbf{0}}} \frac{\mathcal{A}_{\theta_h}(\vec{\boldsymbol{\varkappa}}_h, \vec{\boldsymbol{\varrho}}_h)}{\|\vec{\boldsymbol{\varrho}}_h\|_{\mathbf{X}_1}} \geq \alpha_{\mathcal{A},\mathtt{d}} \, \|\vec{\boldsymbol{\varkappa}}_h\|_{\mathbf{X}_2} \qquad \forall\, \vec{\boldsymbol{\varkappa}}_h \in \mathcal{K}_{2,h}\,, \quad \text{and} \tag{5.13a}$$

$$\sup_{\vec{\boldsymbol{\varkappa}}_h \in \mathcal{K}_{2,h}} \mathcal{A}_{\theta_h}(\vec{\boldsymbol{\varkappa}}_h, \vec{\boldsymbol{\varrho}}_h) > 0 \qquad \forall\, \vec{\boldsymbol{\varrho}}_h \in \mathcal{K}_{1,h}\,,\ \vec{\boldsymbol{\varrho}}_h \neq \mathbf{0}\,, \tag{5.13b}$$

where $\alpha_{\mathcal{A},\mathtt{d}}$ is a positive constant depending only on $\alpha_{\mathbf{a},\mathtt{d}}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$ and $\|\mathbf{a}\|$ (cf. (4.44)).

The well-definedness of $\boldsymbol{\mathcal{R}}_h$ is established now in the following result.

Theorem 5.3. *For each* $\theta_h \in \mathrm{M}_{1,h}$ *there exists a unique* $(\underset{\sim}{\vec{\boldsymbol{\rho}}_h}, \underset{\sim}{\phi_h}) = \big((\underset{\sim}{\boldsymbol{\rho}_h}, \underset{\sim}{\boldsymbol{\xi}_h}), \underset{\sim}{\phi_h}\big) \in \mathbf{X}_{2,h} \times \mathbf{Y}_{1,h}$ *solution of* (5.4), *and hence one can define* $\boldsymbol{\mathcal{R}}_h(\theta_h) := \underset{\sim}{\boldsymbol{\rho}_h} \in \mathbf{X}^{\boldsymbol{\rho}}_{2,h}$. *Moreover, there exists a positive constant* $C_{\boldsymbol{\mathcal{R}},\mathtt{d}}$, *depending only on* $\alpha_{\mathcal{A},\mathtt{d}}$, $\beta_{\mathcal{B},2,\mathtt{d}}$ *and* $\|\mathcal{A}\|$ (*cf.* (4.44)), *such that*

$$\|\boldsymbol{\mathcal{R}}_h(\theta_h)\|_{0,j;\Omega} \leq \|\underset{\sim}{\boldsymbol{\rho}_h}\|_{0,j;\Omega} \leq \|\underset{\sim}{\vec{\boldsymbol{\rho}}_h}\|_{\mathbf{X}_2} \leq C_{\boldsymbol{\mathcal{R}},\mathtt{d}} \, \|g\|_{0,j;\Omega}.$$

Proof. Thanks to the inf-sup conditions of $\mathcal{A}_{\theta_h}$ (cf. (5.13a)-(5.13b)), $\mathcal{B}_1$ and $\mathcal{B}_2$ (cf. (**H.8**)), the proof reduces to a simple application of [5, Corollary 2.2]. Further details are omitted. □

Since the a priori estimation for the second component $\underset{\sim}{\phi_h}$ of the unique solution of (5.4) will be required later on, we deduce, analogously to the derivation of (4.67), that there exists a positive constant $\widetilde{C}_{\boldsymbol{\mathcal{R}},\mathtt{d}}$ depending only on $\alpha_{\mathcal{A},\mathtt{d}}$, $\beta_{\mathcal{B},1,\mathtt{d}}$, $\beta_{\mathcal{B},2,\mathtt{d}}$, and $\|\mathcal{A}\|$, such that

$$\|\underset{\sim}{\phi_h}\|_{\mathbf{Y}_1} \leq \widetilde{C}_{\boldsymbol{\mathcal{R}},\mathtt{d}} \, \|g\|_{0,j;\Omega}\,. \tag{5.14}$$

The well-definedness of $\mathbf{S}_h$, $\mathbf{T}_h$ and $\boldsymbol{\mathcal{R}}_h$ guarantees that of $\boldsymbol{\Xi}_h$ (cf. (5.5)), so that the next section is devoted to establishing the existence of a fixed point for it.

5.4 Solvability analysis of the discrete fixed point equation

In this section we address the solvability of the Galerkin scheme (5.1) by means of the equivalent fixed-point formulation (5.6). The arguments follow closely those of Section 4.5, and therefore most technical details are omitted, with emphasis placed only on the aspects that differ in the discrete setting. We emphasize that only existence is obtained through Brouwer's fixed-point theorem, and the reason for this will be explained later on.

Similarly as for (4.68), but now employing the discrete assumption (5.10), we let $\delta_{\mathtt{d}} = \dfrac{\alpha_{\mathbf{T},\mathtt{d}}}{2}$ and define the closed ball

$$W_{\delta_{\mathtt{d}}} := \Big\{ \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}} : \quad \|\mathbf{z}_h\|_{0,4;\Omega} \,\le\, \delta_{\mathtt{d}} \Big\},$$

which is shown, analogously to the derivation of Lemma 4.14, to be mapped into itself through $\boldsymbol{\Xi}_h$ under the discrete analogue restriction of (4.69), which read exactly as that one, but with $C(\delta_{\mathtt{d}})$ and $\delta_{\mathtt{d}}$ instead of $C(\delta)$ and δ, respectively. Moreover, similarly to $C(\delta)$, $C(\delta_{\mathtt{d}})$ depends only on $C_{\mathbf{S},\mathtt{d}}$, $C_{\mathbf{T},\mathtt{d}}$, $C_{\boldsymbol{\mathcal{R}},\mathtt{d}}$ and $\delta_{\mathtt{d}}$.

Now, the continuity properties of $\mathbf{S}_h$ and $\mathbf{T}_h$ are given as discrete versions of Lemmas 4.15 and 4.16, reading exactly as those but with $L_{\mathbf{S},\mathtt{d}}$ and $L_{\mathbf{T},\mathtt{d}}$ instead of $L_{\mathbf{S}}$ and $L_{\mathbf{T}}$, respectively, where similarly to the latter, $L_{\mathbf{S},\mathtt{d}}$ depends only on n, κ, $C_{\mathbf{S},\mathtt{d}}$, $\boldsymbol{\alpha}_{\mathtt{d}}$, δ_d and δ_t; and $L_{\mathbf{T},\mathtt{d}} := 2\alpha_{\mathbf{T},\mathtt{d}}^{-1}\, C_{\mathbf{T},\mathtt{d}}$.

On the other hand, we require the additionally hypothesis

(H.9) $\mathbf{X}_{2,h}^{\boldsymbol{\rho}} \subseteq \mathbf{L}^\infty(\Omega)$,

to deduce a continuity property for $\boldsymbol{\mathcal{R}}_h$, since (**RH**) is not applicable in the present discrete setting. Certainly, this condition seems to be very restrictive, but, as it will see in Section 6, the specific spaces provided there satisfy this condition trivially.

The continuity property for $\boldsymbol{\mathcal{R}}_h$ is established now.

Lemma 5.4. *There exists a positive constant $L_{\boldsymbol{\mathcal{R}}}$, depending only on L_η (cf. (2.2)) and $\alpha_{\mathcal{A},\mathtt{d}}$ (cf. (5.13a)), such that*

$$\|\boldsymbol{\mathcal{R}}_h(\underset{\sim}{\theta}_h) - \boldsymbol{\mathcal{R}}_h(\underline{\theta}_h)\|_{0,r;\Omega} \,\le\, L_{\boldsymbol{\mathcal{R}},\mathtt{d}}\, \|\boldsymbol{\mathcal{R}}_h(\underline{\theta}_h)\|_{0,\infty;\Omega}\, \|\underset{\sim}{\theta}_h - \underline{\theta}_h\|_{0,r;\Omega} \tag{5.15}$$

for all $\underset{\sim}{\theta}_h, \underline{\theta}_h \in M_{1,h}$.

Proof. Given $\underset{\sim}{\theta}_h, \underline{\theta}_h \in \mathrm{M}_{1,h}$, we let

$$(\underset{\sim}{\vec{\boldsymbol{\rho}}}_h, \underset{\sim}{\phi}_h) \,=\, \big((\underset{\sim}{\boldsymbol{\rho}}_h, \underset{\sim}{\boldsymbol{\xi}}_h), \underset{\sim}{\phi}_h\big) \in \mathbf{X}_{2,h} \times \mathbf{Y}_{1,h} \quad \text{and} \quad (\underline{\vec{\boldsymbol{\rho}}}_h, \underline{\phi}_h) \,=\, \big((\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{\xi}}_h), \underline{\phi}_h\big) \in \mathbf{X}_{2,h} \times \mathbf{Y}_{1,h}$$

be the respective solutions of (5.4), so that $\boldsymbol{\mathcal{R}}_h(\underset{\sim}{\theta}_h) := \underset{\sim}{\boldsymbol{\rho}}_h$ and $\boldsymbol{\mathcal{R}}_h(\underline{\theta}_h) := \underline{\boldsymbol{\rho}}_h$. Now, following analogous arguments to those used in the proof of Lemma 4.17, we derive the discrete version of (4.84), from which, using that $j = r$ (cf. (4.52)) and bounding uniformly $\underline{\boldsymbol{\rho}}_h$ thanks to **(H.9)**, we conclude that

$$\alpha_{\mathcal{A},\mathtt{d}} \|\underset{\sim}{\vec{\boldsymbol{\rho}}}_h - \underline{\vec{\boldsymbol{\rho}}}_h\|_{\mathbf{X}_2} \,\le\, L_\eta \big\| |\underset{\sim}{\theta}_h - \underline{\theta}_h|\, |\underline{\boldsymbol{\rho}}_h| \big\|_{0,r;\Omega} \,\le\, L_\eta\, \|\boldsymbol{\mathcal{R}}_h(\underline{\theta}_h)\|_{0,\infty,\Omega} \big\|\underset{\sim}{\theta}_h - \underline{\theta}_h\big\|_{0,r;\Omega},$$

which yields (5.15) with $L_{\boldsymbol{\mathcal{R}},\mathtt{d}} := L_\eta / \alpha_{\mathcal{A},\mathtt{d}}$. □

Regarding the continuity property of $\boldsymbol{\Xi}_h$, it follows analogously to the derivation of (4.89), except that, instead of (4.86), we use its discrete counterpart given by Lemma 5.4. Indeed, given $\underset{\sim}{\mathbf{z}}_h$, $\underline{\mathbf{z}}_h \in \mathbf{H}_h^{\mathbf{u}}$, the definition of $\mathbf{T}_h$ guarantees that $\mathbf{T}_{2,h}(\underset{\sim}{\mathbf{z}}_h)$, $\mathbf{T}_{2,h}(\underline{\mathbf{z}}_h) \in \mathrm{M}_{1,h}$, so that the aforementioned Lemma yields

$$\|\boldsymbol{\mathcal{R}}_h(\mathbf{T}_{2,h}(\underset{\sim}{\mathbf{z}}_h)) - \boldsymbol{\mathcal{R}}_h(\mathbf{T}_{2,h}(\underline{\mathbf{z}}_h))\|_{0,j;\Omega} \,\le\, L_{\boldsymbol{\mathcal{R}},\mathtt{d}}\, \|\boldsymbol{\mathcal{R}}_h(\mathbf{T}_{2,h}(\underline{\mathbf{z}}_h))\|_{0,\infty;\Omega}\, \|\mathbf{T}_{2,h}(\underset{\sim}{\mathbf{z}}_h) - \mathbf{T}_{2,h}(\underline{\mathbf{z}}_h)\|_{0,r;\Omega}\,.$$

Consequently, the continuity property of $\boldsymbol{\Xi}_h$ reads as

$$\begin{aligned} \|\boldsymbol{\Xi}_h(\underset{\sim}{\mathbf{z}}_h) - \boldsymbol{\Xi}_h(\underline{\mathbf{z}}_h)\|_{0,4;\Omega} \,\le\, L_{\boldsymbol{\Xi},\mathtt{d}} \Big\{ \Big(& \|g\|_{0,j;\Omega}\, \|\boldsymbol{\mathcal{R}}_h(\mathbf{T}_{2,h}(\underline{\mathbf{z}}_h))\|_{0,\infty;\Omega}\, \mathtt{D}(\vartheta_D, f) \,+\, \|g\|_{0,j;\Omega}^2 \\ & + \|\mathbf{g}\|_{0,p;\Omega} \Big) \mathtt{D}(\vartheta_D, f) + \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} \|\underset{\sim}{\mathbf{z}}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega}\,, \end{aligned} \tag{5.16}$$

where $L_{\Xi,\mathtt{d}}$ is a positive constant depending only on $L_{\mathbf{S},\mathtt{d}}$, $L_{\mathbf{T},\mathtt{d}}$, $L_{\boldsymbol{\mathcal{R}},\mathtt{d}}$, $C_{\mathbf{S},\mathtt{d}}$, $C_{\mathbf{T},\mathtt{d}}$, $C_{\boldsymbol{\mathcal{R}},\mathtt{d}}$, and $\delta_{\mathtt{d}}$.

The main result of this section is now established.

Theorem 5.5. *Assume hypotheses* (**H.1**) - (**H.9**)*, and that the data are sufficiently small so that the discrete version of* (4.69) *holds, that is,*

$$C(\delta_{\mathtt{d}})\Big\{\big(\|g\|^2_{0,j;\Omega} + \|\mathbf{g}\|_{0,p;\Omega}\big)\big(\|\vartheta_D\|_{1/s,r;\Gamma} + \|f\|_{0,q;\Omega}\big) + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\} \le \delta_{\mathtt{d}}\,. \tag{5.17}$$

Then, the operator Ξ_h *has at least one fixed point* $\mathbf{u}_h \in W_{\delta_{\mathtt{d}}}$*. Equivalently, the coupled problem* (5.1) *has at least one solution* $(\vec{\mathbf{t}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$*,* $(\boldsymbol{\chi}_h, \vartheta_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$*, and* $\big(\vec{\boldsymbol{\rho}}_h, \phi_h\big) \in \mathbf{X}_{2,h} \times \mathbf{Y}_{1,h}$*, with* $\mathbf{u}_h \in W_{\delta_{\mathtt{d}}}$*. Moreover, there holds the following a priori estimates*

$$\begin{aligned}
\|(\vec{\mathbf{t}}_h, \boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,&\le\, C_{1,\mathtt{d}}\Big\{\big(\|g\|^2_{0,j;\Omega} + \|\mathbf{g}\|_{0,p;\Omega}\big)\big(\|\vartheta_D\|_{1/s,r;\Gamma} + \|f\|_{0,j;\Omega}\big) + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,,\\
\|(\boldsymbol{\chi}_h, \vartheta_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \,&\le\, C_{\mathbf{T},\mathtt{d}}\Big\{\|\vartheta_D\|_{1/s,r;\Gamma} + \|f\|_{0,j;\Omega}\Big\}\,,\\
\|\big(\vec{\boldsymbol{\rho}}_h, \phi_h\big)\|_{\mathbf{X}_2\times\mathbf{Y}_1} \,&\le\, C_{2,\mathtt{d}}\,\|g\|_{0,j;\Omega}\,,
\end{aligned}$$

where $C_{1,\mathtt{d}}$ *is a positive constant depending on* $C_{\mathbf{S},\mathtt{d}}$*,* $\widetilde{C}_{\mathbf{S},\mathtt{d}}$*,* $C_{\mathbf{T},\mathtt{d}}$*,* $C_{\boldsymbol{\mathcal{R}},\mathtt{d}}$ *and* $\delta_{\mathtt{d}}$*; and* $C_{2,\mathtt{d}} := C_{\boldsymbol{\mathcal{R}},\mathtt{d}} + \widetilde{C}_{\boldsymbol{\mathcal{R}},\mathtt{d}}$.

Proof. As it was mentioned at the early stage of this section, we remember that (5.17) guarantees that Ξ_h maps the nonempty, convex, compact closed ball $W_{\delta_{\mathtt{d}}}$ into itself. Then, thanks to the continuity property of Ξ_h (cf. (5.16)), an application of Brouwer's Theorem [10, Theorem 9.9-2] gives the existence of at least one fixed-point $\mathbf{u}_h \in \mathbf{H}^{\mathbf{u}}_h$. Finally, the a priori estimations are obtained straightforwardly by Lemmas 5.1-5.3, and the estimations provided therein, along with (5.9) and (5.14). ☐

To conclude this section, we remark that the lack of a uniform-in-h bound for $\|\boldsymbol{\mathcal{R}}_h(\mathbf{T}_{2,h}(\underline{\mathbf{z}}_h))\|_{0,\infty;\Omega}$ prevents the use of (5.16) to establish a contraction property. Therefore, the Banach fixed-point theorem can not be applied to ensure uniqueness of the discrete solution, even for sufficiently small data.

5.5 A priori error analysis

In this section we consider that $\big((\vec{\mathbf{t}}, \boldsymbol{\sigma}), (\boldsymbol{\chi}, \vartheta), (\vec{\boldsymbol{\rho}}, \phi)\big) \in (\mathbf{H}\times\mathbf{Q}) \times (\mathrm{X}_2\times\mathrm{M}_1) \times (\mathbf{X}_2\times\mathbf{Y}_1)$ is the unique solution of (3.27) with $\mathbf{u} \in W_\delta$, and that $\big((\vec{\mathbf{t}}_h, \boldsymbol{\sigma}_h), (\boldsymbol{\chi}_h, \vartheta_h), (\vec{\boldsymbol{\rho}}_h, \phi_h)\big) \in (\mathbf{H}_h\times\mathbf{Q}_h) \times (\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}) \times (\mathbf{X}_{2,h}\times\mathbf{Y}_{1,h})$ is a solution of (5.1) with $\mathbf{u}_h \in W_{\delta_{\mathtt{d}}}$. Then, we define the global error as

$$\mathtt{E} \,:=\, \|(\vec{\mathbf{t}}, \boldsymbol{\sigma}) - (\vec{\mathbf{t}}_h, \boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\boldsymbol{\chi}, \vartheta) - (\boldsymbol{\chi}_h, \vartheta_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \|(\vec{\boldsymbol{\rho}}, \phi) - (\vec{\boldsymbol{\rho}}_h, \phi_h)\|_{\mathbf{X}_2\times\mathbf{Y}_1}, \tag{5.18}$$

and address an estimation of it by analyzing each of its addends. To this end, given a subspace Z_h of an arbitrary Banach space $(Z, \|\cdot\|_Z)$, we set

$$\mathrm{dist}(z, Z_h) \,:=\, \inf_{z_h\in Z_h} \|z - z_h\|_Z \quad \forall\, z \in Z. \tag{5.19}$$

First, we use the Strang a priori error estimate provided by [11, Lemma 6.1] (see also [28, Theorem 2.2]) applied to the context given by the first two rows of (3.27) and (5.1), from which readily follows that there exists a positive constant $\widehat{C}_S$, depending only on $\boldsymbol{\alpha}_{\mathtt{d}}$, $\boldsymbol{\beta}_{\mathtt{d}}$, $\|\mathbf{A}\|$ and $\|\mathbf{B}\|$, such that

$$\begin{aligned}
\|(\vec{\mathbf{t}}, \boldsymbol{\sigma}) - (\vec{\mathbf{t}}_h, \boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,&\le\, \widehat{C}_S\Big\{\mathrm{dist}(\vec{\mathbf{t}}, \mathbf{H}_h) + \mathrm{dist}(\boldsymbol{\sigma}, \mathbf{Q}_h)\\
&\quad + \|\mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta} - \mathbf{F}_{\boldsymbol{\rho}_h,\boldsymbol{\chi}_h,\vartheta_h}\|_{\mathbf{H}'_h} + \|\mathbf{A}_{\mathbf{u}}(\vec{\mathbf{t}}, \cdot) - \mathbf{A}_{\mathbf{u}_h}(\vec{\mathbf{t}}, \cdot)\|_{\mathbf{H}'_h}\Big\}\,.
\end{aligned} \tag{5.20}$$

Now, using the definition of $\mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta}$ (cf. (3.10)), we obtain

$$\|\mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta} - \mathbf{F}_{\boldsymbol{\rho}_h,\boldsymbol{\chi}_h,\vartheta_h}\|_{\mathbf{H}_h'} \,=\, \sup_{\substack{\vec{\mathbf{s}}_h\in\mathbf{H}_h\\ \vec{\mathbf{s}}_h\neq\mathbf{0}}} \frac{\left|\kappa^{-1}\,\delta_d\displaystyle\int_\Omega\big(|\boldsymbol{\rho}|^2\boldsymbol{\chi} - |\boldsymbol{\rho}_h|^2\boldsymbol{\chi}_h\big)\cdot\mathbf{v}_h \,-\, \delta_t\int_\Omega(\vartheta-\vartheta_h)\,\mathbf{g}\cdot\mathbf{v}_h\right|}{\|\vec{\mathbf{s}}_h\|_{\mathbf{H}}}\,.$$

Since the structure of the terms inside the supremum coincides with that of (4.75) and (4.76), an analogous argument yields

$$\begin{aligned}\|\mathbf{F}_{\boldsymbol{\rho},\boldsymbol{\chi},\vartheta} - \mathbf{F}_{\boldsymbol{\rho}_h,\boldsymbol{\chi}_h,\vartheta_h}\|_{\mathbf{H}_h'} \,\le\, \max\big\{\kappa^{-1}\delta_d,\delta_t\big\}\Big\{&\big(\|\boldsymbol{\rho}\|_{0,j;\Omega} + \|\boldsymbol{\rho}_h\|_{0,j;\Omega}\big)\,\|\boldsymbol{\rho}-\boldsymbol{\rho}_h\|_{0,j;\Omega}\,\|\boldsymbol{\chi}\|_{0,r;\Omega}\\ &+\|\boldsymbol{\rho}_h\|_{0,j;\Omega}^2\,\|\boldsymbol{\chi}-\boldsymbol{\chi}_h\|_{0,r;\Omega} \,+\, \|\vartheta-\vartheta_h\|_{0,r;\Omega}\,\|\mathbf{g}\|_{0,p;\Omega}\Big\}\,.\end{aligned}\tag{5.21}$$

On the other hand, the definitions of $\mathbf{A}_{\mathbf{u}}$ and $\mathbf{A}_{\mathbf{u}_h}$ (cf. (3.9a)) give

$$\|\mathbf{A}_{\mathbf{u}}(\vec{\mathbf{t}},\cdot) - \mathbf{A}_{\mathbf{u}_h}(\vec{\mathbf{t}},\cdot)\|_{\mathbf{H}_h'} \,=\, \sup_{\substack{\vec{\mathbf{s}}_h\in\mathbf{H}_h\\ \vec{\mathbf{s}}_h\neq\mathbf{0}}} \frac{\left|\dfrac{1}{2}\displaystyle\int_\Omega\mathbf{t}\,(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{v}_h \,-\, \dfrac{1}{2}\int_\Omega\big\{\mathbf{u}\otimes(\mathbf{u}-\mathbf{u}_h)\big\}:\mathbf{s}_h\right|}{\|\vec{\mathbf{s}}_h\|_{\mathbf{H}}},$$

so that, arguing as in the derivation of (4.76), we then obtain

$$\|\mathbf{A}_{\mathbf{u}}(\vec{\mathbf{t}},\cdot) - \mathbf{A}_{\mathbf{u}_h}(\vec{\mathbf{t}},\cdot)\|_{\mathbf{H}_h'} \,\le\, \frac{\sqrt{n}}{2}\|\vec{\mathbf{t}}\|_{\mathbf{H}}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,.\tag{5.22}$$

Consequently, inserting (5.21) and (5.22) on (5.20) yields

$$\begin{aligned}\|(\vec{\mathbf{t}},\boldsymbol{\sigma}) - (\vec{\mathbf{t}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\le\, &\widehat{C}_{\mathbf{S}}\Big\{\mathrm{dist}(\vec{\mathbf{t}},\mathbf{H}_h) \,+\, \mathrm{dist}(\boldsymbol{\sigma},\mathbf{Q}_h)\Big\}\\ +\,\underline{C}_{\mathbf{S}}\Big\{&\big(\|\boldsymbol{\rho}\|_{0,j;\Omega} \,+\, \|\boldsymbol{\rho}_h\|_{0,j;\Omega}\big)\,\|\boldsymbol{\chi}\|_{0,r;\Omega}\,\|\boldsymbol{\rho}-\boldsymbol{\rho}_h\|_{0,j;\Omega}\\ +\,\|\boldsymbol{\rho}_h\|_{0,j;\Omega}^2\,&\|\boldsymbol{\chi}-\boldsymbol{\chi}_h\|_{0,r;\Omega} \,+\, \|\mathbf{g}\|_{0,p;\Omega}\,\|\vartheta-\vartheta_h\|_{0,r;\Omega} \,+\, \|\vec{\mathbf{t}}\|_{\mathbf{H}}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,,\end{aligned}\tag{5.23}$$

where $\underline{C}_{\mathbf{S}} \,:=\, \widehat{C}_{\mathbf{S}}\,\max\Big\{\kappa^{-1}\delta_d,\delta_t,\sqrt{n}/2\Big\}$.

Next, we apply the Strang a priori error estimate from [5, Proposition 2.1, Corollary 2.3, Theorem 2.3] (see also [28, Theorem 2.2]) to the context given by the third and fourth rows of (3.27) and (5.1), in which the free terms are considered as part of the respective functionals on the right-hand side. Thus, we deduce the existence of a constant $\widehat{C}_{\mathbf{T}}$, depending on $\alpha_{\mathtt{d}}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$, $\|a\|$, $\|b_1\|$ and $\|b_2\|$, such that

$$\|(\boldsymbol{\chi},\vartheta) - (\boldsymbol{\chi}_h,\vartheta_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \le \widehat{C}_{\mathbf{T}}\left\{\mathrm{dist}(\boldsymbol{\chi},\mathrm{X}_{2,h}) \,+\, \mathrm{dist}(\vartheta,\mathrm{M}_{1,h}) + \sup_{\substack{\omega_h\in\mathrm{M}_{2,h}\\ \omega_h\neq 0}}\frac{\left|\displaystyle\int_\Omega(\mathbf{u}\cdot\boldsymbol{\chi} - \mathbf{u}_h\cdot\boldsymbol{\chi}_h)\omega_h\right|}{\|\omega_h\|_{\mathrm{M}_2}}\right\},$$

from which, adding and subtracting the expression $\mathbf{u}\cdot\boldsymbol{\chi}_h$ in the numerator within the supremum, and proceeding as for the derivation of (3.12), it readily follows that

$$\begin{aligned}\|(\boldsymbol{\chi},\vartheta) - (\boldsymbol{\chi}_h,\vartheta_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \,\le\, &\widehat{C}_{\mathbf{T}}\Big\{\mathrm{dist}(\boldsymbol{\chi},\mathrm{X}_{2,h}) \,+\, \mathrm{dist}(\vartheta,\mathrm{M}_{1,h})\Big\}\\ &+\widehat{C}_{\mathbf{T}}\Big\{\|\mathbf{u}\|_{0,4;\Omega}\|\boldsymbol{\chi}-\boldsymbol{\chi}_h\|_{0,r;\Omega} \,+\, \|\boldsymbol{\chi}_h\|_{0,r;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,.\end{aligned}$$

In turn, applying once again the Strang a priori error estimate from [28, Theorem 2.2], but this time to the last two rows of (3.27) and (5.1), we obtain an estimation for $\|(\vec{\boldsymbol{\rho}},\phi) - (\vec{\boldsymbol{\rho}}_h,\phi_h)\|_{\mathbf{X}_2\times\mathbf{Y}_1}$. The supremum involved is handled by adding and subtracting $\int_\Omega \eta(\vartheta)\boldsymbol{\rho}\cdot\boldsymbol{\varrho}_h$ and $\int_\Omega \eta(\vartheta_h)\boldsymbol{\rho}\cdot\boldsymbol{\varrho}_h$, rearranging the resulting terms, invoking Hölder's inequality with exponents j-ℓ, and using the upper bound for η (cf. (2.2)). This yields the existence of a positive constant $\widehat{C}_{\boldsymbol{\mathcal{R}}}$, depending on $\alpha_{\mathcal{A},\mathtt{d}}$, $\beta_{\mathcal{B},1,\mathtt{d}}$, $\beta_{\mathcal{B},2,\mathtt{d}}$, $\|\mathcal{A}\|$, $\|\mathcal{B}_1\|$, $\|\mathcal{B}_2\|$, and η_2, such that

$$\|(\vec{\boldsymbol{\rho}},\phi)-(\vec{\boldsymbol{\rho}}_h,\phi_h)\|_{\mathbf{X}_2\times\mathbf{Y}_1}\,\le\,\widehat{C}_{\boldsymbol{\mathcal{R}}}\Big\{\mathrm{dist}(\vec{\boldsymbol{\rho}},\mathbf{X}_{2,h})\,+\,\mathrm{dist}(\phi,\mathbf{Y}_{1,h})\,+\,\|(\mathcal{A}_\theta-\mathcal{A}_{\theta_h})(\vec{\boldsymbol{\rho}},\cdot)\|_{\mathbf{X}'_{1,h}}\Big\}\,,$$

which, combined with an estimate of the consistency term $\mathcal{A}_\theta-\mathcal{A}_{\theta_h}$ as in the final part of the proof of Lemma 4.17, yields

$$\|(\vec{\boldsymbol{\rho}},\phi)-(\vec{\boldsymbol{\rho}}_h,\phi_h)\|_{\mathbf{X}_2\times\mathbf{Y}_1}\,\le\,\widehat{C}_{\boldsymbol{\mathcal{R}}}\Big\{\mathrm{dist}(\vec{\boldsymbol{\rho}},\mathbf{X}_{2,h})\,+\,\mathrm{dist}(\phi,\mathbf{Y}_{1,h})\Big\}+\overline{C}_{\boldsymbol{\mathcal{R}}}\,\|g\|_{\epsilon,j;\Omega}\,\|\vartheta-\vartheta_h\|_{0,r;\Omega},\qquad(5.24)$$

where $\overline{C}_{\boldsymbol{\mathcal{R}}}\,:=\,\widehat{C}_{\boldsymbol{\mathcal{R}}}\,L_\eta\,\|\mathbf{i}_\epsilon\|\,C_\epsilon$. Finally, adding (5.23) - (5.24), and employing the a priori estimates for $\|\vec{\boldsymbol{\rho}}\|_{\mathbf{X}_2}$, $\|\vec{\boldsymbol{\rho}}_h\|_{\mathbf{X}_2}$, $\|\boldsymbol{\chi}\|_{\mathrm{X}_2}$, $\|\boldsymbol{\chi}_h\|_{\mathrm{X}_2}$, $\|\vec{\mathbf{t}}\|_{\mathbf{H}}$ provided by Theorems 4.18 and 5.5, we get, in terms of the notation introduced in (4.87), (5.18) and (5.19), that

$$\begin{aligned}\mathtt{E}\le\widehat{C}\Big\{&\mathrm{dist}\big((\vec{\mathbf{t}},\boldsymbol{\sigma}),\mathbf{H}_h\times\mathbf{Q}_h\big)\,+\,\mathrm{dist}\big((\boldsymbol{\chi},\vartheta),\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big)\,+\,\mathrm{dist}\big((\vec{\boldsymbol{\rho}},\phi),\mathbf{X}_{2,h}\times\mathbf{Y}_{1,h}\big)\Big\}\\&+\widehat{C}_0\Big\{\mathtt{D}(\vartheta_D,f)\big(\|g\|_{0,j;\Omega}+1\big)\,+\,\big(\|g\|_{0,j;\Omega}^2\,+\,\|\mathbf{g}\|_{0,p;\Omega}\big)\big(1\,+\,\mathtt{D}(\vartheta_D,f)\big)\\&\qquad+\|g\|_{\epsilon,j;\Omega}+\|\mathbf{f}\|_{0,4/3;\Omega}\,+\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,\mathtt{E}\,,\end{aligned}$$

where $\widehat{C}\,:=\,\max\big\{\widehat{C}_{\mathbf{S}},\widehat{C}_{\mathbf{T}},\widehat{C}_{\boldsymbol{\mathcal{R}}}\big\}$, and $\widehat{C}_0$ is a positive constant depending on $\overline{C}_{\mathbf{S}}$, C_2, $C_{2,\mathtt{d}}$, $C_{\mathbf{T}}$, $C_{\mathbf{T},\mathtt{d}}$, C_1, $\overline{C}_{\mathbf{T}}$ and $\overline{C}_{\boldsymbol{\mathcal{R}}}$.

We can now state the Céa type Lemma as a straightforward consequence of the last inequality.

Theorem 5.6. *In addition to the hypotheses of Theorems 4.18 and 5.5, assume that*

$$\begin{aligned}\widehat{C}_0\Big\{&\mathtt{D}(\vartheta_D,f)\big(\|g\|_{0,j;\Omega}+1\big)\,+\,\big(\|g\|_{0,j;\Omega}^2\,+\,\|\mathbf{g}\|_{0,p;\Omega}\big)\big(1\,+\,\mathtt{D}(\vartheta_D,f)\big)\\&+\|g\|_{\epsilon,j;\Omega}+\|\mathbf{f}\|_{0,4/3;\Omega}\,+\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,\le\,\frac{1}{2}\,.\end{aligned}$$

Then, denoting $\overline{C}\,:=\,2\widehat{C}$, there holds

$$\begin{aligned}&\|(\vec{\mathbf{t}},\boldsymbol{\sigma})-(\vec{\mathbf{t}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}}\,+\,\|(\boldsymbol{\chi},\vartheta)-(\boldsymbol{\chi}_h,\vartheta_h)\|_{\mathrm{X}_2\times\mathrm{M}_1}\,+\,\|(\vec{\boldsymbol{\rho}},\phi)-(\vec{\boldsymbol{\rho}}_h,\phi_h)\|_{\mathbf{X}_2\times\mathbf{Y}_1}\\&\quad\le\,\overline{C}\Big\{\mathrm{dist}\big((\vec{\mathbf{t}},\boldsymbol{\sigma}),\mathbf{H}_h\times\mathbf{Q}_h\big)\,+\,\mathrm{dist}\big((\boldsymbol{\chi},\vartheta),\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big)\,+\,\mathrm{dist}\big((\vec{\boldsymbol{\rho}},\phi),\mathbf{X}_{2,h}\times\mathbf{Y}_{1,h}\big)\Big\}\,.\end{aligned}$$

6 Specific finite element subspaces

We now present finite element subspaces fulfilling the hypotheses **(H.1)**-**(H.9)** introduced in Sections 5.3 and 5.4, followed by the derivation of convergence rates for the corresponding discrete method.

6.1 Preliminaries

Given an integer $k\ge 0$ and a subset $S\subseteq\mathrm{R}^2$, we denote by $\mathrm{P}_k(S)$ and $\widetilde{\mathrm{P}}_k(S)$ the space of polynomial functions on S of degree $\le\,k$ and $=\,k$, respectively. Additionally, using the notations from Section 5.1, we define for each $K\in\mathcal{T}_h$ the corresponding local Raviart-Thomas space of order k as

$$\mathbf{RT}_k(K)\,:=\,\mathbf{P}_k(K)\oplus\widetilde{\mathrm{P}}_k(K)\mathbf{x},$$

where $\mathbf{P}_k(K) := [\mathrm{P}_k(K)]^2$, and $\mathbf{x}$ is the generic vector in R^2. By denoting $\mathbb{P}_k(K)$ and $\mathbb{RT}_k(K)$ the tensor version of $\mathbf{P}_K(K)$ and $\mathbf{RT}_k(K)$, respectively, we let $\mathrm{P}_k(\mathcal{T}_h)$, $\mathbf{P}_k(\mathcal{T}_h)$, $\mathbb{P}_k(\mathcal{T}_h)$, $\mathbf{RT}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_k(K)$, $\mathbf{P}_k(K)$, $\mathbb{P}_k(K)$, $\mathbf{RT}_k(K)$ and $\mathbb{RT}_k(K)$, respectively, that is

$$\begin{aligned}
\mathrm{P}_k(\mathcal{T}_h) &:= \Big\{\phi_h \in \mathrm{L}^2(\Omega): \quad \phi_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\},\\
\mathbf{P}_k(\mathcal{T}_h) &:= \Big\{\mathbf{v}_h \in \mathbf{L}^2(\Omega): \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\},\\
\mathbb{P}_k(\mathcal{T}_h) &:= \Big\{\mathbf{s}_h \in \mathbb{L}^2(\Omega): \quad \mathbf{s}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\},
\end{aligned}$$

$$\begin{aligned}
\mathbf{RT}_k(\mathcal{T}_h) &:= \Big\{\boldsymbol{\eta}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \boldsymbol{\eta}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\},\\
\mathbb{RT}_k(\mathcal{T}_h) &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}.
\end{aligned}$$

We remark here that for each $t, s \in (1, +\infty)$ such that $t \geq s$, there hold

$$\mathrm{P}_k(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)\,, \quad \mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div}_s;\Omega)\,, \quad \text{and} \quad \mathbb{RT}_k(\mathcal{T}_h) \subseteq \mathbb{H}(\mathbf{div}_t;\Omega)\,,$$

inclusions that will be implicitly used below when introducing the specific finite element subspaces. In fact, we now define

$$\begin{gathered}
\mathbb{H}_h^{\mathbf{t}} := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_k(\mathcal{T}_h)\,, \quad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\mathcal{T}_h)\,, \quad \mathbf{H}_h := \mathbb{H}_h^{\mathbf{t}} \times \mathbf{H}_h^{\mathbf{u}}\,,\\
\mathbb{H}_h^{\boldsymbol{\sigma}} := \mathbb{RT}_k(\mathcal{T}_h)\,, \quad \mathbf{Q}_h := \mathbb{H}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\
\mathrm{X}_{2,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \mathrm{M}_{1,h} := \mathrm{P}_k(\mathcal{T}_h)\\
\mathrm{X}_{1,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \mathrm{M}_{2,h} := \mathrm{P}_k(\mathcal{T}_h)\\
\mathbf{X}_{2,h}^{\boldsymbol{\rho}} := \mathbf{P}_k(\mathcal{T}_h)\,, \quad \mathbf{X}_{2,h}^{\boldsymbol{\xi}} := \mathbf{RT}_k(\mathcal{T}_h) \cap \mathbf{H}_0^j(\mathrm{div}_j;\Omega)\,,\\
\mathbf{X}_{2,h} := \mathbf{X}_{2,h}^{\boldsymbol{\rho}} \times \mathbf{X}_{2,h}^{\boldsymbol{\xi}}\,, \quad \mathbf{Y}_{1,h} := \mathrm{P}_k(\mathcal{T}_h) \cap \mathrm{L}_0^j(\Omega)\,,\\
\mathbf{X}_{1,h}^{\boldsymbol{\varrho}} := \mathbf{P}_k(\mathcal{T}_h)\,, \quad \mathbf{X}_{1,h}^{\boldsymbol{\lambda}} := \mathbf{RT}_k(\mathcal{T}_h) \cap \mathbf{H}_0^{\ell}(\mathrm{div}_{\ell};\Omega)\,,\\
\mathbf{X}_{1,h} := \mathbf{X}_{1,h}^{\boldsymbol{\varrho}} \times \mathbf{X}_{1,h}^{\boldsymbol{\lambda}}\,, \quad \mathbf{Y}_{2,h} := \mathrm{P}_k(\mathcal{T}_h) \cap \mathrm{L}_0^{\ell}(\Omega)\,.
\end{gathered} \tag{6.1}$$

6.2 Verification of the stability conditions

Throughout this section, we restrict our attention to convex domains and show that the finite element spaces introduced in (6.1) satisfy hypotheses **(H.1)**-**(H.9)** from Section 5. Since most of these properties have already been established in earlier works, we provide appropriate references and include proofs for completeness when needed. To this end, we introduce the following operators. First, we let $\boldsymbol{\mathcal{P}}_h^k : \mathbf{L}^1(\Omega) \to \mathbf{P}_k(\mathcal{T}_h)$ be the projector that assigns, to each $\mathbf{v} \in \mathbf{L}^1(\Omega)$, the unique element $\boldsymbol{\mathcal{P}}_h^k(\mathbf{v})$ in $\mathbf{P}_k(\mathcal{T}_h)$ satisfying

$$\int_\Omega \boldsymbol{\mathcal{P}}_h^k(\mathbf{v}) \cdot \mathbf{q}_h = \int_\Omega \mathbf{v} \cdot \mathbf{q}_h \qquad \forall\, \mathbf{q}_h \in \mathbf{P}_k(\mathcal{T}_h). \tag{6.2}$$

Now, introducing the space

$$V_h^k := \Big\{\boldsymbol{\eta}_h \in \mathbf{RT}_k(\mathcal{T}_h) : \quad \boldsymbol{\eta}_h \cdot \boldsymbol{\nu} \quad \text{on} \quad \Gamma \quad \text{and} \quad \mathrm{div}(\boldsymbol{\eta}_h) = 0 \quad \text{in} \quad \Omega\Big\}, \tag{6.3}$$

we let similarly $\varTheta_h^k : \mathbf{L}^1(\Omega) \to V_h^k$ be the projector that assigns, to each $\boldsymbol{\varkappa} \in \mathbf{L}^1(\Omega)$, the unique element $\varTheta_h^k(\boldsymbol{\varkappa})$ in V_h^k satisfying

$$\int_\Omega \varTheta_h^k(\boldsymbol{\varkappa}) \cdot \boldsymbol{\lambda}_h = \int_\Omega \boldsymbol{\varkappa} \cdot \boldsymbol{\lambda}_h \qquad \forall\, \boldsymbol{\lambda}_h \in V_h^k. \tag{6.4}$$

At this point, we gather two auxiliary results concerning these projectors that enable us to verify the required inf–sup conditions. Further details can be found in [8, Appendix A] and [31, Section 4.3].

Lemma 6.1. *Given* $t \in (1, +\infty)$ *and an integer* $k \geq 0$*, there exists a positive constant* $C_{\mathcal{P}}$*, independent of* h*, such that*

$$\|\boldsymbol{\mathcal{P}}_h^k(\mathbf{w})\|_{0,t;\Omega} \,\leq\, C_{\mathcal{P}}\, \|\mathbf{w}\|_{0,t;\Omega} \qquad \forall\, \mathbf{w} \in \mathbf{L}^t(\Omega). \tag{6.5}$$

Lemma 6.2. *Assume* Ω *is convex. Given* $t \in (1, +\infty)$ *and an integer* $k \geq 0$*, there exists a positive constant* C_t^k*, independent of* h*, such that*

$$\|\Theta_h^k(\boldsymbol{\omega})\|_{0,t;\Omega} \,\leq\, C_t^k\, \|\boldsymbol{\omega}\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\omega} \in \widetilde{\mathbf{H}}_0^t(\mathrm{div}_t;\Omega). \tag{6.6}$$

Additionally, we state a fundamental property which will be used repeatedly in the sequel. For this, we observe that for every $\boldsymbol{\eta}_h \in V_h^k$ (cf. (6.3)), it holds that $\boldsymbol{\eta}_h \in \mathbf{RT}_k(\mathcal{T}_h)$ and $\mathrm{div}(\boldsymbol{\eta}_h) = 0$, which implies that (cf. proof of [27, Theorem 3.3]) $\boldsymbol{\eta}_h \in \mathbf{P}_k(\mathcal{T}_h)$, showing in this way that

$$V_h^k \subseteq \mathbf{P}_k(\mathcal{T}_h). \tag{6.7}$$

Hypotheses **(H.1)** and **(H.2)** are established in [8, Lemma 5.1], while **(H.3)** follows from [8, Lemma 5.2]. Regarding **(H.4)**, the case $i = 1$ can be obtained as a slight modification of [31, Lemma 4.5], and its proof is therefore omitted, whereas the case $i = 2$ is proved in [8, Lemma 5.3]. Next, **(H.5)** follows directly from the definitions in (6.1).

We now focus on **(H.6)**, starting with (5.12a). Indeed, adding and subtracting η_1 (cf. (2.2)), we obtain (cf. (3.26a))

$$\mathbf{a}_{\theta_h}(\boldsymbol{\varkappa}_h, \boldsymbol{\varrho}_h) \,=\, \eta_1 \int_\Omega \boldsymbol{\varkappa}_h \cdot \boldsymbol{\varrho}_h \,+\, \int_\Omega \big(\eta(\theta_h) - \eta_1\big)\, \boldsymbol{\varkappa}_h \cdot \boldsymbol{\varrho}_h, \qquad \forall\, \boldsymbol{\varkappa}_h \in V_{2,h},\, \forall\, \boldsymbol{\varrho}_h \in V_{1,h}. \tag{6.8}$$

Next, bounding $\eta(\theta_h)$ from above by η_2 (cf. (2.2)) and applying Hölder's inequality with exponents j and ℓ, we find that

$$\left| \int_\Omega \big(\eta(\theta_h) - \eta_1\big)\, \boldsymbol{\varkappa}_h \cdot \boldsymbol{\varrho}_h \right| \,\leq\, \big(\eta_2 - \eta_1\big)\, \|\boldsymbol{\varkappa}_h\|_{0,j;\Omega}\, \|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega} \qquad \forall\, \boldsymbol{\varkappa}_h \in V_{2,h},\, \forall\, \boldsymbol{\varrho}_h \in V_{1,h},$$

which, when combined with (6.8), yields

$$\sup_{\substack{\boldsymbol{\varrho}_h \in \mathrm{V}_{1,h} \\ \boldsymbol{\varrho}_h \neq \mathbf{0}}} \frac{\mathbf{a}_{\theta_h}(\boldsymbol{\varkappa}_h, \boldsymbol{\varrho}_h)}{\|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}} \,\geq\, \eta_1 \sup_{\substack{\boldsymbol{\varrho}_h \in \mathrm{V}_{1,h} \\ \boldsymbol{\varrho}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\varkappa}_h \cdot \boldsymbol{\varrho}_h}{\|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}} \,-\, \big(\eta_2 - \eta_1\big)\, \|\boldsymbol{\varkappa}_h\|_{0,j;\Omega} \quad \forall\, \boldsymbol{\varkappa}_h \in \mathrm{V}_{2,h}. \tag{6.9}$$

This procedure is introduced to overcome the lack of regularity in the discrete setting. The resulting supremum can be handled by adapting the arguments used in the continuous version of **(H.6)** (cf. Lemma 4.10). Indeed, given $\theta_h \in \mathrm{M}_{1,h}$ (cf. (5.1)), we first let $\boldsymbol{\varkappa}_h \in V_{2,h} \subseteq \mathbf{L}^j(\Omega)$ (cf. (5.11)), $\boldsymbol{\varkappa}_h \neq \mathbf{0}$, and define $\boldsymbol{\varkappa}_{h,\ell} \in \mathbf{L}^\ell(\Omega)$ as in (4.31), but with j and ℓ instead of r and s, respectively, which satisfies (cf. (4.32))

$$\int_\Omega \boldsymbol{\varkappa}_{h,\ell} \cdot \boldsymbol{\varkappa}_h \,=\, \|\boldsymbol{\varkappa}_{h,\ell}\|_{0,\ell;\Omega}\, \|\boldsymbol{\varkappa}_h\|_{0,j;\Omega}. \tag{6.10}$$

Now, using Lemma 4.9 with $t = \ell$ and $\xi = 1$, we define $\widehat{\boldsymbol{\varrho}}_h := \boldsymbol{\mathcal{P}}_h^k(\boldsymbol{\varkappa}_{h,\ell}) - \Theta_h^k\big(D_{\ell,\xi}(\boldsymbol{\varkappa}_{h,\ell})\big)$ and observe, according to the definitions of $\boldsymbol{\mathcal{P}}_h^k$ (cf. (6.2)) and Θ_h^k (cf. (6.4)), and in view of (6.7), that $\widehat{\boldsymbol{\varrho}}_h$ belongs to

$\mathbf{P}_k(\mathcal{T}_h)$. Moreover, given $\boldsymbol{\varsigma}_h \in \mathbf{X}^{\boldsymbol{\xi}}_{2,h} \cap \widetilde{\mathbf{H}}^j_0(\mathrm{div}_j;\Omega)$, it is clear that $\boldsymbol{\varsigma}_h \in V^k_h$ (cf. (6.3)), so that, according to (6.2) and (6.4), as well as applying (4.51) with $\mathbf{z} = \boldsymbol{\varsigma}_h$, we get

$$\int_\Omega \widehat{\boldsymbol{\varrho}}_h \cdot \boldsymbol{\varsigma}_h \,=\, \int_\Omega \boldsymbol{\mathcal{P}}^k_h(\boldsymbol{\varkappa}_{h,\ell}) \cdot \boldsymbol{\varsigma}_h \,-\, \Theta^k_h\big(D_{\ell,\xi}(\boldsymbol{\varkappa}_{h,\ell})\big) \cdot \boldsymbol{\varsigma}_h \,=\, \int_\Omega \boldsymbol{\varkappa}_{h,\ell} \cdot \boldsymbol{\varsigma}_h \,-\, \int_\Omega D_{\ell,\xi}(\boldsymbol{\varkappa}_{h,\ell}) \cdot \boldsymbol{\varsigma}_h \,=\, 0,$$

thus showing that $\widehat{\boldsymbol{\varrho}}_h \in V_{1,h}$ (cf. (5.11)). Using the definition of $\widehat{\boldsymbol{\varrho}}_h$, the integral displayed below can be decomposed into two terms. The first is handled by invoking (6.2) since $\boldsymbol{\varkappa}_h \in V_{2,h} \subseteq \mathbf{P}_k(\mathcal{T}_h)$, whereas the second vanishes as a consequence of the defining property of $V_{2,h}$, as applied to $\boldsymbol{\varkappa}_h$ and $\Theta^k_h\big(D_{\ell,\xi}(\boldsymbol{\varkappa}_{h,\ell})\big) \in V^k_h$ (cf. (6.3)), which is contained in $\mathbf{X}^{\boldsymbol{\lambda}}_{1,h} \cap \widetilde{\mathbf{H}}^\ell_0(\mathrm{div}_\ell;\Omega)$. Collecting these contributions and applying (6.10), we obtain

$$\int_\Omega \boldsymbol{\varkappa}_h \cdot \widehat{\boldsymbol{\varrho}}_h \,=\, \int_\Omega \boldsymbol{\varkappa}_h \cdot \boldsymbol{\mathcal{P}}^k_h(\boldsymbol{\varkappa}_{h,\ell}) \,-\, \int_\Omega \boldsymbol{\varkappa}_h \cdot \Theta^k_h\big(D_{\ell,\xi}(\boldsymbol{\varkappa}_{h,\ell})\big) \,=\, \int_\Omega \boldsymbol{\varkappa}_h \cdot \boldsymbol{\varkappa}_{h,\ell} \,=\, \|\boldsymbol{\varkappa}_{h,\ell}\|_{0,\ell;\Omega}\, \|\boldsymbol{\varkappa}_h\|_{0,j;\Omega}, \tag{6.11}$$

which implies that $\widehat{\boldsymbol{\varrho}}_h \neq \mathbf{0}$. On the other hand, Hölder's inequality is applied with j and ℓ, together with the stability properties of $\boldsymbol{\mathcal{P}}^k_h$ (cf. (6.5)) and Θ^k_h (cf. (6.6)), as well as the continuity of the operator $D_{\ell,\xi}$. In this context, defining $\widehat{C}_{\mathcal{P},k} := C_{\mathcal{P}} + \max\big\{C^k_\ell\,\|D_{\ell,\xi}\|,\, C^k_j\,\|D_{j,\xi}\|\big\}$, one obtains

$$\|\widehat{\boldsymbol{\varrho}}_h\|_{0,\ell;\Omega} \,\leq\, \|\boldsymbol{\mathcal{P}}^k_h(\boldsymbol{\varkappa}_{h,\ell})\|_{0,\ell;\Omega} \,+\, \|\Theta^k_h(D_{\ell,\xi}(\boldsymbol{\varkappa}_{h,\ell}))\|_{0,\ell;\Omega} \,\leq\, \widehat{C}_{\mathcal{P},k}\, \|\boldsymbol{\varkappa}_{h,\ell}\|_{0,\ell;\Omega}. \tag{6.12}$$

We now bound the supremum in (6.9) from below by choosing $\widehat{\boldsymbol{\varrho}}_h$ and substituting (6.11) into the numerator, while the denominator is estimated by (6.12), obtaining

$$\sup_{\substack{\boldsymbol{\varrho}_h \in \mathrm{V}_{1,h}\\ \boldsymbol{\varrho}_h \neq \mathbf{0}}} \frac{\mathbf{a}_{\theta_h}(\boldsymbol{\varkappa}_h, \boldsymbol{\varrho}_h)}{\|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}} \,\geq\, \eta_1 \, \frac{\displaystyle\int_\Omega \boldsymbol{\varkappa}_h \cdot \widehat{\boldsymbol{\varrho}}_h}{\|\widehat{\boldsymbol{\varrho}}_h\|_{0,\ell;\Omega}} \,-\, \big(\eta_2 - \eta_1\big)\, \|\boldsymbol{\varkappa}_h\|_{0,j;\Omega} \,\geq\, \left\{ \frac{\eta_1}{\widehat{C}_{\mathcal{P},k}} \,-\, \big(\eta_2 \,-\, \eta_1\big) \right\} \|\boldsymbol{\varkappa}_h\|_{0,j;\Omega}\,,$$

from which, assuming that

$$\eta_2 - \eta_1 \,\leq\, \frac{\eta_1}{2\,\widehat{C}_{\mathcal{P},k}}\,, \tag{6.13}$$

we obtain (5.12a) with $\alpha_{\mathbf{a},\mathbf{d}} := \dfrac{\eta_1}{2\,\widehat{C}_{\mathcal{P},k}}$. The argument leading to (5.12b) is analogous. In fact, given $\boldsymbol{\varrho}_h \in V_{1,h} \subseteq \mathbf{L}^\ell(\Omega)$ (cf. (5.11)), $\boldsymbol{\varrho}_h \neq \mathbf{0}$, we define $\boldsymbol{\varrho}_{h,j} \in \mathbf{L}^j(\Omega)$ as in (4.31), but now with ℓ and j instead of r and s, respectively, and notice that (cf. (4.32))

$$\int_\Omega \boldsymbol{\varrho}_h \cdot \boldsymbol{\varrho}_{h,j} \,=\, \|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}\, \|\boldsymbol{\varrho}_{h,j}\|_{0,j;\Omega}.$$

Then, letting $\widehat{\boldsymbol{\varkappa}}_h := \boldsymbol{\mathcal{P}}^k_h(\boldsymbol{\varrho}_{h,j}) \,-\, \Theta^k_h\big(D_{j,\xi}(\boldsymbol{\varrho}_{h,j})\big) \in \mathbf{P}_k(\mathcal{T}_h)$, and proceeding similarly as before, (6.13) allows us to get

$$\sup_{\boldsymbol{\varkappa}_h \in \mathrm{V}_{2,h}} \mathbf{a}_{\theta_h}(\boldsymbol{\varkappa}_h, \boldsymbol{\varrho}_h) \,\geq\, \mathbf{a}_{\theta_h}(\widehat{\boldsymbol{\varkappa}}_h, \boldsymbol{\varrho}_h) \,\geq\, \Big\{\eta_1 \,-\, \big(\eta_2 \,-\, \eta_1\big)\widehat{C}_{\mathcal{P},k}\Big\}\, \|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}\, \|\boldsymbol{\varrho}_{h,j}\|_{0,j;\Omega} > 0,$$

from which, (5.12b) follows.

We summarize the previous discussion in the following result, which establishes that **(H.6)** holds.

Lemma 6.3. *Assume that the relative variation of* η (cf. (2.2)) *is small enough so that* (6.13) *is satisfied. Then, given* $k \geq 0$, *there exists a positive constant* $\alpha_{\mathbf{a},\mathbf{d}}$, *independent of* h, *such that for each* $\theta_h \in \mathrm{M}_{1,h}$ *there hold*

$$\sup_{\substack{\boldsymbol{\varrho}_h \in \mathrm{V}_{1,h}\\ \boldsymbol{\varrho}_h \neq \mathbf{0}}} \frac{\mathbf{a}_{\theta_h}(\boldsymbol{\varkappa}_h, \boldsymbol{\varrho}_h)}{\|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}} \,\geq\, \alpha_{\mathbf{a},\mathbf{d}}\, \|\boldsymbol{\varkappa}_h\|_{0,j;\Omega} \qquad \forall\, \boldsymbol{\varkappa}_h \in \mathrm{V}_{2,h}, \quad \textit{and}$$

$$\sup_{\boldsymbol{\varkappa}_h \in \mathrm{V}_{2,h}} \mathbf{a}_{\theta_h}(\boldsymbol{\varkappa}_h, \boldsymbol{\varrho}_h) \,>\, 0 \qquad \forall\, \boldsymbol{\varrho}_h \in \mathrm{V}_{1,h},\ \boldsymbol{\varrho}_h \neq \mathbf{0}\,.$$

We now present the result ensuring that (**H.7**) holds.

Lemma 6.4. *There exist positive constants* $\beta_{i,\mathtt{d}}$, $i \in \{1,2\}$, *such that*

$$\sup_{\substack{\boldsymbol{\varrho}_h \in \mathbf{X}^{\boldsymbol{\varrho}}_{1,h}\\ \boldsymbol{\varrho}_h \neq \mathbf{0}}} \frac{\mathbf{b}_1(\boldsymbol{\varrho}_h, \boldsymbol{\varsigma}_h)}{\|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}} \geq \beta_{1,\mathtt{d}} \, \|\boldsymbol{\varsigma}_h\|_{j,\mathrm{div}_j;\Omega} \qquad \forall\, \boldsymbol{\varsigma}_h \in \mathbf{X}^{\boldsymbol{\xi}}_{2,h} \cap \widetilde{\mathbf{H}}^j_0(\mathrm{div}_j;\Omega)\,, \quad \textit{and}$$

$$\sup_{\substack{\boldsymbol{\varkappa}_h \in \mathbf{X}^{\boldsymbol{\rho}}_{2,h}\\ \boldsymbol{\varkappa}_h \neq \mathbf{0}}} \frac{\mathbf{b}_2(\boldsymbol{\varkappa}_h, \boldsymbol{\lambda}_h)}{\|\boldsymbol{\varkappa}_h\|_{0,j;\Omega}} \geq \beta_{2,\mathtt{d}} \, \|\boldsymbol{\lambda}_h\|_{\ell;\mathrm{div}_\ell;\Omega} \qquad \forall\, \boldsymbol{\lambda}_h \in \mathbf{X}^{\boldsymbol{\lambda}}_{1,h} \cap \widetilde{\mathbf{H}}^\ell_0(\mathrm{div}_\ell;\Omega)\,.$$

Proof. For the case $i = 1$, given $\boldsymbol{\varsigma}_h \in \mathbf{X}^{\boldsymbol{\xi}}_{2,h} \cap \widetilde{\mathbf{H}}^j_0(\mathrm{div}_j;\Omega)$, we define $\boldsymbol{\varsigma}_{h,\ell} \in \mathbf{L}^\ell(\Omega)$ as in 4.31 but with j and ℓ instead of r and s, respectively, which satisfies (cf. (4.32))

$$\int_\Omega \boldsymbol{\varsigma}_h \cdot \boldsymbol{\varsigma}_{h,\ell} = \|\boldsymbol{\varsigma}_h\|_{0,j;\Omega} \, \|\boldsymbol{\varsigma}_{h,\ell}\|_{0,\ell;\Omega}. \tag{6.14}$$

Then, defining $\widehat{\boldsymbol{\varrho}}_h := -\boldsymbol{\mathcal{P}}^k_h(\boldsymbol{\varsigma}_{h,\ell}) \in \mathbf{P}_k(\mathcal{T}_h) = \mathbf{X}^{\boldsymbol{\varrho}}_{1,h}$, we bound the supremum from below by $\widehat{\boldsymbol{\varrho}}_h$. Noting from (6.3) and (6.7) that $\boldsymbol{\varsigma}_h \in \mathbf{P}_k(\mathcal{T}_h)$, and then employing (6.14) and the stability of $\boldsymbol{\mathcal{P}}^k_h$ (cf. (6.5)), we deduce that

$$\sup_{\substack{\boldsymbol{\varrho}_h \in \mathbf{X}^{\boldsymbol{\varrho}}_{1,h}\\ \boldsymbol{\varrho}_h \neq \mathbf{0}}} \frac{\mathbf{b}_1(\boldsymbol{\varrho}_h, \boldsymbol{\varsigma}_h)}{\|\boldsymbol{\varrho}_h\|_{0,\ell;\Omega}} \geq \frac{\mathbf{b}_1(\widehat{\boldsymbol{\varrho}}_h, \boldsymbol{\varsigma}_h)}{\|\widehat{\boldsymbol{\varrho}}_h\|_{0,\ell;\Omega}} = \frac{\displaystyle\int_\Omega \boldsymbol{\mathcal{P}}^k_h(\boldsymbol{\varsigma}_{h,\ell}) \cdot \boldsymbol{\varsigma}_h}{\|\widehat{\boldsymbol{\varrho}}_h\|_{0,j;\Omega}} \geq \frac{\displaystyle\int_\Omega \boldsymbol{\varsigma}_{h,\ell} \cdot \boldsymbol{\varsigma}_h}{C_{\mathcal{P}} \, \|\boldsymbol{\varsigma}_{h,\ell}\|_{0,\ell;\Omega}} = C_{\mathcal{P}}^{-1} \, \|\boldsymbol{\varsigma}_h\|_{0,j;\Omega}\,.$$

In this way, using that $\mathrm{div}(\boldsymbol{\varsigma}_h) = 0$, we have $\|\boldsymbol{\varsigma}_h\|_{0,j;\Omega} = \|\boldsymbol{\varsigma}_h\|_{j,\mathrm{div}_j;\Omega}$, thus concluding the proof with $\beta_{1,\mathtt{d}} := C_{\mathcal{P}}^{-1}$. The case $i = 2$ follows an analogous procedure by exchanging the roles of ℓ and j, so we omit further details and just mention that one also obtains $\beta_{2,\mathtt{d}} := C_{\mathcal{P}}^{-1}$. □

Concerning (**H.8**), we recall that $\vec{\boldsymbol{\varkappa}}_h := (\boldsymbol{\varkappa}_h, \boldsymbol{\varsigma}_h)$, and observe that, for $i \in \{1,2\}$, there holds

$$\sup_{\substack{\vec{\boldsymbol{\varkappa}}_h \in \mathbf{X}_{i,h}\\ \vec{\boldsymbol{\varkappa}}_h \neq \mathbf{0}}} \frac{\mathcal{B}_i(\vec{\boldsymbol{\varkappa}}_h, \psi_h)}{\|\vec{\boldsymbol{\varkappa}}_h\|_{\mathbf{X}_i}} \geq \sup_{\substack{(\mathbf{0},\boldsymbol{\varsigma}_h) \in \mathbf{X}_{i,h}\\ \boldsymbol{\varsigma}_h \neq \mathbf{0}}} \frac{\mathcal{B}_i\big((\mathbf{0}, \boldsymbol{\varsigma}_h), \psi_h\big)}{\|(\mathbf{0}, \boldsymbol{\varsigma}_h)\|_{\mathbf{X}_i}} \qquad \forall\, \psi_h \in \mathbf{Y}_{i,h}\,,$$

which coincides with the expression for which the discrete inf–sup condition is established in [31, Lemma 4.4]. In our setting, the admissible indices (cf. (4.52)) $j = 4$ and $\ell = 4/3$ lie within the range covered by that result and pose no restriction for convex domains. By contrast, in the non-convex case, for $i = 1$, one also requires that $\ell > 3/2$, which is not satisfied here. This is the only instance where such a restriction arises, as all previous results hold independently of the geometry of the domain, thereby justifying the assumption of convexity imposed at the beginning of this section. Finally, as it was established in Section 5.4, (**H.9**) is trivially verified from (6.1).

6.3 Rates of convergence

We now collect the approximation properties of the finite element subspaces introduced in (6.1), which follow from those of the Raviart–Thomas interpolator and the orthogonal projector onto piecewise scalar, vector, and tensor polynomials in the corresponding L^p-norms, together with estimates obtained via interpolation between Sobolev spaces (see, for instance, [31, Section 4.5], [8, Section 5.3]). Indeed, for each space defined in (6.1), we have:

$(\mathbf{AP}_h^{\mathbf{t}})$ there exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\mathbf{s} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, there holds

$$\operatorname{dist}(\mathbf{s}, \mathbb{H}_h^{\mathbf{t}}) \,\leq\, C\, h^\ell \, \|\mathbf{s}\|_{\ell,\Omega} \,,$$

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{\ell,4}(\Omega)$, there holds

$$\operatorname{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) \,\leq\, C\, h^\ell \, \|\mathbf{v}\|_{\ell,4;\Omega} \,,$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists a positive constant C, independent of h, such that for each $\ell \in [1, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{\ell,4/3}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\tau}, \mathbf{Q}_h) \,\leq\, C\, h^\ell \left\{ \|\boldsymbol{\tau}\|_{\ell,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{\ell,4/3;\Omega} \right\},$$

$(\mathbf{AP}_h^{\boldsymbol{\chi}})$ there exists a positive constant C, independent of h, such that for each $\ell \in [1, k+1]$, and for each $\boldsymbol{\eta} \in \mathbf{W}^{\ell,r}(\Omega)$ with $\operatorname{div}(\boldsymbol{\eta}) \in \mathrm{W}^{\ell,q}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\eta}, X_{2,h}) \,\leq\, C\, h^\ell \left\{ \|\boldsymbol{\eta}\|_{\ell,r;\Omega} \,+\, \|\operatorname{div}(\boldsymbol{\eta})\|_{\ell,q;\Omega} \right\},$$

$(\mathbf{AP}_h^{\vartheta})$ there exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\omega \in \mathrm{W}^{\ell,r}(\Omega)$, there holds

$$\operatorname{dist}(\omega, M_{1,h}) \,\leq\, C\, h^\ell \, \|\omega\|_{\ell,r;\Omega} \,,$$

$(\mathbf{AP}_h^{\boldsymbol{\rho}})$ there exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\boldsymbol{\varrho} \in \mathbf{W}^{\ell,j}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\varrho}, \mathbf{X}_{2,h}^{\boldsymbol{\rho}}) \,\leq\, C\, h^\ell \, \|\boldsymbol{\varrho}\|_{\ell,j;\Omega} \,,$$

$(\mathbf{AP}_h^{\boldsymbol{\xi}})$ there exists a positive constant C, independent of h, such that for each $\ell \in [1, k+1]$, and for each $\boldsymbol{\lambda} \in \mathbf{W}^{\ell,j}(\Omega)$ with $\operatorname{div}(\boldsymbol{\lambda}) \in \mathrm{W}^{\ell,j}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\lambda}, \mathbf{X}_{2,h}^{\boldsymbol{\xi}}) \,\leq\, C\, h^\ell \left\{ \|\boldsymbol{\lambda}\|_{\ell,j;\Omega} \,+\, \|\operatorname{div}(\boldsymbol{\lambda})\|_{\ell,j;\Omega} \right\},$$

$(\mathbf{AP}_h^{\phi})$ there exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\varphi \in \mathrm{W}^{\ell,j}(\Omega)$, there holds

$$\operatorname{dist}(\varphi, \mathbf{Y}_{1,h}) \,\leq\, C\, h^\ell \, \|\varphi\|_{\ell,j;\Omega} \,.$$

We have the following main theorem.

Theorem 6.5. *Let* $\big((\vec{\mathbf{t}}, \boldsymbol{\sigma}), (\boldsymbol{\chi}, \vartheta), (\vec{\boldsymbol{\rho}}, \phi)\big) \in (\mathbf{H} \times \mathbf{Q}) \times (\mathrm{X}_2 \times \mathrm{M}_1) \times (\mathbf{X}_2 \times \mathbf{Y}_1)$ *be the unique solution of* (3.27) *with* $\mathbf{u} \in W_\delta$, *and let* $\big((\vec{\mathbf{t}}_h, \boldsymbol{\sigma}_h), (\boldsymbol{\chi}_h, \vartheta_h), (\vec{\boldsymbol{\rho}}_h, \phi_h)\big) \in (\mathbf{H}_h \times \mathbf{Q}_h) \times (\mathrm{X}_{2,h} \times \mathrm{M}_{1,h}) \times (\mathbf{X}_{2,h} \times \mathbf{Y}_{1,h})$ *be a solution of* (5.1) *with* $\mathbf{u}_h \in W_{\delta_{\mathtt{d}}}$, *whose existences are guaranteed by Theorems* (4.18) *and* (5.5), *respectively. Assume that hypotheses of Theorem* 5.6 *and Lemma* 6.3 *hold, and that there exists* $\ell \in [1, k+1]$ *such that* $\mathbf{t} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{\ell,4}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{\ell,4/3}(\Omega)$, $\boldsymbol{\chi} \in \mathbf{W}^{\ell,r}(\Omega)$, $\operatorname{div}(\boldsymbol{\chi}) \in \mathrm{W}^{\ell,q}(\Omega)$, $\vartheta \in \mathrm{W}^{\ell,r}(\Omega)$, $\boldsymbol{\rho} \in \mathbf{W}^{\ell,j}(\Omega)$, $\boldsymbol{\xi} \in \mathbf{W}^{\ell,j}(\Omega)$, $\operatorname{div}(\boldsymbol{\xi}) \in \mathrm{W}^{\ell,j}(\Omega)$, *and* $\phi \in \mathrm{W}^{\ell,j}(\Omega)$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\begin{aligned} &\|(\vec{\mathbf{t}}, \boldsymbol{\sigma}) - (\vec{\mathbf{t}}_h, \boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\boldsymbol{\chi}, \vartheta) - (\boldsymbol{\chi}_h, \vartheta_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \|(\vec{\boldsymbol{\rho}}, \phi) - (\vec{\boldsymbol{\rho}}_h, \phi_h)\|_{\mathbf{X}_2\times\mathbf{Y}_1} \\ &\quad \leq C\, h^\ell \Big\{ \|\mathbf{t}\|_{\ell,\Omega} + \|\mathbf{u}\|_{\ell,4;\Omega} + \|\boldsymbol{\sigma}\|_{\ell,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\ell,4/3;\Omega} + \|\boldsymbol{\chi}\|_{\ell,r;\Omega} \\ &\qquad + \|\operatorname{div}(\boldsymbol{\chi})\|_{\ell,q;\Omega} + \|\vartheta\|_{\ell,r;\Omega} + \|\boldsymbol{\rho}\|_{\ell,j;\Omega} + \|\boldsymbol{\xi}\|_{\ell,j;\Omega} + \|\operatorname{div}(\boldsymbol{\xi})\|_{\ell,j;\Omega} + \|\phi\|_{\ell,j;\Omega} \Big\}. \end{aligned}$$

Proof. It follows straightforwardly from Theorem 5.6 and the above approximation properties. $\square$

7 Numerical results

In this section, we present a numerical verification of the convergence properties established in Section 6, together with an illustration of the applicability of the proposed method to a problem of practical interest. The finite element spaces employed throughout the experiments are those introduced in (6.1). In particular, the zero-average condition for the pressure (cf. (2.1)), which in the mixed formulation corresponds to a zero-mean condition on the trace of the Bernoulli stress, is enforced by means of a real Lagrange multiplier. An analogous approach is adopted for the imposition of the zero-average condition for the internal electric potential (cf. (2.1)). Furthermore, unless otherwise stated, all parameters are fixed as $\kappa = \delta_d = \delta_t = \mu = 1$. In addition, motivated by experimental evidence (see, e.g., [17, 19]), the dielectric permittivity is assumed to depend linearly on the temperature. Accordingly, we consider the model

$$\eta(\vartheta) = \eta_2 - \eta_1(\vartheta).$$

All computational tests have been implemented using the open-source finite element library FEniCS [3]. The resulting nonlinear systems are solved via a Newton–Raphson scheme, with an incremental relative tolerance of 10^{-7}, while the associated linear subproblems are handled using the direct solver MUMPS through the PETSc interface with FEniCS.

We now introduce additional notation for the individual errors as:

$$\begin{gathered}\mathsf{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0;\Omega}, \quad \mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}, \quad \mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}, \\ \mathsf{e}(\boldsymbol{\chi}) := \|\boldsymbol{\chi} - \boldsymbol{\chi}_h\|_{r,\mathrm{div}_q;\Omega}, \quad \mathsf{e}(\vartheta) := \|\vartheta - \vartheta_h\|_{0,r;\Omega}, \\ \mathsf{e}(\boldsymbol{\rho}) := \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,j;\Omega}, \quad \mathsf{e}(\boldsymbol{\xi}) := \|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{j,\mathrm{div}_j;\Omega}, \quad \mathsf{e}(\phi) := \|\phi - \phi_h\|_{0,j;\Omega},\end{gathered}$$

where, accordingly to (4.52), $r = j = 4$ and $q = 2$. Moreover, in order to quantify the convergence behavior, we compute local error decay rates through the experimental order of convergence, defined by `EoC` $:= \log(e(\cdot)/\widehat{e}(\cdot))\,[\log(h/\widehat{h})]^{-1}$, where h and $\widehat{h}$ denote two consecutive mesh sizes, with corresponding errors e and $\widehat{e}$. In turn, `DoFs` denotes the number of degrees of freedom, whereas `iter` represents the number of Newton iterations.

Example 1: Convergence for smooth exact solutions in a 2D domain. We first consider a manufactured solution on the unit square $\Omega = (0,1)^2$. The exact solutions for the velocity, pressure, temperature, and electric potential are given by

$$\begin{aligned}\mathbf{u}(x,y) &= \begin{pmatrix} \cos(\pi x)\,\sin(\pi y) \\ -\sin(\pi x)\,\cos(\pi y) \end{pmatrix}, \quad \mathsf{p}(x,y) = \sin\big(2\pi(x+y)\big), \\ \vartheta(x,y) &= \cos(x)\,\exp\big(-(x+y)\big), \quad \phi(x,y) = \sin(\pi x)\,\cos(\pi y).\end{aligned} \tag{7.1}$$

With these closed-form primal variables we can compute exact mixed variables through the constitutive equations, as well as non-homogeneous boundary data and forcing terms. A successively refined family of meshes is constructed and errors are computed on each mesh refinement, using the lowest-order finite element families with $k = 0$ and $k = 1$. The error history reported in Table 7.1 confirms the optimal convergence rates of $O(h^{k+1})$ as predicted by Theorem 6.5. The table also shows the number of Newton iterations that are required in each refinement level. The parametric regime used for the manufactured solution examples does not seem very demanding, as no more than three iterations are required irrespective of the mesh resolution. We also show the approximate solutions for this test in Figure 7.1, generated with the method using $k = 1$, and on a relatively coarse mesh. All fields are well-resolved.

Example 2: Convergence for smooth exact solutions in a 3D domain. Although the theoretical analysis has been developed only for the two-dimensional setting, we include a three-dimensional test to assess the performance of the method in this case. We consider the following closed-form primal solutions

DoFs	h	$\mathtt{e}(\mathbf{t})$	EoC	$\mathtt{e}(\mathbf{u})$	EoC	$\mathtt{e}(\boldsymbol{\sigma})$	EoC	$\mathtt{e}(\boldsymbol{\chi})$	EoC	$\mathtt{e}(\vartheta)$	EoC
$k=0$											
138	0.707	1.67e+00	$\star$	4.30e-01	$\star$	9.64e+00	$\star$	2.38e-01	$\star$	1.31e-01	$\star$
514	0.354	9.19e-01	0.86	2.28e-01	0.92	5.64e+00	0.77	1.22e-01	0.96	6.68e-02	0.97
1986	0.177	5.02e-01	0.87	1.16e-01	0.97	2.94e+00	0.94	6.23e-02	0.98	3.36e-02	0.99
7810	0.088	2.62e-01	0.94	5.85e-02	0.99	1.48e+00	0.99	3.11e-02	1.00	1.68e-02	1.00
30978	0.044	1.33e-01	0.98	2.92e-02	1.00	7.38e-01	1.00	1.55e-02	1.00	8.41e-03	1.00
123394	0.022	6.70e-02	0.99	1.46e-02	1.00	3.69e-01	1.00	7.76e-03	1.00	4.21e-03	1.00
$k=1$											
410	0.707	5.12e-01	$\star$	1.41e-01	$\star$	5.22e+00	$\star$	2.67e-02	$\star$	1.23e-02	$\star$
1570	0.354	1.79e-01	1.51	3.58e-02	1.98	1.35e+00	1.95	7.36e-03	1.86	3.16e-03	1.97
6146	0.177	5.23e-02	1.78	9.08e-03	1.98	3.58e-01	1.91	1.79e-03	2.04	7.95e-04	1.99
24322	0.088	1.39e-02	1.92	2.28e-03	2.00	9.05e-02	1.98	4.48e-04	2.00	1.99e-04	2.00
96770	0.044	3.53e-03	1.97	5.69e-04	2.00	2.27e-02	1.99	1.13e-04	1.99	4.98e-05	2.00
386050	0.022	8.89e-04	1.99	1.42e-04	2.00	5.69e-03	2.00	2.82e-05	1.99	1.24e-05	2.00

DoFs	h	$\mathtt{e}(\boldsymbol{\rho})$	EoC	$\mathtt{e}(\boldsymbol{\xi})$	EoC	$\mathtt{e}(\phi)$	EoC	iter
$k=0$								
138	0.707	1.43e+00	$\star$	7.18e+00	$\star$	3.43e-01	$\star$	3
514	0.354	7.75e-01	0.88	4.00e+00	0.84	1.74e-01	0.98	3
1986	0.177	3.95e-01	0.97	2.04e+00	0.97	8.71e-02	1.00	3
7810	0.088	1.98e-01	0.99	1.02e+00	0.99	4.36e-02	1.00	3
30978	0.044	9.92e-02	1.00	5.12e-01	1.00	2.18e-02	1.00	3
123394	0.022	4.96e-02	1.00	2.56e-01	1.00	1.09e-02	1.00	3
$k=1$								
410	0.707	5.28e-01	$\star$	2.30e+00	$\star$	1.30e-01	$\star$	3
1570	0.354	1.25e-01	2.08	5.68e-01	2.02	2.62e-02	2.31	3
6146	0.177	3.13e-02	2.00	1.44e-01	1.98	6.61e-03	1.99	3
24322	0.088	7.89e-03	1.99	3.64e-02	1.99	1.72e-03	1.95	3
96770	0.044	1.98e-03	1.99	9.13e-03	1.99	4.39e-04	1.97	3
386050	0.022	4.97e-04	2.00	2.29e-03	2.00	1.11e-04	1.98	3

Table 7.1: Example 1. Error history associated with the convergence test against smooth manufactured solutions in 2D, for the two lowest order methods ($k = 0, 1$).

on the unit cube $\Omega = (0,1)^3$:

$$\mathbf{u}(x,y,z) = \begin{pmatrix} \sin(\pi x)\,\cos(\pi y)\,\cos(\pi z) \\ -2\cos(\pi x)\,\sin(\pi y)\,\cos(\pi z) \\ \cos(\pi x)\,\cos(\pi y)\,\sin(\pi z) \end{pmatrix}, \quad \mathtt{p}(x,y,z) = \sin\big(2\pi(x+y+z)\big),$$

$$\vartheta(x,y,z) = \cos(xyz)\,\exp\big(-(x+y+z)\big), \quad \phi(x,y,z) = \sin(\pi x)\,\cos(\pi y)\,\cos(\pi z).$$

For this case we only carry out the experimental error analysis for the lowest-order finite element scheme (setting $k = 0$), where the error is computed using the only feasible indices (cf. (4.52)) available in the two-dimensional case. The outcome of the convergence test is presented in Table 7.2, and the results align with those reported in the 2D case: the method is optimally convergent in every field, however, in this case the displacement takes a bit longer to reach the asymptotic regime. These observations further support the discussion following (4.52), suggesting that the restrictions imposed by the analysis are mainly theoretical, while the method still performs satisfactorily at the numerical level. We plot the numerically computed field variables in Figure 7.2. Similarly as in 2D, the vectors $\boldsymbol{\rho}_h$ and $\boldsymbol{\xi}_h$ have the same direction and are very similar in magnitude.

Example 3: Application to TEHD in cylindrical annuli. Finally, we consider an application to thermo-electro-hydrodynamic (TEHD) flows in cylindrical annuli, leading to buoyancy-like convection in dielectric fluids. In this case, we extend the model to its time-dependent version. More specifically,

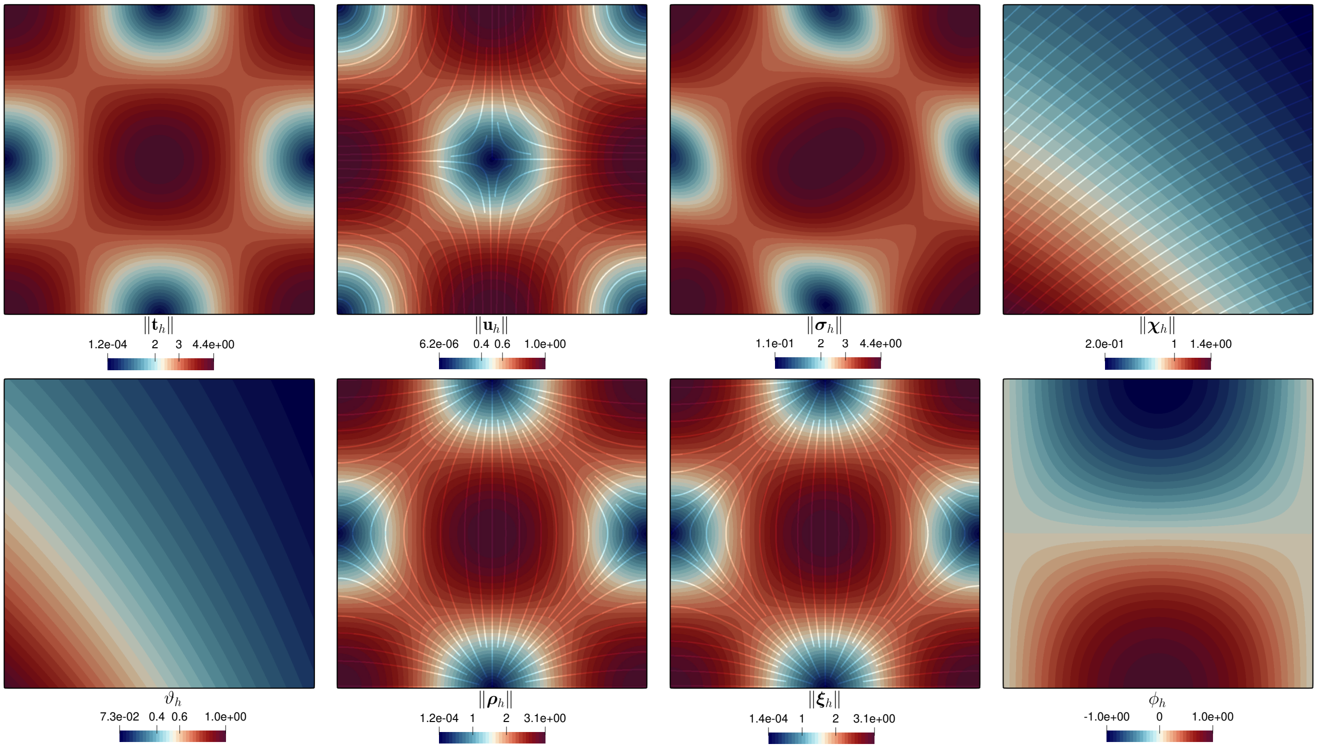

Figure 7.1: Example 1. Approximate numerical solutions (magnitude of the tensor variables, magnitude and evenly spaced streamlines for the vector quantities, and profiles of the scalar unknowns) for the convergence test in 2D, computed with the second-order scheme.

DoFs	h	e($\mathbf{t}$)	EoC	e($\mathbf{u}$)	EoC	e($\boldsymbol{\sigma}$)	EoC	e($\boldsymbol{\chi}$)	EoC	e(ϑ)	EoC	e($\boldsymbol{\rho}$)	EoC	e($\boldsymbol{\xi}$)	EoC	e(ϕ)	EoC	iter
188	1.732	6.20e+00	$\star$	7.12e-01	$\star$	5.70e+01	$\star$	6.37e-01	$\star$	1.83e-01	$\star$	1.91e+00	$\star$	1.43e+01	$\star$	3.90e-01	$\star$	3
1370	0.866	3.79e+00	0.71	4.70e-01	0.60	2.72e+01	1.07	3.43e-01	0.89	1.05e-01	0.80	1.28e+00	0.57	8.84e+00	0.69	2.67e-01	0.55	3
10466	0.433	2.26e+00	0.74	3.45e-01	0.45	1.76e+01	0.63	1.75e-01	0.97	5.47e-02	0.95	6.90e-01	0.89	4.60e+00	0.94	1.35e-01	0.98	3
81794	0.217	1.21e+00	0.90	1.85e-01	0.90	9.20e+00	0.93	8.81e-02	0.99	2.76e-02	0.99	3.56e-01	0.96	2.35e+00	0.97	6.93e-02	0.97	3
646658	0.108	6.23e-01	0.96	9.37e-02	0.98	4.64e+00	0.99	4.40e-02	1.00	1.39e-02	1.00	1.79e-01	0.99	1.18e+00	0.99	3.48e-02	0.99	3

Table 7.2: Error history associated with the convergence test against smooth manufactured solutions in 3D.

the momentum balance and thermal energy equations from (2.1) are modified by incorporating time derivatives of the velocity and temperature fields: $\partial_t\mathbf{u} - \mu\,\Delta\mathbf{u} + \big(\nabla\mathbf{u}\big)\mathbf{u} + \nabla\mathrm{p} - \delta_d\,|\nabla\phi|^2\,\nabla\vartheta = -\delta_t\,\vartheta\,\mathbf{g} + \mathbf{f}$ and $\partial_t\vartheta - \kappa\,\Delta\vartheta + \mathbf{u}\cdot\nabla\vartheta = f$, respectively. The spatial formulation remains unchanged, while time discretization is performed using an implicit backward Euler scheme with constant time step $\Delta t = 10^{-4}$. The domain configuration consists of an annular region of inner radius 0.7 and outer radius 1, thereby preserving the radius ratio reported in [19], and is discretized using a non-structured triangular mesh. The model parametrization adopted herein corresponds to the Rayleigh regime associated with chaotic reversal patterns in heat transfer and flow patterns in annular convection studied in [36], where electric effects are not considered. For the numerical experiment considered here, the model is extended by incorporating the TEHD coupling together with a different set of model parameters and boundary conditions. More

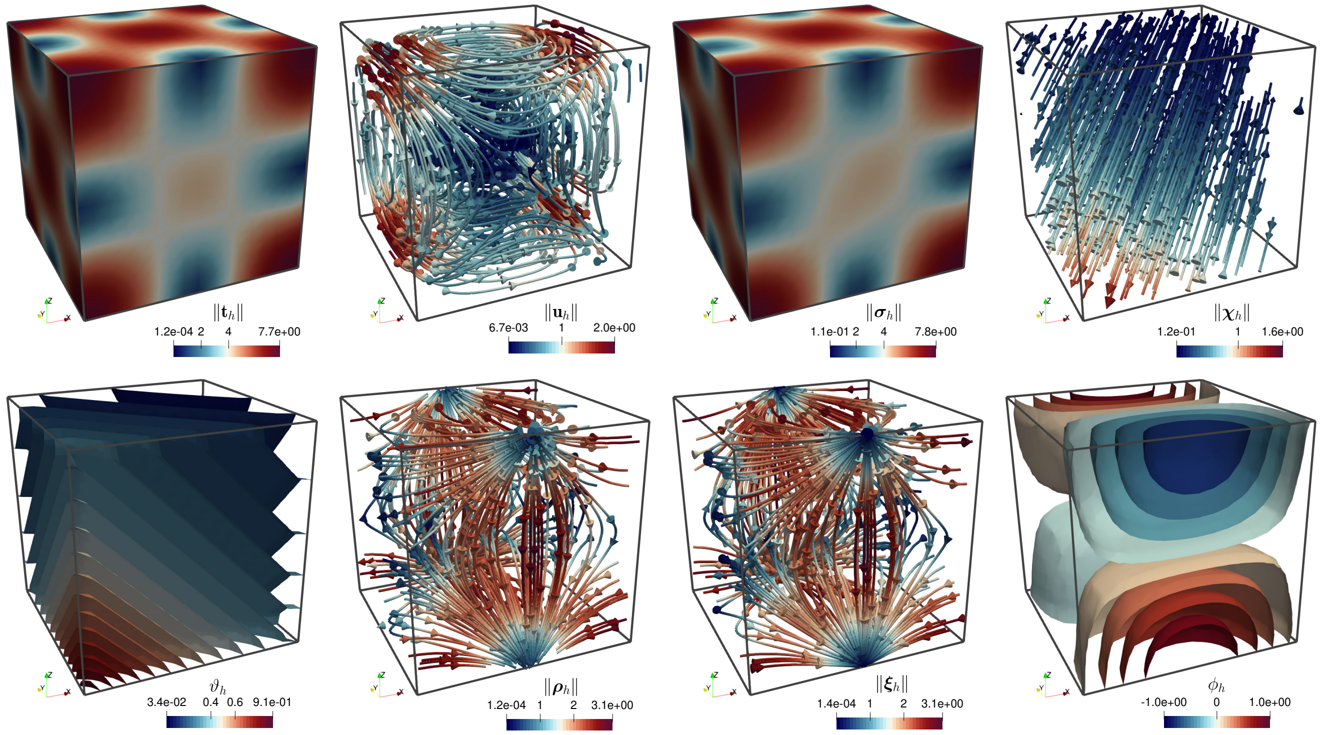

Figure 7.2: Approximate numerical solutions (magnitude of the tensor variables, streamlines for the vector quantities, and iso-contours of the scalar unknowns) for the convergence test in 3D, computed with the lowest-order scheme.

precisely, no-slip conditions are imposed on the whole boundary, namely $\mathbf{u} = \mathbf{0}$. Thermal exchange occurs along the outer boundary with an imposed temperature that decreases linearly with height $\vartheta = \vartheta_{\text{out}}$, whereas the inner boundary is assumed to be adiabatic, i.e., $\boldsymbol{\chi} \cdot \boldsymbol{\nu} = 0$. In addition, a grounded potential ϕ_{in} is imposed on the inner radius, while a transient applied voltage ϕ_{out} is prescribed on the outer radius, which eliminates the need for a Lagrange multiplier to constrain the potential to have zero mean.

Now, using the notation introduced in Section 2, the data are as follows

$$\kappa = \mu = \eta_2 = 1, \quad \eta_1 = 0.1, \quad \delta_t = 5 \cdot 10^6, \quad \delta_d = 1.75, \quad \phi_{\text{in}} = 0,$$
$$\phi_{\text{out}} = \sqrt{2} \sin\left(\frac{\pi}{0.005}\, t\right), \quad \vartheta_{\text{out}} = 1 - \frac{y}{\sqrt{x^2 + y^2}}.$$

The results of the simulation (after 100 time steps) are summarized in Figure 7.3. We present snapshots at different time instants of the radial and angular components of the velocity (computed as $u_r := \mathbf{u}_h \cdot \boldsymbol{e}_r$ and $u_a := \mathbf{u}_h \cdot \boldsymbol{e}_a$, respectively, where $\boldsymbol{e}_r = \frac{1}{|\mathbf{x}|}(x, y)$ and $\boldsymbol{e}_a = \frac{1}{|\mathbf{x}|}(-y, x)$), as well as of the temperature and electric potential. The typical recirculation patterns associated with chaotic thermal conditions are observed, especially in the temperature profiles.

Since the boundary condition for the potential is periodic in time, so it is the potential solution. However, at certain locations the behavior of other fields is not necessarily periodic and it confirms the chaotic regime mentioned above. This is more clearly observed in Figure 7.4, where we plot the numerical solution over time at a sample point near the top center of the annulus. A rapid change in radial velocity is seen near time $t = 0.003$, as well as an increase of thermal activity only after $t = 0.0025$. This application shows that the scheme remains effective on non-convex domains, indicating that the convexity assumption introduced at the beginning of Section 6.2 is a limitation of the analysis rather than of the performance of the numerical method.

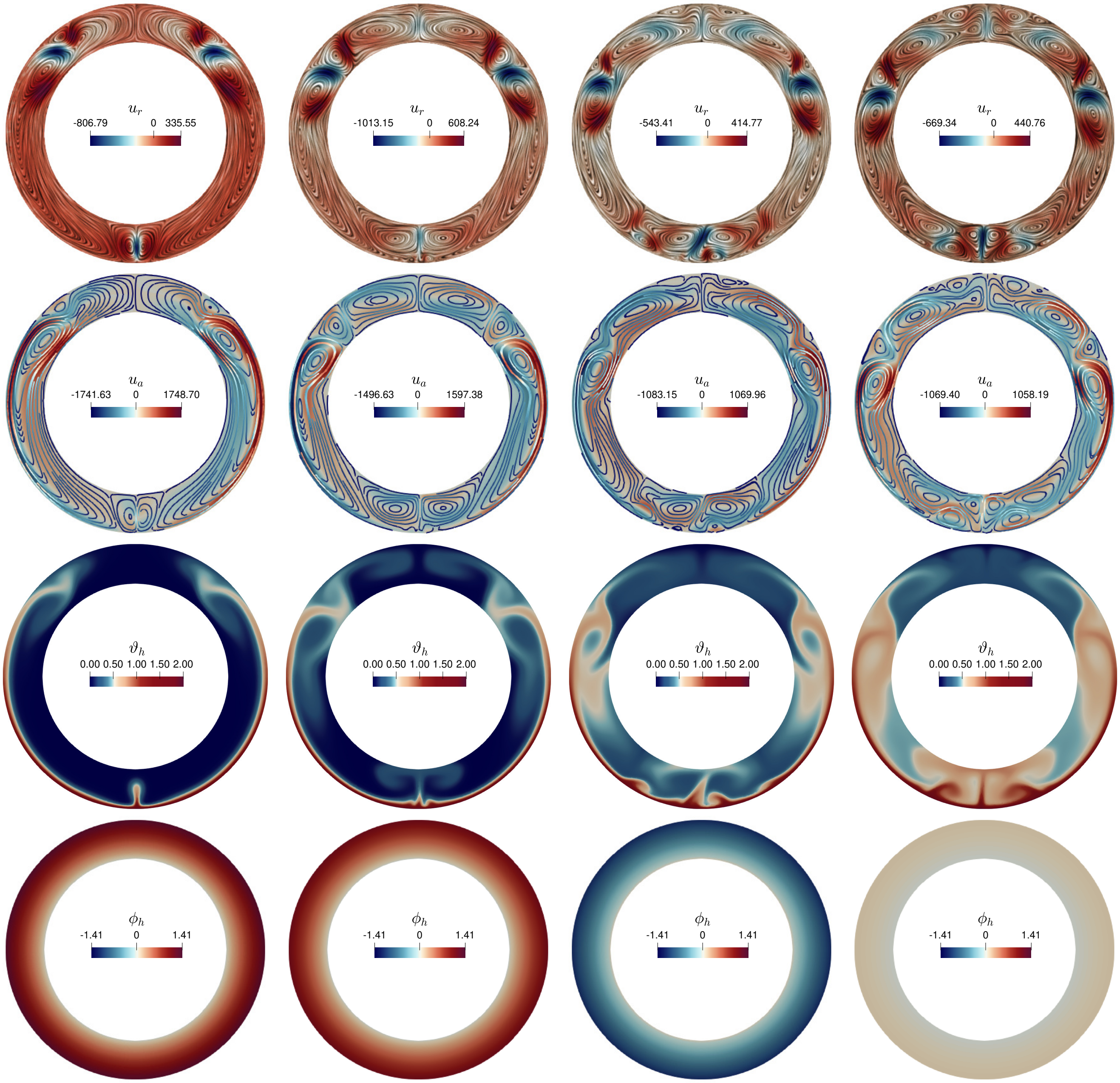

Figure 7.3: Snapshots of approximate solutions for the time-dependent TEHD system on an annular region. From top to bottom: radial component of velocity (with line integral contours), angular component of velocity (with evenly spaced streamlines), temperature profile, and potential distribution. From left to right: adimensional times $t = 0.002, 0.0035, 0.0065, 0.01$.

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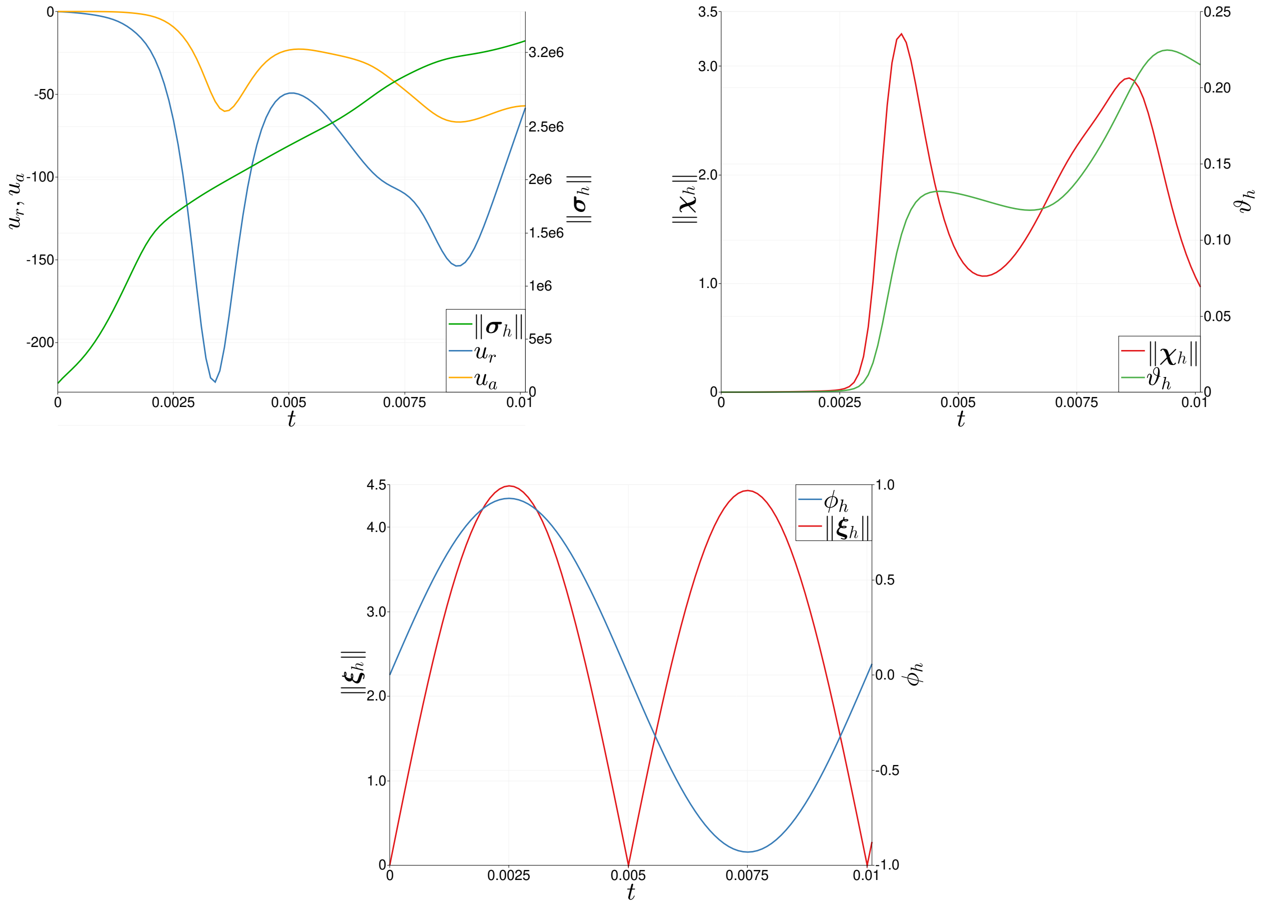

Figure 7.4: Transients of some fields of the numerical solution extracted at the point $(-0.025, 0.884)$.

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