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New Banach spaces-based mixed finite element methods for steady-state flows of magnetic fluids*

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Abstract

In this paper we introduce and analyze new mixed finite element methods for solving the steady-state flows of magnetic fluids. To this end, we consider two different ways of deriving the stationary model from the time-dependent problem, namely, either by dropping the time derivatives or by replacing them by a zero order term. In this way, and motivated by our interest in mixed approaches, we are led to systems of equations for the fluid having as unknowns, in the first case, the velocity, its gradient, and a partial stress magnetic tensor, which is given by the sum of the Maxwell and hydrodynamics stress tensors, and the half of the convective term. Similarly, in the second one they turn out to be the strain rate tensor, the velocity, the full stress magnetic tensor, and the vorticity tensor. Needless to say, in both cases the total pressure is eliminated and computed afterwards via a postprocessing formula in terms of the (partial or full) magnetic tensor and the velocity. Next, we introduce a potential unknown so that the magnetic field satisfies a Neumann boundary value problem with the normal component of the applied magnetic field as the corresponding boundary condition. The resulting mixed variational formulations for the fluid fit the structures of a nonlinear saddle-point problem, and a nonlinear perturbation of, in turn, a perturbed twofold saddle-point problem, respectively, whereas the one for the magnetic field is given by a usual saddle-point setting. Hence, fixed-point strategies, along with the generalized Babuška-Brezzi theory in Banach spaces and recent related abstract results, are employed to establish the well-posedness of the associated continuous and discrete schemes. In particular, piecewise polynomials, Raviart-Thomas, PEERS, and AFW elements are utilized to define the stable Galerkin schemes. Regarding the magnetisation of the fluid, which satisfies the simplified Shliomis equation arising after neglecting the advective term, we employ the mixed finite element solutions of the fluid and potential equations, to propose two iterative schemes approximating this unknown. Finally, several numerical results illustrating the performance of our methods are reported.

Key words: magnetic fluids, stress magnetic tensor, saddle-point problems, mixed finite element methods, fixed point theory, a priori error analysis.

Mathematics subject classifications (2020): 35Q35, 35Q61, 65N12, 65N15, 65N30, 76W05

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1 Introduction

Magnetic fluids, also known as ferrofluids [35], are synthetic suspensions composed of nano-sized magnetizable particles dispersed in a viscous carrier fluid. When subjected to an external magnetic field, these particles become magnetized, inducing a collective, continuum-level response of the fluid to the applied field. This magneto-responsive behavior enables the fluid to be manipulated and actuated through external magnetic control [33], giving rise to a wide range of potential applications across various scientific and technological domains [37, 29, 20, 38]. The macroscopic properties of magnetic fluids, particularly their equilibrium magnetization, have been extensively investigated [35]. The magnetization of magnetic fluids arises from the alignment of the individual magnetic moments of the suspended nanoparticles with the direction of the applied magnetic field. While the equilibrium magnetic behavior of these fluids in quiescent conditions is reasonably well understood [35], the influence of fluid motion on their magnetic response remains an open question [39, 21, 40, 19]. The simplest modeling approach treats magnetic fluids as superparamagnetic, assuming that magnetization is unaffected by flow [35]. However, this assumption is only valid when flow effects are negligible. In regimes where fluid motion plays a significant role – such as in the rheological characterization of these fluids – an evolution equation for magnetization, coupled to the flow field, becomes necessary [39, 21, 19]. Despite the significance of magnetic fluids and numerous attempts to formulate a definitive mathematical framework describing their magnetization under flow [19], a universally accepted governing equation for magnetization dynamics in flowing magnetic fluids has yet to be established. Under flow conditions, the local vorticity of the fluid introduces a competing mechanism that may hinder the alignment of magnetic moments, thus altering the dynamics of the magnetization of the fluid. As such, a comprehensive understanding of the interplay between flow fields and magnetization is essential for the accurate modeling of these materials. A foundational model addressing this coupling was proposed by Shliomis [39], incorporating the essential mechanisms that govern the magnetization dynamics in the presence of flow. This model has served as the basis for many subsequent theoretical and computational studies [33] and is the model adopted in the present work.

Numerical simulations of magnetic fluid flows have been pursued using different numerical techniques. One of the earliest numerical treatments of plane Poiseuille flow using a steady-state version of the Shliomis magnetization evolution equation was presented in [34], in which a finite volume method that incorporated the magnetization of the fluid following the methodology outlined in [20] was proposed. Magnetically responsive biomagnetic fluids, such as blood, have been investigated in [42], where a finite-difference based vorticity-stream function formulation was developed, considering a superparamagnetic magnetization model. A similar model was later solved using a finite-volume method for a driven cavity problem in which magnetohydrodynamic effects were also brought into the problem [43]. Similarly, a finite-volume based method was employed to solve a driven cavity problem of a magnetic fluid with magnetization governed by Shliomis equation [41]. More recently, the steady-state solutions of the driven cavity problem were investigated in detail using a finite-difference scheme on the weakly magnetizable limit and all the effects of the phenomenological terms of the magnetization equation on the flow were isolated and analyzed [44]. In [28], second-order accuracy for the temporal integration of the governing equations was achieved using an unconditionally stable backward difference formula. Furthermore, mixed finite element methods within a Hilbertian framework in space, together with implicit and semi-implicit Euler schemes in time, are considered in [30], [45], and [46] for the simplified ferrohydrodynamics equations and for the ferrofluid flow model with magnetization parallel to the magnetic field.

One of the major difficulties posed by magnetic fluid flows is their dependence on the fluid magnetisation, which satisfies a Shliomis-type equation. This aspect will be described in more detail in the following section. We note in advance that the strong nonlinearity of the Navier–Stokes-type model, together with its coupling with the potential equation, prevents us from guaranteeing the successful application of classical numerical methods such as primal finite element techniques, which are typically better suited to linear problems, particularly when formulated within a Hilbert space framework. In this regard, it is important to emphasize that the suitability of Banach space based approaches for analyzing the continuous and discrete solvability of a wide range of nonlinear problems in continuum mechanics, mainly through mixed formulations, has been confirmed by numerous contributions in recent years. Among the models addressed are Brinkman–Forchheimer, Navier–Stokes, Boussinesq, coupled flow-transport systems and fluidized beds, and a non-exhaustive list of representative references includes [12, 7, 17, 14, 11, 2, 16, 27, 10]. Needless to say, a distinctive feature of mixed formulations is the introduction of additional unknowns, typically related to the original variables of the model, for analytical or physical reasons. Moreover, one of the main advantages of working within a Banach framework is that no artificial augmentation is required, unlike in Hilbert space based formulations where additional terms are often introduced to enforce properties such as ellipticity or strong monotonicity. Consequently, the functional spaces to which the unknowns belong arise naturally from the testing procedure and from the application of the Cauchy–Schwarz and Hölder inequalities. In this way, the resulting formulations remain closer to the original physical model while preserving analytical tractability. Finally, the main benefits of employing a mixed approach include the derivation of momentum conservative numerical schemes and the possibility of obtaining direct approximations of additional variables of physical interest, either by incorporating them explicitly into the formulation or through suitable postprocessing procedures based on the primary unknowns.

According to the previous discussion, the aim of the present work is to introduce and analyze two new mixed finite element methods for the numerical approximation of steady-state magnetic fluid flows. The paper is organized as follows. In the remainder of this section, we collect the notation that will be used throughout the manuscript. In Section 2, we describe the mathematical model and present the two approaches proposed to reformulate the original fluid system. Next, in Section 3, we develop the mixed variational formulations corresponding to each approach, as well as the one associated with the potential equation. In the first case, the fluid formulation fits into the framework of a nonlinear saddle-point problem, whereas in the second it can be interpreted as a nonlinear perturbation of a twofold saddle-point problem. In contrast, the formulation for the potential equation is posed within a standard saddle-point setting. The corresponding solvability analysis of the first mixed formulation for the fluid is carried out in Section 4. The approach relies on a fixed-point argument with respect to the velocity variable, combined with a Banach-space version of the Babuška–Brezzi theory and the classical Banach fixed-point theorem. The associated Galerkin scheme, formulated for general finite element spaces, is also developed in this section. In addition, we establish the corresponding a priori error estimates, which depend on the approximation of both the magnetic field and the fluid magnetisation, and subsequently prove convergence results for specific choices of finite element subspaces. Similarly, in Section 5 we develop the continuous and discrete analyses of the second mixed formulation for the fluid. As in the previous case, the approach is based on a fixed-point argument with respect to the velocity variable, now combined with a recent abstract result for perturbed twofold saddle-point problems and the Banach fixed-point theorem. The analysis of the potential equation is then carried out in Section 6 by applying an appropriate abstract result, and the corresponding rates of convergence are also established. Section 7 is devoted to the iterative approximation of the fluid magnetisation associated with the two mixed approaches. Numerical experiments are finally presented in Section 8, where the theoretical findings are illustrated and, in particular, the optimal rates of convergence obtained through the iterative procedure introduced in Section 7 are confirmed.

Preliminary notations

In what follows $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$, is a given bounded domain with polyhedral boundary Γ, whose outward unit normal vector is denoted by $\boldsymbol{\nu}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. On the other hand, given any generic scalar functional space S, we let $\mathbf{S}$ and $\mathbb{S}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Furthermore, as usual $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in R^n. Also, for any vector field $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we define the tensor product between them as $\mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{j=1,n}$, and denote by $\nabla\mathbf{v}$ and $\operatorname{div}(\mathbf{v})$ the gradient and divergence, respectively. In addition, for any tensor $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathsf{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}\,, \quad \text{and} \quad \boldsymbol{\tau}^{\mathsf{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\,\mathbb{I}\,.$$

Next, given $t \in (1,+\infty)$, we introduce the Banach spaces

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \Big\}\,, \tag{1.1}$$

and

$$\mathbf{H}^t(\operatorname{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^t(\Omega) : \quad \operatorname{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega) \Big\}\,, \tag{1.2}$$

provided, respectively, with the natural norms

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,,$$

and

$$\|\boldsymbol{\tau}\|_{t,\operatorname{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\operatorname{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\operatorname{div}_t;\Omega)\,.$$

Then, we recall that, proceeding as in [23, eq. (1.43), Section 1.3.4] (see also [8, Section 4.1] and [14, Section 3.1]), one can prove that for each $t \in \begin{cases} (1,+\infty] & \text{in } \mathrm{R}^2 \\ [6/5,+\infty] & \text{in } \mathrm{R}^3 \end{cases}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\operatorname{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}(\operatorname{div}_t;\Omega)\times \mathrm{H}^1(\Omega)\,, \tag{1.3}$$

and analogously

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{v}\rangle = \int_\Omega \Big\{ \boldsymbol{\tau} : \nabla\mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega)\times \mathbf{H}^1(\Omega)\,, \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ denotes in (1.3) (resp. (1.4)) the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma)$) and $\mathrm{H}^{-1/2}(\Gamma)$ (resp. $\mathbf{H}^{-1/2}(\Gamma)$). In turn, given t, $t' \in (1,+\infty)$ conjugate to each other, that is such that $\frac{1}{t} + \frac{1}{t'} = 1$, there also holds (cf. [22, Corollary B.57])

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\operatorname{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}^t(\operatorname{div}_t;\Omega)\times \mathrm{W}^{1,t'}(\Omega)\,, \tag{1.5}$$

where $\langle\cdot,\cdot\rangle$ stands here for the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$. Finally, for any pair of normed spaces $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$, we provide the product space $X\times Y$ with the natural norm $\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y \quad \forall\,(x,y)\in X\times Y$.

2 The model problem

In this section we describe the problem of interest, which refers to the flow of a magnetic fluid in the presence of a magnetic field. We first describe the full time-dependent model and then the associated stationary one, adopting for the latter two different approaches.

2.1 The time-dependent problem

Under the assumption that the magnetic forces on the momentum equation are given by the Maxwell stress tensor $\boldsymbol{\sigma}_M$, and that the hydrodynamics stress tensor is the classical symmetrical one, say $\boldsymbol{\sigma}_H$, for an incompressible fluid, the governing equations of the latter are given by [19, 35]:

$$\begin{gathered}
\rho\,\frac{\partial\mathbf{u}}{\partial t} + \rho\,(\nabla\mathbf{u})\,\mathbf{u} = \mathbf{div}(\boldsymbol{\sigma}_M+\boldsymbol{\sigma}_H) \quad\text{in}\quad\Omega\,,\\
\mathrm{div}(\mathbf{u}) = 0 \quad\text{in}\quad\Omega\,,\qquad \mathbf{u} = \mathbf{u}_D \quad\text{on}\quad\Gamma\,,\\
\boldsymbol{\sigma}_M = -p_M\,\mathbb{I} + \frac{1}{2}\Big(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\Big) \quad\text{in}\quad\Omega\,,\\
p_M = \frac{1}{2}\mu_0\,\boldsymbol{\mathcal{H}}\cdot\boldsymbol{\mathcal{H}} \quad\text{in}\quad\Omega\,,\\
\boldsymbol{\sigma}_H = -\,p_H\,\mathbb{I} + 2\mu\,\mathbf{e}(\mathbf{u}) \quad\text{in}\quad\Omega\,,
\end{gathered}\tag{2.1}$$

where ρ is the density of the fluid, μ_0 is the magnetic permeability of the vacuum, μ is the kinematic viscosity of the fluid, $\mathbf{u}$ is the fluid velocity, $\mathbf{u}_D$ is a Dirichlet datum for $\mathbf{u}$, p_M is the magnetic pressure, p_H is the hydrodynamic pressure, $\boldsymbol{\mathcal{B}} := \mu_0(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{M}})$ is the magnetic induction, $\boldsymbol{\mathcal{M}}$ is the magnetisation of the fluid, $\boldsymbol{\mathcal{H}}$ is the magnetic field, and $\mathbf{e}(\mathbf{u}) := \frac{1}{2}\Big(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{t}}\Big)$ is the symmetric part of the velocity gradient.

Next, we consider the Shliomis equation [39] for $\boldsymbol{\mathcal{M}}$, so that assuming small Re numbers, we can neglect the advective magnetic term in it, thus obtaining

$$\frac{\partial\boldsymbol{\mathcal{M}}}{\partial t} = -\frac{1}{\varepsilon}\left(\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{M}}_0\right) + \frac{1}{2}\left(\nabla\times\mathbf{u}\right)\times\boldsymbol{\mathcal{M}} - \frac{\mu_0}{6\,\mu\,\phi_0}\,\boldsymbol{\mathcal{M}}\times\left(\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}\right), \tag{2.2}$$

where ε is a relaxation time (usually very small), ϕ_0 is the magnetic particle volume fraction, and $\boldsymbol{\mathcal{M}}_0$ is the equilibrium magnetisation, which depends on the magnetic field $\boldsymbol{\mathcal{H}}$ and is defined by [35]

$$\boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}}) := M_s\,L(\alpha)\,\frac{\boldsymbol{\mathcal{H}}}{\|\boldsymbol{\mathcal{H}}\|}\,,$$

where L is the Langevin function

$$L(\alpha) := \coth(\alpha) - \alpha^{-1}\,,\quad\text{with}\quad \alpha := \frac{m}{\kappa T}\,\|\boldsymbol{\mathcal{H}}\|\,,$$

and M_s, m, and κ are magnetic parameters of the fluid, whereas T is the temperature of the fluid.

In turn, the magnetic field $\boldsymbol{\mathcal{H}}$ satisfies the Maxwell equations on the magnetostatic limit, in which no electrical field is involved, and hence, based on the irrotationality of $\boldsymbol{\mathcal{H}}$, we introduce a magnetic potential ϕ and arrive to the following boundary value problem with Neumann boundary conditions:

$$\begin{gathered}\boldsymbol{\mathcal{H}} \,=\, -\nabla\phi \quad \text{in} \quad \Omega\,, \quad \operatorname{div}\big(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{M}}\big) \,=\, 0 \quad \text{in} \quad \Omega\,, \\ \big(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{M}}\big)\cdot\boldsymbol{\nu} \,=\, \boldsymbol{\mathcal{H}}_0\cdot\boldsymbol{\nu} \quad \text{on} \quad \Gamma\,,\end{gathered} \tag{2.3}$$

where $\boldsymbol{\mathcal{H}}_0$ is the applied magnetic field, which is assumed to satisfy $\operatorname{div}(\boldsymbol{\mathcal{H}}_0) = 0$ and $\nabla\times\boldsymbol{\mathcal{H}}_0 = \mathbf{0}$.

Before continuing, we find it important to draw a comparison between model (2.1) and the most prominent Rosensweig model. Under the simplifying assumption of no angular velocity, the momentum equation in the Rosensweig model [35, 36] (see also [32]) reads:

$$\rho\,\frac{\partial\mathbf{u}}{\partial t} + \rho\,(\nabla\mathbf{u})\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}_H) = \mu_0(\nabla\boldsymbol{\mathcal{H}})\boldsymbol{\mathcal{M}} \tag{2.4}$$

The forcing term in (2.4) is known as *Kelvin force.* Using that $\operatorname{div}(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{M}}) = 0$ and $\nabla\times\boldsymbol{\mathcal{H}} = \mathbf{0}$, standard vector calculus identities imply that

$$\operatorname{div}(\boldsymbol{\sigma}_M) - \mu_0(\nabla\boldsymbol{\mathcal{H}})\boldsymbol{\mathcal{M}} = \frac{\mu_0}{2}\nabla\times(\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}).$$

That is, the magnetic forcing in our model of interest (2.1) only differs from the classical Kelvin force in a multiple of the curl of the magnetic torque $\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}$. For small enough relaxation time parameter ε, it is expected for $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{H}}$ to be approximately colinear, i.e. , $\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}\approx\mathbf{0}$ and so $\nabla\times(\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}})\approx\mathbf{0}$. All in all, this shows that our model of interest (2.1) is a physically sound approximation of the Rosensweig model (under the no angular velocity assumption).

2.2 The steady-state cases

In what follows we consider stationary cases of (2.1) and (2.2), which arise either by dropping the corresponding time derivatives or by replacing them by a zero order term, thus yielding two different approaches.

2.2.1 A first stationary approach

In this case we replace the first equation of (2.1) by

$$\rho\,(\nabla\mathbf{u})\,\mathbf{u} \,=\, \mathbf{div}(\boldsymbol{\sigma}_M+\boldsymbol{\sigma}_H) \quad \text{in} \quad \Omega\,. \tag{2.5}$$

Then, we introduce the velocity gradient and a partial stress magnetic tensor as further unknowns, that is

$$\mathbf{t} := \nabla\mathbf{u} \qquad \text{and} \qquad \boldsymbol{\sigma} := \boldsymbol{\sigma}_H + \boldsymbol{\sigma}_M - \frac{\rho}{2}(\mathbf{u}\otimes\mathbf{u})\,, \tag{2.6}$$

so that, defining the total pressure

$$p \,:=\, p_M \,+\, p_H \,=\, \frac{1}{2}\mu_0\,\boldsymbol{\mathcal{H}}\cdot\boldsymbol{\mathcal{H}} \,+\, p_H\,, \tag{2.7}$$

we deduce, according to the third and fifth rows in (2.1), that

$$\boldsymbol{\sigma} = 2\mu\,\mathbf{t}_\mathbf{s} \,-\, p\,\mathbb{I} \,-\, \frac{\rho}{2}(\mathbf{u}\otimes\mathbf{u}) \,+\, \frac{1}{2}\big(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\big)\,, \tag{2.8}$$

where $\mathbf{t_s} := \frac{1}{2}(\mathbf{t} + \mathbf{t}^{\mathtt{t}})$ is the symmetric part of $\mathbf{t}$. Now, thanks to the incompressibility condition we have $\mathbf{div}(\mathbf{u} \otimes \mathbf{u}) = (\nabla \mathbf{u})\,\mathbf{u} = \mathbf{t}\,\mathbf{u}$, and therefore (2.5) becomes

$$\mathbf{div}(\boldsymbol{\sigma}) - \frac{\rho}{2}\,\mathbf{t}\mathbf{u} = 0 \quad \text{in} \quad \Omega\,. \tag{2.9}$$

In addition, applying the matrix trace to (2.8) and using that $\operatorname{tr}(\mathbf{t_s}) = \operatorname{div}(\mathbf{u}) = 0$, we get

$$\operatorname{tr}\Big(\boldsymbol{\sigma} + \frac{\rho}{2}\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{2}\,(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}})\Big) = -\,n\,p\,,$$

that is

$$p = -\frac{1}{n}\operatorname{tr}\Big(\boldsymbol{\sigma} + \frac{\rho}{2}\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{2}\,(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}})\Big)\,. \tag{2.10}$$

Moreover, bearing in mind that the deviatoric operator $^{\mathtt{d}}$ is certainly linear, and applying it to (2.8), we arrive at

$$\boldsymbol{\sigma}^{\mathtt{d}} = 2\mu\,\mathbf{t_s} - \frac{\rho}{2}\,(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} + \frac{1}{2}\,(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}})^{\mathtt{d}}\,. \tag{2.11}$$

Conversely, starting from (2.10) and (2.11), we easily recover the incompressibility condition, given by $\operatorname{tr}(\mathbf{t}) = \operatorname{tr}(\mathbf{t_s}) = 0$, and (2.8), whence these pair of equations are equivalent. Furthermore, for uniqueness of the total pressure p, we introduce the condition

$$\int_\Omega p = 0\,,$$

which, according to (2.10), becomes

$$\int_\Omega \operatorname{tr}\Big(\boldsymbol{\sigma} + \frac{\rho}{2}\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{2}\,(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}})\Big) = 0\,. \tag{2.12}$$

In this way, eliminating p from the equations, which can be computed afterwards according to (2.10), a first stationary version of (2.1) is stated as: Find $(\mathbf{t}, \mathbf{u}, \boldsymbol{\sigma})$ in suitable spaces to be defined below, such that

$$\begin{gathered}
\mathbf{t} = \nabla\mathbf{u} \quad \text{in} \quad \Omega\,,\\
2\mu\,\mathbf{t_s} - \boldsymbol{\sigma}^{\mathtt{d}} - \frac{\rho}{2}\,(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} = -\frac{1}{2}\,(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}})^{\mathtt{d}} \quad \text{in} \quad \Omega\,,\\
\mathbf{div}(\boldsymbol{\sigma}) - \frac{\rho}{2}\,\mathbf{t}\mathbf{u} = 0 \quad \text{in} \quad \Omega\,,\\
\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \qquad \int_\Omega \operatorname{tr}\Big(\boldsymbol{\sigma} + \frac{\rho}{2}\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{2}\,(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}})\Big) = 0\,.
\end{gathered} \tag{2.13}$$

On the other hand, the corresponding stationary version of (2.2) becomes

$$\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}}) = \frac{\varepsilon}{2}\,(\nabla\times\mathbf{u})\times\boldsymbol{\mathcal{M}} - \frac{\varepsilon\,\mu_0}{6\,\mu\,\phi_0}\,\boldsymbol{\mathcal{M}}\times(\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}})\,,$$

from which, using that

$$(\nabla\times\mathbf{u})\times\boldsymbol{\mathcal{M}} = (\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}})\,\boldsymbol{\mathcal{M}}\,,$$

and recalling from the first row of (2.13) that $\mathbf{t} = \nabla\mathbf{u}$, we can write

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}}) + \frac{\varepsilon}{2}\,(\mathbf{t} - \mathbf{t}^{\mathtt{t}})\,\boldsymbol{\mathcal{M}} - \frac{\varepsilon\,\mu_0}{6\,\mu\,\phi_0}\,\boldsymbol{\mathcal{M}}\times(\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}})\,. \tag{2.14}$$

2.2.2 A second stationary approach

Motivated by the eventual use of an Euler scheme for (2.1), and without loss of generality, we now replace its first equation by

$$\rho\,\mathbf{u} + \rho\,(\nabla\mathbf{u})\,\mathbf{u} \,=\, \mathbf{div}(\boldsymbol{\sigma}_M + \boldsymbol{\sigma}_H) + \mathbf{f} \quad \text{in} \quad \Omega\,, \tag{2.15}$$

where $\mathbf{f}$ is a given source. Then, we introduce the symmetric part of the velocity gradient, also named strain rate tensor, and the full stress magnetic tensor as further unknowns, that is

$$\mathbf{t} := \mathbf{e}(\mathbf{u}) \,=\, \nabla\mathbf{u} - \boldsymbol{\gamma} \qquad \text{and} \qquad \boldsymbol{\sigma} := \boldsymbol{\sigma}_H + \boldsymbol{\sigma}_M - \rho(\mathbf{u}\otimes\mathbf{u})\,, \tag{2.16}$$

where

$$\boldsymbol{\gamma} \,:=\, \frac{1}{2}\big(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\big) \tag{2.17}$$

is the skew-symmetric part of $\nabla\mathbf{u}$, also named vorticity. In this way, and similarly to (2.9), we find that (2.15) becomes

$$\rho\,\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f} \quad \text{in} \quad \Omega\,.$$

In turn, defining again the total pressure p as in (2.7), we deduce, along with the third and fifth rows in (2.1), that

$$\boldsymbol{\sigma} = 2\mu\,\mathbf{t} - p\mathbb{I} - \rho(\mathbf{u}\otimes\mathbf{u}) + \frac{1}{2}\left(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\right). \tag{2.18}$$

In addition, proceeding analogously to the derivation of (2.10) and (2.11), that is applying matrix trace and deviatoric operator to (2.18), we get

$$p \,=\, -\frac{1}{n}\,\mathrm{tr}\Big(\boldsymbol{\sigma} + \rho\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{2}\left(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\right)\Big)\,, \tag{2.19}$$

and

$$\boldsymbol{\sigma}^{\mathtt{d}} \,=\, 2\mu\,\mathbf{t} - \rho\left(\mathbf{u}\otimes\mathbf{u}\right)^{\mathtt{d}} + \frac{1}{2}\left(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\right)^{\mathtt{d}}. \tag{2.20}$$

In particular, the uniqueness condition for p reads now

$$\int_\Omega \mathrm{tr}\Big(\boldsymbol{\sigma} + \rho\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{2}\left(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\right)\Big) \,=\, 0\,. \tag{2.21}$$

According to the above discussion, we conclude that a second stationary version of (2.1) reduces to: Find $(\mathbf{t},\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\gamma})$ in suitable spaces to be defined later on, such that

$$\begin{gathered}
\mathbf{t} + \boldsymbol{\gamma} \,=\, \nabla\mathbf{u} \quad \text{in} \quad \Omega\,,\\
2\mu\,\mathbf{t} - \boldsymbol{\sigma}^{\mathtt{d}} - \rho\left(\mathbf{u}\otimes\mathbf{u}\right)^{\mathtt{d}} \,=\, -\frac{1}{2}\left(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\right)^{\mathtt{d}} \quad \text{in} \quad \Omega\,,\\
\rho\,\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f} \quad \text{in} \quad \Omega\,,\\
\mathbf{u} \,=\, \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \qquad \int_\Omega \mathrm{tr}\Big(\boldsymbol{\sigma} + \rho\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{2}\left(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\right)\Big) \,=\, 0\,.
\end{gathered} \tag{2.22}$$

Furthermore, the corresponding stationary version of (2.2) is given by

$$\boldsymbol{\mathcal{M}} \,=\, -\frac{1}{\varepsilon}\left(\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}})\right) + \frac{1}{2}\left(\nabla\times\mathbf{u}\right)\times\boldsymbol{\mathcal{M}} - \frac{\mu_0}{6\,\mu\,\phi_0}\,\boldsymbol{\mathcal{M}}\times\left(\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}\right),$$

from which, using now, thanks to (2.17), that

$$\left(\nabla\times\mathbf{u}\right)\times\boldsymbol{\mathcal{M}} \,=\, \left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\right)\boldsymbol{\mathcal{M}} \,=\, 2\,\boldsymbol{\gamma}\,\boldsymbol{\mathcal{M}}\,,$$

we get

$$\boldsymbol{\mathcal{M}} \,=\, \left(1 + \varepsilon\right)^{-1}\Big\{\boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}}) + \varepsilon\,\boldsymbol{\gamma}\,\boldsymbol{\mathcal{M}} - \frac{\varepsilon\mu_0}{6\,\mu\,\phi_0}\,\boldsymbol{\mathcal{M}}\times\left(\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}\right)\Big\}\,. \tag{2.23}$$

3 The continuous formulations

In this section we derive the mixed variational formulations for the two versions of the stationary fluid equations, namely (2.13) and (2.22), and for the potential equations given by (2.3).

3.1 A first mixed formulation for the fluid

We begin with the first equation of (2.13) by assuming, originally, that $\mathbf{u}$ is sought in $\mathbf{H}^1(\Omega)$. Then, applying the integration by parts formula (1.4) with $t \geq \frac{2n}{n+2}$ and $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$, and making use of the Dirichlet boundary condition on $\mathbf{u}$, for which we assume from now on that $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, we get

$$\int_\Omega \mathbf{t} : \boldsymbol{\tau} \,+\, \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,=\, \langle \boldsymbol{\tau\nu}, \mathbf{u}_D\rangle \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,. \tag{3.1}$$

Recall that, as described in [8, eq. (2.5)] (see also [14, eq. (3.2)]), $\boldsymbol{\tau\nu}$ belongs to $\mathbf{H}^{-1/2}(\Omega)$ for each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$. In turn, it is clear that the first term of (3.1) makes sense for $\mathbf{t}$ in $\mathbb{L}^2(\Omega)$, whence, bearing in mind its free trace property, we seek this unknown in the space

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) \,:=\, \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \mathrm{tr}(\mathbf{s}) \,=\, 0 \quad \text{in} \quad \Omega \Big\}\,. \tag{3.2}$$

In addition, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)$, we notice from the second term of (3.1) that it would actually suffice to look for $\mathbf{u}$ in $\mathbf{L}^{t'}(\Omega)$, where t' is the conjugate of t. However, testing the second equation of (2.13) against tensors $\mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and using in this case that $\boldsymbol{\zeta}^{\mathsf{d}} : \mathbf{s} \,=\, \boldsymbol{\zeta} : \mathbf{s}$ for all $\boldsymbol{\zeta} \in \mathbb{L}^2(\Omega)$, and that $\mathbf{r}_{\mathbf{s}} : \mathbf{s} \,=\, \mathbf{r}_{\mathbf{s}} : \mathbf{s}_{\mathbf{s}}$ for all $\mathbf{r} \in \mathbb{L}^2(\Omega)$, we formally obtain

$$2\mu \int_\Omega \mathbf{t}_{\mathbf{s}} : \mathbf{s}_{\mathbf{s}} \,-\, \int_\Omega \boldsymbol{\sigma} : \mathbf{s} \,-\, \frac{\rho}{2}\int_\Omega (\mathbf{u}\otimes\mathbf{u}) : \mathbf{s} \,=\, -\frac{1}{2}\int_\Omega \big(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\big) : \mathbf{s} \quad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.3}$$

from which we see that its third term makes sense for $(\mathbf{u}\otimes\mathbf{u}) \in \mathbb{L}^2(\Omega)$, that is, according to the Cauchy–Schwarz and Hölder inequalities, for $\mathbf{u} \in \mathbf{L}^4(\Omega)$, and hence from now on we choose $t' = 4$, which yields $t = 4/3$. Similarly, the fourth term is well-defined provided $\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}$ belong to $\mathbb{L}^2(\Omega)$ which holds, in particular, for

$$\boldsymbol{\mathcal{H}} \in \mathbf{L}^{2j}(\Omega) \quad \text{and} \quad \boldsymbol{\mathcal{B}} \in \mathbf{L}^{2\ell}(\Omega)\,, \quad \text{with } j,\, \ell \in (1,+\infty) \text{ conjugate to each other.} \tag{3.4}$$

Next, it is clear that the second term of (3.3) makes sense for $\boldsymbol{\sigma} \in \mathbb{L}^2(\Omega)$, so that, in order to utilize the same space for this unknown and its corresponding test functions $\boldsymbol{\tau}$, we look for $\boldsymbol{\sigma}$ in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well. Therefore, having now that $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$, we test the third equation of (2.13) against $\mathbf{v} \in \mathbf{L}^4(\Omega)$, thus obtaining

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \,+\, \frac{\rho}{2}\int_\Omega \mathbf{t}\,\mathbf{u}\cdot\mathbf{v} \,=\, 0 \qquad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,. \tag{3.5}$$

Note that the second term of (3.5) is well-defined as well since $\mathbf{t} \in \mathbb{L}^2(\Omega)$ and $\mathbf{u}$, $\mathbf{v} \in \mathbf{L}^4(\Omega)$.

At this point we consider for the convenience of the subsequent analysis, the decomposition (see, for instance, [7], [14])

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,\oplus \mathrm{R}\,\mathbb{I}, \tag{3.6}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \Big\}\,, \tag{3.7}$$

which means that each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ can be written, in a unique manner, as $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d_0\mathbb{I}$, with $\boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $d_0 := \frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \in \mathrm{R}$. In particular, taking into account the uniqueness condition (2.12) (see, also, the fourth row of (2.13)), we find that the unknown $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ can be decomposed as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0\mathbb{I}, \quad \text{with} \quad \boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$$
$$\text{and} \quad c_0 = -\frac{1}{2\,n\,|\Omega|}\int_\Omega \mathrm{tr}\left(\rho\left(\mathbf{u}\otimes\mathbf{u}\right) - \left(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\right)\right)\,,$$

which says that c_0 can be computed once the velocity $\mathbf{u}$ is known. In addition, we observe that (3.5) and (3.3) remain unchanged if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$, and hence we rename from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Moreover, thanks to the compatibility condition satisfied by the datum $\mathbf{u}_D$ and the fact that $\mathbf{t}$ is sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, we notice that testing (3.1) against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Therefore, denoting

$$\vec{\mathbf{u}} := (\mathbf{u},\mathbf{t}), \quad \vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,,$$

we arrive at the following variational formulation of (2.13): Find $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in (\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega))\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ such that

$$\begin{aligned}\mathbf{a}(\vec{\mathbf{u}},\vec{\mathbf{v}}) + \mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) &= \mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{v}}) \qquad \forall\ \vec{\mathbf{v}} \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,,\\ \mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega))\,,\end{aligned} \tag{3.8}$$

where, given arbitrary $\mathbf{w} \in \mathbf{L}^4(\Omega)$, the bilinear forms $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}(\mathbf{w},\cdot,\cdot)$ are defined by

$$\mathbf{a}(\vec{\mathbf{u}},\vec{\mathbf{v}}) := 2\mu\int_\Omega \mathbf{t}_{\mathbf{s}} : \mathbf{s}_{\mathbf{s}}\,, \quad \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\tau}) := -\int_\Omega \boldsymbol{\tau} : \mathbf{s} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \tag{3.9}$$

$$\mathbf{c}(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) := \frac{\rho}{2}\left\{\int_\Omega \mathbf{t}\,\mathbf{w}\cdot\mathbf{v} - \int_\Omega (\mathbf{u}\otimes\mathbf{w}) : \mathbf{s}\right\}\,, \tag{3.10}$$

for all $\vec{\mathbf{u}} := (\mathbf{u},\mathbf{t}), \vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)$ and $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$; whereas, given $\boldsymbol{\mathcal{B}} \in \mathbf{L}^{2\ell}(\Omega)$ and $\boldsymbol{\mathcal{H}} \in \mathbf{L}^{2j}(\Omega)$, the functionals $\mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}$ and $\mathbf{G}$ are defined, respectively, by

$$\mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{v}}) := -\frac{1}{2}\int_\Omega \left(\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}}\otimes\boldsymbol{\mathcal{B}}\right) : \mathbf{s}\,, \tag{3.11}$$

for all $\vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and

$$\mathbf{G}(\boldsymbol{\tau}) := -\langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{u}_D\rangle_\Gamma \qquad \forall\ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,. \tag{3.12}$$

It is important to stress here that the structure of (3.8) coincides with the one given by the first two rows of [14, eq. (3.17)] (see, also, [9], [15], and [17]). Hence, the corresponding continuous and discrete analyses to be presented below, being closely related to those provided in [14, Sections 3, 4, and 5], will only refer to the main aspects of them.

Now, regarding the boundedness properties of the present forms, and setting

$$\|\vec{\mathbf{v}}\| := \|\mathbf{v}\|_{0,4;\Omega} + \|\mathbf{s}\|_{0,\Omega} \qquad \forall\ \vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,,$$

we notice that Hölder's inequality, the duality between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$, and the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, yield

$$\begin{aligned} |\mathbf{a}(\vec{\mathbf{u}}, \vec{\mathbf{v}})| \,&\le\, 2\mu\, \|\vec{\mathbf{u}}\|\, \|\vec{\mathbf{v}}\| \,, \\ |\mathbf{b}(\vec{\mathbf{v}}, \boldsymbol{\tau})| \,&\le\, \|\vec{\mathbf{v}}\|\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \,, \end{aligned} \tag{3.13}$$

$$|\mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{v}})| \,\le\, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\|_{0,\Omega}\, \|\vec{\mathbf{v}}\| \,, \tag{3.14}$$

and

$$|\mathbf{G}(\boldsymbol{\tau})| \,\le\, \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3},\Omega} \,, \tag{3.15}$$

for all $\vec{\mathbf{u}},\ \vec{\mathbf{v}} \,\in\, \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)$ and $\boldsymbol{\tau} \,\in\, \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, where $\widetilde{\mathbf{u}}_D \,:=\, \max\big\{1, \|\mathbf{i}_4\|\big\}\, \mathbf{u}_D$. We finish this section by recalling from [14, Lemma 3.4] some properties satisfied by $\mathbf{c}$, which are consequences of the Cauchy-Schwarz inequality and the definition of the tensor product, namely

$$\mathbf{c}(\mathbf{w}; \vec{\mathbf{v}}, \vec{\mathbf{v}}) \,=\, 0 \,, \tag{3.16}$$

$$|\mathbf{c}(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}})| \,\le\, \|\mathbf{w}\|_{0,4;\Omega}\, \|\vec{\mathbf{u}}\|\, \|\vec{\mathbf{v}}\| \,, \tag{3.17}$$

$$|\mathbf{c}(\mathbf{w}, \vec{\mathbf{u}}, \vec{\mathbf{v}}) - \mathbf{c}(\mathbf{z}, \vec{\mathbf{u}}, \vec{\mathbf{v}})| \,\le\, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}\, \|\vec{\mathbf{u}}\|\, \|\vec{\mathbf{v}}\| \,, \tag{3.18}$$

for all $\mathbf{w} \in \mathbf{L}^4(\Omega)$ and $\vec{\mathbf{u}},\ \vec{\mathbf{v}} \,\in\, \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)$.

3.2 A second mixed formulation for the fluid

We now address the variational formulation of (2.22) by adapting the analysis from Section 3.1. Indeed, knowing in advance that the term $(\mathbf{u} \otimes \mathbf{u})$ in the second equation of (2.22) will lead us to look for $\mathbf{u}$ in $\mathbf{L}^4(\Omega)$, we now test the first equation of (2.22) directly against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, thus obtaining, thanks again to (1.4),

$$\int_\Omega \mathbf{t} : \boldsymbol{\tau} \,+\, \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} \,+\, \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \,=\, \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D \rangle \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) \,. \tag{3.19}$$

It is clear that the second term of (3.19) is well-defined for $\boldsymbol{\gamma} \in \mathbb{L}^2(\Omega)$, so that, according to the skew-symmetry of the vorticity (cf. (2.17)), we look for this unknown in the space

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) \,:=\, \Big\{ \boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\eta}^{\mathrm{t}} \,=\, -\boldsymbol{\eta} \Big\} \,, \tag{3.20}$$

whereas, as before, $\mathbf{t}$ is certainly sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$. Next, proceeding analogously to the derivation of (3.3) and (3.5), and assuming from now on that $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$, the corresponding testing of the second and third equations of (2.22) yield, respectively,

$$2\mu \int_\Omega \mathbf{t} : \mathbf{s} \,-\, \int_\Omega \boldsymbol{\sigma} : \mathbf{s} \,-\, \rho \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \mathbf{s} \,=\, -\frac{1}{2} \int_\Omega \big(\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}} \otimes \boldsymbol{\mathcal{B}} \big) : \mathbf{s} \quad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) \,, \tag{3.21}$$

and

$$\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) \,-\, \rho \int_\Omega \mathbf{u} \cdot \mathbf{v} \,=\, -\int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega) \,. \tag{3.22}$$

In addition, employing again the decomposition of $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ specified by (3.6) and (3.7), and invoking now the uniqueness condition (2.21), we deduce that $\boldsymbol{\sigma}$ can be decomposed as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0 \mathbb{I}, \quad \text{with} \quad \boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$$

$$\text{and} \quad c_0 = -\frac{1}{n|\Omega|} \int_\Omega \mathrm{tr} \left(\rho\, (\mathbf{u} \otimes \mathbf{u}) - \frac{1}{2} \big(\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}} \otimes \boldsymbol{\mathcal{B}} \big) \right) .$$

Then, exactly as for (3.5) and (3.3), (3.21) and (3.22) remain unchanged as well if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$, so that this unknown is simply renamed from now on $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. In turn, due basically to the same arguments regarding (3.1), we observe that testing (3.19) against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Furthermore, noting from (2.18) that $\boldsymbol{\sigma}$ is a symmetric tensor, we impose this property weakly as

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} \,=\, 0 \qquad \forall\, \boldsymbol{\eta} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)\,. \tag{3.23}$$

We remark here that, similarly as observed for the right hand side of (3.3), though in the present case motivated by the source terms of (3.21) and (3.23), we need to assume that $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}$ satisfy the feasibility condition (3.4).

Consequently, suitably gathering (3.19), (3.21), (3.22), and (3.23), introducing the spaces

$$\mathbf{X} \,:=\, \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad \mathbf{Y} \,:=\, \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)\,,$$

and denoting

$$\vec{\mathbf{t}} \,:=\, (\mathbf{t},\boldsymbol{\sigma})\,, \quad \vec{\mathbf{s}} \,:=\, (\mathbf{s},\boldsymbol{\tau})\,, \quad \vec{\mathbf{r}} \,:=\, (\mathbf{r},\boldsymbol{\zeta}) \,\in\, \mathbf{X}\,,$$

$$\vec{\mathbf{u}} \,:=\, (\mathbf{u},\boldsymbol{\gamma})\,, \quad \vec{\mathbf{v}} \,:=\, (\mathbf{v},\boldsymbol{\eta})\,, \quad \vec{\mathbf{w}} \,:=\, (\mathbf{w},\boldsymbol{\chi}) \,\in\, \mathbf{Y}\,,$$

we arrive at the following variational formulation of (2.22): Find $(\vec{\mathbf{t}},\vec{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y}$ such that

$$\begin{array}{lcll} \mathcal{A}(\vec{\mathbf{t}},\vec{\mathbf{s}}) \,+\, \mathcal{B}(\vec{\mathbf{s}},\vec{\mathbf{u}}) \,+\, \mathcal{D}(\mathbf{u};\mathbf{u},\mathbf{s}) & = & \mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{s}}) & \forall\, \vec{\mathbf{s}} \in \mathbf{X}\,, \\ \mathcal{B}(\vec{\mathbf{t}},\vec{\mathbf{v}}) \,-\, \mathcal{C}(\vec{\mathbf{u}},\vec{\mathbf{v}}) & = & \mathcal{G}(\vec{\mathbf{v}}) & \forall\, \vec{\mathbf{v}} \in \mathbf{Y}\,, \end{array} \tag{3.24}$$

where the bilinear forms $\mathcal{A} : \mathbf{X} \times \mathbf{X} \to \mathrm{R}$, $\mathcal{B} : \mathbf{X} \times \mathbf{Y} \to \mathrm{R}$, and $\mathcal{C} : \mathbf{Y} \times \mathbf{Y} \to \mathrm{R}$, are defined by

$$\mathcal{A}(\vec{\mathbf{r}},\vec{\mathbf{s}}) \,:=\, a(\mathbf{r},\mathbf{s}) + b_1(\mathbf{s},\boldsymbol{\zeta}) + b_2(\mathbf{r},\boldsymbol{\tau}) \qquad \forall\, \vec{\mathbf{r}},\, \vec{\mathbf{s}} \in \mathbf{X}\,, \tag{3.25}$$

with

$$a(\mathbf{r},\mathbf{s}) \,:=\, 2\mu \int_\Omega \mathbf{r} : \mathbf{s}\,, \qquad b_1(\mathbf{s},\boldsymbol{\zeta}) \,:=\, -\int_\Omega \boldsymbol{\zeta} : \mathbf{s}\,, \qquad b_2(\mathbf{r},\boldsymbol{\tau}) \,:=\, \int_\Omega \mathbf{r} : \boldsymbol{\tau}\,, \tag{3.26}$$

$$\mathcal{B}(\vec{\mathbf{r}},\vec{\mathbf{v}}) \,:=\, \int_\Omega \boldsymbol{\eta} : \boldsymbol{\zeta} \,+\, \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\zeta}) \qquad \forall\, (\vec{\mathbf{r}},\vec{\mathbf{v}}) \,\in\, \mathbf{X} \times \mathbf{Y}\,, \tag{3.27}$$

and

$$\mathcal{C}(\vec{\mathbf{w}},\vec{\mathbf{v}}) \,:=\, \rho \int_\Omega \mathbf{w} \cdot \mathbf{v} \qquad \forall\, \vec{\mathbf{w}},\, \vec{\mathbf{v}} \,\in\, \mathbf{Y}\,, \tag{3.28}$$

whereas for each $\mathbf{w} \in \mathbf{L}^4(\Omega)$, $\mathcal{D}(\mathbf{w};\cdot,\cdot) : \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)$ is the bilinear form given by

$$\mathcal{D}(\mathbf{w};\mathbf{v},\mathbf{s}) \,:=\, -\rho \int_\Omega (\mathbf{w} \otimes \mathbf{v}) : \mathbf{s} \qquad \forall\, (\mathbf{v},\mathbf{s}) \,\in\, \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,. \tag{3.29}$$

In addition, similarly to (3.11), given $\boldsymbol{\mathcal{B}} \in \mathbf{L}^{2\ell}(\Omega)$ and $\boldsymbol{\mathcal{H}} \in \mathbf{L}^{2j}(\Omega)$, the functionals $\mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}$ and $\mathcal{G}$ are defined, respectively, by

$$\mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{s}}) \,:=\, -\frac{1}{2}\int_\Omega \big(\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{H}} \otimes \boldsymbol{\mathcal{B}}\big) : \mathbf{s} \,+\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D \rangle \qquad \forall\, \vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau}) \in \mathbf{X}\,, \tag{3.30}$$

and

$$\mathcal{G}(\vec{\mathbf{v}}) \,:=\, -\int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \vec{\mathbf{v}} := (\mathbf{v},\boldsymbol{\eta}) \,\in\, \mathbf{Y}\,. \tag{3.31}$$

Like the respective statement in Section 3.1, we remark here that the structure of (3.24) is basically the same provided in [26, eq. (3.19)] for a fully mixed formulation of the Navier-Stokes-Brinkman equations. Consequently, when discussing on the corresponding continuous and discrete analyses later on, we will omit most details and refer to [26].

In addition, endowing $\mathbf{X}$ and $\mathbf{Y}$ with the norms

$$\begin{aligned}\|\vec{\mathbf{s}}\|_{\mathbf{X}} &:= \|\mathbf{s}\|_{0,\Omega} + \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \vec{\mathbf{s}} = (\mathbf{s},\boldsymbol{\tau}) \in \mathbf{X} \quad \text{and}\\ \|\vec{\mathbf{v}}\|_{\mathbf{Y}} &:= \|\mathbf{v}\|_{0,4;\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega} \qquad \forall\, \vec{\mathbf{v}} = (\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}\,,\end{aligned}$$

we find that the stability properties of the above bilinear forms and linear functionals, which follow from simple applications of the Cauchy-Schwarz and Hölder inequalities, are given by

$$|a(\mathbf{r},\mathbf{s})| \,\le\, 2\mu\,\|\mathbf{r}\|_{0,\Omega}\,\|\mathbf{s}\|_{0,\Omega} \qquad \forall\, \mathbf{r},\,\mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.32}$$

$$|b_i(\mathbf{s},\boldsymbol{\tau})| \,\le\, \|\mathbf{s}\|_{0,\Omega}\,\|\boldsymbol{\tau}\|_{0,\Omega} \qquad \forall\, (\mathbf{s},\boldsymbol{\tau}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \quad \forall\, i \in \{1,2\}\,, \tag{3.33}$$

$$|\mathcal{B}(\vec{\mathbf{r}},\vec{\mathbf{v}})| \,\le\, \|\boldsymbol{\zeta}\|_{\mathbf{div}_{4/3};\Omega}\,\|\vec{\mathbf{v}}\|_{\mathbf{Y}} \,\le\, \|\vec{\mathbf{r}}\|_{\mathbf{X}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Y}} \qquad \forall\, (\vec{\mathbf{r}},\vec{\mathbf{v}}) \in \mathbf{X} \times \mathbf{Y}\,, \tag{3.34}$$

$$|\mathcal{C}(\vec{\mathbf{w}},\vec{\mathbf{v}})| \,\le\, \rho\,|\Omega|^{1/2}\,\|\mathbf{w}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega} \,\le\, \rho\,|\Omega|^{1/2}\,\|\vec{\mathbf{w}}\|_{\mathbf{Y}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Y}} \qquad \forall\, (\vec{\mathbf{w}},\vec{\mathbf{v}}) \in \mathbf{Y} \times \mathbf{Y}\,, \tag{3.35}$$

$$|\mathcal{D}(\mathbf{w};\mathbf{v},\mathbf{s})| \,\le\, \rho\,\|\mathbf{w}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega} \qquad \forall\, (\mathbf{w},\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.36}$$

$$|\mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{s}})| \,\le\, \Big(\|\tilde{\mathbf{u}}_D\|_{1/2,\Gamma} \,+\, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\|_{0,\Omega}\Big)\,\|\vec{\mathbf{s}}\|_{\mathbf{X}} \qquad \forall\, \vec{\mathbf{s}} = (\mathbf{s},\boldsymbol{\tau}) \in \mathbf{X}\,, \tag{3.37}$$

and

$$|\mathcal{G}(\vec{\mathbf{v}})| \,\le\, \|\mathbf{f}\|_{0,4/3;\Omega}\,\|\vec{\mathbf{v}}\|_{\mathbf{Y}} \qquad \forall\, \vec{\mathbf{v}} = (\mathbf{v},\boldsymbol{\eta}) \in \mathbf{Y}\,. \tag{3.38}$$

3.3 The potential equation

Regarding the potential equation (2.3), we introduce the auxiliary unknown

$$\boldsymbol{\vartheta} \,:=\, \mu_0^{-1}\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{H}}_0 \,=\, \boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{H}}_0 \quad \text{in} \quad \Omega\,,$$

whence, using that $\boldsymbol{\mathcal{H}}_0$ is divergence free, we find that (2.3) can be reformulated, equivalently, as

$$\begin{gathered}\boldsymbol{\vartheta} \,-\, \boldsymbol{\mathcal{M}} \,+\, \boldsymbol{\mathcal{H}}_0 \,=\, -\nabla\phi \quad \text{in} \quad \Omega\,, \quad \mathrm{div}(\boldsymbol{\vartheta}) \,=\, 0 \quad \text{in} \quad \Omega\,,\\ \boldsymbol{\vartheta}\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma\,.\end{gathered} \tag{3.39}$$

In this way, once $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{H}}_0$ are given, and (3.39) is solved for $\boldsymbol{\vartheta}$ and ϕ in spaces to be specified below, the required magnetic induction $\boldsymbol{\mathcal{B}}$ and magnetic field $\boldsymbol{\mathcal{H}}$ are computed as

$$\boldsymbol{\mathcal{B}} \,=\, \mu_0\,\big(\boldsymbol{\vartheta} + \boldsymbol{\mathcal{H}}_0\big) \quad \text{and} \quad \boldsymbol{\mathcal{H}} \,=\, \mu_0^{-1}\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{M}}\,. \tag{3.40}$$

More precisely, following (3.4), we consider $j,\,\ell \in (1,+\infty)$ conjugate to each other, and set

$$r \,:=\, 2\,\ell\,, \qquad s \,:=\, \frac{r}{r-1}\ \text{(conjugate of } r)\,, \qquad \text{and} \quad \varrho \,:=\, 2\,j\,, \tag{3.41}$$

so that, under the assumptions that $\boldsymbol{\mathcal{H}}_0 \in \mathbf{L}^r(\Omega)$ and $\boldsymbol{\mathcal{M}} \in \mathbf{L}^r(\Omega)$, we look for $\boldsymbol{\vartheta}$ in $\mathbf{L}^r(\Omega)$, thus guaranteeing that $\boldsymbol{\vartheta} - \boldsymbol{\mathcal{M}} + \boldsymbol{\mathcal{H}}_0$ also belongs to $\mathbf{L}^r(\Omega)$. Notice that, provided $\varrho \ge r$, this is compatible with the assumptions $\boldsymbol{\mathcal{B}} \in \mathbf{L}^r(\Omega)$ and $\boldsymbol{\mathcal{H}} \in \mathbf{L}^\varrho(\Omega)$ (cf. (3.4)), and with the identities in (3.40), because the continuous embedding $\mathbf{L}^\varrho(\Omega) \hookrightarrow \mathbf{L}^r(\Omega)$ imply that $\boldsymbol{\mathcal{H}}$ also belongs to $\mathbf{L}^r(\Omega)$. This fact and the identity for $\nabla\phi$ in (3.39) suggest to originally seek $\phi \in \mathrm{W}^{1,r}(\Omega)$. Consequently, from now on, we assume that $\varrho \geq r$.

Introducing for each $t \in (1,+\infty)$ the space (cf. (1.2))

$$\mathbf{H}_0^t(\mathrm{div}_t;\Omega) \,:=\, \Big\{ \boldsymbol{\xi} \in \mathbf{H}^t(\mathrm{div}_t;\Omega) : \quad \boldsymbol{\xi}\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma \Big\},$$

and applying (1.5) with $(t,t') = (s,r)$ to $(\boldsymbol{\xi},\phi) \in \mathbf{H}_0^s(\mathrm{div}_s;\Omega) \times \mathrm{W}^{1,r}(\Omega)$, we find that

$$\int_\Omega \boldsymbol{\xi}\cdot\nabla\phi \,=\, -\int_\Omega \phi\,\mathrm{div}(\boldsymbol{\xi}),$$

whence the testing of the first equation of (3.39) against $\boldsymbol{\xi} \in \mathbf{H}_0^s(\mathrm{div}_s;\Omega)$ becomes

$$\int_\Omega \boldsymbol{\vartheta}\cdot\boldsymbol{\xi} - \int_\Omega \phi\,\mathrm{div}(\boldsymbol{\xi}) \,=\, \int_\Omega \big(\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{H}}_0\big)\cdot\boldsymbol{\xi} \qquad \forall\, \boldsymbol{\xi} \in \mathbf{H}_0^s(\mathrm{div}_s;\Omega). \tag{3.42}$$

Note here that, belonging $\mathrm{div}(\boldsymbol{\xi})$ to $\mathbf{L}^s(\Omega)$, the second term of (3.42) makes sense even for $\phi \in \mathrm{L}^r(\Omega)$, and hence it would suffice to look for this unknown in this space instead of $\mathrm{W}^{1,r}(\Omega)$. However, due to the Neumann boundary condition satisfied by $\boldsymbol{\xi} \in \mathbf{H}_0^s(\mathrm{div}_s;\Omega)$, that term vanishes when ϕ is constant, and hence, for sake of uniqueness of this unknown, we actually look for it in $\mathrm{L}_0^r(\Omega)$, where

$$\mathrm{L}_0^t(\Omega) \,:=\, \Big\{ \varphi \in \mathrm{L}^t(\Omega) : \quad \int_\Omega \varphi \,=\, 0 \Big\} \qquad \forall\, t \in (1,+\infty). \tag{3.43}$$

In turn, having at first instance $\boldsymbol{\vartheta} \in \mathbf{L}^r(\Omega)$, and bearing in mind the momentum equation and the Neumann boundary condition in (3.39), we realize that this unknown must be sought in $\mathbf{H}_0^r(\mathrm{div}_r;\Omega)$. Thus, knowing now that $\mathrm{div}(\boldsymbol{\vartheta}) \in \mathrm{L}^r(\Omega)$, and employing the decomposition $\mathrm{L}^t(\Omega) \,=\, \mathrm{L}_0^t(\Omega) \oplus \mathrm{R}$, we realize that testing the second equation of (3.39) against $\psi \in \mathrm{L}^s(\Omega)$ is equivalent to doing it against $\psi \in \mathrm{L}_0^s(\Omega)$, which reads

$$\int_\Omega \psi\,\mathrm{div}(\boldsymbol{\vartheta}) \,=\, 0 \qquad \forall\, \psi \in \mathrm{L}_0^s(\Omega). \tag{3.44}$$

Consequently, according to the previous discussion, and gathering (3.42) and (3.44), the variational formulation of (3.39) reduces to: Find $(\boldsymbol{\vartheta},\phi) \in \mathrm{X}_2 \times \mathrm{M}_1$ such that

$$\begin{array}{rcll} \mathrm{A}(\boldsymbol{\vartheta},\boldsymbol{\xi}) \,+\, \mathrm{B}_1(\boldsymbol{\xi},\phi) & = & \mathrm{F}_{\boldsymbol{\mathcal{M}}}(\boldsymbol{\xi}) & \forall\, \boldsymbol{\xi} \in \mathrm{X}_1, \\[2ex] \mathrm{B}_2(\boldsymbol{\vartheta},\psi) & = & 0 & \forall\, \psi \in \mathrm{M}_2, \end{array} \tag{3.45}$$

where

$$\mathrm{X}_2 \,:=\, \mathbf{H}_0^r(\mathrm{div}_r;\Omega), \quad \mathrm{X}_1 \,:=\, \mathbf{H}_0^s(\mathrm{div}_s;\Omega), \quad \mathrm{M}_1 \,:=\, \mathrm{L}_0^r(\Omega), \quad \text{and} \quad \mathrm{M}_2 \,:=\, \mathrm{L}_0^s(\Omega), \tag{3.46}$$

whereas the bilinear forms $\mathrm{A} : \mathrm{X}_2 \times \mathrm{X}_1 \to \mathrm{R}$ and $\mathrm{B}_i : \mathrm{X}_i \times \mathrm{M}_i \to \mathrm{R}$, $i \in \big\{1,2\big\}$, and the functional $\mathrm{F}_{\boldsymbol{\mathcal{M}}} : \mathrm{X}_1 \to \mathrm{R}$, are defined, respectively, by

$$\mathrm{A}(\boldsymbol{\lambda},\boldsymbol{\xi}) \,:=\, \int_\Omega \boldsymbol{\lambda}\cdot\boldsymbol{\xi} \qquad \forall\, (\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathrm{X}_2 \times \mathrm{X}_1, \tag{3.47}$$

$$\mathrm{B}_i(\boldsymbol{\xi},\psi) \,:=\, -\int_\Omega \psi\,\mathrm{div}(\boldsymbol{\xi}) \qquad \forall\, (\boldsymbol{\xi},\psi) \in \mathrm{X}_i \times \mathrm{M}_i, \quad \forall\, i \in \big\{1,2\big\}, \tag{3.48}$$

and

$$\mathrm{F}_{\boldsymbol{\mathcal{M}}}(\boldsymbol{\xi}) \,:=\, \int_\Omega \big(\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{H}}_0\big)\cdot\boldsymbol{\xi} \qquad \forall\, \boldsymbol{\xi} \in \mathrm{X}_1, \tag{3.49}$$

Analogously to respective remarks in Sections 3.1 and 3.2, we stress here that, except for a variable viscosity in [25], the structure of (3.45) coincides with that obtained in [25, eq. (2.29)] for the mixed formulation of Darcy's equation within the context of its coupling with the heat equation. According to it, and while some details will be provided, the continuous and discrete analyses of (3.45), to be discussed later on, will basically reduce to referring to the proper results from [25].

Next, endowing X_2, X_1, M_1, and M_2 with the norms

$$\|\cdot\|_{\mathrm{X}_2} := \|\cdot\|_{r,\mathrm{div}_r;\Omega}\,, \quad \|\cdot\|_{\mathrm{X}_1} := \|\cdot\|_{s,\mathrm{div}_s;\Omega}\,, \quad \|\cdot\|_{\mathrm{M}_1} := \|\cdot\|_{0,r;\Omega}\,, \quad \text{and} \quad \|\cdot\|_{\mathrm{M}_2} := \|\cdot\|_{0,s;\Omega}\,,$$

we readily find that the stability bounds of A, B_1, B_2, and F, which are consequence of the Hölder and triangle inequalities, are stated as follows

$$\big|\mathrm{A}(\boldsymbol{\lambda},\boldsymbol{\xi})\big| \,\le\, \|\boldsymbol{\lambda}\|_{0,r;\Omega}\,\|\boldsymbol{\xi}\|_{0,s;\Omega} \,\le\, \|\boldsymbol{\lambda}\|_{\mathrm{X}_2}\,\|\boldsymbol{\xi}\|_{\mathrm{X}_1} \quad \forall\,(\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathrm{X}_2\times\mathrm{X}_1\,, \tag{3.50}$$

$$\big|\mathrm{B}_1(\boldsymbol{\xi},\psi)\big| \,\le\, \|\mathrm{div}(\boldsymbol{\xi})\|_{0,s;\Omega}\,\|\psi\|_{0,r;\Omega} \,\le\, \|\boldsymbol{\xi}\|_{\mathrm{X}_1}\,\|\psi\|_{\mathrm{M}_1} \quad \forall\,(\boldsymbol{\xi},\psi) \in \mathrm{X}_1\times\mathrm{M}_1\,, \tag{3.51}$$

$$\big|\mathrm{B}_2(\boldsymbol{\lambda},\psi)\big| \,\le\, \|\mathrm{div}(\boldsymbol{\lambda})\|_{0,r;\Omega}\,\|\psi\|_{0,s;\Omega} \,\le\, \|\boldsymbol{\lambda}\|_{\mathrm{X}_2}\,\|\psi\|_{\mathrm{M}_2} \quad \forall\,(\boldsymbol{\lambda},\psi) \in \mathrm{X}_2\times\mathrm{M}_2\,, \tag{3.52}$$

and

$$\big|\mathrm{F}_{\boldsymbol{\mathcal{M}}}(\boldsymbol{\xi})\big| \,\le\, \|\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{H}}_0\|_{0,r;\Omega}\,\|\boldsymbol{\xi}\|_{0,s;\Omega} \,\le\, \Big\{\|\boldsymbol{\mathcal{M}}\|_{0,r;\Omega} + \|\boldsymbol{\mathcal{H}}_0\|_{0,r;\Omega}\Big\}\,\|\boldsymbol{\xi}\|_{\mathrm{X}_1} \quad \forall\,\boldsymbol{\xi}\in\mathrm{X}_1\,. \tag{3.53}$$

4 Analysis of the first mixed formulation for the fluid

In this section we address the continuous and discrete analyses of the variational formulation (3.8).

4.1 The continuous analysis

As already announced in Section 3.1, here we follow [14, Section 3] and analyze the solvability of (3.8) by means of a fixed point approach. To that end, let $\boldsymbol{\Xi} : \mathbf{L}^4(\Omega) \to \mathbf{L}^4(\Omega)$ be the operator defined by:

$$\boldsymbol{\Xi}(\mathbf{w}) \,:=\, \mathbf{u} \qquad \forall\,\mathbf{w}\in\mathbf{L}^4(\Omega)\,,$$

where $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \,:=\, \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big) \in \big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\big)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ is the unique solution (to be confirmed below) of problem (3.8) when $\mathbf{c}(\mathbf{u};\cdot,\cdot)$ is replaced by $\mathbf{c}(\mathbf{w};\cdot,\cdot)$, which reads

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}},\vec{\mathbf{v}}) + \mathbf{c}(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) \;&=\; \mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{v}}) \qquad \forall\;\vec{\mathbf{v}}\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega),\\ \mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) \;&=\; \mathbf{G}(\boldsymbol{\tau}) \qquad\qquad \forall\;\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega). \end{aligned} \tag{4.1}$$

Then, we realize that solving (3.8) is equivalent to seeking a fixed point of $\boldsymbol{\Xi}$, that is: Find $\mathbf{u}\in\mathbf{L}^4(\Omega)$ such that

$$\boldsymbol{\Xi}(\mathbf{u}) \,=\, \mathbf{u}\,. \tag{4.2}$$

In what follows we apply the Banach version of the Babuška-Brezzi theory (see, e.g. [22, Theorem 2.34]) to show that the linearized problem (4.1) is well-posed, which is equivalent to the well-definiteness of the operator $\boldsymbol{\Xi}$. To this end, we first let $\mathbf{V}$ be the kernel of the operator induced by the bilinear form $\mathbf{b}$, that is

$$\mathbf{V} \,:=\, \Big\{\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s})\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\tau}) = 0 \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3;\Omega})\Big\}\,,$$

which, according to [14, Eq. (3.34)], reduces to

$$\mathbf{V} \,=\, \Big\{ \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\rm tr}(\Omega) : \quad \nabla \mathbf{v} = \mathbf{s}, \quad \text{and} \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega) \Big\} .$$

The following two results establish the $\mathbf{V}$-ellipticity of $\mathbf{a}$ and the continuous inf-sup condition for $\mathbf{b}$, whose respective proofs can be found in [14, Lemmas 3.2 and 3.3].

Lemma 4.1. *There exists a positive constant* $\alpha > 0$ *such that*

$$\mathbf{a}(\vec{\mathbf{v}}, \vec{\mathbf{v}}) \,\geq\, \alpha \, \|\vec{\mathbf{v}}\|^2 \qquad \forall \, \vec{\mathbf{v}} \in \mathbf{V} . \tag{4.3}$$

Lemma 4.2. *There exists a positive constant* β *such that*

$$\sup_{\substack{\vec{\mathbf{v}} \in \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\rm tr}(\Omega) \\ \vec{\mathbf{v}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{v}}, \boldsymbol{\tau})}{\|\vec{\mathbf{v}}\|} \,\geq\, \beta \, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall \, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) . \tag{4.4}$$

According to the foregoing results, we now establish the well-definiteness of the operator $\boldsymbol{\Xi}$. From now on, and according to the notations introduced in (3.41), we use r and ϱ instead of 2ℓ and $2j$, respectively.

Lemma 4.3. *For each* $\mathbf{w} \in \mathbf{L}^4(\Omega)$ *there exists a unique* $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in (\mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\rm tr}(\Omega)) \times \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ *solution to the problem* (4.1)*, and hence one can define* $\boldsymbol{\Xi}(\mathbf{w}) := \mathbf{u} \in \mathbf{L}^4(\Omega)$*. Moreover, there exist positive constants* $C_{\boldsymbol{\Xi},1}$ *and* $C_{\boldsymbol{\Xi},2}$*, depending only on* α*,* β*, and* μ*, such that there holds the following a priori estimate*

$$\|\boldsymbol{\Xi}(\mathbf{w})\| \,=\, \|\mathbf{u}\|_{0,4;\Omega} \,\leq\, \|\vec{\mathbf{u}}\| \,\leq\, C_{\boldsymbol{\Xi},1} \, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\|_{0,\Omega} \,+\, C_{\boldsymbol{\Xi},2} \left(1 + \|\mathbf{w}\|_{0,4;\Omega}\right) \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} . \tag{4.5}$$

Proof. Given $\mathbf{w} \in \mathbf{L}^4(\Omega)$, we proceed as in [14, Lemma 3.5] and introduce the auxiliary bilinear form $\mathcal{A}_{\mathbf{w}} : (\mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\rm tr}(\Omega)) \times (\mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\rm tr}(\Omega)) \to \mathrm{R}$ defined by

$$\mathcal{A}_{\mathbf{w}}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) := \mathbf{a}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) + \mathbf{c}(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}), \tag{4.6}$$

which allows us to rewrite problem (4.1) as:

$$\begin{aligned} \mathcal{A}_{\mathbf{w}}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}}, \boldsymbol{\sigma}) \;&=\; \mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{v}}) \qquad \forall \, \vec{\mathbf{v}} \in \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\rm tr}(\Omega), \\ \mathbf{b}(\vec{\mathbf{u}}, \boldsymbol{\tau}) \;&=\; \mathbf{G}(\boldsymbol{\tau}) \qquad\quad\;\; \forall \, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega). \end{aligned} \tag{4.7}$$

Concerning $\mathcal{A}_{\mathbf{w}}$, we notice that from (3.13) and (3.17) it transpires that

$$|\mathcal{A}_{\mathbf{w}}(\vec{\mathbf{u}}, \vec{\mathbf{v}})| \,\leq\, (2\mu + \|\mathbf{w}\|_{0,4;\Omega}) \, \|\vec{\mathbf{u}}\| \, \|\vec{\mathbf{v}}\| \qquad \forall \, \vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\rm tr}(\Omega) . \tag{4.8}$$

Moreover, it is clear that (4.3) combined with (3.16) yield the $\mathbf{V}$-ellipticity of $\mathcal{A}_{\mathbf{w}}$ with the constant α appearing in (4.3), which implies that $\mathcal{A}_{\mathbf{w}}$ satisfies the hypotheses given by [22, Theorem 2.34, eq. (2.28)]. In this way, together with the boundedness of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}$ and $\mathbf{G}$ (cf. (3.13), (3.14), and (3.15)) and the inf-sup condition satisfied by $\mathbf{b}$ with constant β (cf. Lemma 4.2), which corresponds to the hypothesis given by [22, Theorem 2.34, eq. (2.29)], the result follows from a straightforward application of [22, Theorem 2.34] to problem (4.7). In particular, the a priori estimate for $\mathbf{u}$ (cf. first estimate in [22, Theorem 2.34, eq. (2.30)]) reads

$$\|\mathbf{u}\|_{0,4;\Omega} \,\leq\, \|\vec{\mathbf{u}}\| \,\leq\, \frac{1}{\alpha} \, \|\mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}\| \,+\, \frac{1}{\beta} \left(1 + \frac{\|\mathcal{A}_{\mathbf{w}}\|}{\alpha}\right) \|\mathbf{G}\| ,$$

which, combined with (3.14), (3.15), and (4.8), gives (4.5) and completes the proof. □

We remark for later use that applying the second inequality from [22, Theorem 2.34, eq. (2.30)] to the problem defining $\boldsymbol{\Xi}$ (cf. (4.1) or (4.7)), and then utilizing the bounds for $\|\mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}\|$, $\|\mathbf{G}\|$, and $\|\mathcal{A}_{\mathbf{w}}\|$ that arise from (3.14), (3.15), and (4.8), we deduce the existence of a positive constant $C_{\boldsymbol{\Xi},3}$, depending only on α, β, and μ, such that the a priori estimate for $\boldsymbol{\sigma}$ reduces to

$$\|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3};\Omega} \,\le\, C_{\boldsymbol{\Xi},3}\,\big(1+\|\mathbf{w}\|_{0,4;\Omega}\big)\Big\{\|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}\|_{0,\Omega} + \big(1+\|\mathbf{w}\|_{0,4;\Omega}\big)\,\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\Big\}\,. \tag{4.9}$$

Having proved the well-posedness of (4.1), which ensures that operator $\boldsymbol{\Xi}$ is well defined, we now address the solvability analysis of the fixed point equation (4.2) (equivalently (3.8)). To this end, given an arbitrary radius δ, we let

$$W \,:=\, \Big\{\mathbf{w}\in\mathbf{L}^4(\Omega): \quad \|\mathbf{w}\|_{0,4;\Omega}\,\le\,\delta\Big\}\,.$$

Then, we begin by providing suitable conditions under which $\boldsymbol{\Xi}$ maps W into itself.

Lemma 4.4. *Assume that $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{H}}$, and the datum $\mathbf{u}_D$ are sufficiently small so that*

$$C_{\boldsymbol{\Xi},1}\,\|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}\|_{0,\Omega} \,+\, C_{\boldsymbol{\Xi},2}\,\big(1+\delta\big)\,\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\,\le\,\delta\,. \tag{4.10}$$

Then, $\boldsymbol{\Xi}(W)\,\subseteq\,W$.

Proof. It is a direct consequence of the a priori estimate (4.5) and the definition of W. □

A Lipschitz-continuity property of $\boldsymbol{\Xi}$ is established next.

Lemma 4.5. *For all $\mathbf{w}$, $\mathbf{w}_0\in\mathbf{L}^4(\Omega)$ there holds*

$$\begin{aligned}&\|\boldsymbol{\Xi}(\mathbf{w})-\boldsymbol{\Xi}(\mathbf{w}_0)\|_{0,4;\Omega}\\ &\qquad\le\,\alpha^{-1}\Big\{C_{\boldsymbol{\Xi},1}\,\|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}\|_{0,\Omega} \,+\, C_{\boldsymbol{\Xi},2}\,\big(1+\|\mathbf{w}_0\|_{0,4;\Omega}\big)\,\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\Big\}\,\|\mathbf{w}-\mathbf{w}_0\|_{0,4;\Omega}\,.\end{aligned} \tag{4.11}$$

Proof. The proof consists of a simplification of the one of [14, Lemma 3.8]. In fact, given $\mathbf{w}$, $\mathbf{w}_0\in\mathbf{L}^4(\Omega)$, we let $\mathbf{u}:=\boldsymbol{\Xi}(\mathbf{w})$ and $\mathbf{u}_0:=\boldsymbol{\Xi}(\mathbf{w}_0)$, where $\big(\vec{\mathbf{u}},\boldsymbol{\sigma}\big):=\big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big)$ and $\big(\vec{\mathbf{u}}_0,\boldsymbol{\sigma}_0\big):=\big((\mathbf{u}_0,\mathbf{t}_0),\boldsymbol{\sigma}_0\big)$ are the corresponding solutions of (4.1). It is clear from the respective second equations of (4.1) that $\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\in\mathbf{V}$, whence the $\mathbf{V}$-ellipticity of $\mathbf{a}$ (cf. (4.3)), and the respective first equations applied to $\vec{\mathbf{v}}=\vec{\mathbf{u}}-\vec{\mathbf{u}}_0$, give

$$\alpha\,\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|^2\,\le\,\mathbf{a}(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\,=\,\mathbf{c}(\mathbf{w}_0;\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)-\mathbf{c}(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\,.$$

Then, subtracting and adding $\vec{\mathbf{u}}_0$ to the first component of the term $\mathbf{c}(\mathbf{w};\cdot,\cdot)$, using from (3.16) that $\mathbf{c}(\mathbf{w};\vec{\mathbf{u}}-\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)=0$, and applying (3.18), we find that

$$\alpha\,\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|^2\,\le\,\mathbf{c}(\mathbf{w}_0;\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)-\mathbf{c}(\mathbf{w};\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\,\le\,\|\mathbf{w}-\mathbf{w}_0\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}_0\|\,\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|\,,$$

which, after simplifying by $\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|$, and employing the bound for $\|\vec{\mathbf{u}}_0\|$ provided by (4.5), yields (4.11). □

The well-posedness of (3.8) is now established as follows.

Theorem 4.6. *Assume that $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{H}}$, and the datum $\mathbf{u}_D$ are sufficiently small so that*

$$C_{\boldsymbol{\Xi},1}\,\|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}\|_{0,\Omega} \,+\, C_{\boldsymbol{\Xi},2}\,\big(1+\delta\big)\,\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} \,<\, \min\big\{\delta,\alpha\big\}\,. \tag{4.12}$$

Then, problem (3.8) *has a unique solution* $\big(\vec{\mathbf{u}},\boldsymbol{\sigma}\big) := \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big) \in (\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega))\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, *with* $\mathbf{u}\in W$, *and there hold*

$$\|\mathbf{u}\|_{0,4;\Omega} \,\le\, \|\vec{\mathbf{u}}\| \,\le\, C_{\boldsymbol{\Xi},1}\,\|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}\|_{0,\Omega} \,+\, C_{\boldsymbol{\Xi},2}\,\big(1+\delta\big)\,\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\,, \tag{4.13}$$

and

$$\|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3};\Omega} \,\le\, C_{\boldsymbol{\Xi},3}\,\big(1+\delta\big)\Big\{\|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}\|_{0,\Omega} + \big(1+\delta\big)\,\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\Big\}\,. \tag{4.14}$$

Proof. We start by observing that (4.12) implies (4.10), which, in virtue of Lemma 4.4, guarantees that $\Xi(W)\subseteq W$. In turn, it follows from (4.11) that for all $\mathbf{w}$, $\mathbf{w}_0\in W$ there holds

$$\|\boldsymbol{\Xi}(\mathbf{w})-\boldsymbol{\Xi}(\mathbf{w}_0)\|_{0,4;\Omega} \,\le\, L_{\boldsymbol{\Xi}}\big(\mathtt{data}\big)\,\|\mathbf{w}-\mathbf{w}_0\|_{0,4;\Omega}\,,$$

where, thanks to the assumption (4.12),

$$L_{\boldsymbol{\Xi}}\big(\mathtt{data}\big) \,:=\, \alpha^{-1}\Big\{C_{\boldsymbol{\Xi},1}\,\|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}\|_{0,\Omega} \,+\, C_{\boldsymbol{\Xi},2}\,\big(1+\delta\big)\,\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\Big\} \,<\, 1\,.$$

In this way, $\boldsymbol{\Xi}: W\to W$ is a contraction, and hence the classical Banach fixed point theorem (see, e.g., [13, Theorem 3.7-1]) yields the existence of a unique fixed point $\mathbf{u}\in W$ of $\boldsymbol{\Xi}$, and therefore, a unique solution of (3.8). Moreover, the estimates (4.13) and (4.14) follow respectively from (4.5) and (4.9) by bounding $\|\mathbf{u}\|_{0,4;\Omega}$ by δ. □

4.2 Discrete analysis

In this section we resort to the analysis and results from [14, Sections 4 and 5] to address the solvability of a discrete version of (3.8), derive the corresponding a priori error estimates, and introduce specific finite element subspaces satisfying the required stability conditions.

4.2.1 The Galerkin scheme

We first consider arbitrary finite element subspaces $\widetilde{\mathbb{H}}^{\mathbf{t}}_h$, $\widetilde{\mathbb{H}}^{\boldsymbol{\sigma}}_h$, and $\mathbf{H}^{\mathbf{u}}_h$, of $\mathbb{L}^2(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and $\mathbf{L}^4(\Omega)$, respectively, and set

$$\mathbb{H}^{\mathbf{t}}_h \,:=\, \widetilde{\mathbb{H}}^{\mathbf{t}}_h\cap\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,,\quad \mathbb{H}^{\boldsymbol{\sigma}}_h \,:=\, \widetilde{\mathbb{H}}^{\boldsymbol{\sigma}}_h\cap\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\quad\text{and}\quad \mathbf{H}_h := \mathbf{H}^{\mathbf{u}}_h\times\mathbb{H}^{\mathbf{t}}_h\,. \tag{4.15}$$

In addition, we introduce the notations

$$\vec{\mathbf{u}}_h := (\mathbf{u}_h,\mathbf{t}_h), \qquad \vec{\mathbf{v}}_h := (\mathbf{v}_h,\mathbf{s}_h) \,\in\, \mathbf{H}_h\,.$$

Then, given discrete approximations $\boldsymbol{\mathcal{B}}_h$ and $\boldsymbol{\mathcal{H}}_h$ (to be defined later on) of $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}$, respectively, the Galerkin scheme associated with (3.8) reads: Find $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\in\mathbf{H}_h\times\mathbb{H}^{\boldsymbol{\sigma}}_h$ such that

$$\begin{aligned}\mathbf{a}(\vec{\mathbf{u}}_h,\vec{\mathbf{v}}_h)+\mathbf{c}(\mathbf{u}_h;\vec{\mathbf{u}}_h,\vec{\mathbf{v}}_h)+\mathbf{b}(\vec{\mathbf{v}}_h,\boldsymbol{\sigma}_h) &= \mathbf{F}_{\boldsymbol{\mathcal{B}}_h,\boldsymbol{\mathcal{H}}_h}(\vec{\mathbf{v}}_h) \quad \forall\,\vec{\mathbf{v}}_h\in\mathbf{H}_h,\\ \mathbf{b}(\vec{\mathbf{u}}_h,\boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) \qquad\quad \forall\,\boldsymbol{\tau}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_h\,,\end{aligned} \tag{4.16}$$

where the functional $\mathbf{F}_{\boldsymbol{\mathcal{B}}_h,\boldsymbol{\mathcal{H}}_h}$ is defined accordingly to (3.11).

In what follows, we adopt the discrete analogue of the fixed point strategy introduced in Section 4.1 to analyze the solvability of (4.16). Indeed, we now let $\boldsymbol{\Xi}_h : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{u}}$ be the operator defined by:

$$\boldsymbol{\Xi}_h(\mathbf{w}_h) \,=\, \mathbf{u}_h \qquad \forall\, \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}\,,$$

where $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \,:=\, \big((\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\sigma}_h\big) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ is the unique solution (to be confirmed below) of problem (4.16) when $\mathbf{c}(\mathbf{u}_h; \cdot, \cdot)$ is replaced by $\mathbf{c}(\mathbf{w}_h; \cdot, \cdot)$, that is

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{c}(\mathbf{w}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= \mathbf{F}_{\boldsymbol{\mathcal{B}}_h, \boldsymbol{\mathcal{H}}_h}(\vec{\mathbf{v}}_h) \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h, \\ \mathbf{b}(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) \qquad\quad\; \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,. \end{aligned} \tag{4.17}$$

Then, we realize that solving (4.16) is equivalent to seeking a fixed point of $\boldsymbol{\Xi}_h$, that is: Find $\mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that

$$\boldsymbol{\Xi}_h(\mathbf{u}_h) \,=\, \mathbf{u}_h\,. \tag{4.18}$$

The discrete version of [22, Theorem 2.34], namely [22, Proposition 2.42], is employed next to show that, under a pair of suitable hypotheses on the finite element subspaces, problem (4.17) is well-posed, equivalently, $\boldsymbol{\Xi}_h$ is well-defined. In order to set the first of these assumptions, we denote by $\mathbf{s}_{h,\mathtt{s}}$ and $\mathbf{s}_{h,\mathtt{sk}}$ the symmetric and skew-symmetric parts, respectively, of each tensor $\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}}$, so that

$$\mathbf{s}_h \,=\, \mathbf{s}_{h,\mathtt{s}} + \mathbf{s}_{h,\mathtt{sk}} \quad \text{and} \quad \|\mathbf{s}_h\|_{0,\Omega}^2 \,=\, \|\mathbf{s}_{h,\mathtt{s}}\|_{0,\Omega}^2 \,+\, \|\mathbf{s}_{h,\mathtt{sk}}\|_{0,\Omega}^2\,. \tag{4.19}$$

Then, we let $\mathbf{V}_h$ be the discrete kernel of $\mathbf{b}$, that is

$$\mathbf{V}_h \,:=\, \Big\{ \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{H}_h : \quad \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) = 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \Big\}\,,$$

and suppose that

(**H.1**) there exists a positive constant $C_{\mathtt{d}}$, independent of h, such that

$$\|\mathbf{s}_{h,\mathtt{s}}\|_{0,\Omega} \,\geq\, C_{\mathtt{d}}\, \|(\mathbf{v}_h, \mathbf{s}_{h,\mathtt{sk}})\| \qquad \forall\, \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h\,. \tag{4.20}$$

According to the definition of $\mathbf{a}$ (cf. (3.9)), and as a consequence of (**H.1**), we deduce that for each $\vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h$ there holds

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \,=\, 2\mu\, \|\mathbf{s}_{h,\mathtt{s}}\|_{0,\Omega}^2 \,&\geq\, \mu\, \|\mathbf{s}_{h,\mathtt{s}}\|_{0,\Omega}^2 + \mu\, C_{\mathtt{d}}^2\, \|(\mathbf{v}_h, \mathbf{s}_{h,\mathtt{sk}})\|^2 \\ &=\, \mu\, \|\mathbf{s}_{h,\mathtt{s}}\|_{0,\Omega}^2 + \mu\, C_{\mathtt{d}}^2 \Big\{ \|\mathbf{s}_{h,\mathtt{sk}}\|_{0,\Omega}^2 + \|\mathbf{v}_h\|_{0,4;\Omega}^2 \Big\} \end{aligned}$$

which, along with (4.19), readily yields the discrete version of Lemma 4.1, that is

$$\mathbf{a}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \,\geq\, \alpha_{\mathtt{d}}\, \|\vec{\mathbf{v}}_h\|^2 \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{V}_h\,, \tag{4.21}$$

with $\alpha_{\mathtt{d}} \,:=\, \mu \min\big\{1, C_{\mathtt{d}}^2\big\}$.

Furthermore, as the second hypothesis we assume that

(**H.2**) there exists a positive constant $\beta_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{v}}_h \in \mathbf{H}_h \\ \vec{\mathbf{v}}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h)}{\|\vec{\mathbf{v}}_h\|} \,\geq\, \beta_{\mathtt{d}}\, \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,, \tag{4.22}$$

which constitutes, in turn, the discrete version of Lemma 4.2.

We now establish the discrete analogue of Lemma 4.3.

Lemma 4.7. *For each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$ there exists a unique $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ solution to the problem* (4.16)*, and hence one can define $\Xi_h(\mathbf{w}_h) := \mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$. Moreover, there exist positive constants $\widetilde{C}_{\Xi,1}$ and $\widetilde{C}_{\Xi,2}$, depending only on $\alpha_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, and μ, such that there holds the following a priori estimate*

$$\|\Xi_h(\mathbf{w}_h)\| = \|\mathbf{u}_h\|_{0,4;\Omega} \,\leq\, \widetilde{C}_{\Xi,1}\, \|\boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \,+\, \widetilde{C}_{\Xi,2}\, \big(1 + \|\mathbf{w}_h\|_{0,4;\Omega}\big)\, \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}. \tag{4.23}$$

Proof. Recalling the definition (4.6) of $\mathcal{A}_{\mathbf{w}_h}$, we rewrite problem (4.17) as: Find $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ such that

$$\begin{aligned} \mathcal{A}_{\mathbf{w}_h}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= \mathbf{F}_{\boldsymbol{\mathcal{B}}_h, \boldsymbol{\mathcal{H}}_h}(\vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h, \\ \mathbf{b}(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) \qquad\quad\;\; \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}. \end{aligned} \tag{4.24}$$

We already know from (4.8) that $\|\mathcal{A}_{\mathbf{w}_h}\|_{(\mathbf{H}_h)'} \leq 2\mu + \|\mathbf{w}_h\|_{0,4;\Omega}$. Moreover, given $\vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{H}_h$, it follows from (4.21) and (3.16) that $\mathcal{A}_{\mathbf{w}_h}$ is $\mathbf{V}_h$-elliptic with constant $\alpha_{\mathtt{d}}$. In this way, recalling from (3.13), (3.14) (with $\boldsymbol{\mathcal{B}}_h$ and $\boldsymbol{\mathcal{H}}_h$ instead of $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}$, respectively), and (3.15) the boundedness of $\mathbf{b}$, $\mathbf{F}_{\boldsymbol{\mathcal{B}}_h, \boldsymbol{\mathcal{H}}_h}$ and $\mathbf{G}$, and from (**H.2**) the discrete inf-sup condition for $\mathbf{b}$ with constant $\beta_{\mathtt{d}}$, the unique solvability of (4.24) follows from a direct application of [22, Proposition 2.42]. In particular, the corresponding a priori estimate for $\mathbf{u}_h$ (cf. discrete version of the first estimate in [22, Theorem 2.34, eq. (2.30)]) reads:

$$\|\mathbf{u}_h\|_{0,4;\Omega} \,\leq\, \|\vec{\mathbf{u}}_h\| \,\leq\, \frac{1}{\alpha_{\mathtt{d}}}\|\mathbf{F}_{\boldsymbol{\mathcal{B}}_h, \boldsymbol{\mathcal{H}}_h}\|_{(\mathbf{H}_h)'} \,+\, \frac{1}{\beta_{\mathtt{d}}}\left(1 + \frac{\|\mathcal{A}_{\mathbf{w}_h}\|_{(\mathbf{H}_h)'}}{\alpha_{\mathtt{d}}}\right) \|\mathbf{G}\|_{(\mathbf{H}_h)'}\,,$$

which, together with (4.8), (3.14) (with $\boldsymbol{\mathcal{B}}_h$ and $\boldsymbol{\mathcal{H}}_h$), and (3.15), yields (4.23). □

We remark here that, similarly to the derivation of (4.9), we deduce the existence of a positive constant $\widetilde{C}_{\Xi,3}$, depending only on $\alpha_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, and μ, such that the corresponding a priori estimate for $\boldsymbol{\sigma}_h$ becomes

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} \,\leq\, \widetilde{C}_{\Xi,3}\, \big(1 + \|\mathbf{w}_h\|_{0,4;\Omega}\big)\Big\{\|\boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} + \big(1 + \|\mathbf{w}_h\|_{0,4;\Omega}\big)\, \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\Big\}\,.$$

Next, we proceed analogously to the analysis in Section 4.1 to address the well-posedness of the discrete fixed point equation (4.18). Thus, given again the radius δ, we begin by introducing the ball

$$W_h \,:=\, \Big\{\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}} : \quad \|\mathbf{w}_h\|_{0,4;\Omega} \,\leq\, \delta\Big\}\,.$$

Then, the discrete version of Lemma 4.4 is stated as follows.

Lemma 4.8. *Assume that $\boldsymbol{\mathcal{B}}_h$, $\boldsymbol{\mathcal{H}}_h$, and the datum $\mathbf{u}_D$ are sufficiently small so that*

$$\widetilde{C}_{\Xi,1}\, \|\boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \,+\, \widetilde{C}_{\Xi,2}\, \big(1 + \delta\big)\, \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} \,\leq\, \delta\,. \tag{4.25}$$

Then, $\Xi_h(W_h) \subseteq W_h$.

Proof. It is a direct consequence of the a priori estimate (4.23) and the definition of W_h. □

We now prove the discrete analogue of Lemma 4.5.

Lemma 4.9. *For all $\mathbf{w}_h$, $\mathbf{w}_{0,h} \in \mathbf{H}_h^{\mathbf{u}}$ there holds*

$$\begin{aligned} &\|\Xi_h(\mathbf{w}_h) - \Xi_h(\mathbf{w}_{0,h})\|_{0,4;\Omega} \\ &\quad\leq\, \alpha_{\mathtt{d}}^{-1}\, \Big\{\widetilde{C}_{\Xi,1}\, \|\boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \,+\, \widetilde{C}_{\Xi,2}\, \big(1 + \|\mathbf{w}_{0,h}\|_{0,4;\Omega}\big)\, \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\Big\}\, \|\mathbf{w}_h - \mathbf{w}_{0,h}\|_{0,4;\Omega}\,. \end{aligned} \tag{4.26}$$

Proof. The proof follows analogously to that of Lemma 4.5 by starting now from the $\mathbf{V}_h$-ellipticity of $\mathbf{a}$ with constant $\alpha_{\mathtt{d}}$ (cf. (4.21)), and by ending with the bound for $\|\boldsymbol{\Xi}_h(\mathbf{w}_{0,h})\|$ provided by (4.23). We omit further details. $\square$

Having established the Lipschitz-continuity of $\boldsymbol{\Xi}_h$, the well-posedness of (4.16) is addressed next.

Theorem 4.10. *Let $\boldsymbol{\mathcal{B}}_h$ and $\boldsymbol{\mathcal{H}}_h$ be approximations (to be defined later on) of $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}$, respectively, and assume that $\boldsymbol{\mathcal{B}}_h$, $\boldsymbol{\mathcal{H}}_h$, and the datum $\mathbf{u}_D$ satisfy* (4.25). *Then, problem* (4.16) *has a solution $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$, with $\mathbf{u}_h \in W_h$, and there hold*

$$\|\mathbf{u}_h\|_{0,4;\Omega} \,\leq\, \|\vec{\mathbf{u}}_h\| \,\leq\, \widetilde{C}_{\boldsymbol{\Xi},1}\, \|\boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \,+\, \widetilde{C}_{\boldsymbol{\Xi},2}\, \big(1+\delta\big)\, \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\,, \tag{4.27}$$

and

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} \,\leq\, \widetilde{C}_{\boldsymbol{\Xi},3}\, \big(1+\delta\big)\Big\{\|\boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} + \big(1+\delta\big)\, \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\Big\}\,. \tag{4.28}$$

Moreover, under the additional assumption

$$\widetilde{C}_{\boldsymbol{\Xi},1}\, \|\boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \,+\, \widetilde{C}_{\boldsymbol{\Xi},2}\, \big(1+\delta\big)\, \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} \,<\, \alpha_{\mathtt{d}}\,, \tag{4.29}$$

the above solution is unique.

Proof. In virtue of the equivalence between (4.16) and (4.18), and bearing in mind Lemma 4.8 and the continuity of $\boldsymbol{\Xi}_h$ provided by (4.26), the existence of solution follows from a straightforward application of Brouwer's Theorem (see. e.g. [13, Theorem 9.9-2]). In turn, (4.29) and the Banach fixed point theorem (see, e.g., [13, Theorem 3.7-1]) imply the corresponding uniqueness. Further details are omitted. $\square$

We stress here that, similarly to (4.12) (cf. Theorem 4.6), the assumptions (4.25) and (4.29) can be rewritten jointly as

$$\widetilde{C}_{\boldsymbol{\Xi},1}\, \|\boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \,+\, \widetilde{C}_{\boldsymbol{\Xi},2}\, \big(1+\delta\big)\, \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} \,<\, \min\big\{\delta, \alpha_{\mathtt{d}}\big\}\,.$$

4.2.2 A priori error estimates

In what follows, we derive a Céa-type estimate for the error $\|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|$, where $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in (\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega))\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, with $\mathbf{u}\in W$, is the solution of (3.8), and $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\in\mathbf{H}_h\times\mathbb{H}_h^{\boldsymbol{\sigma}}$, with $\mathbf{u}_h\in W_h$, is the solution of (4.16). We begin by rewriting problems (3.8) and (4.16) as we did in the proofs of Lemmas 4.3 and 4.7 for the linearized problems (4.7) and (4.24), that is

$$\begin{aligned} \mathcal{A}_{\mathbf{u}}(\vec{\mathbf{u}},\vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) \;&=\; \mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{v}}) \qquad \forall\ \vec{\mathbf{v}}\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega),\\ \mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) \;&=\; \mathbf{G}(\boldsymbol{\tau}) \qquad\quad \forall\ \boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} \mathcal{A}_{\mathbf{u}_h}(\vec{\mathbf{u}}_h,\vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h,\boldsymbol{\sigma}_h) \;&=\; \mathbf{F}_{\boldsymbol{\mathcal{B}}_h,\boldsymbol{\mathcal{H}}_h}(\vec{\mathbf{v}}_h) \qquad \forall\ \vec{\mathbf{v}}_h\in\mathbf{H}_h,\\ \mathbf{b}(\vec{\mathbf{u}}_h,\boldsymbol{\tau}_h) \;&=\; \mathbf{G}(\boldsymbol{\tau}_h) \qquad\quad \forall\ \boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\,, \end{aligned} \tag{4.31}$$

where, given $\mathbf{z}\in\mathbf{L}^4(\Omega)$,

$$\mathcal{A}_{\mathbf{z}}(\vec{\mathbf{w}},\vec{\mathbf{v}}) \,:=\, \mathbf{a}(\vec{\mathbf{w}},\vec{\mathbf{v}}) + \mathbf{c}(\mathbf{z};\vec{\mathbf{w}},\vec{\mathbf{v}}) \qquad \forall\ \vec{\mathbf{w}},\ \vec{\mathbf{v}}\in(\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega))\,. \tag{4.32}$$

Next, bearing in mind that both $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$ are bounded from above by δ, we notice from (4.8) and (3.13) that

$$\|\mathcal{A}_{\mathbf{u}}\| \,\le\, (2\mu+\delta)\,, \quad \|\mathcal{A}_{\mathbf{u}_h}\| \,\le\, (2\mu+\delta)\,, \quad \text{and} \quad \|\mathbf{b}\| \,\le\, 1\,. \tag{4.33}$$

In addition, we know from the proof of Lemma 4.7 that $\mathcal{A}_{\mathbf{u}_h}$ is $\mathbf{V}_h$-elliptic with constant $\alpha_{\mathtt{d}}$, and it is clear from (**H.2**) that $\mathbf{b}$ satisfies the discrete inf-sup condition with constant $\beta_{\mathtt{d}}$. In this way, applying the Strang-type estimate provided by [14, Lemma 6.1] to the context given by (4.30) and (4.31), we deduce the existence of a positive constant $\widetilde{C}$, depending only on μ, δ, $\alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$, such that

$$\begin{aligned} \|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\| \,+\, \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} \,\le\, \widetilde{C}\,\Big\{&\mathrm{dist}(\vec{\mathbf{u}},\mathbf{H}_h) \,+\, \mathrm{dist}(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}) \\ &+\|\mathcal{A}_{\mathbf{u}}(\vec{\mathbf{u}},\cdot)-\mathcal{A}_{\mathbf{u}_h}(\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'} \,+\, \|\mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}-\mathbf{F}_{\boldsymbol{\mathcal{B}}_h,\boldsymbol{\mathcal{H}}_h}\|_{\mathbf{H}_h'}\Big\}\,. \end{aligned} \tag{4.34}$$

Hereafter, given a subspace X_h of a generic Banach space $\big(X,\|\cdot\|_X\big)$, we set for each $x\in X$

$$\mathrm{dist}(x,X_h) \,:=\, \inf_{x_h\in X_h}\|x-x_h\|_X\,.$$

Now, taking into account the definitions of both $\mathcal{A}_{\mathbf{u}}$ and $\mathcal{A}_{\mathbf{u}_h}$, the estimate (3.18), and the *a priori* estimate for $\|\vec{\mathbf{u}}\|$ (cf. (4.13)), we find that

$$\begin{aligned} \|\mathcal{A}_{\mathbf{u}}(\vec{\mathbf{u}},\cdot)-\mathcal{A}_{\mathbf{u}_h}(\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'} &\,=\, \|\mathbf{c}(\mathbf{u},\vec{\mathbf{u}},\cdot)-\mathbf{c}(\mathbf{u}_h,\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'} \,\le\, \|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}\| \\ &\le\, \Big\{C_{\boldsymbol{\Xi},1}\,\|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}\|_{0,\Omega} \,+\, C_{\boldsymbol{\Xi},2}\,\big(1+\delta\big)\,\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}\Big\}\,\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\|\,. \end{aligned} \tag{4.35}$$

Next, according to the definitions of $\mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}$ and $\mathbf{F}_{\boldsymbol{\mathcal{B}}_h,\boldsymbol{\mathcal{H}}_h}$, and applying the Cauchy–Schwarz inequality, we readily deduce that

$$\|\mathbf{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}-\mathbf{F}_{\boldsymbol{\mathcal{B}}_h,\boldsymbol{\mathcal{H}}_h}\|_{\mathbf{H}_h'} \,\le\, \|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}-\boldsymbol{\mathcal{B}}_h\otimes\boldsymbol{\mathcal{H}}_h\|_{0,\Omega}\,. \tag{4.36}$$

In this way, replacing (4.35) and (4.36) back into (4.34), we arrive at the following result.

Theorem 4.11. *Assume that $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{H}}$, and the datum $\mathbf{u}_D$ satisfy*

$$C_{\boldsymbol{\Xi},1}\,\|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}\|_{0,\Omega} \,+\, C_{\boldsymbol{\Xi},2}\,\big(1+\delta\big)\,\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} \,\le\, \frac{1}{2\widetilde{C}}. \tag{4.37}$$

Then, there exists a positive constant C_{ST}, depending only on μ, δ, $\alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$, such that

$$\begin{aligned} &\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\| \,+\, \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} \\ &\quad\le\, C_{ST}\,\Big\{\mathrm{dist}(\vec{\mathbf{u}},\mathbf{H}_h) \,+\, \mathrm{dist}(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}) +\, \|\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}-\boldsymbol{\mathcal{B}}_h\otimes\boldsymbol{\mathcal{H}}_h\|_{0,\Omega}\Big\}\,. \end{aligned} \tag{4.38}$$

4.2.3 Specific finite element subspaces

In this section we resort to [14, Section 5.2] to introduce explicit discrete spaces satisfying the required stability conditions. To this end, we let $\big\{\mathcal{T}_h\big\}_{h>0}$ be a regular family of triangulations of Ω by triangles K (respectively tetrahedra K in R^3), set $h:=\max\big\{h_K\,:\,K\in\mathcal{T}_h\big\}$, and for each $h>0$ we denote by $\mathcal{T}_h^{\mathtt{b}}$ the barycentric refinement of $\mathcal{T}_h$. In turn, given an integer $\ell\ge 0$ and $S\subseteq\mathrm{R}^n$, we let $\mathrm{P}_\ell(S)$ be the space of polynomials of degree at most ℓ defined on S, and, as indicated in Section 1, let $\mathbf{P}_\ell(S)$ and $\mathbb{P}_\ell(S)$ be its respective vector and tensor versions. Also, we define the local Raviart-Thomas space of order ℓ as $\mathbf{RT}_\ell(S) := \mathbf{P}_\ell(S) \oplus \widetilde{\mathrm{P}}_\ell(S)\,\mathbf{x}$, where $\widetilde{\mathrm{P}}_\ell(S)$ is the space of polynomials of degree equal to ℓ, and $\mathbf{x} := (x_1, \ldots, x_n)^{\mathtt{t}}$ is a generic vector of R^n. In addition, we introduce the global spaces

$$\mathbf{P}_\ell(\mathcal{T}_h) := \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

$$\mathbb{P}_\ell(\mathcal{T}_h) := \Big\{ \boldsymbol{\delta}_h \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\delta}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

$$\mathbf{RT}_\ell(\mathcal{T}_h) := \Big\{ \boldsymbol{\tau}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbf{RT}_\ell(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

and

$$\mathbb{RT}_\ell(\mathcal{T}_h) := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_{h,i} \in \mathbf{RT}_\ell(\mathcal{T}_h) \quad \forall\, i \in \big\{1, \ldots, n\big\} \Big\},$$

with analogue definitions for $\mathbf{P}_\ell(\mathcal{T}_h^{\mathtt{b}})$, $\mathbb{P}_\ell(\mathcal{T}_h^{\mathtt{b}})$, $\mathbf{RT}_\ell(\mathcal{T}_h^{\mathtt{b}})$, and $\mathbb{RT}_\ell(\mathcal{T}_h^{\mathtt{b}})$, where $\boldsymbol{\tau}_{h,i}$ stands for the ith-row of $\boldsymbol{\tau}_h$. Then, for each integer $k \geq n-1$, we define the finite element subspaces of $\mathbf{L}^4(\Omega)$, $\mathbb{L}^2(\Omega)$, and $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, respectively, given by

$$\widetilde{\mathbb{H}}_h^{\mathbf{t}} := \mathbb{P}_k(\mathcal{T}_h^{\mathtt{b}}), \qquad \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} := \mathbb{RT}_k(\mathcal{T}_h^{\mathtt{b}}), \quad \text{and} \quad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\mathcal{T}_h^{\mathtt{b}}), \tag{4.39}$$

which are then utilized in (4.15) to set $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, and $\mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}}$. In this regard, we recall from [14, Section 5.2] that the present restriction on k is needed to guarantee the stability of the Galerkin scheme associated with the primal formulation of the Stokes equations, when Scott-Vogelius elements are utilized. Now, it is proved in [14] that the specific subspaces defined in (4.39) satisfy the assumptions (**H.1**) and (**H.2**) introduced in Section 4.2.1. More precisely, employing the equivalence and sufficiency results provided in [14, Lemmas 5.1 and 5.2], it is shown in [14, Section 5.2] that, in order to conclude the aforementioned hypotheses, it suffices to establish the inf-sup conditions specified in [14, eqs. (5.13), (5.14), and (5.15)], which, in turn, are proved in [14, Sections 5.2 and 5.4]. We omit further details.

We recall next the approximation properties of the present subspaces $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, and $\mathbf{H}_h^{\mathbf{u}}$ (see, e.g. [6], [8], [14], [22], [23]):

($\mathbf{AP}_h^{\mathbf{t}}$): there exists $C > 0$, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\mathbf{s} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, there holds

$$\mathrm{dist}\big(\mathbf{s}, \mathbb{H}_h^{\mathbf{t}}\big) := \inf_{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \leq C\, h^\ell\, \|\mathbf{s}\|_{\ell,\Omega},$$

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$): there exists $C > 0$, independent of h, such that for each $\ell \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{\ell,4/3}(\Omega)$, there holds

$$\mathrm{dist}\big(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}\big) := \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \leq C\, h^\ell \Big\{ \|\boldsymbol{\tau}\|_{\ell,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\ell,4/3;\Omega} \Big\},$$

($\mathbf{AP}_h^{\mathbf{u}}$): there exists $C > 0$, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{\ell,4}(\Omega)$, there holds

$$\mathrm{dist}\big(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}\big) := \inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \leq C\, h^\ell\, \|\mathbf{v}\|_{\ell,4;\Omega},$$

We conclude this section with the following convergence result for the Galerkin scheme (4.16).

Theorem 4.12. *In addition to the hypotheses of Theorems* 4.6*,* 4.10*, and* 4.11*, assume that there exists* $\ell \in (0, k+1]$ *such that* $\mathbf{u} \in \mathbf{W}^{\ell,4}(\Omega)$*,* $\mathbf{t} \in \mathbb{H}^{\ell}(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$*,* $\boldsymbol{\sigma} \in \mathbb{H}^{\ell}(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$*, and* $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{\ell,4/3}(\Omega)$*. Then, there exists a positive constant* C*, independent of* h*, such that*

$$\begin{aligned}\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} \,\le\, C \Big\{ h^{\ell} \Big(\|\mathbf{u}\|_{\ell,4;\Omega} + \|\mathbf{t}\|_{\ell,\Omega} + \|\boldsymbol{\sigma}\|_{\ell,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\ell,4/3;\Omega} \Big) \\ + \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \Big\} .\end{aligned} \tag{4.40}$$

Proof. The result follows from a direct application of Theorem 4.11 and the above approximation properties of the finite element subspaces. Further details are omitted. □

5 Analysis of the second mixed formulation for the fluid

The continuous and discrete analyses of the variational formulation (3.24) are developed in this section.

5.1 The continuous analysis

As remarked in Section 3.2, we now follow [26, Sections 3.2 and 3.3] to address the well-posedness of (3.24) by employing also a fixed point strategy. To this end, we first realize that (3.24) can be equivalently formulated as: Find $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y}$ such that

$$\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) + \mathcal{D}(\mathbf{u}; \mathbf{u}, \mathbf{s}) \,=\, \boldsymbol{\mathcal{F}}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{X} \times \mathbf{Y} , \tag{5.1}$$

where the bilinear form $\boldsymbol{\mathcal{A}} : (\mathbf{X} \times \mathbf{Y}) \times (\mathbf{X} \times \mathbf{Y}) \to \mathrm{R}$ and the functional $\boldsymbol{\mathcal{F}}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}} : \mathbf{X} \times \mathbf{Y} \to \mathrm{R}$ are given by

$$\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{r}}, \vec{\mathbf{w}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) \,:=\, \mathcal{A}(\vec{\mathbf{r}}, \vec{\mathbf{s}}) + \mathcal{B}(\vec{\mathbf{s}}, \vec{\mathbf{w}}) + \mathcal{B}(\vec{\mathbf{r}}, \vec{\mathbf{v}}) - \mathcal{C}(\vec{\mathbf{w}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{r}}, \vec{\mathbf{w}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{X} \times \mathbf{Y} , \tag{5.2}$$

and

$$\boldsymbol{\mathcal{F}}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \,:=\, \mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{s}}) + \mathcal{G}(\vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{X} \times \mathbf{Y} . \tag{5.3}$$

Then, similarly to [26, Section 3.3], we now introduce the operator $\boldsymbol{\Xi} : \mathbf{L}^4(\Omega) \to \mathbf{L}^4(\Omega)$ defined by:

$$\boldsymbol{\Xi}(\mathbf{z}) \,:=\, \mathbf{u} \qquad \forall\, \mathbf{z} \in \mathbf{L}^4(\Omega) ,$$

where $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) := \big((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \gamma)\big) \in \mathbf{X} \times \mathbf{Y}$ is the unique solution (to be confirmed below) of the linear problem arising from (5.1) when $\mathcal{D}(\mathbf{u}; \mathbf{u}, \mathbf{s})$ is replaced by $\mathcal{D}(\mathbf{z}; \mathbf{u}, \mathbf{s})$, that is

$$\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) + \mathcal{D}(\mathbf{z}; \mathbf{u}, \mathbf{s}) \,=\, \boldsymbol{\mathcal{F}}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{X} \times \mathbf{Y} . \tag{5.4}$$

In this way, (5.1) can be rewritten as the fixed point equation: Find $\mathbf{z} \in \mathbf{L}^4(\Omega)$ such that

$$\boldsymbol{\Xi}(\mathbf{z}) \,=\, \mathbf{z} , \tag{5.5}$$

so that the unique solution $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) := \big((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \gamma)\big) \in \mathbf{X} \times \mathbf{Y}$ of (5.4), with the fixed point $\mathbf{z}$, becomes the unique solution of (5.1).

Next, because of the perturbed saddle-point structure of $\boldsymbol{\mathcal{A}}$ (cf. (5.2)), in what follows we resort to the slight variant of [18, Theorem 3.4] provided by [26, Theorem 3.2] to derive the global inf-sup condition satisfied by $\boldsymbol{\mathcal{A}}$. According to it, we first establish the inf-sup conditions for $\mathcal{A}$ in the kernel $\mathcal{V}$ of the bilinear form $\mathcal{B}$ (cf. (3.27)). More precisely, due to the saddle-point structure in Banach spaces of $\mathcal{A}$ (cf. (3.25)), it suffices to verify the assumptions of the corresponding Babuška-Brezzi theory (cf. [3, Theorem 2.1, Corollary 2.1], [26, Theorem 3.1]). To this end, we begin by observing that $\mathcal{V}$ reduces to

$$\mathcal{V} \,:=\, \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathcal{V}_0\,,$$

where

$$\mathcal{V}_0 \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) : \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{t}} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \quad \text{in} \quad \Omega \Big\}\,. \tag{5.6}$$

Then, letting K_i be the kernel of $b_i|_{\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathcal{V}_0}$, for each $i \in \{1,2\}$, and noting from (3.26) that $b_1 = -b_2$, we obtain

$$\mathrm{K}_1 \,=\, \mathrm{K}_2 \,=\, \mathrm{K} \,:=\, \Big\{ \mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \int_\Omega \mathbf{s} : \boldsymbol{\tau} = 0 \quad \forall\, \boldsymbol{\tau} \in \mathcal{V}_0 \Big\}\,. \tag{5.7}$$

In turn, it is also clear from (3.26) that

$$a(\mathbf{s},\mathbf{s}) \,=\, 2\mu\, \|\mathbf{s}\|^2_{0,\Omega} \qquad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{5.8}$$

which says that a is $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$-elliptic with constant $\widetilde{\alpha} := 2\mu$, and hence, in particular, a is obviously K-elliptic as well, thus satisfying the hypotheses i) and ii) of [26, Theorem 3.1]. Note that the explicit condition defining K (cf. (5.7)) is not needed for the above.

On the other hand, the inf-sup conditions for the bilinear forms b_1 and b_2 are stated as follows.

Lemma 5.1. *There exists a positive constant* $\widetilde{\beta}$ *such that for each* $i \in \{1,2\}$ *there holds*

$$\sup_{\substack{\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\\ \mathbf{s}\neq\mathbf{0}}} \frac{b_i(\mathbf{s},\boldsymbol{\tau})}{\|\mathbf{s}\|_{0,\Omega}} \,\geq\, \widetilde{\beta}\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathcal{V}_0\,. \tag{5.9}$$

Proof. We refer to [26, Lemma 3.3] for details. We just mention here that the existence of a positive constant c_1, depending only on Ω, such that (cf. [7, Lemma 3.2])

$$c_1\, \|\boldsymbol{\tau}\|_{0,\Omega} \,\leq\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \tag{5.10}$$

plays a key role in the proof. □

According to (5.8) and Lemma 5.1 we have that a, b_1, and b_2 satisfy the hypotheses of [26, Theorem 3.1] (see also [3, Theorem 2.1, Corollary 2.1]), and thus there exists a positive constant $\alpha_{\mathcal{A}}$, depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, and $\|a\| \,=\, 2\mu$ (cf. (3.32)), such that

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathcal{V}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathcal{A}(\vec{\mathbf{r}},\vec{\mathbf{s}})}{\|\vec{\mathbf{s}}\|_{\mathbf{X}}} \,\geq\, \alpha_{\mathcal{A}}\, \|\vec{\mathbf{r}}\|_{\mathbf{X}} \qquad \forall\, \vec{\mathbf{r}} \in \mathcal{V}\,. \tag{5.11}$$

Moreover, it is easy to see that the hypotheses of [26, Theorem 3.1] are equally satisfied if the roles of b_1 and b_2 are exchanged, and hence with the same constant $\alpha_{\mathcal{A}}$ from (5.11) we get

$$\sup_{\substack{\vec{\mathbf{r}}\in\mathcal{V}\\ \vec{\mathbf{r}}\neq\mathbf{0}}} \frac{\mathcal{A}(\vec{\mathbf{r}},\vec{\mathbf{s}})}{\|\vec{\mathbf{r}}\|_{\mathbf{X}}} \,\geq\, \alpha_{\mathcal{A}}\, \|\vec{\mathbf{s}}\|_{\mathbf{X}} \qquad \forall\, \vec{\mathbf{s}} \in \mathcal{V}\,. \tag{5.12}$$

In turn, bearing in mind the definition of $\mathcal{A}$ (cf. (3.25)), the fact again that $b_1 = -\,b_2$ (cf. (3.26)), and (5.8), we obtain

$$\mathcal{A}(\vec{\mathbf{r}},\vec{\mathbf{r}}) \,=\, a(\mathbf{r},\mathbf{r}) \,=\, 2\mu\, \|\mathbf{r}\|^2_{0,\Omega} \,\geq\, 0 \qquad \forall\, \vec{\mathbf{r}} \in \mathbf{X}\,,$$

which says that $\mathcal{A}$ is positive semi-definite. Similarly, it is clear from (3.28) that $\mathcal{C}$ is symmetric and positive semi-definite.

Furthermore, the continuous inf-sup condition for $\mathcal{B}$ has already been proved in [27] (see also [26]). More precisely, making use of the Poincaré and the first Korn (cf. [31, Theorem 10.1] or [5, Corollaries 9.2.22 and 9.2.25]) inequalities, which establish, respectively, that

$$\|\mathbf{v}\|_{1,\Omega}^2 \,\le\, c_{\mathrm{P}}\, |\mathbf{v}|_{1,\Omega}^2 \quad \text{and} \quad |\mathbf{v}|_{1,\Omega}^2 \,\le\, 2\,\|\mathbf{e}(\mathbf{v})\|_{0,\Omega}^2 \quad \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \tag{5.13}$$

with a positive constant c_{P} depending on Ω, and recalling from Section 3.1 (cf. (3.15)) that $\mathbf{i}_4$ is the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, one is able to show the following result (cf. [27, Lemma 3.5], [26, Lemma 3.4]).

Lemma 5.2. *There exists a positive constant $\beta_{\mathcal{B}}$, depending only on c_{P} and $\|\mathbf{i}_4\|$, such that*

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{X}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathcal{B}(\vec{\mathbf{s}},\vec{\mathbf{v}})}{\|\vec{\mathbf{s}}\|_{\mathbf{X}}} \,\ge\, \beta_{\mathcal{B}}\, \|\vec{\mathbf{v}}\|_{\mathbf{Y}} \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{Y}\,. \tag{5.14}$$

In this way, since $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy the hypotheses of [26, Theorem 3.2], we conclude the existence of a positive constant $\alpha_{\boldsymbol{\mathcal{A}}}$, depending on $\|\mathcal{A}\|$, $\|\mathcal{C}\|$, $\alpha_{\mathcal{A}}$, and $\beta_{\mathcal{B}}$, such that

$$\sup_{\substack{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{X}\times\mathbf{Y}\\ (\vec{\mathbf{s}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big)}{\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{X}\times\mathbf{Y}}} \,\ge\, \alpha_{\boldsymbol{\mathcal{A}}}\, \|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{X}\times\mathbf{Y}} \qquad \forall\, (\vec{\mathbf{r}},\vec{\mathbf{w}}) \in \mathbf{X}\times\mathbf{Y}\,, \tag{5.15}$$

and

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\mathbf{w}})\in\mathbf{X}\times\mathbf{Y}\\ (\vec{\mathbf{r}},\vec{\mathbf{w}})\neq\mathbf{0}}} \frac{\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big)}{\|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{X}\times\mathbf{Y}}} \,\ge\, \alpha_{\boldsymbol{\mathcal{A}}}\, \|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{X}\times\mathbf{Y}} \qquad \forall\, (\vec{\mathbf{s}},\vec{\mathbf{v}}) \in \mathbf{X}\times\mathbf{Y}\,. \tag{5.16}$$

Thus, employing (5.15) and the boundedness property for $\mathcal{D}$ given by (3.36), we find that for each $\mathbf{z} \in \mathbf{L}^4(\Omega)$ there holds

$$\sup_{\substack{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{X}\times\mathbf{Y}\\ (\vec{\mathbf{s}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big) + \mathcal{D}(\mathbf{z};\mathbf{w},\mathbf{s})}{\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{X}\times\mathbf{Y}}} \,\ge\, \big(\alpha_{\boldsymbol{\mathcal{A}}} \,-\, \rho\,\|\mathbf{z}\|_{0,4;\Omega}\big)\, \|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{X}\times\mathbf{Y}} \quad \forall\, (\vec{\mathbf{r}},\vec{\mathbf{w}}) \in \mathbf{X}\times\mathbf{Y}\,,$$

and hence, for each $\mathbf{z} \in \mathbf{L}^4(\Omega)$ such that, say $\|\mathbf{z}\|_{0,4;\Omega} \,\le\, \dfrac{\alpha_{\boldsymbol{\mathcal{A}}}}{2\rho}$, we get

$$\sup_{\substack{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{X}\times\mathbf{Y}\\ (\vec{\mathbf{s}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big) + \mathcal{D}(\mathbf{z};\mathbf{w},\mathbf{s})}{\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{X}\times\mathbf{Y}}} \,\ge\, \frac{\alpha_{\boldsymbol{\mathcal{A}}}}{2}\, \|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{X}\times\mathbf{Y}} \qquad \forall\, (\vec{\mathbf{r}},\vec{\mathbf{w}}) \in \mathbf{X}\times\mathbf{Y}\,. \tag{5.17}$$

Analogously, using (5.16) and (3.36), and assuming the same range for $\mathbf{z}$, we arrive at

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\mathbf{w}})\in\mathbf{X}\times\mathbf{Y}\\ (\vec{\mathbf{r}},\vec{\mathbf{w}})\neq\mathbf{0}}} \frac{\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big) + \mathcal{D}(\mathbf{z};\mathbf{w},\mathbf{s})}{\|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{X}\times\mathbf{Y}}} \,\ge\, \frac{\alpha_{\boldsymbol{\mathcal{A}}}}{2}\, \|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{X}\times\mathbf{Y}} \qquad \forall\, (\vec{\mathbf{s}},\vec{\mathbf{v}}) \in \mathbf{X}\times\mathbf{Y}\,. \tag{5.18}$$

The well-posedness of (5.4), equivalently that the operator $\boldsymbol{\Xi}$ is well-defined, is stated as follows.

Lemma 5.3. *For each* $\mathbf{z} \in \mathbf{L}^4(\Omega)$ *such that* $\|\mathbf{z}\|_{0,4;\Omega} \le \dfrac{\alpha_{\boldsymbol{\mathcal{A}}}}{2\rho}$*, problem* (5.4) *has a unique solution* $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) = \big((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})\big) \in \mathbf{X} \times \mathbf{Y}$*, and hence* $\Xi(\mathbf{z}) := \mathbf{u} \in \mathbf{L}^4(\Omega)$ *is well-defined. Moreover, there holds*

$$\begin{aligned}\|\Xi(\mathbf{z})\|_{0,4;\Omega} \,=\, \|\mathbf{u}\|_{0,4;\Omega} \,&\le\, \|(\vec{\mathbf{t}}, \vec{\mathbf{u}})\|_{\mathbf{X}\times\mathbf{Y}} \\ &\le\, \frac{2}{\alpha_{\boldsymbol{\mathcal{A}}}} \Big\{ \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\|_{0,\Omega} \Big\} .\end{aligned} \tag{5.19}$$

Proof. Given $\mathbf{z}$ as indicated, the inf-sup conditions (5.17) and (5.18), along with a direct application of the Banach–Nečas–Babuška Theorem (cf. [22, Theorem 2.6]), imply the existence of a unique solution of (5.4). In turn, the corresponding a priori estimate and the boundedness of $\boldsymbol{\mathcal{F}}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}$ (cf. (5.3)), which follows from those of $\mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}$ (cf. (3.37)) and $\mathcal{G}$ (cf. (3.38)), yields (5.19) and concludes the proof. □

Having established the well-definedness of Ξ, we now prove that this operator maps a suitable ball into itself. More precisely, introducing

$$\mathbf{W} \,:=\, \Big\{ \mathbf{z} \in \mathbf{L}^4(\Omega) : \quad \|\mathbf{z}\|_{0,4;\Omega} \,\le\, \frac{\alpha_{\boldsymbol{\mathcal{A}}}}{2\rho} \Big\} ,$$

we have the following result.

Lemma 5.4. *Assume that*

$$\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\|_{0,\Omega} \,\le\, \frac{\alpha^2_{\boldsymbol{\mathcal{A}}}}{4\rho} . \tag{5.20}$$

Then, $\Xi(\mathbf{W}) \,\subseteq\, \mathbf{W}$.

Proof. It follows straightforwardly from the a priori estimate (5.19) and the assumption (5.20). □

We are now in position to state next the solvability of the fixed-point equation (5.5), and hence, equivalently, that of (5.1).

Theorem 5.5. *Assume that*

$$\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\|_{0,\Omega} \,<\, \frac{\alpha^2_{\boldsymbol{\mathcal{A}}}}{4\rho} . \tag{5.21}$$

Then, the operator Ξ *has a unique fixed-point* $\mathbf{z} \in \mathbf{W}$*. Equivalently,* (5.1) *has a unique solution* $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) := \big((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})\big) \in \mathbf{X} \times \mathbf{Y}$*, where* $(\vec{\mathbf{t}}, \vec{\mathbf{u}})$ *is the unique solution of* (5.4) *with* $\mathbf{z} = \mathbf{u}$*. Moreover, there holds*

$$\|(\vec{\mathbf{t}}, \vec{\mathbf{u}})\|_{\mathbf{X}\times\mathbf{Y}} \,\le\, \frac{2}{\alpha_{\boldsymbol{\mathcal{A}}}} \Big\{ \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\|_{0,\Omega} \Big\} . \tag{5.22}$$

Proof. We first notice that (5.21) and Lemma 5.4 guarantee that Ξ maps $\mathbf{W}$ into $\mathbf{W}$. Then, given $\mathbf{z}_i \in \mathbf{W}$, $i \in \{1, 2\}$, we let $(\vec{\mathbf{t}}_i, \vec{\mathbf{u}}_i) \in \mathbf{X} \times \mathbf{Y}$ be the solution of (5.4) with $\mathbf{z} = \mathbf{z}_i$, and proceed analogously to the proof of [26, Theorem 3.7]. Indeed, applying the global inf-sup condition (5.17) with $\mathbf{z} = \mathbf{z}_1$ to $(\vec{\mathbf{r}}, \vec{\mathbf{w}}) := (\vec{\mathbf{t}}_1, \vec{\mathbf{u}}_1) - (\vec{\mathbf{t}}_2, \vec{\mathbf{u}}_2)$, and then employing the boundedness property of $\mathcal{D}$ (cf. (3.36)) and the a priori estimate (5.19), we readily find that

$$\|\Xi(\mathbf{z}_1) - \Xi(\mathbf{z}_2)\|_{0,4;\Omega} \,\le\, \frac{4\rho}{\alpha^2_{\boldsymbol{\mathcal{A}}}} \Big\{ \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\|_{0,\Omega} \Big\} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega} ,$$

which, thanks to (5.21) again, says that Ξ is a contraction. In this way, a simple application of the Banach fixed point theorem along with the a priori estimate (5.19) conclude the proof. □

5.2 Discrete analysis

In this section we make use of the analysis and results from [26, Section 4] to establish the well-posedness of a discrete version of (5.1), derive the corresponding a priori error estimates, and specify finite element subspaces accomplishing the stability conditions.

5.2.1 The Galerkin scheme

We first let $\mathbb{H}_h^{\mathbf{t}}$, $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$ be arbitrary finite element subspaces of $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, $\mathbf{L}^4(\Omega)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively. Specific such subspaces satisfying the stability conditions to be introduced along this section, will be defined later on in Section 5.2.3. Then, letting

$$\mathbb{H}_h^{\boldsymbol{\sigma}} \,:=\, \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,$$

defining the product spaces

$$\mathbf{X}_h \,:=\, \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}\,, \qquad \mathbf{Y}_h \,:=\, \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\,,$$

setting the notations

$$\vec{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\sigma}_h)\,,\ \vec{\mathbf{s}}_h := (\mathbf{s}_h, \boldsymbol{\tau}_h)\,,\ \vec{\mathbf{r}}_h := (\mathbf{r}_h, \boldsymbol{\zeta}_h) \in \mathbf{X}_h\,,$$

$$\vec{\mathbf{u}}_h := (\mathbf{u}_h, \boldsymbol{\gamma}_h)\,,\ \vec{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\eta}_h)\,,\ \vec{\mathbf{w}}_h := (\mathbf{w}_h, \boldsymbol{\chi}_h) \in \mathbf{Y}_h\,,$$

and given discrete approximations $\boldsymbol{\mathcal{B}}_h$ and $\boldsymbol{\mathcal{H}}_h$ (to be defined later on) of $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}$, respectively, the Galerkin scheme associated with (5.1) reads as follows: Find $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ such that

$$\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) \,+\, \mathcal{D}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h) \,=\, \boldsymbol{\mathcal{F}}_{\boldsymbol{\mathcal{B}}_h, \boldsymbol{\mathcal{H}}_h}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \qquad \forall\ (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h\,, \tag{5.23}$$

where the functional $\boldsymbol{\mathcal{F}}_{\boldsymbol{\mathcal{B}}_h, \boldsymbol{\mathcal{H}}_h} : \mathbf{X}_h \times \mathbf{Y}_h \to \mathrm{R}$ is defined as in (5.3) with $\boldsymbol{\mathcal{B}}_h$ and $\boldsymbol{\mathcal{H}}_h$ instead of $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}$, respectively.

Then, similarly to [26, Section 4.2], we now let $\boldsymbol{\Xi}_h : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{u}}$ be the operator defined by

$$\boldsymbol{\Xi}_h(\mathbf{z}_h) \,:=\, \mathbf{u}_h \qquad \forall\ \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}\,,$$

where $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) \,:=\, \big((\mathbf{t}_h, \boldsymbol{\sigma}_h)), (\mathbf{u}_h, \boldsymbol{\gamma}_h)\big) \in \mathbf{X}_h \times \mathbf{Y}_h$ is the unique solution (to be confirmed later on) of the discrete problem arising from (5.23) when $\mathcal{D}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h)$ is replaced by $\mathcal{D}(\mathbf{z}_h; \mathbf{u}_h, \mathbf{s}_h)$, that is

$$\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) \,+\, \mathcal{D}(\mathbf{z}_h; \mathbf{u}_h, \mathbf{s}_h) \,=\, \boldsymbol{\mathcal{F}}_{\boldsymbol{\mathcal{B}}_h, \boldsymbol{\mathcal{H}}_h}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \qquad \forall\ (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h\,. \tag{5.24}$$

In order to analyze the solvability of (5.24), in what follows we make use of the discrete versions of the Babuška-Brezzi theory, the solvability result for perturbed saddle point problems, and the Banach-Nečas-Babuška theorem, all them in Banach spaces, which are available in [3, Sections 2.2 and 2.3], [26, Theorem 4.1], and [22, Theorem 2.22], respectively. More precisely, proceeding analogously to Section 5.1, we first notice that the kernel $\mathcal{V}_h$ of $\mathcal{B}|_{\mathbf{X}_h \times \mathbf{Y}_h}$ reduces to

$$\mathcal{V}_h \,:=\, \mathbb{H}_h^{\mathbf{t}} \times \mathcal{V}_{0,h}\,,$$

where

$$\mathcal{V}_{0,h} \,:=\, \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\eta}_h \,=\, 0 \quad \forall\ \boldsymbol{\eta}_h \in \mathbb{H}_h^{\gamma} \text{ and } \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) \,=\, 0 \quad \forall\ \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}} \Big\}\,.$$

In order to continue the discrete analysis, some hypotheses on the arbitrary finite element subspaces are required. More precisely, following exactly as in [26, Section 4.2], we first assume that

(H.0) $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ contains the multiples of the identity tensor $\mathbb{I}$,

(H.1) $\mathbf{div}(\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}) \subseteq \mathbf{H}_h^{\mathbf{u}}$.

Thanks to (**H.0**) we obtain the discrete version of the decomposition given by (3.6) and (3.7), that is $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} = \mathbb{H}_h^{\boldsymbol{\sigma}} \oplus \mathrm{R}\,\mathbb{I}$, which yields

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau}_h - \Big(\frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h)\Big)\,\mathbb{I} : \quad \boldsymbol{\tau}_h \in \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \right\},$$

whereas (**H.1**) allows to show that the tensors of $\mathcal{V}_{0,h}$ are divergence free, and hence

$$\mathcal{V}_{0,h} := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\eta}_h = 0 \quad \forall\, \boldsymbol{\eta}_h \in \mathbb{H}_h^{\gamma} \text{ and } \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \right\}.$$

Then, letting $\mathrm{K}_{i,h}$ be the kernel of $b_i|_{\mathbb{H}_h^{\mathbf{t}}\times\mathcal{V}_{0,h}}$, for each $i \in \{1,2\}$, and bearing in mind from (3.26) that $b_1 = -\,b_2$, we find that

$$\mathrm{K}_{1,h} = \mathrm{K}_{2,h} = \mathrm{K}_h := \left\{ \mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} : \quad \int_\Omega \mathbf{s}_h : \boldsymbol{\tau}_h = 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathcal{V}_{0,h} \right\}, \tag{5.25}$$

so that, being the bilinear form a $\mathbb{L}^2_{\mathbf{tr}}(\Omega)$-elliptic with constant $\widetilde{\alpha} := 2\mu$ (cf (5.8)), it is certainly K_h-elliptic as well, and thus the corresponding hypotheses from the Babuška–Brezzi theory in Banach spaces (cf. [3, eqs. (2.19) and (2.20)]) are verified with the same constant $\widetilde{\alpha}$.

Next, we also assume that

(H.2) $\big(\mathcal{V}_{0,h}\big)^{\mathtt{d}} := \left\{ \boldsymbol{\tau}_h^{\mathtt{d}} : \quad \boldsymbol{\tau}_h \in \mathcal{V}_{0,h} \right\} \subseteq \mathbb{H}_h^{\mathbf{t}}$,

which, along with the inequality (5.10), imply the discrete inf-sup condition for $b_i|_{\mathbb{H}_h^{\mathbf{t}}\times\mathcal{V}_{0,h}}$, $i \in \{1,2\}$, namely

$$\sup_{\substack{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} \\ \mathbf{s}_h \neq \mathbf{0}}} \frac{b_i(\mathbf{s}_h, \boldsymbol{\tau}_h)}{\|\mathbf{s}_h\|_{0,\Omega}} \geq \widetilde{\beta}\, \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathcal{V}_{0,h}\,,$$

with $\widetilde{\beta} = c_1$.

In this way, having a, b_1, and b_2 satisfied the assumptions of [3, Corollary 2.2], we conclude the discrete analogue of (5.11) with the same constant $\alpha_{\mathcal{A}}$, that is

$$\sup_{\substack{\vec{\mathbf{s}}_h \in \mathcal{V}_h \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\mathcal{A}(\vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathbf{X}}} \geq \alpha_{\mathcal{A}}\, \|\vec{\mathbf{r}}_h\|_{\mathbf{X}} \qquad \forall\, \vec{\mathbf{r}}_h \in \mathcal{V}_h\,. \tag{5.26}$$

In addition, it is clear from the continuous analysis (cf. Section 5.1) that $\mathcal{A}$ and $\mathcal{C}$ are positive semi-definite, and that $\mathcal{C}$ is symmetric. Hence, in order for $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ to fully satisfy the hypotheses of [26, Theorem 4.1], it remains to establish the discrete inf-sup condition for $\mathcal{B}$, which is actually assumed as follows:

(H.3) there exists a positive constant $\beta_{\mathcal{B},\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{s}}_h \in \mathbf{X}_h \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\mathcal{B}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathbf{X}}} \geq \beta_{\mathcal{B},\mathtt{d}}\, \|\vec{\mathbf{v}}_h\|_{\mathbf{Y}} \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{Y}_h\,. \tag{5.27}$$

Specific finite element subspaces satisfying (**H.0**) - (**H.3**), are collected later on in Section 5.2.3.

Next, bearing in mind the above, and proceeding analogously to the continuous case (cf. Section 5.1), we derive the existence of a positive constant $\alpha_{\boldsymbol{\mathcal{A}},\mathrm{d}}$, depending on $\|\mathcal{A}\|$, $\|\mathcal{C}\|$, $\alpha_{\mathcal{A}}$, and $\beta_{\mathcal{B},\mathrm{d}}$, and hence independent of h, such that for each $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that, say $\|\mathbf{z}_h\|_{0,4;\Omega} \le \dfrac{\alpha_{\boldsymbol{\mathcal{A}},\mathrm{d}}}{2\rho}$, there holds

$$\sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\in\mathbf{X}_h\times\mathbf{Y}_h\\(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\neq\mathbf{0}}} \frac{\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\big)+\mathcal{D}(\mathbf{z}_h;\mathbf{w}_h,\mathbf{s}_h)}{\|(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\|_{\mathbf{X}\times\mathbf{Y}}} \ge \frac{\alpha_{\boldsymbol{\mathcal{A}},\mathrm{d}}}{2}\|(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathbf{X}\times\mathbf{Y}} \tag{5.28}$$

for all $(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\in\mathbf{X}_h\times\mathbf{Y}_h$.

We are now in position to state next the discrete analogues of Lemmas 5.3 and 5.4, and Theorem 5.5. Being the respective proofs almost verbatim to those of the continuous case, we omit further details.

Lemma 5.6. *For each* $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ *such that* $\|\mathbf{z}_h\|_{0,4;\Omega} \le \dfrac{\alpha_{\boldsymbol{\mathcal{A}},\mathrm{d}}}{2\rho}$*, problem* (5.24) *has a unique solution* $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h) = \big((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\gamma_h)\big)\in\mathbf{X}_h\times\mathbf{Y}_h$*, and hence* $\Xi_h(\mathbf{z}_h) := \mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ *is well-defined. Moreover, there holds*

$$\begin{aligned}\|\Xi_h(\mathbf{z}_h)\|_{0,4;\Omega} \,&=\, \|\mathbf{u}_h\|_{0,4;\Omega} \,\le\, \|(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{X}\times\mathbf{Y}}\\ &\le\, \frac{2}{\alpha_{\boldsymbol{\mathcal{A}},\mathrm{d}}}\Big\{\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}+\|\mathbf{f}\|_{0,4/3;\Omega}+\|\boldsymbol{\mathcal{B}}_h\otimes\boldsymbol{\mathcal{H}}_h\|_{0,\Omega}\Big\}.\end{aligned} \tag{5.29}$$

Lemma 5.7. *Let*

$$\mathbf{W}_h := \Big\{\mathbf{z}_h\in\mathbf{H}_h^{\mathbf{u}}:\quad \|\mathbf{z}_h\|_{0,4;\Omega}\le\frac{\alpha_{\boldsymbol{\mathcal{A}},\mathrm{d}}}{2\rho}\Big\}, \tag{5.30}$$

and assume that

$$\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}+\|\mathbf{f}\|_{0,4/3;\Omega}+\|\boldsymbol{\mathcal{B}}_h\otimes\boldsymbol{\mathcal{H}}_h\|_{0,\Omega}\le\frac{\alpha^2_{\boldsymbol{\mathcal{A}},\mathrm{d}}}{4\rho}. \tag{5.31}$$

Then, $\Xi_h(\mathbf{W}_h)\subseteq\mathbf{W}_h$.

Theorem 5.8. *Let* $\boldsymbol{\mathcal{B}}_h$ *and* $\boldsymbol{\mathcal{H}}_h$ *be approximations (to be defined later on) of* $\boldsymbol{\mathcal{B}}$ *and* $\boldsymbol{\mathcal{H}}$*, respectively, and assume that*

$$\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}+\|\mathbf{f}\|_{0,4/3;\Omega}+\|\boldsymbol{\mathcal{B}}_h\otimes\boldsymbol{\mathcal{H}}_h\|_{0,\Omega}<\frac{\alpha^2_{\boldsymbol{\mathcal{A}},\mathrm{d}}}{4\rho}. \tag{5.32}$$

Then, the operator Ξ_h *has a unique fixed-point* $\mathbf{z}_h\in\mathbf{W}_h$*. Equivalently,* (5.23) *has a unique solution* $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h) := \big((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\gamma_h)\big)\in\mathbf{X}_h\times\mathbf{Y}_h$*, where* $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)$ *is the unique solution of* (5.24) *with* $\mathbf{z}_h=\mathbf{u}_h$*. Moreover, there holds*

$$\|(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{X}\times\mathbf{Y}}\le\frac{2}{\alpha_{\boldsymbol{\mathcal{A}},\mathrm{d}}}\Big\{\|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma}+\|\mathbf{f}\|_{0,4/3;\Omega}+\|\boldsymbol{\mathcal{B}}_h\otimes\boldsymbol{\mathcal{H}}_h\|_{0,\Omega}\Big\}. \tag{5.33}$$

5.2.2 A priori error estimates

In this section we derive a Céa-type estimate for the error $\|(\vec{\mathbf{t}},\vec{\mathbf{u}})-(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{X}\times\mathbf{Y}}$, where $(\vec{\mathbf{t}},\vec{\mathbf{u}})\in\mathbf{X}\times\mathbf{Y}$ and $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\in\mathbf{X}_h\times\mathbf{Y}_h$ are the unique solutions of (5.1) and (5.23), respectively. To this end, we proceed as in [26, Section 4.3] and observe first from these formulations that

$$\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{t}},\vec{\mathbf{u}})-(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\big)=\mathcal{D}\big(\mathbf{u}_h;\mathbf{u}_h,\mathbf{s}_h\big)-\mathcal{D}\big(\mathbf{u};\mathbf{u},\mathbf{s}_h\big)+\big(\mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}-\mathcal{F}_{\boldsymbol{\mathcal{B}}_h,\boldsymbol{\mathcal{H}}_h}\big)(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h) \tag{5.34}$$

for all $(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h$. Then, applying the triangle inequality, employing the discrete inf-sup condition for $\boldsymbol{\mathcal{A}}$, which reads as (5.15), but with $\mathbf{X}_h \times \mathbf{Y}_h$ and $\alpha_{\boldsymbol{\mathcal{A}},\mathtt{d}}$ instead of $\mathbf{X} \times \mathbf{Y}$ and $\alpha_{\boldsymbol{\mathcal{A}}}$, respectively, and then using the boundedness of $\boldsymbol{\mathcal{A}}$, whose norm $\|\boldsymbol{\mathcal{A}}\|$ depends on $\|a\| = 2\mu$ (cf. (3.32)), $\|b_i\| = 1$ (cf. (3.33), $i \in \{1, 2\}$, $\|\mathcal{B}\| = 1$ (cf. (3.34)), and $\|\mathcal{C}\| = \rho\,|\Omega|^{1/2}$ (cf. (3.35)), we deduce the existence of a positive constant $C_{\boldsymbol{\mathcal{A}}}$, depending only on $\|\boldsymbol{\mathcal{A}}\|$ and $\alpha_{\mathbf{A},\mathtt{d}}$, such that, similarly to [26, eq. (4.22)], there holds

$$\begin{aligned} \|(\vec{\mathbf{t}}, \vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{X}\times\mathbf{Y}} \,\le\, C_{\boldsymbol{\mathcal{A}}} \Bigg\{ & \operatorname{dist}(\vec{\mathbf{t}}, \mathbf{X}_h) \,+\, \operatorname{dist}(\vec{\mathbf{u}}, \mathbf{Y}_h) + \|\mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}} - \mathcal{F}_{\boldsymbol{\mathcal{B}}_h,\boldsymbol{\mathcal{H}}_h}\|_{\mathbf{X}_h'} \\ & + \sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\in\mathbf{X}_h\times\mathbf{Y}_h \\ (\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\neq\mathbf{0}}} \frac{\mathcal{D}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h) - \mathcal{D}(\mathbf{u}; \mathbf{u}, \mathbf{s}_h)}{\|(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\|_{\mathbf{X}\times\mathbf{Y}}} \Bigg\}, \end{aligned} \tag{5.35}$$

where, similarly to (4.36), using the definition of $\mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}}$ (cf. (3.30)), and applying the Cauchy–Schwarz inequality, we find that

$$\|\mathcal{F}_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{H}}} - \mathcal{F}_{\boldsymbol{\mathcal{B}}_h,\boldsymbol{\mathcal{H}}_h}\|_{\mathbf{X}_h'} \,\le\, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega}\,. \tag{5.36}$$

In turn, proceeding analogously to the derivation of [26, eq. (4.23)], that is adding and subtracting $\mathbf{u}$ in the second component of $\mathcal{D}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h)$, and then using the boundedness property of $\mathcal{D}$ (cf. (3.36)), along with the a priori estimates for $\|\mathbf{u}\|_{0,4;\Omega}$ (cf. (5.22)) and $\|\mathbf{u}_h\|_{0,4;\Omega}$ (cf. (5.33)), we arrive at

$$\sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\in\mathbf{X}_h\times\mathbf{Y}_h \\ (\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\neq\mathbf{0}}} \frac{\mathcal{D}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h) - \mathcal{D}(\mathbf{u}; \mathbf{u}, \mathbf{s}_h)}{\|(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\|_{\mathbf{X}\times\mathbf{Y}}} \,\le\, \mathcal{L}_h(\mathtt{data})\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,, \tag{5.37}$$

where, denoting $\widetilde{\alpha}_{\boldsymbol{\mathcal{A}}} := \min\{\alpha_{\boldsymbol{\mathcal{A}}}, \alpha_{\boldsymbol{\mathcal{A}},\mathtt{d}}\}$, we set

$$\mathcal{L}_h(\mathtt{data}) \,:=\, \frac{4\rho}{\widetilde{\alpha}_{\boldsymbol{\mathcal{A}}}} \Big\{ \|\widetilde{\mathbf{u}}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\|_{0,\Omega} + \|\boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \Big\}\,. \tag{5.38}$$

In this way, employing (5.36) and (5.37) back into (5.35), we deduce the following preliminary estimate

$$\begin{aligned} \|(\vec{\mathbf{t}}, \vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{X}\times\mathbf{Y}} \,\le\, C_{\boldsymbol{\mathcal{A}}} \Bigg\{ & \operatorname{dist}(\vec{\mathbf{t}}, \mathbf{X}_h) \,+\, \operatorname{dist}(\vec{\mathbf{u}}, \mathbf{Y}_h) \\ & + \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \,+\, \mathcal{L}_h(\mathtt{data})\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \Bigg\}\,. \end{aligned} \tag{5.39}$$

As a direct consequence of (5.39), we obtain the following result.

Theorem 5.1. *Assume that* $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{B}}_h$, $\boldsymbol{\mathcal{H}}_h$, *and the data* $\mathbf{u}_D$ *and* $\mathbf{f}$ *satisfy*

$$\mathcal{L}_h(\mathtt{data}) \,\le\, \frac{1}{2\,C_{\boldsymbol{\mathcal{A}}}}\,. \tag{5.40}$$

Then, there holds

$$\|(\vec{\mathbf{t}}, \vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{X}\times\mathbf{Y}} \,\le\, 2\,C_{\boldsymbol{\mathcal{A}}} \Big\{ \operatorname{dist}(\vec{\mathbf{t}}, \mathbf{X}_h) \,+\, \operatorname{dist}(\vec{\mathbf{u}}, \mathbf{Y}_h) \,+\, \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega} \Big\}\,. \tag{5.41}$$

5.2.3 Specific finite element subspaces

We now resort to [26, Section 4.4], which in turn makes use of the analysis and results from [27, Section 4.4], to specify two examples of finite element subspaces $\mathbb{H}_h^{\mathbf{t}}$, $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$ satisfying the hypotheses (**H.0**), (**H.1**), (**H.2**), and (**H.3**) described in Section 5.2.1. To this end, and besides the notations already introduced in Section 4.2.3, we now we let b_K be the bubble function on each $K \in \mathcal{T}_h$, which is defined as the product of its $n+1$ barycentric coordinates, and introduce the local bubble spaces of order $\ell \geq 0$ as

$$\mathbf{B}_\ell(K) := \operatorname{curl}\big(b_K\, \mathrm{P}_\ell(K)\big) \quad \text{if} \quad n = 2\,, \quad \text{and} \quad \mathbf{B}_\ell(K) := \operatorname{curl}\big(b_K\, \mathbf{P}_\ell(K)\big) \quad \text{if} \quad n = 3\,,$$

where $\operatorname{curl}(v) := \big(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\big)$ if $n = 2$ and $v : K \to \mathrm{R}$, and $\operatorname{curl}(\mathbf{v}) := \nabla \times \mathbf{v}$ if $n = 3$ and $\mathbf{v} : K \to \mathrm{R}^3$. Next, we set the global space

$$\mathbb{B}_\ell(\mathcal{T}_h) := \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_{h,i}|_K \in \mathbf{B}_\ell(K) \quad \forall\, i \in \big\{1,...,n\big\}\,, \quad \forall\, K \in \mathcal{T}_h\Big\}\,.$$

In addition, we let $\mathbb{C}(\bar{\Omega}) := [C(\bar{\Omega})]^{n\times n}$. Then, given an integer $k \geq 0$, and as established in [26, Sections 4.4.2 and 4.4.3], the hypotheses specified in Section 5.2.1, are satisfied by subspaces based on PEERS and Arnold–Falk–Winther (AFW) elements, namely

a) PEERS-BASED FINITE ELEMENT SUBSPACES:

$$\begin{gathered}\mathbb{H}_h^{\mathbf{t}} := \mathbb{P}_{k+n}(\mathcal{T}_h) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \quad \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} := \mathbb{RT}_k(\mathcal{T}_h) \oplus \mathbb{B}_k(\mathcal{T}_h)\,, \quad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\mathcal{T}_h)\,, \\ \text{and} \qquad \mathbb{H}_h^{\gamma} := \mathbb{C}(\bar{\Omega}) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{P}_{k+1}(\mathcal{T}_h)\,,\end{gathered} \tag{5.42}$$

b) AFW-BASED FINITE ELEMENT SUBSPACES:

$$\begin{gathered}\mathbb{H}_h^{\mathbf{t}} := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \quad \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap \mathbb{H}(\mathbf{div};\Omega)\,, \quad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\mathcal{T}_h)\,, \\ \text{and} \qquad \mathbb{H}_h^{\gamma} := \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{P}_k(\mathcal{T}_h)\,.\end{gathered} \tag{5.43}$$

In particular, we stress that the occurrence of the assumption (**H.2**) forces the particular polynomial degree needed by each subspace $\mathbb{H}_h^{\mathbf{t}}$. We refer to [26, last part of Sections 4.4.2 and 4.4.3] for the corresponding details.

Now, the approximation properties of $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$, for PEERS (cf. (5.42)) as well as for AFW (cf. (5.43)), whose derivations, similarly to those in Section 4.2.3, follow basically from the error estimates of the Raviart–Thomas and AFW interpolation operators, and of projectors onto piecewise vector and tensor polynomials (cf. [22, Proposition 1.135]), are provided below. While the first two were already given in Section 4.2.3, we state them again in what follows for sake of clearness of the presentation:

$\big(\mathbf{AP}_h^{\boldsymbol{\sigma}}\big)$ there exists a positive constant C, independent of h, such that for each $\ell \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{\ell,4/3}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}\big) \leq C\, h^\ell \Big\{\|\boldsymbol{\tau}\|_{\ell,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\ell,4/3;\Omega}\Big\}\,,$$

$\big(\mathbf{AP}_h^{\mathbf{u}}\big)$ there exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{\ell,4}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}\big) \leq C\, h^\ell \|\mathbf{v}\|_{\ell,4;\Omega}\,,$$

$\big(\mathbf{AP}_h^{\gamma}\big)$ there exists a positive constant C, independent of h, such that for each $\ell \in [0,k+1]$, and for each $\boldsymbol{\delta} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\delta}, \mathbb{H}_h^{\gamma}\big) \,\le\, C\, h^\ell \|\boldsymbol{\delta}\|_{\ell,\Omega}\,.$$

In turn, denoting $k^* := \begin{cases} k+n & \text{for PEERS-based} \\ k+1 & \text{for AFW-based} \end{cases}$, the approximation property for $\mathbb{H}_h^{\mathbf{t}}$ is similar to that of $\mathbf{H}_h^{\mathbf{u}}$, that is:

$\big(\mathbf{AP}_h^{\mathbf{t}}\big)$ there exists a positive constant C, independent of h, such that for each $\ell \in [0,k^*+1]$, and for each $\mathbf{s} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{s}, \mathbb{H}_h^{\mathbf{t}}\big) \,\le\, C\, h^\ell \|\mathbf{s}\|_{\ell,\Omega}\,.$$

Finally, the convergence result for the Galerkin scheme (5.23) is stated as follows.

Theorem 5.2. *In addition to the hypotheses of Lemma* 5.1*, and Theorems* 5.5 *and* 5.8*, assume that there exists* $\ell \in (0,k+1]$ *such that* $\mathbf{t} \in \mathbb{H}^\ell(\Omega)\cap\mathbb{L}^2_{\mathrm{tr}}(\Omega)$*,* $\boldsymbol{\sigma} \in \mathbb{H}^\ell(\Omega)\cap\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$*,* $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{\ell,4/3}(\Omega)$*,* $\mathbf{u} \in \mathbf{W}^{\ell,4}(\Omega)$*, and* $\boldsymbol{\gamma} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$*. Then, there exists a positive constant* C*, independent of* h*, such that*

$$\begin{aligned} \|(\vec{\mathbf{t}},\vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{X}\times\mathbf{Y}} \,\le\, C\,\Big\{ h^\ell \Big(\|\mathbf{t}\|_{\ell,\Omega} + \|\boldsymbol{\sigma}\|_{\ell,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\ell,4/3;\Omega} + \|\mathbf{u}\|_{\ell,4;\Omega} + \|\boldsymbol{\gamma}\|_{\ell,\Omega}\Big) \\ + \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega}\Big\}\,. \end{aligned} \tag{5.44}$$

Proof. It follows from a direct application of Theorem 5.1 and the above approximation properties of the finite element subspaces. $\square$

6 Analysis of the potential equation

Similarly as in Section 5, here we apply again the generalized Babuška-Brezzi theory in Banach spaces (cf. [3, Theorem 2.1, Corollary 2.1]) to perform now the continuous and discrete analyses of the variational formulation (3.45). To this end, and as already announced in Section 3.3, we resort to some of the corresponding results provided in [25, Sections 2, 3, and 4].

6.1 The continuous analysis

We begin by letting $\mathcal{K}_i$, $i \in \{1,2\}$, be the kernel of the bilinear form B_i (cf. (3.48)), that is

$$\mathcal{K}_i \,:=\, \Big\{ \boldsymbol{\xi} \in X_i : \quad \mathrm{B}_i(\boldsymbol{\xi},\psi) \,=\, 0 \qquad \forall\, \psi \in \mathrm{M}_i \Big\}\,,$$

which, according to the definitions of the spaces involved (cf. (3.46)), yields

$$\begin{aligned} \mathcal{K}_1 \quad &:= \quad \Big\{ \boldsymbol{\xi} \in \mathbf{H}_0^s(\mathrm{div}_s;\Omega) : \quad \mathrm{div}(\boldsymbol{\xi}) = 0 \quad \text{in} \quad \Omega \Big\}, \quad \text{and} \\ \mathcal{K}_2 \quad &:= \quad \Big\{ \boldsymbol{\xi} \in \mathbf{H}_0^r(\mathrm{div}_r;\Omega) : \quad \mathrm{div}(\boldsymbol{\xi}) = 0 \quad \text{in} \quad \Omega \Big\}\,. \end{aligned}$$

Next, in order to establish the continuous inf-sup condition involving A and the kernels $\mathcal{K}_i$, $i \in \{1,2\}$, as required by [3, Theorem 2.1], we recall from the key result provided by [25, Lemma 2.3, Section 2.1] that the index $r := 2\,\ell$ and its conjugate s (cf. (3.41)) must both lie in $[4/3, 4]$ when $n = 2$, and in $[3/2, 3]$ when $n = 3$, which we assume from now on. This means, as indicated in (3.41), that ℓ must lie in $(1, 2]$ when $n = 2$, and in $(1, 3/2]$ when $n = 3$, so that its conjugate j lies in $[2, +\infty)$ and $[3, +\infty)$, respectively, thus ensuring that $\varrho := 2\,j \geq 2\,\ell =: r$, as requested right after (3.41).

In this way, having precised the feasible range for r and s, we now state the following results taken basically from [25].

Lemma 6.1. *There exists a positive constant* $\widetilde{\alpha}$ *such that*

$$\sup_{\substack{\boldsymbol{\xi}\in\mathcal{K}_1\\ \boldsymbol{\xi}\neq\mathbf{0}}} \frac{\mathrm{A}(\boldsymbol{\lambda},\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|_{\mathrm{X}_1}} \,\geq\, \widetilde{\alpha}\,\|\boldsymbol{\lambda}\|_{\mathrm{X}_2} \qquad \forall\, \boldsymbol{\lambda}\in\mathcal{K}_2\,. \tag{6.1a}$$

In addition, there holds

$$\sup_{\boldsymbol{\lambda}\in\mathcal{K}_2} \mathrm{A}(\boldsymbol{\lambda},\boldsymbol{\xi}) \,>\, 0 \qquad \forall\, \boldsymbol{\xi}\in\mathcal{K}_1\,,\ \boldsymbol{\xi}\neq\mathbf{0}\,. \tag{6.1b}$$

Proof. It reduces to the particular case of [25, Lemma 2.6, Section 2.4.2] when the temperature-dependent viscosity factor appearing in the definition of the bilinear form in there, is equal to 1. □

Lemma 6.2. *There exist positive constants* $\widetilde{\beta}_1$, $\widetilde{\beta}_2$ *such that for each* $i \in \{1, 2\}$ *there holds*

$$\sup_{\substack{\boldsymbol{\xi}\in\mathrm{X}_i\\ \boldsymbol{\xi}\neq\mathbf{0}}} \frac{\mathrm{B}_i(\boldsymbol{\xi},\psi)}{\|\boldsymbol{\xi}\|_{\mathrm{X}_i}} \,\geq\, \widetilde{\beta}_i\,\|\psi\|_{\mathrm{M}_i} \qquad \forall\, \psi\in\mathrm{M}_i\,. \tag{6.2}$$

Proof. It corresponds exactly to the proof of [25, Lemma 2.7, Section 2.4.2]. □

Consequently, we are now in position to establish the well-posedness of (3.45).

Theorem 6.1. *There exists a unique* $(\boldsymbol{\vartheta}, \phi) \in \mathrm{X}_2 \times \mathrm{M}_1$ *solution to* (3.45)*. Moreover, there holds*

$$\begin{aligned} \|\boldsymbol{\vartheta}\|_{r,\mathrm{div}_r;\Omega} \,&\leq\, \frac{1}{\widetilde{\alpha}}\Big\{\|\boldsymbol{\mathcal{M}}\|_{0,r;\Omega} + \|\boldsymbol{\mathcal{H}}_0\|_{0,r;\Omega}\Big\}\,, \quad \textit{and}\\ \|\phi\|_{0,r;\Omega} \,&\leq\, \frac{1}{\widetilde{\beta}_1}\left(1 + \frac{1}{\widetilde{\alpha}}\right)\Big\{\|\boldsymbol{\mathcal{M}}\|_{0,r;\Omega} + \|\boldsymbol{\mathcal{H}}_0\|_{0,r;\Omega}\Big\}\,. \end{aligned} \tag{6.3}$$

Proof. Knowing from (3.50) up to (3.53) that A, B_1, B_2, and $\mathrm{F}_{\boldsymbol{\mathcal{M}}}$ are all bounded with $\|\mathrm{A}\| \leq 1$, $\|\mathrm{B}_i\| \leq 1$, $i \in \big\{1, 2\big\}$, and $\|F_{\boldsymbol{\mathcal{M}}}\| \leq \|\boldsymbol{\mathcal{M}}\|_{0,r;\Omega} + \|\boldsymbol{\mathcal{H}}_0\|_{0,r;\Omega}$, and bearing in mind Lemmas 6.1 and 6.2, the unique solvability of (3.45) arises from a straightforward application of [3, Theorem 2.1]. In turn, the a priori bounds given by (6.3) follow from [3, eqs. (2.15) and (2.16), Corollary 2.1]. □

6.2 The discrete analysis

We begin by recalling from [25, Section 4.2] the finite element subspaces to be employed, which, given an integer $k \geq 0$, and making use of the same notations from Section 4.2.3, are defined as

$$\begin{aligned} \mathrm{X}_{2,h} \,&:=\, \mathbf{H}^r_0(\mathrm{div}_r;\Omega) \cap \mathbf{RT}_k(\mathcal{T}_h)\,, \qquad \mathrm{M}_{1,h} \,:=\, \mathrm{L}^r_0(\Omega) \cap \mathrm{P}_k(\mathcal{T}_h)\,,\\ \mathrm{X}_{1,h} \,&:=\, \mathbf{H}^s_0(\mathrm{div}_s;\Omega) \cap \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \text{and} \quad \mathrm{M}_{2,h} \,:=\, \mathrm{L}^s_0(\Omega) \cap \mathrm{P}_k(\mathcal{T}_h)\,. \end{aligned} \tag{6.4}$$

Then, given a discrete approximation $\boldsymbol{\mathcal{M}}_h$ (to be defined later on) of $\boldsymbol{\mathcal{M}}$, the Galerkin scheme associated with (3.45) reads: Find $(\boldsymbol{\vartheta}_h, \phi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ such that

$$\begin{aligned} \mathrm{A}(\boldsymbol{\vartheta}_h, \boldsymbol{\xi}_h) + \mathrm{B}_1(\boldsymbol{\xi}_h, \phi_h) \quad &= \quad \mathrm{F}_{\boldsymbol{\mathcal{M}}_h}(\boldsymbol{\xi}_h) \qquad \forall\, \boldsymbol{\xi}_h \in \mathrm{X}_{1,h}\,, \\ \mathrm{B}_2(\boldsymbol{\vartheta}_h, \psi_h) \quad &= \quad 0 \qquad \forall\, \psi_h \in \mathrm{M}_{2,h}\,, \end{aligned} \tag{6.5}$$

where, according to (3.49),

$$\mathrm{F}_{\boldsymbol{\mathcal{M}}_h}(\boldsymbol{\xi}_h) := \int_\Omega \big(\boldsymbol{\mathcal{M}}_h - \boldsymbol{\mathcal{H}}_0\big) \cdot \boldsymbol{\xi}_h \qquad \forall\, \boldsymbol{\xi}_h \in \mathrm{X}_{1,h}\,. \tag{6.6}$$

It readily follows from (6.4) that $\mathrm{div}\big(\mathrm{X}_{i,h}\big) \subseteq \mathrm{M}_{i,h}$ for all $i \in \{1,2\}$, and hence, the corresponding discrete kernels of the bilinear forms B_1 and B_2 algebraically coincide, which are easily seen to become the space

$$\mathcal{K}_h^k := \Big\{ \boldsymbol{\xi}_h \in \mathbf{RT}_k(\mathcal{T}_h) : \quad \boldsymbol{\xi}_h \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma \quad \text{and} \quad \mathrm{div}(\boldsymbol{\xi}_h) = 0 \quad \text{in} \quad \Omega \Big\}\,.$$

Then, we resort again to [25] to state next the required discrete inf-sup conditions for A, and for B_1 and B_2.

Lemma 6.3. *Assume that $n = 2$. Then, there exists a positive constant $\widetilde{\alpha}_{\mathtt{d}}$, independent of h, except when $k = 0$ and Ω is non-convex, such that*

$$\sup_{\substack{\boldsymbol{\xi}_h \in \mathcal{K}_h^k \\ \boldsymbol{\xi}_h \neq \mathbf{0}}} \frac{\mathrm{A}(\boldsymbol{\lambda}_h, \boldsymbol{\xi}_h)}{\|\boldsymbol{\xi}_h\|_{\mathrm{X}_1}} \,\geq\, \widetilde{\alpha}_{\mathtt{d}}\, \|\boldsymbol{\lambda}_h\|_{\mathrm{X}_2} \qquad \forall\, \boldsymbol{\lambda}_h \in \mathcal{K}_h^k\,. \tag{6.7a}$$

In addition, there holds

$$\sup_{\boldsymbol{\lambda}_h \in \mathcal{K}_h^k} \mathrm{A}(\boldsymbol{\lambda}_h, \boldsymbol{\xi}_h) \,>\, 0 \qquad \forall\, \boldsymbol{\xi}_h \in \mathcal{K}_h^k\,,\ \boldsymbol{\xi}_h \neq \mathbf{0}\,. \tag{6.7b}$$

Proof. Analogously to the proof of Lemma 6.1, it reduces to the particular case of [25, Lemma 4.3, Section 4.4] when the temperature-dependent viscosity factor appearing in the definition of the bilinear form in there, is equal to 1. □

Some remarks on Lemma 6.3 come into place now. To this end, we first let $\Theta_h^k : \mathrm{L}^s(\Omega) \to \mathcal{K}_h^k$ be the $\mathrm{L}^2(\Omega)$-type orthogonal projector, which is defined, for each $\boldsymbol{\xi} \in \mathrm{L}^s(\Omega)$, as the unique element $\Theta_h^k(\boldsymbol{\xi}) \in \mathcal{K}_h^k$, such that

$$\int_\Omega \Theta_h^k(\boldsymbol{\xi}) \cdot \boldsymbol{\lambda}_h \,=\, \int_\Omega \boldsymbol{\xi} \cdot \boldsymbol{\lambda}_h \qquad \forall\, \boldsymbol{\lambda}_h \in \mathcal{K}_h^k\,,$$

and which is easily seen to be bounded when $\mathcal{K}_h^k$ is endowed with the norm of $\mathrm{L}^s(\Omega)$ as well. Then, we recall from [25, Lemma 4.3, Section 4.4] that $\widetilde{\alpha}_{\mathtt{d}}$ (cf. (6.7a)) depends on the reciprocal of $\|\Theta_h^k\|_{\mathcal{L}\left(\mathrm{L}^s(\Omega),\mathcal{K}_h^k\right)}$. Moreover, it is shown in [25, Lemma 4.2, Section 4.3] that $\|\Theta_h^k\|_{\mathcal{L}\left(\mathrm{L}^s(\Omega),\mathcal{K}_h^k\right)}$ is independent of h, except when Ω is non-convex and $k = 0$, which explains the corresponding sentence on the inf-sup constant indicated in the statement of Lemma 6.3. However, this dependence, given by the factor $\big(-\log(h)\big)^{|1-2/s|}$, grows very slowly as h approaches 0, and hence $\widetilde{\alpha}_{\mathtt{d}}$ remains reasonably bounded by below. In turn, it is also commented in the proof of [25, Lemma 4.2, Section 4.3] that the aforementioned properties of Θ_h^k depend strongly on the particular relation between the differential operators curl and ∇ in $2D$, which does not hold in $3D$. This is the reason why, up to the authors' knowledge, establishing Lemma 6.3 in $3D$ with a constant $\widetilde{\alpha}_{\mathtt{d}}$ independent of the meshsize, is still an open issue. Certainly, proceeding analogously to the proof of [25, Lemma 4.3, Section 4.4]), one can show that the inequality (6.7a) still holds in $3D$ with a constant depending also on the reciprocal of $\|\Theta_h^k\|_{\mathcal{L}\left(\mathrm{L}^s(\Omega),\mathcal{K}_h^k\right)}$, but the eventual independence on h of the later is precisely what has not been proved yet.

The discrete version of Lemma 6.2, which is valid in both $2D$ and $3D$, is stated as follows.

Lemma 6.4. *There exist positive constants $\widetilde{\beta}_{1,\mathtt{d}}$, $\widetilde{\beta}_{2,\mathtt{d}}$, independent of h, such that for each $i \in \{1,2\}$ there holds*

$$\sup_{\substack{\boldsymbol{\xi}_h\in \mathrm{X}_{i,h}\\ \boldsymbol{\xi}_h\neq \mathbf{0}}} \frac{\mathrm{B}_i(\boldsymbol{\xi}_h,\psi_h)}{\|\boldsymbol{\xi}_h\|_{\mathrm{X}_i}} \,\geq\, \widetilde{\beta}_{i,\mathtt{d}}\,\|\psi_h\|_{\mathrm{M}_i} \qquad \forall\ \psi_h\in \mathrm{M}_{i,h}\,. \tag{6.8}$$

Proof. It corresponds exactly to the proof of [25, Lemma 4.4, Section 4.4]. □

Regarding Lemma 6.4, we just remark, as explained in the proof of [25, Lemma 4.4, Section 4.4], that when Ω is non-convex, the discrete inf-sup condition for B_1, and hence our whole discrete analysis, is restricted to $s > 3/2$ and to those polygonal regions with largest interior angle strictly less than $s\pi$. These constraints arise from the application of a boundedness property of the Raviart-Thomas interpolation operator (cf. [25, eq. (C.12b)]), which, in turn, requires the regularity result provided by [25, Lemma C.1].

As a consequence of the previous results and discussions, and, in particular, assuming the hypotheses and constraints arising along them, we are now in position to establish the well-posedness of (6.5).

Theorem 6.2. *Let $\boldsymbol{\mathcal{M}}_h$ be an approximation (to be defined later on) of $\boldsymbol{\mathcal{M}}$. Then, there exists a unique $(\boldsymbol{\vartheta}_h,\phi_h)\in \mathrm{X}_{2,h}\times \mathrm{M}_{1,h}$ solution to (6.5). Moreover, there holds*

$$\begin{aligned}\|\boldsymbol{\vartheta}_h\|_{r,\mathrm{div}_r;\Omega} &\leq \frac{1}{\widetilde{\alpha}_{\mathtt{d}}}\Big\{\|\boldsymbol{\mathcal{M}}_h\|_{0,r;\Omega}+\|\boldsymbol{\mathcal{H}}_0\|_{0,r;\Omega}\Big\}, \quad \textit{and}\\ \|\phi_h\|_{0,r;\Omega} &\leq \frac{1}{\widetilde{\beta}_{1,\mathtt{d}}}\left(1+\frac{1}{\widetilde{\alpha}_{\mathtt{d}}}\right)\Big\{\|\boldsymbol{\mathcal{M}}_h\|_{0,r;\Omega}+\|\boldsymbol{\mathcal{H}}_0\|_{0,r;\Omega}\Big\}\,.\end{aligned} \tag{6.9}$$

Proof. As stated in the proof of Theorem 6.1, we already know that all the forms involved are bounded. Hence, invoking additionally Lemmas 6.3 and 6.4, the proof follows by applying the discrete version of the generalized Babuška-Brezzi theory (cf. [3, Corollary 2.2] or [24, Theorem 1.2]). In particular, the a priori bounds given by (6.9) arise from [3, eqs. (2.24) and (2.25), Corollary 2.2] (see, also [24, eqs. (1.7) and (1.8), Theorem 1.2]) and the fact that $\|\mathrm{F}_{\boldsymbol{\mathcal{M}}_h}\|\,\leq\,\|\boldsymbol{\mathcal{M}}_h\|_{0,r;\Omega}+\|\boldsymbol{\mathcal{H}}_0\|_{0,r;\Omega}$. □

Next, we establish a Céa-type estimate for the Galerkin error associated with (3.45) and (6.5).

Theorem 6.3. *There exists a positive constant $C_{\mathtt{ST}}$, depending only on $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{1,\mathtt{d}}$, and $\widetilde{\beta}_{2,\mathtt{d}}$, such that*

$$\|(\boldsymbol{\vartheta},\phi)-(\boldsymbol{\vartheta}_h,\phi_h)\|\,\leq\, C_{\mathtt{ST}}\Big\{\mathrm{dist}\big(\boldsymbol{\vartheta},\mathrm{X}_{2,h}\big)\,+\,\mathrm{dist}\big(\phi,\mathrm{M}_{1,h}\big)\,+\,\|\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{M}}_h\|_{0,r;\Omega}\Big\}\,. \tag{6.10}$$

Proof. Applying the Strang estimate for the generalized Babuška-Brezzi theory (cf. [24, eqs. (2.20) and (2.21),Theorem 2.1]) to (3.45) and (6.5), and recalling that the norms of A, B_1, and B_2 are all bounded by 1, we deduce the existence of such constant $C_{\mathtt{ST}}$ satisfying

$$\|(\boldsymbol{\vartheta},\phi)-(\boldsymbol{\vartheta}_h,\phi_h)\| \,\le\, C_{\mathtt{ST}}\Big\{\mathrm{dist}\big(\boldsymbol{\vartheta},\mathrm{X}_{2,h}\big) \,+\, \mathrm{dist}\big(\phi,\mathrm{M}_{1,h}\big) \,+\, \|\mathrm{F}_{\boldsymbol{\mathcal{M}}}-\mathrm{F}_{\boldsymbol{\mathcal{M}}_h}\|_{\mathrm{X}'_{1,h}}\Big\}\,.$$

Then, it is readily seen from (3.49) and (6.6) that $\|\mathrm{F}_{\boldsymbol{\mathcal{M}}}-\mathrm{F}_{\boldsymbol{\mathcal{M}}_h}\|_{\mathrm{X}'_{1,h}} \,\le\, \|\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{M}}_h\|_{0,r;\Omega}$, which, replaced back in the foregoing equation, yields (6.10) and ends the proof. □

We now aim to derive the rates of convergence of the Galerkin scheme (6.5), for which we previously collect the approximation properties of $\mathrm{X}_{2,h}$ and $\mathrm{M}_{1,h}$ (cf. [25, Section 4.5]):

$(\mathbf{AP}_h^{\boldsymbol{\vartheta}})$ there exists $C>0$, independent of h, such that for each $\ell\in[1,k+1]$, and for each $\boldsymbol{\xi}\in\mathbf{W}^{\ell,r}(\Omega)$ with $\mathrm{div}(\boldsymbol{\xi})\in\mathrm{W}^{\ell,r}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\xi},\mathrm{X}_{2,h}) \,:=\, \inf_{\boldsymbol{\xi}_h\in\mathrm{X}_{2,h}}\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{r,\mathrm{div}_r;\Omega} \,\le\, C\,h^{\ell}\,\Big\{\|\boldsymbol{\xi}\|_{\ell,r;\Omega} \,+\, \|\mathrm{div}(\boldsymbol{\xi})\|_{\ell,r;\Omega}\Big\}\,,$$

$(\mathbf{AP}_h^{\phi})$ there exists $C>0$, independent of h, such that for each $\ell\in[0,k+1]$, and for each $\psi\in\mathrm{W}^{\ell,r}(\Omega)$, there holds

$$\mathrm{dist}(\psi,\mathrm{M}_{1,h}) \,:=\, \inf_{\psi_h\in\mathrm{M}_{1,h}}\|\psi-\psi_h\|_{0,r;\Omega} \,\le\, C\,h^{\ell}\,\|\psi\|_{\ell,r;\Omega}\,.$$

The rates of convergence of (6.5) are now stated as follows.

Theorem 6.4. *Assume that there exists* $\ell\in[0,k+1]$ *such that* $\boldsymbol{\vartheta}\in\mathbf{W}^{\ell,r}(\Omega)$, $\mathrm{div}(\boldsymbol{\vartheta})\in\mathrm{W}^{\ell,r}(\Omega)$, *and* $\phi\in\mathrm{W}^{\ell,r}(\Omega)$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\|(\boldsymbol{\vartheta},\phi)-(\boldsymbol{\vartheta}_h,\phi_h)\| \le C_{\mathtt{ST}}\Big\{C\,h^{\ell}\,\Big(\|\boldsymbol{\vartheta}\|_{\ell,r;\Omega}+\|\mathrm{div}(\boldsymbol{\vartheta})\|_{\ell,r;\Omega}+\|\phi\|_{\ell,r;\Omega}\Big)+\|\boldsymbol{\mathcal{M}}-\boldsymbol{\mathcal{M}}_h\|_{0,r;\Omega}\Big\}\,. \quad (6.11)$$

Proof. It is a straightforward consequence of Theorem 6.3 and the above approximation properties. □

7 Iterative approximations of the magnetisation of the fluid

Having established the well-posedness of the continuous and discrete formulations for the fluid and potential equations, we now propose two iterative procedures, motivated by the stationary approaches suggested in Sections 2.2.1 and 2.2.2, to approximate $\boldsymbol{\mathcal{M}}$. More precisely, we employ the Galerkin schemes (4.16), (5.23), and (6.5), along with (2.14) and (2.23), to generate sequential updates aiming to the aforementioned goal. The corresponding algorithms, both taking $\boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}}_0)$ as the initial guess for $\boldsymbol{\mathcal{M}}_h$, are described as follows:

FIRST ITERATIVE METHOD TO APPROXIMATE $\boldsymbol{\mathcal{M}}$

i) let $\boldsymbol{\mathcal{M}}_h \,:=\, \boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}}_0)$,

ii) solve (6.5), that is: Find $(\boldsymbol{\vartheta}_h,\phi_h)\in\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}$ such that

$$\begin{array}{lcll}\mathrm{A}(\boldsymbol{\vartheta}_h,\boldsymbol{\xi}_h) \,+\, \mathrm{B}_1(\boldsymbol{\xi}_h,\phi_h) & = & \mathrm{F}_{\boldsymbol{\mathcal{M}}_h}(\boldsymbol{\xi}_h) & \forall\,\boldsymbol{\xi}_h\in\mathrm{X}_{1,h}\,,\\ \mathrm{B}_2(\boldsymbol{\vartheta}_h,\psi_h) & = & 0 & \forall\,\psi_h\in\mathrm{M}_{2,h}\,,\end{array}$$

iii) compute $\boldsymbol{\mathcal{B}}_h$ and $\boldsymbol{\mathcal{H}}_h$ as suggested by the discrete version of (3.40), that is

$$\boldsymbol{\mathcal{B}}_h \,:=\, \mu_0\,(\boldsymbol{\vartheta}_h + \boldsymbol{\mathcal{H}}_0) \quad \text{and} \quad \boldsymbol{\mathcal{H}}_h \,:=\, \mu_0^{-1}\,\boldsymbol{\mathcal{B}}_h \,-\, \boldsymbol{\mathcal{M}}_h\,, \tag{7.1}$$

iv) solve (4.16), that is: Find $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) := \big((\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\sigma}_h\big) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ such that

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{c}(\mathbf{u}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= \mathbf{F}_{\boldsymbol{\mathcal{B}}_h, \boldsymbol{\mathcal{H}}_h}(\vec{\mathbf{v}}_h) \quad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h, \\ \mathbf{b}(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) \qquad\quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,, \end{aligned}$$

v) update $\boldsymbol{\mathcal{M}}_h$ as suggested by the discrete version of (2.14), that is, compute

$$\boldsymbol{\mathcal{M}}_h^{\texttt{new}} \,:=\, \boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}}_h) \,+\, \frac{\varepsilon}{2}\big(\mathbf{t}_h - \mathbf{t}_h^{\mathtt{t}}\big)\,\boldsymbol{\mathcal{M}}_h \,-\, \frac{\varepsilon\,\mu_0}{6\,\mu\,\phi_0}\,\boldsymbol{\mathcal{M}}_h \times \big(\boldsymbol{\mathcal{M}}_h \times \boldsymbol{\mathcal{H}}_h\big)\,, \tag{7.2}$$

vi) check the stopping criterion, say $\|\boldsymbol{\mathcal{M}}_h - \boldsymbol{\mathcal{M}}_h^{\texttt{new}}\|_{0,r;\Omega} \,\le\, \texttt{TOL}$, where TOL is a given tolerance,

vii) if criterion from vi) is satisfied, end the iterations, otherwise, let $\boldsymbol{\mathcal{M}}_h \,:=\, \boldsymbol{\mathcal{M}}_h^{\texttt{new}}$ and go to ii).

SECOND ITERATIVE METHOD TO APPROXIMATE $\boldsymbol{\mathcal{M}}$

i) let $\boldsymbol{\mathcal{M}}_h \,:=\, \boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}}_0)$,

ii) solve (6.5), that is: Find $(\boldsymbol{\vartheta}_h, \phi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ such that

$$\begin{aligned} \mathrm{A}(\boldsymbol{\vartheta}_h, \boldsymbol{\xi}_h) \,+\, \mathrm{B}_1(\boldsymbol{\xi}_h, \phi_h) &= \mathrm{F}_{\boldsymbol{\mathcal{M}}_h}(\boldsymbol{\xi}_h) \qquad \forall\, \boldsymbol{\xi}_h \in \mathrm{X}_{1,h}\,, \\ \mathrm{B}_2(\boldsymbol{\vartheta}_h, \psi_h) &= 0 \qquad\qquad\quad \forall\, \psi_h \in \mathrm{M}_{2,h}\,, \end{aligned}$$

iii) compute $\boldsymbol{\mathcal{B}}_h$ and $\boldsymbol{\mathcal{H}}_h$ as suggested by the discrete version of (3.40), that is

$$\boldsymbol{\mathcal{B}}_h \,:=\, \mu_0\,(\boldsymbol{\vartheta}_h + \boldsymbol{\mathcal{H}}_0) \quad \text{and} \quad \boldsymbol{\mathcal{H}}_h \,:=\, \mu_0^{-1}\,\boldsymbol{\mathcal{B}}_h \,-\, \boldsymbol{\mathcal{M}}_h\,,$$

iv) solve (5.23), that is: Find $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) := \big((\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h)\big) \in \mathbf{X}_h \times \mathbf{Y}_h$ such that

$$\boldsymbol{\mathcal{A}}\big((\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) \,+\, \mathcal{D}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h) \,=\, \boldsymbol{\mathcal{F}}_{\boldsymbol{\mathcal{B}}_h, \boldsymbol{\mathcal{H}}_h}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \qquad \forall\, (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h\,,$$

[PEERS-based (cf. (5.42)) and AFW-based (cf. (5.43)) are feasible choices for $\mathbf{X}_h \times \mathbf{Y}_h$].

v) update $\boldsymbol{\mathcal{M}}_h$ as suggested by the discrete version of (2.23), that is, compute

$$\boldsymbol{\mathcal{M}}_h^{\texttt{new}} \,:=\, \big(1 \,+\, \varepsilon\,\rho\big)^{-1}\Big\{\boldsymbol{\mathcal{M}}_0(\boldsymbol{\mathcal{H}}_h) \,+\, \varepsilon\,\boldsymbol{\gamma}_h\,\boldsymbol{\mathcal{M}}_h \,-\, \frac{\varepsilon\,\mu_0}{6\,\mu\,\phi_0}\,\boldsymbol{\mathcal{M}}_h \times \big(\boldsymbol{\mathcal{M}}_h \times \boldsymbol{\mathcal{H}}_h\big)\Big\}\,, \tag{7.3}$$

vi) check the stopping criterion, say $\|\boldsymbol{\mathcal{M}}_h - \boldsymbol{\mathcal{M}}_h^{\texttt{new}}\|_{0,r;\Omega} \,\le\, \texttt{TOL}$, where TOL is a given tolerance,

vii) if criterion from vi) is satisfied, end the iterations, otherwise, let $\boldsymbol{\mathcal{M}}_h \,:=\, \boldsymbol{\mathcal{M}}_h^{\texttt{new}}$ and go to ii).

8 Numerical results

In this section, we consider the two iterative approximations of the fluid magnetisation detailed in Section 7 and present three examples illustrating the performance of the mixed finite element method (4.16) on barycentric triangulations and of the method (5.23), coupled with (6.5), on quasi-uniform triangulations of the respective domains. In what follows, the sets of finite element subspaces generated on barycentric meshes with $k \geq n-1$ and on quasi-uniform meshes with $k \in \{0,1\}$ will be simply referred to as the $\mathbf{P}_k - \mathbb{P}_k - \mathbb{RT}_k - \mathbf{RT}_k - \mathrm{P}_k$ and $\big\{\mathrm{PEERS}_k, \mathrm{AFW}_k\big\} - \mathbf{RT}_k - \mathrm{P}_k$ discretizations, respectively. The numerical methods are implemented using the open-source finite element library `FEniCS` [1]. The Navier–Stokes problems arising in the two iterative methods used to approximate $\boldsymbol{\mathcal{M}}$ in Section 7, namely (4.16) and (5.23), are solved by means of a Newton method with the same fixed tolerance $\mathtt{TOL} = 1\mathrm{E}{-}06$ considered for the outer iterative schemes. Since the Newton method exhibits fast convergence, requiring only two to three iterations at each coupling step, we report only the total number of outer iterations required to approximate $\boldsymbol{\mathcal{M}}$.

We now introduce some additional notation. The individual errors are denoted by

$$\mathsf{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}\,, \quad \mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}\,, \quad \mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,, \quad \mathsf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega}\,,$$

$$\mathsf{e}(p) := \|p - p_h\|_{0,\Omega}\,, \quad \mathsf{e}(\boldsymbol{\vartheta}) := \|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_h\|_{r,\mathrm{div}_r;\Omega}\,, \quad \mathsf{e}(\phi) := \|\phi - \phi_h\|_{0,r;\Omega}\,, \quad \mathsf{e}(\boldsymbol{\mathcal{H}}) := \|\boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{H}}_h\|_{0,\varrho;\Omega}\,,$$

$$\mathsf{e}(\boldsymbol{\mathcal{M}}) := \|\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}}_h\|_{0,r;\Omega}\,, \quad \text{and} \quad \mathsf{e}(\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}) := \|\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{B}}_h \otimes \boldsymbol{\mathcal{H}}_h\|_{0,\Omega}\,,$$

where r and ϱ are defined in Section 6.1 and will be specified in the examples below. Next, as usual, for each $\star \in \big\{\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, p, \boldsymbol{\vartheta}, \phi, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}\big\}$ we let $\mathsf{r}(\star)$ be the experimental rate of convergence given by $\mathsf{r}(\star) := \log(\mathsf{e}(\star)/\widehat{\mathsf{e}}(\star))/\log(h/\widehat{h})$, where h and $\widehat{h}$ denote two consecutive meshsizes with errors e and $\widehat{\mathsf{e}}$, respectively. We emphasize that the magnetic induction field $\boldsymbol{\mathcal{B}}$ can be computed by the postprocessing procedure detailed in (7.1). However, to avoid overloading this section, in the examples below we present only plots of this unknown.

The examples to be considered in this section are described next. In the first two examples, for the sake of simplicity, we take $\mu = 1$, $\rho = 1$, $\mu_0 = 1$, $\phi_0 = 1$, $M_s = 1$, $m = 1$, $\kappa = 1$, and $T = 1$. In addition, the zero mean values of $\mathrm{tr}(\boldsymbol{\sigma}_h)$ and ϕ over Ω are imposed using real Lagrange multipliers.

Example 1: Convergence against smooth exact solutions in a 2D domain

In this test, we corroborate the convergence rates on a two-dimensional domain and study the behaviour of the error $\mathsf{e}(\boldsymbol{\mathcal{M}})$ for different values of the relaxation time parameter ε. The computational domain is the unit square $\Omega = (0,1)^2$. We choose the parameters $r = 4$ and $\varrho = 4$. The initial meshes are shown in Figure 8.1, whereas the manufactured solution is given by

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \qquad \begin{aligned} p_H &= \cos(\pi x_1)\,\exp(x_2)\,, \\ \phi &= \exp(x_1 + x_2) - c_0\,, \end{aligned}$$

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \exp(x_1 + x_2) + \sin(x_1)\cos(x_2) \\ \exp(x_1 + x_2) - \cos(x_1)\sin(x_2) \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\mathcal{H}}_0 = \begin{pmatrix} x_2^2 \\ x_1^2 \end{pmatrix}.$$

Notice that both the magnetic potential ϕ and the total pressure $p = p_M + p_H - p_0$, computed as in (2.7), belong to $\mathrm{L}_0^2(\Omega)$, with $c_0 = (\exp(1) - 1)^2$ and $p_0 = \frac{\mu_0}{4}\,(\exp(2) - 1)^2$. The model problem is then complemented with the appropriate right-hand side and boundary conditions in accordance with (2.3), (2.13), and (2.22). In turn, the Shliomis equations (2.14) and (2.23) are adjusted to the

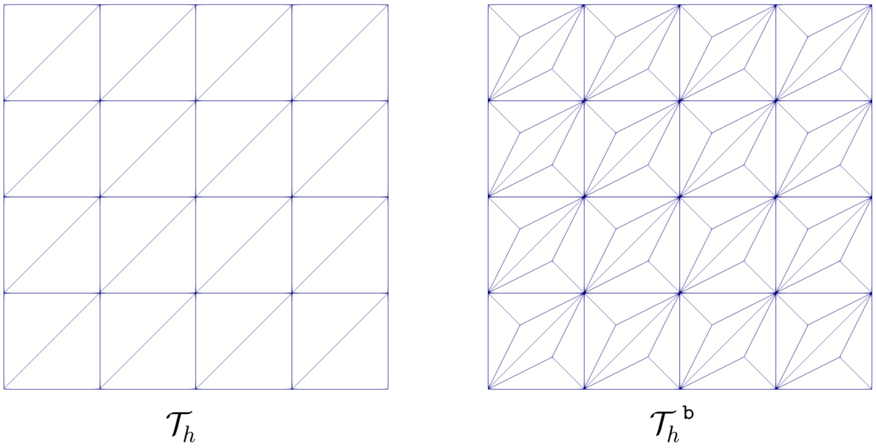
$\mathcal{T}_h$ $\mathcal{T}_h^{\mathsf{b}}$

Figure 8.1: [Example 1] Left: initial quasi-uniform mesh, right: initial barycentric mesh.

current two-dimensional setting, and a source term is added so that the aforementioned manufactured solutions satisfy the corresponding equations.

Table 8.1 illustrates the effect of the relaxation time parameter ε on the approximation error of $\boldsymbol{\mathcal{M}}$ when using the first iterative method described in Section 7 on barycentric meshes with $k = 1$. The results obtained with $k = 2$, as well as those produced by the second iterative method for $k \in \{0, 1\}$, are nearly identical or even slightly better, and are therefore omitted for the sake of brevity. We observe optimal convergence rates for all values of ε across all meshes. In particular, the most demanding case corresponds to $\varepsilon = 1\text{E}{-02}$, whereas smaller values of ε lead to an easier recovery of the optimal convergence behaviour. Moreover, the number of iterations remains constant and equal to 11 for all mesh sizes and all values of ε considered. Accordingly, Table 8.2 reports the convergence history for sequences of barycentric mesh refinements in the case $\varepsilon = 1\text{E}{-02}$, together with the corresponding iteration counts. In turn, Tables 8.3–8.4 present the corresponding results obtained with the second approach on sequences of quasi-uniform mesh refinements for $\varepsilon = 1\text{E}{-01}$, showing that this approach is slightly superior to the first one, in the sense that smaller values of ε can be considered with fewer degrees of freedom. As already announced, we emphasize that both approaches allow us not only to approximate the original unknowns, but also to recover the pressure field through the formulae (2.10) and (2.19), and the magnetic field and the magnetization of the fluid through (7.1) and (7.2)–(7.3), respectively. The results confirm that the optimal convergence rates $\mathcal{O}(h^{k+1})$ predicted by Theorems 4.12, 5.2, and 6.4 are attained for $k \in \{1, 2\}$ and $k \in \{0, 1\}$, respectively, for both the $\mathbf{P}_k - \mathbb{P}_k - \mathbb{RT}_k - \mathbf{RT}_k - \mathrm{P}_k$ and the $\big\{\mathrm{PEERS}_k, \mathrm{AFW}_k\big\} - \mathbf{RT}_k - \mathrm{P}_k$ discrete schemes. We emphasize that, for $k = 1$, the first iterative approach is more expensive than the second in terms of degrees of freedom, as it requires nearly twice as many degrees of freedom on the finest mesh. Accordingly, in the following examples we focus on the second iterative approach. In Figure 8.2 we display some solutions obtained with the mixed $\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathbf{RT}_2 - \mathrm{P}_2$ approximation on barycentric meshes with mesh size $h = 0.0236$ and $21,600$ triangular elements (corresponding to $1,459,082$ `DOF`).

Example 2: Convergence in convex and non-convex 3D domains

In the second example, we study the rates of convergence in three-dimensional convex and non-convex domains. Specifically, we consider the domains $\Omega_{\mathrm{C}} = (0,1)^3$ and $\Omega_{\mathrm{NC}} = (0,1)^3 \setminus (0.5,1)^3$, with relaxation time parameter $\varepsilon = 1\text{E}{-03}$, and the indexes $r = 3$ and $\varrho = 6$. The same manufactured

	$\varepsilon = 1\text{E}{-}02$		$\varepsilon = 1\text{E}{-}03$		$\varepsilon = 1\text{E}{-}04$		$\varepsilon = 1\text{E}{-}05$		$\varepsilon = 1\text{E}{-}06$	
h	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)
0.3536	3.7E-02	–	3.0E-02	–	3.0E-02	–	3.0E-02	–	3.0E-02	–
0.1768	1.5E-02	1.324	7.7E-03	1.980	7.6E-03	1.988	7.6E-03	1.988	7.6E-03	1.988
0.1010	7.1E-03	1.333	2.5E-03	1.981	2.5E-03	1.996	2.5E-03	1.997	2.5E-03	1.997
0.0643	3.6E-03	1.475	1.0E-03	1.979	1.0E-03	1.999	1.0E-03	1.999	1.0E-03	1.999
0.0393	1.6E-03	1.615	3.9E-04	1.979	3.8E-04	1.999	3.8E-04	2.000	3.8E-04	2.000
0.0236	6.7E-04	1.739	1.4E-04	1.981	1.4E-04	2.000	1.4E-04	2.000	1.4E-04	2.000

Table 8.1: [Example 1, $k = 1$] Meshsizes, errors, and rates of convergence for the first fully mixed approach on barycentric meshes for different values of ε.

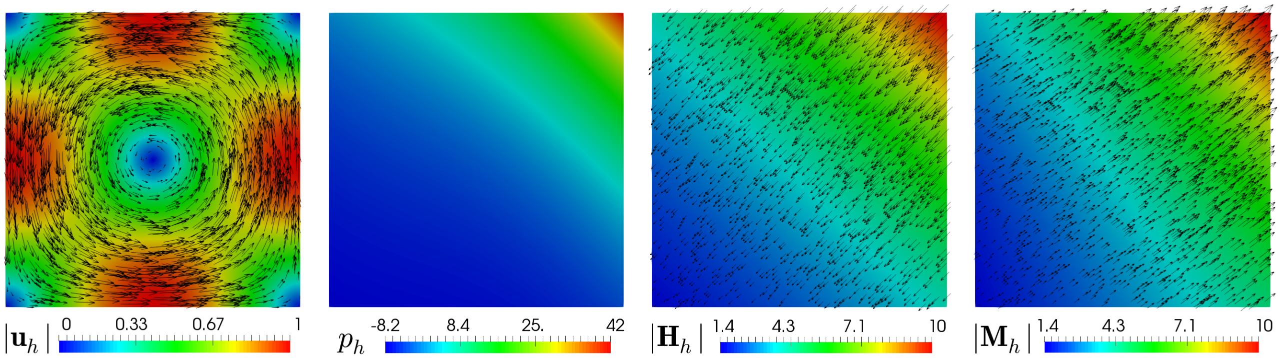

Figure 8.2: [Example 1] Computed velocity magnitude, pressure field, and magnetic field and fluid magnetisation magnitudes with $\varepsilon = 1\text{E}{-}02$.

solution is used in both domains and is given by

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix}, \qquad \begin{aligned} p_H &= \cos(\pi x_1)\exp(x_2 + x_3)\,, \\ \phi &= \exp(x_1 + x_2 + x_3) - c_0\,, \end{aligned}$$

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \exp(x_1 + x_2 + x_3) + \sin(x_1)\cos(x_2)\cos(x_3) \\ \exp(x_1 + x_2 + x_3) - 2\,\cos(x_1)\sin(x_2)\cos(x_3) \\ \exp(x_1 + x_2 + x_3) + \cos(x_1)\cos(x_2)\sin(x_3) \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\mathcal{H}}_0 = \begin{pmatrix} x_3^2 \\ x_1^2 \\ x_2^2 \end{pmatrix},$$

where, both the magnetic potential ϕ and the total pressure $p = p_M + p_H - p_0$ belong to $\mathrm{L}_0^2(\Omega)$, with $c_0 = (\exp(1) - 1)^3$ and $\frac{3}{16}\mu_0\,(\exp(2) - 1)^3$. Similarly to the first example, the right-hand side, the Shliomis relationships (2.14) and (2.23), and the boundary conditions are adjusted according to (2.3), (2.13), and (2.22) using the above exact solution. The convergence history for a set of quasi-uniform mesh refinements using $k = 0$ in the second iterative method is shown in Tables 8.5 and 8.6. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$, both in convex and non-convex 3D domains, as it was proved by Theorems 5.2 and 6.4. We observe a considerable increasing of degrees of freedom in the PEERS$_0$-based scheme compared to the AFW$_0$ one. This is justified mainly by the fact that the symmetric part of the velocity gradient is approximated with $\mathbb{P}_3(\Omega)$ and $\mathbb{P}_1(\Omega)$, respectively. The latter and for the sake of simplicity we only show results for the AFW$_0$ approach in Table 8.6. In addition, some components of the numerical solution are displayed in Figures 8.3 and 8.4 for the convex and non-convex domains, respectively. These results were obtained

$\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{RT}_1 - \mathrm{P}_1$ approximation on barycentric meshes

DOF	h	iter	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(p)	r(p)
3,218	0.3536	11	3.6E-02	–	5.0E-01	–	1.2E+00	–	3.9E-01	–
12,770	0.1768	11	8.9E-03	2.011	2.1E-01	1.232	3.4E-01	1.771	1.2E-01	1.719
38,978	0.1010	11	2.8E-03	2.081	9.0E-02	1.536	1.2E-01	1.836	4.3E-02	1.790
960,98	0.0643	11	1.1E-03	2.063	4.2E-02	1.693	5.2E-02	1.884	1.9E-02	1.843
257,042	0.0393	11	4.0E-04	2.034	1.7E-02	1.794	2.0E-02	1.920	7.4E-03	1.886
713,522	0.0236	11	1.4E-04	2.014	6.7E-03	1.868	7.4E-03	1.948	2.8E-03	1.923

e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{H}}$)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)
1.5E-02	–	2.0E-02	–	3.8E-02	–	3.7E-02	–	4.8E-02	–
4.7E-03	1.621	5.0E-03	1.985	1.4E-02	1.447	1.5E-02	1.324	1.8E-02	1.415
2.0E-03	1.556	1.6E-03	1.997	6.6E-03	1.346	7.1E-03	1.333	7.8E-03	1.485
9.5E-04	1.604	6.6E-04	2.000	3.4E-03	1.467	3.6E-03	1.475	3.8E-03	1.601
4.1E-04	1.692	2.5E-04	2.000	1.5E-03	1.607	1.6E-03	1.615	1.6E-03	1.709
1.7E-04	1.783	8.9E-05	2.000	6.3E-04	1.734	6.7E-04	1.739	6.5E-04	1.803

$\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathbf{RT}_2 - \mathrm{P}_2$ approximation on barycentric meshes

DOF	h	iter	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(p)	r(p)
6,554	0.3536	11	3.9E-03	–	3.2E-02	–	8.0E-02	–	2.6E-02	–
26,066	0.1768	11	5.0E-04	2.972	5.0E-03	2.676	1.0E-02	2.903	3.5E-03	2.879
79,634	0.1010	11	9.3E-05	2.993	1.0E-03	2.834	2.1E-03	2.950	6.8E-04	2.935
196,418	0.0643	11	2.4E-05	2.998	2.7E-04	2.903	5.4E-04	2.970	1.8E-04	2.959
525,530	0.0393	11	5.5E-06	2.999	6.4E-05	2.940	1.2E-04	2.981	4.1E-05	2.974
1,459,082	0.0236	11	1.2E-06	3.000	1.4E-05	2.964	2.7E-05	2.989	9.0E-06	2.984

e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{H}}$)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)
7.6E-04	–	6.7E-04	–	2.3E-03	–	2.5E-03	–	2.3E-03	–
1.4E-04	2.485	8.5E-05	2.981	4.3E-04	2.404	4.7E-04	2.412	4.1E-04	2.461
2.9E-05	2.752	1.6E-05	2.995	9.7E-05	2.672	1.1E-04	2.686	8.8E-05	2.742
8.0E-06	2.849	4.1E-06	2.998	2.7E-05	2.800	3.0E-05	2.809	2.4E-05	2.850
1.9E-06	2.903	9.3E-07	2.999	6.6E-06	2.873	7.2E-06	2.879	5.8E-06	2.906
4.3E-07	2.939	2.0E-07	2.982	1.5E-06	2.921	1.6E-06	2.925	1.3E-06	2.932

Table 8.2: [Example 1, $k \in \{1, 2\}$] Number of degrees of freedom, meshsizes, iteration count, errors, and rates of convergence for the first fully mixed approach on barycentric meshes with $\varepsilon = 1\mathrm{E}{-}02$.

using the mixed PEERS$_0$ and AFW$_0$ approximations, respectively, with mesh sizes $h = 0.0962/0.0961$ and $34,992/42540$ tetrahedral elements, corresponding to $6,361,907/2,535,332$ degrees of freedom.

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PEERS$_0$ – $\mathbf{RT}_0$ – P$_0$ approximation on quasi-uniform meshes

`DOF`	h	`iter`	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
643	0.3536	14	1.6E+00	–	1.1E+01	–	2.8E-01	–	6.5E-01	–
2,499	0.1768	14	8.7E-01	0.845	5.8E+00	0.959	1.3E-01	1.118	2.7E-01	1.272
7,563	0.1010	14	5.2E-01	0.918	3.3E+00	0.999	6.9E-02	1.087	1.3E-01	1.369
18,571	0.0643	14	3.4E-01	0.956	2.1E+00	1.009	4.3E-02	1.044	6.4E-02	1.468
49,539	0.0393	15	2.1E-01	0.976	1.3E+00	1.009	2.6E-02	1.020	3.0E-02	1.528
137,283	0.0236	17	1.3E-01	0.988	7.7E-01	1.007	1.6E-02	1.008	1.4E-02	1.566

e(p)	r(p)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)
2.9E+00	–	2.0E-01	–	4.6E-01	–	7.1E-01	–	7.2E-01	–
1.5E+00	0.983	1.0E-01	0.958	2.3E-01	0.988	3.6E-01	0.970	3.9E-01	0.903
8.2E-01	1.054	5.7E-02	1.006	1.3E-01	0.998	2.1E-01	1.013	2.2E-01	0.993
5.1E-01	1.060	3.6E-02	1.029	8.5E-02	1.000	1.3E-01	1.035	1.4E-01	1.038
3.0E-01	1.048	2.2E-02	1.037	5.2E-02	1.001	7.7E-02	1.040	8.2E-02	1.058
1.8E-01	1.031	1.3E-02	1.039	3.1E-02	1.000	4.5E-02	1.034	4.8E-02	1.065

AFW$_0$ – $\mathbf{RT}_0$ – P$_0$ approximation on quasi-uniform meshes

`DOF`	h	`iter`	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
698	0.3536	14	4.6E-01	–	8.1E+00	–	2.3E-01	–	5.9E-01	–
2,706	0.1768	14	2.2E-01	1.053	4.0E+00	1.012	1.2E-01	0.966	3.0E-01	0.988
8,178	0.1010	14	1.3E-01	1.019	2.3E+00	1.005	6.7E-02	0.990	1.7E-01	0.996
20,066	0.0643	14	8.0E-02	1.007	1.5E+00	1.002	4.3E-02	0.997	1.1E-01	0.998
53,498	0.0393	14	4.9E-02	1.003	8.9E-01	1.001	2.6E-02	0.999	6.6E-02	0.999
148,202	0.0236	14	2.9E-02	1.001	5.3E-01	1.000	1.6E-02	0.999	4.0E-02	1.000

e(p)	r(p)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)
2.6E+00	–	1.8E-01	–	4.6E-01	–	6.6E-01	–	6.3E-01	–
1.3E+00	0.998	8.8E-02	0.997	2.3E-01	0.989	3.3E-01	0.993	3.2E-01	0.990
7.4E-01	1.000	5.1E-02	1.001	1.3E-01	0.997	1.9E-01	0.998	1.8E-01	0.999
4.7E-01	1.000	3.2E-02	1.001	8.5E-02	0.999	1.2E-01	0.999	1.2E-01	1.000
2.9E-01	1.000	2.0E-02	1.000	5.2E-02	1.000	7.3E-02	1.000	7.1E-02	1.000
1.7E-01	1.000	1.2E-02	1.000	3.1E-02	1.000	4.4E-02	1.000	4.3E-02	1.000

Table 8.3: [Example 1, $k = 0$] Number of degrees of freedom, meshsizes, iteration count, errors, and rates of convergence for the second fully mixed approach on quasi-uniform meshes with $\varepsilon = 1\text{E}{-}01$.

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PEERS$_1$ – $\mathbf{RT}_1$ – P$_1$ approximation on quasi-uniform meshes

DOF	h	iter	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
1,667	0.3536	14	1.5E-01	–	1.3E+00	–	4.1E-02	–	6.3E-02	–
6,531	0.1768	14	4.4E-02	1.734	3.3E-01	1.945	1.1E-02	1.968	2.3E-02	1.430
19,827	0.1010	14	1.6E-02	1.830	1.1E-01	1.968	3.5E-03	1.992	9.5E-03	1.607
48,755	0.0643	14	6.7E-03	1.888	4.5E-02	1.980	1.4E-03	1.997	4.3E-03	1.745
130,179	0.0393	14	2.6E-03	1.926	1.7E-02	1.987	5.2E-04	1.999	1.7E-03	1.834
360,963	0.0236	14	9.6E-04	1.954	6.1E-03	1.992	1.9E-04	2.000	6.6E-04	1.897

e(p)	r(p)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)
3.0E-01	–	1.4E-02	–	2.6E-02	–	4.3E-02	–	4.5E-02	–
7.7E-02	1.977	3.7E-03	1.937	6.7E-03	1.974	1.3E-02	1.743	1.3E-02	1.773
2.5E-02	1.994	1.3E-03	1.937	2.2E-03	1.991	5.2E-03	1.625	4.7E-03	1.815
1.0E-02	1.997	5.2E-04	1.945	9.0E-04	1.997	2.5E-03	1.644	2.0E-03	1.875
3.8E-03	1.996	2.0E-04	1.950	3.3E-04	1.999	1.1E-03	1.711	7.8E-04	1.945
1.4E-03	1.997	7.4E-05	1.943	1.2E-04	1.999	4.2E-04	1.796	2.8E-04	2.011

AFW$_1$ – $\mathbf{RT}_1$ – P$_1$ approximation on quasi-uniform meshes

DOF	h	iter	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
1,666	0.3536	14	6.3E-02	–	9.6E-01	–	4.1E-02	–	8.8E-02	–
6,530	0.1768	14	1.5E-02	2.069	2.4E-01	2.004	1.1E-02	1.958	2.3E-02	1.968
19,826	0.1010	14	4.8E-03	2.045	7.8E-02	2.004	3.5E-03	1.988	7.4E-03	1.984
48,754	0.0643	14	1.9E-03	2.027	3.2E-02	2.003	1.4E-03	1.996	3.0E-03	1.991
130,178	0.0393	14	7.1E-04	2.016	1.2E-02	2.002	5.2E-04	1.998	1.1E-03	1.995
360,962	0.0236	14	2.5E-04	2.010	4.2E-03	2.001	1.9E-04	1.999	4.1E-04	1.997

e(p)	r(p)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)
2.9E-01	–	1.3E-02	–	2.7E-02	–	4.0E-02	–	3.8E-02	–
7.3E-02	1.980	3.2E-03	2.022	6.8E-03	1.982	1.0E-02	2.000	9.1E-03	2.058
2.4E-02	1.994	1.1E-03	2.014	2.2E-03	1.995	3.3E-03	2.001	2.9E-03	2.033
9.6E-03	1.997	4.2E-04	2.009	9.0E-04	1.998	1.3E-03	2.001	1.2E-03	2.022
3.6E-03	1.999	1.6E-04	2.004	3.4E-04	1.999	5.0E-04	2.001	4.3E-04	2.014
1.3E-03	1.999	5.8E-05	1.968	1.2E-04	2.000	1.8E-04	2.000	1.6E-04	2.008

Table 8.4: [Example 1, $k = 1$] Number of degrees of freedom, meshsizes, iteration count, errors, and rates of convergence for the second fully mixed approach on quasi-uniform meshes with $\varepsilon = 1\text{E}{-01}$.

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$\mathrm{PEERS}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation on quasi-uniform meshes

`DOF`	h	`iter`	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(γ)	r(γ)	e(p)	r(p)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}$)
8,867	0.8660	9	1.4E+01	–	1.5E+02	–	2.5E+00	–	1.0E+01	–	4.0E+01	–	3.5E-01	–	1.8E+00	–	3.1E+00	–	2.9E+00	–
70,265	0.4330	9	9.3E+00	0.567	8.2E+01	0.855	1.2E+00	1.076	4.7E+00	1.075	2.2E+01	0.881	1.7E-01	0.993	9.1E-01	0.964	1.6E+00	0.966	1.5E+00	0.983
559,757	0.2165	9	5.4E+00	0.782	4.2E+01	0.962	4.6E-01	1.386	1.9E+00	1.293	1.1E+01	1.004	8.7E-02	1.002	4.6E-01	0.991	7.9E-01	0.991	7.5E-01	0.991
1,886,657	0.1443	9	3.8E+00	0.881	2.8E+01	1.012	2.4E-01	1.571	1.1E+00	1.435	7.0E+00	1.082	5.8E-02	1.004	3.0E-01	0.997	5.3E-01	0.997	5.0E-01	1.001
6,361,907	0.0962	9	2.6E+00	0.927	1.8E+01	1.024	1.3E-01	1.644	5.9E-01	1.513	4.5E+00	1.098	3.9E-02	1.004	2.0E-01	0.999	3.5E-01	0.999	3.3E-01	1.006

$\mathrm{AFW}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation on quasi-uniform meshes

`DOF`	h	`iter`	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(γ)	r(γ)	e(p)	r(p)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{H}}$)
3,074	0.8660	9	3.9E+00	–	1.1E+02	–	6.4E-01	–	2.7E+00	–	3.7E+01	–	3.5E-01	–	1.8E+00	–	3.1E+00	–	2.9E+00	–
23,618	0.4330	9	1.6E+00	1.293	5.0E+01	1.073	3.1E-01	1.054	1.2E+00	1.148	1.8E+01	1.021	1.7E-01	0.997	9.0E-01	0.964	1.6E+00	0.966	1.5E+00	0.993
185,090	0.2165	9	5.7E-01	1.490	2.4E+01	1.071	1.6E-01	0.984	5.3E-01	1.166	9.1E+00	1.010	8.7E-02	1.003	4.6E-01	0.991	7.9E-01	0.991	7.3E-01	1.001
620,354	0.1443	9	3.2E-01	1.383	1.6E+01	1.048	1.0E-01	0.991	3.4E-01	1.086	6.1E+00	1.005	5.8E-02	1.003	3.0E-01	0.997	5.3E-01	0.997	4.9E-01	1.002
2,083,970	0.0962	9	2.0E-01	1.253	1.0E+01	1.031	6.9E-02	0.995	2.3E-01	1.045	4.0E+00	1.002	3.9E-02	1.002	2.0E-01	0.999	3.5E-01	0.999	3.3E-01	1.002

Table 8.5: [Example 2, $k = 0$] Number of degrees of freedom, meshsizes, iteration count, errors, and rates of convergence for the second fully mixed approach in the convex 3D domain with $\varepsilon = 1\mathrm{E}{-}03$.

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AFW$_0$ − $\mathbf{RT}_0$ − P$_0$ approximation on quasi-uniform meshes										
`DOF`	h	`iter`	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
15,773	0.7075	9	1.1E+00	–	2.5E+01	–	3.5E-01	–	1.2E+00	–
31,471	0.4026	9	7.1E-01	0.726	1.9E+01	0.517	2.5E-01	0.558	8.0E-01	0.649
209,874	0.2121	9	3.5E-01	1.103	9.4E+00	1.074	1.3E-01	0.997	4.3E-01	0.959
655,925	0.1484	9	2.3E-01	1.195	6.5E+00	1.033	9.0E-02	1.105	2.9E-01	1.150
2,535,332	0.0961	9	1.4E-01	1.094	4.1E+00	1.085	5.8E-02	1.030	1.8E-01	1.061

e(p)	r(p)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\mathcal{M}}$)	r($\boldsymbol{\mathcal{M}}$)	e($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)	r($\boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{H}}$)
7.5E+00	–	2.2E-01	–	5.8E-01	–	1.0E+00	–	1.5E+00	–
6.1E+00	0.364	1.5E-01	0.626	4.3E-01	0.544	7.5E-01	0.536	1.0E+00	0.604
3.1E+00	1.037	8.0E-02	1.009	2.2E-01	1.014	3.9E-01	1.023	5.5E-01	0.975
2.2E+00	0.952	5.3E-02	1.131	1.6E-01	0.993	2.7E-01	0.992	3.7E-01	1.094
1.4E+00	1.096	3.4E-02	1.045	9.8E-02	1.081	1.7E-01	1.083	2.4E-01	1.047

Table 8.6: [Example 2, $k = 0$] Number of degrees of freedom, meshsizes, iteration count, errors, and rates of convergence for the second fully mixed approach in the non-convex 3D domain with $\varepsilon = 1\text{E}{-}03$.

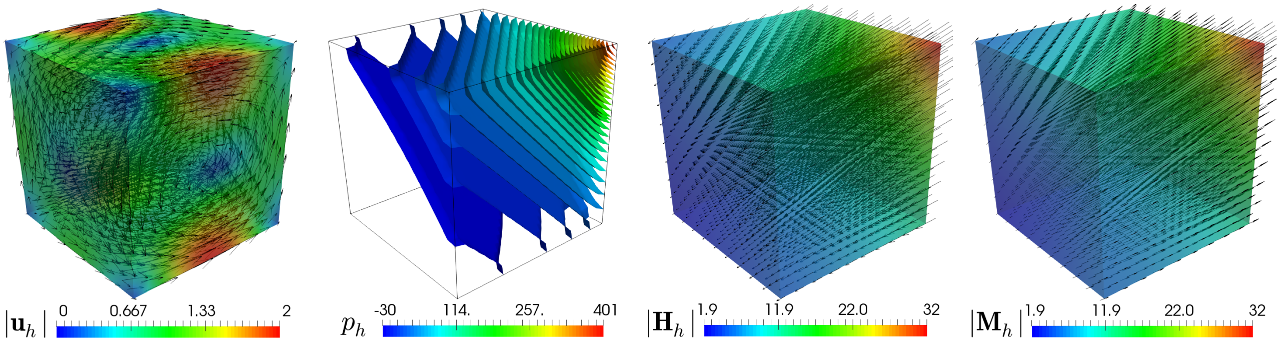

Figure 8.3: [Example 2] Computed velocity magnitude, pressure field, and magnetic field and fluid magnetisation magnitudes with $\varepsilon = 1\text{E}{-}03$.

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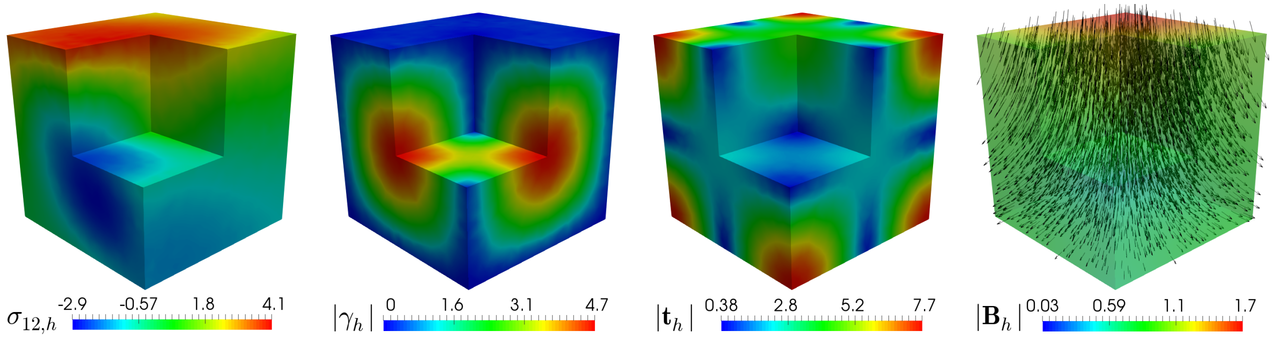

Figure 8.4: [Example 2] Computed stress magnetic tensor component, vorticity and strain rate tensor magnitudes, and magnetic induction with $\varepsilon = 1\text{E}{-}03$.

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