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A Lagrange multiplier approach for fluid-flow problem with inhomogeneous mixed boundary conditions *

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Abstract

In this work, we propose a Lagrange multiplier-based numerical scheme for fluid-flow problems with inhomogeneous mixed boundary conditions. More precisely, we consider the Stokes and Navier–Stokes equations in a bounded polyhedral domain subject to an inhomogeneous Dirichlet condition on a portion Γ_D of the boundary, together with a prescribed Neumann condition on the complementary portion Γ_N for the Stokes problem, and a directional do-nothing condition on Γ_N for the Navier–Stokes problem. The Dirichlet datum is imposed weakly by means of a Lagrange multiplier, which avoids the need for the discrete velocity space to conform exactly to the Dirichlet data. For the Stokes problem, we derive the corresponding mixed variational formulation and establish its well-posedness, for arbitrary finite element subspaces satisfying a set of abstract discrete inf-sup and stability hypotheses, via the classical Babuška–Brezzi theory. For the Navier–Stokes problem, the presence of the convective term requires a more delicate analysis: we identify an explicit correction of the convective trilinear form, involving the Dirichlet datum on Γ_D, and introduce an abstract hypothesis on the resulting discrete convective form guaranteeing coercivity. We prove well-posedness of both the Stokes and the Navier–Stokes discrete schemes and derive the corresponding Céa estimates, all stated in terms of the abstract hypotheses on the finite element subspaces. We then introduce concrete finite element subspaces satisfying these hypotheses and derive the corresponding theoretical rates of convergence. Finally, numerical experiments confirm the predicted convergence rates and illustrate the performance and robustness of the proposed method, including an application to a reverse osmosis membrane channel with spacer filaments in submerged and zigzag configurations.

Key words: Stokes; Navier–Stokes; inhomogeneous mixed boundary conditions; do-nothing condition; Lagrange multiplier; finite element method; numerical analysis; error estimates

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1 Introduction

The numerical resolution of fluid flow problems modeled by the Stokes and Navier–Stokes equations is a topic of major interest within the numerical analysis community, given the wide

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range of applications in which they arise, from classical hydrodynamics to membrane filtration, biomedical flows, and industrial processes (see, e.g., [18, 27, 22]). In most of these applications, the boundary conditions imposed on the flow play a decisive role in the fidelity of the model: no-slip conditions represent solid, impermeable walls, prescribed inflow and outflow velocities or pressures account for interaction with the surroundings, and do-nothing conditions are used to truncate otherwise unbounded regions. Several of these conditions must often be imposed simultaneously on complementary portions of the boundary, giving rise to mixed boundary value problems.

A standard technique for treating an inhomogeneous Dirichlet datum is to introduce a lifting, that is, an extension of the Dirichlet datum into the domain, and to seek the solution as the sum of this lifting and an unknown vanishing on the Dirichlet boundary, thereby reducing the problem to one with homogeneous boundary data. For the Stokes and Navier–Stokes equations, however, this lifting must itself be divergence-free, so as not to introduce a spurious forcing term into the incompressibility constraint; while such an extension can be constructed at the continuous level, no analogous construction is generally available at the discrete level, since standard finite element velocity spaces rarely contain a discretely divergence-free extension of an arbitrary boundary datum. Even when a discrete lifting can be explicitly constructed, doing so typically requires solving an auxiliary Stokes problem for each admissible datum, as illustrated in the closely related reverse osmosis model of [1], where the permeation velocity depends nonlinearly on the solute concentration, so that the lifting must be reconstructed at every step of the underlying fixed-point iteration. A similar difficulty arises even in the case of a single Dirichlet boundary, i.e., when the Dirichlet condition is prescribed on the entire boundary. In this direction, recently, Jeßberger and Kaltenbach [21] constructed an explicit discrete divergence-preserving lifting for the generalized Navier–Stokes equations with inhomogeneous Dirichlet and divergence data, for standard inf-sup stable finite element pairs, though at the cost of a smallness condition on the data for the existence of discrete solutions in the shear-thinning and Newtonian regimes.

An alternative approach to treat inhomogeneous Dirichlet data on part of the boundary, combined with Neumann or do-nothing conditions on the remainder, relies on Nitsche-type penalty formulations, following the generalized Nitsche approach of Juntunen and Stenberg [23] for Robin-type conditions, later extended to genuinely partitioned boundaries (see, e.g., [3]). These formulations impose the essential condition only asymptotically and require a mesh-dependent stabilization parameter; moreover, their rigorous error analysis is typically restricted to the homogeneous case, with inhomogeneous data handled only at the discrete or numerical level via a lifting [3].

Another approach is the Lagrange multiplier framework, where, unlike the approaches described above, the Dirichlet datum is enforced exactly through a saddle-point condition rather than penalized or lifted. Within this framework, however, the mixed case has received comparatively little attention at the discrete level. The closest related works, discussed in detail below, are [19] and [9], both of which prescribe the Dirichlet datum on the entire boundary and therefore do not encounter the difficulties inherent to the mixed case. The mixed case is treated at the continuous level in [25] to derive well-posedness and first-order optimality conditions for a boundary control problem, but only the wall boundary carries an inhomogeneous datum, and at the discrete level the multiplier is not approximated, a standard lifting-based scheme being used instead; the gap concerning a genuinely discretized multiplier for the mixed case therefore persists. A related use of multipliers in a mixed-boundary setting appears in [5], where a discrete multiplier enforces a no-tangential-flow condition on part of the boundary while a pressure datum is prescribed on the same portion; since the Dirichlet portion of the boundary remains homogeneous there, the difficulties associated with a genuinely inhomogeneous, partially prescribed velocity trace do not arise. This gap is not incidental: since the Dirichlet datum is only prescribed on part of the boundary, it can no longer be handled through the trace of the full boundary as a whole, but requires a more delicate treatment of the traces on each boundary portion and of how they relate to one another (cf. Section 1.1), which determines the space in which the multiplier is sought, the duality pairing enforcing the datum on its portion of the boundary alone, and, ultimately, the discrete inf–sup condition required of the multiplier finite element space.

Gunzburger and Hou [19] developed such a framework for the Stokes and Navier–Stokes equations, identifying the multiplier with the boundary traction; building on the coupled approach of Babuška [2], they took the multiplier space to be the trace on the boundary of the velocity finite element space itself, so that the boundary stress uncouples from the velocity and pressure and is obtained through an inexpensive postprocessing step, and derived optimal error estimates, including for the boundary stress, under minimal regularity assumptions on the data. More recently, Chrysafinos and Hou [9] extended this coupled approach to the evolutionary Stokes equations with inhomogeneous boundary and divergence data, developing a parabolic analog of the classical saddle-point theory together with a decoupled multiplier scheme analogous to the one above. Both works, however, restrict their analysis to a convex domain Ω with the Dirichlet condition prescribed on the entire boundary.

According to the above discussion, in this paper we propose a numerical scheme for the Stokes and Navier–Stokes equations subject to an inhomogeneous Dirichlet condition on part of the boundary, together with a prescribed Neumann datum on the complementary portion for the Stokes problem, and a directional do-nothing condition on that same portion for the Navier–Stokes problem. Following the approach in [19], we introduce a Lagrange multiplier, defined as the trace of the boundary traction on the Dirichlet portion of the boundary and, following the idea in [17, Section 4.3], approximated on an independent triangulation thereof; we then enforce the Dirichlet datum through a saddle-point condition rather than through the discrete velocity space itself, which allows the primal finite element space to be chosen without conforming exactly to the prescribed datum. In contrast to [19, 9], our analysis does not require the domain to be convex.

The paper is organized as follows: in Section 2 we derive the mixed variational formulation of the Stokes problem and, via the classical Babuška–Brezzi theory, establish its well-posedness together with a Céa-type estimate, under a set of abstract hypotheses on the finite element subspaces. In Section 3 we extend the analysis to Navier–Stokes: we show that a suitable redundant correction term, arising from the fact that the velocity coincides with the prescribed datum on the Dirichlet boundary, restores coercivity of the convective term at the discrete level, and derive an abstract hypothesis on the discrete convective form under which well-posedness of the continuous and discrete nonlinear problems follows from a fixed-point strategy, together with a Strang-type Céa estimate (Lemma 3.7). In Section 4 we verify these hypotheses for the Hood–Taylor and MINI elements, with the multiplier approximated by discontinuous piecewise polynomials on an independent boundary triangulation; in contrast with the smallness condition on the data required in [21] for the existence of discrete solutions in the shear-thinning and Newtonian regimes, coercivity of the discrete convective form in our setting is achieved, for arbitrarily large data, by taking the multiplier mesh sufficiently fine, yielding explicit rates of convergence. Finally, in Section 5 we present numerical experiments confirming the predicted convergence rates via a manufactured solution, and illustrate the performance and robustness of the method on a reverse osmosis membrane channel with spacer filaments in submerged and zigzag configurations.

1.1 Preliminaries

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with polyhedral boundary Γ, and assume that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, with $\Gamma_1 \cap \Gamma_2 = \emptyset$. Throughout the paper we use standard notation for Sobolev spaces $\mathrm{W}^{m,p}(\Omega)$, where $p \in [1, \infty]$ and $m \geq 0$, and for the Lebesgue spaces $\mathrm{L}^p(\Omega) := \mathrm{W}^{0,p}(\Omega)$, equipped with the norms $\|\cdot\|_{\mathrm{W}^{m,p}(\Omega)}$ and $\|\cdot\|_{\mathrm{L}^p(\Omega)}$, respectively.

When $p = 2$, we write $\mathrm{H}^m(\Omega)$ instead of $\mathrm{W}^{m,2}(\Omega)$, and for simplicity, for $\mathrm{H}^m(\Omega)$, $\mathbf{H}^m(\Omega) := [\mathrm{H}^m(\Omega)]^2$, and $\mathbb{H}^m(\Omega) := [\mathrm{H}^m(\Omega)]^{2\times 2}$, the corresponding norms and seminorms will be denoted by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$.

Next, we let $\gamma_0 : \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Gamma)$ be the well-known trace operator, satisfying

$$\|\varphi\|_{0,\Gamma} \leq C_\Gamma \|\varphi\|_{1,\Omega} \quad \forall \varphi \in \mathrm{H}^1(\Omega), \tag{1.1}$$

with $C_\Gamma > 0$ and define the trace space of $\mathrm{H}^1(\Omega)$ as

$$\mathrm{H}^{1/2}(\Gamma) := \gamma_0(\mathrm{H}^1(\Omega)),$$

endowed with the norm

$$\|\psi\|_{1/2,\Gamma} := \inf \left\{ \|w\|_{1,\Omega} : w \in \mathrm{H}^1(\Omega),\ \gamma_0(w) = \psi \right\}. \tag{1.2}$$

In the sequel, the dual space of $\mathrm{H}^{1/2}(\Gamma)$ is denoted by $\mathrm{H}^{-1/2}(\Gamma)$ and we employ $\langle \cdot, \cdot \rangle_\Gamma$ to denote the duality pairing on $\mathrm{H}^{-1/2}(\Gamma) \times \mathrm{H}^{1/2}(\Gamma)$, which coincides with the $\mathrm{L}^2(\Gamma)$-inner product when applied to functions in $\mathrm{L}^2(\Gamma)$. The vector-valued counterparts are denoted by $\mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma)$, corresponding to $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$, respectively.

In what follows, we use a vector-valued version of $\tilde{\gamma}_0$, denoted by $\tilde{\boldsymbol{\gamma}}_{\mathbf{0}}$, which is defined component-wise by $\tilde{\gamma}_0$. Furthermore, by the Sobolev embedding $\mathrm{H}^{1/2}(\Gamma) \hookrightarrow \mathrm{L}^4(\Gamma)$, we have

$$\|\psi\|_{\mathrm{L}^4(\Gamma)} \leq \widehat{C}_\Gamma \|\psi\|_{1/2,\Gamma} \quad \forall \psi \in \mathrm{H}^{1/2}(\Gamma), \tag{1.3}$$

with $\widehat{C}_\Gamma > 0$.

We now recall some definitions and technical results on extension operators (see, e.g., [14, **?**]). To that end, we let

$$E_{0,\Gamma_2} : \mathrm{H}^{1/2}(\Gamma_2) \to \mathrm{L}^2(\Gamma)$$

be the extension operator given by

$$E_{0,\Gamma_2}(\xi) := \begin{cases} \xi & \text{on } \Gamma_2, \\ 0 & \text{on } \Gamma_1, \end{cases} \qquad \forall\, \xi \in \mathrm{H}^{1/2}(\Gamma_2).$$

We then introduce the space

$$\mathrm{H}^{1/2}_{00}(\Gamma_2) := \left\{ \xi \in \mathrm{H}^{1/2}(\Gamma_2) : E_{0,\Gamma_2}(\xi) \in \mathrm{H}^{1/2}(\Gamma) \right\},$$

endowed with the norm

$$\|\xi\|_{1/2,00,\Gamma_2} := \left\|E_{0,\Gamma_2}(\xi)\right\|_{1/2,\Gamma}, \tag{1.4}$$

where $\|\cdot\|_{1/2,\Gamma}$ is the norm defined in (1.2).

We also define the extension operator

$$E_{\Gamma_1} : \mathrm{H}^{1/2}(\Gamma_1) \longrightarrow \mathrm{H}^{1/2}(\Gamma)\,, \quad \xi \longmapsto E_{\Gamma_1}(\xi) := z|_\Gamma,$$

where $z \in \mathrm{H}^1(\Omega)$ is the unique solution of the boundary value problem

$$-\Delta z = 0 \text{ in } \Omega, \qquad z = \xi \text{ on } \Gamma_1, \qquad \nabla z \cdot \mathbf{n} = 0 \text{ on } \Gamma_2.$$

This operator satisfies

$$\|E_{\Gamma_1}(\xi)\|_{1/2,\Gamma} \leq C\,\|\xi\|_{1/2,\Gamma_1}, \tag{1.5}$$

where $C > 0$ is a constant independent of ξ. In addition, one can show (see [16, Lemma 2.2]) that for all $\varrho \in \mathrm{H}^{1/2}(\Gamma)$, there exist unique elements

$$\varrho_1 = \varrho|_{\Gamma_1} \in \mathrm{H}^{1/2}(\Gamma_1) \quad \text{and} \quad \varrho_2 := (\varrho - \varrho|_{\Gamma_1})|_{\Gamma_2} \in \mathrm{H}^{1/2}_{00}(\Gamma_2), \tag{1.6}$$

such that

$$\varrho = E_{\Gamma_1}(\varrho_1) + E_{0,\Gamma_2}(\varrho_2) \tag{1.7}$$

and

$$c_1(\|E_{\Gamma_1}(\varrho_1)\|_{1/2,\Gamma_1} + \|E_{0,\Gamma_2}(\varrho_2)\|_{1/2,00,\Gamma_2}) \leq \|\varrho\|_{1/2,\Gamma} \leq c_2(\|E_{\Gamma_1}(\varrho_1)\|_{1/2,\Gamma_1} + \|E_{0,\Gamma_2}(\varrho_2)\|_{1/2,00,\Gamma_2}), \tag{1.8}$$

with c_1, $c_2 > 0$ only depending on Ω. The latter implies that the following decomposition holds and is stable:

$$\mathrm{H}^{1/2}(\Gamma) = E_{\Gamma_1}(\mathrm{H}^{1/2}(\Gamma_1)) \oplus E_{0,\Gamma_2}(\mathrm{H}^{1/2}_{00}(\Gamma_2)).$$

Now, given $\xi \in \mathrm{H}^{-1/2}(\Gamma)$ and $g \in \mathrm{L}^2(\Gamma_2)$, we say that $\xi = g$ on Γ_2 in the weak sense if

$$\langle \xi, E_{0,\Gamma_2}(\phi)\rangle_\Gamma = \int_{\Gamma_2} g\,\phi, \qquad \forall\,\phi \in \mathrm{H}^{1/2}_{00}(\Gamma_2). \tag{1.9}$$

Using the preceding definition, we introduce the subspace of functionals in $\mathrm{H}^{-1/2}(\Gamma)$ that vanish on Γ_2 in the weak sense:

$$\mathrm{H}^{-1/2}_{\Gamma_2}(\Gamma) := \left\{\xi \in \mathrm{H}^{-1/2}(\Gamma) : \langle \xi, E_{0,\Gamma_2}(\phi)\rangle_\Gamma = 0, \quad \forall\,\phi \in \mathrm{H}^{1/2}_{00}(\Gamma_2)\right\}. \tag{1.10}$$

Notice that, given $\lambda \in \mathrm{H}^{-1/2}(\Gamma)$ satisfying (1.9), it admits the decomposition

$$\lambda = \lambda_1 + \lambda_2, \tag{1.11}$$

where $\lambda_2 \in \mathrm{H}^{-1/2}(\Gamma)$ is defined by

$$\langle \lambda_2, \phi\rangle_\Gamma := \int_{\Gamma_2} g\,\phi, \qquad \forall\,\phi \in \mathrm{H}^{1/2}(\Gamma),$$

and $\lambda_1 := \lambda - \lambda_2$. Observe that from (1.9) it follows that

$$\langle \lambda_1, E_{0,\Gamma_2}(\psi)\rangle_\Gamma = \langle \lambda - \lambda_2, E_{0,\Gamma_2}(\psi)\rangle_\Gamma = 0, \qquad \forall\, \psi \in \mathrm{H}^{1/2}_{00}(\Gamma_2),$$

and hence

$$\lambda_1 \in \mathrm{H}^{-1/2}_{\Gamma_2}(\Gamma).$$

In addition, the following identity holds:

$$\langle \lambda, \phi\rangle_\Gamma = \langle \lambda_1, \phi\rangle_\Gamma + \int_{\Gamma_2} g\,\phi, \qquad \forall\, \phi \in \mathrm{H}^{1/2}(\Gamma). \tag{1.12}$$

Finally, given $u \in \mathrm{H}^1(\Omega)$ and $\varphi \in \mathrm{H}^{1/2}(\Gamma_1)$, we say that $u = \varphi$ on Γ_1 if

$$\langle \xi, u\rangle_\Gamma = \langle \xi, E_{\Gamma_1}(\varphi)\rangle_\Gamma, \qquad \forall\, \xi \in \mathrm{H}^{-1/2}_{\Gamma_2}(\Gamma). \tag{1.13}$$

In particular, if $\varphi = 0$ on Γ_1, we deduce that (1.13) implies that $u \in \mathrm{H}^1_{\Gamma_1}(\Omega)$.

In what follows, $\mathbf{E}_{0,\Gamma_2}$ and $\mathbf{E}_{\Gamma_1}$ denote the vector-valued counterparts of E_{0,Γ_2} and E_{Γ_1}, respectively, acting component-wise on vector-valued functions, and let

$$\mathbf{H}^{-1/2}(\Gamma_2) := \left\{ \boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma) : \langle \boldsymbol{\xi}, \mathbf{E}_{0,\Gamma_2}(\boldsymbol{\varrho})\rangle_\Gamma = 0, \quad \forall \boldsymbol{\varrho} \in \mathbf{H}^{1/2}_{00}(\Gamma_2) \right\},$$

$$\mathrm{H}^1_{\Gamma_1}(\Omega) := \left\{ v \in \mathrm{H}^1(\Omega) : v|_{\Gamma_1} = 0 \right\} \quad \text{and} \quad \mathbf{H}^1_{\Gamma_1}(\Omega) := \left[\mathrm{H}^1_{\Gamma_1}(\Omega)\right]^2.$$

Throughout the paper, the symbol $\mathbf{0}$ stands for a generic null vector, and, when no ambiguity arises, $|\cdot|$ denotes the absolute value in $\mathbb{R}$ and the Euclidean norm in $\mathbb{R}^2$ or $\mathbb{R}^{2\times 2}$.

2 The Stokes problem with inhomogeneous mixed boundary conditions

We begin by considering the Stokes problem with mixed boundary conditions:

$$\begin{aligned} -\mathbf{div}\big(\nu\nabla\mathbf{u} - p\mathrm{I}\big) = \mathbf{f} \qquad &\text{in} \quad \Omega, \qquad \operatorname{div}\mathbf{u} = 0 \qquad \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{u}_D \qquad \text{on} \quad \Gamma_D, \qquad &\big(\nu\nabla\mathbf{u} - p\mathrm{I}\big)\mathbf{n} = \mathbf{g} \qquad \text{on} \quad \Gamma_N, \end{aligned} \tag{2.1}$$

where $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma_D)$ and $\mathbf{g} \in \mathbf{L}^2(\Gamma_N)$ are prescribed boundary data. The aim of this section is to develop a Lagrange multiplier formulation for (2.1) and to establish its well-posedness. To this end, we first derive the corresponding weak formulation and then analyze its solvability.

2.1 Weak formulation

To deduce the weak formulation associated to (2.1), we first proceed as in [19] and introduce the auxiliary function

$$\boldsymbol{\chi} := -(\nu\nabla\mathbf{u} - p\mathrm{I})\mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma).$$

Observe that the boundary condition on Γ_N can be expressed in terms of $\boldsymbol{\chi}$ as

$$\langle \boldsymbol{\chi}, \mathbf{E}_{0,\Gamma_N}(\boldsymbol{\phi})\rangle_\Gamma = -\int_{\Gamma_N} \mathbf{g}\cdot\boldsymbol{\phi}, \qquad \forall\, \boldsymbol{\phi} \in \mathbf{H}^{1/2}_{00}(\Gamma_N).$$

Therefore, by (1.12), it follows that

$$\langle \boldsymbol{\chi}, \boldsymbol{\phi} \rangle_\Gamma = \langle \boldsymbol{\lambda}, \boldsymbol{\phi} \rangle_\Gamma - \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\phi}, \qquad \forall\, \boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma), \tag{2.2}$$

with $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}_{\Gamma_N}(\Gamma)$.

Now, multiplying the first equation of (2.1) by a test function $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and integrating by parts, we obtain

$$\nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \operatorname{div} \mathbf{v} + \langle \boldsymbol{\chi}, \boldsymbol{\gamma}_0(\mathbf{v}) \rangle_\Gamma = \int_\Omega \mathbf{f} \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega),$$

which together with (2.2) implies:

$$\nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \operatorname{div} \mathbf{v} + \langle \boldsymbol{\lambda}, \boldsymbol{\gamma}_0(\mathbf{v}) \rangle_\Gamma = \int_\Omega \mathbf{f} \cdot \mathbf{v} \,+\, \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\gamma}_0(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega). \tag{2.3}$$

In turn, the incompressibility constraint, given by the second equation of (2.1), and the Dirichlet boundary condition $\mathbf{u} = \mathbf{u}_D$ on Γ_D are imposed weakly as

$$\int_\Omega q \operatorname{div} \mathbf{u} = 0 \quad \forall\, q \in \mathrm{L}^2(\Omega), \qquad \langle \boldsymbol{\xi}, \mathbf{u} \rangle_\Gamma = \langle \boldsymbol{\xi}, \mathbf{E}_{\Gamma_D}(\mathbf{u}_D) \rangle_\Gamma \quad \forall\, \boldsymbol{\xi} \in \mathbf{H}^{-1/2}_{\Gamma_N}(\Gamma), \tag{2.4}$$

where the latter identity follows from the definition (1.13).

In this way, from (2.3) and (2.4) we arrive at the following variational formulation for (2.1): Find $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times \mathrm{M}$, such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, (p, \boldsymbol{\lambda})) &= F(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \\ b(\mathbf{u}, (q, \boldsymbol{\xi})) &= G(q, \boldsymbol{\xi}) \quad \forall\, (q, \boldsymbol{\xi}) \in \mathrm{M}, \end{aligned} \tag{2.5}$$

where

$$\mathrm{M} := \mathrm{L}^2(\Omega) \times \mathbf{H}^{-1/2}_{\Gamma_N}(\Gamma),$$

is endowed with the norm $\|(q, \boldsymbol{\xi})\|_{\mathrm{M}} = \|q\|_{0,\Omega} + \|\boldsymbol{\xi}\|_{-1/2,\Gamma}$, a and b are given by

$$a(\mathbf{u}, \mathbf{v}) := \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}, \quad b(\mathbf{v}, (q, \boldsymbol{\xi})) := b_1(\mathbf{v}, q) + b_2(\mathbf{v}, \boldsymbol{\xi}), \tag{2.6}$$

with

$$b_1(\mathbf{v}, q) := -\int_\Omega q \operatorname{div} \mathbf{v} \quad \text{and} \quad b_2(\mathbf{v}, \boldsymbol{\xi}) := \langle \boldsymbol{\xi}, \boldsymbol{\gamma}_0(\mathbf{v}) \rangle_\Gamma\,,$$

and F and G are the functionals given by

$$F(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} \,+\, \int_{\Gamma_N} \mathbf{g} \cdot \boldsymbol{\gamma}_0(\mathbf{v}) \quad \text{and} \quad G(q, \boldsymbol{\xi}) := \langle \boldsymbol{\xi}, \mathbf{E}_{\Gamma_D}(\mathbf{u}_D) \rangle_\Gamma\,. \tag{2.7}$$

To establish the well-posedness of (2.5), we employ the classical Babuska-Brezzi theory. To do this, we first observe that the following estimates hold for a, b_1 and b_2:

$$\begin{aligned} |a(\mathbf{u}, \mathbf{v})| &\le \nu \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, && \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega), \\ |b_1(\mathbf{v}, q)| &\le C_{b,1} \|q\|_{0,\Omega} \|\mathbf{v}\|_{1,\Omega}, && \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \forall q \in \mathrm{L}^2(\Omega), \\ |b_2(\mathbf{v}, \boldsymbol{\xi})| &\le C_{b,2} \|\boldsymbol{\xi}\|_{-1/2,\Gamma} \|\mathbf{v}\|_{1,\Omega}, && \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \forall \boldsymbol{\xi} \in \mathbf{H}^{-1/2}_{\Gamma_N}(\Gamma). \end{aligned} \tag{2.8}$$

In addition, using the trace inequality and estimates (1.5) and (1.8), we readily obtain the following

$$\begin{aligned} |F(\mathbf{v})| &\leq C_{F,S}(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Gamma_N})\|\mathbf{v}\|_{1,\Omega}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \\ |G(q,\boldsymbol{\xi})| &\leq C_G\|\mathbf{u}_D\|_{1/2,\Gamma_D}\|\boldsymbol{\xi}\|_{-1/2,\Gamma}, \qquad \forall (q,\boldsymbol{\xi}) \in \mathrm{M}. \end{aligned} \tag{2.9}$$

The following lemma establishes the inf-sup estimate of b

Lemma 2.1 *There exists $\beta > 0$, such that:*

$$S := \sup_{\substack{\mathbf{v}\in\mathbf{H}^1(\Omega)\\ \mathbf{v}\neq\mathbf{0}}} \frac{|b(\mathbf{v},(q,\boldsymbol{\xi}))|}{\|\mathbf{v}\|_{1,\Omega}} \geq \beta(\|q\|_{0,\Omega} + \|\boldsymbol{\xi}\|_{-1/2,\Gamma}), \qquad \forall (q,\xi) \in \mathrm{M}. \tag{2.10}$$

Proof. Let $(q,\boldsymbol{\xi}) \in \mathrm{M}$. We first observe that, given $\mathbf{v} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$, it follows that $\boldsymbol{\gamma}_0(\mathbf{v}) \in \mathbf{H}^{1/2}_{00}(\Gamma_N)$, thus $b_2(\mathbf{v},\boldsymbol{\xi}) = \langle \boldsymbol{\xi}, \boldsymbol{\gamma}_0(\mathbf{v})\rangle_\Gamma = 0$. Then surjectivity of the operator

$$\mathrm{div}\, : \mathbf{H}^1_0(\Omega) \to \mathrm{L}^2_0(\Omega) := \left\{ q \in \mathrm{L}^2(\Omega) : \int_\Omega q = 0 \right\},$$

the orthogonal decomposition $\mathrm{L}^2(\Omega) = \mathrm{L}^2_0(\Omega) \oplus \mathbb{R}$ and the fact that $\mathbf{H}^1_{\Gamma_D}(\Omega) \subseteq \mathbf{H}^1(\Omega)$, imply

$$S \geq \sup_{\substack{\mathbf{v}\in\mathbf{H}^1_{\Gamma_D}(\Omega)\\ \mathbf{v}\neq\mathbf{0}}} \frac{|b_1(\mathbf{v},q)|}{\|\mathbf{v}\|_{1,\Omega}} \geq C_1\|q\|_{0,\Omega}, \qquad \forall q \in \mathrm{L}^2(\Omega), \tag{2.11}$$

with $C_1 > 0$ depending on Ω. In turn, applying the Cauchy-Schwarz inequality and proceeding analogously to [17, Section 2.4.4], we deduce that

$$S \geq \sup_{\substack{\mathbf{v}\in\mathbf{H}^1(\Omega)\\ \mathbf{v}\neq\mathbf{0}}} \frac{|b(\mathbf{v},(q,\boldsymbol{\xi}))|}{\|\mathbf{v}\|_{1,\Omega}} \geq \sup_{\substack{\mathbf{v}\in\mathbf{H}^1(\Omega)\\ \mathbf{v}\neq\mathbf{0}}} \frac{|\langle \boldsymbol{\xi}, \boldsymbol{\gamma}_0(\mathbf{v})\rangle_\Gamma|}{\|\mathbf{v}\|_{1,\Omega}} - C_3\|q\|_{0,\Omega} \geq C_2\|\boldsymbol{\xi}\|_{-1/2,\Gamma} - C_3\|q\|_{0,\Omega}, \tag{2.12}$$

with $C_2, C_3 > 0$. In this way, combining (2.11) and (2.12), we obtain

$$\left(1 + \frac{2C_3}{C_1}\right) S \geq C_2\|\boldsymbol{\xi}\|_{-1/2,\Gamma} + C_3\|q\|_{0,\Omega},$$

which clearly implies the result. □

Now, let $\mathbf{V}$ be the kernel of the bilinear form b, that is

$$\mathbf{V} := \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) : b(\mathbf{v},(q,\boldsymbol{\xi})) = 0 \quad \forall (q,\boldsymbol{\xi}) \in \mathrm{M} \right\}.$$

Given $\mathbf{v} \in \mathbf{V}$ we observe that, in particular, it satisfies

$$\langle \boldsymbol{\xi}, \boldsymbol{\gamma}_0(\mathbf{v})\rangle_\Gamma = 0 \quad \forall \boldsymbol{\xi} \in \mathbf{H}^{-1/2}_{\Gamma_N}(\Gamma),$$

which according to (1.13) implies that $\mathbf{v} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$. In this way, it is not difficult to see that $\mathbf{V}$ can be characterized as follows

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}^1_{\Gamma_D}(\Omega) \,:\, \mathrm{div}\,\mathbf{v} = 0 \quad \text{in} \quad \Omega \right\}. \tag{2.13}$$

In this way, owing to the Generalized Poincaré inequality (see e.g. [6, Lemma B.63]), we obtain that

$$a(\mathbf{v},\mathbf{v}) \geq \nu\alpha \|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.14}$$

with $\alpha > 0$ only depending on Ω.

Now we are in position of establishing the well-posedness of problem (2.5).

Theorem 2.2 *There exists a unique* $(\mathbf{u},(p,\boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times \mathrm{M}$*, solution to* (2.5)*. In addition, the solution satisfies*

$$\begin{aligned} \|\mathbf{u}\|_{1,\Omega} &\leq \frac{c_1}{\nu}(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Gamma_N}) + c_2\|\mathbf{u}_D\|_{1/2,\Gamma_D}, \\ \|(p,\boldsymbol{\lambda})\|_{\mathrm{M}} &\leq c_3(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Gamma_N}) + \nu c_4\|\mathbf{u}_D\|_{1/2,\Gamma_D}, \end{aligned} \tag{2.15}$$

with c_1, c_2, c_3, $c_4 > 0$ *depending on* α, β, $C_{F,S}$ *and* C_G.

Proof. The existence and uniqueness of the solution follow directly from (2.10), (2.14), and the Babuška–Brezzi theory; see, for instance, [15, Theorem 2.34]. □

2.2 Galerkin scheme

In this section we introduce the Galerkin scheme of the variational problem (2.5), analyze its solvability and provide the corresponding Cea's estimate.

We begin by taking arbitrary finite dimensional subspaces

$$\mathbf{H}_h \subseteq \mathbf{H}^1(\Omega), \quad \mathrm{Q}_h \subseteq \mathrm{L}^2(\Omega), \quad \boldsymbol{\Lambda}_h \subseteq \mathbf{H}_{\Gamma_N}^{-1/2}(\Gamma) \quad \text{and} \quad \mathrm{M}_h := \mathrm{Q}_h \times \boldsymbol{\Lambda}_h.$$

Hereafter, h stands for the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $d=2$) or tetrahedra K, (when $d=3$) of diameter h_K, that is $h := \max\{h_K : K \in \mathcal{T}_h\}$.

In this way, the Galerkin scheme associated to (2.5) reads: Find $(\mathbf{u}_h,(p_h,\boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times \mathrm{M}_h$, such that

$$\begin{aligned} a(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,(p_h,\boldsymbol{\lambda}_h)) &= F(\mathbf{v}_h) & \forall\, \mathbf{v} \in \mathbf{H}_h, \\ b(\mathbf{u}_h,(q,\boldsymbol{\xi}_h)) &= G(q_h,\boldsymbol{\xi}_h) & \forall\,(q_h,\boldsymbol{\xi}_h) \in \mathrm{M}_h. \end{aligned} \tag{2.16}$$

In what follows we assume that $\mathbf{H}_h$, Q_h and $\boldsymbol{\Lambda}_h$ satisfy the following hypotheses:

(**H.0**) $\mathcal{P}_0(\Omega) \subseteq \mathrm{Q}_h$, where $\mathcal{P}_0(\Omega)$ denotes the space of constant functions on Ω.

(**H.1**) $\boldsymbol{\xi}_h|_{\Gamma_D} \in \mathbf{L}^2(\Gamma_D)$, $\boldsymbol{\xi}_h|_{\Gamma_N} = 0$, $\forall \boldsymbol{\xi} \in \boldsymbol{\Lambda}_h$ and $\mathbf{e}_{\Gamma_D}^i \in \boldsymbol{\Lambda}_h$ for all $i \in \{1,\ldots,d\}$ where

$$\mathbf{e}_{\Gamma_D}^i = \begin{cases} \mathbf{e}^i, & \text{on } \Gamma_D, \\ \mathbf{0}, & \text{on } \Gamma_N, \end{cases}$$

with $\mathbf{e}^i$ denoting the i-th canonical basis vector of $\mathbb{R}^d$.

(**H.2**) There exists $\mathbf{v}_0 \in \mathbf{H}_{\Gamma_D}^1(\Omega)$, such that

$$\mathbf{v}_0 \in \mathbf{H}_{h,\Gamma_D} := \mathbf{H}_h \cap \mathbf{H}_{\Gamma_D}^1(\Omega) \quad \forall h > 0 \quad \text{and} \quad \int_{\Gamma_N} \mathbf{v}_0 \cdot \mathbf{n} \neq 0.$$

($\mathbf{H.3}$) There exists $\widetilde{\beta}_1 > 0$, independent of the discretization parameter h, such that

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \cap \mathbf{H}_0^1(\Omega) \\ \mathbf{v}_h \neq 0}} \frac{b_1(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \widetilde{\beta}_1 \|q_h\|_{0,\Omega}, \quad \forall q_h \in \mathrm{Q}_{h,0} := \mathrm{Q}_h \cap \mathrm{L}_0^2(\Omega). \tag{2.17}$$

($\mathbf{H.4}$) There exists $\widetilde{\beta}_2 > 0$, independent of the discretization parameter h, such that

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq 0}} \frac{b_2(\mathbf{v}_h, \boldsymbol{\xi}_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \widetilde{\beta}_2 \|\boldsymbol{\xi}_h\|_{-1/2,\Gamma}, \quad \forall \boldsymbol{\xi}_h \in \boldsymbol{\Lambda}_h. \tag{2.18}$$

Notice that ($\mathbf{H.0}$) implies the orthogonal decomposition

$$\mathrm{Q}_h = \mathrm{Q}_{h,0} \oplus \mathbb{R}. \tag{2.19}$$

Moreover, ($\mathbf{H.1}$) implies that the kernel of the bilinear form b, namely

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}_h : b(\mathbf{v}_h, (q_h, \boldsymbol{\xi}_h)) = 0, \quad \forall\, (q_h, \boldsymbol{\xi}_h) \in \mathrm{M}_h\}, \tag{2.20}$$

is contained in

$$\widetilde{\mathbf{H}}_h := \left\{ \mathbf{v}_h = (v_{h,1}, \ldots, v_{h,d}) \in \mathbf{H}_h : \int_{\Gamma_D} v_{h,i} = 0, \quad i = 1, \ldots, d \right\}. \tag{2.21}$$

Indeed, if $\mathbf{v}_h = (v_{h,1}, \ldots, v_{h,d}) \in \mathbf{V}_h$, then

$$b(\mathbf{v}_h, (0, \mathbf{e}^i_{\Gamma_D})) = \langle \mathbf{e}^i_{\Gamma_D}, \mathbf{v}_h \rangle_\Gamma = \int_{\Gamma_D} v_{h,i} = 0, \qquad i = 1, \ldots, d,$$

and therefore $\mathbf{V}_h \subseteq \widetilde{\mathbf{H}}_h$. Then, noticing that the Generalized Poincaré inequality (see e.g. [6, Lemma B.63]) implies that $|\cdot|_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$ are equivalent norms on $\widetilde{\mathbf{H}}_h$, it follows that the bilinear form a satisfies:

$$a(\mathbf{v}_h, \mathbf{v}_h) \geq \nu \tilde{\alpha} \|\mathbf{v}_h\|^2_{1,\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.22}$$

with $\tilde{\alpha} > 0$ only depending on Ω.

Hypotheses ($\mathbf{H.0}$) and ($\mathbf{H.2}$)–($\mathbf{H.4}$) imply that the bilinear form b satisfies a discrete inf–sup condition on $\mathbf{H}_h \times \mathrm{Q}_h \times \boldsymbol{\Lambda}_h$, which constitutes the discrete counterpart of (2.10). More precisely, the following result holds.

Lemma 2.3 *Assume that hypotheses* ($\mathbf{H.0}$), ($\mathbf{H.2}$), ($\mathbf{H.3}$) *and* ($\mathbf{H.4}$) *hold. Then, the following estimate holds:*

$$S_h := \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{|b(\mathbf{v}_h, (q_h, \boldsymbol{\xi}_h))|}{\|\mathbf{v}_h\|_{1,\Omega}} \gtrsim \widetilde{\beta} (\|q_h\|_{0,\Omega} + \|\boldsymbol{\xi}_h\|_{-1/2,\Gamma}), \quad \forall (q_h, \xi_h) \in \mathrm{M}_h. \tag{2.23}$$

Proof. Given $(q_h, \boldsymbol{\xi}_h) \in \mathrm{M}_h$, we first employ the decomposition (2.19) and write $q_h = q_{h,0} + \bar{q}_h$, where $q_{h,0} \in \mathrm{Q}_{h,0}$ and $\bar{q}_h \in \mathbb{R}$. Then, noticing that $b_1(\mathbf{v}_h, \bar{q}_h) = 0$ and $b_2(\mathbf{v}_h, \boldsymbol{\xi}_h) = 0$, for all $\mathbf{v}_h \in \mathbf{H}_h \cap \mathbf{H}_0^1(\Omega)$, from (2.17) we deduce that

$$S_h \geq \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \cap \mathbf{H}_0^1(\Omega) \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{|b_1(\mathbf{v}_h, q_{h,0})|}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \widetilde{\beta}_1 \|q_{h,0}\|_{0,\Omega}. \tag{2.24}$$

Now, we let $\mathbf{v}_0 \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ satisfying $(\mathbf{H.2})$ and observe that

$$S_h \geq \frac{|b_1(\mathbf{v}_0, q_h)|}{\|\mathbf{v}_0\|_{1,\Omega}} = \frac{\left|\displaystyle\int_\Omega q_{h,0} \mathrm{div}\, \mathbf{v}_0 + \bar{q}_h \int_{\Gamma_N} \mathbf{v}_0 \cdot \mathbf{n}\right|}{\|\mathbf{v}_0\|_{1,\Omega}} \geq c_0 \bar{q}_h - c_1 \|q_{h,0}\|_{0,\Omega} \tag{2.25}$$

with $c_0 := \dfrac{\left|\displaystyle\int_{\Gamma_N} \mathbf{v}_0 \cdot \mathbf{n}\right|}{\|\mathbf{v}_0\|_{1,\Omega}}$ and $c_1 > 0$ independent of h. Then, combining (2.24) and (2.25) we obtain

$$\left(1 + \frac{2c_1}{\widetilde{\beta}_1}\right) S_h \geq c_0 \bar{q}_h + c_1 \|q_{h,0}\|_{0,\Omega},$$

which implies that

$$S_h \geq c_2 \|q_h\|_{0,\Omega}, \tag{2.26}$$

with $c_2 > 0$, independent of h.

On the other hand, using $(\mathbf{H.4})$, we deduce that

$$S_h \geq \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{|b(\mathbf{v}_h, (q_h, \boldsymbol{\xi}_h))|}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{|b_2(\mathbf{v}_h, \boldsymbol{\xi}_h)|}{\|\mathbf{v}_h\|_{1,\Omega}} - c_3 \|q_h\|_{0,\Omega} \geq \widetilde{\beta}_2 \|\boldsymbol{\xi}_h\|_{-1/2,\Gamma} - c_3 \|q_h\|_{0,\Omega}, \tag{2.27}$$

with $c_3 > 0$, independent of h.

Therefore, proceeding as in the proof of Lemma 2.1 we combine (2.26) and (2.27) to arrive at (2.23), which concludes the proof. $\square$

Now we are in position of establishing the well-posedness of problem (2.16) and the corresponding Cea's estimate.

Theorem 2.4 *Assume that hypotheses* $(\mathbf{H.0})$–$(\mathbf{H.4})$ *hold. Then, there exists a unique* $(\mathbf{u}_h, p_h, \boldsymbol{\lambda}_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \boldsymbol{\Lambda}_h$*, solution to* (2.16)*. In addition, the solution satisfies*

$$\begin{aligned} \|\mathbf{u}_h\|_{1,\Omega} &\leq \frac{\tilde{c}_1}{\nu}(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Gamma_N}) + \tilde{c}_2 \|\mathbf{u}_D\|_{1/2,\Gamma_D}, \\ \|p_h\|_{0,\Omega} + \|\boldsymbol{\lambda}_h\|_{-1/2,\Gamma} &\leq \tilde{c}_3(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Gamma_N}) + \nu \tilde{c}_4 \|\mathbf{u}_D\|_{1/2,\Gamma_D}, \end{aligned} \tag{2.28}$$

with $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$, $\tilde{c}_4 > 0$ *depending on* $\tilde{\alpha}$, $\widetilde{\beta}$, $C_{F,S}$ *and* C_G*. Finally, if* $(\mathbf{u}, p, \boldsymbol{\lambda}) \in \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \times \mathbf{H}^{-1/2}_{\Gamma_N}(\Gamma)$ *is the unique solution of* (2.5)*, then*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} &\leq \tilde{c}_5 \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \frac{\tilde{c}_6}{\nu} \inf_{(q_h, \boldsymbol{\xi}_h) \in \mathrm{M}_h} \|(p - q_h, \boldsymbol{\lambda} - \boldsymbol{\xi}_h)\|_{\mathrm{M}}, \\ \|(p - p_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)\|_{\mathrm{M}} &\leq \tilde{c}_7 \nu \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \tilde{c}_8 \inf_{(q_h, \boldsymbol{\xi}_h) \in \mathrm{M}_h} \|(p - q_h, \boldsymbol{\lambda} - \boldsymbol{\xi}_h)\|_{\mathrm{M}}, \end{aligned} \tag{2.29}$$

with $\tilde{c}_5$, $\tilde{c}_6$, $\tilde{c}_7$, $\tilde{c}_8 > 0$ *depending on* $\tilde{\alpha}$, $\widetilde{\beta}$, $C_{b,1}$ *and* $C_{b,2}$.

Proof. The existence and uniqueness of the solution, as well as estimate (2.15), follow directly from (2.10), (2.14), and the Babuška–Brezzi theory; see [15, Theorem 2.34]. Moreover, estimate (2.29) follows from the classical Céa-type estimate for mixed finite element methods; see [15, Lemma 2.44]. $\square$

3 The Navier–Stokes problem with inohomgeneous Dirichlet and directional do-nothing boundary conditions

In this section, we turn to the numerical approximation of the incompressible Navier–Stokes equations subject to an inhomogeneous Dirichlet boundary condition on Γ_D and a *directional do-nothing condition* on Γ_N; see, e.g., [22, Section 2.4]. More precisely, given a body force $\mathbf{f}$ and a Dirichlet datum $\mathbf{u}_D$, we seek the velocity $\mathbf{u}$ and the pressure p satisfying:

$$
\begin{aligned}
-\mathbf{div}\big(\nu\nabla\mathbf{u} - p\mathrm{I}\big) + \rho\,(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} \qquad &\text{in} \quad \Omega, \qquad \operatorname{div}\mathbf{u} = 0 \qquad \text{in} \quad \Omega,\\
\mathbf{u} = \mathbf{u}_D \qquad \text{on} \quad \Gamma_D, \qquad &\big(\nu\nabla\mathbf{u} - p\mathrm{I}\big)\mathbf{n} = \frac{\rho}{2}(\mathbf{u}\cdot\mathbf{n})^{-}\mathbf{u} \qquad \text{on} \quad \Gamma_N.
\end{aligned}
\tag{3.1}
$$

Here, $(\mathbf{u}\cdot\mathbf{n})^-$ denotes the negative part of $(\mathbf{u}\cdot\mathbf{n})$. More generally, for any scalar function ϕ, we define its positive and negative parts by

$$
\phi^+ := \frac{1}{2}\big(\phi + |\phi|\big), \qquad \phi^- := \frac{1}{2}\big(\phi - |\phi|\big), \tag{3.2}
$$

so that

$$
\phi = \phi^+ + \phi^-.
$$

3.1 Weak formulation

Analogously to Section 2, we proceed as in [19] and introduce the auxiliary function

$$
\boldsymbol{\chi} := -(\nu\nabla\mathbf{u} - p\mathrm{I})\mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma).
$$

Assuming sufficient regularity of $\mathbf{u}$ so that $(\mathbf{u}\cdot\mathbf{n})^-\mathbf{u}|_{\Gamma_N} \in \mathbf{L}^2(\Gamma_N)$, the boundary condition on Γ_N can be expressed in terms of $\boldsymbol{\chi}$ as

$$
\langle \boldsymbol{\chi}, \mathbf{E}_{0,\Gamma_N}(\boldsymbol{\phi})\rangle_\Gamma = -\frac{\rho}{2}\int_{\Gamma_N} (\mathbf{u}\cdot\mathbf{n})^-\,\mathbf{u}\cdot\boldsymbol{\phi}, \qquad \forall\,\boldsymbol{\phi} \in \mathbf{H}^{1/2}_{00}(\Gamma_N).
$$

Therefore, by (1.12), we conclude that there exists $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}_{\Gamma_N}(\Gamma)$ such that

$$
\langle \boldsymbol{\chi}, \boldsymbol{\phi}\rangle_\Gamma = \langle \boldsymbol{\lambda}, \boldsymbol{\phi}\rangle_\Gamma - \frac{\rho}{2}\int_{\Gamma_N} (\mathbf{u}\cdot\mathbf{n})^-\,\mathbf{u}\cdot\boldsymbol{\phi}, \qquad \forall\,\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma). \tag{3.3}
$$

Then, multiplying the first equation of (3.1) by a test function $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and integrating by parts and employing (3.3), we obtain

$$
\nu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} + \rho\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \frac{\rho}{2}\int_{\Gamma_N} (\mathbf{u}\cdot\mathbf{n})^-\mathbf{u}\cdot\mathbf{v} - \int_\Omega p\operatorname{div}\mathbf{v} + \langle \boldsymbol{\lambda}, \mathbf{v}\rangle_\Gamma = \int_\Omega \mathbf{f}\cdot\mathbf{v} \quad \forall\,\mathbf{v} \in \mathbf{H}^1(\Omega). \tag{3.4}
$$

For technical reasons that will become clear in Section 4.1, and using the fact that $\mathbf{u} = \mathbf{u}_D$ on Γ_D, we introduce the following redundant term:

$$
-\frac{\rho}{2}\int_{\Gamma_D} (\mathbf{u}_D\cdot\mathbf{n})\,\mathbf{u}\cdot\mathbf{v} = -\frac{\rho}{2}\int_{\Gamma_D} (\mathbf{u}_D\cdot\mathbf{n})\,\mathbf{u}_D\cdot\mathbf{v}, \qquad \forall\,\mathbf{v} \in \mathbf{H}^1(\Omega). \tag{3.5}
$$

Finally, analogously to Section 2, the incompressibility constraint and the Dirichlet boundary condition on Γ_D are imposed weakly as

$$\int_\Omega q \operatorname{div} \mathbf{u} = 0 \quad \forall\, q \in \mathrm{L}^2(\Omega), \qquad \langle \boldsymbol{\xi}, \mathbf{u}\rangle_\Gamma = \langle \boldsymbol{\xi}, \mathbf{E}_{\Gamma_D}(\mathbf{u}_D)\rangle_\Gamma \quad \forall\, \boldsymbol{\xi} \in \mathbf{H}^{-1/2}_{\Gamma_N}(\Gamma). \tag{3.6}$$

In this way, from (3.4) and (3.6) we arrive at the following variational formulation for (3.1): Find $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times \mathrm{M}$, such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + O(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, (p, \boldsymbol{\lambda})) &= F_N(\mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \\ b(\mathbf{u}, (q, \boldsymbol{\xi})) &= G(q, \boldsymbol{\xi}), \quad \forall\, (q, \boldsymbol{\xi}) \in \mathrm{M}. \end{aligned} \tag{3.7}$$

Here, a and b denote the bilinear forms introduced in (2.6), while G is the linear functional defined in (2.7). Furthermore, the functional F and the nonlinear form O are given by

$$\begin{aligned} F_N(\mathbf{v}) :=&\ \int_\Omega \mathbf{f} \cdot \mathbf{v} - \frac{\rho}{2} \int_{\Gamma_D} (\mathbf{u}_D \cdot \mathbf{n}) \mathbf{u}_D \cdot \mathbf{v}, \\ O(\mathbf{w}; \mathbf{u}, \mathbf{v}) :=&\ \rho \int_\Omega (\mathbf{w} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} - \frac{\rho}{2} \int_{\Gamma_D} (\mathbf{u}_D \cdot \mathbf{n}) \mathbf{u} \cdot \mathbf{v} - \frac{\rho}{2} \int_{\Gamma_N} (\mathbf{w} \cdot \mathbf{n})^- \mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

We observe that, since $\mathbf{u} \in \mathbf{H}^1(\Omega)$, the trace theorem and the Sobolev embedding $\mathrm{H}^{1/2}(\Gamma) \hookrightarrow \mathrm{L}^4(\Gamma)$ imply that $\mathbf{u}|_\Gamma \in \mathbf{L}^4(\Gamma)$. Consequently, $(\mathbf{u} \cdot \mathbf{n})^- \,\mathbf{u}|_{\Gamma_N} \in \mathbf{L}^2(\Gamma_N)$.

3.2 Well-posedness

Now we apply a fixed point strategy to prove existence and uniqueness of solution of the variational formulation (3.7). To do this, in addition to the properties established in Section 2.1 for the bilinear forms a and b and the functional G, namely (2.8), (2.9) and (2.10), we observe that the nonlinear form O, satisfies

$$\begin{aligned} O(\mathbf{w}; \mathbf{v}, \mathbf{v}) =&\ \frac{\rho}{2} \left(-\int_\Omega |\mathbf{v}|^2 \operatorname{div}(\mathbf{w}) + \int_\Gamma (\mathbf{w} \cdot \mathbf{n}) |\mathbf{v}|^2 \right) - \frac{\rho}{2} \int_{\Gamma_D} (\mathbf{u}_D \cdot \mathbf{n}) |\mathbf{v}|^2 - \frac{\rho}{2} \int_{\Gamma_N} (\mathbf{w} \cdot \mathbf{n})^- |\mathbf{v}|^2, \\ =&\ \frac{\rho}{2} \left(-\int_\Omega |\mathbf{v}|^2 \operatorname{div}(\mathbf{w}) + \int_{\Gamma_D} ((\mathbf{w} - \mathbf{u}_D) \cdot \mathbf{n}) |\mathbf{v}|^2 \right) + \frac{\rho}{2} \int_{\Gamma_N} (\mathbf{w} \cdot \mathbf{n})^+ |\mathbf{v}|^2, \end{aligned} \tag{3.8}$$

for all $\mathbf{w}, \mathbf{v} \in \mathbf{H}^1(\Omega)$. In turn, estimates (1.1) and (1.3), the Sobolev embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^6(\Omega)$ (see [26, Theorem 1.3.4]) and the inequality $||t| - |s|| \le |t - s|$, imply

$$|O(\mathbf{w}; \mathbf{u}, \mathbf{v}) - O(\tilde{\mathbf{w}}; \mathbf{u}, \mathbf{v})| \le C_{O,1} \rho (\|\mathbf{w} - \tilde{\mathbf{w}}\|_{\mathbf{L}^3(\Omega)} + \|\mathbf{w} - \tilde{\mathbf{w}}\|_{\mathbf{L}^3(\Gamma)}) \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \tag{3.9}$$

for all $\mathbf{w}, \tilde{\mathbf{w}}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$, with $C_{O,1} > 0$ only depending on Ω. Moreover, according to the Sobolev embedding $\mathrm{H}^{1/2}(\Gamma) \hookrightarrow \mathrm{L}^3(\Gamma)$ (see [13, Theorem 1.1]) and using $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^3(\Omega)$, estimates (3.9) and (1.1) imply

$$|O(\mathbf{w}; \mathbf{u}, \mathbf{v}) - O(\tilde{\mathbf{w}}; \mathbf{u}, \mathbf{v})| \le C_{O,2} \rho \|\mathbf{w} - \tilde{\mathbf{w}}\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \tag{3.10}$$

with $C_{O,2} > 0$ only depending on Ω.

Regarding the functional F_N, applying the Cauchy-Schwarz inequality and estimate (1.1), we observe that

$$|F_N(\mathbf{v})| \le C_{F,N} (\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D}) \|\mathbf{v}\|_{1,\Omega}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{3.11}$$

with $C_{F,N} > 0$ only depending on Ω.

We next introduce the auxiliary operator

$$\mathcal{J} : \mathbf{H}^1(\Omega) \longrightarrow \mathbf{H}^1(\Omega), \qquad \mathbf{w} \longmapsto \mathcal{J}(\mathbf{w}) := \mathbf{u},$$

where $\mathbf{u} \in \mathbf{H}^1(\Omega)$ is the velocity component of the unique solution, whose existence will be established below in Lemma 3.1, of the following linearized problem: Find $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times \mathrm{M}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + O(\mathbf{w}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, (p, \boldsymbol{\lambda})) &= F_N(\mathbf{v}), \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \\ b(\mathbf{u}, (q, \boldsymbol{\xi})) &= G((q, \boldsymbol{\xi})), \quad \forall\, (q, \boldsymbol{\xi}) \in \mathrm{M}. \end{aligned} \tag{3.12}$$

Therefore, it is readily seen that the well-posedness analysis of (3.7) reduces to the study of the following fixed-point problem: Find $\mathbf{u} \in \mathbf{H}^1(\Omega)$ such that

$$\mathcal{J}(\mathbf{u}) = \mathbf{u}. \tag{3.13}$$

As a consequence, in what follows we focus on establishing the unique solvability of the fixed-point problem (3.13). To this end, we first observe that the fixed-point formulation is meaningful only if the operator $\mathcal{J}$ is well defined. Therefore, we begin by analyzing the well-posedness of problem (3.12), which in turn guarantees that $\mathcal{J}$ is well defined. This result is stated in the following lemma.

Lemma 3.1 *Let $\mathbf{w} \in \mathbf{H}^1(\Omega)$, such that $\operatorname{div} \mathbf{w} = 0$ in Ω. Then, there exists a unique solution $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times \mathrm{M}$ to problem (3.12). Moreover, the following estimate holds:*

$$\begin{aligned} \|\mathbf{u}\|_{1,\Omega} &\leq \frac{c_1}{\nu}(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D}) + c_2\|\mathbf{u}_D\|_{1/2,\Gamma_D}, \\ \|(p, \boldsymbol{\lambda})\|_{\mathrm{M}} &\leq c_3(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D}) + \nu c_4\|\mathbf{u}_D\|_{1/2,\Gamma_D}, \end{aligned} \tag{3.14}$$

with c_1, c_2, c_3, $c_4 > 0$ depending on $C_{F,N}$, C_G, α and β.

Proof. Let $\mathbf{w} \in \mathbf{H}^1(\Omega)$ and let $\mathbf{V}$ denote the kernel of the bilinear form b, characterized by (2.13). From (3.8) it is clear that

$$O(\mathbf{w}; \mathbf{v}, \mathbf{v}) \geq 0 \quad \forall\, \mathbf{v} \in \mathbf{V}, \tag{3.15}$$

which together with (2.14), implies

$$a(\mathbf{v}, \mathbf{v}) + O(\mathbf{w}; \mathbf{v}, \mathbf{v}) \geq \nu\alpha\, \|\mathbf{v}\|^2_{1,\Omega} \qquad \forall\, \mathbf{v} \in \mathbf{V}, \tag{3.16}$$

where $\alpha > 0$ depends only on Ω. Hence, the bilinear form $(\mathbf{u}, \mathbf{v}) \mapsto a(\mathbf{u}, \mathbf{v}) + O(\mathbf{w}; \mathbf{u}, \mathbf{v})$ is elliptic on $\mathbf{V}$. Consequently, the above estimate together with the inf–sup condition (2.10) implies the well-posedness of problem (3.12) and estimate (3.14) through a straightforward application of the Babuška–Brezzi theory (see [15, Theorem 2.34]). We omit the details. □

Now we are in position of establishing the well-posedness of (3.7).

Theorem 3.2 *There exists at least one solution $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times \mathrm{M}$ to problem (3.7) satisfying (3.14). Moreover, the solution is unique whenever*

$$C_{O,2} M(\mathrm{data}) < 1, \tag{3.17}$$

with

$$M(\mathrm{data}) := \frac{\rho}{\nu\alpha}\left(\frac{c_1}{\nu}\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D}\right) + c_2\|\mathbf{u}_D\|_{1/2,\Gamma_D}\right),$$

where α, $C_{O,2}$, c_1 and c_2 are the positive constants satisfying (2.14), (3.10) and (3.14).

Proof. Let us first define the bounded and convex set

$$\mathbf{W} := \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \text{ and } \|\mathbf{v}\|_{1,\Omega} \leq \frac{c_1}{\nu} \left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma_D}^2 \right) + c_2 \|\mathbf{u}_D\|_{1/2,\Gamma_D} \right\}.$$

According to Lemma 3.1, it is clear that $\mathcal{J}(\mathbf{W}) \subseteq \mathbf{W}$.

Now, let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$ be given, and for each $i \in \{1,2\}$ let $(\mathbf{u}_i, (p_i, \boldsymbol{\lambda}_i)) \in \mathbf{H}^1(\Omega) \times \mathrm{M}$ denote the unique solution of (3.12) corresponding to $\mathbf{w} = \mathbf{w}_i$, that is,

$$\begin{aligned} a(\mathbf{u}_i, \mathbf{v}) + O(\mathbf{w}_i; \mathbf{u}_i, \mathbf{v}) + b(\mathbf{v}, (p_i, \boldsymbol{\lambda}_i)) &= F_N(\mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \\ b(\mathbf{u}_i, (q, \boldsymbol{\xi})) &= G(q, \boldsymbol{\xi}), \quad \forall\, (q, \boldsymbol{\xi}) \in \mathrm{M}. \end{aligned} \tag{3.18}$$

Subtracting both systems for $i \in \{1,2\}$, we deduce that

$$\begin{aligned} a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}) + O(\mathbf{w}_1; \mathbf{u}_1, \mathbf{v}) - O(\mathbf{w}_2; \mathbf{u}_2, \mathbf{v}) + b(\mathbf{v}, (p_1 - p_2, \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)) &= 0, \\ b(\mathbf{u}_1 - \mathbf{u}_2, (q, \boldsymbol{\xi})) &= 0, \end{aligned} \tag{3.19}$$

for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $(q, \boldsymbol{\xi}) \in \mathrm{M}$. Notice that, the second equation of (3.19) implies that $\mathbf{u}_1 - \mathbf{u}_2 \in \mathbf{V}$. Then, taking $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ in the first equation of (3.19), adding and subtracting $O(\mathbf{w}_1; \mathbf{u}_2, \mathbf{v})$ and using (3.9) and (3.16), we deduce that

$$\begin{aligned} \nu\alpha \, \|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega}^2 &\leq |O(\mathbf{w}_1; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) - O(\mathbf{w}_2; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)| \\ &\leq \rho C_{O,1} (\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^3(\Omega)} + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^3(\Gamma)}) \|\mathbf{u}_2\|_{1,\Omega} \|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} \end{aligned} \tag{3.20}$$

which together with the fact that $\mathbf{u}_2 = \mathcal{J}(\mathbf{w}_2) \in \mathbf{W}$, implies that

$$\|\mathcal{J}(\mathbf{w}_1) - \mathcal{J}(\mathbf{w}_2)\|_{1,\Omega} \leq C_{O,1} M(\text{data}) \left(\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^3(\Omega)} + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^3(\Gamma)} \right). \tag{3.21}$$

Next, let $\{\mathbf{z}_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbf{W}$. Since $\mathbf{W}$ is bounded in $\mathbf{H}^1(\Omega)$, there exist a subsequence $\{\widetilde{\mathbf{z}}_k\}_{k \in \mathbb{N}}$ and $\mathbf{z} \in \mathbf{H}^1(\Omega)$, such that $\widetilde{\mathbf{z}}_k \rightharpoonup \mathbf{z}$ in $\mathbf{H}^1(\Omega)$. Furthermore, since the Sobolev embeddings $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^3(\Omega)$ and $\mathrm{H}^{1/2}(\Gamma) \hookrightarrow \mathrm{L}^3(\Gamma)$ are compact, and the trace operator $\boldsymbol{\gamma}_0$ is continuous, it follows that

$$\|\widetilde{\mathbf{z}}_k - \mathbf{z}\|_{\mathbf{L}^3(\Omega)} \xrightarrow{k \to \infty} 0 \quad \text{and} \quad \|\widetilde{\mathbf{z}}_k - \mathbf{z}\|_{\mathbf{L}^3(\Gamma)} \xrightarrow{k \to \infty} 0. \tag{3.22}$$

Therefore, combining (3.21) and (3.22), we deduce that $\|\mathcal{J}(\widetilde{\mathbf{z}}_k) - \mathcal{J}(\mathbf{z})\|_{1,\Omega} \xrightarrow{k \to \infty} 0$. Hence, $\mathcal{J}(\widetilde{\mathbf{z}}_k) \to \mathcal{J}(\mathbf{z})$ strongly in $\mathbf{H}^1(\Omega)$, which shows that $\mathcal{J}(\mathbf{W})$ is relatively compact in $\mathbf{H}^1(\Omega)$. Consequently, $\overline{\mathcal{J}(\mathbf{W})}$ is compact. Therefore, applying Schauder's theorem (see e.g. [10, Theorem 9.12-1(b)]) we deduce that there exists $\mathbf{u} \in \mathbf{W}$ solution to (3.13). Equivalently, there exists $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times \mathrm{M}$, solution to (3.7). Moreover, according to Lemma 3.1, $(\mathbf{u}, (p, \boldsymbol{\lambda}))$ satisfies (3.14).

Finally, to establish uniqueness, we observe from the first inequality in (3.20) and estimate (3.10) that

$$\|\mathcal{J}(\mathbf{w}_1) - \mathcal{J}(\mathbf{w}_2)\|_{1,\Omega} \leq C_{O,2} M(\text{data}) \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega}, \tag{3.23}$$

for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$. Therefore, assumption (3.17) implies that $\mathcal{J}$ is a contraction on $\mathbf{W}$. Therefore, the uniqueness of the fixed point, and hence of the solution to (3.7), follows from the Banach fixed-point theorem. □

3.3 Galerkin scheme

Analogously to Section 2.2, we let $\mathbf{H}_h \subseteq \mathbf{H}^1(\Omega)$, $\mathrm{Q}_h \subseteq \mathrm{L}^2(\Omega)$ and $\boldsymbol{\Lambda}_h \subseteq \mathbf{H}^{-1/2}_{\Gamma_N}(\Gamma)$ be arbitrary finite-dimensional subspaces, define $\mathrm{M}_h := \mathrm{Q}_h \times \boldsymbol{\Lambda}_h$, and assume that these spaces satisfy hypotheses (**H.0**)–(**H.4**) introduced in Section 2.2.

As will be discussed in Section 4, the pair $(\mathbf{H}_h, \mathrm{Q}_h)$ will be chosen as a stable finite element pair for the Stokes problem, which does not necessarily yield exactly divergence-free discrete velocities. Consequently, the coercivity property (3.16), which holds at the continuous level, may fail in the discrete setting. To overcome this difficulty, from now on we assume that:

(**H.5**) There exists $O_h : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$, such that:

(i) For all $\mathbf{w} \in \mathbf{H}^1(\Omega)$, $O_h(\mathbf{w}; \cdot, \cdot)$ is a bilinear form.

(ii) $\big|O_h(\mathbf{w}; \mathbf{u}, \mathbf{v}) - O_h(\widetilde{\mathbf{w}}; \mathbf{u}, \mathbf{v})\big| \le \widetilde{C}_O\, \rho\, \|\mathbf{w} - \widetilde{\mathbf{w}}\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}$, for all $\mathbf{w}, \widetilde{\mathbf{w}}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$, where $\widetilde{C}_O > 0$ depends only on Ω.

(iii) $O_h(\mathbf{w}; \mathbf{u}, \mathbf{v}) = O(\mathbf{w}; \mathbf{u}, \mathbf{v})$, for all $\mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$, with $\operatorname{div} \mathbf{w} = 0$ in Ω.

(iv) There exist $\gamma > 0$, independent of h, and a subset $\mathbf{W}_h \subseteq \mathbf{H}_h$ consisting of elements $\mathbf{w}_h \in \mathbf{H}_h$ satisfying
$$b_2(\mathbf{w}_h, \boldsymbol{\xi}_h) = \int_{\Gamma_D} \mathbf{u}_D \cdot \boldsymbol{\xi}_h, \qquad \forall\, \boldsymbol{\xi}_h \in \boldsymbol{\Lambda}_h, \tag{3.24}$$
such that
$$a(\mathbf{v}_h, \mathbf{v}_h) + O_h(\mathbf{w}_h; \mathbf{v}_h, \mathbf{v}_h) \ge \gamma \nu \|\mathbf{v}_h\|^2_{1,\Omega}, \tag{3.25}$$
for all $\mathbf{v}_h \in \widetilde{\mathbf{H}}_h$ (cf. (2.21)) and for all $\mathbf{w}_h \in \mathbf{W}_h$.

With these definitions, the Galerkin approximation of (3.7) reads as follows: Find $(\mathbf{u}_h, (p_h, \boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times \mathrm{M}_h$ such that
$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + O_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, (p_h, \boldsymbol{\lambda}_h)) &= F_N(\mathbf{v}_h), \quad && \forall\, \mathbf{v}_h \in \mathbf{H}_h, \\ b(\mathbf{u}_h, (q_h, \boldsymbol{\xi}_h)) &= G(q_h, \boldsymbol{\xi}_h), && \forall\, (q_h, \boldsymbol{\xi}_h) \in \mathrm{M}_h. \end{aligned} \tag{3.26}$$

To analyze the well-posedness of the Galerkin scheme (3.26), we proceed analogously to the continuous setting and introduce the auxiliary operator
$$\mathcal{J}_h : \mathbf{H}_h \longrightarrow \mathbf{H}_h, \qquad \mathbf{w}_h \longmapsto \mathcal{J}_h(\mathbf{w}_h) := \mathbf{u}_h,$$
where $\mathbf{u}_h \in \mathbf{H}_h$ is the velocity component of the unique solution, of the following linearized problem: Find $(\mathbf{u}_h, (p_h, \boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times \mathrm{M}_h$ such that
$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + O_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, (p_h, \boldsymbol{\lambda}_h)) &= F_N(\mathbf{v}_h), \quad && \forall\, \mathbf{v}_h \in \mathbf{H}_h, \\ b(\mathbf{u}_h, (q_h, \boldsymbol{\xi}_h)) &= G(q_h, \boldsymbol{\xi}_h), && \forall\, (q_h, \boldsymbol{\xi}_h) \in \mathrm{M}_h. \end{aligned} \tag{3.27}$$

Then, the analysis of problem (3.26) reduces to studying the following fixed-point problem: Find $\mathbf{u}_h \in \mathbf{H}_h$ such that
$$\mathcal{J}_h(\mathbf{u}_h) = \mathbf{u}_h. \tag{3.28}$$

The following lemma establishes that the operator $\mathcal{J}_h$ is well defined or, equivalently, that the linearized problem (3.27) is well posed, for data $\mathbf{w}_h$ lying in the set $\mathbf{W}_h$ postulated in (**H.5**)(iv).

Lemma 3.3 *Assume that hypotheses* $(\mathbf{H.0})$–$(\mathbf{H.5})$ *hold. Then, for every* $\mathbf{w}_h \in \mathbf{W}_h$ *(cf.* $(\mathbf{H.5})$*(iv)), there exists a unique solution* $(\mathbf{u}_h, (p_h, \boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times \mathrm{M}_h$ *to problem* (3.27)*. Moreover, the following estimate holds:*

$$\begin{aligned} \|\mathbf{u}_h\|_{1,\Omega} &\le \frac{\tilde{c}_1}{\nu}(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D}) + \tilde{c}_2\|\mathbf{u}_D\|_{1/2,\Gamma_D}, \\ \|(p_h, \boldsymbol{\lambda}_h)\|_{\mathrm{M}} &\le \tilde{c}_3(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D}) + \nu\tilde{c}_4\|\mathbf{u}_D\|_{1/2,\Gamma_D}, \end{aligned} \tag{3.29}$$

with $\tilde{c}_1$*,* $\tilde{c}_2$*,* $\tilde{c}_3$*,* $\tilde{c}_4 > 0$ *depending on* $C_{F,N}$*,* C_G*,* γ*, and* $\widetilde{\beta}$*.*

Proof. Let $\mathbf{w}_h \in \mathbf{W}_h$, and let $\mathbf{V}_h$ be the discrete kernel of the bilinear form b (cf. (2.20)). Then, according to $(\mathbf{H.1})$, there holds $\mathbf{V}_h \subseteq \widetilde{\mathbf{H}}_h$ (cf. (2.21)), which combined with (iv) in $(\mathbf{H.5})$, implies that $a(\cdot,\cdot) + O_h(\mathbf{w}_h;\cdot,\cdot)$ is elliptic on $\mathbf{V}_h$. Moreover, by Lemma 2.3, hypotheses $(\mathbf{H.0})$, $(\mathbf{H.2})$, $(\mathbf{H.3})$, and $(\mathbf{H.4})$ guarantee that the bilinear form b satisfies the discrete inf-sup condition (2.23). Hence, the well-posedness of (3.27) and estimate (3.29) follow directly from the Babuška–Brezzi theory; see [15, Theorem 2.34]. □

Remark 3.4 *The set* $\mathbf{W}_h$ *postulated in* $(\mathbf{H.5})$*(iv) is not arbitrary: since the constants* $\tilde{c}_1, \tilde{c}_2$ *in* (3.29) *arise from the abstract Babuška–Brezzi theory applied with coercivity constant* $\gamma\nu$ *and inf-sup constant* $\widetilde{\beta}$ *(cf.* $(\mathbf{H.4})$*), they are determined solely by* γ*,* $\widetilde{\beta}$*,* $C_{F,N}$*, and* C_G*, independently of any particular choice of* $\mathbf{W}_h$*. Consequently, one must take* $\mathbf{W}_h$ *to be precisely*

$$\begin{aligned} \mathbf{W}_h := \Big\{ \mathbf{v}_h \in \mathbf{H}_h \,:\, &\mathbf{v}_h \text{ satisfies } (3.24) \text{ and} \\ &\|\mathbf{v}_h\|_{1,\Omega} \le \frac{\tilde{c}_1}{\nu}(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D}) + \tilde{c}_2\|\mathbf{u}_D\|_{1/2,\Gamma_D} \Big\}, \end{aligned} \tag{3.30}$$

for this self-consistency to hold. In the concrete setting of Section 4, Lemma 4.3 confirms that this specific choice of $\mathbf{W}_h$ *indeed satisfies* $(\mathbf{H.5})$*(iv), under the additional mesh restriction* (4.10)*.*

Corollary 3.5 $\mathcal{J}_h(\mathbf{W}_h) \subseteq \mathbf{W}_h$*.*

Proof. Let $\mathbf{w}_h \in \mathbf{W}_h$ and set $\mathbf{u}_h := \mathcal{J}_h(\mathbf{w}_h)$. By Lemma 3.3, $\mathbf{u}_h$ satisfies estimate (3.29), that is, $\mathbf{u}_h$ satisfies the norm bound defining $\mathbf{W}_h$. On the other hand, the second equation of (3.27), together with the definitions of b_2 and G (cf. (2.7)), shows that $\mathbf{u}_h$ itself satisfies (3.24). Hence $\mathbf{u}_h = \mathcal{J}_h(\mathbf{w}_h) \in \mathbf{W}_h$, which proves the assertion. □

Now we are in position of establishing the well-posedness of (3.26).

Theorem 3.6 *Assume that hypotheses* $(\mathbf{H.0})$–$(\mathbf{H.5})$ *hold. Then, there exists at least one solution* $(\mathbf{u}_h, (p_h, \boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times \mathrm{M}_h$ *to problem* (3.26) *satisfying* (3.29)*. Moreover, the solution is unique whenever*

$$\frac{\rho\widetilde{C}_O}{\nu\gamma}\left(\frac{\tilde{c}_1}{\nu}(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D}) + \tilde{c}_2\|\mathbf{u}_D\|_{1/2,\Gamma_D}\right) < 1, \tag{3.31}$$

where $\widetilde{C}_O$ *and* γ *are the positive constants satisfying (ii) and (iv) in* $(\mathbf{H.5})$*, respectively, while* $\tilde{c}_1$ *and* $\tilde{c}_2$ *are the constants satisfying the first estimate in* (3.29)*.*

Proof. Analogously to the proof of Theorem 3.2, it suffices to show that the operator $\mathscr{J}_h$ admits a fixed point on $\mathbf{W}_h$ (cf. (3.30)), and that this fixed point is unique under hypothesis (3.31). By Corollary 3.5, $\mathscr{J}_h(\mathbf{W}_h) \subseteq \mathbf{W}_h$. In turn, given $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}_h$, denoting by $\mathbf{u}_1 = \mathscr{J}_h(\mathbf{w}_1)$ and $\mathbf{u}_2 = \mathscr{J}_h(\mathbf{w}_2)$, and proceeding analogously to the proof of Theorem 3.2, from (ii) in (**H.5**) and (3.25), we obtain

$$\begin{aligned}\nu\gamma\, \|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega}^2 &\leq |O_h(\mathbf{w}_1; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) - O_h(\mathbf{w}_2; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)| \\ &\leq \rho \widetilde{C}_O \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega} \|\mathbf{u}_2\|_{1,\Omega} \|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega}\end{aligned}$$

which, together with the fact that $\mathbf{u}_2 \in \mathbf{W}_h$, implies that

$$\|\mathscr{J}_h(\mathbf{w}_1) - \mathscr{J}_h(\mathbf{w}_2)\|_{1,\Omega} \leq \frac{\rho \widetilde{C}_O}{\nu\gamma} \left(\frac{\tilde{c}_1}{\nu} (\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma_D}^2) + \tilde{c}_2 \|\mathbf{u}_D\|_{1/2,\Gamma_D} \right) \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega}. \tag{3.32}$$

Hence, the operator $\mathscr{J}_h$ is Lipschitz continuous and, under assumption (3.31), it is a contraction on $\mathbf{W}_h$. Therefore, Brouwer's fixed-point theorem guarantees the existence of a fixed point of $\mathscr{J}_h$ on $\mathbf{W}_h$, while its uniqueness follows from the Banach fixed-point theorem. Finally, taking $\mathbf{w}_h = \mathbf{u}_h$ in Lemma 3.3, we conclude that the solution of (3.26) satisfies the stability estimate (3.29). $\square$

We now derive a Céa-type error estimate for the Galerkin approximation (3.26). To this end, we first establish the following Strang-type estimate for mixed problems.

Lemma 3.7 *Let* H *and* Q *be reflexive Banach spaces, and let* $\{\mathrm{H}_h\}_{h>0}$ *and* $\{\mathrm{Q}_h\}_{h>0}$ *be families of finite-dimensional subspaces of* H *and* Q*, respectively. Furthermore, let* $a : \mathrm{H} \times \mathrm{H} \to \mathbb{R}$ *and* $b : \mathrm{H} \times \mathrm{Q} \to \mathbb{R}$ *be bounded bilinear forms. Suppose that*

- *There exists* $\mu > 0$*, such that*

$$a(v, v) \geq \mu \|v\|_{\mathrm{H}}^2, \quad \forall v \in V := \{v \in \mathrm{H} : b(v, q) = 0, \quad \forall\, q \in \mathrm{Q}\}.$$

- *There exists* $\tilde{\mu} > 0$*, independent of* h*, such that*

$$a(v_h, v_h) \geq \tilde{\mu} \|v_h\|_{\mathrm{H}}^2, \quad \forall v_h \in V_h := \{v_h \in \mathrm{H}_h : b(v_h, q_h) = 0, \quad \forall\, q_h \in \mathrm{Q}_h\}.$$

- *There exist* $\beta > 0$ *and* $\tilde{\beta} > 0$*, independent of* h*, such that*

$$\sup_{\substack{v \in \mathrm{H} \\ v \neq \mathbf{0}}} \frac{|b(v, q)|}{\|v\|_{\mathrm{H}}} \geq \beta \|q\|_{\mathrm{Q}}, \quad \forall q \in \mathrm{Q} \quad \text{and} \quad \sup_{\substack{v_h \in \mathrm{H}_h \\ v_h \neq \mathbf{0}}} \frac{|b(v_h, q_h)|}{\|v_h\|_{\mathrm{H}}} \geq \tilde{\beta} \|q_h\|_{\mathrm{Q}}, \quad \forall q_h \in \mathrm{Q}_h.$$

Let $F \in \mathrm{H}'$ *and* $G \in \mathrm{Q}'$*, and for each* $h > 0$*, let* $\{F_h\}_{h>0}$*, with* $F_h \in \mathrm{H}_h'$*, be a uniformly bounded family of linear functionals. In addition, let* $(u, p) \in \mathrm{H} \times \mathrm{Q}$ *and* $(u_h, p_h) \in \mathrm{H}_h \times \mathrm{Q}_h$*, satisfying*

$$\begin{aligned} a(u, v) + b(v, p) &= F(v) \quad \forall v \in \mathrm{H}, \\ b(u, q) &= G(q) \quad \forall q \in \mathrm{Q}, \end{aligned} \quad \text{and} \quad \begin{aligned} a(u_h, v_h) + b(v_h, p_h) &= F_h(v_h) \quad \forall v_h \in \mathrm{H}_h, \\ b(u_h, q_h) &= G(q_h) \quad \forall q_h \in \mathrm{Q}_h. \end{aligned}$$

Then, the following estimates hold

$$
\begin{aligned}
\|u-u_h\|_{\mathrm{H}} \leq \quad & c_1 \inf_{v_h\in \mathrm{H}_h} \|u-v_h\|_{\mathrm{H}} + c_2 \inf_{q_h\in \mathrm{Q}_h} \|p-q_h\|_{\mathrm{Q}} + \frac{1}{\tilde{\mu}} \sup_{\substack{z_h\in \mathrm{H}_h\\ z_h\neq \mathbf{0}}} \frac{|F(z_h)-F_h(z_h)|}{\|z_h\|_{\mathrm{H}}}, \\
\|p-p_h\|_{\mathrm{Q}} \leq \quad & c_3 \inf_{v_h\in \mathrm{H}_h} \|u-v_h\|_{\mathrm{H}} + c_4 \inf_{q_h\in \mathrm{Q}_h} \|p-q_h\|_{\mathrm{Q}} + c_5 \sup_{\substack{z_h\in \mathrm{H}_h\\ z_h\neq \mathbf{0}}} \frac{|F(z_h)-F_h(z_h)|}{\|z_h\|_{\mathrm{H}}},
\end{aligned}
\tag{3.33}
$$

with $c_1 := \left(1+\frac{\|a\|}{\tilde{\mu}}\right)\left(1+\frac{\|b\|}{\tilde{\beta}}\right)$, $c_2 := \frac{\|b\|}{\tilde{\mu}}$ *if* $V_h \not\subset V$ *and* $c_2 := 0$ *otherwise,* $c_3 := \frac{\|a\|}{\tilde{\beta}} c_1$, $c_4 := 1 + \frac{\|b\|}{\tilde{\beta}} + \frac{\|a\|}{\tilde{\beta}} c_2$ *and* $c_5 := \frac{1}{\tilde{\beta}}\left(1+\frac{\|a\|}{\tilde{\mu}}\right)$, *where* $\|a\|$ *and* $\|b\|$ *denote the boundedness constants of* a *and* b.

Proof. Noting that, for each $w_h \in \mathrm{H}_h$ satisfying $b(w_h, q_h) = G(q_h)$ for all $q_h \in \mathrm{Q}_h$, and for every $y_h \in V_h$, there holds

$$
a(u_h - w_h, y_h) = a(u - w_h, y_h) + b(y_h, p - p_h) + F_h(y_h) - F(y_h),
$$

the proof of (3.33) follows along the same lines as that of [15, Lemma 2.44]. We therefore omit the remaining details. □

We conclude this section by establishing the corresponding Cea's estimate.

Theorem 3.8 *Assume that hypotheses* (**H.0**)–(**H.5**) *hold. Let* $(\mathbf{u}_h,(p_h,\boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times \mathrm{M}_h$ *be a solution of problem* (3.26)*. Moreover, assume that* (3.17) *holds, and let* $(\mathbf{u},(p,\boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega)\times \mathrm{M}$ *be the unique solution of* (3.7)*. Assume further that*

$$
\frac{\rho}{\tilde{\alpha}\nu}\left\{ \widetilde{C}_O \left[\frac{c_1+\tilde{c}_1}{\nu}\Big(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D}\Big) + (c_2+\tilde{c}_2)\|\mathbf{u}_D\|_{1/2,\Gamma_D} \right] + \frac{\widehat{C}_\Gamma^2}{2}\|\mathbf{u}_D\|_{1/2,\Gamma_D} \right\} \leq \frac{1}{2},
\tag{3.34}
$$

with c_1, c_2 *and* $\tilde{c}_1, \tilde{c}_2$ *being the positive constants satisfying* (3.14) *and* (3.29)*, respectively, and* $\widehat{C}_\Gamma > 0$ *the constant of the Sobolev embedding* (1.3)*. Then, the following error estimates hold:*

$$
\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega} \leq \hat{c}_1 \inf_{\mathbf{v}_h\in \mathbf{H}_h} \|\mathbf{u}-\mathbf{v}_h\|_{1,\Omega} + \frac{\hat{c}_2}{\nu} \inf_{(q_h,\boldsymbol{\xi}_h)\in \mathrm{M}_h} \|(p-q_h, \boldsymbol{\lambda}-\boldsymbol{\xi}_h)\|_{\mathrm{M}}, \tag{3.35}
$$

$$
\|(p-p_h, \boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|_{\mathrm{M}} \leq \hat{c}_3 \nu \inf_{\mathbf{v}_h\in \mathbf{H}_h} \|\mathbf{u}-\mathbf{v}_h\|_{1,\Omega} + \hat{c}_4 \inf_{(q_h,\boldsymbol{\xi}_h)\in \mathrm{M}_h} \|(p-q_h, \boldsymbol{\lambda}-\boldsymbol{\xi}_h)\|_{\mathrm{M}}. \tag{3.36}
$$

The positive constants $\hat{c}_1$, $\hat{c}_2$, $\hat{c}_3$, *and* $\hat{c}_4$ *depend only on* Ω.

Proof. Given $\mathbf{z} \in \mathbf{H}^1(\Omega)$, we let $F_{\mathbf{z}} : \mathbf{H}^1(\Omega) \to \mathbb{R}$ and $F^h_{\mathbf{z}} : \mathbf{H}_h \to \mathbb{R}$ be the linear functionals given by

$$
F_{\mathbf{z}}(\mathbf{v}) := F_N(\mathbf{v}) - O(\mathbf{z};\mathbf{z},\mathbf{v}) \quad \text{and} \quad F^h_{\mathbf{z}}(\mathbf{v}_h) := F_N(\mathbf{v}_h) - O_h(\mathbf{z};\mathbf{z},\mathbf{v}_h), \tag{3.37}
$$

and observe that, according to (3.7) and (3.26), $(\mathbf{u},(p,\boldsymbol{\lambda}))$ and $(\mathbf{u}_h,(p_h,\boldsymbol{\lambda}_h))$ satisfy

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},(p,\boldsymbol{\lambda})) &= F_{\mathbf{u}}(\mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \\
b(\mathbf{u},(q,\boldsymbol{\xi})) &= G(q,\boldsymbol{\xi}), \quad \forall\, (q,\boldsymbol{\xi}) \in \mathrm{M},
\end{aligned}
\tag{3.38}
$$

and

$$
\begin{aligned}
a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, (p_h, \boldsymbol{\lambda}_h)) &= F^h_{\mathbf{u}_h}(\mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{H}_h, \\
b(\mathbf{u}_h, (q_h, \boldsymbol{\xi}_h)) &= G(q_h, \boldsymbol{\xi}_h), \quad \forall\, (q_h, \boldsymbol{\xi}_h) \in \mathrm{M}_h.
\end{aligned} \tag{3.39}
$$

Then, noticing that a and b satisfy the hypotheses of Lemma 3.7, we deduce from (3.33) that the following estimates hold

$$
\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \leq \quad C_1 \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \frac{C_2}{\nu} \inf_{(q_h, \boldsymbol{\xi}_h) \in \mathrm{M}_h} \|(p - q_h, \boldsymbol{\lambda} - \boldsymbol{\xi}_h)\|_{\mathrm{M}}
$$

$$
+ \frac{1}{\nu \tilde{\alpha}} \sup_{\mathbf{v}_h \in \mathbf{H}_h} \frac{|F_{\mathbf{u}}(\mathbf{v}_h) - F^h_{\mathbf{u}_h}(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{1,\Omega}}, \tag{3.40}
$$

$$
\|(p - p_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)\|_{\mathrm{M}} \leq \quad C_3 \nu \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + C_4 \inf_{(q_h, \boldsymbol{\xi}_h) \in \mathrm{M}_h} \|(p - q_h, \boldsymbol{\lambda} - \boldsymbol{\xi}_h)\|_{\mathrm{M}}
$$

$$
+ C_5 \sup_{\mathbf{v}_h \in \mathbf{H}_h} \frac{|F_{\mathbf{u}}(\mathbf{v}_h) - F^h_{\mathbf{u}_h}(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{1,\Omega}}, \tag{3.41}
$$

with $C_1, \ldots, C_5 > 0$ depending only on Ω. In turn, since $\operatorname{div} \mathbf{u} = 0$ in Ω, property (iii) in (**H.5**) gives $O(\mathbf{u}; \mathbf{u}, \mathbf{v}_h) = O_h(\mathbf{u}; \mathbf{u}, \mathbf{v}_h)$, so that, adding and subtracting $O_h(\mathbf{u}_h; \mathbf{u}, \mathbf{v}_h)$,

$$
\begin{aligned}
|F_{\mathbf{u}}(\mathbf{v}_h) - F^h_{\mathbf{u}_h}(\mathbf{v}_h)| &= |O_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - O_h(\mathbf{u}; \mathbf{u}, \mathbf{v}_h)| \\
&\leq \big|O_h(\mathbf{u}_h; \mathbf{u}, \mathbf{v}_h) - O_h(\mathbf{u}; \mathbf{u}, \mathbf{v}_h)\big| + \big|O_h(\mathbf{u}_h; \mathbf{u}_h - \mathbf{u}, \mathbf{v}_h)\big|.
\end{aligned} \tag{3.42}
$$

The first term is bounded directly by property (ii) in (**H.5**),

$$
\big|O_h(\mathbf{u}_h; \mathbf{u}, \mathbf{v}_h) - O_h(\mathbf{u}; \mathbf{u}, \mathbf{v}_h)\big| \leq \widetilde{C}_O \rho\, \|\mathbf{u}_h - \mathbf{u}\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}. \tag{3.43}
$$

For the second term, adding and subtracting $O_h(\mathbf{0}; \mathbf{u}_h - \mathbf{u}, \mathbf{v}_h)$ and using property (ii) once more,

$$
\big|O_h(\mathbf{u}_h; \mathbf{u}_h - \mathbf{u}, \mathbf{v}_h)\big| \leq \widetilde{C}_O \rho\, \|\mathbf{u}_h\|_{1,\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega} + \big|O_h(\mathbf{0}; \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h)\big|. \tag{3.44}
$$

Since $O_h(\mathbf{0}; \mathbf{v}, \mathbf{v}_h) = O(\mathbf{0}; \mathbf{v}, \mathbf{v}_h) = -\dfrac{\rho}{2} \displaystyle\int_{\Gamma_D} (\mathbf{u}_D \cdot \mathbf{n})\, \mathbf{v} \cdot \mathbf{v}_h$ for all $\mathbf{v}, \mathbf{v}_h \in \mathbf{H}^1(\Omega)$ (cf. the definitions of O and O_h), the Cauchy–Schwarz inequality and the Sobolev embedding (1.3) give

$$
\big|O_h(\mathbf{0}; \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h)\big| \leq \frac{\rho}{2}\, \widehat{C}_\Gamma^2\, \|\mathbf{u}_D\|_{1/2,\Gamma_D}\, \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}. \tag{3.45}
$$

Combining (3.42)–(3.45), we obtain

$$
|F_{\mathbf{u}}(\mathbf{v}_h) - F^h_{\mathbf{u}_h}(\mathbf{v}_h)| \;\leq\; \rho \left\{ \widetilde{C}_O \big(\|\mathbf{u}\|_{1,\Omega} + \|\mathbf{u}_h\|_{1,\Omega}\big) + \frac{\widehat{C}_\Gamma^2}{2} \|\mathbf{u}_D\|_{1/2,\Gamma_D} \right\} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}. \tag{3.46}
$$

Thus, recalling from (3.14) and (3.29) that

$$
\|\mathbf{u}\|_{1,\Omega} + \|\mathbf{u}_h\|_{1,\Omega} \;\leq\; \frac{c_1 + \tilde{c}_1}{\nu} \Big(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D} \Big) + (c_2 + \tilde{c}_2) \|\mathbf{u}_D\|_{1/2,\Gamma_D},
$$

we deduce from (3.46) that

$$\begin{aligned}\frac{1}{\tilde{\alpha}\nu}\sup_{\mathbf{v}_h\in\mathbf{H}_h}\frac{|F_{\mathbf{u}}(\mathbf{v}_h)-F^h_{\mathbf{u}_h}(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{1,\Omega}} &\le \frac{\rho}{\tilde{\alpha}\nu}\Bigg\{\widetilde{C}_O\Bigg[\frac{c_1+\tilde{c}_1}{\nu}\Big(\|\mathbf{f}\|_{0,\Omega}+\|\mathbf{u}_D\|^2_{1/2,\Gamma_D}\Big)\\ &\quad+(c_2+\tilde{c}_2)\|\mathbf{u}_D\|_{1/2,\Gamma_D}\Bigg]+\frac{\widehat{C}^2_\Gamma}{2}\|\mathbf{u}_D\|_{1/2,\Gamma_D}\Bigg\}\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}.\end{aligned} \tag{3.47}$$

In this way, estimate (3.35) follows directly from (3.47), estimate (3.40), and assumption (3.34). Moreover, combining (3.35), (3.41), and (3.47), we obtain (3.36). This completes the proof. □

4 Concrete finite element spaces

In this section we provide concrete examples of finite elements subspaces satisfying hypotheses (**H.0**)–(**H.4**). Additionally, we introduce a particular choice for O_h satisfying (**H.5**).

Among all the choices of finite element subspaces that can be utilized to approximate the solution of (2.5) and (3.7), in what follows we introduce two particular examples. To do that, given an integer $k \ge 0$ and a set S of $\mathbb{R}^d$, in the sequel we denote by $\mathrm{P}_k(S)$ the space of polynomial functions on S of degree $\le k$.

4.1 Hood-Taylor + discontinuous piecewise polynomials

In order to approximate the velocity and pressure fields in Ω, we consider the well known Hood–Taylor finite element pair (see, e.g. [18]), which provides a stable and widely used discretization for the Stokes and Navier–Stokes equations. More precisely, given an integer $k \ge 1$, we define $\mathbf{H}_h$ and Q_h as the finite element subspaces given by continuous piecewise polynomials of degree $k+1$ and k, respectively, that is,

$$\mathbf{H}_h := \Big\{\mathbf{v}_h \in [C(\overline{\Omega})]^d : \mathbf{v}_h\big|_K \in [\mathrm{P}_{k+1}(K)]^n \quad \forall K \in \mathcal{T}_h\Big\} \tag{4.1}$$

and

$$\mathrm{Q}_h := \big\{q_h \in C(\overline{\Omega}) : q_h\big|_K \in \mathrm{P}_k(K) \quad \forall K \in \mathcal{T}_h\big\}. \tag{4.2}$$

Now, to approximate the Lagrange multiplier $\boldsymbol{\lambda}$, and for the technical reasons detailed in [17, Section 4.3], we introduce an independent triangulation $\{\overline{\Gamma}_1, \overline{\Gamma}_2, \dots, \overline{\Gamma}_m\}$ of Γ, made up of triangles in $\mathbb{R}^3$ or straight segments in $\mathbb{R}^2$, and set $\tilde{h} := \max\{|\Gamma_j| : j \in \{1, \dots, m\}\}$. Based on this independent mesh, given $l \ge 0$, we define the finite element subspace used to approximate the Lagrange multiplier $\boldsymbol{\lambda}$ as

$$\boldsymbol{\Lambda}^l_{\tilde{h}} := \Big\{\boldsymbol{\xi}_{\tilde{h}} \in \mathbf{L}^2(\Gamma) : \boldsymbol{\xi}_{\tilde{h}}\big|_{\overline{\Gamma}_j} \in [\mathrm{P}_l(\overline{\Gamma}_j)]^d \quad \forall j \in \{1, \dots, m\} \quad \text{and} \quad \boldsymbol{\xi}_{\tilde{h}}|_{\Gamma_N} = \mathbf{0}\Big\}. \tag{4.3}$$

It is clear that these finite element spaces satisfy hypotheses (**H.0**)–(**H.3**). In particular, the construction of the function $\mathbf{v}_0$ satisfying (**H.2**) can be found in [7, Lemma 3.2]. The following lemma shows that the pair $(\mathbf{H}_h, \boldsymbol{\Lambda}^k_{\tilde{h}})$ satisfies hypothesis (**H.4**).

Lemma 4.1 *There exist $\varepsilon_0 > 0$ and $\widehat{\beta}_{\mathrm{T}} > 0$, independent of h and $\tilde{h}$, such that for each $h \le \varepsilon_0 \tilde{h}$, there holds*

$$\sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq 0}} \frac{b_2(\mathbf{v}_h, \boldsymbol{\xi}_{\tilde{h}})}{\|\mathbf{v}_h\|_{1,\Omega}} \ge \widehat{\beta}_{\mathrm{T}}\,\|\boldsymbol{\xi}_{\tilde{h}}\|_{-1/2,\Gamma} \quad \forall \boldsymbol{\xi}_{\tilde{h}} \in \boldsymbol{\Lambda}^k_{\tilde{h}}. \tag{4.4}$$

Proof. The proof follows essentially the same arguments employed in [17, Lemma 4.7], where the approximating spaces for $\mathbf{v}$ and $\boldsymbol{\lambda}$ are defined analogously but with $k = 0$ and $l = 0$. Indeed, it suffices to replace the orthogonal projector from $\mathbf{H}^1(\Omega)$ onto the continuous piecewise polynomials of degree ≤ 1, employed therein, by the one onto the continuous piecewise polynomials of degree $\leq k + 1$, as required here. We omit further details. $\square$

Let us now recall the approximation properties of the finite element subspaces introduced above.

($\mathbf{AP}_h^{\mathbf{u}}$) There exists $C > 0$, independent of h, such that for all $\mathbf{u} \in \mathbf{H}^{k+2}(\Omega)$, there holds

$$\inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} \leq C h^{k+1} \|\mathbf{u}\|_{k+2,\Omega}.$$

($\mathbf{AP}_h^{p}$) There exists $C > 0$, independent of h, such that for all $p \in \mathrm{H}^{k+1}(\Omega)$, there holds

$$\inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \leq C h^{k+1} \|p\|_{k+1,\Omega}.$$

($\mathbf{AP}_{\tilde{h}}^{\boldsymbol{\lambda}}$) There exists $C > 0$, independent of $\tilde{h}$, such that for all $\boldsymbol{\lambda} \in \mathbf{H}^{k+\frac{1}{2}}(\Gamma)$, there holds

$$\inf_{\boldsymbol{\xi}_{\tilde{h}} \in \boldsymbol{\Lambda}_{\tilde{h}}^k} \|\boldsymbol{\lambda} - \boldsymbol{\xi}_{\tilde{h}}\|_{-1/2,\Gamma} \leq C \tilde{h}^{k+1} \|\boldsymbol{\lambda}\|_{k+\frac{1}{2},\Gamma}.$$

The following result establishes the theoretical rate of convergence of the Galerkin scheme (2.16) associated to the Stokes problem (2.5).

Theorem 4.2 *Let $k \geq 1$ be an integer, and let $\mathbf{H}_h$, Q_h, and $\boldsymbol{\Lambda}_{\tilde{h}}$ be the finite element spaces defined in* (4.1), (4.2), *and* (4.3), *respectively, and assume that $h \leq \varepsilon_0 \tilde{h}$, with $\varepsilon_0 > 0$ the constant established in Lemma 4.1. Then there exists a unique $(\mathbf{u}_h, p_h, \boldsymbol{\lambda}_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \boldsymbol{\Lambda}_{\tilde{h}}^k$ solving* (2.16), *satisfying* (2.28). *Moreover, if the exact solution of* (2.5) *satisfies $\mathbf{u} \in \mathbf{H}^{k+2}(\Omega)$, $p \in \mathrm{H}^{k+1}(\Omega)$, and $\boldsymbol{\lambda} \in \mathbf{H}^{k+1/2}(\Gamma)$, then there holds*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} &\leq C_1 \, h^{k+1} \|\mathbf{u}\|_{k+2,\Omega} + \frac{C_2}{\nu} \big\{ h^{k+1} \|p\|_{k+1,\Omega} + \tilde{h}^{k+1} \|\boldsymbol{\lambda}\|_{k+\frac{1}{2},\Gamma} \big\}, \\ \|(p - p_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)\|_{\mathrm{M}} &\leq C_3 \, \nu \, h^{k+1} \|\mathbf{u}\|_{k+2,\Omega} + C_4 \big\{ h^{k+1} \|p\|_{k+1,\Omega} + \tilde{h}^{k+1} \|\boldsymbol{\lambda}\|_{k+\frac{1}{2},\Gamma} \big\}, \end{aligned} \tag{4.5}$$

with C_1, C_2, C_3, $C_4 > 0$ depending only on Ω.

Proof. The well-posedness of (2.16) follows from the fact that the aforementioned discrete spaces satisfy hypotheses (**H.0**)–(**H.4**), while estimates (4.5) follow from the Céa estimates (2.29) together with the approximation properties ($\mathbf{AP}_h^{\mathbf{u}}$), ($\mathbf{AP}_h^{p}$), and ($\mathbf{AP}_{\tilde{h}}^{\boldsymbol{\lambda}}$). We omit further details. $\square$

Now we focus on establishing the well-posedness of (3.26), considering the spaces (4.1), (4.2), and (4.3), together with a specific form O_h satisfying (**H.5**). More precisely, we consider here the skew-symmetric form introduced in [27],

$$O_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) := O(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) + \frac{\rho}{2} \int_\Omega (\operatorname{div} \mathbf{w}_h) \, \mathbf{u}_h \cdot \mathbf{v}_h \qquad \forall \, \mathbf{w}_h, \, \mathbf{u}_h, \, \mathbf{v}_h \in \mathbf{H}_h; \tag{4.6}$$

for further choices of discrete convective forms, we refer to, e.g., [24].

It is clear that O_h satisfies i), ii), and iii) in (**H.5**). We now turn to establishing that O_h satisfies iv) in (**H.5**).

To that end, we let $l \geq 0$ be an integer, and let $\Pi^l_{\tilde{h}} : \mathbf{L}^2(\Gamma_D) \to \mathcal{P}^l_{\tilde{h}}(\Gamma_D)$ denote the $\mathbf{L}^2(\Gamma_D)$-orthogonal projection onto

$$\mathcal{P}^l_{\tilde{h}}(\Gamma_D) := \Big\{ \mathbf{v} \in \mathbf{L}^2(\Gamma_D) \,:\, \mathbf{v}|_{\overline{\Gamma}_j} \in [\mathrm{P}_l(\overline{\Gamma}_j)]^n \quad \forall\, j \in \mathcal{J}_D \Big\}, \qquad \mathcal{J}_D := \{ j \in \{1, \dots, m\} : \overline{\Gamma}_j \subset \overline{\Gamma}_D \}.$$

Proceeding as in the proof of [17, Lemma 4.8], one shows that there exists $C_P > 0$, independent of $\tilde{h}$, such that

$$\|\mathbf{z} - \Pi^l_{\tilde{h}}(\mathbf{z})\|_{0,\Gamma_D} \;\leq\; C_P\, \tilde{h}^{1/2}\, \|\mathbf{z}\|_{1/2,\Gamma_D}, \qquad \forall\, \mathbf{z} \in \mathbf{H}^{1/2}(\Gamma_D). \tag{4.7}$$

Lemma 4.3 *Let* $\gamma := \dfrac{\tilde{\alpha}}{2}$, *with* $\tilde{\alpha} > 0$ *being the positive constant satisfying* (2.22), *and let* $\mathbf{w}_h \in \mathbf{H}_h$ *be such that*

$$b_2(\mathbf{w}_h, \boldsymbol{\xi}_h) = \int_{\Gamma_D} \mathbf{u}_D \cdot \boldsymbol{\xi}_h, \qquad \forall\, \boldsymbol{\xi}_h \in \boldsymbol{\Lambda}^k_{\tilde{h}}, \tag{4.8}$$

and

$$\|\mathbf{w}_h\|_{1,\Omega} \leq \frac{\tilde{c}_1}{\nu} \big(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D} \big) + \tilde{c}_2 \|\mathbf{u}_D\|_{1/2,\Gamma_D}, \tag{4.9}$$

with $\tilde{c}_1$, $\tilde{c}_2 > 0$ *the constants satisfying the first estimate in* (3.29), *which depend on* $C_{F,N}$, C_G, γ, *and* $\widetilde{\beta}$. *Define*

$$\widetilde{M}(\text{data}) := \frac{\rho}{\nu\gamma} \left[\frac{\tilde{c}_1}{\nu} \Big(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D} \Big) + (\tilde{c}_2 + 1) \|\mathbf{u}_D\|_{1/2,\Gamma_D} \right].$$

Then, there exists $\varepsilon_1 > 0$, *independent of* h, $\tilde{h}$, *and the physical parameters, such that, for each* $\tilde{h} > 0$ *satisfying*

$$\tilde{h} \;\leq\; \frac{\varepsilon_1}{\widetilde{M}(\text{data})^2}, \tag{4.10}$$

the following estimate holds

$$a(\mathbf{v}_h, \mathbf{v}_h) + O_h(\mathbf{w}_h; \mathbf{v}_h, \mathbf{v}_h) \geq \gamma\nu \|\mathbf{v}_h\|^2_{1,\Omega}, \quad \forall \mathbf{v}_h \in \widetilde{\mathbf{H}}_h, \tag{4.11}$$

with $\widetilde{\mathbf{H}}_h \subseteq \mathbf{H}_h$ *defined in* (2.21).

Proof. Let $\mathbf{v}_h \in \widetilde{\mathbf{H}}_h$ and let $\mathbf{w}_h \in \mathbf{H}_h$ satisfy (4.8). First, from (3.8) and (4.6) we observe that

$$O_h(\mathbf{w}_h; \mathbf{v}_h, \mathbf{v}_h) = \frac{\rho}{2} \int_{\Gamma_D} \big((\mathbf{w}_h - \mathbf{u}_D) \cdot \mathbf{n} \big) |\mathbf{v}_h|^2 + \frac{\rho}{2} \int_{\Gamma_N} (\mathbf{w}_h \cdot \mathbf{n})^+ |\mathbf{v}_h|^2 \geq \frac{\rho}{2} \int_{\Gamma_D} \big((\mathbf{w}_h - \mathbf{u}_D) \cdot \mathbf{n} \big) |\mathbf{v}_h|^2. \tag{4.12}$$

In turn, from (4.8) it readily follows that

$$\int_{\Gamma_D} (\mathbf{w}_h - \mathbf{u}_D) \cdot \boldsymbol{\xi}_h = 0, \qquad \forall\, \boldsymbol{\xi}_h \in \boldsymbol{\Lambda}^k_{\tilde{h}}|_{\Gamma_D} = \mathcal{P}^k_{\tilde{h}}(\Gamma_D),$$

which means that $\Pi^k_{\tilde{h}}\big((\mathbf{w}_h - \mathbf{u}_D)|_{\Gamma_D} \big) = \mathbf{0}$, and consequently,

$$\mathbf{w}_h - \mathbf{u}_D = (\mathbf{w}_h - \mathbf{u}_D) - \Pi^k_{\tilde{h}}(\mathbf{w}_h - \mathbf{u}_D) \qquad \text{in } \Gamma_D.$$

In this way, applying (4.7) with $l = k$ to $\mathbf{z} := (\mathbf{w}_h - \mathbf{u}_D)|_{\Gamma_D} \in \mathbf{H}^{1/2}(\Gamma_D)$, and employing the triangle inequality and the trace inequality $\|\mathbf{w}_h\|_{1/2,\Gamma} \leq C\|\mathbf{w}_h\|_{1,\Omega}$, we obtain

$$\|\mathbf{w}_h - \mathbf{u}_D\|_{0,\Gamma_D} \;\leq\; C_1\, \tilde{h}^{1/2} \left(\|\mathbf{w}_h\|_{1,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma_D}\right), \tag{4.13}$$

with $C_1 > 0$, independent of h, $\tilde{h}$, and the physical parameters. This estimate, together with the Cauchy–Schwarz inequality and the Sobolev embedding (1.3), implies

$$\begin{aligned} \left|\int_{\Gamma_D} \big((\mathbf{w}_h - \mathbf{u}_D)\cdot\mathbf{n}\big)|\mathbf{v}_h|^2\right| \;&\leq\; \|\mathbf{w}_h - \mathbf{u}_D\|_{0,\Gamma_D}\, \|\mathbf{v}_h\|^2_{\mathbf{L}^4(\Gamma_D)} \\ &\leq\; C_2\, \tilde{h}^{1/2}\big(\|\mathbf{w}_h\|_{1,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma_D}\big)\, \|\mathbf{v}_h\|^2_{1,\Omega}, \end{aligned} \tag{4.14}$$

with $C_2 > 0$, independent of h, $\tilde{h}$, and the physical parameters. Therefore, from (4.12), (4.14), the coercivity of a given by (2.22), and recalling that $\tilde{\alpha} = 2\gamma$, we deduce

$$a(\mathbf{v}_h,\mathbf{v}_h) + O_h(\mathbf{w}_h;\mathbf{v}_h,\mathbf{v}_h) \;\geq\; \left\{2\gamma\nu - \frac{1}{2}\, C_2\, \tilde{h}^{1/2}\, \rho\big(\|\mathbf{w}_h\|_{1,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma_D}\big)\right\} \|\mathbf{v}_h\|^2_{1,\Omega}. \tag{4.15}$$

This estimate, inequality (4.9), and hypothesis (4.10) (with $\varepsilon_1 := 4/C_2^2$) yield (4.11), which concludes the proof. □

Having verified the hypothesis on O_h, we finally establish the theoretical rate of convergence of the Galerkin scheme (3.26) associated to the Navier-Stokes problem (3.7).

Theorem 4.4 *Let $k \geq 1$ be an integer, and let $\mathbf{H}_h$, Q_h, and $\boldsymbol{\Lambda}^k_{\tilde{h}}$ be the finite element spaces defined in* (4.1), (4.2), *and* (4.3), *respectively, and assume that $h \leq \varepsilon_0 \tilde{h}$, with $\varepsilon_0 > 0$ the constant established in Lemma 4.1. Assume further that estimate* (4.10) *holds. Then there exists at least one $(\mathbf{u}_h, p_h, \boldsymbol{\lambda}_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \boldsymbol{\Lambda}^k_{\tilde{h}}$ solving* (3.26), *satisfying* (3.29), *which is unique under assumption* (3.31). *Moreover, if the exact solution of* (3.7) *satisfies $\mathbf{u} \in \mathbf{H}^{k+2}(\Omega)$, $p \in \mathrm{H}^{k+1}(\Omega)$, and $\boldsymbol{\lambda} \in \mathbf{H}^{k+1/2}(\Gamma)$, and assumption* (3.34) *holds, then there holds*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \quad &\leq C_1\, h^{k+1}\|\mathbf{u}\|_{k+2,\Omega} + \frac{C_2}{\nu}\big\{h^{k+1}\|p\|_{k+1,\Omega} + \tilde{h}^{k+1}\|\boldsymbol{\lambda}\|_{k+\frac{1}{2},\Gamma}\big\}, \\ \|(p - p_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)\|_{\mathrm{M}} \quad &\leq C_3\, \nu\, h^{k+1}\|\mathbf{u}\|_{k+2,\Omega} + C_4\big\{h^{k+1}\|p\|_{k+1,\Omega} + \tilde{h}^{k+1}\|\boldsymbol{\lambda}\|_{k+\frac{1}{2},\Gamma}\big\}, \end{aligned} \tag{4.16}$$

with C_1, C_2, C_3, $C_4 > 0$ depending only on Ω.

Proof. The well-posedness of (3.26) follows from the fact that the aforementioned discrete spaces satisfy hypotheses (**H.0**)–(**H.4**) and O_h satisfies hypothesis (**H.5**), while estimates (4.16) follow from the Céa estimates (3.35) and (3.36), together with the approximation properties $(\mathbf{AP}^{\mathbf{u}}_h)$, $(\mathbf{AP}^p_h)$, and $(\mathbf{AP}^{\boldsymbol{\lambda}}_{\tilde{h}})$. We omit further details. □

4.2 MINI-element + piecewise constants

In what follows, for the sake of simplicity we restrict ourselves to the 2D case. For each $T \in \mathcal{T}_h$, we let $\mathrm{P}_{1,b}(T)$ be the space (see, e.g. [18])

$$\mathrm{P}_{1,b}(T) := [\mathrm{P}_1(T) \oplus \mathrm{span}\{b_T\}]^2,$$

where $b_T := \varphi_1\varphi_2\varphi_3$ is a P_3 bubble function on T, and $\varphi_1, \varphi_2, \varphi_3$ are the barycentric coordinates of T. Then, the MINI-element (see, e.g. [18]) consists of the pair $(\mathbf{H}_h, \mathrm{Q}_h)$, where

$$\mathbf{H}_h := \left\{ \mathbf{v}_h \in [C(\overline{\Omega})]^2 : \mathbf{v}_h\big|_K \in \mathrm{P}_{1,b}(K) \quad \forall K \in \mathcal{T}_h \right\} \tag{4.17}$$

and

$$\mathrm{Q}_h := \left\{ q_h \in C(\overline{\Omega}) \,:\, q_h\big|_K \in \mathrm{P}_1(K) \quad \forall K \in \mathcal{T}_h \right\}. \tag{4.18}$$

Since the bubble function b_T vanishes on ∂T, the trace of any $\mathbf{v}_h \in \mathbf{H}_h$ on Γ is piecewise linear. Accordingly, to approximate $\boldsymbol{\lambda}$ we proceed as in Section 4.1 and choose the discrete space

$$\boldsymbol{\Lambda}^0_{\tilde{h}} := \left\{ \boldsymbol{\xi}_{\tilde{h}} \in [\mathrm{L}^2(\Gamma)]^2 \,:\, \boldsymbol{\xi}_{\tilde{h}}\big|_{\overline{\Gamma}_j} \in [\mathrm{P}_0(\overline{\Gamma}_j)]^2 \quad \forall j \in \{1, \ldots, m\} \quad \text{and} \quad \boldsymbol{\xi}_{\tilde{h}}|_{\Gamma_N} = \mathbf{0} \right\}, \tag{4.19}$$

where $\{\overline{\Gamma}_1, \overline{\Gamma}_2, \ldots, \overline{\Gamma}_m\}$ and $\tilde{h}$ are defined as in Section 4.1. Proceeding as in Section 4.1 (with $l = 0$ in place of $l = k$), one shows that these spaces satisfy hypotheses (**H.0**)–(**H.4**), and that Lemma 4.3 holds with $\mathcal{P}^0_{\tilde{h}}(\Gamma_D)$ in place of $\mathcal{P}^k_{\tilde{h}}(\Gamma_D)$, so that (**H.5**) is satisfied as well. In particular, this implies that the Galerkin scheme (2.16) is well posed and estimates (2.15) and (2.29) hold. In addition, considering the convective term O_h introduced in (4.6), we conclude, analogously to Theorem 4.4, that the discrete scheme (3.26) is well posed and convergent.

The approximation properties of these subspaces are as follows.

$(\widehat{\mathbf{AP}}^{\mathbf{u}}_h)$ There exists $C > 0$, independent of h, such that for all $\mathbf{u} \in \mathbf{H}^2(\Omega)$, there holds

$$\inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} \,\leq\, Ch\|\mathbf{u}\|_{2,\Omega}.$$

$(\widehat{\mathbf{AP}}^{p}_h)$ There exists $C > 0$, independent of h, such that for all $p \in \mathrm{H}^1(\Omega)$, there holds

$$\inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \,\leq\, Ch\|p\|_{1,\Omega}.$$

$(\widehat{\mathbf{AP}}^{\boldsymbol{\lambda}}_{\tilde{h}})$ There exists $C > 0$, independent of $\tilde{h}$, such that for all $\boldsymbol{\lambda} \in \mathbf{H}^{1/2}(\Gamma)$, there holds

$$\inf_{\boldsymbol{\xi}_{\tilde{h}} \in \boldsymbol{\Lambda}^0_{\tilde{h}}} \|\boldsymbol{\lambda} - \boldsymbol{\xi}_{\tilde{h}}\|_{-1/2,\Gamma} \,\leq\, C\tilde{h}\|\boldsymbol{\lambda}\|_{1/2,\Gamma}.$$

Thanks to these approximation properties, and provided $h \leq \varepsilon_0 \tilde{h}$, with $\varepsilon_0 > 0$ the constant established in the analogue of Lemma 4.1 for $l = 0$, the following theoretical rates of convergence are attained for the Stokes and Navier–Stokes problems, proceeding as in the proof of Theorems 4.2 and 4.4, respectively:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} &\leq C_1\, h\|\mathbf{u}\|_{2,\Omega} + \frac{C_2}{\nu}\big\{h\|p\|_{1,\Omega} + \tilde{h}\|\boldsymbol{\lambda}\|_{1/2,\Gamma}\big\}, \\ \|(p - p_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)\|_{\mathrm{M}} &\leq C_3\,\nu\, h\|\mathbf{u}\|_{2,\Omega} + C_4\big\{h\|p\|_{1,\Omega} + \tilde{h}\|\boldsymbol{\lambda}\|_{1/2,\Gamma}\big\}, \end{aligned} \tag{4.20}$$

with C_1, C_2, C_3, $C_4 > 0$ depending only on Ω, provided $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $p \in \mathrm{H}^1(\Omega)$, and $\boldsymbol{\lambda} \in \mathbf{H}^{1/2}(\Gamma)$, and, for the Navier–Stokes case, assuming further that the mesh threshold (4.10) (with $l = 0$) and conditions (3.31) and (3.34) hold.

5 Computational results

In this section, we present four numerical examples illustrating the accuracy of our finite element schemes (2.16) and (3.26), and confirming the theoretical results. Our implementation is based on the *FreeFem++* software (see [20]), in conjunction with the direct linear solver UMFPACK (see [12]). For the Navier–Stokes problem, in particular, we employ a Newton method with a fixed tolerance $\mathrm{tol} = 10^{-8}$, and the iterations are terminated once the relative error between the coefficient vectors of two consecutive iterates is sufficiently small, i.e.,

$$\frac{|\mathrm{coeff}^{m+1} - \mathrm{coeff}^{m}|}{|\mathrm{coeff}^{m+1}|} \leq \mathrm{tol},$$

where $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^{dof}$, with dof the total number of degrees of freedom defining the discrete spaces $\mathbf{H}_h$, Q_h, and $\boldsymbol{\Lambda}_{\tilde{h}}$ described in Section 4.

The section is divided into two parts, each addressing a different objective. In the first, we consider a manufactured solution and investigate the convergence and stability of the proposed schemes: unstructured, uniformly refined meshes are employed, and the asymptotic convergence behavior is analyzed for the Hood–Taylor element with discontinuous edge polynomials of degree $k = 1$, presented in Section 4.1, and for the MINI element with piecewise constants, introduced in Section 4.2. In the second, we consider a more realistic scenario in which a fluid flows through an open channel with permeable membrane walls, in the spirit of the Berman flow studied in [4], here applied to the context of reverse osmosis desalination processes (see [1, 8]).

5.1 Convergence and stability test

In our first example, we illustrate the accuracy of our method by considering a manufactured exact solution defined on the unit square $\Omega = (0,1) \times (0,1)$, whose boundary is decomposed as

$$\Gamma_D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \qquad \Gamma_N = \{(1,y) : 0 \leq y \leq 1\},$$

where $\Gamma_1 = \{(x,0) : 0 \leq x \leq 1\}$, $\Gamma_2 = \{(x,1) : 0 \leq x \leq 1\}$, and $\Gamma_3 = \{(0,y) : 0 \leq y \leq 1\}$. We consider a viscosity $\nu = 1$ and adjust the terms on the right-hand side of (2.1) and (3.1) so that the exact solution is given by the functions

$$\mathbf{u} = \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ -\cos(\pi y)\sin(\pi x) \end{pmatrix} \quad \text{and} \quad p = \cos(\pi x)e^{y}.$$

The corresponding inhomogeneous Dirichlet datum is then given by

$$\mathbf{u}_D = \begin{cases} (0, -\sin(\pi x))^T & \text{on } \Gamma_1, \\ (0, \sin(\pi x))^T & \text{on } \Gamma_2, \\ (\sin(\pi y), 0)^T & \text{on } \Gamma_3, \end{cases}$$

while, on Γ_N, the prescribed datum $\mathbf{g}$ in (2.1) is given by $\mathbf{g} = (-e^{y}, -\nu\pi\cos(\pi y))^T$. Since the manufactured solution is smooth, the restriction of the exact Lagrange multiplier $\boldsymbol{\lambda}$ to Γ_D reduces to $\boldsymbol{\lambda}|_{\Gamma_D} = -(\nu\nabla\mathbf{u} - p\mathrm{I})\mathbf{n}|_{\Gamma_D}$.

The approximation errors for the velocity and pressure are measured, respectively, by

$$\mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}, \qquad \mathsf{e}(p) := \|p - p_h\|_{0,\Omega},$$

whereas, since the $\mathbf{H}^{-1/2}(\Gamma)$-norm is not directly computable, the error for the Lagrange multiplier is measured in the $\mathbf{L}^2$-norm on its support Γ_D, that is,

$$\mathsf{e}(\boldsymbol{\lambda}) := \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{0,\Gamma_D}.$$

The experimental order of convergence is computed from two successive mesh sizes h and h' according to

$$r(\%) := \frac{\log((\mathsf{e}(\%)/\mathsf{e}'(\%))}{\log(h/h')},$$

where $\mathsf{e}(\%)$ and $\mathsf{e}'(\%)$ denote the corresponding errors associated with the mesh sizes h and h', respectively, with $\% \in \{\mathbf{u}, p, \boldsymbol{\lambda}\}$.

Tables 5.1 and 5.2 report the convergence history of the discrete schemes (2.16) and (3.26), respectively, over a sequence of uniformly refined meshes. In both cases, the theoretical rates of convergence are attained for all unknowns. In particular, in Table 5.2 we observe that the Newton method converges in 7 iterations for each mesh. In addition, in both tables we notice that, for the MINI element, the convergence rate of the pressure is approximately 1.5, a superconvergence phenomenon already documented in the literature (see, e.g., [1, 11]).

Table 5.1: Degrees of freedom, mesh sizes, errors and rates of convergence and for the Galerkin scheme (2.16)

h	DoFs	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	$e(\boldsymbol{\lambda})$	$r(\boldsymbol{\lambda})$
$\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_1$							
0.1768	775	4.7632e-02	1.99	1.4646e-02	2.28	2.1741e-01	2.06
0.0884	2711	1.1996e-02	2.00	3.0172e-03	2.25	5.2248e-02	2.07
0.0442	10039	2.9966e-03	2.00	6.3229e-04	2.14	1.2449e-02	2.04
0.0221	38519	7.4802e-04	2.00	1.4322e-04	2.07	3.0290e-03	2.02
0.0110	150775	1.8682e-04	2.00	3.4097e-05	2.03	7.4760e-04	2.01
$\mathbb{P}_{1,b} - \mathbb{P}_1 - \mathbb{P}_0$							
0.4714	94	1.8475e+00	1.10	2.6603e+00	1.65	9.2744e+00	1.09
0.1088	1354	3.6612e-01	1.01	2.3588e-01	1.55	1.3045e+00	1.05
0.0615	4014	2.0533e-01	1.01	9.7574e-02	1.55	6.9201e-01	1.05
0.0429	8074	1.4270e-01	1.01	5.5660e-02	1.55	4.6729e-01	1.01
0.0329	13534	1.0936e-01	1.00	3.6973e-02	1.54	3.5542e-01	1.02

5.2 Study of a membrane channel

In our second example, we are interested in studying a fluid flow in a rectangular open channel with permeable membrane walls. The computational domain is given by $\Omega := (0, a) \times (0, b)$, where $a = 20\,\mathrm{mm}$ and $b = 2\,\mathrm{mm}$. Three circular spacers of radius $r = 0.4\,\mathrm{mm}$ are embedded inside the channel, and we consider two configurations. In the first, referred to as the submerged configuration, the spacers are centered at $(7\,\mathrm{mm}, 1.0\,\mathrm{mm})$, $(10\,\mathrm{mm}, 1.0\,\mathrm{mm})$, and

Table 5.2: Degrees of freedom, mesh sizes, errors, rates of convergence and number of iterations for the Galerkin scheme (3.26)

h	DoFs	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	$e(\boldsymbol{\lambda})$	$r(\boldsymbol{\lambda})$	IT
				$\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_1$				
0.1768	775	4.8043e-02	2.00	1.1269e-02	2.16	1.1782e-01	2.04	7
0.0884	2711	1.2043e-02	2.00	2.5265e-03	2.13	2.8684e-02	2.05	7
0.0442	10039	3.0019e-03	2.00	5.7615e-04	2.07	6.9386e-03	2.03	7
0.0221	38519	7.4865e-04	2.00	1.3692e-04	2.04	1.7032e-03	2.01	7
0.0110	150775	1.8689e-04	2.00	3.3364e-05	2.02	4.2218e-04	2.01	7
				$\mathbb{P}_{1,b} - \mathbb{P}_1 - \mathbb{P}_0$				
0.4714	94	2.0366e+00	1.17	1.3954e+00	1.68	5.1435e+00	1.11	7
0.1088	1354	3.6656e-01	1.01	1.1933e-01	1.55	7.0989e-01	1.04	7
0.0615	4014	2.0547e-01	1.01	4.9142e-02	1.56	3.7888e-01	1.04	7
0.0429	8074	1.4275e-01	1.01	2.7991e-02	1.55	2.5653e-01	1.00	7
0.0329	13534	1.0939e-01	1.00	1.8593e-02	1.55	1.9538e-01	1.02	7

$(13\,\mathrm{mm}, 1.0\,\mathrm{mm})$, that is, aligned along the channel centerline. In the second, referred to as the zigzag configuration, the spacers are instead centered at $(7\,\mathrm{mm}, 0.4\,\mathrm{mm})$, $(10\,\mathrm{mm}, 1.6\,\mathrm{mm})$, $(13\,\mathrm{mm}, 0.4\,\mathrm{mm})$, that is, they are alternately positioned close to the lower and upper channel walls, so as to generate a more complex flow pattern, as illustrated in Figure 5.1. These obstacles represent spacer filaments commonly employed in reverse osmosis membrane modules to promote mixing and reduce concentration polarization (see [1, 8]).

The boundary of the channel, Γ, is decomposed into disjoint parts according to their physical role: the inlet boundary Γ_{in}, located at $x = 0$; the membrane boundary Γ_{m}, located at $y = 0$ and $y = b$; the spacer surfaces Γ_{w}, associated with the three circular filaments embedded inside the channel; and the outflow boundary Γ_N, located at $x = a$. The Dirichlet boundary is then defined as $\Gamma_D := \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{m}} \cup \Gamma_{\mathrm{w}}$.

We consider the Navier–Stokes equations (3.1) with null forcing datum $\mathbf{f} = \mathbf{0}$, dynamic viscosity $\nu = 8.9 \times 10^{-7}$ kg mm^{-1}s^{-1}, density $\rho = 1027.2 \times 10^{-9}$ kg/mm^3, and Dirichlet datum

$$\mathbf{u}_D := \begin{cases} \mathbf{u}_{\mathrm{in}} & \text{on } \Gamma_{\mathrm{in}}, \\ v_w \mathbf{n} & \text{on } \Gamma_{\mathrm{m}}, \\ \mathbf{0} & \text{on } \Gamma_{\mathrm{w}}, \end{cases}$$

with

$$\mathbf{u}_{\mathrm{in}} := \begin{pmatrix} \left[u_0 - \dfrac{2x}{b} v_w\right] \left[1.5(1-\lambda^2)\right] \left[1 - \dfrac{\mathrm{Re}}{420}(2 - 7\lambda^2 - 7\lambda^4)\right] \\ v_w \left[0.5\lambda(3-\lambda^2) - \dfrac{\mathrm{Re}}{280}\lambda(2 - 3\lambda^2 + \lambda^6)\right] \end{pmatrix}, \tag{5.1}$$

where v_w is the permeate velocity, $\lambda = 2y/b - 1$ is a scaling parameter with $y \in [0, b]$, $u_0 = 120\,\mathrm{mm/s}$ is the average velocity at the channel entrance, and Re is the Reynolds number, computed as

$$\mathrm{Re} = \frac{\rho\, v_w (b/2)}{\nu}.$$

This choice is inspired by Berman-type flows (see, e.g., [4]), and the physical parameters are taken to describe seawater (see [1, 8]).

Notice that the channel length is chosen sufficiently large to allow the inflow profile to evolve into a fully developed parabolic regime.

Figure 5.1 displays the streamlines and velocity magnitude obtained for $u_0 = 120\,\mathrm{mm/s}$ under the submerged (top) and zigzag (bottom) spacer configurations, while Figures 5.2 and 5.3 show the corresponding pressure fields and the pressure drop along the membrane wall, $\Delta p(x, 0) := p(0, 0) - p(x, 0)$. In the submerged configuration, the main flow splits symmetrically around each spacer, generating closed recirculation zones downstream of every filament, and reaches a peak velocity of about $6.0 \times 10^2\,\mathrm{mm/s}$; in the zigzag configuration, the flow instead follows a pronounced serpentine path, alternately deflected toward each membrane wall, with a lower peak velocity of about $3.1 \times 10^2\,\mathrm{mm/s}$. This behavior is consistent with the trends reported in [8] for analogous spacer arrangements: alternating spacer placement promotes a dominant, meandering flow path that enhances mixing near the membrane, at the expense of a less well-defined recirculation pattern behind each obstacle.

This contrast in flow structure is also reflected in the pressure loss. The submerged configuration exhibits a substantially larger pressure range and an overall pressure drop of about 0.068, roughly 1.5 times larger than the drop of about 0.045 observed for the zigzag configuration. This is consistent with the mechanism identified in [8]: each spacer produces two narrower flow constrictions in the submerged configuration, one along each membrane wall, but only a single, wider constriction in the zigzag configuration, so that the submerged arrangement incurs a larger hydraulic penalty despite using spacers of the same size and number. The spatial pressure pattern differs accordingly: in the submerged configuration the pressure decreases smoothly and nearly symmetrically about the centerline, whereas in the zigzag configuration it follows the meandering flow path, with a localized swirling pattern between the second and third spacers reflecting the stronger local velocity gradients there.

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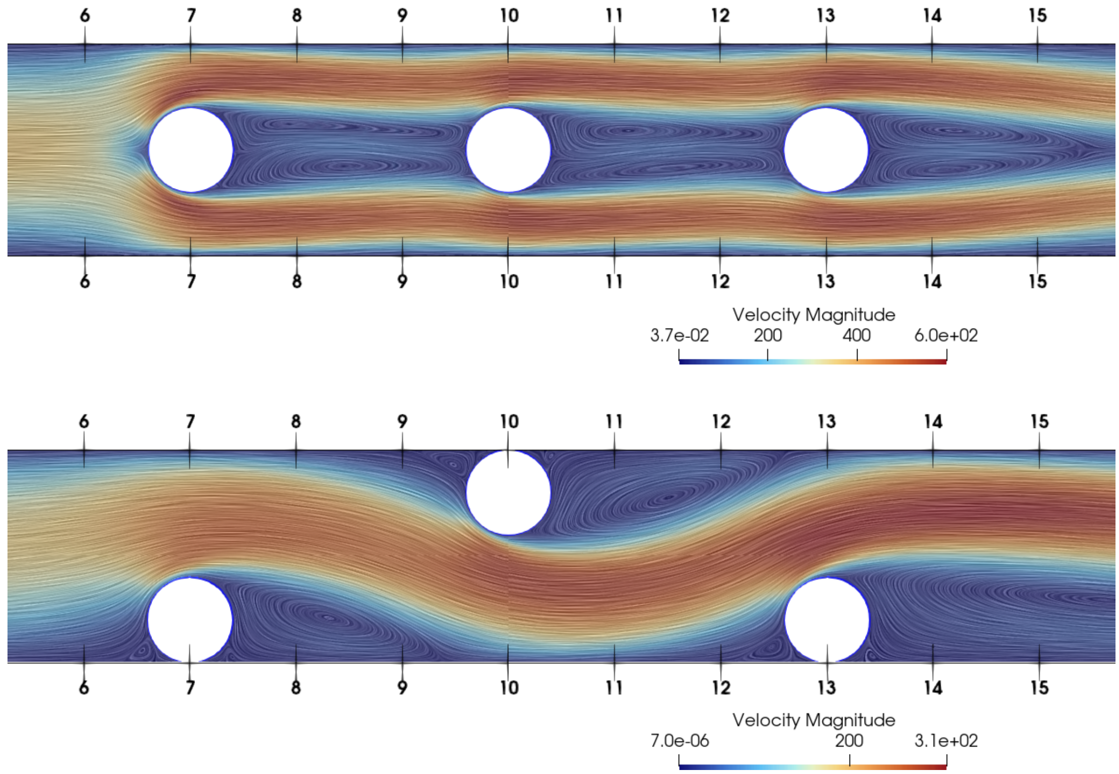

Figure 5.1: Example 5.2. Comparison of streamlines for an inlet velocity profile $u_0 = 120\,\mathrm{mm\,s^{-1}}$ obtained for the submerged spacer configuration (top) and the zigzag spacer configuration (bottom).

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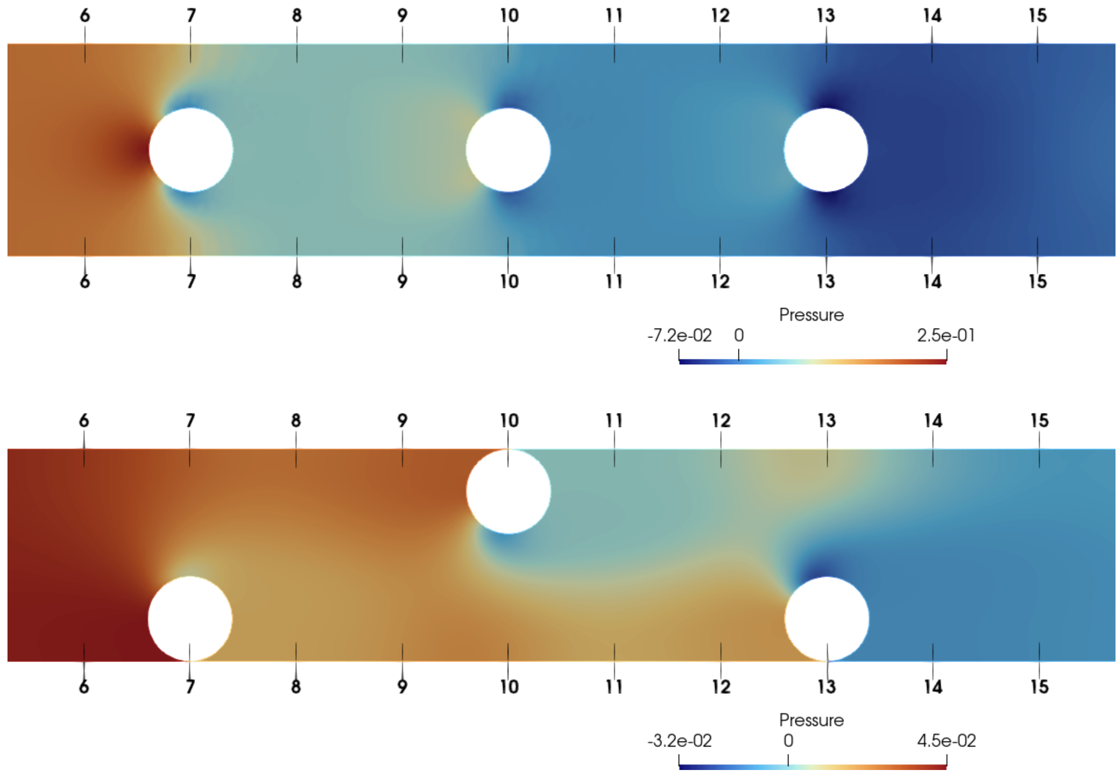

Figure 5.2: Example 5.2. Comparison of pressure contours for an inlet velocity profile $u_0 = 120\,\mathrm{mm\,s^{-1}}$ obtained for the submerged spacer configuration (top) and the zigzag spacer configuration (bottom).

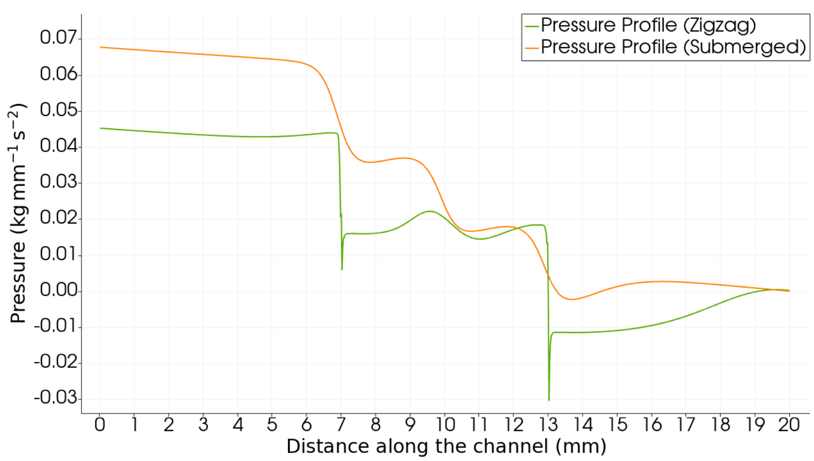

Figure 5.3: Example 5.2. Pressure drop along the channel for the submerged and zigzag spacer configurations with an inlet velocity profile of $u_0 = 120\,\mathrm{mm\,s^{-1}}$.

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