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Abstract

In this paper, we introduce and analyze a new skew-symmetry-based mixed formulation for the Navier–Stokes equations with variable viscosity, which leads to the derivation of two families of stable mixed finite element methods for their numerical approximation. Motivated by the need to employ the symmetric stress tensor in this setting, our approach introduces the strain rate and vorticity tensors as independent unknowns. Their average, acting on the velocity, provides one half of the convective term, while the remaining contribution is accounted for in the momentum equation by introducing an incomplete symmetric stress tensor containing half of the tensor product of the velocity with itself. The resulting variational system exhibits a typical saddle-point structure in Banach spaces, but with a nonlinear main operator. A key step in its derivation is the introduction of a null expression given by the tensor inner product of suitably chosen symmetric and skew-symmetric tensors. We then show that the variational system is equivalent to a reduced problem posed on the kernel of the coupling bilinear form. Existence and uniqueness of a solution to the latter are obtained by means of a fixed-point argument that linearizes the nonlinear part of the aforementioned operator, yielding a skew-symmetric bilinear form in its place, and by applying the Schauder and Banach fixed-point theorems. In contrast, the solvability analysis of the Galerkin scheme is carried out directly in the original saddle-point setting. Under suitable hypotheses on the finite element subspaces, existence and uniqueness of a discrete solution are proved through a similar fixed-point strategy, using the Brouwer and Banach fixed-point theorems. A priori error estimates are then derived, including a corresponding bound for the pressure approximation, with this unknown having been eliminated from the continuous formulation and subsequently recovered by postprocessing. PEERS- and AFW-based finite element subspaces of order $k \geq 0$, defined on shape-regular simplicial meshes, are shown to satisfy the required stability conditions, and the associated convergence rates are established. Finally, several numerical experiments illustrating the performance of the methods and confirming the theoretical findings are reported.

Key words: Navier-Stokes equations, skew-symmetry, stress-based formulation, Banach spaces, mixed finite element methods, a priori error analysis

Mathematics subject classifications (2000): 35J66, 65J15, 65N12, 65N15, 65N30, 47J26, 76D07.

1 Introduction

The diffusion term in the Navier–Stokes equations, given by the divergence of twice the viscosity times the strain rate tensor, reduces to the viscosity times the Laplacian of the velocity, provided that the

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viscosity is constant and the flow is incompressible. Thus, it is only in this case that the velocity gradient, rather than its symmetric part, can be used to define a total or partial pseudostress for the purpose of deriving a mixed variational formulation of the model. Indeed, we begin by referring to [8], where a pseudostress tensor combining the velocity gradient and the convective term is employed to introduce and analyze a new momentum-conservative mixed finite element method for the Navier–Stokes equations, with the pseudostress and velocity as the main unknowns. Building on this pseudostress formulation under the assumptions of constant viscosity and incompressibility, the approach has been extended, for instance, to fully-mixed formulations for the Boussinesq model [12] and the Navier–Stokes/Darcy coupled problem [13].

Concerning the literature on Navier–Stokes models in which the symmetric part of the velocity gradient is considered in the constitutive equation, and the Cauchy stress tensor is therefore introduced, we begin by referring to [23] as a pioneering contribution. In that work, the authors propose and analyze a skew-symmetric dual-mixed finite element method in which the Cauchy stress tensor is treated as a primary unknown. They construct finite element spaces satisfying the corresponding inf-sup conditions, including those associated with the augmented PEERS and AFW elements. This construction is restricted to the two-dimensional setting. The same work also introduces a new family of finite elements for both two- and three-dimensional domains, based on the div-stable elements of Scott and Vogelius [25, 27]. This framework was subsequently extended to the stationary Boussinesq problem in [17], where both the velocity gradient and a Bernoulli stress tensor are introduced as auxiliary unknowns. In particular, the discrete spaces for the fluid variables are constructed on triangulations obtained through barycentric refinements (Alfeld splits), which provide the macroelement structure required to establish the inf-sup condition and a discrete Korn-type inequality. More recently, [15] developed a fully-mixed formulation for the Boussinesq problem with temperature-dependent viscosity within a Banach space framework. In that work, the velocity gradient and Bernoulli stress, together with the corresponding variables associated with the heat equation, are incorporated as additional unknowns, and the resulting discrete spaces are again built on barycentrically refined meshes. Finally, we mention [16], where the aforementioned approach was applied to the Oberbeck–Boussinesq system.

On the other hand, several works have addressed the design and analysis of numerical schemes based on skew-symmetric formulations, which allow for less restrictive data assumptions in the analysis of existence and uniqueness, see, e.g., [10, 2, 7, 3, 11]. In [10], a fully-mixed formulation for the Brinkman–Forchheimer and double-diffusion equations is analyzed using a skew-symmetric approach. The resulting problem is reformulated as a fixed-point equation, and its unique solvability, for both the continuous and discrete problems, is established by combining the Banach fixed-point theorem with classical results on nonlinear monotone operators and the Babuška–Brezzi theory in reflexive Banach spaces. Similar arguments are subsequently extended in [7] to a Banach-space-based fully-mixed finite element method for the convective Brinkman–Forchheimer equations coupled with a nonlinear transport phenomenon. In [3], the skew-symmetry of the operators arising in the convective Brinkman–Forchheimer model and its coupling with Maxwell’s equations is exploited to establish the unique solvability of the continuous and discrete problems. This is achieved by recasting the mixed formulation as a fixed-point equation and applying Banach’s fixed-point theorem, together with classical results on nonlinear monotone operators and a suitable small-data assumption. More recently, this approach has been extended in [11] to a transient Oseen-type Kelvin–Voigt–Brinkman–Forchheimer model for viscoelastic flows at high velocities in highly porous media. Besides the velocity field, the formulation introduces the velocity gradient and the viscoelastic pseudostress as auxiliary unknowns, leading to a three-field mixed variational formulation within a Banach space framework that exploits the skew-symmetric structure of one of the operators involved.

Motivated by the aforementioned developments, the purpose of the present paper is to develop and analyze a new mixed finite element method for the Navier–Stokes problem that combines the use of the symmetric part of the velocity gradient in the constitutive equation with the skew-symmetry-based

approach described above, while avoiding barycentric mesh refinements. To this end, we introduce the strain rate and vorticity tensors as additional unknowns, along with an incomplete symmetric pseudostress tensor incorporating one half of the tensor product of the velocity with itself. We then eliminate the pressure unknown by exploiting the incompressibility condition, and subsequently recover it through a post-processing procedure. The nonlinearity of the model and the mixed nature of the resulting formulation naturally lead to a variational problem posed in a Banach space framework, as in [9, 8, 12, 22, 13]. The incorporation of a vanishing term arising from the tensor inner product of suitably chosen symmetric and skew-symmetric tensors plays a key role in its derivation.

We then introduce a reduced problem and prove its equivalence with the original mixed formulation. Following [8, 13, 15], we combine a fixed-point argument, the abstract results developed in [15] to establish the required inf-sup condition and ellipticity bound needed to apply the well-known Banach–Nečas–Babuška theorem, suitable small-data assumptions, and the Schauder and Banach fixed-point theorems to establish the existence and uniqueness of the solution to the reduced formulation, and hence to the original mixed formulation. We stress that the small-data assumption required for the existence of a solution is imposed only on the boundary data. However, this does not constitute a restrictive condition, since, by means of a suitable solenoidal lifting, the analysis can be reduced to the case of homogeneous Dirichlet boundary conditions, for which the corresponding small-data assumption is trivially satisfied. We then turn to the discrete problem and derive a stable Galerkin scheme for general conforming finite-dimensional subspaces on general shape-regular triangulations. The existence and uniqueness analysis follows a strategy similar to that employed in the continuous case, but is now carried out directly for the original discrete saddle-point problem, with Brouwer’s theorem used to establish the existence of a discrete solution. Moreover, by applying an ad hoc Strang-type lemma in Banach spaces established in [20], we derive the corresponding *a priori* error estimates for these general discrete spaces. A main novelty of our approach is that the required stability properties are obtained without resorting to barycentric mesh refinements, allowing for a direct finite element construction on general triangulations. For specific choices of finite element spaces, we employ PEERS and AFW elements of order $k \geq 0$ for the velocity, vorticity, and pseudostress variables. The strain rate tensor is approximated by piecewise polynomials of degree $\leq k + n$ and $\leq k + 1$ for the PEERS and AFW approaches, respectively. For these finite element spaces, we prove that the resulting discrete methods converge with optimal rates, including the optimal convergence of the postprocessed pressure. Finally, we provide numerical experiments in 2D and 3D to illustrate the performance of the proposed method.

We have organized the rest of this article as follows. The remainder of this section introduces the notation and functional spaces used throughout the analysis. Section 2 presents the Navier–Stokes model and derives its skew-symmetry-based reformulation, while Section 3 introduces the corresponding mixed variational formulation. Section 4 establishes the well-posedness of the continuous formulation, and Section 5 introduces and analyzes the Galerkin scheme, including the corresponding stability and error estimates and the particular finite element spaces considered. Finally, Section 6 presents numerical examples in 2D and 3D, with and without manufactured solutions, illustrating the performance of the proposed mixed finite element method.

Preliminary notations

In this paper Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \in \{2, 3\}$, whose outward normal at $\Gamma := \partial\Omega$ is denoted by $\boldsymbol{\nu}$. Standard notation is adopted for Lebesgue spaces $L^t(\Omega)$ and Sobolev spaces $W^{l,t}(\Omega)$, with $l \geq 0$ and $t \in [1, +\infty)$, whose corresponding norms, either for the scalar and vector case, are denoted by $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{l,t;\Omega}$, respectively. Note, in particular, that $W^{0,t}(\Omega) = L^t(\Omega)$, and if $t = 2$ we write $H^l(\Omega)$ instead of $W^{l,2}(\Omega)$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$, respectively. Furthermore, $H^{1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma)$) is the space of traces of functions of $H^1(\Omega)$

(resp. $\mathbf{H}^1(\Omega)$) with dual $\mathbf{H}^{-1/2}(\Gamma)$ (resp. $\mathbf{H}^{-1/2}(\Gamma)$). In turn, given any generic scalar functional space $\mathbb{M}$, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vector and tensor counterparts, whereas $\|\cdot\|$ is employed for the norm of any element or operator whenever there is no confusion about the spaces to which they belong. On the other hand, as usual, $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathbb{R}^{n \times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathbb{R}^n$. In addition, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, respectively, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j} \right)_{i,j=1,n}, \quad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^n \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Furthermore, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product operators, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^\mathbf{t} = (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) = \sum_{i=1}^n \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij} \zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^\mathbf{d} := \boldsymbol{\tau} - \frac{1}{n} \operatorname{tr}(\boldsymbol{\tau}) \mathbb{I}.$$

Moreover, given $t \in (1, +\infty)$, we also introduce the Banach space

$$\mathbb{H}(\mathbf{div}_t; \Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \right\}, \quad (1.1)$$

which is endowed with the natural norm defined by

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t; \Omega} := \|\boldsymbol{\tau}\|_{0, \Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0, t; \Omega} \quad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t; \Omega). \quad (1.2)$$

Then, proceeding as in [19, eq. (1.43), Section 1.3.4] (see also [9, Section 4.1] and [15, Section 3.1]), one can prove that for each $t \in \begin{cases} (1, +\infty] & \text{if } n = 2 \\ [6/5, +\infty] & \text{if } n = 3 \end{cases}$, there holds

$$\langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{v} \rangle = \int_{\Omega} \left\{ \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \right\} \quad \forall (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t; \Omega) \times \mathbf{H}^1(\Omega), \quad (1.3)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. It is important to remark here that the second term on the right-hand side of (1.3) is well-defined because of the continuous embedding of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^{t'}(\Omega)$, where t' is the conjugate of t , which reduces to $t' \in \begin{cases} [1, +\infty) & \text{if } n = 2 \\ [1, 6] & \text{if } n = 3 \end{cases}$.

2 The Navier-Stokes equations

The Navier-Stokes problem consists of finding the velocity $\mathbf{u}$ and the pressure p of a fluid occupying the region Ω , such that

$$\begin{aligned} -\mathbf{div}(2\mu \mathbf{e}(\mathbf{u})) + \delta (\nabla \mathbf{u}) \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}(\mathbf{u}) &= 0 \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma, \quad \int_{\Omega} p = 0, \end{aligned} \quad (2.1)$$

where $\mathbf{e}(\mathbf{u}) := \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^\mathbf{t})$ is the symmetric part of $\nabla \mathbf{u}$, also known as the strain-rate tensor, and the given data are a function $\mu : \Omega \rightarrow \mathbb{R}^+$ describing the viscosity, a constant δ denoting the density, a volume force $\mathbf{f}$, and the boundary velocity $\mathbf{u}_D$. The right spaces to which $\mathbf{f}$ and $\mathbf{u}_D$ need to belong are derived later on. We also note that the mean value condition on the pressure p , which means

$$p \in \mathbf{L}_0^2(\Omega) := \left\{ z \in \mathbf{L}^2(\Omega) : \int_{\Omega} z = 0 \right\}, \quad (2.2)$$

is imposed for the uniqueness of this unknown. In turn, due to the incompressibility condition of the fluid (first eq. in the second row of (2.1)), the Dirichlet datum $\mathbf{u}_D$ must formally satisfy the compatibility condition

$$\int_{\Gamma} \mathbf{u}_D \cdot \boldsymbol{\nu} = 0. \quad (2.3)$$

Furthermore, we assume that there exist constants $\mu_1, \mu_2 > 0$, such that

$$\mu_1 \leq \mu(x) \leq \mu_2 \quad \forall x \in \Omega. \quad (2.4)$$

Since we are interested in a mixed variational formulation of our problem, we decompose the velocity gradient as

$$\nabla \mathbf{u} = \boldsymbol{\rho} + \boldsymbol{\gamma} \quad \text{in } \Omega, \quad (2.5)$$

where

$$\boldsymbol{\rho} := \mathbf{e}(\mathbf{u}) \quad \text{and} \quad \boldsymbol{\gamma} := \frac{1}{2} (\nabla \mathbf{u} - (\nabla \mathbf{u})^t), \quad (2.6)$$

known as the strain rate and vorticity tensors, respectively, constitute auxiliary unknowns that, according to the fact that $\text{tr}(\mathbf{e}(\mathbf{u})) = \text{div}(\mathbf{u}) = 0$, and the skew-symmetry of $\boldsymbol{\gamma}$, will be sought, respectively, in

$$\begin{aligned} \mathbb{L}_{\text{tr}}^2(\Omega) &:= \left\{ \boldsymbol{\rho} \in \mathbb{L}^2(\Omega) : \text{tr}(\boldsymbol{\rho}) = 0 \right\} \quad \text{and} \\ \mathbb{L}_{\text{sk}}^2(\Omega) &:= \left\{ \boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \boldsymbol{\eta}^t = -\boldsymbol{\eta} \right\}. \end{aligned} \quad (2.7)$$

Next, it is easily seen, resorting again to the incompressibility of the fluid, that $\mathbf{div}(\mathbf{u} \otimes \mathbf{u}) = (\nabla \mathbf{u})\mathbf{u}$, which, along with (2.5), yields

$$(\nabla \mathbf{u})\mathbf{u} = \frac{1}{2} \mathbf{div}(\mathbf{u} \otimes \mathbf{u}) + \frac{1}{2} (\boldsymbol{\rho} + \boldsymbol{\gamma}) \mathbf{u}.$$

In this way, defining the symmetric tensor

$$\boldsymbol{\sigma} := 2\mu \boldsymbol{\rho} - \frac{\delta}{2} (\mathbf{u} \otimes \mathbf{u}) - p \mathbb{I} \quad \text{in } \Omega, \quad (2.8)$$

we find that the first equation of (2.1) can be rewritten as

$$-\mathbf{div}(\boldsymbol{\sigma}) + \frac{\delta}{2} (\boldsymbol{\rho} + \boldsymbol{\gamma}) \mathbf{u} = \mathbf{f} \quad \text{in } \Omega. \quad (2.9)$$

We stress here that, for convenience of the continuous and discrete analyses to be performed later on, the strong symmetry of $\boldsymbol{\rho}$ has not been incorporated explicitly in the definition of the space $\mathbb{L}_{\text{tr}}^2(\Omega)$ (cf. (2.7)). Instead of it, we impose later on the weak symmetry of $\boldsymbol{\sigma}$ (cf. (3.7)), which, bearing in mind (2.8), is equivalent to doing it to $\mu \boldsymbol{\rho}$, and hence to $\boldsymbol{\rho}$. Now, applying deviatoric operator $^{\text{d}}$ and the matrix trace to (2.8), which can be seen as the constitutive equation expressing $\boldsymbol{\sigma}$ in terms of $\mathbf{u}$ and p , we can easily show that the latter and the incompressibility condition are equivalent to the pair of equations

$$\begin{aligned} \boldsymbol{\sigma}^{\text{d}} &= 2\mu \boldsymbol{\rho} - \frac{\delta}{2} (\mathbf{u} \otimes \mathbf{u})^{\text{d}} \quad \text{in } \Omega, \quad \text{and} \\ p &= -\frac{1}{n} \text{tr} \left(\boldsymbol{\sigma} + \frac{\delta}{2} (\mathbf{u} \otimes \mathbf{u}) \right) \quad \text{in } \Omega. \end{aligned} \quad (2.10)$$

Thus, since the pressure can be computed later on by the postprocessed formula suggested by the second row of (2.10), we eliminate this unknown from the system, thus yielding the following set of

equations with unknowns $\mathbf{u}$, $\boldsymbol{\rho}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\sigma}$

$$\begin{aligned} \nabla \mathbf{u} &= \boldsymbol{\rho} + \boldsymbol{\gamma} \quad \text{in } \Omega, \quad \boldsymbol{\sigma}^d = 2\mu \boldsymbol{\rho} - \frac{\delta}{2}(\mathbf{u} \otimes \mathbf{u})^d \quad \text{in } \Omega, \\ -\operatorname{div}(\boldsymbol{\sigma}) + \frac{\delta}{2}(\boldsymbol{\rho} + \boldsymbol{\gamma}) \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_D \quad \text{on } \Gamma, \quad \int_{\Omega} \operatorname{tr}\left(\boldsymbol{\sigma} + \frac{\delta}{2}(\mathbf{u} \otimes \mathbf{u})\right) = 0. \end{aligned} \quad (2.11)$$

We notice here that the incompressibility condition $0 = \operatorname{div}(\mathbf{u}) = \operatorname{tr}(\boldsymbol{\rho})$ is implicitly contained in the second equation of the first row of (2.11) since the matrix trace of each deviatoric tensor is 0.

3 The mixed variational formulation

We now proceed to derive the variational formulation of (2.11). Multiplying the first equation in the first row of (2.11) by $\boldsymbol{\tau} \in \mathbb{H}(\operatorname{div}_t; \Omega)$, with $t \in \begin{cases} (1, +\infty) & \text{if } n = 2 \\ [6/5, +\infty) & \text{if } n = 3 \end{cases}$, and assuming originally that $\mathbf{u} \in \mathbf{H}^1(\Omega)$, which implicitly requires to consider $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, we can apply the integration by parts formula (1.3) to deduce that

$$\int_{\Omega} (\boldsymbol{\rho} + \boldsymbol{\gamma}) : \boldsymbol{\tau} + \int_{\Omega} \mathbf{u} \cdot \operatorname{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u}_D \rangle \quad \forall \boldsymbol{\tau} \in \mathbb{H}(\operatorname{div}_t; \Omega). \quad (3.1)$$

It is clear, thanks to Cauchy-Schwarz's inequality, that the first term of (3.1) is well defined since $\boldsymbol{\rho}$, $\boldsymbol{\tau}$, and $\boldsymbol{\gamma}$ belong to $\mathbb{L}^2(\Omega)$. In turn, knowing that $\operatorname{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)$, we realize from the second term and Hölder's inequality that, instead of $\mathbf{u} \in \mathbf{H}^1(\Omega)$, it suffices to look for $\mathbf{u} \in \mathbf{L}^{t'}(\Omega)$, where t' is the conjugate of t . Now, testing the second equation in the first row of (2.11) with $\boldsymbol{\varrho} \in \mathbb{L}_{\operatorname{tr}}^2(\Omega)$, using, thanks to the null trace of $\boldsymbol{\varrho}$, that $\boldsymbol{\varrho} : \boldsymbol{\zeta}^d = \boldsymbol{\varrho} : \boldsymbol{\zeta}$ for any tensor field $\boldsymbol{\zeta}$, and observing additionally that for any vector fields $\mathbf{v}$, $\mathbf{w}$, there holds

$$\boldsymbol{\zeta} : (\mathbf{v} \otimes \mathbf{w}) = \boldsymbol{\zeta} \mathbf{w} \cdot \mathbf{v}, \quad (3.2)$$

we formally get

$$2 \int_{\Omega} \mu \boldsymbol{\rho} : \boldsymbol{\varrho} - \int_{\Omega} \boldsymbol{\varrho} : \boldsymbol{\sigma} - \frac{\delta}{2} \int_{\Omega} \boldsymbol{\varrho} \mathbf{u} \cdot \mathbf{u} = 0 \quad \forall \boldsymbol{\varrho} \in \mathbb{L}_{\operatorname{tr}}^2(\Omega). \quad (3.3)$$

Being both $\boldsymbol{\rho}$ and $\boldsymbol{\varrho}$ in $\mathbb{L}^2(\Omega)$, we notice here that the first term is well-defined, whereas the second one makes sense if $\boldsymbol{\sigma}$ is sought in $\mathbb{L}^2(\Omega)$. Furthermore, invoking again Cauchy-Schwarz's inequality, and using the same notations from (3.2), we obtain

$$\left| \int_{\Omega} \boldsymbol{\zeta} : (\mathbf{v} \otimes \mathbf{w}) \right| = \left| \int_{\Omega} \boldsymbol{\zeta} \mathbf{w} \cdot \mathbf{v} \right| \leq \sqrt{n} \|\boldsymbol{\zeta}\|_{0,\Omega} \|\mathbf{w}\|_{0,4;\Omega} \|\mathbf{v}\|_{0,4;\Omega}, \quad (3.4)$$

so that, in particular, for the third term of (3.3) there holds

$$\left| \int_{\Omega} \boldsymbol{\varrho} \mathbf{u} \cdot \mathbf{u} \right| \leq \sqrt{n} \|\boldsymbol{\varrho}\|_{0,\Omega} \|\mathbf{u}\|_{0,4;\Omega}^2, \quad (3.5)$$

which necessarily yields $t' = 4$, that is, equivalently, $t = 4/3$, whence $\mathbf{u}$ must be actually sought in $\mathbf{L}^4(\Omega)$, whereas $\boldsymbol{\tau}$ lies in $\mathbb{H}(\operatorname{div}_{4/3}; \Omega)$. On the other hand, looking for $\boldsymbol{\sigma}$ in the same space of its corresponding test function $\boldsymbol{\tau}$, we test the equation in the second row of (2.11) against $\mathbf{v} \in \mathbf{L}^4(\Omega)$, which, under the assumption that $\mathbf{f}$ belongs to $\mathbf{L}^{4/3}(\Omega)$, gives

$$- \int_{\Omega} \mathbf{v} \cdot \operatorname{div}(\boldsymbol{\sigma}) + \frac{\delta}{2} \int_{\Omega} (\boldsymbol{\rho} + \boldsymbol{\gamma}) \mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{L}^4(\Omega). \quad (3.6)$$

Thanks to Hölder's inequality and the fact that $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$, the first term of (3.6) is clearly well-defined, whereas, employing again (3.4), there hold

$$\left| \int_{\Omega} (\boldsymbol{\rho} + \boldsymbol{\gamma}) \mathbf{u} \cdot \mathbf{v} \right| \leq \sqrt{n} (\|\boldsymbol{\rho}\|_{0,\Omega} + \|\boldsymbol{\gamma}\|_{0,\Omega}) \|\mathbf{u}\|_{0,4;\Omega} \|\mathbf{v}\|_{0,4;\Omega},$$

which shows that the second term makes sense as well. Finally, as previously announced, the weak symmetry of $\boldsymbol{\sigma}$ is imposed as

$$\int_{\Omega} \boldsymbol{\eta} : \boldsymbol{\sigma} = 0 \quad \forall \boldsymbol{\eta} \in \mathbb{L}_{\text{sk}}^2(\Omega). \quad (3.7)$$

Next, we consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3}; \Omega) = \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \oplus \mathbb{R}\mathbb{I}, \quad (3.8)$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega) : \int_{\Omega} \text{tr}(\boldsymbol{\tau}) = 0 \right\}.$$

In particular, there exist unique $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ and $c_0 \in \mathbb{R}$ such that $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0 \mathbb{I}$. Moreover, according to the second equation in the third row of (2.11), we get

$$c_0 = \frac{1}{n|\Omega|} \int_{\Omega} \text{tr}(\boldsymbol{\sigma}) = -\frac{\delta}{2n|\Omega|} \int_{\Omega} \text{tr}(\mathbf{u} \otimes \mathbf{u}). \quad (3.9)$$

Hence, the constant c_0 can be computed once the velocity $\mathbf{u}$ is known, so that it only remains to obtain $\boldsymbol{\sigma}_0$. In this regard, we notice that (3.3), (3.6), and (3.7) remain unchanged if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$. In addition, due to the fact that $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$ are sought in $\mathbb{L}_{\text{tr}}^2(\Omega)$ and $\mathbb{L}_{\text{sk}}^2(\Omega)$, respectively, and using the compatibility condition (2.3), we realize that testing (3.1) against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ (recall that $t = 4/3$) is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$, which, multiplied by -1 , becomes

$$-\int_{\Omega} (\boldsymbol{\rho} + \boldsymbol{\gamma}) : \boldsymbol{\tau} - \int_{\Omega} \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) = -\langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u}_D \rangle \quad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega). \quad (3.10)$$

Thus, redenoting from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$, adding (3.3) and (3.6), and subtracting (3.7), we arrive at

$$\begin{aligned} & 2 \int_{\Omega} \mu \boldsymbol{\rho} : \boldsymbol{\varrho} + \frac{\delta}{2} \int_{\Omega} (\boldsymbol{\rho} + \boldsymbol{\gamma}) \mathbf{u} \cdot \mathbf{v} - \frac{\delta}{2} \int_{\Omega} \boldsymbol{\varrho} \mathbf{u} \cdot \mathbf{u} \\ & - \int_{\Omega} (\boldsymbol{\varrho} + \boldsymbol{\eta}) : \boldsymbol{\sigma} - \int_{\Omega} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \end{aligned} \quad (3.11)$$

for all $\vec{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{L}^4(\Omega) \times \mathbb{L}_{\text{sk}}^2(\Omega)$, for all $\boldsymbol{\varrho} \in \mathbb{L}_{\text{tr}}^2(\Omega)$. Then, besides the novel incorporation of (2.9), another key feature of the present approach is the introduction of the expression

$$-\frac{\delta}{2} \int_{\Omega} \boldsymbol{\eta} \mathbf{u} \cdot \mathbf{u}$$

at the end of the first row of (3.11). Since this term vanishes as a consequence of the identity $\boldsymbol{\eta} \mathbf{u} \cdot \mathbf{u} = \boldsymbol{\eta} : (\mathbf{u} \otimes \mathbf{u})$, the skew-symmetry of $\boldsymbol{\eta}$, and the symmetry of $(\mathbf{u} \otimes \mathbf{u})$, its addition does not alter the meaning of the equation. In this way, (3.11) becomes

$$\begin{aligned} & 2 \int_{\Omega} \mu \boldsymbol{\rho} : \boldsymbol{\varrho} + \frac{\delta}{2} \int_{\Omega} \left\{ (\boldsymbol{\rho} + \boldsymbol{\gamma}) \mathbf{u} \cdot \mathbf{v} - (\boldsymbol{\varrho} + \boldsymbol{\eta}) \mathbf{u} \cdot \mathbf{u} \right\} \\ & - \int_{\Omega} (\boldsymbol{\varrho} + \boldsymbol{\eta}) : \boldsymbol{\sigma} - \int_{\Omega} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \end{aligned} \quad (3.12)$$

for all $\vec{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{L}^4(\Omega) \times \mathbb{L}_{\text{sk}}^2(\Omega)$, for all $\boldsymbol{\varrho} \in \mathbb{L}_{\text{tr}}^2(\Omega)$. We show below that the aforementioned incorporation allows to obtain a partial skew-symmetry property of the bilinear form arising from the first row of (3.12) after performing a suitable linearization. Finally, gathering (3.12) and (3.10), we arrive at the following variational formulation of (2.11): Find $(\vec{\mathbf{u}}, \boldsymbol{\rho}) := ((\mathbf{u}, \boldsymbol{\gamma}), \boldsymbol{\rho}) \in \mathbf{X} \times \mathbf{Y}$ and $\boldsymbol{\sigma} \in \mathbf{Z}$, such that

$$\begin{aligned} \mathbf{A}_{\mathbf{u}}((\vec{\mathbf{u}}, \boldsymbol{\rho}), (\vec{\mathbf{v}}, \boldsymbol{\varrho})) + \mathbf{B}((\vec{\mathbf{v}}, \boldsymbol{\varrho}), \boldsymbol{\sigma}) &= \mathbf{F}((\vec{\mathbf{v}}, \boldsymbol{\varrho})) \quad \forall (\vec{\mathbf{v}}, \boldsymbol{\varrho}) := ((\mathbf{v}, \boldsymbol{\eta}), \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}, \\ \mathbf{B}((\vec{\mathbf{u}}, \boldsymbol{\rho}), \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbf{Z}, \end{aligned} \quad (3.13)$$

where $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are the Banach spaces

$$\mathbf{X} := \mathbf{L}^4(\Omega) \times \mathbb{L}_{\text{sk}}^2(\Omega), \quad \mathbf{Y} := \mathbb{L}_{\text{tr}}^2(\Omega), \quad \text{and} \quad \mathbf{Z} := \mathbb{H}_0(\text{div}_{4/3}; \Omega), \quad (3.14)$$

endowed with the usual norms of the individual spaces, and with the product norms

$$\begin{aligned} \|\vec{\mathbf{v}}\| &:= \|\mathbf{v}\|_{0,4;\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega} \quad \forall \vec{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{X}, \\ \|(\vec{\mathbf{v}}, \boldsymbol{\varrho})\| &:= \|\vec{\mathbf{v}}\| + \|\boldsymbol{\varrho}\|_{0,\Omega} \quad \forall (\vec{\mathbf{v}}, \boldsymbol{\varrho}) := ((\mathbf{v}, \boldsymbol{\eta}), \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}, \end{aligned} \quad (3.15)$$

whereas $\mathbf{A}_{\mathbf{z}} : (\mathbf{X} \times \mathbf{Y}) \times (\mathbf{X} \times \mathbf{Y}) \rightarrow \mathbb{R}$, for each $\mathbf{z} \in \mathbf{L}^4(\Omega)$, and $\mathbf{B} : (\mathbf{X} \times \mathbf{Y}) \times \mathbf{Z} \rightarrow \mathbb{R}$ are the bilinear forms defined as

$$\mathbf{A}_{\mathbf{z}}((\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho})) := 2 \int_{\Omega} \mu \boldsymbol{\vartheta} : \boldsymbol{\varrho} + \mathbf{D}(\mathbf{z}; (\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho})), \quad (3.16)$$

with

$$\mathbf{D}(\mathbf{z}; (\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho})) := \frac{\delta}{2} \int_{\Omega} \left\{ (\boldsymbol{\vartheta} + \boldsymbol{\xi}) \mathbf{z} \cdot \mathbf{v} - (\boldsymbol{\varrho} + \boldsymbol{\eta}) \mathbf{z} \cdot \mathbf{w} \right\}, \quad (3.17)$$

for all $(\vec{\mathbf{w}}, \boldsymbol{\vartheta}) := ((\mathbf{w}, \boldsymbol{\xi}), \boldsymbol{\vartheta})$, $(\vec{\mathbf{v}}, \boldsymbol{\varrho}) := ((\mathbf{v}, \boldsymbol{\eta}), \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}$, and

$$\mathbf{B}((\vec{\mathbf{v}}, \boldsymbol{\varrho}), \boldsymbol{\tau}) := - \int_{\Omega} (\boldsymbol{\varrho} + \boldsymbol{\eta}) : \boldsymbol{\tau} - \int_{\Omega} \mathbf{v} \cdot \text{div}(\boldsymbol{\tau}), \quad (3.18)$$

for all $(\vec{\mathbf{v}}, \boldsymbol{\varrho}) := ((\mathbf{v}, \boldsymbol{\eta}), \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}$, for all $\boldsymbol{\tau} \in \mathbf{Z}$. In turn, $\mathbf{F} : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbb{R}$ and $\mathbf{G} : \mathbf{Z} \rightarrow \mathbb{R}$ are the linear functionals given by

$$\mathbf{F}((\vec{\mathbf{v}}, \boldsymbol{\varrho})) := \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \quad \forall (\vec{\mathbf{v}}, \boldsymbol{\varrho}) := ((\mathbf{v}, \boldsymbol{\eta}), \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}, \quad (3.19)$$

and

$$\mathbf{G}(\boldsymbol{\tau}) := -\langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u}_D \rangle \quad \forall \boldsymbol{\tau} \in \mathbf{Z}. \quad (3.20)$$

4 The continuous solvability analysis

In this section, we analyze the solvability of (3.13) by means of an equivalent reduced problem on the kernel of $\mathbf{B}$ and a fixed-point strategy. In this way, a particular property of this kernel, together with a compactness argument and the Schauder fixed-point theorem, yields the existence of a solution, whereas a smallness assumption on the data and the classical Banach fixed-point theorem imply its uniqueness.

4.1 Preliminaries

We begin by establishing the stability properties of the forms and functionals involved. Indeed, employing the Cauchy-Schwarz and Hölder inequalities, and, in particular, the consequence of them given by (3.4), as well as the upper bound of μ (cf. (2.4)), the integration by parts formula (1.3) with $t = 4/3$ and $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{v} = \mathbf{u}_D$ on Γ and $\|\mathbf{v}\|_{1,\Omega} = \|\mathbf{u}_D\|_{1/2,\Gamma}$, and the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \rightarrow \mathbf{L}^4(\Omega)$, we readily obtain (cf. (3.15))

$$\begin{aligned} |\mathbf{D}(\mathbf{z}; (\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho}))| &\leq \sqrt{n} \delta \|\mathbf{z}\|_{0,4;\Omega} \|(\vec{\mathbf{w}}, \boldsymbol{\vartheta})\| \|(\vec{\mathbf{v}}, \boldsymbol{\varrho})\|, \\ |\mathbf{A}_{\mathbf{z}}((\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho}))| &\leq (2\mu_2 + \sqrt{n} \delta \|\mathbf{z}\|_{0,4;\Omega}) \|(\vec{\mathbf{w}}, \boldsymbol{\vartheta})\| \|(\vec{\mathbf{v}}, \boldsymbol{\varrho})\|, \\ |\mathbf{B}((\vec{\mathbf{v}}, \boldsymbol{\varrho}), \boldsymbol{\tau})| &\leq \|(\vec{\mathbf{v}}, \boldsymbol{\varrho})\| \|\boldsymbol{\tau}\|_{\text{div}_{4/3;\Omega}}, \\ |\mathbf{F}((\vec{\mathbf{v}}, \boldsymbol{\varrho}))| &\leq \|\mathbf{f}\|_{0,4/3;\Omega} \|(\vec{\mathbf{v}}, \boldsymbol{\varrho})\|, \\ |\mathbf{G}(\boldsymbol{\tau})| &\leq \max\{1, \|\mathbf{i}_4\|\} \|\mathbf{u}_D\|_{1/2,\Gamma} \|\boldsymbol{\tau}\|_{\text{div}_{4/3;\Omega}}, \end{aligned} \quad (4.1)$$

for all $(\vec{\mathbf{w}}, \boldsymbol{\vartheta}) := ((\mathbf{w}, \boldsymbol{\xi}), \boldsymbol{\vartheta})$, $(\vec{\mathbf{v}}, \boldsymbol{\varrho}) := ((\mathbf{v}, \boldsymbol{\eta}), \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}$, for all $\boldsymbol{\tau} \in \mathbf{Z}$.

Furthermore, we now observe the partial skew-symmetry property of $\mathbf{A}_{\mathbf{z}}$, which consists of that of its component $\mathbf{D}(\mathbf{z}; \cdot, \cdot)$, that is

$$\mathbf{D}(\mathbf{z}; (\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho})) = -\mathbf{D}(\mathbf{z}; (\vec{\mathbf{v}}, \boldsymbol{\varrho}), (\vec{\mathbf{w}}, \boldsymbol{\vartheta})) \quad \forall (\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}, \quad (4.2)$$

which clearly yields

$$\mathbf{D}(\mathbf{z}; (\vec{\mathbf{v}}, \boldsymbol{\varrho}), (\vec{\mathbf{v}}, \boldsymbol{\varrho})) = 0 \quad \forall (\vec{\mathbf{v}}, \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}. \quad (4.3)$$

In addition, we also notice the continuity of $\mathbf{D}(\mathbf{z}; \cdot, \cdot)$ with respect to $\mathbf{z}$, which is a consequence of its linearity with respect to $\mathbf{z}$ and the boundedness property from the first row of (4.1), namely

$$\begin{aligned} |\mathbf{D}(\mathbf{z}; (\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho})) - \mathbf{D}(\tilde{\mathbf{z}}; (\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho}))| &= |\mathbf{D}(\mathbf{z} - \tilde{\mathbf{z}}; (\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho}))| \\ &\leq \sqrt{n} \delta \|\mathbf{z} - \tilde{\mathbf{z}}\|_{0,4;\Omega} \|(\vec{\mathbf{w}}, \boldsymbol{\vartheta})\| \|(\vec{\mathbf{v}}, \boldsymbol{\varrho})\|, \end{aligned} \quad (4.4)$$

for all $\mathbf{z}, \tilde{\mathbf{z}} \in \mathbf{L}^4(\Omega)$, for all $(\vec{\mathbf{w}}, \boldsymbol{\vartheta}), (\vec{\mathbf{v}}, \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}$.

Our goal throughout the rest of this section is to prove a result providing a connection between the eventual solutions of (3.13) and those of a kind of simplified version of it, for which we previously need to show that $\mathbf{B}$ verifies its continuous inf-sup condition. To this end, we recall next [20, Lemma 5.41], which is a particular case of [20, Lemma 5.39].

Lemma 4.1. *Let X , Y , and Z be reflexive Banach spaces, and let $b : (X \times Y) \times Z \rightarrow \mathbb{R}$ be a bounded bilinear form with boundedness constant denoted by $\|b\|$. In addition, define*

$$V := \left\{ (x, y) \in X \times Y : \quad b((x, y), z) = 0 \quad \forall z \in Z \right\}, \quad (4.5)$$

which corresponds to the null space of the operator induced by b , and

$$Z_0 := \left\{ z \in Z : \quad b((x, 0), z) = 0 \quad \forall x \in X \right\}. \quad (4.6)$$

Then, the following statements are equivalent:

1) *there exist positive constants β_1 and β_2 such that*

$$\sup_{\substack{(x,y) \in X \times Y \\ (x,y) \neq 0}} \frac{b((x,y),z)}{\|(x,y)\|} \geq \beta_1 \|z\| \quad \forall z \in Z, \quad (4.7)$$

and

$$\|y\| \geq \beta_2 \|x\| \quad \forall (x,y) \in V, \quad (4.8)$$

2) *there exist positive constants β_3 and β_4 such that*

$$\sup_{\substack{z \in Z \\ z \neq 0}} \frac{b((x,0),z)}{\|z\|} \geq \beta_3 \|x\| \quad \forall x \in X, \quad (4.9)$$

and

$$\sup_{\substack{y \in Y \\ y \neq 0}} \frac{b((0,y),z)}{\|y\|} \geq \beta_4 \|z\| \quad \forall z \in Z_0. \quad (4.10)$$

Moreover, in the implication $1) \Rightarrow 2)$, the constants β_3 and β_4 depend only on β_1 and β_2 , whereas in the converse implication $2) \Rightarrow 1)$, the constants β_1 and β_2 depend only on β_3 , β_4 , and $\|b\|$.

In what follows we apply Lemma 4.1 to the present context of the bilinear form $\mathbf{B}$ (cf. (3.18)) and the spaces $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ (cf. (3.14)). Indeed, denoting by $\mathcal{B} \in \mathcal{L}(\mathbf{X} \times \mathbf{Y}, \mathbf{Z}')$ the operator induced by $\mathbf{B}$, we first let $\mathbf{V}$ be its null space, that is

$$\mathbf{V} := \left\{ (\vec{\mathbf{v}}, \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y} : \quad \mathbf{B}((\vec{\mathbf{v}}, \boldsymbol{\varrho}), \boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{Z} \right\}. \quad (4.11)$$

In order to derive an explicit characterization of $\mathbf{V}$, we now observe from (3.18) that $\mathbf{B}((\vec{\mathbf{v}}, \boldsymbol{\varrho}), \mathbb{I}) = 0$ for all $(\vec{\mathbf{v}}, \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y}$, which, in view of the decomposition (3.8), implies that $\mathbf{V}$ can equivalently be defined as

$$\mathbf{V} := \left\{ (\vec{\mathbf{v}}, \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y} : \quad \mathbf{B}((\vec{\mathbf{v}}, \boldsymbol{\varrho}), \boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega) \right\}. \quad (4.12)$$

Then, using distributional arguments together with the integration by parts formula (1.3), and the surjectivity of the normal trace operator from $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ onto $\mathbf{H}^{-1/2}(\Gamma)$ – the latter following similarly to the proof of [19, Theorem 1.7] – we deduce that

$$\mathbf{V} := \left\{ (\vec{\mathbf{v}}, \boldsymbol{\varrho}) := ((\mathbf{v}, \boldsymbol{\eta}), \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y} : \quad \nabla \mathbf{v} = \boldsymbol{\varrho} + \boldsymbol{\eta} \quad \text{and} \quad \mathbf{v} \in \mathbf{H}_0^1(\Omega) \right\}. \quad (4.13)$$

In turn, we now let

$$\mathbf{Z}_0 := \left\{ \boldsymbol{\tau} \in \mathbf{Z} : \quad \mathbf{B}((\vec{\mathbf{v}}, 0), \boldsymbol{\tau}) = 0 \quad \forall \vec{\mathbf{v}} \in \mathbf{X} \right\},$$

that is

$$\mathbf{Z}_0 := \left\{ \boldsymbol{\tau} \in \mathbf{Z} : \quad \int_{\Omega} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) + \int_{\Omega} \boldsymbol{\eta} : \boldsymbol{\tau} = 0 \quad \forall \vec{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{X} \right\}, \quad (4.14)$$

which readily yields

$$\mathbf{Z}_0 := \left\{ \boldsymbol{\tau} \in \mathbf{Z} : \quad \mathbf{div}(\boldsymbol{\tau}) = 0 \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\tau}^t \quad \text{in} \quad \Omega \right\}. \quad (4.15)$$

Next, letting c_P be the positive constant guaranteeing Poincaré's inequality, that is $\|\mathbf{v}\|_{1,\Omega}^2 \leq c_P \|\mathbf{v}\|_{1,\Omega}^2$ $\forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)$, we recall from [22, Lemma 3.5] that there exists a positive constant β_3 , depending only on c_P and $\|\mathbf{i}_4\|$, such that

$$\sup_{\substack{\boldsymbol{\tau} \in \mathbf{Z} \\ \boldsymbol{\tau} \neq 0}} \frac{\mathbf{B}((\vec{\mathbf{v}}, 0), \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\text{div}_{4/3}; \Omega}} = \sup_{\substack{\boldsymbol{\tau} \in \mathbf{Z} \\ \boldsymbol{\tau} \neq 0}} \frac{-\int_{\Omega} \boldsymbol{\eta} : \boldsymbol{\tau} - \int_{\Omega} \mathbf{v} \cdot \text{div}(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\text{div}_{4/3}; \Omega}} \geq \beta_3 \|\vec{\mathbf{v}}\|, \quad (4.16)$$

for all $\vec{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{X}$. Furthermore, proceeding similarly to the proof of [19, Lemma 2.3], we can prove that there exists a positive constant c_1 , depending only on Ω , such that

$$\|\boldsymbol{\tau}^{\text{d}}\|_{0,\Omega} + \|\text{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega} \geq c_1 \|\boldsymbol{\tau}\|_{0,\Omega} \quad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\text{div}_{4/3}; \Omega), \quad (4.17)$$

which yields, in particular,

$$\|\boldsymbol{\tau}^{\text{d}}\|_{0,\Omega} \geq c_1 \|\boldsymbol{\tau}\|_{0,\Omega} = c_1 \|\boldsymbol{\tau}\|_{\text{div}_{4/3}; \Omega} \quad \forall \boldsymbol{\tau} \in \mathbf{Z}_0. \quad (4.18)$$

Thus, given $\boldsymbol{\tau} \in \mathbf{Z}_0$, we bound from below by choosing $\boldsymbol{\varrho} := -\boldsymbol{\tau}^{\text{d}} \in \mathbf{Y} := \mathbb{L}_{\text{tr}}^2(\Omega)$, to obtain

$$\sup_{\substack{\boldsymbol{\varrho} \in \mathbf{Y} \\ \boldsymbol{\varrho} \neq 0}} \frac{\mathbf{B}((\vec{0}, \boldsymbol{\varrho}), \boldsymbol{\tau})}{\|\boldsymbol{\varrho}\|_{0,\Omega}} = \sup_{\substack{\boldsymbol{\varrho} \in \mathbf{Y} \\ \boldsymbol{\varrho} \neq 0}} \frac{-\int_{\Omega} \boldsymbol{\varrho} : \boldsymbol{\tau}}{\|\boldsymbol{\varrho}\|_{0,\Omega}} \geq \frac{\int_{\Omega} \boldsymbol{\tau}^{\text{d}} : \boldsymbol{\tau}}{\|\boldsymbol{\tau}^{\text{d}}\|_{0,\Omega}} = \|\boldsymbol{\tau}^{\text{d}}\|_{0,\Omega},$$

which, along with (4.18), implies

$$\sup_{\substack{\boldsymbol{\varrho} \in \mathbf{Y} \\ \boldsymbol{\varrho} \neq 0}} \frac{\mathbf{B}((\vec{0}, \boldsymbol{\varrho}), \boldsymbol{\tau})}{\|\boldsymbol{\varrho}\|_{0,\Omega}} \geq \beta_4 \|\boldsymbol{\tau}\|_{\text{div}_{4/3}; \Omega}, \quad (4.19)$$

with $\beta_4 = c_1$. We find it important to stress here that, if the space $\mathbb{L}_{\text{tr}}^2(\Omega)$, to which the unknown $\boldsymbol{\rho}$ belongs, had been required to satisfy a strong symmetry condition, then, since $\boldsymbol{\tau}$, and hence $\boldsymbol{\tau}^{\text{d}}$, is not, in general, strongly symmetric, we could not have taken $\boldsymbol{\varrho} := -\boldsymbol{\tau}^{\text{d}}$ in the above proof, thus preventing us from proving (4.19).

The previous analysis allows us to establish the following result.

Lemma 4.2. *There exist positive constants β_1 and β_2 , depending only on β_3 , β_4 , and $\|\mathbf{B}\| \leq 1$ (cf. (4.1)), such that*

$$\sup_{\substack{(\vec{\mathbf{v}}, \boldsymbol{\varrho}) \in \mathbf{X} \times \mathbf{Y} \\ (\vec{\mathbf{v}}, \boldsymbol{\varrho}) \neq 0}} \frac{\mathbf{B}((\vec{\mathbf{v}}, \boldsymbol{\varrho}), \boldsymbol{\tau})}{\|(\vec{\mathbf{v}}, \boldsymbol{\varrho})\|} \geq \beta_1 \|\boldsymbol{\tau}\|_{\text{div}_{4/3}; \Omega} \quad \forall \boldsymbol{\tau} \in \mathbf{Z}, \quad (4.20)$$

and

$$\|\boldsymbol{\varrho}\|_{0,\Omega} \geq \beta_2 \|\vec{\mathbf{v}}\| \quad \forall (\vec{\mathbf{v}}, \boldsymbol{\varrho}) \in \mathbf{V}. \quad (4.21)$$

Proof. Bearing in mind the inequalities (4.16) and (4.19), which constitute a particular case of the statements (4.9) and (4.10), the proof follows from a straightforward application of Lemma 4.1, thus yielding (4.20) and (4.21) as the corresponding particular cases of (4.7) and (4.8), respectively. $\square$

Having shown the continuous inf-sup condition for $\mathbf{B}$ with constant β_1 (cf. (4.20)), equivalently the surjectivity of $\mathcal{B}$, we deduce, in virtue of the converse of [18, Lemma A.42] (see, also [20, Lemma 3.65]), that there exists $(\vec{\mathbf{u}}_g, \boldsymbol{\rho}_g) := ((\mathbf{u}_g, \boldsymbol{\gamma}_g), \boldsymbol{\rho}_g) \in \mathbf{X} \times \mathbf{Y}$ such that

$$\mathcal{B}((\vec{\mathbf{u}}_g, \boldsymbol{\rho}_g)) = \mathbf{G}, \quad \text{and} \quad \beta_1 \|(\vec{\mathbf{u}}_g, \boldsymbol{\rho}_g)\| \leq \|\mathbf{G}\|, \quad (4.22)$$

so that, employing from the last row of (4.1) that $\|\mathbf{G}\| \leq \max\{1, \|\mathbf{i}_4\|\} \|\mathbf{u}_D\|_{1/2, \Gamma}$, we arrive at

$$\|(\tilde{\mathbf{u}}_g, \rho_g)\| \leq c_D \|\mathbf{u}_D\|_{1/2, \Gamma} \quad \text{with} \quad c_D := \beta_1^{-1} \max\{1, \|\mathbf{i}_4\|\}. \quad (4.23)$$

Then, we consider what is usually called the reduced problem with respect to (3.13): Find $(\underline{\mathbf{u}}, \underline{\rho}) := ((\underline{\mathbf{u}}, \underline{\gamma}), \underline{\rho}) \in \mathbf{V}$ such that

$$\mathbf{A}_{\underline{\mathbf{u}}+\mathbf{u}_g}((\underline{\mathbf{u}}, \underline{\rho}) + (\tilde{\mathbf{u}}_g, \rho_g), (\vec{\mathbf{v}}, \varrho)) = \mathbf{F}((\vec{\mathbf{v}}, \varrho)) \quad \forall (\vec{\mathbf{v}}, \varrho) \in \mathbf{V}. \quad (4.24)$$

As previously announced, we are now in position to establish a suitable connection between the solutions of (3.13) and (4.24).

Lemma 4.3. *Let $(\tilde{\mathbf{u}}_g, \rho_g) \in \mathbf{X} \times \mathbf{Y}$ satisfy (4.22). Then, given a solution $((\tilde{\mathbf{u}}, \rho), \sigma) \in (\mathbf{X} \times \mathbf{Y}) \times \mathbf{Z}$ of (3.13), it follows that $(\underline{\mathbf{u}}, \underline{\rho}) := (\tilde{\mathbf{u}}, \rho) - (\tilde{\mathbf{u}}_g, \rho_g)$ is a solution of (4.24). Conversely, given a solution $(\underline{\mathbf{u}}, \underline{\rho}) \in \mathbf{V}$ of (4.24), there exists a unique $\sigma \in \mathbf{Z}$ such that, upon defining $(\tilde{\mathbf{u}}, \rho) := (\underline{\mathbf{u}}, \underline{\rho}) + (\tilde{\mathbf{u}}_g, \rho_g)$, the triple $((\tilde{\mathbf{u}}, \rho), \sigma)$ is a solution of (3.13). Moreover, there holds*

$$\|\sigma\|_{\text{div}_{4/3}; \Omega} \leq \beta_1^{-1} \left\{ \|\mathbf{f}\|_{0, 4/3; \Omega} + (2\mu_2 + \sqrt{n}\delta \|(\tilde{\mathbf{u}}, \rho)\|) \|(\tilde{\mathbf{u}}, \rho)\| \right\}. \quad (4.25)$$

Proof. Let $((\tilde{\mathbf{u}}, \rho), \sigma) \in (\mathbf{X} \times \mathbf{Y}) \times \mathbf{Z}$ be a solution of (3.13). It is clear from its second equation that $\mathcal{B}((\tilde{\mathbf{u}}, \rho)) = \mathbf{G}$, whereas restricting the first equation of (3.13) to test functions in $\mathbf{V}$, we obtain

$$\mathbf{A}_{\mathbf{u}}((\tilde{\mathbf{u}}, \rho), (\vec{\mathbf{v}}, \varrho)) = \mathbf{F}((\vec{\mathbf{v}}, \varrho)) \quad \forall (\vec{\mathbf{v}}, \varrho) \in \mathbf{V}. \quad (4.26)$$

Then, defining $(\underline{\mathbf{u}}, \underline{\rho}) := (\tilde{\mathbf{u}}, \rho) - (\tilde{\mathbf{u}}_g, \rho_g)$, which clearly belongs to $\mathbf{V}$, (4.26) can be rewritten as

$$\mathbf{A}_{\underline{\mathbf{u}}+\mathbf{u}_g}((\underline{\mathbf{u}}, \underline{\rho}) + (\tilde{\mathbf{u}}_g, \rho_g), (\vec{\mathbf{v}}, \varrho)) = \mathbf{F}((\vec{\mathbf{v}}, \varrho)) \quad \forall (\vec{\mathbf{v}}, \varrho) \in \mathbf{V},$$

thus showing that $(\underline{\mathbf{u}}, \underline{\rho})$ is solution of (4.24). Conversely, let $(\underline{\mathbf{u}}, \underline{\rho}) \in \mathbf{V}$ be a solution of (4.24), which means, equivalently, that

$$\mathbf{F} - \mathbf{A}_{\underline{\mathbf{u}}+\mathbf{u}_g}((\underline{\mathbf{u}}, \underline{\rho}) + (\tilde{\mathbf{u}}_g, \rho_g), \cdot) \in \mathbf{V}^\circ, \quad (4.27)$$

where $\mathbf{V}^\circ$, the annihilator of $\mathbf{V}$, is defined as

$$\mathbf{V}^\circ := \left\{ \Theta \in (\mathbf{X} \times \mathbf{Y})' : \Theta((\vec{\mathbf{v}}, \varrho)) = 0 \quad \forall (\vec{\mathbf{v}}, \varrho) \in \mathbf{V} \right\}.$$

Next, knowing that $\mathcal{B} \in \mathcal{L}(\mathbf{X} \times \mathbf{Y}, \mathbf{Z}')$ is surjective, it follows, equivalently, that its adjoint operator $\mathcal{B}' \in \mathcal{L}(\mathbf{Z}', (\mathbf{X} \times \mathbf{Y})')$ is injective with range $R(\mathcal{B}') = \mathbf{V}^\circ$. Hence, bearing in mind (4.27), we deduce, in particular, that there exists a unique $\mathcal{G} \in \mathbf{Z}''$ such that

$$\mathcal{B}'(\mathcal{G}) = \mathbf{F} - \mathbf{A}_{\underline{\mathbf{u}}+\mathbf{u}_g}((\underline{\mathbf{u}}, \underline{\rho}) + (\tilde{\mathbf{u}}_g, \rho_g), \cdot). \quad (4.28)$$

Moreover, since $\mathbf{Z} := \mathbb{H}_0(\text{div}_{4/3}; \Omega)$ is easily shown to be reflexive, there exists a unique $\sigma \in \mathbf{Z}$ such that $\mathcal{G} = \mathcal{J}_{\mathbf{Z}}(\sigma)$, where $\mathcal{J}_{\mathbf{Z}} : \mathbf{Z} \rightarrow \mathbf{Z}''$ is the canonical isomorphism characterizing the aforementioned property of $\mathbf{Z}$. Then, noting that for each $(\vec{\mathbf{v}}, \varrho) \in \mathbf{X} \times \mathbf{Y}$ there holds

$$\mathcal{B}'(\mathcal{G})((\vec{\mathbf{v}}, \varrho)) = \mathcal{J}_{\mathbf{Z}}(\sigma)(\mathcal{B}((\vec{\mathbf{v}}, \varrho))) = \mathcal{B}((\vec{\mathbf{v}}, \varrho))(\sigma) = \mathbf{B}((\vec{\mathbf{v}}, \varrho), \sigma), \quad (4.29)$$

and defining $(\tilde{\mathbf{u}}, \rho) := (\underline{\mathbf{u}}, \underline{\rho}) + (\tilde{\mathbf{u}}_g, \rho_g)$, the identity (4.28) becomes the first equation of (3.13). In turn, it is clear that $\mathcal{B}((\tilde{\mathbf{u}}, \rho)) = \mathcal{B}((\tilde{\mathbf{u}}_g, \rho_g)) = \mathbf{G}$, which constitutes the second one, thus showing that $((\tilde{\mathbf{u}}, \rho), \sigma)$ is a solution of (3.13). Finally, applying (4.20) to $\tau = \sigma$, and using (4.29) and (4.28), we readily deduce that

$$\beta_1 \|\sigma\|_{\text{div}_{4/3}; \Omega} \leq \|\mathbf{F} - \mathbf{A}_{\underline{\mathbf{u}}+\mathbf{u}_g}((\underline{\mathbf{u}}, \underline{\rho}) + (\tilde{\mathbf{u}}_g, \rho_g), \cdot)\|,$$

from which, invoking the corresponding estimates from (4.1), and using that

$$\|\underline{\mathbf{u}} + \mathbf{u}_g\| \leq \|(\underline{\tilde{\mathbf{u}}}, \underline{\rho}) + (\underline{\tilde{\mathbf{u}}}_g, \rho_g)\| = \|(\underline{\tilde{\mathbf{u}}}, \rho)\|,$$

we arrive at (4.25) and end the proof. $\square$

We emphasize that, in other words, Lemma 4.3 establishes the equivalence between the existence of solutions to problems (3.13) and (4.24). Moreover, the following lemma shows that the same equivalence holds for uniqueness of solutions.

Lemma 4.4. *The formulation (3.13) has a unique solution if and only if (4.24) does.*

Proof. Let us first assume that (3.13) has a unique solution, and let $(\underline{\tilde{\mathbf{u}}}, \underline{\rho}), (\underline{\tilde{\mathbf{w}}}, \underline{\vartheta}) \in \mathbf{V}$ be solutions of (4.24). Then, defining $(\underline{\tilde{\mathbf{u}}}, \underline{\rho}) := (\underline{\tilde{\mathbf{u}}}, \underline{\rho}) + (\underline{\tilde{\mathbf{u}}}_g, \rho_g)$, and $(\underline{\tilde{\mathbf{w}}}, \underline{\vartheta}) := (\underline{\tilde{\mathbf{w}}}, \underline{\vartheta}) + (\underline{\tilde{\mathbf{u}}}_g, \rho_g)$, it follows from Lemma 4.3 that there exist unique $\sigma, \zeta \in \mathbf{Z}$, such that the triples $((\underline{\tilde{\mathbf{u}}}, \underline{\rho}), \sigma)$ and $((\underline{\tilde{\mathbf{w}}}, \underline{\vartheta}), \zeta)$ are solutions of (3.13). Hence, in view of the uniqueness of the latter we deduce, in particular, that $(\underline{\tilde{\mathbf{u}}}, \underline{\rho}) = (\underline{\tilde{\mathbf{w}}}, \underline{\vartheta})$, thus confirming the uniqueness of solution to (4.24). Conversely, let us assume that (4.24) has a unique solution, and let $((\underline{\tilde{\mathbf{u}}}, \underline{\rho}), \sigma), ((\underline{\tilde{\mathbf{w}}}, \underline{\vartheta}), \zeta) \in (\mathbf{X} \times \mathbf{Y}) \times \mathbf{Z}$ be solutions of (3.13). It follows again from Lemma 4.3 that $(\underline{\tilde{\mathbf{u}}}, \underline{\rho}) := (\underline{\tilde{\mathbf{u}}}, \underline{\rho}) - (\underline{\tilde{\mathbf{u}}}_g, \rho_g)$ and $(\underline{\tilde{\mathbf{w}}}, \underline{\vartheta}) := (\underline{\tilde{\mathbf{w}}}, \underline{\vartheta}) - (\underline{\tilde{\mathbf{u}}}_g, \rho_g)$ are solutions of (4.24), from whose uniqueness we deduce that $(\underline{\tilde{\mathbf{u}}}, \underline{\rho}) = (\underline{\tilde{\mathbf{w}}}, \underline{\vartheta})$. This identity and the first row of (3.13) for each one of the aforementioned solutions readily yield $\mathbf{B}((\underline{\tilde{\mathbf{v}}}, \underline{\varrho}), \sigma - \zeta) = 0$ for all $(\underline{\tilde{\mathbf{v}}}, \underline{\varrho}) \in \mathbf{X} \times \mathbf{Y}$, or equivalently (cf. (4.29)) $\mathcal{B}'(\mathcal{J}_Z(\sigma - \zeta)) = 0$. Hence, thanks to the injectivity of both $\mathcal{B}'$ and $\mathcal{J}_Z$ we obtain $\sigma = \zeta$, thus concluding the uniqueness of solution to (3.13). $\square$

4.2 The fixed-point strategy

In this section, we introduce a fixed-point strategy to analyze the solvability of the reduced problem (4.24). Since the corresponding solution is sought in $\mathbf{V}$, whose characterization (cf. (4.13)) implies that its first component $\underline{\mathbf{u}}$ belongs to $\mathbf{H}_0^1(\Omega)$, we define the operator $\Xi : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$ by $\Xi(\underline{\mathbf{z}}) := \underline{\mathbf{u}}$, where $(\underline{\tilde{\mathbf{u}}}, \underline{\rho}) := ((\underline{\mathbf{u}}, \underline{\gamma}), \underline{\rho}) \in \mathbf{V}$ is the unique solution (to be confirmed below) of the linear problem arising from (4.24) after replacing $\mathbf{A}_{\underline{\mathbf{u}}+\mathbf{u}_g}$ by $\mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}$, that is

$$\mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}((\underline{\tilde{\mathbf{u}}}, \underline{\rho}) + (\underline{\tilde{\mathbf{u}}}_g, \rho_g), (\underline{\tilde{\mathbf{v}}}, \underline{\varrho})) = \mathbf{F}((\underline{\tilde{\mathbf{v}}}, \underline{\varrho})) \quad \forall (\underline{\tilde{\mathbf{v}}}, \underline{\varrho}) \in \mathbf{V}. \quad (4.30)$$

Note that (4.30) can be rephrased as follows: Find $(\underline{\tilde{\mathbf{u}}}, \underline{\rho}) := ((\underline{\mathbf{u}}, \underline{\gamma}), \underline{\rho}) \in \mathbf{V}$ such that

$$\mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}((\underline{\tilde{\mathbf{u}}}, \underline{\rho}), (\underline{\tilde{\mathbf{v}}}, \underline{\varrho})) = \mathbf{F}_{\underline{\mathbf{z}}}((\underline{\tilde{\mathbf{v}}}, \underline{\varrho})) \quad \forall (\underline{\tilde{\mathbf{v}}}, \underline{\varrho}) \in \mathbf{V}, \quad (4.31)$$

where

$$\mathbf{F}_{\underline{\mathbf{z}}}((\underline{\tilde{\mathbf{v}}}, \underline{\varrho})) := \mathbf{F}((\underline{\tilde{\mathbf{v}}}, \underline{\varrho})) - \mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}((\underline{\tilde{\mathbf{u}}}_g, \rho_g), (\underline{\tilde{\mathbf{v}}}, \underline{\varrho})) \quad \forall (\underline{\tilde{\mathbf{v}}}, \underline{\varrho}) \in \mathbf{V}. \quad (4.32)$$

In this way, (4.24) is equivalent to the following fixed-point equation: Find $\underline{\mathbf{z}} \in \mathbf{H}_0^1(\Omega)$ such that

$$\Xi(\underline{\mathbf{z}}) = \underline{\mathbf{z}}. \quad (4.33)$$

4.3 Well-definedness of the operator Ξ

We now address the well-posedness of (4.31) (equivalently, (4.30)), thereby proving that Ξ is well defined. Indeed, we first observe, in virtue of the definition of $\mathbf{A}_{\underline{\mathbf{z}}}$ (cf. (3.16)) and the skew-symmetry property of $\mathbf{D}$ (cf. (4.3)), that for each $(\underline{\tilde{\mathbf{v}}}, \underline{\varrho}) \in \mathbf{V}$ there holds

$$\mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}((\underline{\tilde{\mathbf{v}}}, \underline{\varrho}), (\underline{\tilde{\mathbf{v}}}, \underline{\varrho})) = 2 \int_{\Omega} \mu |\underline{\varrho}|^2 + \mathbf{D}(\underline{\mathbf{z}} + \mathbf{u}_g; (\underline{\tilde{\mathbf{v}}}, \underline{\varrho}), (\underline{\tilde{\mathbf{v}}}, \underline{\varrho})) = 2 \int_{\Omega} \mu |\underline{\varrho}|^2.$$

Hence, employing the lower bound of μ (cf. (2.4)) along with the estimate (4.21) (cf. Lemma 4.2), we readily deduce from the foregoing equation that

$$\mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}((\vec{\mathbf{v}}, \underline{\boldsymbol{\varrho}}), (\vec{\mathbf{v}}, \underline{\boldsymbol{\varrho}})) \geq \mu_1 \beta_2^2 \|\vec{\mathbf{v}}\|^2 + \mu_1 \|\underline{\boldsymbol{\varrho}}\|_{0,\Omega}^2,$$

which yields

$$\mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}((\vec{\mathbf{v}}, \underline{\boldsymbol{\varrho}}), (\vec{\mathbf{v}}, \underline{\boldsymbol{\varrho}})) \geq \alpha_{\mathbf{A}} \|(\vec{\mathbf{v}}, \underline{\boldsymbol{\varrho}})\|^2 \quad \forall (\vec{\mathbf{v}}, \underline{\boldsymbol{\varrho}}) \in \mathbf{V}, \quad (4.34)$$

with $\alpha_{\mathbf{A}} := \mu_1 \min\{\beta_2^2, 1\}$. In turn, it is clear from the boundedness property of $\mathbf{A}_{\underline{\mathbf{z}}}$ (cf. second row of (4.1)) that $\mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}$ is certainly bounded as well with

$$\|\mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}\| \leq 2\mu_2 + \sqrt{n}\delta (\|\underline{\mathbf{z}}\|_{0,4;\Omega} + \|\mathbf{u}_g\|_{0,4;\Omega}). \quad (4.35)$$

Thus, we are in position to establish the following result.

Lemma 4.5. *For each $\underline{\mathbf{z}} \in \mathbf{H}_0^1(\Omega)$, there exists a unique $(\underline{\mathbf{u}}, \underline{\boldsymbol{\rho}}) := ((\underline{\mathbf{u}}, \underline{\boldsymbol{\gamma}}), \underline{\boldsymbol{\rho}}) \in \mathbf{V}$ solution of (4.31) (equivalently, (4.30)), and hence one can define $\Xi(\underline{\mathbf{z}}) := \underline{\mathbf{u}}$. Moreover, there exist positive constants C_0 and c_0 , depending only on $\alpha_{\mathbf{A}}$, μ_2 , c_D (cf. (4.23)), δ , and n , such that*

$$\|(\underline{\mathbf{u}}, \underline{\boldsymbol{\rho}})\| \leq C_0 \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} + c_0 \|\mathbf{u}_D\|_{1/2,\Gamma} \|\underline{\mathbf{z}}\|_{0,4;\Omega}, \quad (4.36)$$

and

$$\|\Xi(\underline{\mathbf{z}})\|_{1,\Omega} \leq C_{\Xi} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} + c_{\Xi} \|\mathbf{u}_D\|_{1/2,\Gamma} \|\underline{\mathbf{z}}\|_{1,\Omega}, \quad (4.37)$$

with

$$C_{\Xi} := \max\{1, |\Omega|^{1/4}\} C_0 \quad \text{and} \quad c_{\Xi} := \max\{1, |\Omega|^{1/4}\} \|\mathbf{i}_4\| c_0. \quad (4.38)$$

Proof. It is readily seen from (4.34) and (4.35) that the bilinear form $\mathbf{A}_{\underline{\mathbf{z}}+\mathbf{u}_g}$ satisfies the hypotheses of the Banach-Nečas-Babuška theorem (cf. [18, Theorem 2.6]) on $\mathbf{V}$, which, being a closed subspace of the reflexive Banach space $\mathbf{X} \times \mathbf{Y}$, is itself reflexive. Thus, a straightforward application of this result implies the existence of a unique solution $(\underline{\mathbf{u}}, \underline{\boldsymbol{\rho}}) := ((\underline{\mathbf{u}}, \underline{\boldsymbol{\gamma}}), \underline{\boldsymbol{\rho}}) \in \mathbf{V}$ to (4.31) (equivalently, (4.30)). Moreover, the corresponding a priori estimate (cf. [18, eq. (BNB2), Theorem 2.6]) reads

$$\|(\underline{\mathbf{u}}, \underline{\boldsymbol{\rho}})\| \leq \frac{1}{\alpha_{\mathbf{A}}} \|\mathbf{F}_{\underline{\mathbf{z}}}\|,$$

whereas the definition of $\mathbf{F}_{\underline{\mathbf{z}}}$ (cf. (4.32)) and the stability estimates in (4.1) yield

$$\|\mathbf{F}_{\underline{\mathbf{z}}}\| \leq \|\mathbf{f}\|_{0,4/3;\Omega} + \left(2\mu_2 + \sqrt{n}\delta (\|\underline{\mathbf{z}}\|_{0,4;\Omega} + \|\mathbf{u}_g\|_{0,4;\Omega}) \right) \|(\underline{\mathbf{u}}_g, \underline{\boldsymbol{\rho}}_g)\|.$$

Substituting the latter into the former, and bounding both $\|\mathbf{u}_g\|_{0,4;\Omega}$ and $\|(\underline{\mathbf{u}}_g, \underline{\boldsymbol{\rho}}_g)\|$ according to (4.23), we arrive at (4.36) with

$$C_0 := \alpha_{\mathbf{A}}^{-1} \max\{1, 2\mu_2 c_D, \sqrt{n}\delta c_D^2\} \quad \text{and} \quad c_0 := \alpha_{\mathbf{A}}^{-1} \sqrt{n}\delta c_D. \quad (4.39)$$

Next, noting from the characterization of $\mathbf{V}$ (cf. (4.13)) that $\nabla \underline{\mathbf{u}} = \underline{\boldsymbol{\rho}} + \underline{\boldsymbol{\gamma}}$, and applying Hölder's inequality, we find that

$$\begin{aligned} \|\Xi(\underline{\mathbf{z}})\|_{1,\Omega} = \|\underline{\mathbf{u}}\|_{1,\Omega} &\leq \|\underline{\mathbf{u}}\|_{0,\Omega} + \|\nabla \underline{\mathbf{u}}\|_{0,\Omega} \leq |\Omega|^{1/4} \|\underline{\mathbf{u}}\|_{0,4;\Omega} + \|\underline{\boldsymbol{\rho}}\|_{0,\Omega} + \|\underline{\boldsymbol{\gamma}}\|_{0,\Omega} \\ &\leq \max\{1, |\Omega|^{1/4}\} \|(\underline{\mathbf{u}}, \underline{\boldsymbol{\rho}})\|, \end{aligned} \quad (4.40)$$

which, along with (4.36) and the fact that $\|\underline{\mathbf{z}}\|_{0,4;\Omega} \leq \|\mathbf{i}_4\| \|\underline{\mathbf{z}}\|_{1,\Omega}$, implies (4.37) and ends the proof. $\square$

4.4 Solvability analysis of the fixed-point scheme

Having shown that Ξ is well defined, our next goal is to analyze the solvability of the fixed-point equation (4.33) (equivalently, that of the reduced problem (4.24)). We begin by proving that Ξ maps a nonempty, closed, bounded, and convex subset of $\mathbf{H}_0^1(\Omega)$ into itself. Indeed, denoting

$$\mathbf{r} := 2C_{\Xi} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\}, \quad (4.41)$$

we introduce the ball

$$W(\mathbf{r}) := \left\{ \mathbf{z} \in \mathbf{H}_0^1(\Omega) : \|\mathbf{z}\|_{1,\Omega} \leq \mathbf{r} \right\}, \quad (4.42)$$

and establish the following result.

Lemma 4.6. *Assume that the Dirichlet datum $\mathbf{u}_D$ is sufficiently small so that*

$$c_{\Xi} \|\mathbf{u}_D\|_{1/2,\Gamma} \leq \frac{1}{2}. \quad (4.43)$$

Then $\Xi(W(\mathbf{r})) \subseteq W(\mathbf{r})$.

Proof. Given $\mathbf{z} \in W(\mathbf{r})$, it follows directly from (4.37), the assumption (4.43), and the definition of $\mathbf{r}$ (cf. (4.41)) that $\|\Xi(\mathbf{z})\|_{1,\Omega} \leq \mathbf{r}$, which implies that $\Xi(\mathbf{z}) \in W(\mathbf{r})$, thus completing the proof. $\square$

Next, we establish a compactness property of Ξ , and hence its continuity.

Lemma 4.7. *Assume that (4.43) holds. Then, there exists a positive constant L_{Ξ} , depending only on $\alpha_{\mathbf{A}}$, n , C_{Ξ} , δ , c_D , and $|\Omega|$, such that*

$$\|\Xi(\mathbf{z}) - \Xi(\mathbf{z}')\|_{1,\Omega} \leq L_{\Xi} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} \|\mathbf{z} - \mathbf{z}'\|_{0,4;\Omega}, \quad (4.44)$$

for all $\mathbf{z}, \mathbf{z}' \in W(\mathbf{r})$.

Proof. Given $\mathbf{z}, \mathbf{z}' \in W(\mathbf{r})$, we let $\Xi(\mathbf{z}) := \underline{\mathbf{u}}$ and $\Xi(\mathbf{z}') := \underline{\mathbf{u}}'$, where $(\underline{\mathbf{u}}, \underline{\rho}) := ((\underline{\mathbf{u}}, \underline{\gamma}), \underline{\rho})$ and $(\underline{\mathbf{u}}', \underline{\rho}') := ((\underline{\mathbf{u}}', \underline{\gamma}'), \underline{\rho}')$, both in $\mathbf{V}$, are the unique solutions – guaranteed by Lemma 4.5 – of the problems

$$\mathbf{A}_{\mathbf{z}+\mathbf{u}_g}((\underline{\mathbf{u}}, \underline{\rho}) + (\underline{\mathbf{u}}_g, \underline{\rho}_g), (\underline{\mathbf{v}}, \underline{\varrho})) = \mathbf{F}((\underline{\mathbf{v}}, \underline{\varrho})) \quad \forall (\underline{\mathbf{v}}, \underline{\varrho}) \in \mathbf{V}, \quad (4.45)$$

and

$$\mathbf{A}_{\mathbf{z}'+\mathbf{u}_g}((\underline{\mathbf{u}}', \underline{\rho}') + (\underline{\mathbf{u}}_g, \underline{\rho}_g), (\underline{\mathbf{v}}, \underline{\varrho})) = \mathbf{F}((\underline{\mathbf{v}}, \underline{\varrho})) \quad \forall (\underline{\mathbf{v}}, \underline{\varrho}) \in \mathbf{V}, \quad (4.46)$$

respectively. Then, applying the $\mathbf{V}$ -ellipticity of $\mathbf{A}_{\mathbf{z}+\mathbf{u}_g}$ (cf. (4.34)) to $(\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{u}}', \underline{\rho}')$, adding and subtracting $(\underline{\mathbf{u}}_g, \underline{\rho}_g)$ in the first component of this bilinear form, and then employing (4.45) and (4.46), we find that

$$\begin{aligned} \alpha_{\mathbf{A}} \|(\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{u}}', \underline{\rho}')\|^2 &\leq \mathbf{A}_{\mathbf{z}+\mathbf{u}_g}((\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{u}}', \underline{\rho}'), (\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{u}}', \underline{\rho}')) \\ &= \mathbf{A}_{\mathbf{z}+\mathbf{u}_g}((\underline{\mathbf{u}}, \underline{\rho}) + (\underline{\mathbf{u}}_g, \underline{\rho}_g) - ((\underline{\mathbf{u}}', \underline{\rho}') + (\underline{\mathbf{u}}_g, \underline{\rho}_g)), (\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{u}}', \underline{\rho}')) \\ &= \mathbf{F}((\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{u}}', \underline{\rho}')) - \mathbf{A}_{\mathbf{z}+\mathbf{u}_g}((\underline{\mathbf{u}}, \underline{\rho}) + (\underline{\mathbf{u}}_g, \underline{\rho}_g), (\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{u}}', \underline{\rho}')) \\ &= (\mathbf{A}_{\mathbf{z}+\mathbf{u}_g} - \mathbf{A}_{\mathbf{z}'+\mathbf{u}_g})((\underline{\mathbf{u}}, \underline{\rho}) + (\underline{\mathbf{u}}_g, \underline{\rho}_g), (\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{u}}', \underline{\rho}')). \end{aligned}$$

Next, by the definitions of $\mathbf{A}_{\mathbf{z}+\mathbf{u}_g}$ and $\mathbf{A}_{\mathbf{z}'+\mathbf{u}_g}$ (cf. (3.16)), and in view of (4.4), the last term above is bounded by

$$\sqrt{n} \delta \|\mathbf{z} - \mathbf{z}'\|_{0,4;\Omega} \|(\underline{\mathbf{u}}, \underline{\rho}) + (\underline{\mathbf{u}}_g, \underline{\rho}_g)\| \|(\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{u}}', \underline{\rho}')\|,$$

which, inserted into the foregoing inequality, gives

$$\|(\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{v}}, \underline{\rho})\| \leq \alpha_{\mathbf{A}}^{-1} \sqrt{n} \delta \|\underline{\mathbf{z}} - \underline{\mathbf{z}}\|_{0,4;\Omega} \|(\underline{\mathbf{u}}, \underline{\rho}) + (\underline{\mathbf{u}}_g, \underline{\rho}_g)\|. \quad (4.47)$$

In turn, noting from (4.38) that $C_0 \leq C_{\Xi}$ and

$$c_0 \|\mathbf{u}_D\|_{1/2,\Gamma} \|\mathbf{z}\|_{0,4;\Omega} \leq c_0 \|\mathbf{i}_4\| \|\mathbf{u}_D\|_{1/2,\Gamma} \|\mathbf{z}\|_{1,\Omega} \leq c_{\Xi} \|\mathbf{u}_D\|_{1/2,\Gamma} \|\mathbf{z}\|_{1,\Omega}, \quad (4.48)$$

and using (4.41) and (4.43), we deduce from (4.36), applied to $(\underline{\mathbf{u}}, \underline{\rho})$, that

$$\|(\underline{\mathbf{u}}, \underline{\rho})\| \leq 2 C_{\Xi} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\}. \quad (4.49)$$

This estimate, together with (4.47) and the bound (4.23) for $\|(\underline{\mathbf{u}}_g, \underline{\rho}_g)\|$, yields

$$\|(\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{v}}, \underline{\rho})\| \leq \ell_{\Xi} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} \|\underline{\mathbf{z}} - \underline{\mathbf{z}}\|_{0,4;\Omega},$$

where $\ell_{\Xi} := \alpha_{\mathbf{A}}^{-1} \sqrt{n} \delta (2 C_{\Xi} + c_D)$. Finally, proceeding as in (4.40), we obtain

$$\|\Xi(\underline{\mathbf{z}}) - \Xi(\underline{\mathbf{z}})\|_{1,\Omega} \leq \max \{1, |\Omega|^{1/4}\} \|(\underline{\mathbf{u}}, \underline{\rho}) - (\underline{\mathbf{v}}, \underline{\rho})\|,$$

and, combining this with the previous inequality, we arrive at (4.44) with $L_{\Xi} := \max \{1, |\Omega|^{1/4}\} \ell_{\Xi}$. $\square$

The aforementioned property of Ξ refers precisely to the estimate (4.44) and the fact that $\mathbf{H}_0^1(\Omega)$ is compactly embedded in $\mathbf{L}^4(\Omega)$. In fact, as shown in what follows, the above and the Schauder fixed-point theorem (cf. [14, Theorem 9.12-1(b)]) imply the existence of solution of (4.33), and hence of (4.24).

Theorem 4.8. *Assume that (4.43) holds. Then, there exists a fixed point $\underline{\mathbf{u}} \in W(\mathbf{r})$ of Ξ , and hence a solution $(\underline{\mathbf{u}}, \underline{\rho}) := ((\underline{\mathbf{u}}, \underline{\gamma}), \underline{\rho}) \in \mathbf{V}$ of the reduced problem (4.24), which satisfies*

$$\|(\underline{\mathbf{u}}, \underline{\rho})\| \leq 2 C_{\Xi} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\}. \quad (4.50)$$

Proof. Knowing from Lemma 4.6 that, under the assumption (4.43), Ξ maps the ball $W(\mathbf{r})$ into itself, and bearing in mind that compact sets are closed, it remains to establish that $\Xi(W(\mathbf{r}))$ is compact and then conclude by [14, Theorem 9.12-1(b)] the existence result. Indeed, given a sequence $\{\Xi(\mathbf{z}_k)\}_{k \in \mathbb{N}} \subseteq \Xi(W(\mathbf{r}))$, with $\{\mathbf{z}_k\}_{k \in \mathbb{N}} \subseteq W(\mathbf{r})$, the boundedness of the latter and the reflexivity of $\mathbf{H}_0^1(\Omega)$ imply the existence of a subsequence $\{\mathbf{z}_k^{(1)}\}_{k \in \mathbb{N}} \subseteq \{\mathbf{z}_k\}_{k \in \mathbb{N}}$ and an element $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ such that $\mathbf{z}_k^{(1)} \xrightarrow{w} \mathbf{z}$ in $\mathbf{H}_0^1(\Omega)$. Note that, being $W(\mathbf{r})$ convex and closed, it is also weakly closed (cf. [20, Lemma 7.40]), and hence $\mathbf{z} \in W(\mathbf{r})$. Then, by the compact embedding of $\mathbf{H}_0^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, it follows that $\mathbf{z}_k^{(1)} \rightarrow \mathbf{z}$ in $\mathbf{L}^4(\Omega)$, which, using (4.44), implies that $\Xi(\mathbf{z}_k^{(1)}) \rightarrow \Xi(\mathbf{z})$ in $\mathbf{H}_0^1(\Omega)$. This shows that the sequence $\Xi(\mathbf{z}_k)_{k \in \mathbb{N}}$ admits a convergent subsequence, thereby proving the compactness of $\Xi(W(\mathbf{r}))$. Finally, proceeding similarly to the derivation of (4.49), which means using again the fact that $C_0 \leq C_{\Xi}$ along with (4.48) and (4.43), the estimate (4.50) follows straightforwardly from (4.36). $\square$

Having proved Theorem 4.8, we are ready now to address the solvability of (3.13).

Theorem 4.9. Assume that (4.43) holds, and let $(\vec{\mathbf{u}}_g, \boldsymbol{\rho}_g) \in \mathbf{X} \times \mathbf{Y}$ satisfy (4.22). Then, there exists a solution $(\vec{\mathbf{u}}, \boldsymbol{\rho}) := ((\mathbf{u}, \boldsymbol{\gamma}), \boldsymbol{\rho}) \in \mathbf{X} \times \mathbf{Y}$ and $\boldsymbol{\sigma} \in \mathbf{Z}$ of the original problem (3.13), and there exist positive constants $\tilde{C}_\Xi$ and $\tilde{c}_\Xi$, depending only on C_Ξ , c_D , β_1 , μ_2 , n , and δ , such that

$$\|(\vec{\mathbf{u}}, \boldsymbol{\rho})\| \leq \tilde{C}_\Xi \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} \quad (4.51)$$

and

$$\|\boldsymbol{\sigma}\|_{\text{div}_{4/3;\Omega}} \leq \tilde{c}_\Xi \sum_{j=1}^2 \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\}^j. \quad (4.52)$$

Moreover, letting $\tilde{L}_\Xi := L_\Xi \|\mathbf{i}_4\|$, and assuming, additionally, that

$$\tilde{L}_\Xi \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} < 1, \quad (4.53)$$

this solution $((\vec{\mathbf{u}}, \boldsymbol{\rho}), \boldsymbol{\sigma}) \in (\mathbf{X} \times \mathbf{Y}) \times \mathbf{Z}$ of (3.13), with $\mathbf{u} = \underline{\mathbf{u}} + \mathbf{u}_g$ and $\underline{\mathbf{u}} \in W(\mathbf{r})$, is unique.

Proof. It follows from (4.43) and Theorem 4.8 that there exists a solution $(\vec{\mathbf{u}}, \boldsymbol{\rho}) := ((\underline{\mathbf{u}}, \boldsymbol{\gamma}), \boldsymbol{\rho}) \in \mathbf{V}$ of the reduced problem (4.24), which satisfies (4.50). Thus, defining $(\vec{\mathbf{u}}, \boldsymbol{\rho}) := (\vec{\mathbf{u}}, \boldsymbol{\rho}) + (\vec{\mathbf{u}}_g, \boldsymbol{\rho}_g)$ and applying Lemma 4.3, we deduce that there exists a unique $\boldsymbol{\sigma} \in \mathbf{Z}$, satisfying (4.25), such that $((\vec{\mathbf{u}}, \boldsymbol{\rho}), \boldsymbol{\sigma})$ is a solution of (3.13). Then, noting from (4.23) that

$$\|(\vec{\mathbf{u}}, \boldsymbol{\rho})\| \leq \|(\vec{\mathbf{u}}, \boldsymbol{\rho})\| + \|(\vec{\mathbf{u}}_g, \boldsymbol{\rho}_g)\| \leq \|(\vec{\mathbf{u}}, \boldsymbol{\rho})\| + c_D \|\mathbf{u}_D\|_{1/2,\Gamma},$$

the a priori estimates (4.51) and (4.52) follow from simple algebraic computations involving the above inequality, together with the upper bounds (4.50) and (4.25) for $\|(\vec{\mathbf{u}}, \boldsymbol{\rho})\|$ and $\|\boldsymbol{\sigma}\|_{\text{div}_{4/3;\Omega}}$, respectively. In turn, (4.44) (cf. Lemma 4.7) yields

$$\|\Xi(\underline{\mathbf{z}}) - \Xi(\underline{\mathbf{z}})\|_{1,\Omega} \leq \tilde{L}_\Xi \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} \|\underline{\mathbf{z}} - \underline{\mathbf{z}}\|_{1,\Omega}, \quad (4.54)$$

for all $\underline{\mathbf{z}}, \underline{\mathbf{z}} \in W(\mathbf{r})$. Hence, (4.53) and the classical Banach fixed-point theorem imply the existence of a unique fixed point of Ξ in $W(\mathbf{r})$, and thus of a unique solution of the reduced problem (4.24) with $\underline{\mathbf{u}} \in W(\mathbf{r})$. In this way, the uniqueness of the solution of (3.13) is a direct consequence of the equivalence result provided by Lemma 4.4. $\square$

4.5 On the smallness condition on $\mathbf{u}_D$

We end this section with a clarifying remark on (4.43). Although this inequality may appear to impose a severe restriction on the admissible Dirichlet boundary data for establishing the existence of a solution to the continuous formulation, we show next that suitable solenoidal liftings of a broad class of such data allow us to reduce the analysis to the case of a homogeneous Dirichlet boundary condition, for which (4.43) is trivially satisfied. Indeed, given $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ satisfying $\mathbf{u}_D \cdot \boldsymbol{\nu} = 0$ on Γ , we know from [21, Lemma 3.1] that, for $t = 4$ and for each $\varepsilon > 0$, there exists $\mathbf{u}_\varepsilon \in \mathbf{H}^1(\Omega)$ such that

$$\text{div}(\mathbf{u}_\varepsilon) = 0 \quad \text{in } \Omega, \quad \mathbf{u}_\varepsilon = \mathbf{u}_D \quad \text{on } \Gamma, \quad \text{and} \quad \|\mathbf{u}_\varepsilon\|_{0,4;\Omega} \leq \varepsilon. \quad (4.55)$$

Introducing the translated unknown $\underline{\mathbf{u}} := \mathbf{u} - \mathbf{u}_\varepsilon$, we obtain $\text{div}(\underline{\mathbf{u}}) = 0$ in Ω and $\underline{\mathbf{u}} = 0$ on Γ . Thus, replacing $\mathbf{u}$ by $\underline{\mathbf{u}} + \mathbf{u}_\varepsilon$ in (2.1), observing that

$$(\nabla \mathbf{u}) \mathbf{u} = \text{div}(\mathbf{u} \otimes \mathbf{u}) = \text{div} \left(\frac{1}{2} (\underline{\mathbf{u}} \otimes \underline{\mathbf{u}}) + (\underline{\mathbf{u}} \otimes \mathbf{u}_\varepsilon) + (\mathbf{u}_\varepsilon \otimes \underline{\mathbf{u}}) + (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \right) + \frac{1}{2} (\nabla \underline{\mathbf{u}}) \underline{\mathbf{u}},$$

defining

$$\underline{\sigma} := 2\mu \mathbf{e}(\underline{\mathbf{u}}) - \frac{\delta}{2}(\underline{\mathbf{u}} \otimes \underline{\mathbf{u}}) - \delta(\underline{\mathbf{u}} \otimes \underline{\mathbf{u}}_\varepsilon) - \delta(\underline{\mathbf{u}}_\varepsilon \otimes \underline{\mathbf{u}}) - \delta(\underline{\mathbf{u}}_\varepsilon \otimes \underline{\mathbf{u}}_\varepsilon) - p\mathbb{I} + 2\mu \mathbf{e}(\underline{\mathbf{u}}_\varepsilon),$$

and denoting by $\underline{\rho}$ and $\underline{\gamma}$ the symmetric and skew-symmetric components, respectively, of $\nabla \underline{\mathbf{u}}$, we deduce, instead of (2.11), the system

$$\begin{aligned} \nabla \underline{\mathbf{u}} &= \underline{\rho} + \underline{\gamma} \quad \text{in } \Omega, \\ \underline{\sigma}^d &= 2\mu \underline{\rho} - \frac{\delta}{2}(\underline{\mathbf{u}} \otimes \underline{\mathbf{u}})^d - \delta(\underline{\mathbf{u}} \otimes \underline{\mathbf{u}}_\varepsilon)^d - \delta(\underline{\mathbf{u}}_\varepsilon \otimes \underline{\mathbf{u}})^d - \delta(\underline{\mathbf{u}}_\varepsilon \otimes \underline{\mathbf{u}}_\varepsilon)^d + 2\mu \mathbf{e}(\underline{\mathbf{u}}_\varepsilon) \quad \text{in } \Omega, \\ -\operatorname{div}(\underline{\sigma}) + \frac{\delta}{2}(\underline{\rho} + \underline{\gamma}) \underline{\mathbf{u}} &= \mathbf{f} \quad \text{in } \Omega, \end{aligned} \tag{4.56}$$

$$\underline{\mathbf{u}} = 0 \quad \text{on } \Gamma, \quad \int_{\Omega} \operatorname{tr} \left(\underline{\sigma} + \frac{\delta}{2}(\underline{\mathbf{u}} \otimes \underline{\mathbf{u}}) + \delta(\underline{\mathbf{u}} \otimes \underline{\mathbf{u}}_\varepsilon) + \delta(\underline{\mathbf{u}}_\varepsilon \otimes \underline{\mathbf{u}}) + \delta(\underline{\mathbf{u}}_\varepsilon \otimes \underline{\mathbf{u}}_\varepsilon) \right) = 0.$$

Thus, proceeding as in Section 3, we arrive at the following variational formulation of (4.56): Find $(\underline{\mathbf{u}}, \underline{\rho}) := ((\underline{\mathbf{u}}, \underline{\gamma}), \underline{\rho}) \in \mathbf{X} \times \mathbf{Y}$ and $\underline{\sigma} \in \mathbf{Z}$, such that

$$\begin{aligned} (\mathbf{A}_{\underline{\mathbf{u}}} + \mathbf{E}_\varepsilon)((\underline{\mathbf{u}}, \underline{\rho}), (\underline{\mathbf{v}}, \underline{\varrho})) + \mathbf{B}((\underline{\mathbf{v}}, \underline{\varrho}), \underline{\sigma}) &= \mathbf{F}_\varepsilon((\underline{\mathbf{v}}, \underline{\varrho})) \quad \forall (\underline{\mathbf{v}}, \underline{\varrho}) \in \mathbf{X} \times \mathbf{Y}, \\ \mathbf{B}((\underline{\mathbf{u}}, \underline{\rho}), \underline{\tau}) &= 0 \quad \forall \underline{\tau} \in \mathbf{Z}, \end{aligned} \tag{4.57}$$

where $\mathbf{A}_{\mathbf{z}}$, for each $\mathbf{z} \in \mathbf{L}^4(\Omega)$, and $\mathbf{B}$ are the bilinear forms introduced in (3.16) and (3.18), respectively, whereas $\mathbf{E}_\varepsilon : (\mathbf{X} \times \mathbf{Y}) \times (\mathbf{X} \times \mathbf{Y}) \rightarrow \mathbb{R}$ is the bilinear form defined as

$$\mathbf{E}_\varepsilon((\underline{\mathbf{w}}, \underline{\vartheta}), (\underline{\mathbf{v}}, \underline{\varrho})) := -\delta \int_{\Omega} \left((\underline{\mathbf{w}} \otimes \underline{\mathbf{u}}_\varepsilon) + (\underline{\mathbf{u}}_\varepsilon \otimes \underline{\mathbf{w}}) \right) : \underline{\varrho}, \tag{4.58}$$

for all $(\underline{\mathbf{w}}, \underline{\vartheta}) := ((\underline{\mathbf{w}}, \underline{\xi}), \underline{\vartheta})$, $(\underline{\mathbf{v}}, \underline{\varrho}) := ((\underline{\mathbf{v}}, \underline{\eta}), \underline{\varrho}) \in \mathbf{X} \times \mathbf{Y}$. In turn, $\mathbf{F}_\varepsilon : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbb{R}$ is the linear functional given by

$$\mathbf{F}_\varepsilon((\underline{\mathbf{v}}, \underline{\varrho})) := \int_{\Omega} \mathbf{f} \cdot \underline{\mathbf{v}} + \delta \int_{\Omega} (\underline{\mathbf{u}}_\varepsilon \otimes \underline{\mathbf{u}}_\varepsilon) : \underline{\varrho} - 2 \int_{\Omega} \mu \mathbf{e}(\underline{\mathbf{u}}_\varepsilon) : \underline{\varrho}, \tag{4.59}$$

for all $(\underline{\mathbf{v}}, \underline{\varrho}) := ((\underline{\mathbf{v}}, \underline{\eta}), \underline{\varrho}) \in \mathbf{X} \times \mathbf{Y}$. It follows from the Cauchy–Schwarz inequality, (4.58), and (4.55) that

$$|\mathbf{E}_\varepsilon((\underline{\mathbf{w}}, \underline{\vartheta}), (\underline{\mathbf{v}}, \underline{\varrho}))| \leq 2\delta\sqrt{n}\varepsilon \|(\underline{\mathbf{w}}, \underline{\vartheta})\| \|(\underline{\mathbf{v}}, \underline{\varrho})\|,$$

which means that, by a suitable choice of ε , $\mathbf{A}_{\underline{\mathbf{u}}} + \mathbf{E}_\varepsilon$ can be made an arbitrarily small perturbation of $\mathbf{A}_{\underline{\mathbf{u}}}$. Consequently, the solvability analysis developed in Sections 4.1–4.4 can be easily adapted to the pair $(\mathbf{A}_{\underline{\mathbf{u}}} + \mathbf{E}_\varepsilon, \mathbf{B})$. In particular, applying the Cauchy–Schwarz and Hölder inequalities, and using the estimate in (4.55) and the upper bound for μ (cf. (2.4)), we deduce from (4.59) that

$$\|\mathbf{F}_\varepsilon\| \leq \|\mathbf{f}\|_{0,4/3;\Omega} + \delta\sqrt{n}\varepsilon^2 + 2\mu_2 \|\mathbf{e}(\underline{\mathbf{u}}_\varepsilon)\|_{0,\Omega},$$

whence the corresponding estimate (4.37) takes the form

$$\|\Xi(\underline{\mathbf{z}})\|_{1,\Omega} \leq C_\Xi \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \delta\sqrt{n}\varepsilon^2 + 2\mu_2 \|\mathbf{e}(\underline{\mathbf{u}}_\varepsilon)\|_{0,\Omega} \right\}. \tag{4.60}$$

To proceed with the analysis, it suffices to define the ball $W(\mathbf{r})$ (cf. (4.42)) with $\mathbf{r}$ equal to the right-hand side of (4.60), without imposing any further restriction on $\mathbf{f}$. We omit the details and emphasize that, in view of the preceding discussion, the existence analysis extends to Dirichlet data $\underline{\mathbf{u}}_D \in \mathbf{H}^{1/2}(\Gamma)$ satisfying $\underline{\mathbf{u}}_D \cdot \underline{\boldsymbol{\nu}} = 0$ on Γ , with no smallness condition on $\underline{\mathbf{u}}_D$.

5 The Galerkin scheme

In this section, we introduce the Galerkin scheme associated with (3.13), address its solvability analysis, establish the corresponding a priori error estimates, and derive rates of convergence for finite element subspaces ensuring the stability of the discrete scheme.

5.1 The discrete problem

Let $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^\gamma$, $\mathbb{H}_h^\rho$, and $\mathbb{H}_h^\sigma$ be generic finite-dimensional subspaces of $\mathbf{L}^4(\Omega)$, $\mathbb{L}_{\text{sk}}^2(\Omega)$, $\mathbb{L}_{\text{tr}}^2(\Omega)$, and $\mathbb{H}(\text{div}_{4/3}; \Omega)$, respectively. As usual, the subscript h employed here will acquire a geometric meaning when introducing specific finite element subspaces later in Section 5.4. Then, defining

$$\mathbf{X}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^\gamma, \quad \mathbf{Y}_h := \mathbb{H}_h^\rho, \quad \text{and} \quad \mathbf{Z}_h := \mathbb{H}_h^\sigma \cap \mathbb{H}_0(\text{div}_{4/3}; \Omega), \quad (5.1)$$

the Galerkin scheme associated with (3.13) reads: Find $(\vec{\mathbf{u}}_h, \rho_h) := ((\mathbf{u}_h, \gamma_h), \rho_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ and $\sigma_h \in \mathbf{Z}_h$, such that

$$\begin{aligned} \mathbf{A}_{\mathbf{u}_h}((\vec{\mathbf{u}}_h, \rho_h), (\vec{\mathbf{v}}_h, \varrho_h)) + \mathbf{B}((\vec{\mathbf{v}}_h, \varrho_h), \sigma_h) &= \mathbf{F}((\vec{\mathbf{v}}_h, \varrho_h)), \\ \mathbf{B}((\vec{\mathbf{u}}_h, \rho_h), \tau_h) &= \mathbf{G}(\tau_h), \end{aligned} \quad (5.2)$$

for all $(\vec{\mathbf{v}}_h, \varrho_h) := ((\mathbf{v}_h, \eta_h), \varrho_h) \in \mathbf{X}_h \times \mathbf{Y}_h$, and $\tau_h \in \mathbf{Z}_h$.

The solvability of (5.2) is addressed below in Section 5.2 by proceeding similarly to the analysis in Section 4. More precisely, we employ a fixed-point strategy here as well and apply again Lemma 4.1, but, unlike in that section, and due to the use of Brouwer's fixed-point theorem instead of Schauder's, neither an equivalent reduced problem nor a compactness argument, such as that employed in Theorem 4.8, is required to establish the existence of a discrete solution. Then, as in the continuous case, sufficiently small data and the Banach fixed-point theorem allow us to conclude uniqueness.

5.2 Discrete solvability analysis

We begin by letting $\mathbf{T}_h : \mathbf{H}_h^{\mathbf{u}} \rightarrow \mathbf{H}_h^{\mathbf{u}}$ be the operator defined by

$$\mathbf{T}_h(\mathbf{z}_h) := \underline{\mathbf{u}}_h \quad \forall \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}, \quad (5.3)$$

where $(\vec{\mathbf{u}}_h, \rho_h) := ((\underline{\mathbf{u}}_h, \underline{\gamma}_h), \rho_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ and $\underline{\sigma}_h \in \mathbf{Z}_h$ constitute the unique solution (to be confirmed below) of the linear problem arising from (5.2) after replacing $\mathbf{A}_{\mathbf{u}_h}$ by $\mathbf{A}_{\underline{\mathbf{u}}_h}$, that is

$$\begin{aligned} \mathbf{A}_{\underline{\mathbf{u}}_h}((\vec{\mathbf{u}}_h, \rho_h), (\vec{\mathbf{v}}_h, \varrho_h)) + \mathbf{B}((\vec{\mathbf{v}}_h, \varrho_h), \underline{\sigma}_h) &= \mathbf{F}((\vec{\mathbf{v}}_h, \varrho_h)), \\ \mathbf{B}((\vec{\mathbf{u}}_h, \rho_h), \tau_h) &= \mathbf{G}(\tau_h), \end{aligned} \quad (5.4)$$

for all $(\vec{\mathbf{v}}_h, \varrho_h) := ((\mathbf{v}_h, \eta_h), \varrho_h) \in \mathbf{X}_h \times \mathbf{Y}_h$, and $\tau_h \in \mathbf{Z}_h$. Then, it is easily seen that solving (5.2) is equivalent to finding a fixed point of $\mathbf{T}_h$, that is: $\mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that

$$\mathbf{T}_h(\mathbf{u}_h) = \mathbf{u}_h. \quad (5.5)$$

We now aim to apply the discrete Babuška–Brezzi theory in Banach spaces (cf. [20, Section 5.2.4]) to prove that (5.4) is well-posed, or equivalently, that $\mathbf{T}_h$ is well-defined. To this end, we first employ

the equivalence result provided by Lemma 4.1 in the present discrete context. Indeed, let $\mathbf{V}_h$ and $\mathbf{Z}_{0,h}$ be the discrete versions of $\mathbf{V}$ (cf. (4.11)) and $\mathbf{Z}_0$ (cf. (4.14)), respectively, that is,

$$\mathbf{V}_h := \left\{ (\vec{\mathbf{v}}_h, \boldsymbol{\varrho}_h) \in \mathbf{X}_h \times \mathbf{Y}_h : \quad \mathbf{B}((\vec{\mathbf{v}}_h, \boldsymbol{\varrho}_h), \boldsymbol{\tau}_h) = 0 \quad \forall \boldsymbol{\tau}_h \in \mathbf{Z}_h \right\}, \quad (5.6)$$

and

$$\mathbf{Z}_{0,h} := \left\{ \boldsymbol{\tau}_h \in \mathbf{Z}_h : \quad \int_{\Omega} \mathbf{v}_h \cdot \operatorname{div}(\boldsymbol{\tau}_h) + \int_{\Omega} \boldsymbol{\eta}_h : \boldsymbol{\tau}_h = 0 \quad \forall \vec{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{X}_h \right\}. \quad (5.7)$$

Then, under the assumption

(H.1) $\mathbb{H}_h^{\sigma}$ contains the multiples of the identity tensor $\mathbb{I}$,

it follows that

$$\mathbf{Z}_h = \left\{ \boldsymbol{\tau}_h - \left(\frac{1}{n|\Omega|} \int_{\Omega} \operatorname{tr}(\boldsymbol{\tau}_h) \right) \mathbb{I} : \quad \boldsymbol{\tau}_h \in \mathbb{H}_h^{\sigma} \right\}.$$

In turn, introducing the hypothesis

(H.2) $\operatorname{div}(\mathbb{H}_h^{\sigma}) \subseteq \mathbf{H}_h^u$,

we readily deduce from (5.1) and (5.7) that

$$\mathbf{Z}_{0,h} = \left\{ \boldsymbol{\tau}_h \in \mathbf{Z}_h : \quad \operatorname{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in } \Omega, \quad \text{and} \quad \int_{\Omega} \boldsymbol{\eta}_h : \boldsymbol{\tau}_h = 0 \quad \forall \boldsymbol{\eta}_h \in \mathbb{H}_h^{\gamma} \right\}. \quad (5.8)$$

Next, we consider the particular case of (4.9) in Lemma 4.1 as a third assumption, namely

(H.3) there exists a positive constant $\beta_{3,d}$, independent of h , such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{Z}_h \\ \boldsymbol{\tau}_h \neq 0}} \frac{\mathbf{B}((\vec{\mathbf{v}}_h, 0), \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\operatorname{div}_{4/3}; \Omega}} = \sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{Z}_h \\ \boldsymbol{\tau}_h \neq 0}} \frac{- \int_{\Omega} \boldsymbol{\eta}_h : \boldsymbol{\tau}_h - \int_{\Omega} \mathbf{v}_h \cdot \operatorname{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\operatorname{div}_{4/3}; \Omega}} \geq \beta_{3,d} \|\vec{\mathbf{v}}_h\|, \quad (5.9)$$

for all $\vec{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{X}_h$. On the other hand, we also assume that

(H.4) $\{\boldsymbol{\tau}_h^d : \boldsymbol{\tau}_h \in \mathbf{Z}_{0,h}\} \subseteq \mathbb{H}_h^{\rho}$.

Hence, proceeding analogously to the derivation of (4.19), we take $\boldsymbol{\tau}_h \in \mathbf{Z}_{0,h}$ and, bounding from below by choosing $\boldsymbol{\varrho}_h := -\boldsymbol{\tau}_h^d \in \mathbf{Y}_h := \mathbb{H}_h^{\rho}$, we find that

$$\sup_{\substack{\boldsymbol{\varrho}_h \in \mathbf{Y}_h \\ \boldsymbol{\varrho}_h \neq 0}} \frac{\mathbf{B}((\vec{0}, \boldsymbol{\varrho}_h), \boldsymbol{\tau}_h)}{\|\boldsymbol{\varrho}_h\|_{0,\Omega}} \geq \frac{\int_{\Omega} \boldsymbol{\tau}_h^d : \boldsymbol{\tau}_h}{\|\boldsymbol{\tau}_h^d\|_{0,\Omega}} = \|\boldsymbol{\tau}_h^d\|_{0,\Omega},$$

which, together with (4.17) and the fact that $\boldsymbol{\tau}_h$ is divergence-free, implies

$$\sup_{\substack{\boldsymbol{\varrho}_h \in \mathbf{Y}_h \\ \boldsymbol{\varrho}_h \neq 0}} \frac{\mathbf{B}((\vec{0}, \boldsymbol{\varrho}_h), \boldsymbol{\tau}_h)}{\|\boldsymbol{\varrho}_h\|_{0,\Omega}} \geq \beta_{4,d} \|\boldsymbol{\tau}_h\|_{\operatorname{div}_{4/3}; \Omega}, \quad (5.10)$$

with $\beta_{4,d} = c_1$. This constitutes the corresponding particular case of (4.10) in Lemma 4.1. Similarly to what we observed when deriving (4.19), which is the continuous version of (5.10), we remark here that we could not have taken $\varrho_h := -\tau_h^d$ in the above proof if $\mathbf{Y}_h$ had been required to satisfy a strong symmetry condition.

As a consequence of the preceding discussion, we obtain the discrete analogue of Lemma 4.2.

Lemma 5.1. *There exist positive constants $\beta_{1,d}$ and $\beta_{2,d}$, depending only on $\beta_{3,d}$, $\beta_{4,d}$, and $\|\mathbf{B}\| \leq 1$ (cf. (4.1)), such that*

$$\sup_{\substack{(\vec{\mathbf{v}}_h, \varrho_h) \in \mathbf{X}_h \times \mathbf{Y}_h \\ (\vec{\mathbf{v}}_h, \varrho_h) \neq 0}} \frac{\mathbf{B}((\vec{\mathbf{v}}_h, \varrho_h), \tau_h)}{\|(\vec{\mathbf{v}}_h, \varrho_h)\|} \geq \beta_{1,d} \|\tau_h\|_{\text{div}_{4/3}; \Omega} \quad \forall \tau_h \in \mathbf{Z}_h, \quad (5.11)$$

and

$$\|\varrho_h\|_{0,\Omega} \geq \beta_{2,d} \|\vec{\mathbf{v}}_h\| \quad \forall (\vec{\mathbf{v}}_h, \varrho_h) \in \mathbf{V}_h. \quad (5.12)$$

Proof. Bearing in mind the particular cases of (4.9) and (4.10) given by (5.9) and (5.10), respectively, the proof follows from a straightforward application of Lemma 4.1 to the present discrete setting. This yields (5.11) and (5.12) as the corresponding particular cases of (4.7) and (4.8), respectively. $\square$

Then, similarly to the derivation of (4.34), that is, by invoking the definition of $\mathbf{A}_z$ (cf. (3.16)), the skew-symmetry property of $\mathbf{D}$ (cf. (4.3)), the recently proved estimate (5.12), and the lower bound of μ (cf. (2.4)), we deduce that, for all $\mathbf{z} \in \mathbf{L}^4(\Omega)$ and $(\vec{\mathbf{v}}_h, \varrho_h) \in \mathbf{V}_h$, there holds

$$\mathbf{A}_z((\vec{\mathbf{v}}_h, \varrho_h), (\vec{\mathbf{v}}_h, \varrho_h)) = 2 \int_{\Omega} \mu |\varrho_h|^2 \geq \mu_1 \beta_{2,d}^2 \|\vec{\mathbf{v}}_h\|^2 + \mu_1 \|\varrho_h\|_{0,\Omega}^2,$$

which yields

$$\mathbf{A}_z((\vec{\mathbf{v}}_h, \varrho_h), (\vec{\mathbf{v}}_h, \varrho_h)) \geq \alpha_{\mathbf{A},d} \|(\vec{\mathbf{v}}_h, \varrho_h)\|^2 \quad \forall (\vec{\mathbf{v}}_h, \varrho_h) \in \mathbf{V}_h, \quad (5.13)$$

with $\alpha_{\mathbf{A},d} := \mu_1 \min\{\beta_{2,d}^2, 1\}$.

We are now in a position to prove the announced well-definedness of $\mathbf{T}_h$.

Lemma 5.2. *For each $\mathbf{z}_h \in \mathbf{H}_h^u$, there exist unique $(\vec{\mathbf{u}}_h, \underline{\rho}_h) := ((\underline{\mathbf{u}}_h, \underline{\gamma}_h), \underline{\rho}_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ and $\underline{\sigma}_h \in \mathbf{Z}_h$ solving (5.4), and hence one can define $\mathbf{T}_h(\mathbf{z}_h) := \underline{\mathbf{u}}_h$. Moreover, there exist positive constants $C_{\mathbf{T},d}$, $c_{\mathbf{T},d}$, $\tilde{C}_{\mathbf{T},d}$, and $\tilde{c}_{\mathbf{T},d}$, depending only on $\alpha_{\mathbf{A},d}$, $\beta_{1,d}$, $\|\mathbf{i}_4\|$, μ_2 , $\sqrt{n}$, and δ , such that*

$$\|\mathbf{T}_h(\mathbf{z}_h)\|_{0,4;\Omega} \leq \|(\vec{\mathbf{u}}_h, \underline{\rho}_h)\| \leq C_{\mathbf{T},d} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right\} + c_{\mathbf{T},d} \|\mathbf{u}_D\|_{1/2,\Gamma} \|\mathbf{z}_h\|_{0,4;\Omega} \quad (5.14)$$

and

$$\|\underline{\sigma}_h\|_{\text{div}_{4/3}; \Omega} \leq \tilde{C}_{\mathbf{T},d} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right\} (1 + \|\mathbf{z}_h\|_{0,4;\Omega}) + \tilde{c}_{\mathbf{T},d} \|\mathbf{u}_D\|_{1/2,\Gamma} \|\mathbf{z}_h\|_{0,4;\Omega}^2. \quad (5.15)$$

Proof. Having proved (5.11) and (5.13), both under the assumption that the finite element subspaces satisfy the hypotheses (H.1) up to (H.4), the existence and uniqueness of the solution of (5.4) follow from a straightforward application of the discrete Babuška–Brezzi theory in Banach spaces (see, e.g., [20, Theorem 5.10, Section 5.2.4]). In particular, the a priori estimates for $(\vec{\mathbf{u}}_h, \underline{\rho}_h)$ and $\underline{\sigma}_h$ read

$$\|(\vec{\mathbf{u}}_h, \underline{\rho}_h)\| \leq \frac{1}{\alpha_{\mathbf{A},d}} \|\mathbf{F}\| + \frac{1}{\beta_{1,d}} \left(1 + \frac{\|\mathbf{A}_{\mathbf{z}_h}\|}{\alpha_{\mathbf{A},d}} \right) \|\mathbf{G}\|,$$

and

$$\|\underline{\sigma}_h\|_{\text{div}_{4/3};\Omega} \leq \frac{1}{\beta_{1,d}} \left(1 + \frac{\|\mathbf{A}_{\underline{z}_h}\|}{\alpha_{\mathbf{A},d}}\right) \|\mathbf{F}\| + \frac{\|\mathbf{A}_{\underline{z}_h}\|}{\beta_{1,d}^2} \left(1 + \frac{\|\mathbf{A}_{\underline{z}_h}\|}{\alpha_{\mathbf{A},d}}\right) \|\mathbf{G}\|,$$

respectively, which, together with the upper bounds for $\|\mathbf{A}_{\underline{z}_h}\|$, $\|\mathbf{F}\|$, and $\|\mathbf{G}\|$ (cf. (4.1)), and some algebraic manipulations, imply (5.14) and (5.15), thus ending the proof. $\square$

Next, in order to analyze the solvability of the discrete fixed-point equation (5.5), and similarly to the continuous case (cf. Section 4.4), we first show that $\mathbf{T}_h$ maps a nonempty, closed, bounded, and convex subset of $\mathbf{H}_h^{\mathbf{u}}$ into itself. In fact, letting

$$\mathbf{r}_d := 2 C_{\mathbf{T},d} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right\}, \quad (5.16)$$

we define the ball

$$W(\mathbf{r}_d) := \left\{ \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}} : \|\mathbf{z}_h\|_{0,4;\Omega} \leq \mathbf{r}_d \right\}, \quad (5.17)$$

and establish the following result

Lemma 5.3. *Assume that the Dirichlet datum $\mathbf{u}_D$ is sufficiently small so that*

$$c_{\mathbf{T},d} \|\mathbf{u}_D\|_{1/2;\Gamma} \leq \frac{1}{2}. \quad (5.18)$$

Then $\mathbf{T}_h(W(\mathbf{r}_d)) \subseteq W(\mathbf{r}_d)$.

Proof. Similarly to the proof of Lemma 4.6, it follows straightforwardly from (5.14), the assumption (5.18), and the definition of $\mathbf{r}_d$. $\square$

The continuity property of $\mathbf{T}_h$ is provided next.

Lemma 5.4. *Assume that (5.18) holds. Then, there exists a positive constant $L_{\mathbf{T},d}$, depending only on $\alpha_{\mathbf{A},d}$, n , δ , and $C_{\mathbf{T},d}$, such that*

$$\|\mathbf{T}_h(\underline{\mathbf{z}}_h) - \mathbf{T}_h(\underline{\mathbf{z}}_h)\|_{0,4;\Omega} \leq L_{\mathbf{T},d} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right\} \|\underline{\mathbf{z}}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega}, \quad (5.19)$$

for all $\underline{\mathbf{z}}_h, \underline{\mathbf{z}}_h \in W(\mathbf{r}_d)$.

Proof. Given $\underline{\mathbf{z}}_h, \underline{\mathbf{z}}_h \in W(\mathbf{r}_d)$, we let $\mathbf{T}_h(\underline{\mathbf{z}}_h) := \underline{\mathbf{u}}_h$ and $\mathbf{T}_h(\underline{\mathbf{z}}_h) := \underline{\mathbf{u}}_h$, where $(\underline{\mathbf{u}}_h, \underline{\rho}_h) := ((\underline{\mathbf{u}}_h, \underline{\gamma}_h), \underline{\rho}_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ and $\underline{\sigma}_h \in \mathbf{Z}_h$, along with $(\underline{\mathbf{u}}_h, \underline{\rho}_h) := ((\underline{\mathbf{u}}_h, \underline{\gamma}_h), \underline{\rho}_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ and $\underline{\sigma}_h \in \mathbf{Z}_h$, constitute the unique solutions of the problems defined by (5.4) using $\underline{\mathbf{z}}_h$ and $\underline{\mathbf{z}}_h$, respectively, that is,

$$\begin{aligned} \mathbf{A}_{\underline{\mathbf{z}}_h}((\underline{\mathbf{u}}_h, \underline{\rho}_h), (\underline{\mathbf{v}}_h, \underline{\rho}_h)) + \mathbf{B}((\underline{\mathbf{v}}_h, \underline{\rho}_h), \underline{\sigma}_h) &= \mathbf{F}((\underline{\mathbf{v}}_h, \underline{\rho}_h)), \\ \mathbf{B}((\underline{\mathbf{u}}_h, \underline{\rho}_h), \tau_h) &= \mathbf{G}(\tau_h), \end{aligned} \quad (5.20)$$

and

$$\begin{aligned} \mathbf{A}_{\underline{\mathbf{z}}_h}((\underline{\mathbf{u}}_h, \underline{\rho}_h), (\underline{\mathbf{v}}_h, \underline{\rho}_h)) + \mathbf{B}((\underline{\mathbf{v}}_h, \underline{\rho}_h), \underline{\sigma}_h) &= \mathbf{F}((\underline{\mathbf{v}}_h, \underline{\rho}_h)), \\ \mathbf{B}((\underline{\mathbf{u}}_h, \underline{\rho}_h), \tau_h) &= \mathbf{G}(\tau_h), \end{aligned} \quad (5.21)$$

for all $(\underline{\mathbf{v}}_h, \underline{\rho}_h) := ((\mathbf{v}_h, \underline{\eta}_h), \underline{\rho}_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ and $\tau_h \in \mathbf{Z}_h$. It follows from the second equations of (5.20) and (5.21) that $(\underline{\mathbf{u}}_h, \underline{\rho}_h) - (\underline{\mathbf{u}}_h, \underline{\rho}_h)$ belongs to $\mathbf{V}_h$ (cf. (5.6)). Hence, subtracting the corresponding first equations and testing with $(\underline{\mathbf{v}}_h, \underline{\rho}_h) := (\underline{\mathbf{u}}_h, \underline{\rho}_h) - (\underline{\mathbf{u}}_h, \underline{\rho}_h)$, we readily deduce that

$$\mathbf{A}_{\underline{\mathbf{z}}_h}((\underline{\mathbf{u}}_h, \underline{\rho}_h), (\underline{\mathbf{u}}_h, \underline{\rho}_h) - (\underline{\mathbf{u}}_h, \underline{\rho}_h)) = \mathbf{A}_{\underline{\mathbf{z}}_h}((\underline{\mathbf{u}}_h, \underline{\rho}_h), (\underline{\mathbf{u}}_h, \underline{\rho}_h) - (\underline{\mathbf{u}}_h, \underline{\rho}_h)). \quad (5.22)$$

Next, applying the ellipticity property (5.13) with $\mathbf{z} = \underline{\mathbf{z}}_h$ and $(\vec{\mathbf{v}}_h, \underline{\boldsymbol{\rho}}_h)$ as defined above, and then using (5.22), the definition of $\mathbf{A}_{\mathbf{z}}$ (cf. (3.16)), and the inequality (4.4), we obtain

$$\begin{aligned} \alpha_{\mathbf{A},d} \|(\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h) - (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)\|^2 &\leq \mathbf{A}_{\underline{\mathbf{z}}_h}((\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h) - (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h), (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h) - (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)) \\ &= (\mathbf{A}_{\underline{\mathbf{z}}_h} - \mathbf{A}_{\underline{\mathbf{z}}_h})((\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h), (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h) - (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)) \\ &= \mathbf{D}(\underline{\mathbf{z}}_h - \underline{\mathbf{z}}_h; (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h), (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h) - (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)) \\ &\leq \sqrt{n} \delta \|\underline{\mathbf{z}}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega} \|(\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)\| \|(\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h) - (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)\|, \end{aligned} \quad (5.23)$$

from which it follows that

$$\begin{aligned} \|\mathbf{T}_h(\underline{\mathbf{z}}_h) - \mathbf{T}_h(\underline{\mathbf{z}}_h)\|_{0,4;\Omega} &= \|\underline{\mathbf{u}}_h - \underline{\mathbf{u}}_h\|_{0,4;\Omega} \leq \|(\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h) - (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)\| \\ &\leq \alpha_{\mathbf{A},d}^{-1} \sqrt{n} \delta \|(\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)\| \|\underline{\mathbf{z}}_h - \underline{\mathbf{z}}_h\|_{0,4;\Omega}. \end{aligned} \quad (5.24)$$

Finally, noting from (5.14) and (5.18) that

$$\|(\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)\| \leq \mathbf{r}_d := 2 C_{\mathbf{T},d} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right\},$$

and replacing this bound in (5.24), we arrive at (5.19) with $L_{\mathbf{T},d} = 2 \alpha_{\mathbf{A},d}^{-1} \sqrt{n} \delta C_{\mathbf{T},d}$. $\square$

Having established Lemma 5.4, we now address the solvability of (5.2).

Theorem 5.5. *Assume that (5.18) holds. Then, there exists a fixed point $\mathbf{u}_h \in W(\mathbf{r}_d)$ of $\mathbf{T}_h$, and hence a solution $(\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h) := ((\mathbf{u}_h, \gamma_h), \underline{\boldsymbol{\rho}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ and $\sigma_h \in \mathbf{Z}_h$ of (5.2), which satisfies*

$$\|(\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)\| \leq \mathbf{r}_d := 2 C_{\mathbf{T},d} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right\}, \quad (5.25)$$

and

$$\|\sigma_h\|_{\text{div}_{4/3;\Omega}} \leq \tilde{C}_{\mathbf{T},d} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right\} (1 + \mathbf{r}_d) + \tilde{c}_{\mathbf{T},d} \|\mathbf{u}_D\|_{1/2,\Gamma} \mathbf{r}_d^2. \quad (5.26)$$

Moreover, assuming that

$$L_{\mathbf{T},d} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right\} < 1, \quad (5.27)$$

this solution $((\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h), \sigma_h) \in (\mathbf{X}_h \times \mathbf{Y}_h) \times \mathbf{Z}_h$ of (5.2), is unique.

Proof. Knowing from Lemma 5.3 that, under the assumption (5.18), $\mathbf{T}_h$ maps the ball $W(\mathbf{r}_d)$ into itself, and using the continuity property of $\mathbf{T}_h$ provided by Lemma 5.4, the existence of a fixed point of $\mathbf{T}_h$, and hence of a solution of (5.2), follows from a direct application of Brouwer's fixed-point theorem (cf. [14, Theorem 9.9-2]). In addition, the a priori estimates (5.25) and (5.26) arise from (5.14) and (5.15) by taking $\underline{\mathbf{z}}_h = \mathbf{u}_h$ and noting, for the former, that, due again to (5.18), the right-hand side of (5.14) is bounded by $\mathbf{r}_d$ (cf. (5.16)). Finally, the estimate (5.19), together with the assumption (5.27), guarantees that $\mathbf{T}_h : W(\mathbf{r}_d) \rightarrow W(\mathbf{r}_d)$ is a contraction, and therefore the classical Banach theorem yields the existence of a unique fixed point of $\mathbf{T}_h$, and hence of a unique solution of (5.2). $\square$

5.3 A priori error analysis

In this section, we consider arbitrary finite element subspaces $\mathbf{X}_h$, $\mathbf{Y}_h$, and $\mathbf{Z}_h$ (cf. (5.1)) satisfying the hypotheses (H.1) up to (H.4) introduced in Section 5.2, and derive a Céa estimate for the global error

$$\|(\vec{\mathbf{u}}, \underline{\boldsymbol{\rho}}) - (\vec{\mathbf{u}}_h, \underline{\boldsymbol{\rho}}_h)\| + \|\sigma - \sigma_h\|_{\text{div}_{4/3;\Omega}},$$

where, as guaranteed by Theorems 4.9 and 5.5, $(\vec{\mathbf{u}}, \boldsymbol{\rho}) := ((\mathbf{u}, \boldsymbol{\gamma}), \boldsymbol{\rho}) \in \mathbf{X} \times \mathbf{Y}$ and $\boldsymbol{\sigma} \in \mathbf{Z}$ constitute the unique solution of (3.13), whereas $(\vec{\mathbf{u}}_h, \boldsymbol{\rho}_h) := ((\mathbf{u}_h, \boldsymbol{\gamma}_h), \boldsymbol{\rho}_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ and $\boldsymbol{\sigma}_h \in \mathbf{Z}_h$ form the unique solution of (5.2). Throughout the rest of the paper, given a subspace X_h of an arbitrary Banach space $(X, \|\cdot\|_X)$, we set

$$\text{dist}(x, X_h) := \inf_{x_h \in X_h} \|x - x_h\|_X \quad \forall x \in X.$$

We begin by applying the Strang-type estimates provided by [20, Theorem 5.12] in the setting of (3.13) and (5.2), thus obtaining

$$\begin{aligned} \|(\vec{\mathbf{u}}, \boldsymbol{\rho}) - (\vec{\mathbf{u}}_h, \boldsymbol{\rho}_h)\| &\leq \left(1 + \frac{\|\mathbf{A}_{\mathbf{u}_h}\|}{\alpha_{\mathbf{A},d}}\right) \left(1 + \frac{\|\mathbf{B}\|}{\beta_{1,d}}\right) \text{dist}((\vec{\mathbf{u}}, \boldsymbol{\rho}), \mathbf{X}_h \times \mathbf{Y}_h) \\ &+ \frac{\|\mathbf{B}\|}{\alpha_{\mathbf{A},d}} \text{dist}(\boldsymbol{\sigma}, \mathbf{Z}_h) + \frac{1}{\alpha_{\mathbf{A},d}} \|(\mathbf{A}_{\mathbf{u}} - \mathbf{A}_{\mathbf{u}_h})((\vec{\mathbf{u}}, \boldsymbol{\rho}), \cdot)\|_{\mathbf{V}'_h}, \end{aligned} \quad (5.28)$$

and

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\text{div}_{4/3};\Omega} &\leq C_d \|\mathbf{A}_{\mathbf{u}_h}\| \left(1 + \frac{\|\mathbf{B}\|}{\beta_{1,d}}\right) \text{dist}((\vec{\mathbf{u}}, \boldsymbol{\rho}), \mathbf{X}_h \times \mathbf{Y}_h) \\ &+ (1 + C_d \|\mathbf{B}\|) \text{dist}(\boldsymbol{\sigma}, \mathbf{Z}_h) + C_d \|(\mathbf{A}_{\mathbf{u}} - \mathbf{A}_{\mathbf{u}_h})((\vec{\mathbf{u}}, \boldsymbol{\rho}), \cdot)\|_{(\mathbf{X}_h \times \mathbf{Y}_h)'}, \end{aligned} \quad (5.29)$$

where

$$C_d := \left(1 + \frac{\|\mathbf{A}_{\mathbf{u}_h}\|}{\alpha_{\mathbf{A},d}}\right) \frac{1}{\beta_{1,d}}. \quad (5.30)$$

Since

$$\|(\mathbf{A}_{\mathbf{u}} - \mathbf{A}_{\mathbf{u}_h})((\vec{\mathbf{u}}, \boldsymbol{\rho}), \cdot)\|_{\mathbf{V}'_h} \leq \|(\mathbf{A}_{\mathbf{u}} - \mathbf{A}_{\mathbf{u}_h})((\vec{\mathbf{u}}, \boldsymbol{\rho}), \cdot)\|_{(\mathbf{X}_h \times \mathbf{Y}_h)'}, \quad (5.31)$$

we proceed next to bound the latter norm. Indeed, employing the definition of $\mathbf{A}_{\mathbf{z}}$ (cf. (3.16)) and the inequality (4.4), we find that for all $(\vec{\mathbf{v}}_h, \boldsymbol{\varrho}_h) \in \mathbf{X}_h \times \mathbf{Y}_h$ there holds

$$\begin{aligned} |(\mathbf{A}_{\mathbf{u}} - \mathbf{A}_{\mathbf{u}_h})((\vec{\mathbf{u}}, \boldsymbol{\rho}), (\vec{\mathbf{v}}_h, \boldsymbol{\varrho}_h))| &= |\mathbf{D}(\mathbf{u} - \mathbf{u}_h; (\vec{\mathbf{u}}, \boldsymbol{\rho}), (\vec{\mathbf{v}}_h, \boldsymbol{\varrho}_h))| \\ &\leq \sqrt{n} \delta \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \|(\vec{\mathbf{u}}, \boldsymbol{\rho})\| \|(\vec{\mathbf{v}}_h, \boldsymbol{\varrho}_h)\|, \end{aligned}$$

which, upon bounding $\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}$ by $\|(\vec{\mathbf{u}}, \boldsymbol{\rho}) - (\vec{\mathbf{u}}_h, \boldsymbol{\rho}_h)\|$, yields

$$\|(\mathbf{A}_{\mathbf{u}} - \mathbf{A}_{\mathbf{u}_h})((\vec{\mathbf{u}}, \boldsymbol{\rho}), \cdot)\|_{(\mathbf{X}_h \times \mathbf{Y}_h)'} \leq \sqrt{n} \delta \|(\vec{\mathbf{u}}, \boldsymbol{\rho})\| \|(\vec{\mathbf{u}}, \boldsymbol{\rho}) - (\vec{\mathbf{u}}_h, \boldsymbol{\rho}_h)\|. \quad (5.32)$$

Then, invoking the a priori estimate for $\|(\vec{\mathbf{u}}, \boldsymbol{\rho})\|$ given by (4.51), and substituting the resulting bound from (5.32) into (5.28) and (5.29), which means using previously (5.31) in the former, we arrive, respectively, at

$$\begin{aligned} \|(\vec{\mathbf{u}}, \boldsymbol{\rho}) - (\vec{\mathbf{u}}_h, \boldsymbol{\rho}_h)\| &\leq \left(1 + \frac{\|\mathbf{A}_{\mathbf{u}_h}\|}{\alpha_{\mathbf{A},d}}\right) \left(1 + \frac{\|\mathbf{B}\|}{\beta_{1,d}}\right) \text{dist}((\vec{\mathbf{u}}, \boldsymbol{\rho}), \mathbf{X}_h \times \mathbf{Y}_h) \\ &+ \frac{\|\mathbf{B}\|}{\alpha_{\mathbf{A},d}} \text{dist}(\boldsymbol{\sigma}, \mathbf{Z}_h) + \hat{C} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} \|(\vec{\mathbf{u}}, \boldsymbol{\rho}) - (\vec{\mathbf{u}}_h, \boldsymbol{\rho}_h)\|, \end{aligned} \quad (5.33)$$

with $\hat{C} := \alpha_{\mathbf{A},d}^{-1} \sqrt{n} \delta \tilde{C}_{\Xi}$, and

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\text{div}_{4/3};\Omega} &\leq C_d \|\mathbf{A}_{\mathbf{u}_h}\| \left(1 + \frac{\|\mathbf{B}\|}{\beta_{1,d}}\right) \text{dist}((\vec{\mathbf{u}}, \boldsymbol{\rho}), \mathbf{X}_h \times \mathbf{Y}_h) \\ &+ (1 + C_d \|\mathbf{B}\|) \text{dist}(\boldsymbol{\sigma}, \mathbf{Z}_h) + \hat{c} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} \|(\vec{\mathbf{u}}, \boldsymbol{\rho}) - (\vec{\mathbf{u}}_h, \boldsymbol{\rho}_h)\|, \end{aligned} \quad (5.34)$$

with $\hat{c} := C_d \sqrt{n} \delta \tilde{C}_\Xi$.

The foregoing inequalities allow us to establish now the announced Céa estimate.

Theorem 5.6. *In addition to the hypotheses of Theorems 4.9 and 5.5, assume that*

$$\hat{C} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}^2 \right\} \leq \frac{1}{2}. \quad (5.35)$$

Then there exists a positive constant C , depending only on $\alpha_{\mathbf{A},d}$, $\beta_{1,d}$, μ_2 , $\sqrt{n}$, δ , $\tilde{C}_\Xi$ (cf. (4.51)), $C_{\mathbf{T},d}$ (cf. (5.14)), $\|\mathbf{f}\|_{0,4/3;\Omega}$, and $\|\mathbf{u}_D\|_{1/2,\Gamma}$, such that

$$\|(\bar{\mathbf{u}}, \boldsymbol{\rho}) - (\bar{\mathbf{u}}_h, \boldsymbol{\rho}_h)\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\text{div}_{4/3;\Omega}} \leq C \left\{ \text{dist}((\bar{\mathbf{u}}, \boldsymbol{\rho}), \mathbf{X}_h \times \mathbf{Y}_h) + \text{dist}(\boldsymbol{\sigma}, \mathbf{Z}_h) \right\}. \quad (5.36)$$

Proof. We first employ the assumption (5.35) in (5.33), which yields

$$\|(\bar{\mathbf{u}}, \boldsymbol{\rho}) - (\bar{\mathbf{u}}_h, \boldsymbol{\rho}_h)\| \leq 2 \left(1 + \frac{\|\mathbf{A}_{\mathbf{u}_h}\|}{\alpha_{\mathbf{A},d}} \right) \left(1 + \frac{\|\mathbf{B}\|}{\beta_{1,d}} \right) \text{dist}((\bar{\mathbf{u}}, \boldsymbol{\rho}), \mathbf{X}_h \times \mathbf{Y}_h) + 2 \frac{\|\mathbf{B}\|}{\alpha_{\mathbf{A},d}} \text{dist}(\boldsymbol{\sigma}, \mathbf{Z}_h).$$

Then, the foregoing estimate, together with the one resulting from (5.34) after substituting the former into it, leads to (5.36). In particular, the indicated dependence of the constant C on parameters, other constants, and data follows from the explicit appearance of $\alpha_{\mathbf{A},d}$ and $\beta_{1,d}$ in (5.33) and (5.34), and of $\|\mathbf{f}\|_{0,4/3;\Omega}$ and $\|\mathbf{u}_D\|_{1/2,\Gamma}$ in (5.34), as well as from the definition of C_d (cf. (5.30)) and the bounds (cf. (4.1), (5.25))

$$\|\mathbf{B}\| \leq 1, \quad \|\mathbf{A}_{\mathbf{u}_h}\| \leq (2\mu_2 + \sqrt{n}\delta \|\mathbf{u}_h\|_{0,4;\Omega}), \quad \text{and} \quad \|\mathbf{u}_h\|_{0,4;\Omega} \leq 2C_{\mathbf{T},d} \left\{ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right\}.$$

□

We end this section by deriving an a priori estimate for the error $\|p - p_h\|_{0,\Omega}$, where p_h is the discrete pressure computed according to the postprocessing formula suggested by the second identity in (2.10). More precisely, bearing in mind the decomposition of the original stress according to (3.8) and (3.9), we define

$$p_h = -\frac{1}{n} \text{tr} \left(\boldsymbol{\sigma}_h + c_{0,h} \mathbb{I} + \frac{\delta}{2} (\mathbf{u}_h \otimes \mathbf{u}_h) \right), \quad (5.37)$$

with

$$c_{0,h} = -\frac{\delta}{2n|\Omega|} \int_{\Omega} \text{tr}(\mathbf{u}_h \otimes \mathbf{u}_h). \quad (5.38)$$

Then, applying the Cauchy–Schwarz inequality, performing elementary manipulations, and employing the a priori bounds for $\|\mathbf{u}\|_{0,4;\Omega}$ (cf. (4.51)) and $\|\mathbf{u}_h\|_{0,4;\Omega}$ (cf. (5.25)), we deduce the existence of a positive constant c , depending only on constants introduced in the preceding analysis and the data norms $\|\mathbf{f}\|_{0,4/3;\Omega}$ and $\|\mathbf{u}_D\|_{1/2,\Gamma}$, and hence independent of h , such that

$$\|p - p_h\|_{0,\Omega} \leq c \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \right\}. \quad (5.39)$$

In this way, combining Theorem 5.6 with the estimate (5.39), we obtain the following result.

Theorem 5.7. *Assume the hypotheses of Theorem 5.6. Then there exists a positive constant C , independent of h , such that*

$$\begin{aligned} & \|(\bar{\mathbf{u}}, \boldsymbol{\rho}) - (\bar{\mathbf{u}}_h, \boldsymbol{\rho}_h)\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\text{div}_{4/3;\Omega}} + \|p - p_h\|_{0,\Omega} \\ & \leq C \left\{ \text{dist}((\bar{\mathbf{u}}, \boldsymbol{\rho}), \mathbf{X}_h \times \mathbf{Y}_h) + \text{dist}(\boldsymbol{\sigma}, \mathbf{Z}_h) \right\}. \end{aligned} \quad (5.40)$$

5.4 Specific finite element subspaces

In this section, we provide two examples of concrete finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^\gamma$, $\mathbb{H}_h^\rho$, and $\mathbb{H}_h^\sigma$ satisfying the hypotheses (H.1) up to (H.4) specified in Section 5.2, collect their approximation properties, and establish the rates of convergence of the associated Galerkin schemes. To this end, we first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$ consisting of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K , and set $h := \max\{h_K : K \in \mathcal{T}_h\}$. Then, given an integer $\ell \geq 0$ and $K \in \mathcal{T}_h$, we let $\mathbf{P}_\ell(K)$ be the space of polynomials of degree $\leq \ell$ defined on K , whose vector and tensor versions are denoted by $\mathbf{P}_\ell(K) := [\mathbf{P}_\ell(K)]^n$ and $\mathbb{P}_\ell(K) := [\mathbf{P}_\ell(K)]^{n \times n}$, respectively. Also, we let $\mathbf{RT}_\ell(K) := \mathbf{P}_\ell(K) \oplus \tilde{\mathbf{P}}_\ell(K) \mathbf{x}$ be the local Raviart–Thomas space of order ℓ defined on K , where $\mathbf{x}$ stands for a generic vector in $\mathbb{R}^n$, and $\tilde{\mathbf{P}}_\ell(K)$ is the space of homogeneous polynomials of degree ℓ defined on K . Furthermore, we let b_K be the bubble function on K , defined as the product of its $n + 1$ barycentric coordinates, so that $b_K \in \mathbf{P}_{n+1}(K)$, and introduce the local bubble spaces of order ℓ as

$$\mathbf{b}_\ell(K) := \text{curl}(b_K \mathbf{P}_\ell(K)) \quad \text{if } n = 2, \quad \text{and} \quad \mathbf{b}_\ell(K) := \text{curl}(b_K \mathbf{P}_\ell(K)) \quad \text{if } n = 3,$$

where $\text{curl}(v) := (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})$ if $n = 2$ and $v : K \rightarrow \mathbb{R}$, and $\text{curl}(\mathbf{v}) := \nabla \times \mathbf{v}$ if $n = 3$ and $\mathbf{v} : K \rightarrow \mathbb{R}^3$. In addition, we define the global spaces

$$\mathbf{P}_\ell(\mathcal{T}_h) := \left\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \mathbf{v}_h|_K \in \mathbf{P}_\ell(K) \quad \forall K \in \mathcal{T}_h \right\},$$

$$\mathbb{P}_\ell(\mathcal{T}_h) := \left\{ \boldsymbol{\eta}_h \in \mathbf{L}^2(\Omega) : \boldsymbol{\eta}_h|_K \in \mathbb{P}_\ell(K) \quad \forall K \in \mathcal{T}_h \right\},$$

$$\mathbf{RT}_\ell(\mathcal{T}_h) := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}; \Omega) : \boldsymbol{\tau}_{h,i}|_K \in \mathbf{RT}_\ell(K) \quad \forall i \in \{1, \dots, n\}, \quad \forall K \in \mathcal{T}_h \right\},$$

and

$$\mathbb{B}_\ell(\mathcal{T}_h) := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}; \Omega) : \boldsymbol{\tau}_{h,i}|_K \in \mathbf{b}_\ell(K) \quad \forall i \in \{1, \dots, n\}, \quad \forall K \in \mathcal{T}_h \right\},$$

where $\boldsymbol{\tau}_{h,i}$ stands for the i th row of $\boldsymbol{\tau}_h$. We remark here that $\mathbf{P}_\ell(\mathcal{T}_h)$ is also a subspace of $\mathbf{L}^4(\Omega)$. In addition, since $\mathbf{L}^2(\Omega)$ is continuously embedded in $\mathbf{L}^{4/3}(\Omega)$, it follows that $\mathbb{H}(\mathbf{div}; \Omega)$ is, in turn, continuously embedded in $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$, and hence $\mathbf{RT}_\ell(\mathcal{T}_h)$ and $\mathbb{B}_\ell(\mathcal{T}_h)$ are both subspaces of $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ as well. Furthermore, we denote $\mathbb{C}(\bar{\Omega}) := [C(\bar{\Omega})]^{n \times n}$.

Given an integer $k \geq 0$, the announced examples of stable finite element subspaces are defined next.

a) PEERS-BASED FINITE ELEMENT SUBSPACES

This example is based on PEERS_k , the originally named plane elasticity element with reduced symmetry of order k , whose stability for the Hilbertian mixed formulation of the linear elasticity problem was first established in [4] for $k = 0$ and $n = 2$, and later extended to $k \geq 0$ and $n \in \{2, 3\}$ in [24]. The corresponding subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^\gamma$, and $\mathbb{H}_h^\sigma$ are defined as follows:

$$\mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\mathcal{T}_h), \quad \mathbb{H}_h^\gamma := \mathbb{C}(\bar{\Omega}) \cap \mathbf{P}_{k+1}(\mathcal{T}_h) \cap \mathbf{L}_{\text{sk}}^2(\Omega), \quad \mathbb{H}_h^\sigma := \mathbf{RT}_k(\mathcal{T}_h) + \mathbb{B}_k(\mathcal{T}_h). \quad (5.41)$$

Since the identity matrix $\mathbb{I}$ belongs to $\mathbf{RT}_k(\mathcal{T}_h)$, it is immediate that (H.1) holds. In turn, noting that the tensors in $\mathbb{B}_k(\mathcal{T}_h)$ are divergence-free and that $\mathbf{div}(\mathbf{RT}_k(\mathcal{T}_h)) \subseteq \mathbf{P}_k(\mathcal{T}_h) = \mathbf{H}_h^{\mathbf{u}}$, we deduce that (H.2) is fulfilled by $\mathbb{H}_h^\sigma$ and $\mathbf{H}_h^{\mathbf{u}}$. In addition, as proved by [22, Lemma 4.8] and the subsequent discussion in [22, Section 4.4.3], $\mathbf{X}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^\gamma$ and $\mathbf{Z}_h := \mathbb{H}_h^\sigma \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ also satisfy (H.3). Finally, bearing in mind the resulting subspace $\mathbf{Z}_{0,h}$ (cf. (5.8)), recalling that the divergence free tensors of $\mathbf{RT}_k(\mathcal{T}_h)$ are contained in $\mathbf{P}_k(\mathcal{T}_h)$ (cf. [19, proof of Theorem 3.3]), and observing that $\mathbb{B}_k(\mathcal{T}_h) \subseteq \mathbf{P}_{k+n}(\mathcal{T}_h)$, we deduce that $\mathbf{Z}_{0,h} \subseteq \mathbf{P}_k(\mathcal{T}_h) + \mathbb{B}_k(\mathcal{T}_h) \subseteq \mathbf{P}_{k+n}(\mathcal{T}_h)$, so that in order to accomplish (H.4), it suffices to choose

$$\mathbb{H}_h^\rho := \mathbf{P}_{k+n}(\mathcal{T}_h) \cap \mathbf{L}_{\text{tr}}^2(\Omega). \quad (5.42)$$

b) AFW-BASED FINITE ELEMENT SUBSPACES

This example is based on the Arnold–Falk–Winther (AFW) element of order $k \geq 0$, whose stability for the Hilbertian mixed formulation of the linear elasticity problem was proved in [5]. The corresponding subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\gamma}$, and $\mathbb{H}_h^{\sigma}$ are defined as follows:

$$\mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\mathcal{T}_h), \quad \mathbb{H}_h^{\gamma} := \mathbb{P}_k(\mathcal{T}_h) \cap \mathbb{L}_{\mathbf{sk}}^2(\Omega), \quad \mathbb{H}_h^{\sigma} := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap \mathbb{H}(\mathbf{div}; \Omega). \quad (5.43)$$

It is also straightforward to see here that $\mathbb{H}_h^{\sigma}$ and $\mathbf{H}_h^{\mathbf{u}}$ satisfy **(H.1)** and **(H.2)**. Moreover, as established in [22, Lemma 4.8] and the subsequent discussion in [22, Section 4.4.3], **(H.3)** holds for the present subspaces $\mathbf{X}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}$ and $\mathbf{Z}_h := \mathbb{H}_h^{\sigma} \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ as well. Furthermore, regarding **(H.4)**, and since the definition of $\mathbf{Z}_{0,h}$ (cf. (5.8)) does not seem to admit any further simplification in this case, the inclusion $\mathbf{Z}_{0,h} \subseteq \mathbf{Z}_h := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ suggests choosing

$$\mathbb{H}_h^{\rho} := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap \mathbb{L}_{\mathbf{tr}}^2(\Omega). \quad (5.44)$$

It is worth highlighting that, as established in [22, Lemma 4.8], further examples of finite element subspaces $\mathbf{X}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}$ and $\mathbf{Z}_h := \mathbb{H}_h^{\sigma} \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ satisfying **(H.3)** can be obtained from any stable triplet $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\gamma}$, and $\mathbb{H}_h^{\sigma}$ for the Hilbertian formulation of linear elasticity, provided that, for some integer $\ell \geq 0$, $\mathbb{RT}_{\ell}(\mathcal{T}_h) \subseteq \mathbb{H}_h^{\sigma}$ and $\mathbf{H}_h^{\mathbf{u}} \subseteq \mathbf{P}_{\ell}(\mathcal{T}_h)$.

Now, the approximation properties of $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\gamma}$, and $\mathbf{Z}_h := \mathbb{H}_h^{\sigma} \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$, for both PEERS (cf. (5.41)) and AFW (cf. (5.43)), are derived from the error estimates for the Raviart–Thomas and AFW interpolation operators and the projectors onto piecewise vector and tensor polynomial spaces, together with the corresponding commuting diagram properties and Sobolev space interpolation estimates (cf. [6], [15, eqs. (5.37) and (5.40)], [18, Proposition 1.135]). Their statements are as follows:

(AP_h^u) there exists a positive constant C , independent of h , such that for each $r \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{r,4}(\Omega)$, there holds

$$\text{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) \leq C h^r \|\mathbf{v}\|_{r,4;\Omega},$$

(AP_h^γ) there exists a positive constant C , independent of h , such that for each $r \in [0, k+1]$, and for each $\boldsymbol{\eta} \in \mathbb{H}^r(\Omega) \cap \mathbb{L}_{\mathbf{skew}}^2(\Omega)$, there holds

$$\text{dist}(\boldsymbol{\eta}, \mathbb{H}_h^{\gamma}) \leq C h^r \|\boldsymbol{\eta}\|_{r,\Omega},$$

(AP_h^σ) there exists a positive constant C , independent of h , such that for each $r \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^r(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{r,4/3}(\Omega)$, there holds

$$\text{dist}(\boldsymbol{\tau}, \mathbb{H}_h^{\sigma}) \leq C h^r \left\{ \|\boldsymbol{\tau}\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{r,4/3;\Omega} \right\}.$$

In turn, the approximation property of $\mathbb{H}_h^{\rho}$, which is analogous to that of $\mathbf{H}_h^{\mathbf{u}}$, can also be unified for PEERS (cf. (5.41)) and AFW (cf. (5.43)) by defining $\bar{k} := \begin{cases} k+n & \text{for PEERS-based} \\ k+1 & \text{for AFW-based} \end{cases}$. The corresponding statement reads as follows:

(AP_h^ρ) there exists a positive constant C , independent of h , such that for each $\bar{r} \in [0, \bar{k}+1]$, and for each $\boldsymbol{\varrho} \in \mathbb{H}^{\bar{r}}(\Omega) \cap \mathbb{L}_{\mathbf{tr}}^2(\Omega)$, there holds

$$\text{dist}(\boldsymbol{\varrho}, \mathbb{H}_h^{\rho}) \leq C h^{\bar{r}} \|\boldsymbol{\varrho}\|_{\bar{r},\Omega}.$$

The rates of convergence of the Galerkin scheme (5.2) with the PEERS- and AFW-based finite element subspaces specified in this section are provided in the following theorem.

Theorem 5.8. Assume the hypotheses of Theorem 5.6, and let $((\bar{\mathbf{u}}, \boldsymbol{\rho}), \boldsymbol{\sigma}) \in (\mathbf{X} \times \mathbf{Y}) \times \mathbf{Z}$ and $((\bar{\mathbf{u}}_h, \boldsymbol{\rho}_h), \boldsymbol{\sigma}_h) \in (\mathbf{X}_h \times \mathbf{Y}_h) \times \mathbf{Z}_h$ be the unique solutions of (3.13) and (5.2), whose existence is guaranteed by Theorems 4.9 and 5.5, respectively. Also, let p and p_h be the exact and approximate pressures defined by the second identity in (2.10), and by (5.37) and (5.38), respectively. In turn, given an integer $k \geq 0$, suppose that there exist $r \in (0, k+1]$ and $\bar{r} \in (0, \bar{k}+1]$ such that $\mathbf{u} \in \mathbf{W}^{r,4}(\Omega)$, $\boldsymbol{\gamma} \in \mathbb{H}^r(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^r(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{r,4/3}(\Omega)$, and $\boldsymbol{\rho} \in \mathbb{H}^{\bar{r}}(\Omega)$. Then there exists a positive constant C , independent of h , such that

$$\begin{aligned} & \|(\bar{\mathbf{u}}, \boldsymbol{\rho}) - (\bar{\mathbf{u}}_h, \boldsymbol{\rho}_h)\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} + \|p - p_h\|_{0,\Omega} \\ & \leq C \left\{ h^r \left(\|\mathbf{u}\|_{r,4;\Omega} + \|\boldsymbol{\gamma}\|_{r,\Omega} + \|\boldsymbol{\sigma}\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{r,4/3;\Omega} \right) + h^{\bar{r}} \|\boldsymbol{\rho}\|_{\bar{r},\Omega} \right\}. \end{aligned} \quad (5.45)$$

Proof. It suffices to apply the estimate (5.40), first decomposing

$$\text{dist}((\bar{\mathbf{u}}, \boldsymbol{\rho}), \mathbf{X}_h \times \mathbf{Y}_h) = \text{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) + \text{dist}(\boldsymbol{\gamma}, \mathbb{H}_h^{\boldsymbol{\gamma}}) + \text{dist}(\boldsymbol{\rho}, \mathbb{H}_h^{\boldsymbol{\rho}}),$$

and then employing the approximation properties established above. $\square$

6 Computational results

In this section we consider the two pairs of finite element subspaces detailed in Section 5.4 to present four examples illustrating the performance of the mixed finite element method (5.2) on a set of quasi-uniform triangulations of the respective domains. In what follows, for $k = \{0, 1\}$ and $n = \{2, 3\}$, we simply refer to the corresponding finite element subspace pairs as $\text{PEERS}_k - \mathbb{P}_{k+n}$ and $\text{AFW}_k - \mathbb{P}_{k+1}$, respectively. The numerical methods have been implemented using the open source finite element library FEniCS [1]. A Newton–Raphson algorithm with a fixed tolerance $\text{tol} = 1\text{E} - 06$ is used for the resolution of the nonlinear problem (5.2). As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\text{DoF}}}{\|\mathbf{coeff}^{m+1}\|_{\text{DoF}}} \leq \text{tol},$$

where $\|\cdot\|_{\text{DoF}}$ stands for the usual Euclidean norm in $\mathbf{R}^{\text{DoF}}$ with DoF denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}, \mathbb{H}_h^{\boldsymbol{\gamma}}, \mathbb{H}_h^{\boldsymbol{\sigma}}$, and $\mathbb{H}_h^{\boldsymbol{\rho}}$ (cf. (5.41)–(5.44)).

We now introduce some additional notation. The individual errors are denoted by

$$\mathbf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}, \quad \mathbf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega}, \quad \mathbf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega},$$

$$\mathbf{e}(\boldsymbol{\rho}) := \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\Omega}, \quad \text{and} \quad \mathbf{e}(p) := \|p - p_h\|_{0,\Omega},$$

where p_h stands for the post-processed pressure given by (5.37). Moreover, for each $\star \in \{\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\sigma}, \boldsymbol{\rho}, p\}$ we let $\mathbf{r}(\star)$ be the experimental rate of convergence given by $\mathbf{r}(\star) := \log(\mathbf{e}(\star)/\hat{\mathbf{e}}(\star))/\log(h/\hat{h})$, where h and $\hat{h}$ denote two consecutive meshsizes with errors $\mathbf{e}$ and $\hat{\mathbf{e}}$, respectively.

The examples considered in this section are described below. In all cases, for simplicity, we set $\delta = 1$ and choose the viscosity function as

$$\mu(\mathbf{x}) = \exp\left(-\prod_{i=1}^n x_i\right),$$

which satisfies (2.4). Furthermore, the zero-mean condition for $\text{tr}(\boldsymbol{\sigma}_h)$ over Ω is enforced using a real-valued Lagrange multiplier, which involves adding one row and one column to the matrix system associated with (5.2).

Example 1: Convergence against smooth exact solutions in a 2D domain

In this test we corroborate the rates of convergence in a two-dimensional domain. The domain is the square $\Omega = (0, 1)^2$. We adjust the data $\mathbf{f}$ and $\mathbf{u}_D$ in (2.11) such that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \sin(\pi x_1) \cos(\pi x_2) \\ -\cos(\pi x_1) \sin(\pi x_2) \end{pmatrix} \quad \text{and} \quad p(\mathbf{x}) = \cos(\pi x_1) \sin(0.5 \pi x_2).$$

Tables 6.1 and 6.2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations. As already announced, we stress that we are able not only to approximate the original unknowns but also the pressure field through the formula (5.37). The results confirm that the optimal rates of convergence $\mathcal{O}(h^{k+1})$ predicted by Theorem 5.8, are attained for $k = \{0, 1\}$ by both the $\text{PEERS}_k - \mathbb{P}_{k+2}$ and $\text{AFW}_k - \mathbb{P}_{k+1}$ schemes. The Newton method exhibits a behavior independent of the meshsize, converging in three iterations in all cases. In Figure 6.1 we display some solutions obtained with the mixed $\text{PEERS}_1 - \mathbb{P}_3$ approximation with meshsize $h = 0.013$ and 24,200 triangle elements (actually representing 1,308,122 DoF).

PEERS _k - $\mathbb{P}_{k+2}$ discretization with $k = 0$												
DoF	h	it	$\mathbf{e}(\mathbf{u})$	$\mathbf{r}(\mathbf{u})$	$\mathbf{e}(\boldsymbol{\gamma})$	$\mathbf{r}(\boldsymbol{\gamma})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{r}(\boldsymbol{\sigma})$	$\mathbf{e}(\boldsymbol{\rho})$	$\mathbf{r}(\boldsymbol{\rho})$	$\mathbf{e}(p)$	$\mathbf{r}(p)$
842	0.354	3	2.3E-01	—	2.2E-01	—	5.5E-01	—	4.0E-00	—	4.5E-01	—
3,314	0.177	3	1.2E-01	0.965	6.7E-02	1.694	2.8E-01	0.998	2.0E-00	0.981	2.4E-01	0.916
13,154	0.088	3	5.8E-02	0.991	2.4E-02	1.489	1.4E-01	0.998	1.0E-00	0.996	1.2E-01	0.986
46,082	0.047	3	3.1E-02	0.998	9.4E-03	1.478	7.4E-02	1.000	5.3E-01	0.999	6.5E-02	0.998
183,962	0.024	3	1.6E-02	0.999	3.3E-03	1.494	3.7E-02	1.000	2.7E-01	1.000	3.2E-02	1.000
617,762	0.013	3	8.5E-03	1.000	1.3E-03	1.500	2.0E-02	1.000	1.5E-01	1.000	1.8E-02	1.000

AFW _k - $\mathbb{P}_{k+1}$ discretization with $k = 0$												
DoF	h	it	$\mathbf{e}(\mathbf{u})$	$\mathbf{r}(\mathbf{u})$	$\mathbf{e}(\boldsymbol{\gamma})$	$\mathbf{r}(\boldsymbol{\gamma})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{r}(\boldsymbol{\sigma})$	$\mathbf{e}(\boldsymbol{\rho})$	$\mathbf{r}(\boldsymbol{\rho})$	$\mathbf{e}(p)$	$\mathbf{r}(p)$
609	0.354	3	2.3E-01	—	5.8E-01	—	4.2E-01	—	3.5E-00	—	1.7E-01	—
2,369	0.177	3	1.2E-01	0.966	2.9E-01	0.986	2.1E-01	1.011	1.8E-00	0.998	6.7E-02	1.360
9,345	0.088	3	5.8E-02	0.991	1.5E-01	0.998	1.0E-01	1.002	8.8E-01	1.001	2.9E-02	1.207
32,641	0.047	3	3.1E-02	0.998	7.8E-02	0.999	5.5E-02	1.000	4.7E-01	1.000	1.5E-02	1.072
130,081	0.024	3	1.6E-02	0.999	3.9E-02	1.000	2.8E-02	1.000	2.4E-01	1.000	7.3E-03	1.021
436,481	0.013	3	8.5E-03	1.000	2.1E-02	1.000	1.5E-02	1.000	1.3E-01	1.000	4.0E-03	1.006

Table 6.1: [Example 1, $k = 0$] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the mixed approximations.

Example 2: Convergence against smooth exact solutions in a 3D domain

In the second example we consider the cubic domain $\Omega = (0, 1)^3$. The manufactured solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \sin(\pi x_1) \cos(\pi x_2) \cos(\pi x_3) \\ -2 \cos(\pi x_1) \sin(\pi x_2) \cos(\pi x_3) \\ \cos(\pi x_1) \cos(\pi x_2) \sin(\pi x_3) \end{pmatrix} \quad \text{and} \quad p(\mathbf{x}) = \cos(\pi x_1) \exp(x_2 + x_3).$$

Similarly to the first example, the data $\mathbf{f}$ and $\mathbf{u}_D$ are computed from (2.11) using the above solution. The convergence history for a set of quasi-uniform mesh refinements using $k = 0$ is shown in Table 6.3. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$, as it was proved by Theorem 5.8. We observe a considerable increasing of degrees of freedom in the $\text{PEERS}_0 - \mathbb{P}_3$ scheme compared to the $\text{AFW}_0 - \mathbb{P}_1$ one. This is justified mainly by the fact that the symmetric part of the velocity gradient is approximated with $\mathbb{P}_3(\Omega)$ and $\mathbb{P}_1(\Omega)$, respectively. In addition, some components

PEERS _k – $\mathbb{P}_{k+2}$ discretization with $k = 1$												
DoF	h	it	e(u)	r(u)	e(γ)	r(γ)	e(σ)	r(σ)	e(ρ)	r(ρ)	e(p)	r(p)
1,778	0.354	3	4.1E-02	–	7.9E-02	–	7.9E-02	–	6.0E-01	–	5.7E-02	–
7,010	0.177	3	1.1E-02	1.969	2.2E-02	1.883	2.1E-02	1.900	1.5E-01	1.956	1.7E-02	1.781
27,842	0.088	3	2.6E-03	1.992	6.2E-03	1.793	5.6E-03	1.926	3.9E-02	1.969	4.6E-03	1.854
97,562	0.047	3	7.5E-04	1.998	1.9E-03	1.885	1.6E-03	1.963	1.1E-02	1.985	1.4E-03	1.927
389,522	0.024	3	1.9E-04	1.999	4.9E-04	1.957	4.1E-04	1.985	2.8E-03	1.994	3.5E-04	1.968
1,308,122	0.013	3	5.6E-05	2.000	1.5E-04	1.986	1.2E-04	1.994	8.4E-04	1.998	1.1E-04	1.986

AFW _k – $\mathbb{P}_{k+1}$ discretization with $k = 1$												
DoF	h	it	e(u)	r(u)	e(γ)	r(γ)	e(σ)	r(σ)	e(ρ)	r(ρ)	e(p)	r(p)
1,393	0.354	3	4.1E-02	–	8.8E-02	–	6.2E-02	–	5.4E-01	–	2.0E-02	–
5,473	0.177	3	1.1E-02	1.956	2.3E-02	1.969	1.5E-02	2.065	1.4E-01	2.002	4.6E-03	2.146
21,697	0.088	3	2.6E-03	1.988	5.7E-03	1.985	3.6E-03	2.042	3.4E-02	2.002	1.2E-03	1.950
75,961	0.047	3	7.5E-04	1.997	1.6E-03	1.993	1.0E-03	2.022	9.6E-03	2.002	3.5E-04	1.952
303,121	0.024	3	1.9E-04	1.999	4.1E-04	1.997	2.5E-04	2.011	2.4E-03	2.001	8.9E-05	1.972
1,017,721	0.013	3	5.6E-05	2.000	1.2E-04	1.999	7.5E-05	2.005	7.1E-04	2.001	2.7E-05	1.985

Table 6.2: [Example 1, $k = 1$] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the mixed approximations.

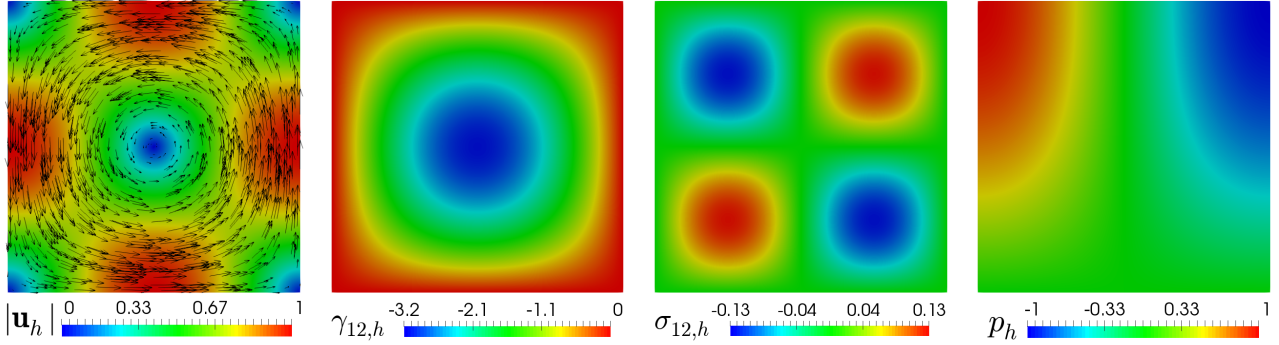


Figure 6.1: [Example 1, $k = 1$] Computed velocity magnitude, vorticity component, pseudostress tensor component, and pressure field.

of the numerical solution are displayed in Figure 6.2, which were built using the mixed AFW₀ – $\mathbb{P}_1$ approximation with meshsize $h = 0.096$ and 34,992 tetrahedral elements (actually representing 1,977,049 DoF).

Example 3: Fluid flow through a rectangular domain with a circular hole.

In the third example, inspired by [26, Chapter 1], we study the behavior of the model for fluid flow in a rectangular domain with a circular hole and non-manufactured solution. More precisely, we consider the domain $\Omega = (0, 1.6) \times (0, 0.4) \setminus \Omega_c$, where

$$\Omega_c = \left\{ (x_1, x_2) : (x_1 - 0.2)^2 + (x_2 - 0.2)^2 < 0.05^2 \right\},$$

with boundary $\Gamma = \partial\Omega$, where the inflow and outflow boundaries are defined as $\Gamma_{\text{in}} = \{0\} \times (0, 0.4)$ and $\Gamma_{\text{out}} = \{1.6\} \times (0, 0.4)$, respectively. The body force is $\mathbf{f} = \mathbf{0}$, and the boundary conditions are

PEERS _k - $\mathbb{P}_{k+3}$ discretization with $k = 0$												
DoF	h	it	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$e(p)$	$r(p)$
8,698	0.866	4	5.7E-01	—	1.4E-00	—	1.8E-00	—	1.5E+01	—	1.2E-00	—
69,016	0.433	4	3.0E-01	0.906	5.0E-01	1.516	9.6E-01	0.923	8.0E-00	0.896	7.3E-01	0.689
550,156	0.217	4	1.6E-01	0.963	1.4E-01	1.847	4.9E-01	0.985	4.1E-00	0.978	3.8E-01	0.945
1,854,688	0.144	4	1.0E-01	0.989	7.3E-02	1.601	3.3E-01	0.990	2.7E-00	0.995	2.5E-01	1.009
6,254,986	0.096	4	6.9E-02	0.995	3.9E-02	1.545	2.2E-01	0.995	1.8E-00	0.999	1.7E-01	1.015

AFW _k - $\mathbb{P}_{k+1}$ discretization with $k = 0$												
DoF	h	it	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$e(p)$	$r(p)$
2,905	0.866	3	5.7E-01	—	1.7E-00	—	1.5E-00	—	1.4E+01	—	1.1Ee-00	—
22,369	0.433	4	3.0E-01	0.918	9.4E-01	0.880	7.3E-01	1.009	7.3E-00	0.948	5.3E-01	1.095
175,489	0.217	4	1.6E-01	0.960	4.8E-01	0.978	3.6E-01	1.017	3.7E-00	1.004	2.3E-01	1.215
588,385	0.144	4	1.0E-01	0.988	3.2E-01	0.996	2.4E-01	1.008	2.4E-00	1.004	1.5E-01	1.119
1,977,049	0.096	4	6.9E-02	0.995	2.1E-01	0.999	1.6E-01	1.005	1.6E-00	1.002	9.5E-02	1.062

Table 6.3: [Example 2, $k = 0$] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the mixed approximations.

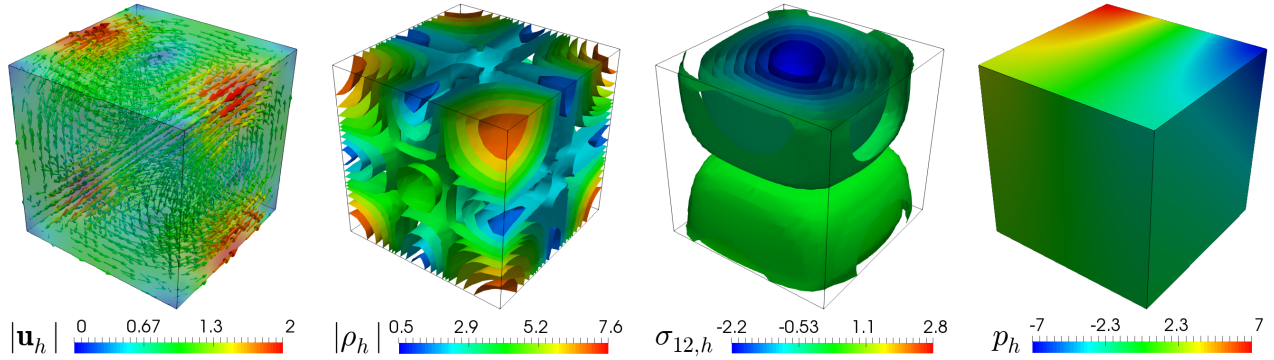


Figure 6.2: [Example 2] Computed velocity magnitude, magnitude of the strain rate tensor, pseudostress tensor component, and pressure field.

given by

$$\mathbf{u} = (-100 x_2 (x_2 - 0.4), 0)^t \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}}),$$

which corresponds to the inflow/outflow driven by a parabolic fluid velocity from the left to the right boundary of the rectangular domain. On the other hand, Figure 6.3 shows the computed velocity magnitude, the magnitudes of the strain rate and vorticity tensors, and the pressure field obtained using the PEERS₀ - $\mathbb{P}_3$ scheme on a mesh with a mesh size of $h = 0.012$ and 21,040 triangular elements, corresponding to 1,138,231 DoF. The nonlinear system converged in four Newton iterations, and the post-processed pressure has a numerically vanishing mean, with $\int_{\Omega} p_h = 2.09 \times 10^{-13}$. This provides further evidence of the numerical consistency of the implementation. As expected, the flow proceeds from left to right across the rectangular domain. Moreover, since the flow cannot enter or leave through the upper and lower boundaries or pass through the circular obstacle, higher velocities are observed around the obstacle. This behavior is consistent with the large gradients observed in the same region and with the pressure distribution.

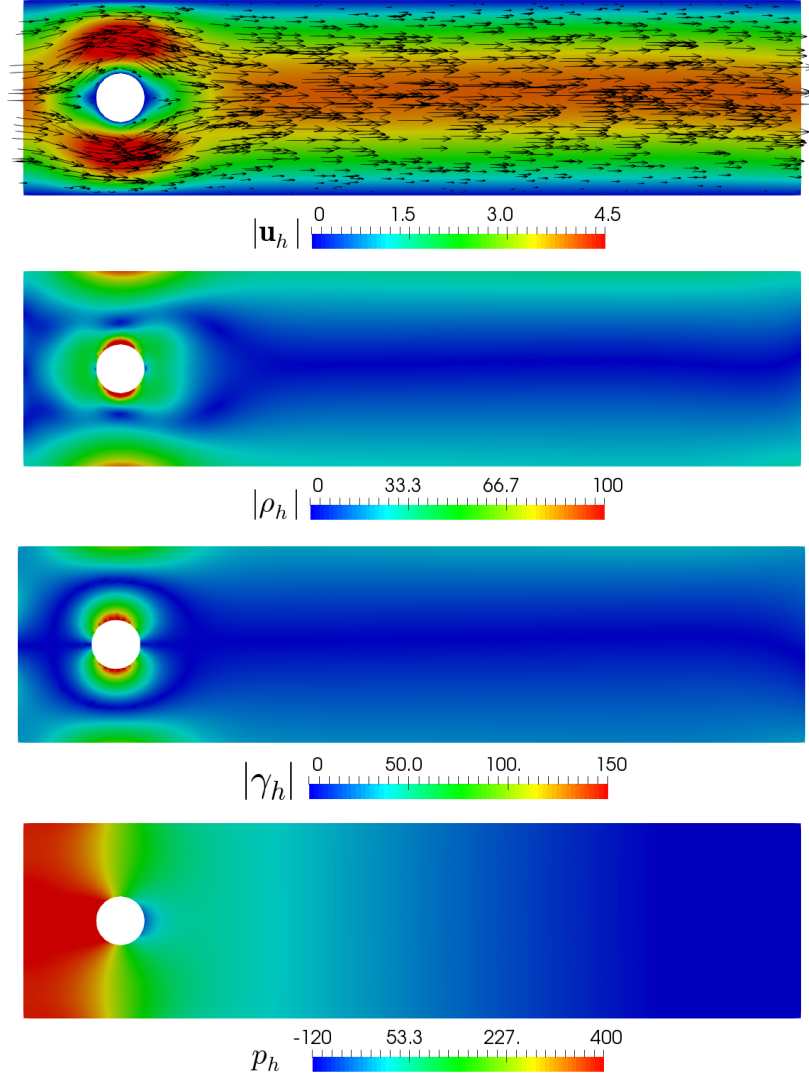


Figure 6.3: [Example 3] Computed velocity magnitude, magnitudes of the strain rate and vorticity tensors, and pressure field.

Example 4: Fluid flow in a cubic lid-driven cavity.

Finally, we consider the three-dimensional lid-driven cavity flow in the unit cube $\Omega = (0, 1)^3$. On the top lid, $x_3 = 1$, the tangential velocity is prescribed as $\mathbf{u} = (1, 0, 0)^t$, while no-slip boundary conditions are imposed on the remaining walls. The body force is set to $\mathbf{f} = \mathbf{0}$. Figure 6.4 displays the computed velocity magnitude, the magnitude of the strain rate tensor, and the pressure field obtained using the mixed $\text{AFW}_0 - \mathbb{P}_1$ scheme on a mesh with mesh size $h = 0.072$ and 82,944 tetrahedral elements, corresponding to 4,675,969 DoF. The nonlinear system converged in three Newton iterations, and the post-processed pressure satisfies the corresponding zero-mean condition, with $\int_{\Omega} p_h = 4.76 \times 10^{-14}$, which is numerically zero and provides an additional consistency check for the computed results. The velocity field exhibits the characteristic large-scale recirculation driven by the motion of the top lid, with the largest velocities occurring near the moving lid. The magnitude of the strain rate tensor attains its largest values along the edges of the top lid, reflecting the strong velocity gradients in these regions. The pressure field reaches its minimum near the left edge of the top lid and its maximum near the right edge, extending along the corresponding side of the cube. In contrast, the pressure

remains nearly constant over the central region of the top lid and throughout most of the interior of the cavity. These observations are consistent with the computed flow pattern.

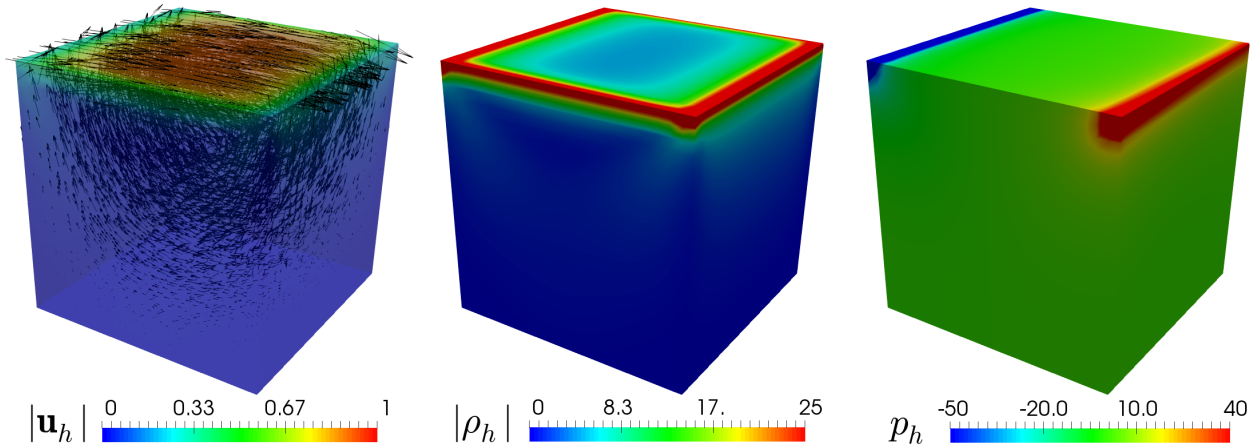


Figure 6.4: [Example 4] Computed velocity magnitude, magnitude of the strain rate tensor, and pressure field.

References

- [1] M.S. ALNAES, J. BLECHTA, J. HAKE, A. JOHANSSON, B. KEHLET, A. LOGG, C. RICHARDSON, J. RING, M.E. ROGNES AND G.N. WELLS, *The FEniCS project version 1.5*. Arch. Numer. Softw. 3 (2015), 9–23.
- [2] L. ANGELO, J. CAMAÑO AND S. CAUCAO, *A five-field mixed formulation for stationary magnetohydrodynamic flows in porous media*. Comput. Methods Appl. Mech. Engrg. 414 (2023), Paper No. 116158.
- [3] L. ANGELO, J. CAMAÑO AND S. CAUCAO, *A skew-symmetric-based mixed FEM for stationary MHD flows in highly porous media*. J. Comput. Appl. Math. 477 (2026), Paper No. 117176.
- [4] D.N. ARNOLD, F. BREZZI AND J. DOUGLAS, PEERS: *A new mixed finite element method for plane elasticity*. Jpn. J. Appl. Math. 1 (1984), 347–367.
- [5] D.N. ARNOLD, R.S. FALK AND R. WINTHER, *Mixed finite element methods for linear elasticity with weakly imposed symmetry*. Math. Comp. 76 (2007), no. 260, 1699–1723.
- [6] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, 1991.
- [7] A.J. BUSTOS, S. CAUCAO, G.N. GATICA AND B.N. VENEGAS, *New fully mixed finite element methods for the coupled convective Brinkman–Forchheimer and nonlinear transport equations*. J. Sci. Comput. 104 (2025), no. 2, Paper No. 64.
- [8] J. CAMAÑO, C. GARCÍA AND R. OYARZÚA, *Analysis of a momentum conservative mixed-FEM for the stationary Navier-Stokes problem*. Numer. Methods Partial Differential Equations 37 (2021), no. 5, 2895–2923.
- [9] J. CAMAÑO, C. MUÑOZ AND R. OYARZÚA, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces*. Electron. Trans. Numer. Anal. 48 (2018), 114–130.

- [10] S. CAUCAO, G.N. GATICA AND J.P. ORTEGA, *A fully-mixed formulation in Banach spaces for the coupling of the steady Brinkman–Forchheimer and double-diffusion equations*. ESAIM Math. Model. Numer. Anal. 5 (2021), no. 6, 2725–2758.
- [11] S. CAUCAO, G.N. GATICA, A. SUAREZ AND I. YOTOV, *A skew-symmetry-based mixed formulation for an Oseen-type Kelvin–Voigt–Brinkman–Forchheimer model*. Preprint 2026-03, Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción, (2026).
[available at <https://www.ci2ma.udec.cl/publicaciones/prepublicaciones/en.php>]
- [12] S. CAUCAO, R. OYARZÚA AND S. VILLA-FUENTES, *A new mixed-FEM for steady-state natural convection models allowing conservation of momentum and thermal energy*. Calcolo 57 (2020), no. 4, Paper No. 36.
- [13] S. CAUCAO, R. OYARZÚA AND S. VILLA-FUENTES, *A priori and a posteriori error analyses of a fully-mixed finite element method for the coupled Navier–Stokes/Darcy problem*. Comput. Methods Appl. Mech. Engrg. 450 (2026), Paper No. 118598.
- [14] P.G. CIARLET, *Linear and Nonlinear Functional Analysis with Applications*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.
- [15] E. COLMENARES, G.N. GATICA AND S. MORAGA, *A Banach spaces-based analysis of a new fully-mixed finite element method for the Boussinesq problem*. ESAIM Math. Model. Numer. Anal. 54 (2020), no. 5, 1525–1568.
- [16] E. COLMENARES, G.N. GATICA, S. MORAGA AND R. RUIZ-BAIER, *A fully-mixed finite element method for the steady state Oberbeck–Boussinesq system*. SMAI J. Comput. Math. 6 (2020), 125–157.
- [17] E. COLMENARES AND M. NEILAN, *Dual-mixed finite element methods for the stationary Boussinesq problem*. Comput. Math. Appl. 72 (2016), no. 7, 1828–1850.
- [18] A. ERN AND J.-L. GUERMOND, *Theory and Practice of Finite Elements*. Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.
- [19] G.N. GATICA, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications*. SpringerBriefs in Mathematics. Springer, Cham, 2014.
- [20] G.N. GATICA, *Introduction to Functional Analysis: Theory and Applications*. Springer, Cham, 2026.
- [21] G.N. GATICA, *A note on arbitrarily small solenoidal liftings of Dirichlet data*. Preprint 2026-31, Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción (2026).
[available at <https://www.ci2ma.udec.cl/publicaciones/prepublicaciones/en.php>]
- [22] G.N. GATICA, R. OYARZÚA, R. RUIZ-BAIER AND Y.D. SOBRAL, *Banach spaces-based analysis of a fully-mixed finite element method for the steady-state model of fluidized beds*. Comput. Math. Appl. 84 (2021), 244–276.
- [23] J.S. HOWELL AND N. WALKINGTON, *Dual mixed finite element methods for the Navier–Stokes equations*. ESAIM Math. Modell. Numer. Anal. 47 (2013), 789–805.
- [24] M. LONSING AND R. VERFÜRTH, *On the stability of BDMS and PEERS elements*. Numer. Math. 99 (2004), no. 1, 131–140.

- [25] L.R. SCOTT AND M. VOGELIUS, *Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials*. RAIRO Modél. Math. Anal. Numér. 19 (1985), no. 1, 111–143.
- [26] S. TUREK, *Efficient solvers for incompressible flow problems*. In: Lecture Notes in Computational Science and Engineering, vol. 6. Springer, Berlin, 1999.
- [27] S. ZHANG, *A new family of stable mixed finite elements for the 3D Stokes equations*. Math. Comp. 74 (2005), no. 250, 543–554.

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