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Analysis of an unfitted HDG method for the coupled linear elasticity/Stokes problem

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Abstract

This work analyzes a Hybridizable discontinuous Galerkin (HDG) method for the fluid/structure interaction problem, where the computational domain does not necessarily fit the interface. We consider a steady-state system arising from the time discretization of a Stokes fluid interacting with a linear elastic solid. We employ the transfer path method to impose the transmission conditions of the fluid velocity and displacement of the solid, and the continuity of the normal component of the stress tensor is approximated by local extrapolations of their polynomial approximation. Under certain assumptions, we prove that the theoretical order of convergence of the error in L^2 -norm is, at least, h^k for all the variables, where h is the meshsize and k the polynomial degree of the HDG spaces, and we identify the specific gap conditions required to recover quasi-optimal or optimal rates. We present numerical experiments supporting the theory, as well as results for a benchmark fluid-structure interaction problem solving the incompressible Navier–Stokes equations.

Keywords: Stokes/Elasticity coupling; curved interfaces; Transfer Path Method; hybridizable discontinuous Galerkin; geometric error.

1 Introduction

The mathematical description of a variety of physical and engineering problems requires partial differential equations (PDEs) in which the domain of interest is partitioned into several distinct subregions, each with its own PDE. In many of these problems, the coupling is enforced through an interface via suitable transmission conditions. This is the case in Fluid-Structure Interaction (FSI) problems. In this context, the fluid flow, typically governed by the Stokes or Navier–Stokes equations, interacts with a deformable solid governed by the equations of linear or nonlinear elasticity, requiring suitable interface conditions such as kinematic continuity and dynamic stress balance across the shared interface.

In the literature, there is a vast amount of research related to FSI problems (see, for instance, [13, 19]). More recently, Hybridizable Discontinuous Galerkin (HDG) methods have been successfully adapted for these coupled systems. For example, [12] presented a monolithic divergence-conforming HDG scheme that uses a single $H(\text{div})$ -conforming finite element space across the entire computational domain for both fluid and structure velocities, ensuring that the interface coupling conditions are satisfied automatically by design. Following a similar monolithic approach, [20] developed an hp -version of the HDG method based on a global velocity-stress formulation, which avoids the traditional displacement variables in the solid to simplify the treatment of the transmission conditions at the discrete level. Recently, a coupled HDG discretization for the interaction between acoustic and elastic waves has been analyzed [2]. In addition, for recent advances and a comprehensive review of finite element discretizations specifically applied to interface and FSI problems, we refer the reader to [1].

When dealing with complex or evolving interfaces, numerical strategies can be broadly classified into fitted and unfitted approaches. Fitted methods are based on a computational mesh that strictly conforms to the interface, which allows them to provide highly accurate solutions. However, in the context of FSI problems involving large displacements or structural deformations, the continuous update of the domain can break the shape regularity of the elements. As highlighted in [1], this geometric degradation requires the generation of a new mesh, which is a computationally costly operation in general.

On the other hand, in unfitted methods, the moving or curved domain is simply immersed within a background grid. Several unfitted strategies have been successfully developed to impose the required transmission conditions. For instance, the Cut Finite Element Method (CutFEM) [4] employs a Nitsche-type approach to enforce these conditions on the elements cut by the interface. A different family of methods handles the interface mismatch using Taylor expansions, such as the Shifted Boundary Method (SBM) [17] and the Transfer Path Method (TPM) [9, 22, 23]. Unlike the SBM, which uses a primal formulation, the TPM relies on a mixed formulation. This makes the TPM a highly effective technique for robustly enforcing the dynamic stress balance and kinematic continuity in FSI problems, even when the fluid and solid subdomains are discretized by completely independent meshes.

This work is closely related to recent works [3, 18, 26] that utilize the TPM for handling transmission conditions suitable in the context of interface problems. Building on these references, our work proposes and analyzes a new method for modeling the coupling in unfitted meshes of a linearly elastic solid and slow fluid flow modeled by the Stokes equations. The model equations are expressed in terms of displacement, velocity, stress, and rotation. To this end, we will employ similar techniques to those developed in the context of HDG schemes for the uncoupled Stokes and elasticity equations [5, 8] (see also [2, 14]). Furthermore, our approach extends the methodology introduced in [10] for an unfitted HDG method for linear elasticity, where the inclusion of the rotation as unknown allows the TPM to suitably handle the boundary data of the problem.

Through a model problem governed by the coupled Stokes/elasticity equations, the main focus of this manuscript is mostly theoretical. That is, we establish sufficient conditions that guarantee the well-posedness and optimal order of convergence of the unfitted HDG method. In parallel, we are preparing a second manuscript with focus on applications. The main contributions of this work can be summarized as follows. First, we propose a high-order numerical framework capable of handling curved interfaces. Through a rigorous *a priori* error analysis, we explicitly track the influence of the geometric gap between the computational and physical interfaces. We demonstrate that, under specific gap-smallness hypotheses, the method successfully preserves high-order accuracy and optimal convergence rates can be recovered. Second, the adoption of a mixed HDG formulation significantly simplifies the treatment of the transmission conditions. By formulating the coupled system in terms of displacement, velocity, stress, and rotation, both the kinematic continuity and the dynamic stress balance are naturally and efficiently incorporated into the discrete numerical traces via the Transfer Path Method. Finally, the proposed methodology provides a highly versatile framework that can be readily extended or adapted to other types of interface problems, generalizing the techniques explored in previous work.

The remainder of this manuscript is organized as follows. In Section 2, we introduce the FSI model. Next, in Section 3, we provide some preliminaries and definitions related to the computational domain, the approximation spaces, and the necessary notation. Then, we present the HDG method for solving the proposed problem and explain the discrete transmission conditions. In Section 4, we demonstrate the stability of the method, followed by the corresponding error analysis in Section 5. Finally, in Section 6, we validate our theoretical results through numerical experiments.

2 Fluid-structure interaction model

The physical domain Ω consists of two disjoint open subdomains Ω_f (fluid) and Ω_s (solid) in $\mathbb{R}^n$ ($n = 2, 3$), with outward unit normal vectors $\boldsymbol{\nu}_s$ and $\boldsymbol{\nu}_f$, respectively. We denote by $\mathcal{I} := \overline{\Omega_f} \cap \overline{\Omega_s}$ the interface that separates the two subdomains. After a semi-implicit Euler discretization in time, for each time step, we obtain a system of partial differential equations that can be written in the following unified mixed form for each subdomain $i \in \{f, s\}$:

$$\mathcal{A}(\boldsymbol{\sigma}_i) - (\nabla \mathbf{u}_i - \boldsymbol{\gamma}_i) = \mathbf{0} \quad \text{in } \Omega_i, \quad (1a)$$

$$-\nabla \cdot \boldsymbol{\sigma}_i + \beta_i \mathbf{u}_i = \mathbf{f}_i \quad \text{in } \Omega_i, \quad (1b)$$

$$\mathbf{u}_i = \mathbf{0} \quad \text{on } \Gamma_i, \quad (1c)$$

where $\Gamma_i := \partial\Omega_i \setminus \mathcal{I}$. Here, $\boldsymbol{\sigma}_i$ represents the stress tensor, $\mathbf{u}_i$ is the unknown velocity (for fluid) or displacement (for solid), $\boldsymbol{\gamma}_i := \frac{1}{2}(\nabla \mathbf{u}_i - (\nabla \mathbf{u}_i)^\top)$ is the rotation tensor, $\beta_i > 0$ is a parameter related to the time step of the discretization, and $\mathbf{f}_i \in \mathbf{L}^2(\Omega_i)$ is a given source term to a neighborhood containing $\mathcal{I}$. The operator $\mathcal{A}$, acting on a tensor $\boldsymbol{\tau}$, characterizes the constitutive behavior of each medium. For the linearly elastic structure ($i = s$), $\mathcal{A}$ denotes the compliance tensor defined by:

$$\mathcal{A}(\boldsymbol{\tau}_s) := \frac{1}{2\mu_s} \left(\boldsymbol{\tau}_s - \frac{\lambda_s}{n\lambda_s + 2\mu_s} \text{tr}(\boldsymbol{\tau}_s) \mathbb{I} \right), \quad (2)$$

where λ_s and μ_s are the Lamé constants. For the Stokes fluid ($i = f$), the operator corresponds to the deviatoric part scaled by the dynamic viscosity μ_f :

$$\mathcal{A}(\tau_f) := \frac{1}{2\mu_f} \tau_f^d = \frac{1}{2\mu_f} \left(\tau_f - \frac{1}{n} \text{tr}(\tau_f) \mathbb{I} \right). \quad (3)$$

For simplicity, we assume the external boundaries Γ_f and Γ_s to be piecewise linear, focusing our attention on the curved interface $\mathcal{I}$. On this interface, the transmission conditions are given by:

$$\alpha_s \mathbf{u}_s - \alpha_f \mathbf{u}_f = \mathbf{g} \quad \text{on } \mathcal{I}, \quad (4a)$$

$$\boldsymbol{\sigma}_s \boldsymbol{\nu}_s + \boldsymbol{\sigma}_f \boldsymbol{\nu}_f = \mathbf{g}_\nu \quad \text{on } \mathcal{I}. \quad (4b)$$

Here, α_s is a positive parameter derived from the time discretization, and $\mathbf{g}$ is a known source term. To facilitate a symmetric analysis, we have introduced the coefficient $\alpha_f := 1$ associated with the fluid velocity. Finally, for the sake of generality, we include a source term $\mathbf{g}_\nu$ representing the jump of normal forces across the interface.

Since the triangulation of the domain does not necessarily fit the interface, we assume that the source terms $\mathbf{f}_i$ can be extended beyond Ω_i .

3 Preliminaries

The computational domain. Let $i \in \{s, f\}$ and let Ω_h^i be a simplicial triangulation of the domain Ω_i with boundary $\mathcal{S}_{i,h}$. We denote the set of all faces of the triangulation Ω_h^i by $\mathcal{E}_h^i$. For the sake of simplicity, we drop the sub-index i and denote the global mesh parameter by h when no confusion arises. We assume that the external boundary Γ_i of the original physical domain is polygonal. This allows us to construct the triangulation such that the discrete external boundary exactly matches the physical one, i.e., $\Gamma_{i,h} = \Gamma_i$. In a general setting, the computational subdomains Ω_h^s and Ω_h^f can be meshed independently. This implies that their respective discrete interfaces, $\mathcal{I}_h^s := \mathcal{S}_{s,h} \setminus \Gamma_{s,h}$ and $\mathcal{I}_h^f := \mathcal{S}_{f,h} \setminus \Gamma_{f,h}$, are not necessarily coincident. As a result, an explicit computational gap or overlap may arise between the discrete subdomains, meaning that the intersection $\overline{\Omega}_h^s \cap \overline{\Omega}_h^f$ does not generally equal the exact physical interface $\mathcal{I}$. In fact, each $\mathcal{I}_h^i$ acts as an independent polygonal approximation of the curved interface $\mathcal{I}$. Bearing this in mind, we define the outward unit normal vectors $\mathbf{n}_s$ and $\mathbf{n}_f$ on their respective discrete interfaces $\mathcal{I}_h^s$ and $\mathcal{I}_h^f$. Due to the geometric discrepancy between the meshes and the physical domain, these discrete normals $\mathbf{n}_i$ generally differ from the exact outward unit normal vector $\boldsymbol{\nu}_i$ of the curved physical interface $\mathcal{I}$. Furthermore, in the presence of an explicit computational gap, $\mathbf{n}_s$ and $\mathbf{n}_f$ are not necessarily collinear. An illustration of this configuration is shown in Figure 1. The family of triangulations $\{\Omega_h^i\}_{h>0}$ is assumed to

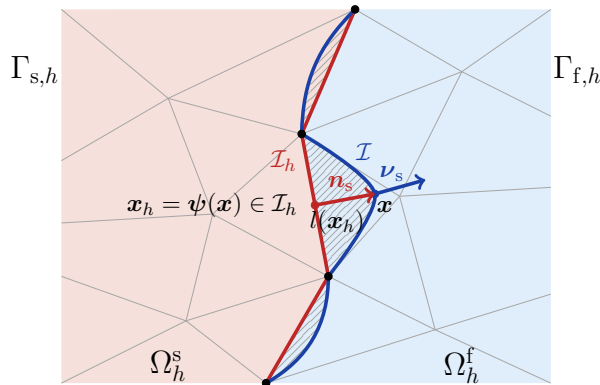


Figure 1: Computational domain. The physical interface $\mathcal{I}$ (blue) is approximated by a piecewise linear discrete interface $\mathcal{I}_h$ (red). A geometric gap (shaded region) arises between $\mathcal{I}_h$ and $\mathcal{I}$. The Transfer Path Method evaluates variables along the segment $l(\mathbf{x}_h)$ parallel to the discrete normal $\mathbf{n}_s$.

be shape-regular, i.e., there exists a constant $\kappa_i > 0$ such that for all elements $K \in \Omega_h^i$ and all $h > 0$, $h_K/\rho_K \leq \kappa_i$, where h_K is the diameter of K and ρ_K is the diameter of the largest ball contained in K . For every element K , we denote by $\mathbf{n}_K$ the outward unit normal vector to K , writing $\mathbf{n}$ instead of $\mathbf{n}_K$ when there is no confusion.

Transfer paths, patches and operators. For $i \in \{s, f\}$, we introduce a bijective mapping $\psi_i : \mathcal{I} \rightarrow \mathcal{I}_h^i$. To connect both discrete boundaries, we define the composition $\psi := \psi_f \circ \psi_s^{-1} : \mathcal{I}_h^s \rightarrow \mathcal{I}_h^f$, which means that for each discrete node $\mathbf{x}_h^s \in \mathcal{I}_h^s$, we associate a unique point $\mathbf{x}_h^f = \psi(\mathbf{x}_h^s) \in \mathcal{I}_h^f$ through their shared physical point $\mathbf{x} := \psi_i^{-1}(\mathbf{x}_h^i) \in \mathcal{I}$. In Figure 1) we show an example of this configuration. We denote by $l_i(\mathbf{x}_h^i)$ and $\mathbf{m}_i(\mathbf{x}_h^i)$ the length and the unit tangent vector of the segment starting at $\mathbf{x}_h^i$ and ending at $\mathbf{x}$. This segment is called a *transfer path* and motivates the name of the method [25]. Now, given a face $e \in \mathcal{I}_h^i$ belonging to the element $K_e \in \Omega_h^i$, we define the *transfer patch* associated to this element as:

$$K_e^{\text{ext}} := \left\{ \mathbf{x}_h^i + \mathbf{m}_i(\mathbf{x}_h^i)r : 0 \leq r \leq l_i(\mathbf{x}_h^i), \mathbf{x}_h^i \in e \right\}. \quad (5)$$

We note that, depending on the local convexity of the physical interface relative to the element K_e , the transfer patch could lie entirely outside K_e or intersect it. In the former case, a polynomial p defined in K_e will be extrapolated to K_e^{ext} and we will denote by $\mathbf{E}(p)$ its canonical extrapolation. As mentioned in Section 1, the source terms $\mathbf{f}_i$ are assumed known in a neighborhood containing these patches. If the transfer patch intersects K_e , in abuse of notation, we will still use the notation $\mathbf{E}(p)$, even though in the intersection $\mathbf{E}(p)$ is equal to p . For vector- or tensor-valued polynomials, $\mathbf{E}$ is applied component-wise.

The canonical extrapolation will be used to evaluate, in the transfer patches, the piecewise polynomials that approximate all the unknowns in (1), except $\mathbf{u}_f$ and $\mathbf{u}_s$. Since their gradients are part of the equations, we can use the transfer path method. More precisely, from (1), we observe that $\nabla \mathbf{u}_i$ can be written as $\nabla \mathbf{u}_i = \mathbf{q}_i$, where $\mathbf{q}_i := \mathcal{A}(\boldsymbol{\sigma}_i) + \boldsymbol{\gamma}_i$. We integrate $\nabla \mathbf{u}_i = \mathbf{q}_i$ along the transfer path connecting $\mathbf{x}_h^i$ and $\mathbf{x}$ to obtain:

$$\mathbf{u}_i(\mathbf{x}) = \mathbf{u}_i(\mathbf{x}_h^i) + l_i(\mathbf{x}_h^i) \int_0^1 \mathbf{q}_i(\mathbf{x}_h^i + (\mathbf{x} - \mathbf{x}_h^i)r) \mathbf{m}_i(\mathbf{x}_h^i) dr.$$

In other words, we can define the following transfer operator $\mathbf{T}$:

$$\mathbf{T}(\mathbf{u}_i, \mathbf{q}_i)(\mathbf{x}) := (\mathbf{u}_i \circ \psi_i)(\mathbf{x}) + l_i(\psi_i(\mathbf{x})) \int_0^1 \mathbf{q}_i(\psi_i(\mathbf{x}) + (\mathbf{x} - \psi_i(\mathbf{x}))r) \mathbf{m}_i(\psi_i(\mathbf{x})) dr \quad \forall \mathbf{x} \in \mathcal{I}. \quad (6)$$

Now, since at the discrete level $\mathbf{q}_i$ will be locally approximated by a polynomial $\mathbf{q}_{i,h}$ and the above integrand is evaluated at points in the transfer patch, we define the following discrete transfer operator:

$$\mathbf{T}_h(\mathbf{w}_{i,h}, \mathbf{q}_{i,h})(\mathbf{x}) := (\mathbf{w}_{i,h} \circ \psi_i)(\mathbf{x}) + l_i(\psi_i(\mathbf{x})) \int_0^1 \mathbf{E}(\mathbf{q}_{i,h})(\psi_i(\mathbf{x}) + (\mathbf{x} - \psi_i(\mathbf{x}))r) \mathbf{m}_i(\psi_i(\mathbf{x})) dr, \quad (7)$$

where $\mathbf{w}_{i,h}$ can be any approximation of $\mathbf{u}_i$. In the context of HDG, $\mathbf{w}_{i,h}$ can be $\mathbf{u}_{i,h}$ or $\hat{\mathbf{u}}_{i,h}$. Since $\mathbf{w}_{i,h}$ is the function being transferred from $\mathcal{I}_h^i$ to $\mathcal{I}$, to shorten the notation, we will simply write $\mathbf{T}_h(\mathbf{w}_{i,h}) = \mathbf{T}_h(\mathbf{w}_{i,h}, \mathbf{q}_{i,h})$.

Spaces and norms. We use the standard notation for Sobolev spaces and their associated norms and seminorms, where vector-valued functions and their corresponding spaces are denoted in boldface font, and blackboard bold font is used for the tensor-valued case. In addition, let D be an open bounded region of $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$. We denote by $(\cdot, \cdot)_D$ and $\langle \cdot, \cdot \rangle_{\partial D}$ the $L^2(D)$ and $L^2(\partial D)$ inner products, respectively, with induced norms $\|\cdot\|_D$ and $\|\cdot\|_{\partial D}$. Given an integer $k \geq 0$, we use the usual notation to denote the space of polynomials of degree at most k as $P_k(D)$, and set $\mathbf{P}_k(D) := [P_k(D)]^n$ and $\mathbb{P}_k(D) := [P_k(D)]^{n \times n}$. In addition, given $K \in \Omega_h^s \cup \Omega_h^f$, we let $\mathbb{A}(K) := \mathbb{P}_k(K) \cap \mathbb{L}_{\text{skew}}^2(K)$, where $\mathbb{L}_{\text{skew}}^2(K) := \{\boldsymbol{\eta} \in \mathbb{L}^2(K) : \boldsymbol{\eta} + \boldsymbol{\eta}^t = \mathbf{0}\}$ is the subspace of skew-symmetric tensors of $\mathbb{L}^2(K)$. For two-dimensions, we set $\mathbb{B}(K) := \text{curl}(\text{curl}(\mathbb{A}(K))\mathbf{b}_K)$, where $\mathbf{b}_K := \prod_{e \subset \partial K} \eta_e$, and η_e is the barycentric coordinate associated to the edge e of K . In the three-dimensional case,

$$\mathbb{B}(K) := \text{curl}(\text{curl}(\mathbb{A}(K))\mathbf{b}_K), \quad \text{with} \quad \mathbf{b}_K := \sum_{e \subset \partial K} \left(\prod_{e' \subset \partial K \setminus \{e\}} \eta_{e'} \right) \nabla \eta_e \otimes \nabla \eta_e.$$

In the two-dimensional case, given vector and tensor-valued fields $\mathbf{v} := (v_1, v_2)$ and $\boldsymbol{\tau} := (\tau_{ij})$, respectively, we let

$$\text{curl}(\mathbf{v}) := \begin{pmatrix} \frac{\partial v_1}{\partial x_2} & -\frac{\partial v_1}{\partial x_1} \\ \frac{\partial v_2}{\partial x_2} & -\frac{\partial v_2}{\partial x_1} \end{pmatrix} \quad \text{and} \quad \text{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} & -\frac{\partial \tau_{11}}{\partial x_2} \\ \frac{\partial \tau_{22}}{\partial x_1} & -\frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Then, for a given polynomial degree k , we introduce the finite-dimensional spaces

$$\begin{aligned}\mathbb{S}_h^i &:= \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_h^i) : \boldsymbol{\tau}|_K \in \mathbb{P}_k(K) + \mathbb{B}(K) \quad \forall K \in \Omega_h^i\}, & \mathbf{V}_h^i &:= \{\mathbf{v} \in \mathbf{L}^2(\Omega_h^i) : \mathbf{v}|_K \in \mathbf{P}_k(K) \quad \forall K \in \Omega_h^i\}, \\ \mathbb{A}_h^i &:= \{\boldsymbol{\eta} \in \mathbb{L}^2(\Omega_h^i) : \boldsymbol{\eta}|_K \in \mathbb{A}(K) \quad \forall K \in \Omega_h^i\}, & \mathbf{M}_h^i &:= \{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h^i) : \boldsymbol{\mu}|_e \in \mathbf{P}_k(e) \quad \forall e \in \mathcal{E}_h^i\}.\end{aligned}$$

Observe that the discrete space $\mathbb{S}_h^i$ can be decomposed as $\mathbb{S}_h^i = \tilde{\mathbb{S}}_h^i \oplus \mathbb{B}_{i,h}$, where

$$\tilde{\mathbb{S}}_h^i := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_h^i) : \boldsymbol{\tau}|_K \in \mathbb{P}_k(K) \quad \forall K \in \Omega_h^i\} \quad \text{and} \quad \mathbb{B}_{i,h} := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_h^i) : \boldsymbol{\tau}|_K \in \mathbb{B}(K) \quad \forall K \in \Omega_h^i\}.$$

We can observe that any function $\boldsymbol{\tau} \in \mathbb{B}_{i,h}$ is such that

$$\nabla \cdot \boldsymbol{\tau}|_K = 0 \quad \forall K \in \Omega_h^i \quad \text{and} \quad \boldsymbol{\tau} \mathbf{n}|_e = 0 \quad \forall e \in \mathcal{E}_h^i.$$

The inner products for the triangulation Ω_h^i ($i \in \{s, f\}$) are given by

$$(\cdot, \cdot)_{\Omega_h^i} := \sum_{K \in \Omega_h^i} (\cdot, \cdot)_K, \quad \langle \cdot, \cdot \rangle_{\partial \Omega_h^i} := \sum_{K \in \Omega_h^i} \langle \cdot, \cdot \rangle_{\partial K}, \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\mathcal{I}_h^i} := \sum_{e \in \mathcal{I}_h^i} \langle \cdot, \cdot \rangle_e,$$

and their corresponding norms will be denoted, respectively, by

$$\|\cdot\|_{\Omega_h^i} := \left(\sum_{K \in \Omega_h^i} \|\cdot\|_K^2 \right)^{1/2}, \quad \|\cdot\|_{\partial \Omega_h^i} := \left(\sum_{K \in \Omega_h^i} \|\cdot\|_{\partial K}^2 \right)^{1/2}, \quad \text{and} \quad \|\cdot\|_{\mathcal{I}_h^i} := \left(\sum_{e \in \mathcal{I}_h^i} \|\cdot\|_e^2 \right)^{1/2}.$$

To avoid proliferation of unimportant constants, we will use the terminology $a \lesssim b$ whenever $a \leq Cb$ and C is a positive constant independent of h and the gap between both discrete interfaces.

The HDG scheme. To approximate the solution to the Stokes equations, we consider the HDG scheme proposed in [5, 21], whereas for the linear elasticity equation, we use [8]. More precisely, for each $i \in \{s, f\}$, the HDG formulation of the coupled system (1) seeks an approximation $(\boldsymbol{\sigma}_{i,h}, \mathbf{u}_{i,h}, \boldsymbol{\gamma}_{i,h}, \hat{\mathbf{u}}_{i,h}) \in \mathbb{S}_h^i \times \mathbf{V}_h^i \times \mathbb{A}_h^i \times \mathbf{M}_h^i$ such that:

$$(\mathcal{A}(\boldsymbol{\sigma}_{i,h}), \boldsymbol{\tau}_{i,h})_{\Omega_h^i} + (\mathbf{u}_{i,h}, \nabla \cdot \boldsymbol{\tau}_{i,h})_{\Omega_h^i} + (\boldsymbol{\gamma}_{i,h}, \boldsymbol{\tau}_{i,h})_{\Omega_h^i} - \langle \hat{\mathbf{u}}_{i,h}, \boldsymbol{\tau}_{i,h} \mathbf{n}_i \rangle_{\partial \Omega_h^i} = 0, \quad (8a)$$

$$(\boldsymbol{\sigma}_{i,h}, \nabla \mathbf{v}_{i,h})_{\Omega_h^i} - \langle \hat{\boldsymbol{\sigma}}_{i,h} \mathbf{n}_i, \mathbf{v}_{i,h} \rangle_{\partial \Omega_h^i} + (\beta_i \mathbf{u}_{i,h}, \mathbf{v}_{i,h})_{\Omega_h^i} = (\mathbf{f}_i, \mathbf{v}_{i,h})_{\Omega_h^i}, \quad (8b)$$

$$(\boldsymbol{\sigma}_{i,h}, \boldsymbol{\eta}_{i,h})_{\Omega_h^i} = 0, \quad (8c)$$

$$\langle \hat{\boldsymbol{\sigma}}_{i,h} \mathbf{n}_i, \boldsymbol{\mu}_{i,h} \rangle_{\partial \Omega_h^i \setminus (\Gamma_{i,h} \cup \mathcal{I}_h^i)} = 0, \quad (8d)$$

$$\langle \hat{\mathbf{u}}_{i,h}, \boldsymbol{\mu}_{i,h} \rangle_{\Gamma_{i,h}} = 0, \quad (8e)$$

for all $(\boldsymbol{\tau}_{i,h}, \mathbf{v}_{i,h}, \boldsymbol{\eta}_{i,h}, \boldsymbol{\mu}_{i,h}) \in \mathbb{S}_h^i \times \mathbf{V}_h^i \times \mathbb{A}_h^i \times \mathbf{M}_h^i$, where

$$\hat{\boldsymbol{\sigma}}_{i,h} \mathbf{n}_i = \boldsymbol{\sigma}_{i,h} \mathbf{n}_i - \tau_i^* (\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}) \quad \text{on} \quad \partial \Omega_h^i, \quad (9)$$

with $\tau_s^* := \tau$ and $\tau_f^* := \tau \mu_f$. Here, τ is a positive stabilization function defined on $\partial \Omega_h^s \cup \partial \Omega_h^f$, assumed to be constant everywhere. The above equations must be complemented with suitable transmission conditions to couple the subdomains across the geometric gap.

According to (4) and (7), we weakly impose “ $\alpha_s \mathbf{T}(\hat{\mathbf{u}}_{s,h}) - \alpha_f \mathbf{T}(\hat{\mathbf{u}}_{f,h}) = \mathbf{g}$ ” at the physical interface $\mathcal{I}$. Since the integration must be performed computationally on a discrete boundary, we select $\mathcal{I}_h^s$ as the reference interface and pull the condition back using ψ_s^{-1} . Thus, we propose the following equation:

$$\langle (\alpha_s \mathbf{T}_h(\hat{\mathbf{u}}_{s,h}) - \alpha_f \mathbf{T}_h(\hat{\mathbf{u}}_{f,h})) \circ \psi_s^{-1}, \boldsymbol{\mu} \rangle_e = \langle \mathbf{g} \circ \psi_s^{-1}, \boldsymbol{\mu} \rangle_e \quad \forall \boldsymbol{\mu} \in \mathbf{P}_k(e) \quad \forall e \in \mathcal{I}_h^s. \quad (10a)$$

Now, inspired by the form of the numerical fluxes (9) and (7), we propose the following discretization of (4b):

$$\langle (\hat{\mathbf{E}}(\hat{\boldsymbol{\sigma}}_{s,h}) + \hat{\mathbf{E}}(\hat{\boldsymbol{\sigma}}_{f,h})) \circ \psi_s^{-1}, \boldsymbol{\mu} \rangle_e = \langle \mathbf{g}_\nu \circ \psi_s^{-1}, \boldsymbol{\mu} \rangle_e \quad \forall \boldsymbol{\mu} \in \mathbf{P}_k(e) \quad \forall e \in \mathcal{I}_h^s, \quad (10b)$$

where the “extrapolation” $\widehat{\mathbf{E}}$ of $\widehat{\boldsymbol{\sigma}}_{i,h}$ for $i \in \{s, f\}$ is defined on $\mathcal{I}$ as

$$\widehat{\mathbf{E}}(\widehat{\boldsymbol{\sigma}}_{i,h}) := \mathbf{E}(\boldsymbol{\sigma}_{i,h})\boldsymbol{\nu}_i - \tau_i^*(\mathbf{T}_h(\mathbf{u}_{i,h}) - \mathbf{T}_h(\widehat{\mathbf{u}}_{i,h})) = \mathbf{E}(\boldsymbol{\sigma}_{i,h})\boldsymbol{\nu}_i - \tau_i^*(\mathbf{u}_{i,h} - \widehat{\mathbf{u}}_{i,h}) \circ \boldsymbol{\psi}_i.$$

In the particular case of matching interfaces with no geometric error, namely $\mathcal{I} = \mathcal{I}_h^s = \mathcal{I}_h^f := \mathcal{I}_h$, the transmission conditions (10) recover the standard form:

$$\begin{aligned} \langle \alpha_s \widehat{\mathbf{u}}_{s,h} - \alpha_f \widehat{\mathbf{u}}_{f,h}, \boldsymbol{\mu} \rangle_e &= \langle \mathbf{g}, \boldsymbol{\mu} \rangle_e \quad \forall \boldsymbol{\mu} \in \mathbf{P}_k(e) \quad \forall e \subset \mathcal{I}_h, \\ \langle \widehat{\boldsymbol{\sigma}}_{s,h} \mathbf{n}_s + \widehat{\boldsymbol{\sigma}}_{f,h} \mathbf{n}_f, \boldsymbol{\mu} \rangle_e &= \langle \mathbf{g}_\nu, \boldsymbol{\mu} \rangle_e \quad \forall \boldsymbol{\mu} \in \mathbf{P}_k(e) \quad \forall e \subset \mathcal{I}_h. \end{aligned}$$

4 Stability analysis

In this section, we establish a stability estimate for the HDG scheme. To that end, we need certain assumptions regarding the mesh configuration and geometry.

Assumptions. We divide the set of assumptions into two groups. The first group consists of geometric assumptions that allow us to simplify the analysis and present the proofs in a more readable manner. The second group establishes the geometric relations required to ensure convergence and optimality of the method.

(A.1) The subdomains are disjoint ($\Omega_h^s \cap \Omega_h^f = \emptyset$) but joined in such a way that their discrete interfaces are geometrically coincident as in Figure 1, i.e., $\mathcal{I}_h^s = \mathcal{I}_h^f := \mathcal{I}_h$.

(A.2) The mapping $\boldsymbol{\psi} : \mathcal{I} \rightarrow \mathcal{I}_h$ is a bijection.

(A.3) For each face $e \subset \mathcal{I}_h$, the direction of the transfer path $\mathbf{m}_i$ is parallel to the discrete normal $\mathbf{n}_i$.

Next, we state the assumptions related to the geometric gap δ :

(A.4) The discrepancy between the discrete normal $\mathbf{n}_i$ and the exact physical normal $\boldsymbol{\nu}_i$ is bounded by $\|\mathbf{n}_i - \boldsymbol{\nu}_i\|_{\infty,e} \leq C_{\text{geo}} \delta_e^\beta$ for $i \in \{s, f\}$.

(A.5) The following condition is satisfied:

$$\mathcal{M}(\delta, h, \text{data}) := \max_{i \in \{s, f\}} \max \{ \mathcal{M}_i(\boldsymbol{\sigma}), \mathcal{M}_i(\partial), \mathcal{M}_i(\widehat{\mathbf{u}}) \} \leq \frac{1}{4},$$

where

$$\mathcal{M}_i(\boldsymbol{\sigma}) := CC(\boldsymbol{\sigma})\mu_i, \quad \mathcal{M}_i(\partial) := C(\partial) + CC(\boldsymbol{\sigma})h\tau_i^*, \quad \mathcal{M}_i(\widehat{\mathbf{u}}) := \frac{CC(\boldsymbol{\sigma})}{\tau_s^*} \frac{(n\lambda_s + 2\mu_s)^2}{|\Omega_h^s|} \max_{e \subset \mathcal{I}_h} \{ h_e \delta_e^{-2/7} \},$$

and, denoting $C_{\text{geo}}^* := 5C_{\text{geo}}^2 C_e^{\text{tr}} + 8(C_e^{\text{ext}})^2$, we define:

$$\begin{aligned} C(\partial) &:= 4 \max_{e \subset \mathcal{I}_h} \{ \delta_e^{2/7} \} + C_{\delta_i, h} \max_{e \subset \mathcal{I}_h} \left\{ \left(\frac{1}{2} \delta_e^{2/7} + 2\alpha_i \tau_i^* \delta_e \right) h_e^{-1} (\tau_i^*)^{-1} \right\}, \\ C(\boldsymbol{\sigma}) &:= C_{\text{geo}}^* \max_{e \subset \mathcal{I}_h} \left\{ \delta_e^{2\beta-2/7} (\tau_i^*)^{-1} h_e^{-1}, \delta_e^{12/7} (\tau_f^*)^{-1} h_e^{-3}, \delta_e^{5/7} h_e^{-1} \right\} \\ &\quad + \min \left\{ \frac{1}{24}, \frac{1}{32C\mu_s \max_{e \subset \mathcal{I}_h} \{ C_e^{\text{tr}} \}} \right\} \max_{e \subset \mathcal{I}_h} \{ C_e^{\text{tr}} \} + C_{\delta_i, h} \max_{i \in \{s, f\}} \left\{ \left(\frac{1}{2} \delta_e^{2/7} + 2\alpha_i \tau_i^* \delta_e \right) (2\mu_i^2)^{-1} \right\}. \end{aligned}$$

(A.6) We define the following constant:

$$C_I(\delta, h) := \max_{i \in \{s, f\}} \max_{e \subset \mathcal{I}_h} \left\{ \delta_e^{\min\{1, \beta\}/2} h_e^{-1/2} + C_e^{\text{ext}} \delta_e h_e^{-3/2}, \delta_e^{1/2} C_{\delta_i, h}^{1/2} \left(h^{-1} + h_e^{-1/2} (\tau_i^*)^{-1/2} \right) \right\}.$$

Furthermore, we define the terms:

$$\begin{aligned}\mathcal{N}(\boldsymbol{\sigma}) &:= \frac{h^2}{\mu_i} + C_I^2(\delta, h) \left(\mu_i + \frac{1}{\mu_i} \right), \quad \mathcal{N}(\partial) := C_I^2(\delta, h) \tau_i^* h, \\ \mathcal{N}(\widehat{\mathbf{u}}) &:= C_I^2(\delta, h) \left(1 + \frac{(n\lambda_s + 2\mu_s)^2}{\tau_s^* |\Omega_h^s|} \max_{e \in \mathcal{I}_h} \{h_e \delta_e^{-2/7}\} \right),\end{aligned}$$

and set $\mathcal{N}(\delta, h, \text{data}) := \max \{\mathcal{N}(\boldsymbol{\sigma}), \mathcal{N}(\partial), \mathcal{N}(\widehat{\mathbf{u}})\}$. We assume that the following condition is satisfied:

$$h^2 \beta_i^2 + C_I^2(\delta, h) (1 + \beta_i)^2 \leq 1/2.$$

(A.7) Let $\tilde{C}$ be the hidden generic constant from the duality estimate in Lemma 9. Denoting

$$C_u^* := \max_{j \in \{s, f\}} \left\{ C(\mathbf{u}) + C C(\boldsymbol{\sigma}) \beta_j^2 + \frac{1}{2} \right\},$$

we define the constant

$$C(\delta, h, \text{data}) := \tilde{C} C_u^* \mathcal{N}(\delta, h, \text{data}),$$

and assume that $C(\delta, h, \text{data}) \leq \frac{1}{4}$.

Let us briefly comment on the assumptions stated above. Note that Assumption (A.1) holds true, for instance, in the geometric configuration illustrated in Figure 1. This ensures that there is neither overlap nor computational gap between discrete subdomains, which is intended to simplify the analysis. In fact, the outward unit normal vectors for the shared discrete interface satisfy $\mathbf{n}_s = -\mathbf{n}_f$ on $\mathcal{I}_h$. On the other hand, Assumption (A.2) is the key to “tie” the shared discrete interface $\mathcal{I}_h$ with the curved interface $\mathcal{I}$. Both assumptions will be invoked in the subsequent lemmas. Assumption (A.4) dictates the discrepancy between the discrete and continuous unit normal vectors of the computational and continuous interfaces, and it is directly applied in the proof of Lemma 7. Thus, by explicitly retaining the parameter β in the definition of $C_I(\delta, h)$ from Assumption (A.6) and the subsequent constants, we find that $\beta \geq 1/2$ is required to preserve the convergence rates. In turn, we observe that the terms in $C(\boldsymbol{\sigma})$ and $C(\partial)$ involve factors such as $\delta_e h_e^{-1}$, $\delta_e^{12/7} h_e^{-3}$, and $\delta_e^{2\beta-2/7} h_e^{-1}$. By assuming that the maximum gap size is $\delta = \mathcal{O}(h^{1+\aleph})$ with $\aleph \in [3/4, 1]$, the most restrictive terms (e.g., $\delta_e^{12/7} h_e^{-3}$) become $\mathcal{O}(1)$ or positive powers of h . This condition is essential for Assumption (A.5), which is required in Lemma 7 to ensure $\mathcal{M}(\delta, h, \text{data}) \leq \frac{1}{4}$ holds for a sufficiently fine mesh, successfully allowing the absorption of the corresponding norms into the left-hand side. Similarly, Assumption (A.6) establishes the necessary geometric bounds for the duality argument developed in Lemma 9. Finally, Assumption (A.7) imposes the condition $C(\delta, h, \text{data}) \leq \frac{1}{4}$, which consequently guarantees the validity of Theorem 1.

Bearing in mind the geometric simplifications introduced above, we consider the associated problem with general source terms to obtain both well-posedness and error bounds in a unified manner. Namely, we look for a solution satisfying (8), but where equations (8a) and (8c) are replaced by:

$$(\mathcal{A}(\boldsymbol{\sigma}_{i,h}), \boldsymbol{\tau}_{i,h})_{\Omega_h^i} + (\mathbf{u}_{i,h}, \nabla \cdot \boldsymbol{\tau}_{i,h})_{\Omega_h^i} + (\boldsymbol{\gamma}_{i,h}, \boldsymbol{\tau}_{i,h})_{\Omega_h^i} - \langle \widehat{\mathbf{u}}_{i,h}, \boldsymbol{\tau}_{i,h} \mathbf{n}_i \rangle_{\partial \Omega_h^i} = (\mathbb{H}_{i,1}, \boldsymbol{\tau}_{i,h})_{\Omega_h^i} + (\mathbb{H}_{i,2}, \boldsymbol{\tau}_{i,h})_{\Omega_h^i}, \quad (11a)$$

$$(\boldsymbol{\sigma}_{i,h}, \boldsymbol{\eta}_{i,h})_{\Omega_h^i} = (\mathbb{E}_i, \boldsymbol{\eta}_{i,h})_{\Omega_h^i}, \quad (11b)$$

where $\mathbb{E}_i \in \mathbb{L}^2(\Omega_h^i)$ and $\mathbb{H}_{i,1}, \mathbb{H}_{i,2} \in \mathbb{L}^2(\Omega_h^i)$ are given functions orthogonal to polynomials of degree $k-1$. We assume that $\mathbb{H}_{i,2} \in \mathbb{A}_h^i$.

Furthermore, due to (A.1), the integration domain for the transmission conditions reduces to the shared discrete interface $\mathcal{I}_h$. By mapping the conditions from $\mathcal{I}$ using the single bijection $\boldsymbol{\psi}^{-1}$, we propose the following simplified equations:

$$\langle (\alpha_s \mathbf{T}_h(\widehat{\mathbf{u}}_{s,h}) - \alpha_f \mathbf{T}_h(\widehat{\mathbf{u}}_{f,h})) \circ \boldsymbol{\psi}^{-1}, \boldsymbol{\mu} \rangle_e = \langle \mathbf{g} \circ \boldsymbol{\psi}^{-1}, \boldsymbol{\mu} \rangle_e \quad \forall \boldsymbol{\mu} \in \mathbf{P}_k(e) \quad \forall e \in \mathcal{I}_h, \quad (12a)$$

$$\langle (\widehat{\mathbf{E}}(\widehat{\boldsymbol{\sigma}}_{s,h}) + \widehat{\mathbf{E}}(\widehat{\boldsymbol{\sigma}}_{f,h})) \circ \boldsymbol{\psi}^{-1}, \boldsymbol{\mu} \rangle_e = \langle \mathbf{g}_\nu \circ \boldsymbol{\psi}^{-1}, \boldsymbol{\mu} \rangle_e \quad \forall \boldsymbol{\mu} \in \mathbf{P}_k(e) \quad \forall e \in \mathcal{I}_h, \quad (12b)$$

where the extrapolation $\widehat{\mathbf{E}}$ of $\widehat{\boldsymbol{\sigma}}_{i,h}$ for $i \in \{s, f\}$ simplifies to:

$$\widehat{\mathbf{E}}(\widehat{\boldsymbol{\sigma}}_{i,h}) := \mathbf{E}(\boldsymbol{\sigma}_{i,h}) \boldsymbol{\nu}_i - \tau_i^* (\mathbf{T}_h(\mathbf{u}_{i,h}) - \mathbf{T}_h(\widehat{\mathbf{u}}_{i,h})) = \mathbf{E}(\boldsymbol{\sigma}_{i,h}) \boldsymbol{\nu}_i - \tau_i^* (\mathbf{u}_{i,h} - \widehat{\mathbf{u}}_{i,h}) \circ \boldsymbol{\psi}.$$

4.1 Preliminaries

This section is devoted to recalling some tools available in the HDG literature that will allow us to show the stability estimate and subsequently provide the error analysis. Bearing in mind the simplified geometric setting introduced by our Assumptions, we adapt these tools to the unified discrete interface $\mathcal{I}_h$.

L^2 -projections [11]. For each subdomain $i \in \{s, f\}$, let us define the local L^2 -orthogonal projections. First, for any facet $e \in \mathcal{E}_h^i$, let $\mathcal{P}_M : \mathbf{L}^2(e) \rightarrow \mathbf{P}_k(e)$ be the projection onto the polynomial space of degree k on the face. Standard approximation results state that for any $\boldsymbol{\mu} \in \mathbf{H}^{l+1}(e)$ with $l \geq 0$:

$$\|\boldsymbol{\mu} - \mathcal{P}_M(\boldsymbol{\mu})\|_e \lesssim h_e^{l+1} |\boldsymbol{\mu}|_{\mathbf{H}^{l+1}(e)}.$$

Similarly, on each element $K \in \Omega_h^i$, we let $\Pi_{\mathbb{A}} : \mathbb{L}^2(K) \rightarrow \mathbb{A}(K)$ be the orthogonal projection onto the local space of skew-symmetric tensor polynomials. If $\boldsymbol{\gamma} \in \mathbb{H}^{l+1}(K)$ with $l \in [0, k]$, then:

$$\|\boldsymbol{\gamma} - \Pi_{\mathbb{A}}(\boldsymbol{\gamma})\|_K \lesssim h_K^{l+1} |\boldsymbol{\gamma}|_{\mathbb{H}^{l+1}(K)}. \quad (13)$$

HDG projections [5, 6, 14]. For each subdomain $i \in \{s, f\}$ and for each element $K \in \Omega_h^i$, let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}^1(K) \times \mathbf{H}^1(K)$. We define the HDG projection $(\Pi\boldsymbol{\sigma}, \Pi\mathbf{u})$ as the unique element in $\mathbb{P}_k(K) \times \mathbf{P}_k(K)$ satisfying the following equations:

$$(\Pi\boldsymbol{\sigma}, \boldsymbol{\tau})_K = (\boldsymbol{\sigma}, \boldsymbol{\tau})_K \quad \forall \boldsymbol{\tau} \in \mathbb{P}_{k-1}(K), \quad (14a)$$

$$(\Pi\mathbf{u}, \mathbf{v})_K = (\mathbf{u}, \mathbf{v})_K \quad \forall \mathbf{v} \in \mathbf{P}_{k-1}(K), \quad (14b)$$

$$\langle \Pi\boldsymbol{\sigma} \mathbf{n}_i - \tau_i^* \Pi\mathbf{u}, \boldsymbol{\mu} \rangle_e = \langle \boldsymbol{\sigma} \mathbf{n}_i - \tau_i^* \mathbf{u}, \boldsymbol{\mu} \rangle_e \quad \forall \boldsymbol{\mu} \in \mathbf{P}_k(e), \quad \forall e \subseteq \partial K. \quad (14c)$$

Assuming that $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}^{l_\sigma+1}(K) \times \mathbf{H}^{l_u+1}(K)$ for $l_u, l_\sigma \in [0, k]$, the following approximation properties hold:

$$\|\mathbf{u} - \Pi\mathbf{u}\|_K \lesssim h_K^{l_u+1} |\mathbf{u}|_{\mathbf{H}^{l_u+1}(K)} + h_K^{l_\sigma+1} |\nabla \cdot \boldsymbol{\sigma}|_{\mathbf{H}^{l_\sigma}(K)}, \quad (15a)$$

$$\|\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}\|_K \lesssim h_K^{l_\sigma+1} |\boldsymbol{\sigma}|_{\mathbb{H}^{l_\sigma+1}(K)} + h_K^{l_u+1} |\mathbf{u}|_{\mathbf{H}^{l_u+1}(K)}. \quad (15b)$$

Estimates on extension patches. Let $i \in \{s, f\}$. We denote by $h_e^\perp$ the largest distance of a point inside an element $K_e \in \Omega_h^i$ to the plane determined by its boundary face $e \subset \mathcal{I}_h$. Similarly, we define δ_e as the maximum length of the transfer paths originating from the face e . In other words,

$$h_e^\perp = \max_{\mathbf{x} \in K_e} |\text{dist}(\mathbf{x}, e)|, \quad \delta_e = \max_{\mathbf{x}_h \in e} l(\mathbf{x}_h),$$

where $\text{dist}(\mathbf{x}, e)$ denotes the distance from $\mathbf{x}$ to the face e , and $l(\mathbf{x}_h)$ is the length of the segment connecting $\mathbf{x}_h \in \mathcal{I}_h$ to its continuous counterpart $\boldsymbol{\psi}^{-1}(\mathbf{x}_h) \in \mathcal{I}$. We note that δ_e is a measure of the local size of the geometric gap, and $\delta := \max_{e \in \mathcal{I}_h} \delta_e$ is a global upper bound of the gap size. We define the ratio $r_e := \delta_e / h_e^\perp$ and its maximum $R_i := \max_{e \in \mathcal{I}_h} r_e$.

For a boundary face $e \subset \mathcal{I}_h$ and its associated transfer patch K_e^{ext} , we define

$$\mathcal{N}^k := \{\mathbf{q} \in \mathbf{P}_k(K_e^{\text{ext}}) : \mathbf{q} \cdot \mathbf{n}_i \neq 0 \text{ on each } e \subset \partial K_e^{\text{ext}}\},$$

where we recall that $\mathbf{n}_i$ is the outward normal vector to the element K_e pointing in the direction of the patch. We can then introduce the constants

$$C_e^{\text{ext}} := \frac{1}{\sqrt{r_e}} \sup_{\mathbf{q} \in \mathcal{N}^k} \frac{\|\mathbf{q} \cdot \mathbf{n}_i\|_{K_e^{\text{ext}}}}{\|\mathbf{q} \cdot \mathbf{n}_i\|_{K_e}}, \quad C_e^{\text{inv}} := h_e^\perp \sup_{\mathbf{q} \in \mathcal{N}^k} \frac{\|\partial_{\mathbf{n}_i} \mathbf{q}\|_{K_e^{\text{ext}}}}{\|\mathbf{q} \cdot \mathbf{n}_i\|_{K_e}}.$$

As proved in [7, Lemma A.2], these constants are independent of the meshsize, but depend on the polynomial degree k . Note that the constants C_e^{ext} and C_e^{inv} could theoretically be different for the solid and fluid sides, but for the sake of simplicity in the analysis, we will consider them as uniform upper bounds.

On the other hand, following the ideas in [7], it will be useful to introduce the following auxiliary functions. Let $e \subset \mathcal{I}_h$ be a face belonging to K_e and its extension K_e^{ext} . For a polynomial function $\mathbf{q}$, we define the average variation along the transfer path:

$$\Lambda_{\mathbf{q}|_{K_e}}(\mathbf{x}_h) := \frac{1}{l(\mathbf{x}_h)} \int_0^{l(\mathbf{x}_h)} (\mathbf{q}(\mathbf{x}_h + \mathbf{n}_i(\mathbf{x}_h)y) - \mathbf{q}(\mathbf{x}_h)) \cdot \mathbf{n}_i dy, \quad (16)$$

where we have explicitly used the discrete normal $\mathbf{n}_i$ as the path direction due to Assumption (A.3). This variation satisfies (cf. [7, Lemma 5.2]),

$$\|l^{1/2}\Lambda_{\mathbf{q}|_{K_e}}\|_e \leq \frac{1}{\sqrt{3}} r_e^{3/2} C_e^{\text{ext}} C_e^{\text{inv}} \|\mathbf{q}\|_{K_e} \quad \forall \mathbf{q} \in \mathbf{P}_k(K_e).$$

The next lemma, adapted from [26, Lemma 2.1] to our case, will be crucial in the error analysis and the development of the duality argument to estimate the approximation error on the physical interface.

Lemma 1. Suppose Assumption (A.2). The following assertions hold true for $i \in \{\text{s}, \text{f}\}$: If $\phi_i \in \mathbf{H}^2(\Omega_i)$ and $\Phi_i := \nabla \phi_i$ then

$$\|l^{-1/2}(\phi_i - \phi_i \circ \psi^{-1})\|_{\mathcal{I}_h} \lesssim \delta^{1/2} \|\phi_i\|_{\mathbf{H}^2(\Omega_i)}. \quad (17)$$

If $\Phi_i \in \mathbb{H}^1(\Omega_i)$ then

$$\|l^{-1/2}(\Phi_i - \Phi_i \circ \psi^{-1})\mathbf{n}_i\|_{\mathcal{I}_h} \lesssim \|\Phi_i\|_{\mathbb{H}^1(\Omega_i)}. \quad (18)$$

Let $e \subset \mathcal{I}$ and $e_h = \psi(e) \subset \mathcal{I}_h$, and let K_e be the element adjacent to e_h . If $p \in P_k(K_e)$, then

$$\|p - p \circ \psi^{-1}\|_{e_h} \lesssim C_e^{\text{ext}} \delta_e h_e^{-3/2} \|p\|_{K_e}. \quad (19)$$

Finally, we recall the discrete trace inequality (cf. [11, Lemma 1.46]): if ϕ is a scalar, vector or tensor-valued polynomial in K_e , then

$$\|\phi\|_e^2 \leq C_e^{\text{tr}} h_e^{-1} \|\phi\|_{K_e}^2, \quad (20)$$

where C_e^{tr} is independent of the meshsize but depends on the polynomial degree k .

We also recall the following results. Some of them are related to the difference in norm between the numerical trace and its approximation using the transfer path method on the discrete and continuous interfaces, whereas others are volumetric estimates obtained in previous works.

Lemma 2. Suppose that Assumptions (A.1)-(A.3) are satisfied. Let $\mathbf{q}_{i,h} := \mathcal{A}(\boldsymbol{\sigma}_{i,h}) + \boldsymbol{\gamma}_{i,h}$. Then, for $i \in \{\text{s}, \text{f}\}$, it holds that

$$\|l^{-1/2}(\mathbf{T}_h(\widehat{\mathbf{u}}_{i,h}) \circ \psi^{-1} - \widehat{\mathbf{u}}_{i,h})\|_{\mathcal{I}_h} \leq C_{\delta_i,h}^{1/2} \|\mathbf{q}_{i,h}\|_{\Omega_h^i}, \quad (21)$$

where

$$C_{\delta_i,h} := 2 \max_{e \in \mathcal{I}_h} \left(\delta_e h_e^{-1} C_e^{\text{tr}} + \frac{1}{3} \delta_e^3 h_e^{-3} \kappa_i^3 (C_e^{\text{ext}} C_e^{\text{inv}})^2 \right).$$

Proof. Details are omitted since this result is a direct adaptation of the estimates established in [3, Lemma 2]. $\square$

Lemma 3. For a given region $D \subset \mathbb{R}^d$, it holds that

$$2\mu_i \|\mathcal{A}(\boldsymbol{\tau})\|_D^2 \leq (\mathcal{A}(\boldsymbol{\tau}), \boldsymbol{\tau})_D \leq \frac{1}{\mu_i} \|\boldsymbol{\tau}\|_D^2 \quad \forall \boldsymbol{\tau} \in \mathbb{L}^2(D).$$

Proof. See [10, Lemma 3.1] for details. $\square$

Lemma 4. There exists a constant $C > 0$, independent of h , such that:

$$\begin{aligned} \|\boldsymbol{\sigma}_{\text{s},h}\|_{\Omega_h^{\text{s}}}^2 &\leq C \left\{ \mu_{\text{s}} (\mathcal{A}(\boldsymbol{\sigma}_{\text{s},h}), \boldsymbol{\sigma}_{\text{s},h})_{\Omega_h^{\text{s}}} + \tau_{\text{s}}^* h \|(\tau_{\text{s}}^*)^{1/2} (\mathbf{u}_{\text{s},h} - \widehat{\mathbf{u}}_{\text{s},h})\|_{\partial\Omega_h^{\text{s}}}^2 + \beta_{\text{s}}^2 \|\mathbf{u}_{\text{s},h}\|_{\Omega_h^{\text{s}}}^2 + \|\mathbf{f}_{\text{s}}\|_{\Omega_h^{\text{s}}}^2 \right. \\ &\quad \left. + \frac{(n\lambda_{\text{s}} + 2\mu_{\text{s}})^2}{|\Omega_h^{\text{s}}|} \left(\max_{e \in \mathcal{I}_h} \{h_e \delta_e^{-2/7}\} \|\delta_e^{1/7} \widehat{\mathbf{u}}_{\text{s},h}\|_{\mathcal{I}_h}^2 + \|\mathbb{H}_{\text{s},1}\|_{\Omega_h^{\text{s}}}^2 \right) \right\}, \end{aligned} \quad (22)$$

$$\|\boldsymbol{\sigma}_{\text{f},h}\|_{\Omega_h^{\text{f}}}^2 \leq C \left\{ \mu_{\text{f}} (\mathcal{A}(\boldsymbol{\sigma}_{\text{f},h}), \boldsymbol{\sigma}_{\text{f},h})_{\Omega_h^{\text{f}}} + \tau_{\text{f}}^* h \|(\tau_{\text{f}}^*)^{1/2} (\mathbf{u}_{\text{f},h} - \widehat{\mathbf{u}}_{\text{f},h})\|_{\partial\Omega_h^{\text{f}}}^2 + \beta_{\text{f}}^2 \|\mathbf{u}_{\text{f},h}\|_{\Omega_h^{\text{f}}}^2 + \|\mathbf{f}_{\text{f}}\|_{\Omega_h^{\text{f}}}^2 \right\}. \quad (23)$$

Proof. We proceed following the arguments in [10, Lemma 3.3] and present the proof for $i = s$, as the case $i = f$ follows analogously. We first decompose the solid stress tensor as $\boldsymbol{\sigma}_{s,h} := \boldsymbol{\sigma}_{s,h}^0 + c_s \mathbb{I}$, where $\text{tr}(\boldsymbol{\sigma}_{s,h}^0) \in L_0^2(\Omega_h^s)$ and $c_s := \frac{1}{n|\Omega_h^s|} \int_{\Omega_h^s} \text{tr}(\boldsymbol{\sigma}_{s,h})$. By orthogonality, we have

$$\|\boldsymbol{\sigma}_{s,h}\|_{\Omega_h^s}^2 = \|\boldsymbol{\sigma}_{s,h}^0\|_{\Omega_h^s}^2 + c_s^2 n |\Omega_h^s| = \|(\boldsymbol{\sigma}_{s,h}^0)^d\|_{\Omega_h^s}^2 + \frac{1}{n} \|\text{tr}(\boldsymbol{\sigma}_{s,h}^0)\|_{\Omega_h^s}^2 + c_s^2 n |\Omega_h^s|. \quad (24)$$

Let $\mathbf{w} \in \mathbf{H}_0^1(\Omega_h^s)$. It is easy to see that $(\boldsymbol{\sigma}_{s,h}, \nabla \mathbf{w})_{\Omega_h^s} = (\boldsymbol{\sigma}_{s,h}^0, \nabla \mathbf{w})_{\Omega_h^s}$ and also that

$$\frac{1}{n} (\text{tr}(\boldsymbol{\sigma}_{s,h}^0), \nabla \cdot \mathbf{w})_{\Omega_h^s} = (\boldsymbol{\sigma}_{s,h}^0, \nabla \mathbf{w})_{\Omega_h^s} - ((\boldsymbol{\sigma}_{s,h}^0)^d, \nabla \mathbf{w})_{\Omega_h^s}.$$

By using the HDG equation (8b), this implies the following identity

$$\begin{aligned} \frac{1}{n} (\text{tr}(\boldsymbol{\sigma}_{s,h}^0), \nabla \cdot \mathbf{w})_{\Omega_h^s} + ((\boldsymbol{\sigma}_{s,h}^0)^d, \nabla \mathbf{w})_{\Omega_h^s} &= (\boldsymbol{\sigma}_{s,h}, \nabla \mathbf{w})_{\Omega_h^s} \\ &= -(\nabla \cdot \boldsymbol{\sigma}_{s,h}, \mathcal{P}_{k-1}(\mathbf{w}))_{\Omega_h^s} + \langle \boldsymbol{\sigma}_{s,h} \mathbf{n}_s, \mathbf{w} \rangle_{\partial \Omega_h^s} \\ &= (\mathbf{f}_s, \mathcal{P}_{k-1}(\mathbf{w}))_{\Omega_h^s} - \beta_s (\mathbf{u}_{s,h}, \mathcal{P}_{k-1}(\mathbf{w}))_{\Omega_h^s} \\ &\quad + \langle \hat{\boldsymbol{\sigma}}_{s,h} \mathbf{n}_s - \boldsymbol{\sigma}_{s,h} \mathbf{n}_s, \mathbf{w} - \mathcal{P}_{k-1}(\mathbf{w}) \rangle_{\partial \Omega_h^s}, \end{aligned}$$

where $\mathcal{P}_{k-1}$ is the L^2 projection over space $\mathbf{P}_{k-1}(\Omega_h^s)$. Using the approximation properties of the L^2 -projection, the fact that

$$\mathbf{w} \in \mathbf{H}_0^1(\Omega_h^s) \quad \text{and} \quad \|(\boldsymbol{\sigma}_{s,h}^0)^d\|_{\Omega_h^s}^2 \leq 2\mu_s (\mathcal{A}(\boldsymbol{\sigma}_{s,h}^0), \boldsymbol{\sigma}_{s,h}^0)_{\Omega_h^s},$$

we obtain

$$\begin{aligned} (\text{tr}(\boldsymbol{\sigma}_{s,h}^0), \nabla \cdot \mathbf{w})_{\Omega_h^s} &\lesssim n \left(h^{1/2} \|\tau_s^*(\mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h})\|_{\partial \Omega_h^s} + (2\mu_s)^{1/2} (\mathcal{A}(\boldsymbol{\sigma}_{s,h}^0), \boldsymbol{\sigma}_{s,h}^0)_{\Omega_h^s}^{1/2} \right. \\ &\quad \left. + \beta_s \|\mathbf{u}_{s,h}\|_{\Omega_h^s} + \|\mathbf{f}_s\|_{\Omega_h^s} \right) \|\mathbf{w}\|_{\mathbf{H}^1(\Omega_h^s)}. \end{aligned}$$

In this way, since $\text{tr}(\boldsymbol{\sigma}_{s,h}^0) \in L_0^2(\Omega_h^s)$, we know that (see, e.g. [15, Corollary 2.4 in Chapter II])

$$\|\text{tr}(\boldsymbol{\sigma}_{s,h}^0)\|_{\Omega_h^s} \lesssim \sup_{\substack{\mathbf{w} \in \mathbf{H}_0^1(\Omega_h^s) \\ \mathbf{w} \neq \mathbf{0}}} \frac{(\nabla \cdot \mathbf{w}, \text{tr}(\boldsymbol{\sigma}_{s,h}^0))}{\|\mathbf{w}\|_{\mathbf{H}^1(\Omega_h^s)}}.$$

Thus,

$$\|\text{tr}(\boldsymbol{\sigma}_{s,h}^0)\|_{\Omega_h^s} \lesssim n h^{1/2} \|\tau_s^*(\mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h})\|_{\partial \Omega_h^s} + n (2\mu_s)^{1/2} (\mathcal{A}(\boldsymbol{\sigma}_{s,h}^0), \boldsymbol{\sigma}_{s,h}^0)_{\Omega_h^s}^{1/2} + n \beta_s \|\mathbf{u}_{s,h}\|_{\Omega_h^s} + n \|\mathbf{f}_s\|_{\Omega_h^s}. \quad (25)$$

For the constant component c_s , we test the first HDG equation (11a) with $\tau_{s,h} = \mathbb{I}$. Because $\gamma_{s,h}$ and $\mathbb{H}_{s,2}$ belong to $\mathbb{A}_h^s$, and thanks to (8e), the equation reduces to:

$$(\mathcal{A}(\boldsymbol{\sigma}_{s,h}), \mathbb{I})_{\Omega_h^s} = \langle \hat{\mathbf{u}}_{s,h}, \mathbb{I} \mathbf{n}_s \rangle_{\partial \Omega_h^s} + (\mathbb{H}_{s,1}, \mathbb{I})_{\Omega_h^s} = \langle \hat{\mathbf{u}}_{s,h}, \mathbb{I} \mathbf{n}_s \rangle_{\mathcal{I}_h} + (\mathbb{H}_{s,1}, \mathbb{I})_{\Omega_h^s}.$$

Since $\text{tr}(\mathcal{A}(\boldsymbol{\sigma}_{s,h})) = \frac{1}{n\lambda_s + 2\mu_s} \text{tr}(\boldsymbol{\sigma}_{s,h})$, by the Cauchy-Schwarz inequality we obtain:

$$c_s^2 n |\Omega_h^s| \lesssim \frac{(n\lambda_s + 2\mu_s)^2}{|\Omega_h^s|} \left(\max_{e \in \mathcal{I}_h} \{h_e \delta_e^{-2/7}\} \|\delta_e^{1/7} \hat{\mathbf{u}}_{s,h}\|_{\mathcal{I}_h}^2 + \|\mathbb{H}_{s,1}\|_{\Omega_h^s}^2 \right). \quad (26)$$

Substituting (25) and (26) back into (24), and noting that $(\mathcal{A}(\boldsymbol{\sigma}_{s,h}^0), \boldsymbol{\sigma}_{s,h}^0)_{\Omega_h^s} \leq (\mathcal{A}(\boldsymbol{\sigma}_{s,h}), \boldsymbol{\sigma}_{s,h})_{\Omega_h^s}$, we obtain (22). $\square$

Let us now decompose the discrete rotation as $\gamma_{i,h} = \gamma_{i,h}^0 + \gamma_{i,h}^c$, where $\gamma_{i,h}^c$ is such that $\gamma_{i,h}^c|_K := \frac{1}{|K|} \int_K \gamma_{i,h}$ for all $K \in \Omega_h^i$, and $\gamma_{i,h}^0 := \gamma_{i,h} - \gamma_{i,h}^c$. By assumption, the source term $\mathbb{E}_i$ is orthogonal to polynomials of degree $k-1$. Thus, for $k \geq 1$, $\mathbb{E}_i$ is orthogonal to piecewise constant polynomials, which directly implies that

$$(\mathbb{E}_i, \gamma_{i,h})_{\Omega_h^i} = (\mathbb{E}_i, \gamma_{i,h}^0)_{\Omega_h^i} + (\mathbb{E}_i, \gamma_{i,h}^c)_{\Omega_h^i} = (\mathbb{E}_i, \gamma_{i,h}^0)_{\Omega_h^i}.$$

Lemma 5. There exists a constant $C > 0$, independent of h , such that

$$\|\gamma_{i,h}^0\|_{\Omega_h^i}^2 \leq C \left(\|\mathcal{A}(\sigma_{i,h})\|_{\Omega_h^i}^2 + \|\mathbb{H}_{i,1}\|_{\Omega_h^i}^2 + \|\mathbb{H}_{i,2}\|_{\Omega_h^i}^2 \right), \quad (27)$$

$$\|\gamma_{i,h}\|_{\Omega_h^i}^2 \leq C \left\{ \|\mathcal{A}(\sigma_{i,h})\|_{\Omega_h^i}^2 + \|\mathbb{H}_{i,1}\|_{\Omega_h^i}^2 + \|\mathbb{H}_{i,2}\|_{\Omega_h^i}^2 + h^{-2} \|\mathbf{u}_{i,h}\|_{\Omega_h^i}^2 + \max_{e \in \mathcal{I}_h} \{h_e^{-1}(\tau_i^*)^{-1}\} \tau_i^* \|\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h}^2 \right\}. \quad (28)$$

Proof. The estimate (27) is obtained from [8, Theorem 3.6] (See also, [14, Lemma 4.2]). Subsequently, the bound (28) follows by bounding the piecewise constant component $\gamma_{i,h}^c$ and invoking the arguments presented in [2, Lemma 5.3]. $\square$

Lemma 6. It holds that

$$\begin{aligned} \mathbb{T} = & \langle \sigma_{s,h}(\mathbf{n}_s - \nu_s), \alpha_s \hat{\mathbf{u}}_{s,h} \rangle_{\mathcal{I}_h} + \langle \sigma_{s,h} \nu_s, \alpha_s (\hat{\mathbf{u}}_{s,h} - \mathbf{T}_h(\hat{\mathbf{u}}_{s,h}) \circ \psi^{-1}) \rangle_{\mathcal{I}_h} + \langle \sigma_{s,h} \nu_s, \mathbf{g} \circ \psi^{-1} \rangle_{\mathcal{I}_h} \\ & - \langle \alpha_f(\hat{\mathbf{u}}_{f,h} - \mathbf{T}_h(\hat{\mathbf{u}}_{f,h}) \circ \psi^{-1}), \sigma_{s,h} \nu_s \rangle_{\mathcal{I}_h} - \langle \alpha_f \hat{\mathbf{u}}_{f,h}, (\mathbf{E}(\sigma_{s,h}) \circ \psi^{-1} - \sigma_{s,h}) \nu_s \rangle_{\mathcal{I}_h} \\ & - \tau_s^* \langle \mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h}, \alpha_s (\hat{\mathbf{u}}_{s,h} - \mathbf{T}_h(\hat{\mathbf{u}}_{s,h}) \circ \psi^{-1}) \rangle_{\mathcal{I}_h} - \tau_s^* \langle \mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h}, \mathbf{g} \circ \psi^{-1} \rangle_{\mathcal{I}_h} \\ & - \tau_s^* \langle \mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h}, \alpha_f(\mathbf{T}_h(\hat{\mathbf{u}}_{f,h}) \circ \psi^{-1} - \hat{\mathbf{u}}_{f,h}) \rangle_{\mathcal{I}_h} \\ & + \langle \sigma_{f,h}(\mathbf{n}_f - \nu_f), \alpha_f \hat{\mathbf{u}}_{f,h} \rangle_{\mathcal{I}_h} + \langle \mathbf{g}_\nu \circ \psi^{-1}, \alpha_f \hat{\mathbf{u}}_{f,h} \rangle_{\mathcal{I}_h} + \langle (\sigma_{f,h} - \mathbf{E}(\sigma_{f,h}) \circ \psi^{-1}) \nu_f, \alpha_f \hat{\mathbf{u}}_{f,h} \rangle_{\mathcal{I}_h} \\ & - \sum_{i \in \{s,f\}} \left(\alpha_i \tau_i^* \langle \delta_e^{1/7} \hat{\mathbf{u}}_{i,h}, \delta_e^{1/7} \hat{\mathbf{u}}_{i,h} \rangle_{\mathcal{I}_h} + \langle \delta_e^{2/7} \tau_i^* (\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}), \alpha_i \hat{\mathbf{u}}_{i,h} \rangle_{\mathcal{I}_h} - \langle \delta_e^{2/7} \tau_i^* \mathbf{u}_{i,h}, \alpha_i \hat{\mathbf{u}}_{i,h} \rangle_{\mathcal{I}_h} \right). \end{aligned} \quad (29)$$

Proof. The proof follows from a direct algebraic manipulation by adding and subtracting suitable terms, combined with the definition of the numerical fluxes (9) and the transmission conditions (12). Furthermore, to facilitate the stability analysis, specifically to obtain the term $\sum_{i \in \{s,f\}} \alpha_i \tau_i^* \|\delta_e^{1/7} \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h}^2$ on the left-hand side of the estimate, we add the following zero-sum term:

$$0 = - \sum_{i \in \{s,f\}} \left(\langle \delta_e^{2/7} \tau_i^* \hat{\mathbf{u}}_{i,h}, \alpha_i \hat{\mathbf{u}}_{i,h} \rangle_{\mathcal{I}_h} + \langle \delta_e^{2/7} \tau_i^* (\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}), \alpha_i \hat{\mathbf{u}}_{i,h} \rangle_{\mathcal{I}_h} - \langle \delta_e^{2/7} \tau_i^* \mathbf{u}_{i,h}, \alpha_i \hat{\mathbf{u}}_{i,h} \rangle_{\mathcal{I}_h} \right),$$

which leads to (29). $\square$

For convenience of notation, let us define the energy norm $\mathcal{Q}(\sigma_h, \mathbf{u}_h, \hat{\mathbf{u}}_h)$ as:

$$\mathcal{Q}(\sigma_h, \mathbf{u}_h, \hat{\mathbf{u}}_h) := \left\{ \sum_{i \in \{s,f\}} \alpha_i \left((\mathcal{A}(\sigma_{i,h}), \sigma_{i,h})_{\Omega_h^i} + \beta_i \|\mathbf{u}_{i,h}\|_{\Omega_h^i}^2 + \tau_i^* \|\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}\|_{\partial \Omega_h^i}^2 + \tau_i^* \|\delta_e^{1/7} \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h}^2 \right) \right\}^{1/2},$$

where $\sigma_h := (\sigma_{s,h}, \sigma_{f,h})$, $\mathbf{u}_h := (\mathbf{u}_{s,h}, \mathbf{u}_{f,h})$, and $\hat{\mathbf{u}}_h := (\hat{\mathbf{u}}_{s,h}, \hat{\mathbf{u}}_{f,h})$. Here, δ_e represents a measure of the distance between the face $e \in \mathcal{I}_h$ and the curved interface $\mathcal{I}$.

The next lemma establishes an upper bound for the energy term $\mathcal{Q}(\sigma_h, \mathbf{u}_h, \hat{\mathbf{u}}_h)$. As we will see, the right-hand side of this estimate depends on the source terms as well as the norms of the discrete variables $\mathbf{u}_h$ and σ_h , which will be subsequently bounded to close the stability argument. More precisely, the result is stated as follows.

Lemma 7. Suppose Assumptions (A.1)–(A.5) hold and let $\gamma \in \mathbb{R}$. Then, we have

$$\begin{aligned} \frac{1}{2} \mathcal{Q}(\sigma_h, \mathbf{u}_h, \hat{\mathbf{u}}_h)^2 \leq & \sum_{i \in \{s,f\}} \alpha_i \left(C(\mathbf{u}) + C C(\sigma) \beta_i^2 + \frac{1}{2} \right) \|\mathbf{u}_{i,h}\|_{\Omega_h^i}^2 \\ & + \sum_{i \in \{s,f\}} \alpha_i \left(C_{\mathbb{H}_{i,1}} \|\mathbb{H}_{i,1}\|_{\Omega_h^i}^2 + C_{\mathbb{H}_{i,2}} \|\mathbb{H}_{i,2}\|_{\Omega_h^i}^2 + \frac{2C}{\mu_i} \|\mathbb{E}_i\|_{\Omega_h^i}^2 + \left(\frac{1}{2} + C C(\sigma) \right) \|\mathbf{f}_i\|_{\Omega_h^i}^2 \right) \\ & + C(\mathbf{g}) \|\mathbf{g} \circ \psi^{-1}\|_{\mathcal{I}_h}^2 + C(\mathbf{g}_\nu) \alpha_f \|\mathbf{g}_\nu \circ \psi^{-1}\|_{\mathcal{I}_h}^2, \end{aligned} \quad (30)$$

where

$$\begin{aligned} C_{\mathbb{H}_{s,1}} &:= \left(\frac{2}{\mu_s} + 8\mu_s \right) (n\lambda_s + 2\mu_s)^2 + \max_{e \in \mathcal{I}_h} \{\delta_e^{2/7}\} C_{\delta_{s,h}} + 2\mu_s^{-1} + C C(\sigma) \frac{(n\lambda_s + 2\mu_s)^2}{|\Omega_h^s|}, \\ C_{\mathbb{H}_{f,1}} &:= \left(\frac{2}{\mu_f} + 8\mu_f \right) 4\mu_f^2 + \max_{e \in \mathcal{I}_h} \{\delta_e^{2/7}\} C_{\delta_{f,h}} + 2\mu_f^{-1}, \\ C_{\mathbb{H}_{i,2}} &:= 8\mu_i + \max_{e \in \mathcal{I}_h} \{\delta_e^{2/7}\} C_{\delta_{i,h}} + 4\mu_i^{-1}, \end{aligned} \quad (31)$$

and

$$\begin{aligned} C(\mathbf{g}) &:= \alpha_s^{-1} \max_{e \in \mathcal{I}_h} \left\{ \left(\max \left\{ 6, 8C\mu_s C_e^{\text{tr}} \right\} + 6\tau_s^* \right) h_e^{-1} \right\}, \quad C(\mathbf{g}_\nu) := 6 \max_{e \in \mathcal{I}_h} \left\{ \left((\tau_f^*)^{-1} + C_e^{\text{tr}} \right) h_e^{-2\gamma} \right\}, \\ C(\mathbf{u}) &:= \max_{e \in \mathcal{I}_h} \left\{ \delta_e^{2/7} \left(C_{\delta_{i,h}} h^{-2} + 4\tau_i^* C_e^{\text{tr}} h_e^{-1} \right) + \frac{1}{24} h_e^{2\gamma-1} \right\}. \end{aligned} \quad (32)$$

Proof. Taking $\boldsymbol{\tau}_{i,h} = \alpha_i \boldsymbol{\sigma}_{i,h}$, $\mathbf{v}_{i,h} = \alpha_i \mathbf{u}_{i,h}$, $\boldsymbol{\eta}_{i,h} = \alpha_i \boldsymbol{\gamma}_{i,h}$, and $\boldsymbol{\mu}_{i,h} = \begin{cases} \alpha_i \hat{\mathbf{u}}_{i,h} & \text{on } \partial\Omega_h^i \setminus (\Gamma_{i,h} \cup \mathcal{I}_h) \\ \alpha_i \hat{\boldsymbol{\sigma}}_{i,h} \mathbf{n}_i & \text{on } \Gamma_{i,h} \end{cases}$ in equations (11a), (8b), (11b), (8d), (8e), we obtain

$$\begin{aligned} & \sum_{i \in \{s,f\}} \alpha_i \left((\mathcal{A}(\boldsymbol{\sigma}_{i,h}), \boldsymbol{\sigma}_{i,h})_{\Omega_h^i} + \beta_i \|\mathbf{u}_{i,h}\|_{\Omega_h^i}^2 + \tau_i^* \|\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}\|_{\partial\Omega_h^i}^2 \right) \\ &= \mathbb{T} + \sum_{i \in \{s,f\}} \alpha_i \left((\mathbb{H}_{i,1}, \boldsymbol{\sigma}_{i,h})_{\Omega_h^i} + (\mathbf{f}_i, \mathbf{u}_{i,h})_{\Omega_h^i} + (\mathbb{H}_{i,2}, \boldsymbol{\sigma}_{i,h})_{\Omega_h^i} + (\mathbb{E}_i, \boldsymbol{\gamma}_{i,h})_{\Omega_h^i} \right). \end{aligned} \quad (33)$$

Combining this identity with the expression for $\mathbb{T}$ in Lemma 6, we can rewrite (33) as

$$\mathcal{Q}(\boldsymbol{\sigma}_h, \mathbf{u}_h, \hat{\mathbf{u}}_h)^2 = \sum_{j=1}^8 I_j + \sum_{i \in \{s,f\}} \alpha_i \left((\mathbb{H}_{i,1} + \mathbb{H}_{i,2}, \boldsymbol{\sigma}_{i,h})_{\Omega_h^i} + (\mathbf{f}_i, \mathbf{u}_{i,h})_{\Omega_h^i} + (\mathbb{E}_i, \boldsymbol{\gamma}_{i,h}^0)_{\Omega_h^i} \right), \quad (34)$$

where the terms I_j correspond exactly to the grouping of terms in (29):

$$\begin{aligned} I_1 &= \langle \boldsymbol{\sigma}_{s,h}(\mathbf{n}_s - \boldsymbol{\nu}_s), \alpha_s \hat{\mathbf{u}}_{s,h} \rangle_{\mathcal{I}_h} + \langle \boldsymbol{\sigma}_{f,h}(\mathbf{n}_f - \boldsymbol{\nu}_f), \alpha_f \hat{\mathbf{u}}_{f,h} \rangle_{\mathcal{I}_h}, \\ I_2 &= -\langle \alpha_f \hat{\mathbf{u}}_{f,h}, (\mathbf{E}(\boldsymbol{\sigma}_{s,h}) \circ \boldsymbol{\psi}^{-1} - \boldsymbol{\sigma}_{s,h}) \boldsymbol{\nu}_s \rangle_{\mathcal{I}_h} + \langle \boldsymbol{\sigma}_{f,h} - \mathbf{E}(\boldsymbol{\sigma}_{f,h}) \circ \boldsymbol{\psi}^{-1} \boldsymbol{\nu}_f, \alpha_f \hat{\mathbf{u}}_{f,h} \rangle_{\mathcal{I}_h}, \\ I_3 &= \langle \boldsymbol{\sigma}_{s,h} \boldsymbol{\nu}_s, \alpha_s (\hat{\mathbf{u}}_{s,h} - \mathbf{T}_h(\hat{\mathbf{u}}_{s,h}) \circ \boldsymbol{\psi}^{-1}) \rangle_{\mathcal{I}_h} - \langle \alpha_f (\hat{\mathbf{u}}_{f,h} - \mathbf{T}_h(\hat{\mathbf{u}}_{f,h}) \circ \boldsymbol{\psi}^{-1}), \boldsymbol{\sigma}_{s,h} \boldsymbol{\nu}_s \rangle_{\mathcal{I}_h}, \\ I_4 &= -\tau_s^* \langle \mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h}, \alpha_s (\hat{\mathbf{u}}_{s,h} - \mathbf{T}_h(\hat{\mathbf{u}}_{s,h}) \circ \boldsymbol{\psi}^{-1}) \rangle_{\mathcal{I}_h} - \tau_s^* \langle \mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h}, \alpha_f (\mathbf{T}_h(\hat{\mathbf{u}}_{f,h}) \circ \boldsymbol{\psi}^{-1} - \hat{\mathbf{u}}_{f,h}) \rangle_{\mathcal{I}_h}, \\ I_5 &= \langle \boldsymbol{\sigma}_{s,h} \boldsymbol{\nu}_s, \mathbf{g} \circ \boldsymbol{\psi}^{-1} \rangle_{\mathcal{I}_h} + \langle \mathbf{g}_\nu \circ \boldsymbol{\psi}^{-1}, \alpha_f (\hat{\mathbf{u}}_{f,h} - \mathbf{u}_{f,h}) \rangle_{\mathcal{I}_h} + \langle \mathbf{g}_\nu \circ \boldsymbol{\psi}^{-1}, \alpha_f \mathbf{u}_{f,h} \rangle_{\mathcal{I}_h}, \\ I_6 &= -\tau_s^* \langle \mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h}, \mathbf{g} \circ \boldsymbol{\psi}^{-1} \rangle_{\mathcal{I}_h}, \\ I_7 &= -\sum_{i \in \{s,f\}} \langle \delta_e^{2/7} \tau_i^* (\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}), \alpha_i \hat{\mathbf{u}}_{i,h} \rangle_{\mathcal{I}_h}, \\ I_8 &= \sum_{i \in \{s,f\}} \langle \delta_e^{2/7} \tau_i^* \mathbf{u}_{i,h}, \alpha_i \hat{\mathbf{u}}_{i,h} \rangle_{\mathcal{I}_h}. \end{aligned}$$

Now, according to the properties of τ_i^* , and applying the Cauchy–Schwarz and Young’s inequalities, we bound each of these terms as follows.

First, for I_1 , we apply the Cauchy–Schwarz inequality and Assumption (A.4). Multiplying and dividing by $\delta_e^{1/7} (\tau_i^*)^{1/2}$, using Young’s inequality with parameter $\varepsilon = \frac{1}{8}$, and the discrete trace inequality $\|\boldsymbol{\sigma}_{i,h}\|_e^2 \leq C_e^{\text{tr}} h_e^{-1} \|\boldsymbol{\sigma}_{i,h}\|_{K_e}^2$ on each boundary element K_e , we obtain:

$$I_1 \leq \sum_{i \in \{s,f\}} \left(4C_{\text{geo}}^2 C_e^{\text{tr}} \max_{e \in \mathcal{I}_h} \left\{ \delta_e^{2\beta-2/7} (\tau_i^*)^{-1} h_e^{-1} \right\} \alpha_i \|\boldsymbol{\sigma}_{i,h}\|_{\Omega_h^i}^2 \right) + \frac{1}{16} \sum_{i \in \{s,f\}} \alpha_i \tau_i^* \|\delta_e^{1/7} \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h}^2. \quad (35)$$

Next, we bound the term I_2 . We apply again the Cauchy–Schwarz inequality, the polynomial estimate from Lemma 1, i.e., $\|(\mathbf{E}(\boldsymbol{\sigma}_{i,h}) \circ \boldsymbol{\psi}^{-1} - \boldsymbol{\sigma}_{i,h}) \boldsymbol{\nu}_i\|_e \leq C_e^{\text{ext}} \delta_e h_e^{-3/2} \|\boldsymbol{\sigma}_{i,h}\|_{K_e}$, and Young’s inequality with parameter $\varepsilon = \frac{1}{16}$ to get:

$$I_2 \leq \sum_{i \in \{s,f\}} \left(8 \max_{e \in \mathcal{I}_h} \left\{ (C_e^{\text{ext}})^2 \delta_e^{12/7} (\tau_i^*)^{-1} h_e^{-3} \right\} \alpha_f \|\boldsymbol{\sigma}_{i,h}\|_{\Omega_h^i}^2 \right) + \frac{1}{16} \alpha_f \tau_f^* \|\delta_e^{1/7} \hat{\mathbf{u}}_{f,h}\|_{\mathcal{I}_h}^2. \quad (36)$$

For the term I_3 , we apply the Cauchy–Schwarz inequality, the discrete trace inequalities, Lemma 2, and Young’s inequality with parameter $\varepsilon = 1$ to find that:

$$I_3 \leq \frac{1}{2} \max_{e \in \mathcal{I}_h} \{C_e^{\text{tr}} \delta_e^{5/7} h_e^{-1}\} \alpha_s \|\boldsymbol{\sigma}_{s,h}\|_{\Omega_h^s}^2 + \frac{1}{2} \max_{e \in \mathcal{I}_h} \{\delta_e^{2/7}\} \sum_{i \in \{s,f\}} \alpha_i C_{\delta_i,h} \|\mathbf{q}_{i,h}\|_{\Omega_h^i}^2, \quad (37)$$

and similarly, using Young’s inequality with parameter $\varepsilon = \frac{1}{8}$, we bound I_4 as:

$$I_4 \leq \frac{1}{8} \tau_s^* \|\mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h}\|_{\mathcal{I}_h}^2 + 2 \max_{e \in \mathcal{I}_h} \{\delta_e \tau_s^*\} \sum_{i \in \{s,f\}} \alpha_i^2 C_{\delta_i,h} \|\mathbf{q}_{i,h}\|_{\Omega_h^i}^2. \quad (38)$$

For the terms I_5 and I_6 , we apply the Cauchy–Schwarz inequality, Young’s inequality, and the discrete trace inequality (20). We also introduce a parameter $\gamma \in \mathbb{R}$ to distribute the weight of the mesh size h_e between the source term and the velocity. For the first inner product in I_5 , we apply Young’s inequality suitable to control the viscosity dependence on the solid domain, while for the remaining terms in I_5 and for I_6 , we choose the parameter to be $\frac{1}{24}$. This yields:

$$\begin{aligned} I_5 &\leq \min \left\{ \frac{1}{24}, \frac{1}{32C\mu_s \max_{e \in \mathcal{I}_h} \{C_e^{\text{tr}}\}} \right\} \max_{e \in \mathcal{I}_h} \{C_e^{\text{tr}}\} \alpha_s \|\boldsymbol{\sigma}_{s,h}\|_{\Omega_h^s}^2 \\ &\quad + \max \left\{ 6, 8C\mu_s \max_{e \in \mathcal{I}_h} \{C_e^{\text{tr}}\} \right\} \max_{e \in \mathcal{I}_h} \{h_e^{-1}\} \alpha_s^{-1} \|\mathbf{g} \circ \boldsymbol{\psi}^{-1}\|_{\mathcal{I}_h}^2 \\ &\quad + \frac{1}{24} \max_{e \in \mathcal{I}_h} \{h_e^{2\gamma-1}\} \alpha_f \|\mathbf{u}_{f,h}\|_{\Omega_h^f}^2 + 6 \max_{e \in \mathcal{I}_h} \left\{ (\tau_f^*)^{-1} + C_e^{\text{tr}} h_e^{-2\gamma} \right\} \alpha_f \|\mathbf{g}_\nu \circ \boldsymbol{\psi}^{-1}\|_{\mathcal{I}_h}^2 + \frac{1}{24} \alpha_f \tau_f^* \|\mathbf{u}_{f,h} - \hat{\mathbf{u}}_{f,h}\|_{\mathcal{I}_h}^2, \\ I_6 &\leq \frac{1}{24} \alpha_s \tau_s^* \|\mathbf{u}_{s,h} - \hat{\mathbf{u}}_{s,h}\|_{\mathcal{I}_h}^2 + 6 \alpha_s^{-1} \tau_s^* \|\mathbf{g} \circ \boldsymbol{\psi}^{-1}\|_{\mathcal{I}_h}^2. \end{aligned} \quad (39)$$

Finally, for I_7 and I_8 , we distribute $\delta_e^{2/7}$ properly, apply Young’s inequality with parameter $\varepsilon = \frac{1}{16}$, and the discrete trace inequality to find that:

$$\begin{aligned} I_7 &\leq \frac{1}{16} \sum_{i \in \{s,f\}} \alpha_i \tau_i^* \|\delta_e^{1/7} \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h}^2 + 4 \max_{e \in \mathcal{I}_h} \{\delta_e^{2/7}\} \sum_{i \in \{s,f\}} \alpha_i \tau_i^* \|\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h}^2, \\ I_8 &\leq \frac{1}{16} \sum_{i \in \{s,f\}} \alpha_i \tau_i^* \|\delta_e^{1/7} \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h}^2 + 4 \max_{e \in \mathcal{I}_h} \{\delta_e^{2/7} C_e^{\text{tr}} h_e^{-1}\} \sum_{i \in \{s,f\}} \alpha_i \tau_i^* \|\mathbf{u}_{i,h}\|_{\Omega_h^i}^2. \end{aligned} \quad (40)$$

Summing the bounds from I_1 to I_8 , bounding $\|\mathbf{q}_{i,h}\|_{\Omega_h^i}^2$ through the triangle inequality, applying Lemma 5, and invoking Lemma 3 to absorb the compliance norm, we deduce that

$$\begin{aligned} \sum_{j=1}^8 I_j &\leq \sum_{i \in \{s,f\}} \alpha_i \left(C(\boldsymbol{\sigma}) \|\boldsymbol{\sigma}_{i,h}\|_{\Omega_h^i}^2 + C(\mathbf{u}) \|\mathbf{u}_{i,h}\|_{\Omega_h^i}^2 + \max_{e \in \mathcal{I}_h} \{\delta_e^{2/7}\} C_{\delta_i,h} \left(\|\mathbb{H}_{i,1}\|_{\Omega_h^i}^2 + \|\mathbb{H}_{i,2}\|_{\Omega_h^i}^2 \right) \right. \\ &\quad \left. + C(\partial) \tau_i^* \|\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h}^2 + \frac{1}{4} \tau_i^* \|\delta_e^{1/7} \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h}^2 \right) + C(\mathbf{g}) \|\mathbf{g} \circ \boldsymbol{\psi}^{-1}\|_{\mathcal{I}_h}^2 + C(\mathbf{g}_\nu) \alpha_f \|\mathbf{g}_\nu \circ \boldsymbol{\psi}^{-1}\|_{\mathcal{I}_h}^2, \end{aligned} \quad (41)$$

where the constants $C(\boldsymbol{\sigma})$ and $C(\partial)$ have been defined in Assumption (A.5), whereas $C(\mathbf{u})$, $C(\mathbf{g})$ and $C(\mathbf{g}_\nu)$ have been defined in (32). We now bound the volumetric source terms on the right-hand side of (34). First, we exploit the symmetry and positive definiteness of $\mathcal{A}$, apply the Cauchy–Schwarz inequality, Young’s inequality with parameter $\varepsilon = \frac{1}{16} \left(\frac{1}{4\mu_i} + \mu_i \right)^{-1}$, and the equivalence from Lemma 3, to obtain:

$$\begin{aligned} \sum_{i \in \{s,f\}} \alpha_i (\mathbb{H}_{i,1} + \mathbb{H}_{i,2}, \boldsymbol{\sigma}_{i,h})_{\Omega_h^i} &= \sum_{i \in \{s,f\}} \alpha_i \left((\mathcal{A}^{-1} \mathbb{H}_{i,1}, \mathcal{A} \boldsymbol{\sigma}_{i,h})_{\Omega_h^i} + (\mathbb{H}_{i,2}, 2\mu_i \mathcal{A} \boldsymbol{\sigma}_{i,h})_{\Omega_h^i} \right) \\ &\leq \sum_{i \in \{s,f\}} \alpha_i \left(\left(\frac{2}{\mu_i} + 8\mu_i \right) \left(\|\mathcal{A}^{-1} \mathbb{H}_{i,1}\|_{\Omega_h^i}^2 + \|\mathbb{H}_{i,2}\|_{\Omega_h^i}^2 \right) + \frac{1}{16} (\mathcal{A}(\boldsymbol{\sigma}_{i,h}), \boldsymbol{\sigma}_{i,h})_{\Omega_h^i} \right). \end{aligned} \quad (42)$$

Noting that by the definition of the compliance tensor $\mathcal{A}$, we have $\|\mathcal{A}^{-1} \mathbb{H}_{s,1}\|_{\Omega_h^s}^2 \leq (n\lambda_s + 2\mu_s)^2 \|\mathbb{H}_{s,1}\|_{\Omega_h^s}^2$ and $\|\mathcal{A}^{-1} \mathbb{H}_{f,1}\|_{\Omega_h^f}^2 \leq 4\mu_f^2 \|\mathbb{H}_{f,1}\|_{\Omega_h^f}^2$, these terms can be explicitly bounded.

Next, we apply the bound given in Lemma 5, and Young's inequality with parameter $\varepsilon = \frac{\mu_i}{4C}$, to get:

$$\sum_{i \in \{s, f\}} \alpha_i (\mathbb{E}_i, \gamma_{i,h}^0)_{\Omega_h^i} \leq \sum_{i \in \{s, f\}} \alpha_i \left(\frac{2C}{\mu_i} \|\mathbb{E}_i\|_{\Omega_h^i}^2 + \frac{1}{16} (\mathcal{A}(\sigma_{i,h}), \sigma_{i,h})_{\Omega_h^i} + 2\mu_i^{-1} \left(\|\mathbb{H}_{i,1}\|_{\Omega_h^i}^2 + \|\mathbb{H}_{i,2}\|_{\Omega_h^i}^2 \right) \right). \quad (43)$$

Also, by the Cauchy–Schwarz and Young's inequalities with parameter $\varepsilon = 1$, we bound the source term f_i as:

$$\sum_{i \in \{s, f\}} \alpha_i (f_i, u_{i,h})_{\Omega_h^i} \leq \sum_{i \in \{s, f\}} \alpha_i \left(\frac{1}{2} \|f_i\|_{\Omega_h^i}^2 + \frac{1}{2} \|u_{i,h}\|_{\Omega_h^i}^2 \right). \quad (44)$$

We now insert the above bounds into the identity (34) and invoke Lemma 4 to expand the term $C(\sigma) \|\sigma_{i,h}\|^2$, and collect the coefficients corresponding to $C_{\mathbb{H}_{i,1}}$ and $C_{\mathbb{H}_{i,2}}$ defined in (31).

Finally, by grouping the remaining terms into the constants $\mathcal{M}_i(\sigma)$, $\mathcal{M}_i(\partial)$, and $\mathcal{M}_i(\hat{u})$ defined in Assumption (A.5), and applying the condition $\mathcal{M}(\delta, h, \text{data}) \leq \frac{1}{4}$, the total coefficients multiplying the energy components on the right-hand side are bounded by $\frac{1}{2}$. Subtracting $\frac{1}{2} \mathcal{Q}(\sigma_h, u_h, \hat{u}_h)^2$ from both sides directly yields the desired estimate (30), which finishes the proof. $\square$

It is worth noting that the L^2 -norm of the velocity, $\|u_{i,h}\|_{\Omega_h^i}^2$, has been left on the right-hand side of (30) despite being part of $\mathcal{Q}^2$. Indeed, absorbing it directly would require overly restrictive conditions, such as $\gamma \geq 1/2$ and $\delta = \mathcal{O}(h^{7/2})$ (to ensure that the term $\delta_e^{2/7} h_e^{-1}$ in $C(u)$ remains bounded), which severely limits the geometric flexibility of the discretization. Instead, we control this term via a duality argument to gain optimal scaling with respect to h .

4.2 A duality argument

We now provide a bound for $\sum_{i \in \{s, f\}} \|u_{i,h}\|_{\Omega_h^i}^2$. To that end, we will employ a duality argument. More precisely, in what follows we consider the following continuous auxiliary problem for a given $\Theta \in \mathbf{L}^2(\Omega_s \cup \Omega_f)$:

$$\mathcal{A}(\Phi_i) - (\nabla \phi_i - \rho_i) = \mathbf{0} \quad \text{in } \Omega_i, \quad (45a)$$

$$-\nabla \cdot \Phi_i + \beta_i \phi_i = \Theta \quad \text{in } \Omega_i, \quad (45b)$$

$$\phi_i = \mathbf{0} \quad \text{on } \Gamma_i, \quad (45c)$$

$$\alpha_s \phi_s - \alpha_f \phi_f = \mathbf{0} \quad \text{on } \mathcal{I}, \quad (45d)$$

$$\Phi_s \nu_s + \Phi_f \nu_f = \mathbf{0} \quad \text{on } \mathcal{I}. \quad (45e)$$

Here, $\rho_i := \frac{1}{2}(\nabla \phi_i - (\nabla \phi_i)^\top)$ for $i \in \{s, f\}$.

Let us suppose that $\mathcal{I}$ is sufficiently smooth, such that

$$\|\Phi_i\|_{\mathbb{H}^1(\Omega_i)} + \|\phi_i\|_{\mathbf{H}^2(\Omega_i)} \leq C_{\text{reg}} \|\Theta\|_{\Omega_s \cup \Omega_f}. \quad (46)$$

Remark. We note that this type of regularity result holds, for example, for elliptic interface problems if the domain Ω is convex and the exact interface $\mathcal{I}$ is of class C^2 (cf. [16, Theorem 4.6]).

In addition, we assume that there exists a sufficiently large and smooth extended domain $\tilde{\Omega}_i$ satisfying $\Omega_i \cup \Omega_h^i \subset \tilde{\Omega}_i$, such that (45a) and (45b) also hold in $\tilde{\Omega}_i$ and the solution in the extended domain satisfies

$$\sum_{i \in \{s, f\}} \left(\|\Phi_i\|_{\mathbb{H}^1(\tilde{\Omega}_i)} + \|\phi_i\|_{\mathbf{H}^2(\tilde{\Omega}_i)} \right) \leq C_{\text{reg}} \|\Theta\|_{\Omega_s \cup \Omega_f}. \quad (47)$$

Here, to pose the dual problem over the whole domain $\tilde{\Omega}_i$, the source term $\Theta \in \mathbf{L}^2(\Omega_h^i)$ is extended by zero onto $\tilde{\Omega}_i \setminus \Omega_h^i$. The constant $C_{\text{reg}} > 0$ is independent of the Lamé parameters and the meshsize.

Lemma 8. Suppose that Assumptions (A.1)–(A.3) hold. Then, the following identities are satisfied:

$$\begin{aligned}
\sum_{i \in \{s, f\}} \alpha_i \langle \hat{\sigma}_{i, h} \nu_i, \mathcal{P}_M(\phi_i) \rangle_{\mathcal{I}_h} &= \sum_{i \in \{s, f\}} \alpha_i \left[\langle \sigma_{i, h} \nu_i - \mathbf{E}(\sigma_{i, h}) \circ \psi^{-1}, \mathcal{P}_M(\phi_i \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \right. \\
&\quad \left. + \langle \hat{\sigma}_{i, h} \nu_i, \mathcal{P}_M(\phi_i - \phi_i \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \right] + \langle \mathbf{g}_\nu \circ \psi^{-1}, \mathcal{P}_M(\alpha_s \phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h}, \\
\sum_{i \in \{s, f\}} \alpha_i \langle \hat{\mathbf{u}}_{i, h}, \mathcal{P}_M(\Phi_i \nu_i) \rangle_{\mathcal{I}_h} &= \langle \mathbf{g} \circ \psi^{-1}, \mathcal{P}_M((\Phi_s \nu_s) \circ \psi^{-1}) \rangle_{\mathcal{I}_h} + \sum_{i \in \{s, f\}} \alpha_i \left[\langle \hat{\mathbf{u}}_{i, h}, \mathcal{P}_M(\Phi_i \nu_i - (\Phi_i \nu_i) \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \right. \\
&\quad \left. - \langle \mathbf{T}_h(\hat{\mathbf{u}}_{i, h}) \circ \psi^{-1} - \hat{\mathbf{u}}_{i, h}, \mathcal{P}_M((\Phi_i \nu_i) \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \right].
\end{aligned}$$

Proof. We restrict the proof to the first identity. The second identity follows analogously; therefore, we omit the details. For $\langle \hat{\sigma}_{s, h} \nu_s, \mathcal{P}_M(\phi_s) \rangle_{\mathcal{I}_h}$, we add and subtract $\phi_s \circ \psi^{-1}$ to obtain:

$$\alpha_s \langle \hat{\sigma}_{s, h} \nu_s, \mathcal{P}_M(\phi_s) \rangle_{\mathcal{I}_h} = \langle \hat{\sigma}_{s, h} \nu_s, \alpha_s \mathcal{P}_M(\phi_s - \phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h} + \langle \hat{\sigma}_{s, h} \nu_s, \alpha_s \mathcal{P}_M(\phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h}.$$

By adding and subtracting $\hat{\mathbf{E}}(\hat{\sigma}_{s, h}) \circ \psi^{-1}$ in the second term and using (10b), we have:

$$\begin{aligned}
\alpha_s \langle \hat{\sigma}_{s, h} \nu_s, \mathcal{P}_M(\phi_s) \rangle_{\mathcal{I}_h} &= \langle \hat{\sigma}_{s, h} \nu_s, \alpha_s \mathcal{P}_M(\phi_s - \phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h} + \langle \hat{\sigma}_{s, h} \nu_s - \hat{\mathbf{E}}(\hat{\sigma}_{s, h}) \circ \psi^{-1}, \alpha_s \mathcal{P}_M(\phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \\
&\quad + \langle \hat{\mathbf{E}}(\hat{\sigma}_{s, h}) \circ \psi^{-1}, \alpha_s \mathcal{P}_M(\phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \\
&= \langle \hat{\sigma}_{s, h} \nu_s, \alpha_s \mathcal{P}_M(\phi_s - \phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h} + \langle \hat{\sigma}_{s, h} \nu_s - \hat{\mathbf{E}}(\hat{\sigma}_{s, h}) \circ \psi^{-1}, \alpha_s \mathcal{P}_M(\phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \\
&\quad - \langle \hat{\mathbf{E}}(\hat{\sigma}_{f, h}) \circ \psi^{-1}, \mathcal{P}_M(\alpha_s \phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h} + \langle \mathbf{g}_\nu \circ \psi^{-1}, \mathcal{P}_M(\alpha_s \phi_s \circ \psi^{-1}) \rangle_{\mathcal{I}_h}.
\end{aligned}$$

Similarly, for $\langle \hat{\sigma}_{f, h} \nu_f, \mathcal{P}_M(\phi_f) \rangle_{\mathcal{I}_h}$, we obtain:

$$\begin{aligned}
\alpha_f \langle \hat{\sigma}_{f, h} \nu_f, \mathcal{P}_M(\phi_f) \rangle_{\mathcal{I}_h} &= \langle \hat{\sigma}_{f, h} \nu_f, \alpha_f \mathcal{P}_M(\phi_f - \phi_f \circ \psi^{-1}) \rangle_{\mathcal{I}_h} + \langle \hat{\sigma}_{f, h} \nu_f - \hat{\mathbf{E}}(\hat{\sigma}_{f, h}) \circ \psi^{-1}, \alpha_f \mathcal{P}_M(\phi_f \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \\
&\quad + \langle \hat{\mathbf{E}}(\hat{\sigma}_{f, h}) \circ \psi^{-1}, \mathcal{P}_M(\alpha_f \phi_f \circ \psi^{-1}) \rangle_{\mathcal{I}_h}.
\end{aligned}$$

Finally, summing these quantities and applying the continuous dual transmission condition (45d), which implies that $\alpha_s \phi_s \circ \psi^{-1} = \alpha_f \phi_f \circ \psi^{-1}$ on $\mathcal{I}_h$, yields the desired identity. $\square$

Lemma 9. Suppose that Assumptions (A.1)–(A.4) and (A.6) hold, and that the elliptic regularity of the dual problem holds. Then, the following estimate is satisfied:

$$\begin{aligned}
\sum_{i \in \{s, f\}} \alpha_i \|\mathbf{u}_{i, h}\|_{\Omega_h^i}^2 &\lesssim \mathcal{N}(\delta, h, \text{data}) \mathcal{Q}(\sigma_h, \mathbf{u}_h, \hat{\mathbf{u}}_h)^2 + h^2 \sum_{i \in \{s, f\}} \alpha_i \|\mathbb{E}_i\|_{\Omega_h^i}^2 \\
&\quad + \sum_{i \in \{s, f\}} \alpha_i \left(h^2 + C_I^2(\delta, h) \right) \left(\|\mathbb{H}_{i, 1}\|_{\Omega_h^i}^2 + \|\mathbb{H}_{i, 2}\|_{\Omega_h^i}^2 + \|\mathbf{f}_i\|_{\Omega_h^i}^2 \right) \\
&\quad + C_I^2(\delta, h) \alpha_s \frac{(n\lambda_s + 2\mu_s)^2}{|\Omega_h^s|} \|\mathbb{H}_{s, 1}\|_{\Omega_h^s}^2 + C_I^2(\delta, h) \left(\|\mathbf{g} \circ \psi^{-1}\|_{\mathcal{I}_h}^2 + \alpha_f^2 \|\mathbf{g}_\nu \circ \psi^{-1}\|_{\mathcal{I}_h}^2 \right).
\end{aligned} \tag{48}$$

Proof. We proceed as in [14, Lemma 4.6]. From the dual equations (45a) and (45b) tested against $\mathbf{u}_{i, h}$, we have:

$$(\mathbf{u}_{i, h}, \boldsymbol{\Theta})_{\Omega_h^i} = -(\mathbf{u}_{i, h}, \nabla \cdot \Phi_i)_{\Omega_h^i} + \beta_i(\mathbf{u}_{i, h}, \phi_i)_{\Omega_h^i} - (\sigma_{i, h}, \mathcal{A}(\Phi_i) - \nabla \phi_i + \rho_i)_{\Omega_h^i}.$$

Integrating by parts the terms involving $\nabla \cdot \Phi_i$ and $\nabla \phi_i$, adding and subtracting the HDG projections $\Pi \Phi_i$ and $\Pi \phi_i$, as well as the local L^2 -projection $\Pi_{\mathbb{A}} \rho_i$, and performing algebraic manipulations together with the equations of the HDG scheme (8), (11a)–(11b) and dual problem, we deduce that

$$\sum_{i \in \{s, f\}} \alpha_i (\mathbf{u}_{i, h}, \boldsymbol{\Theta})_{\Omega_h^i} = \mathbb{S}_V + \mathbb{S}_I,$$

where

$$\begin{aligned}
\mathbb{S}_V &= \sum_{i \in \{s, f\}} \alpha_i \left[(\mathcal{A}(\sigma_{i, h}), \Pi \Phi_i - \Phi_i)_{\Omega_h^i} + (\mathcal{A}(\sigma_{i, h}), \Pi_{\mathbb{A}} \rho_i - \rho_i)_{\Omega_h^i} + (\gamma_{i, h}, \Pi \Phi_i - \Phi_i)_{\Omega_h^i} + (\mathbb{H}_{i, 2}, \Phi_i - \Pi \Phi_i)_{\Omega_h^i} \right. \\
&\quad \left. + (\mathbb{E}_i, \rho_i - \Pi_{\mathbb{A}} \rho_i)_{\Omega_h^i} + (\mathbb{E}_i, \mathcal{P}_{k-1}(\nabla \phi_i) - \nabla \phi_i)_{\Omega_h^i} + \beta_i(\mathbf{u}_{i, h}, \phi_i - \Pi \phi_i)_{\Omega_h^i} + (\mathbf{f}_i, \Pi \phi_i)_{\Omega_h^i} \right]
\end{aligned}$$

and

$$\mathbb{S}_I = \sum_{i \in \{s, f\}} \alpha_i [\langle \sigma_{i,h}(\mathbf{n}_i - \boldsymbol{\nu}_i), \phi_i \rangle_{\mathcal{I}_h} + \langle \hat{\mathbf{u}}_{i,h}, \boldsymbol{\Phi}_i(\boldsymbol{\nu}_i - \mathbf{n}_i) \rangle_{\mathcal{I}_h} + \langle \hat{\sigma}_{i,h} \boldsymbol{\nu}_i, \mathcal{P}_M(\phi_i) \rangle_{\mathcal{I}_h} - \langle \hat{\mathbf{u}}_{i,h}, \mathcal{P}_M(\boldsymbol{\Phi}_i \boldsymbol{\nu}_i) \rangle_{\mathcal{I}_h}].$$

Here $\mathcal{P}_{k-1}|_K$ is the L^2 -projection onto $\mathbf{P}_{k-1}(K)$ for each $K \in \Omega_h^i$. Similarly to the ideas behind the proof of Lemma 6, it is possible to explicitly write $\mathbb{S}_I$ in terms of quantities related to the mismatch between $\mathcal{I}$ and $\mathcal{I}_h$. In this sense, by Lemma 8, we find that:

$$\begin{aligned} \mathbb{S}_I = & \sum_{i \in \{s, f\}} \alpha_i \left[\langle \sigma_{i,h}(\mathbf{n}_i - \boldsymbol{\nu}_i), \phi_i \rangle_{\mathcal{I}_h} + \langle \hat{\mathbf{u}}_{i,h}, \boldsymbol{\Phi}_i(\boldsymbol{\nu}_i - \mathbf{n}_i) \rangle_{\mathcal{I}_h} - \langle \hat{\mathbf{u}}_{i,h}, \mathcal{P}_M(\boldsymbol{\Phi}_i \boldsymbol{\nu}_i - (\boldsymbol{\Phi}_i \boldsymbol{\nu}_i) \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \right. \\ & + \langle \mathbf{T}_h(\hat{\mathbf{u}}_{i,h}) \circ \psi^{-1} - \hat{\mathbf{u}}_{i,h}, \mathcal{P}_M((\boldsymbol{\Phi}_i \boldsymbol{\nu}_i) \circ \psi^{-1}) \rangle_{\mathcal{I}_h} + \langle \sigma_{i,h} \boldsymbol{\nu}_i - \mathbf{E}(\sigma_{i,h}) \circ \psi^{-1}, \mathcal{P}_M(\phi_i \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \\ & \left. + \langle \hat{\sigma}_{i,h} \boldsymbol{\nu}_i, \mathcal{P}_M(\phi_i - \phi_i \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \right] - \langle \mathbf{g} \circ \psi^{-1}, \mathcal{P}_M((\boldsymbol{\Phi}_s \boldsymbol{\nu}_s) \circ \psi^{-1}) \rangle_{\mathcal{I}_h} + \langle \mathbf{g}_\nu \circ \psi^{-1}, \mathcal{P}_M(\alpha_f \phi_f \circ \psi^{-1}) \rangle_{\mathcal{I}_h}. \end{aligned}$$

To bound the volumetric term $\mathbb{S}_V$, we apply the Cauchy–Schwarz inequality, the approximation properties of the HDG (15) and L^2 projections, and the elliptic regularity assumption (47). Furthermore, by invoking the bound (27) and the equivalence given in Lemma 3, we realize that:

$$\begin{aligned} \mathbb{S}_V \leq & \left\{ \sum_{i \in \{s, f\}} \alpha_i \left(\|\mathcal{A}(\sigma_{i,h})\|_{\Omega_h^i} + \|\gamma_{i,h}^0\|_{\Omega_h^i} + \|\mathbb{H}_{i,2}\|_{\Omega_h^i} + \|\mathbb{E}_i\|_{\Omega_h^i} + \beta_i \|\mathbf{u}_{i,h}\|_{\Omega_h^i} \right) \right\} \\ & \cdot \left\{ \sum_{i \in \{s, f\}} \left(\|\Pi \boldsymbol{\Phi}_i - \boldsymbol{\Phi}_i\|_{\Omega_h^i} + \|\Pi_{\mathbb{A}} \boldsymbol{\rho}_i - \boldsymbol{\rho}_i\|_{\Omega_h^i} + \|\mathcal{P}_{k-1}(\nabla \phi_i) - \nabla \phi_i\|_{\Omega_h^i} + \|\phi_i - \Pi \phi_i\|_{\Omega_h^i} \right) \right\} \\ & + \sum_{i \in \{s, f\}} \alpha_i \|\mathbf{f}_i\|_{\Omega_h^i} \|\Pi \phi_i\|_{\Omega_h^i} \\ \lesssim & h \left\{ \sum_{i \in \{s, f\}} \alpha_i \left((2\mu_i)^{-1/2} (\mathcal{A}(\sigma_{i,h}), \sigma_{i,h})_{\Omega_h^i}^{1/2} + \|\mathbb{H}_{i,1}\|_{\Omega_h^i} + \|\mathbb{H}_{i,2}\|_{\Omega_h^i} + \|\mathbb{E}_i\|_{\Omega_h^i} + \|\mathbf{f}_i\|_{\Omega_h^i} \right. \right. \\ & \left. \left. + \beta_i \|\mathbf{u}_{i,h}\|_{\Omega_h^i} \right) \right\} \|\boldsymbol{\Theta}\|_{\Omega_h^s \cup \Omega_h^f}. \end{aligned} \quad (49)$$

On the other hand, we decompose the interface term as $\mathbb{S}_I = \sum_{j=1}^7 \mathbb{B}_j$, where

$$\begin{aligned} \mathbb{B}_1 &= \sum_{i \in \{s, f\}} \alpha_i \langle \sigma_{i,h}(\mathbf{n}_i - \boldsymbol{\nu}_i), \phi_i \rangle_{\mathcal{I}_h}, & \mathbb{B}_5 &= \sum_{i \in \{s, f\}} \alpha_i \langle \hat{\sigma}_{i,h} \boldsymbol{\nu}_i, \mathcal{P}_M(\phi_i - \phi_i \circ \psi^{-1}) \rangle_{\mathcal{I}_h}, \\ \mathbb{B}_2 &= \sum_{i \in \{s, f\}} \alpha_i \langle \sigma_{i,h} \boldsymbol{\nu}_i - \mathbf{E}(\sigma_{i,h}) \circ \psi^{-1}, \mathcal{P}_M(\phi_i \circ \psi^{-1}) \rangle_{\mathcal{I}_h}, & \mathbb{B}_6 &= -\langle \mathbf{g} \circ \psi^{-1}, \mathcal{P}_M((\boldsymbol{\Phi}_s \boldsymbol{\nu}_s) \circ \psi^{-1}) \rangle_{\mathcal{I}_h}, \\ \mathbb{B}_3 &= \sum_{i \in \{s, f\}} \alpha_i \langle \mathbf{T}_h(\hat{\mathbf{u}}_{i,h}) \circ \psi^{-1} - \hat{\mathbf{u}}_{i,h}, \mathcal{P}_M((\boldsymbol{\Phi}_i \boldsymbol{\nu}_i) \circ \psi^{-1}) \rangle_{\mathcal{I}_h}, & \mathbb{B}_7 &= \langle \mathbf{g}_\nu \circ \psi^{-1}, \mathcal{P}_M(\alpha_f \phi_f \circ \psi^{-1}) \rangle_{\mathcal{I}_h}, \\ \mathbb{B}_4 &= \sum_{i \in \{s, f\}} \alpha_i \left[\langle \hat{\mathbf{u}}_{i,h}, \boldsymbol{\Phi}_i(\boldsymbol{\nu}_i - \mathbf{n}_i) - \mathcal{P}_M(\boldsymbol{\Phi}_i \boldsymbol{\nu}_i - (\boldsymbol{\Phi}_i \boldsymbol{\nu}_i) \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \right], \end{aligned}$$

We bound each of these terms employing the Cauchy–Schwarz and discrete trace inequalities (20), the approximation estimates from Lemma 1, and the regularity Assumption (47):

First, using the estimates in Lemma 1 and the discrete trace inequality, we have:

$$\mathbb{B}_1 \lesssim \max_{e \in \mathcal{I}_h} (\delta_e^\beta h_e^{-1/2}) \sum_{i \in \{s, f\}} \alpha_i \|\sigma_{i,h}\|_{\Omega_h^i} \|\boldsymbol{\Theta}\|_{\Omega_h^i}, \quad \text{and} \quad \mathbb{B}_2 \lesssim \max_{e \in \mathcal{I}_h} (C_e^{\text{ext}} \delta_e h_e^{-3/2}) \sum_{i \in \{s, f\}} \alpha_i \|\sigma_{i,h}\|_{\Omega_h^i} \|\boldsymbol{\Theta}\|_{\Omega_h^i}.$$

Second, using the Cauchy–Schwarz inequality, we directly obtain:

$$\mathbb{B}_6 \lesssim \|\mathbf{g} \circ \psi^{-1}\|_{\mathcal{I}_h} \|\boldsymbol{\Theta}\|_{\Omega_h^s}, \quad \text{and} \quad \mathbb{B}_7 \lesssim \alpha_f \|\mathbf{g}_\nu \circ \psi^{-1}\|_{\mathcal{I}_h} \|\boldsymbol{\Theta}\|_{\Omega_h^f}.$$

Third, recalling that $\hat{\sigma}_{i,h}\nu_i = \sigma_{i,h}\nu_i - \tau_i^*(\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h})$, we bound $\mathbb{B}_5$ as:

$$\mathbb{B}_5 \lesssim \max_{e \in \mathcal{I}_h} (\delta_e^{1/2}) \sum_{i \in \{s,f\}} \alpha_i \left(h_e^{-1/2} \|\sigma_{i,h}\|_{\Omega_h^i} + \tau_i^* \|\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h} \right) \|\Theta\|_{\Omega_h^i}.$$

Fourth, we bound the term involving the transfer path operator by invoking Lemma 2 (with $\mathbf{q}_{i,h} := \mathcal{A}(\sigma_{i,h}) + \gamma_{i,h}$). Then, by applying the bound for the rotation given in Lemma 5, we obtain:

$$\begin{aligned} \mathbb{B}_3 &\lesssim \max_{e \in \mathcal{I}_h} (\delta_e^{1/2}) \sum_{i \in \{s,f\}} \alpha_i C_{\delta_i,h}^{1/2} \left(\|\mathcal{A}(\sigma_{i,h})\|_{\Omega_h^i} + \|\mathbb{H}_{i,1}\|_{\Omega_h^i} + \|\mathbb{H}_{i,2}\|_{\Omega_h^i} \right. \\ &\quad \left. + h^{-1} \|\mathbf{u}_{i,h}\|_{\Omega_h^i} + \max_{e \in \mathcal{I}_h} \{h_e^{-1/2} (\tau_i^*)^{-1/2}\} (\tau_i^*)^{1/2} \|\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h} \right) \|\Theta\|_{\Omega_h^i}. \end{aligned}$$

Finally, to bound $\mathbb{B}_4$, we make use of Lemma 1. For that, by adding and subtracting $\Phi_i \mathbf{n}_i$ and $(\Phi_i \circ \psi^{-1}) \mathbf{n}_i$, we note that

$$\Phi_i \nu_i - (\Phi_i \nu_i) \circ \psi^{-1} = \Phi_i (\nu_i - \mathbf{n}_i) + (\Phi_i - \Phi_i \circ \psi^{-1}) \mathbf{n}_i + (\Phi_i \circ \psi^{-1}) (\mathbf{n}_i - \nu_i \circ \psi^{-1}),$$

which yields

$$\mathbb{B}_4 = - \sum_{i \in \{s,f\}} \alpha_i \left[\langle \hat{\mathbf{u}}_{i,h}, \mathcal{P}_M(\Phi_i - \Phi_i \circ \psi^{-1}) \mathbf{n}_i \rangle_{\mathcal{I}_h} + \langle \hat{\mathbf{u}}_{i,h}, \mathcal{P}_M(\Phi_i \circ \psi^{-1}) (\mathbf{n}_i - \nu_i \circ \psi^{-1}) \rangle_{\mathcal{I}_h} \right].$$

Thus, we obtain:

$$\begin{aligned} \mathbb{B}_4 &\lesssim \max_{e \in \mathcal{I}_h} (\delta_e^{1/2} + \delta_e^\beta) \sum_{i \in \{s,f\}} \alpha_i \|\hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h} \|\Theta\|_{\Omega_h^i} \\ &\lesssim \max_{e \in \mathcal{I}_h} \left((\tau_i^*)^{-1/2} (\delta_e^{5/14} + \delta_e^{\beta-1/7}) \right) \sum_{i \in \{s,f\}} \alpha_i (\tau_i^*)^{1/2} \|\delta_e^{1/7} \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h} \|\Theta\|_{\Omega_h^i}. \end{aligned}$$

Summing the bounds for $\mathbb{B}_1$ to $\mathbb{B}_7$, substituting the bounds for $\|\sigma_{i,h}\|_{\Omega_h^i}$, and invoking the equivalence in Lemma 3, we deduce that:

$$\begin{aligned} \mathbb{S}_I &\lesssim C_I(\delta, h) \left\{ \sum_{i \in \{s,f\}} \alpha_i \left[\left(\mu_i^{1/2} + (2\mu_i)^{-1/2} \right) (\mathcal{A}(\sigma_{i,h}), \sigma_{i,h})_{\Omega_h^i}^{1/2} + (1 + \beta_i) \|\mathbf{u}_{i,h}\|_{\Omega_h^i} + \|\mathbb{H}_{i,1}\|_{\Omega_h^i} + \|\mathbb{H}_{i,2}\|_{\Omega_h^i} \right. \right. \\ &\quad \left. \left. + (\tau_i^*)^{1/2} \|(\tau_i^*)^{1/2} (\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h})\|_{\partial\Omega_h^i} + (\tau_i^*)^{1/2} \|\delta_e^{1/7} \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h} + \|\mathbf{f}_i\|_{\Omega_h^i} \right] + \|\mathbf{g} \circ \psi^{-1}\|_{\mathcal{I}_h} + \alpha_f \|\mathbf{g}_\nu \circ \psi^{-1}\|_{\mathcal{I}_h} \right. \\ &\quad \left. + \alpha_s \frac{n\lambda_s + 2\mu_s}{|\Omega_h^s|^{1/2}} \max_{e \in \mathcal{I}_h} \{h_e^{1/2} \delta_e^{-1/7}\} \left(\|\delta_e^{1/7} \hat{\mathbf{u}}_{s,h}\|_{\mathcal{I}_h} + \|\mathbb{H}_{s,1}\|_{\Omega_h^s} \right) \right\} \sum_{i \in \{s,f\}} \|\Theta\|_{\Omega_h^i}, \end{aligned} \quad (50)$$

where $C_I(\delta, h)$ is the constant defined in Assumption (A.6). In this way, from the bounds (49) and (50), we find that

$$\begin{aligned} \sum_{i \in \{s,f\}} \alpha_i (\mathbf{u}_{i,h}, \Theta)_{\Omega_h^i} &\lesssim \left\{ h \sum_{i \in \{s,f\}} \alpha_i \left((2\mu_i)^{-1/2} (\mathcal{A}(\sigma_{i,h}), \sigma_{i,h})_{\Omega_h^i}^{1/2} + \|\mathbb{H}_{i,1}\|_{\Omega_h^i} + \|\mathbb{H}_{i,2}\|_{\Omega_h^i} + \|\mathbb{E}_i\|_{\Omega_h^i} + \|\mathbf{f}_i\|_{\Omega_h^i} \right. \right. \\ &\quad \left. \left. + \beta_i \|\mathbf{u}_{i,h}\|_{\Omega_h^i} \right) + C_I(\delta, h) \left[\sum_{i \in \{s,f\}} \alpha_i \left(\left(\mu_i^{1/2} + (2\mu_i)^{-1/2} \right) (\mathcal{A}(\sigma_{i,h}), \sigma_{i,h})_{\Omega_h^i}^{1/2} + (1 + \beta_i) \|\mathbf{u}_{i,h}\|_{\Omega_h^i} \right. \right. \right. \\ &\quad \left. \left. + \|\mathbb{H}_{i,1}\|_{\Omega_h^i} + \|\mathbb{H}_{i,2}\|_{\Omega_h^i} + \|\mathbf{f}_i\|_{\Omega_h^i} + (\tau_i^*)^{1/2} \|(\tau_i^*)^{1/2} (\mathbf{u}_{i,h} - \hat{\mathbf{u}}_{i,h})\|_{\partial\Omega_h^i} + (\tau_i^*)^{1/2} \|\delta_e^{1/7} \hat{\mathbf{u}}_{i,h}\|_{\mathcal{I}_h} \right) \right. \\ &\quad \left. \left. + \alpha_s \frac{n\lambda_s + 2\mu_s}{|\Omega_h^s|^{1/2}} \left(\max_{e \in \mathcal{I}_h} \{h_e^{1/2} \delta_e^{-1/7}\} \|\delta_e^{1/7} \hat{\mathbf{u}}_{s,h}\|_{\mathcal{I}_h} + \|\mathbb{H}_{s,1}\|_{\Omega_h^s} \right) + \|\mathbf{g} \circ \psi^{-1}\|_{\mathcal{I}_h} + \alpha_f \|\mathbf{g}_\nu \circ \psi^{-1}\|_{\mathcal{I}_h} \right] \right\} \sum_{i \in \{s,f\}} \|\Theta\|_{\Omega_h^i}. \end{aligned}$$

Hence, taking $\Theta = \mathbf{u}_{s,h}$ in Ω_h^s and $\Theta = \mathbf{u}_{f,h}$ in Ω_h^f , employing algebraic manipulations and invoking Assumption (A.6), we obtain (48). $\square$

With these results, we can now state the main theorem of this section.

Theorem 1. Suppose Assumptions (A.1)–(A.7) and the elliptic regularity of the dual problem hold true. Then, the following holds:

$$\begin{aligned} \mathcal{Q}(\sigma_h, \mathbf{u}_h, \hat{\mathbf{u}}_h)^2 &\leq \sum_{i \in \{s, f\}} \alpha_i \left(\tilde{C}_{\mathbb{H}_{i,1}} \|\mathbb{H}_{i,1}\|_{\Omega_h^i}^2 + \tilde{C}_{\mathbb{H}_{i,2}} \|\mathbb{H}_{i,2}\|_{\Omega_h^i}^2 + \tilde{C}_{\mathbb{E}_i} \|\mathbb{E}_i\|_{\Omega_h^i}^2 + \tilde{C}_{f_i} \|\mathbf{f}_i\|_{\Omega_h^i}^2 \right) \\ &\quad + \tilde{C}_g \|\mathbf{g} \circ \psi^{-1}\|_{\mathcal{I}_h}^2 + \tilde{C}_{g_\nu} \alpha_f \|\mathbf{g}_\nu \circ \psi^{-1}\|_{\mathcal{I}_h}^2, \end{aligned} \quad (51)$$

where

$$\begin{aligned} \tilde{C}_{\mathbb{H}_{s,1}} &:= 4 \left(C_{\mathbb{H}_{s,1}} + C_u^* (h^2 + C_I^2(\delta, h)) + C_u^* C_I^2(\delta, h) \frac{(n\lambda_s + 2\mu_s)^2}{|\Omega_h^s|} \max_{e \in \mathcal{I}_h} \{h_e \delta_e^{-2/7}\} \right), \\ \tilde{C}_{\mathbb{H}_{f,1}} &:= 4 \left(C_{\mathbb{H}_{f,1}} + C_u^* (h^2 + C_I^2(\delta, h)) \right), \quad \tilde{C}_{\mathbb{H}_{i,2}} := 4 \left(C_{\mathbb{H}_{i,2}} + C_u^* (h^2 + C_I^2(\delta, h)) \right), \\ \tilde{C}_{\mathbb{E}_i} &:= 4 \left(\frac{2C}{\mu_i} + C_u^* h^2 \right), \quad \tilde{C}_{f_i} := 4 \left(\left(\frac{1}{2} + C C(\sigma) \right) + C_u^* (h^2 + C_I^2(\delta, h)) \right), \\ \tilde{C}_g &:= 4 \left(C(\mathbf{g}) + C_u^* C_I^2(\delta, h) \right), \quad \tilde{C}_{g_\nu} := 4 \left(C(\mathbf{g}_\nu) + C_u^* C_I^2(\delta, h) \alpha_f \right). \end{aligned}$$

Moreover, estimate (48) is satisfied.

Proof. We first employ the duality estimate obtained in Lemma 9 to bound the first term on the right-hand side of (30). Thus, we obtain:

$$\begin{aligned} \left(\frac{1}{2} - C(\delta, h, \text{data}) \right) \mathcal{Q}(\sigma_h, \mathbf{u}_h, \hat{\mathbf{u}}_h)^2 &\leq \sum_{i \in \{s, f\}} \alpha_i \frac{1}{4} \left(\tilde{C}_{\mathbb{H}_{i,1}} \|\mathbb{H}_{i,1}\|_{\Omega_h^i}^2 + \tilde{C}_{\mathbb{H}_{i,2}} \|\mathbb{H}_{i,2}\|_{\Omega_h^i}^2 + \tilde{C}_{\mathbb{E}_i} \|\mathbb{E}_i\|_{\Omega_h^i}^2 + \tilde{C}_{f_i} \|\mathbf{f}_i\|_{\Omega_h^i}^2 \right) \\ &\quad + \frac{1}{4} \tilde{C}_g \|\mathbf{g} \circ \psi^{-1}\|_{\mathcal{I}_h}^2 + \frac{1}{4} \tilde{C}_{g_\nu} \alpha_f \|\mathbf{g}_\nu \circ \psi^{-1}\|_{\mathcal{I}_h}^2. \end{aligned}$$

Since Assumption (A.7) guarantees that $C(\delta, h, \text{data}) \leq \frac{1}{4}$, we immediately obtain the estimate (51). Finally, estimate (48) is established in Lemma 9. $\square$

We end this section with the following result.

Corollary 1. The HDG formulation has a unique solution.

Proof. The proof follows by setting all the source data to zero in (51), which yields $\mathcal{Q}(\sigma_h, \mathbf{u}_h, \hat{\mathbf{u}}_h)^2 \leq 0$ and thus only the trivial solution of the system, which implies existence and uniqueness. $\square$

5 Error Analysis

In this section, we employ the stability estimate obtained in previous sections in order to deduce the error estimates in Theorem 1. To that end, for each subdomain $i \in \{s, f\}$, we introduce the projection of the errors $e^{\sigma_i} := \Pi \sigma_i - \sigma_{i,h}$, $e^{\gamma_i} := \Pi_{\mathbb{A}} \gamma_i - \gamma_{i,h}$, $e^{u_i} := \Pi u_i - u_{i,h}$, $e^{\hat{u}_i} := \mathcal{P}_{\mathbf{M}} u_i - \hat{u}_{i,h}$, and the error of the projections $I^{\sigma_i} := \sigma_i - \Pi \sigma_i$, $I^{\gamma_i} := \gamma_i - \Pi_{\mathbb{A}} \gamma_i$, $I^{u_i} := u_i - \Pi u_i$.

Lemma 10. The projection of the errors satisfies

$$(\mathcal{A}(e^{\sigma_i}), \tau_{i,h})_{\Omega_h^i} + (e^{u_i}, \nabla \cdot \tau_{i,h})_{\Omega_h^i} + (e^{\gamma_i}, \tau_{i,h})_{\Omega_h^i} - \langle e^{\hat{u}_i}, \tau_{i,h} \mathbf{n}_i \rangle_{\partial \Omega_h^i} = -(\mathcal{A}(I^{\sigma_i}), \tau_{i,h})_{\Omega_h^i} - (I^{\gamma_i}, \tau_{i,h})_{\Omega_h^i}, \quad (52a)$$

$$(e^{\sigma_i}, \nabla v_{i,h})_{\Omega_h^i} - \langle e^{\hat{u}_i} \mathbf{n}_i, v_{i,h} \rangle_{\partial \Omega_h^i} + (\beta_i e^{u_i}, v_{i,h})_{\Omega_h^i} = -\beta_i (I^{u_i}, v_{i,h})_{\Omega_h^i}, \quad (52b)$$

$$(e^{\sigma_i}, \boldsymbol{\eta}_{i,h})_{\Omega_h^i} = -(I^{\sigma_i}, \boldsymbol{\eta}_{i,h})_{\Omega_h^i}, \quad (52c)$$

$$\langle e^{\hat{u}_i} \mathbf{n}_i, \boldsymbol{\mu}_{i,h} \rangle_{\partial \Omega_h^i \setminus (\Gamma_{i,h} \cup \mathcal{I}_h)} = 0, \quad (52d)$$

$$\langle e^{\hat{u}_i}, \boldsymbol{\mu}_{i,h} \rangle_{\Gamma_{i,h}} = 0, \quad (52e)$$

for all $(\tau_{i,h}, \mathbf{v}_{i,h}, \boldsymbol{\eta}_{i,h}, \boldsymbol{\mu}_{i,h}) \in \mathbb{S}_h^i \times \mathbf{V}_h^i \times \mathbb{A}_h^i \times \mathbf{M}_h^i$, where

$$e^{\widehat{\sigma}_i} \mathbf{n}_i = e^{\sigma_i} \mathbf{n}_i - \tau_i^* (e^{u_i} - e^{\widehat{u}_i}) \quad \text{on} \quad \partial\Omega_h^i. \quad (53)$$

Moreover, letting $e^{q_i} := \mathcal{A}(e^{\sigma_i}) + e^{\gamma_i}$, for $\mathbf{x}_h \in \mathcal{I}_h$, we have

$$\mathbf{T}_h(e^{\widehat{u}_i})(\psi^{-1}(\mathbf{x}_h)) = e^{\widehat{u}_i}(\mathbf{x}_h) + l(\mathbf{x}_h) \int_0^1 \mathbf{E}(e^{q_i})(\mathbf{y}_i(r)) \mathbf{m}(\mathbf{x}_h) dr, \quad (54a)$$

$$\widehat{\mathbf{E}}(e^{\widehat{\sigma}_i})(\psi^{-1}(\mathbf{x}_h)) = \mathbf{E}(e^{\sigma_i}) \boldsymbol{\nu}_i(\psi^{-1}(\mathbf{x}_h)) - \tau_i^* (e^{u_i} - e^{\widehat{u}_i})(\mathbf{x}_h), \quad (54b)$$

for $i \in \{s, f\}$, where $\mathbf{y}_i(r) := \mathbf{x}_h + (\psi^{-1}(\mathbf{x}_h) - \mathbf{x}_h)r$ parameterizes the transfer path.

They satisfy (with $\mathbf{m} = \mathbf{n}_s = -\mathbf{n}_f$)

$$\langle \alpha_s \mathbf{T}_h(e^{\widehat{u}_s}) \circ \psi^{-1} - \alpha_f \mathbf{T}_h(e^{\widehat{u}_f}) \circ \psi^{-1}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h} = - \sum_{i \in \{s, f\}} \alpha_i \langle l(I^{q_i}) \mathbf{n}_i + l\Lambda_{(I^{q_i})}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h}, \quad (55a)$$

$$\begin{aligned} \langle \widehat{\mathbf{E}}(e^{\widehat{\sigma}_s}) \circ \psi^{-1} + \widehat{\mathbf{E}}(e^{\widehat{\sigma}_f}) \circ \psi^{-1}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h} = & - \sum_{i \in \{s, f\}} \left\langle (I^{\sigma_i} \circ \psi^{-1} - I^{\sigma_i}) \boldsymbol{\nu}_i \circ \psi^{-1} \right. \\ & \left. + I^{\sigma_i} (\boldsymbol{\nu}_i \circ \psi^{-1} - \mathbf{n}_i), \boldsymbol{\mu} \right\rangle_{\mathcal{I}_h}, \end{aligned} \quad (55b)$$

for all $\boldsymbol{\mu} \in \mathbf{P}_k(e)$ defined on each face $e \subset \mathcal{I}_h$.

Proof. The identities (52)–(53) follow directly from the definition of the projection of the errors in [8, Lemma 3.1] and [6, Lemma 3.1]. Now, let $\mathbf{x}_h \in \mathcal{I}_h$. We rewrite (54a) for $e^{\widehat{u}_i}$ as

$$\mathbf{T}_h(e^{\widehat{u}_i}) \circ \psi^{-1}(\mathbf{x}_h) := \mathbf{T}_h(\mathcal{P}_M \mathbf{u}_i) \circ \psi^{-1}(\mathbf{x}_h) - \mathbf{T}_h(\widehat{\mathbf{u}}_{i,h}) \circ \psi^{-1}(\mathbf{x}_h).$$

By adding and subtracting the exact solution $\mathbf{u}_i \circ \psi^{-1}(\mathbf{x}_h) = \mathbf{u}_i(\mathbf{x}_h) + l(\mathbf{x}_h) \int_0^1 \mathbf{q}_i(\mathbf{y}_i(r)) \mathbf{m} dr$, we note that

$$\mathbf{T}_h(\mathcal{P}_M \mathbf{u}_i) \circ \psi^{-1}(\mathbf{x}_h) - \mathbf{u}_i \circ \psi^{-1}(\mathbf{x}_h) = (\mathcal{P}_M \mathbf{u}_i - \mathbf{u}_i)(\mathbf{x}_h) - l(\mathbf{x}_h) \int_0^1 \mathbf{E}(I^{q_i})(\mathbf{y}_i(r)) \mathbf{m} dr.$$

Proceeding in a similar way as in the proof of Lemma 2, using (16) we deduce that

$$l(\mathbf{x}_h) \int_0^1 \mathbf{E}(I^{q_i})(\mathbf{y}_i(r)) \mathbf{m} dr = l(\mathbf{x}_h) \Lambda_{(I^{q_i})}(\mathbf{x}_h) + l(\mathbf{x}_h) I^{q_i}(\mathbf{x}_h) \mathbf{m}.$$

Consequently, noting that $\langle \mathcal{P}_M \mathbf{u}_i - \mathbf{u}_i, \boldsymbol{\mu} \rangle_{\mathcal{I}_h} = 0$ for all $\boldsymbol{\mu} \in \mathbf{M}_h^i(\mathcal{I}_h)$, recalling that $\mathbf{m} = \mathbf{n}_s = -\mathbf{n}_f$, and using the transmission conditions (4a) and (10a), we easily obtain

$$\langle \alpha_s \mathbf{T}_h(e^{\widehat{u}_s}) \circ \psi^{-1} - \alpha_f \mathbf{T}_h(e^{\widehat{u}_f}) \circ \psi^{-1}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h} = - \sum_{i \in \{s, f\}} \alpha_i \langle l(\mathbf{x}_h) I^{q_i} \mathbf{n}_i + l(\mathbf{x}_h) \Lambda_{(I^{q_i})}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h},$$

which proves (55a).

On the other hand, expanding (54b) for $i \in \{s, f\}$ by adding and subtracting the flux $\sigma_i \circ \psi^{-1}$, we obtain:

$$\begin{aligned} \widehat{\mathbf{E}}(e^{\widehat{\sigma}_i}) \circ \psi^{-1} &:= \mathbf{E}(\Pi \sigma_i) \circ \psi^{-1} \boldsymbol{\nu}_i \circ \psi^{-1} - \tau_i^* (\Pi \mathbf{u}_i - \mathcal{P}_M \mathbf{u}_i) - \widehat{\mathbf{E}}(\widehat{\sigma}_{i,h}) \circ \psi^{-1} \\ &= - (I^{\sigma_i} \circ \psi^{-1}) \boldsymbol{\nu}_i \circ \psi^{-1} + (\sigma_i \circ \psi^{-1}) \boldsymbol{\nu}_i \circ \psi^{-1} \\ &\quad + \tau_i^* I^{u_i} + \tau_i^* (\mathcal{P}_M \mathbf{u}_i - \mathbf{u}_i) - \widehat{\mathbf{E}}(\widehat{\sigma}_{i,h}) \circ \psi^{-1}. \end{aligned}$$

Summing these expressions, multiplying by test functions $\boldsymbol{\mu} \in \mathbf{M}_h^i$, and using the conditions (4b) and (10b), we find that

$$\langle \widehat{\mathbf{E}}(e^{\widehat{\sigma}_s}) \circ \psi^{-1} + \widehat{\mathbf{E}}(e^{\widehat{\sigma}_f}) \circ \psi^{-1}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h} = \sum_{i \in \{s, f\}} \langle - (I^{\sigma_i} \circ \psi^{-1}) \boldsymbol{\nu}_i \circ \psi^{-1} + \tau_i^* I^{u_i} + \tau_i^* (\mathcal{P}_M \mathbf{u}_i - \mathbf{u}_i), \boldsymbol{\mu} \rangle_{\mathcal{I}_h}.$$

To exploit the properties of the HDG projection, we add and subtract $I^{\sigma_i} \mathbf{n}_i$ and $I^{\sigma_i} \boldsymbol{\nu}_i \circ \boldsymbol{\psi}^{-1}$ on each face $e \subset \mathcal{I}_h$, to obtain

$$\begin{aligned} -(I^{\sigma_i} \circ \boldsymbol{\psi}^{-1}) \boldsymbol{\nu}_i \circ \boldsymbol{\psi}^{-1} + \tau_i^* I^{\mathbf{u}_i} + \tau_i^* (\mathcal{P}_M \mathbf{u}_i - \mathbf{u}_i) &= -(I^{\sigma_i} \circ \boldsymbol{\psi}^{-1} - I^{\sigma_i}) \boldsymbol{\nu}_i \circ \boldsymbol{\psi}^{-1} \\ &\quad - I^{\sigma_i} (\boldsymbol{\nu}_i \circ \boldsymbol{\psi}^{-1} - \mathbf{n}_i) - (I^{\sigma_i} \mathbf{n}_i - \tau_i^* I^{\mathbf{u}_i}) + \tau_i^* (\mathcal{P}_M \mathbf{u}_i - \mathbf{u}_i). \end{aligned}$$

Finally, noticing from (14c) that $\langle I^{\sigma_i} \mathbf{n}_i - \tau_i^* I^{\mathbf{u}_i}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h} = 0$, and from the definition of the L^2 -projection onto $\mathbf{M}_h^i$ that $\langle \mathcal{P}_M \mathbf{u}_i - \mathbf{u}_i, \boldsymbol{\mu} \rangle_{\mathcal{I}_h} = 0$, (55b) follows. $\square$

We note that the error equations (52a)-(52c) are similar to those in our original HDG scheme, where the term $-(\mathcal{A}(I^{\sigma_i}) + I^{\gamma_i})$ plays the role of the volumetric source $\mathbb{H}_{i,1}$ (with $\mathbb{H}_{i,2} = \mathbf{0}$), while $-\beta_i I^{\mathbf{u}_i}$ and $-I^{\sigma_i}$ play the roles of $\mathbf{f}_i$ and $\mathbb{E}_i$, respectively. In addition, the right-hand sides of (55a) and (55b) play the role of $\mathbf{g}$ and $\mathbf{g}_\nu$. That is, for all $\boldsymbol{\mu} \in \mathbf{P}_k(e)$ defined on each face $e \subset \mathcal{I}_h$, it holds:

$$\begin{aligned} \langle \mathbf{g} \circ \boldsymbol{\psi}^{-1}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h} &= - \sum_{i \in \{s,f\}} \alpha_i \langle l I^{\mathbf{q}_i} \mathbf{n}_i + l \Lambda_{(I^{\mathbf{q}_i})}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h}, \\ \langle \mathbf{g}_\nu \circ \boldsymbol{\psi}^{-1}, \boldsymbol{\mu} \rangle_{\mathcal{I}_h} &= - \sum_{i \in \{s,f\}} \left\langle (I^{\sigma_i} \circ \boldsymbol{\psi}^{-1} - I^{\sigma_i}) \boldsymbol{\nu}_i \circ \boldsymbol{\psi}^{-1} + I^{\sigma_i} (\boldsymbol{\nu}_i \circ \boldsymbol{\psi}^{-1} - \mathbf{n}_i), \boldsymbol{\mu} \right\rangle_{\mathcal{I}_h}. \end{aligned}$$

On the other hand, following the proof of [26, Theorem 4.2], we deduce from a scaling argument to bound the $L^2(\mathcal{I}_h)$ -norm in terms of its $L^2(\Omega_h)$ -norm and $\mathbb{H}^1(\Omega_h)$ -seminorm, and Lemma 1, that

$$\|\mathbf{g} \circ \boldsymbol{\psi}^{-1}\|_{\mathcal{I}_h}^2 \lesssim h \left(\max_{e \in \mathcal{I}_h} (\delta_e^4 h_e^{-4}) + \max_{e \in \mathcal{I}_h} (\delta_e h_e^{-1})^2 \right) \sum_{i \in \{s,f\}} \alpha_i^2 \|I^{\mathbf{q}_i}\|_{\Omega_h^i}^2, \quad (56)$$

and

$$\alpha_f \|\mathbf{g}_\nu \circ \boldsymbol{\psi}^{-1}\|_{\mathcal{I}_h}^2 \lesssim h \max_{e \in \mathcal{I}_h} (\delta_e^\beta h_e^{-1})^2 \sum_{i \in \{s,f\}} \alpha_f \|I^{\sigma_i}\|_{\Omega_h^i}^2 + h \max_{e \in \mathcal{I}_h} (\delta_e h_e^{-1}) \sum_{i \in \{s,f\}} \alpha_f |I^{\sigma_i}|_{\mathbb{H}^1(\Omega_h^i)}^2. \quad (57)$$

Next, from Assumption (A.6), we recall that $C_g \lesssim \max\{h^{-1}, \delta^3 h^{-6}\}$ and $C_{g_\nu} \lesssim \max\{h^{-2\gamma}, \delta^3 h^{-6}\}$. Thus, by combining these bounds, it follows that

$$\begin{aligned} C_g \|\mathbf{g} \circ \boldsymbol{\psi}^{-1}\|_{\mathcal{I}_h}^2 &\lesssim \max\{h^{-1}, \delta^3 h^{-6}\} \left[h \left(\max_{e \in \mathcal{I}_h} (\delta_e^4 h_e^{-4}) + \max_{e \in \mathcal{I}_h} (\delta_e h_e^{-1})^2 \right) \right] \sum_{i \in \{s,f\}} \alpha_i^2 \|I^{\mathbf{q}_i}\|_{\Omega_h^i}^2 \\ &\lesssim \max\{h^{-1}, \delta^3 h^{-6}\} \left(\delta^4 h^{-3} + \delta^2 h^{-1} \right) \sum_{i \in \{s,f\}} \alpha_i^2 \|I^{\mathbf{q}_i}\|_{\Omega_h^i}^2, \end{aligned} \quad (58)$$

and

$$\begin{aligned} C_{g_\nu} \alpha_f \|\mathbf{g}_\nu \circ \boldsymbol{\psi}^{-1}\|_{\mathcal{I}_h}^2 &\lesssim \max\{h^{-2\gamma}, \delta^3 h^{-6}\} \left(h \max_{e \in \mathcal{I}_h} (\delta_e^\beta h_e^{-1})^2 \sum_{i \in \{s,f\}} \alpha_f \|I^{\sigma_i}\|_{\Omega_h^i}^2 + h \max_{e \in \mathcal{I}_h} (\delta_e h_e^{-1}) \sum_{i \in \{s,f\}} \alpha_f |I^{\sigma_i}|_{\mathbb{H}^1(\Omega_h^i)}^2 \right) \\ &\lesssim \max\{h^{-2\gamma}, \delta^3 h^{-6}\} \left(\delta^{2\beta} h^{-1} \sum_{i \in \{s,f\}} \alpha_f \|I^{\sigma_i}\|_{\Omega_h^i}^2 + \delta \sum_{i \in \{s,f\}} \alpha_f |I^{\sigma_i}|_{\mathbb{H}^1(\Omega_h^i)}^2 \right). \end{aligned} \quad (59)$$

At this stage, we now set $\delta := C_{\text{gap}} h^{1+\aleph}$ with $C_{\text{gap}} \geq 0$. In addition, we assume that there exists $h_0 \in (0, 1)$ such that $h < h_0$. Then, inserting the estimates (58) and (59) back into Theorem 1 applied to the error equations (52a)-(52e), we deduce that

$$\begin{aligned} \mathcal{Q}(e^\sigma, e^{\mathbf{u}}, e^{\widehat{\mathbf{u}}})^2 &\lesssim \sum_{i \in \{s,f\}} \alpha_i \left(\max\{1, C_{\text{gap}}^3\} h^{3\aleph-3} (\|I^{\mathbf{q}_i}\|_{\Omega_h^i}^2 + \beta_i^2 \|I^{\mathbf{u}_i}\|_{\Omega_h^i}^2) + \|I^{\sigma_i}\|_{\Omega_h^i}^2 \right) \\ &\quad + \max\{C_{\text{gap}}^2, C_{\text{gap}}^5\} h^{2\aleph} \sum_{i \in \{s,f\}} \alpha_i^2 \|I^{\mathbf{q}_i}\|_{\Omega_h^i}^2 \\ &\quad + \max\{C_{\text{gap}}^{2\beta}, C_{\text{gap}}^{2\beta+3}\} h^{2\beta(1+\aleph)-1+\min\{-2\gamma, 3\aleph-3\}} \sum_{i \in \{s,f\}} \alpha_f \|I^{\sigma_i}\|_{\Omega_h^i}^2 \\ &\quad + \max\{C_{\text{gap}}, C_{\text{gap}}^4\} h^{\min\{1+\aleph-2\gamma, 4\aleph-2\}} \sum_{i \in \{s,f\}} \alpha_f |I^{\sigma_i}|_{\mathbb{H}^1(\Omega_h^i)}^2, \end{aligned} \quad (60)$$

and

$$\begin{aligned}
\sum_{i \in \{s, f\}} \alpha_i \|e^{u_i}\|_{\Omega_h^i}^2 &\lesssim \mathcal{N}(h, \text{data}) \mathcal{Q}(e^\sigma, e^u, e^{\hat{u}})^2 + h^2 \sum_{i \in \{s, f\}} \alpha_i \|I^{\sigma_i}\|_{\Omega_h^i}^2 \\
&+ \sum_{i \in \{s, f\}} \alpha_i \left(h^2 + C_{\text{gap}}^2 h^{2\aleph-1} \right) \left(\|I^{q_i}\|_{\Omega_h^i}^2 + \beta_i^2 \|I^{u_i}\|_{\Omega_h^i}^2 \right) \\
&+ C_{\text{gap}}^2 h^{2\aleph-1} \alpha_s \frac{(n\lambda_s + 2\mu_s)^2}{|\Omega_h^s|} \|I^{q_s}\|_{\Omega_h^s}^2 + C_{\text{gap}}^4 h^{4\aleph} \sum_{i \in \{s, f\}} \alpha_i^2 \|I^{q_i}\|_{\Omega_h^i}^2 \\
&+ C_{\text{gap}}^{2\beta+2} h^{2\beta(1+\aleph)+2\aleph-2} \sum_{i \in \{s, f\}} \alpha_f^2 \|I^{\sigma_i}\|_{\Omega_h^i}^2 + C_{\text{gap}}^3 h^{3\aleph} \sum_{i \in \{s, f\}} \alpha_f^2 |I^{\sigma_i}|_{\mathbb{H}^1(\Omega_h^i)}^2. \tag{61}
\end{aligned}$$

Theorem 2. Let us assume that Assumptions (A) and elliptic regularity are satisfied. Let $\delta = C_{\text{gap}} h^{1+\aleph}$ with $C_{\text{gap}} \geq 0$, and let $\aleph \in [3/4, 1]$, $\beta \geq 1/2$, and $\gamma \in [1/4, 1/2]$. If τ_i^* is of order one, $k \geq 1$, and $(\sigma_i, u_i, \gamma_i) \in \mathbb{H}^{l_\sigma+1}(\Omega_i) \times \mathbf{H}^{l_u+1}(\Omega_i) \times \mathbb{H}^{l_\gamma+1}(\Omega_i)$, for $l_\sigma, l_u, l_\gamma \in [0, k]$, with $i \in \{s, f\}$, then there exists $h_0 \in (0, 1)$, such that for all $h < h_0$, the following holds:

$$\begin{aligned}
\sum_{i \in \{s, f\}} \|\sigma_i - \sigma_{i,h}\|_{\Omega_h^i} &\lesssim \sum_{i \in \{s, f\}} \left[\left(h^{l_\sigma+1} + h^{l_\sigma} \max \left\{ C_{\text{gap}}^{1/2} h^{\frac{1+\aleph-2\gamma}{2}}, C_{\text{gap}}^{3/2} h^{\frac{3\aleph-1}{2}}, C_{\text{gap}}^\beta h^{\frac{2\beta(1+\aleph)-2\gamma+1}{2}} \right\} \right) |\sigma_i|_{\mathbb{H}^{l_\sigma+1}(\Omega_i)} \right. \\
&+ (1 + \beta_i) \left(h^{l_u+1} + h^{l_u} \max \left\{ C_{\text{gap}}^{1/2} h^{\frac{1+\aleph-2\gamma}{2}}, C_{\text{gap}}^{3/2} h^{\frac{3\aleph-1}{2}}, C_{\text{gap}}^\beta h^{\frac{2\beta(1+\aleph)-2\gamma+1}{2}} \right\} \right) |u_i|_{\mathbf{H}^{l_u+1}(\Omega_i)} \\
&+ C_{\text{gap}} h^{l_\sigma+\frac{2\aleph+1}{2}} \alpha_s^{1/2} \frac{n\lambda_s + 2\mu_s}{|\Omega_h^s|^{1/2}} |\sigma_s|_{\mathbb{H}^{l_\sigma+1}(\Omega_s)} + C_{\text{gap}} h^{l_u+\frac{2\aleph+1}{2}} \alpha_s^{1/2} \frac{n\lambda_s + 2\mu_s}{|\Omega_h^s|^{1/2}} |u_s|_{\mathbf{H}^{l_u+1}(\Omega_s)}, \\
&\left. \left(\sum_{i \in \{s, f\}} \alpha_i \|e^{u_i}\|_{\Omega_h^i}^2 \right)^{1/2} \lesssim \max \left\{ h, C_{\text{gap}} h^{\frac{2\aleph-1}{2}} \right\} \mathcal{Q}(e^\sigma, e^u, e^{\hat{u}}) \right. \\
&+ \sum_{i \in \{s, f\}} \left[\left(h^{2+l_\sigma} + h^{l_\sigma} \max \left\{ C_{\text{gap}}^{3/2} h^{\frac{3\aleph}{2}}, C_{\text{gap}}^{\beta+1} h^{\beta(1+\aleph)+\aleph} \right\} \right) |\sigma_i|_{\mathbb{H}^{l_\sigma+1}(\Omega_i)} \right. \\
&+ (1 + \beta_i) \left(h^{2+l_u} + h^{l_u} \max \left\{ C_{\text{gap}}^{3/2} h^{\frac{3\aleph}{2}}, C_{\text{gap}}^{\beta+1} h^{\beta(1+\aleph)+\aleph} \right\} \right) |u_i|_{\mathbf{H}^{l_u+1}(\Omega_i)} \\
&+ C_{\text{gap}} h^{l_\sigma+\frac{2\aleph+1}{2}} \alpha_s^{1/2} \frac{n\lambda_s + 2\mu_s}{|\Omega_h^s|^{1/2}} |\sigma_s|_{\mathbb{H}^{l_\sigma+1}(\Omega_s)} + C_{\text{gap}} h^{l_u+\frac{2\aleph+1}{2}} \alpha_s^{1/2} \frac{n\lambda_s + 2\mu_s}{|\Omega_h^s|^{1/2}} |u_s|_{\mathbf{H}^{l_u+1}(\Omega_s)}, \\
&\left. \sum_{i \in \{s, f\}} \|u_i - u_{i,h}\|_{\Omega_h^i} \lesssim \left(\sum_{i \in \{s, f\}} \alpha_i \|e^{u_i}\|_{\Omega_h^i}^2 \right)^{1/2} + \left(\sum_{i \in \{s, f\}} h^{2(l_u+1)} |u_i|_{\mathbf{H}^{l_u+1}(\Omega_i)}^2 \right)^{1/2}, \right. \\
&\left. \sum_{i \in \{s, f\}} \|\gamma_i - \gamma_{i,h}\|_{\Omega_h^i} \lesssim h^{-1/2} \mathcal{Q}(e^\sigma, e^u, e^{\hat{u}}) + h^{-1} \left(\sum_{i \in \{s, f\}} \alpha_i \|e^{u_i}\|_{\Omega_h^i}^2 \right)^{1/2} \right. \\
&\left. + \sum_{i \in \{s, f\}} \left(h^{l_\sigma+1} |\sigma_i|_{\mathbb{H}^{l_\sigma+1}(\Omega_i)} + h^{l_\gamma+1} |\gamma_i|_{\mathbb{H}^{l_\gamma+1}(\Omega_i)} \right). \right.
\end{aligned}$$

We now explain the consequences of Theorem 2 for some particular cases. In what follows, we assume that $l_\sigma = l_u = l_\gamma = k$ and the elliptic regularity holds for the dual problem.

- (C.1) *Gap δ of order $h^{7/4}$.* In this case $\aleph = 3/4$. Here $\gamma = 1/4$ and $\beta = 1/2$. Then, the order of convergence is $h^{k+5/8}$ for the variable σ_h , $h^{k+7/8}$ for the variable $\mathbf{u}_h$ and the projection of the error $e^{\mathbf{u}}$, and $h^{k+1/8}$ for the variable γ_h .
- (C.2) *Gap δ of order h^2 .* In this case $\aleph = 1$. Here $\gamma = 1/4$, which implies that we can take $\beta = 1/2$. This yields an order of convergence of $h^{k+3/4}$ for the variable σ_h , an optimal order of h^{k+1} for the variable $\mathbf{u}_h$, order $h^{k+1/4}$ for the variable γ_h , and order $h^{k+5/4}$ for the projection of the error $e^{\mathbf{u}}$.
- (C.3) *Matching interfaces (no gap).* In this case, $C_{\text{gap}} = 0$, and our result shows optimal order of convergence of h^{k+1} for the variables σ_h and $\mathbf{u}_h$, order $h^{k+1/2}$ for the variable γ_h , and order h^{k+2} for the projection of the error $e^{\mathbf{u}}$, as expected.

6 Numerical experiments

In this section, we present numerical experiments for different interface configurations in two dimensions. We study the convergence rates of the method under various gap sizes and exterior normal misalignment. Finally, by modifying our method to solve the Oseen equations [27], we are able to extend our results to the fluid-structure coupling using the incompressible Navier–Stokes equations, which we validate on the FSI1 benchmark of [28]. All stabilization parameters are set to $\tau \equiv 1$ unless stated otherwise.

For the convergence experiments, we are interested in the following error quantities:

$$\begin{aligned}
e(\sigma) &:= \sqrt{\sum_{\star \in \{\text{f}, \text{s}\}} \|\sigma_{\star} - \sigma_{\star, h}\|_{L^2(\Omega_{\star, h})}^2}, & e(\gamma) &:= \sqrt{\sum_{\star \in \{\text{f}, \text{s}\}} \|\gamma_{\star} - \gamma_{\star, h}\|_{L^2(\Omega_{\star, h})}^2}, \\
e(\mathbf{u}) &:= \sqrt{\sum_{\star \in \{\text{f}, \text{s}\}} \|\mathbf{u}_{\star} - \mathbf{u}_{\star, h}\|_{L^2(\Omega_{\star, h})}^2}, & \hat{e}(\mathbf{u}) &:= \sqrt{\sum_{\star \in \{\text{f}, \text{s}\}} \sum_{e \in \mathcal{E}_{\star, h}} h_e \|\hat{\mathbf{u}}_{\star} - \mathcal{P}_{\text{M}} \mathbf{u}_{\star, h}\|_{L^2(e)}^2}, \\
e(\nu) &:= \max_{\star \in \{\text{f}, \text{s}\}} \max_{e \in \mathcal{I}_h^{\star}} \max_{\mathbf{x} \in \mathfrak{G}_k(e)} \|\nu_h - \nu \circ \psi(\mathbf{x})\|,
\end{aligned}$$

where $\mathfrak{G}_k(e)$ is the set of $k+1$ Gauss nodes on the edge e .

In Sections 6.1 and 6.2, we study the convergence rates via the method of manufactured solutions with

$$\begin{aligned}
\mathbf{u}_{\text{f}}(x_1, x_2) &= (-\pi \sin(\pi x_1) \cos(\pi x_2), \pi \cos(\pi x_1) \sin(\pi x_2))^{\text{t}}, \\
-\frac{1}{2} \text{tr}(\sigma_{\text{f}}(x_1, x_2)) &= \exp(-(x_1 - 0.5)^2 - (x_2 - 0.5)^2), \\
\mathbf{u}_{\text{s}}(x_1, x_2) &= (x_1 \sin(x_2), x_2 \cos(x_1))^{\text{t}},
\end{aligned}$$

and the source terms are computed so that the above functions solve (1) with

$$\rho_{\text{f}} = \rho_{\text{s}} = 1, \quad \beta_{\text{f}}^2 = \beta_{\text{s}} = \alpha_{\text{f}}^2 = 1, \quad \alpha_{\text{s}} = 1, \quad \mu_{\text{f}} = 1,$$

and the Lamé parameters of the solid given by

$$\lambda_{\text{s}} = \frac{E_{\text{s}} \nu_{\text{s}}}{(1 + \nu_{\text{s}})(1 - 2\nu_{\text{s}})}, \quad \mu_{\text{s}} = \frac{E_{\text{s}}}{2(1 + \nu_{\text{s}})},$$

with Young's modulus $E = 1$ and Poisson ratio $\nu = 0.3333$.

6.1 Example 1: Validation of convergence rates with a flat interface

Here, we consider the following solid subdomain:

$$\Omega_{\text{s}} = (0, 1) \times (0, 0.5), \quad (\text{flat})$$

immersed in the domain $\Omega = (0, 1)^2$. We use the `flat` interface as a benchmark for the best performance of the method, as in this case, for $i \in \{\text{s}, \text{f}\}$, $\nu_i = \mathbf{n}_i = \mathbf{m}_i$ everywhere, thus removing other sources of error and isolating the effect of

the gap size. In the rest of this section, we will explore the convergence rates for symmetric and asymmetric gaps, as well as the performance of the method in the near-incompressible solid case with increasing values of λ_s .

The discrete subdomains are constructed by imposing a controlled gap δ symmetrically:

$$\Omega_h^s := (0, 1) \times (0, 0.5 - \delta), \quad \Omega_h^f := (0, 1) \times (0.5 + \delta, 1).$$

First, in view of the remarks after Theorem 2, we consider gaps of order h^2 and $h^{7/4}$. In Figures 2a and 2b, while some rates seem to be in pre-asymptotical regime, we observe a tendency towards optimal rates in the volume variables as well as $k + 2$ superconvergence of the trace variables are obtained. These rates are higher than those expected according to Theorem 2. In addition, we also consider the case of $\mathcal{O}(h)$ gap, although it does not satisfy the assumptions of Theorem 2. We observe in Figure 2c optimal convergence in all variables, losing superconvergence of the trace variable.

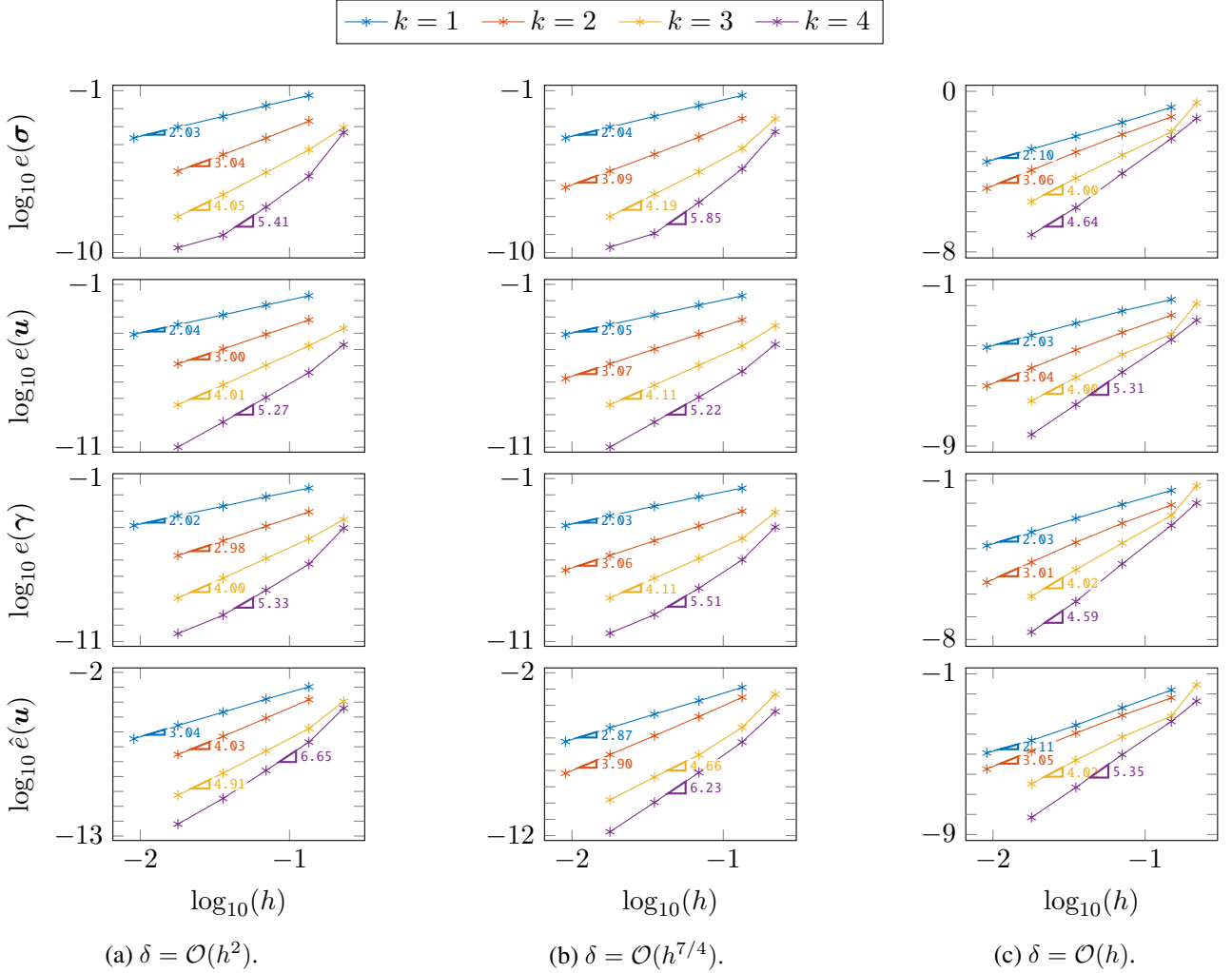


Figure 2: Convergence rates for the flat interface using symmetric gaps.

To show that the method is robust in the incompressible limit of the elasticity equations, we fix $\mu_s = 1$ and let λ_s vary. In order to avoid the right-hand side from increasing in magnitude due to the increasing values of λ_s , we consider a modified manufactured solution in which the solid displacement is divergence-free:

$$\mathbf{u}_s = (\sin x \sin y, \cos x \cos y)^\top.$$

Results can be found in Table 1. For every value of λ_s , the rates are optimal in the volume variables and superconvergent in the trace, and the errors in σ , $\mathbf{u}$ and γ change by less than 2% over the whole range.

Table 1: Convergence rates for the flat interface with $k = 2$ and $\delta = \mathcal{O}(h^2)$ for different values of λ_s .

λ_s	h	$e(\sigma)$	eoc	$e(u)$	eoc	$e(\gamma)$	eoc	$\hat{e}(u)$	eoc
10^3	1.71e-01	4.501650e-03	—	1.342032e-03	—	1.850258e-03	—	2.409474e-04	—
	8.88e-02	4.988216e-04	3.34	1.773999e-04	3.08	2.365802e-04	3.13	1.325114e-05	4.41
	4.38e-02	5.631196e-05	3.09	2.165146e-05	2.98	3.025328e-05	2.91	7.672233e-07	4.03
	2.19e-02	6.555108e-06	3.10	2.582534e-06	3.07	3.660654e-06	3.05	4.600367e-08	4.06
	1.13e-02	8.140251e-07	3.16	3.224981e-07	3.15	4.646476e-07	3.13	2.818135e-09	4.23
10^6	1.71e-01	4.502417e-03	—	1.342030e-03	—	1.850250e-03	—	2.408677e-04	—
	8.88e-02	4.988561e-04	3.35	1.773999e-04	3.08	2.365799e-04	3.13	1.324852e-05	4.41
	4.38e-02	5.631316e-05	3.09	2.165146e-05	2.98	3.025327e-05	2.91	7.671434e-07	4.03
	2.19e-02	6.555148e-06	3.10	2.582534e-06	3.07	3.660653e-06	3.05	4.600196e-08	4.06
	1.13e-02	8.140266e-07	3.16	3.224981e-07	3.15	4.646476e-07	3.13	2.817279e-09	4.23
10^8	1.71e-01	4.502418e-03	—	1.342030e-03	—	1.850250e-03	—	2.408676e-04	—
	8.88e-02	4.988562e-04	3.35	1.773999e-04	3.08	2.365799e-04	3.13	1.324852e-05	4.41
	4.38e-02	5.631316e-05	3.09	2.165146e-05	2.98	3.025327e-05	2.91	7.671433e-07	4.03
	2.19e-02	6.555148e-06	3.10	2.582534e-06	3.07	3.660653e-06	3.05	4.600148e-08	4.06
	1.13e-02	8.140266e-07	3.16	3.224981e-07	3.15	4.646476e-07	3.13	2.818252e-09	4.23

6.2 Example 2: Validation of convergence rates with a curved interface

We consider the following solid subdomain embedded in $\Omega = (-1, 1)^2$:

$$\Omega_s = \{(x_1, x_2) \in \Omega : (2(x_1 + 0.5))^2 + 2x_2^2 - x_1 - 0.5)^2 - (x_1 + 0.5)^2 - x_2^2 + 0.1 < 0\}. \quad (\text{kidney})$$

For constructing each mesh, we utilize two alternative strategies: linear interpolation and active meshes with respect to a background grid; see Figure 3. In the first, we construct contiguous submeshes by linearly interpolating the curved interface and assign opposite elements on each interpolating edge to each subdomain. In this way, we safely land in the case of gap of order h^2 and converging normals in L^∞ -norm, at the cost of having to extend source terms beyond the respective physical domains; see Table 2a. On the other hand, the second strategy consists in utilizing a structured background and defining the submeshes as the union of all elements that are *completely contained* in the physical domain, thus avoiding the extension problem on data while offering the flexibility of working with a domain immersed in a fixed background mesh. However, in the latter approach, the assumptions of Theorem 2 are not satisfied. In fact, this introduces the pathology of non-converging normal vectors, as well as only ensuring a gap of order h . The results are presented in Table 2b.

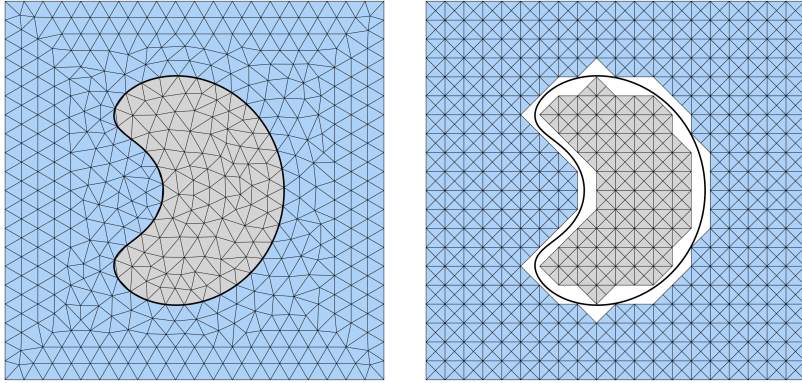


Figure 3: Mesh configurations for the kidney interface. The left configuration corresponds to the linear interpolation approach, and the right one to the embedding in a background grid.

For the linearly interpolated interfaces, we observe optimal rates in the volume unknowns as well as superconvergence of the trace unknown, whereas the embedded background grids present erratic rates that seem to converge to $k + 1$, without superconvergence. This behaviour is consistent with Theorem 2: the normal deviation decreases linearly in h for the interpolated interface ($\beta = 1$), whereas it stagnates around 0.75 for the background grid (assumption (A.4) is violated). Nonetheless, even in the background grid case, the error in all variables decreases monotonically when decreasing the mesh size (except for $k = 1$) or increasing the polynomial degree. In other words, even though it is not possible to theoretically guarantee convergence of the method, numerically we still observe convergence.

Table 2: Convergence rates for the kidney interface.

(a) Linearly interpolated.

k	h	$e(\nu)$	eoc	$e(\sigma)$	eoc	$e(u)$	eoc	$e(\gamma)$	eoc	$\hat{e}(u)$	eoc
1	2.94e-01	5.39e-01	—	5.98e-01	—	2.30e-01	—	2.68e-01	—	2.14e-02	—
	1.49e-01	2.91e-01	0.91	1.50e-01	2.04	5.62e-02	2.08	7.35e-02	1.91	2.80e-03	2.99
	7.44e-02	1.47e-01	0.98	3.91e-02	1.92	1.44e-02	1.96	2.03e-02	1.85	3.86e-04	2.84
	3.71e-02	7.58e-02	0.95	9.98e-03	1.96	3.62e-03	1.98	5.36e-03	1.91	5.11e-05	2.90
2	2.94e-01	6.74e-01	—	4.21e-02	—	1.52e-02	—	1.86e-02	—	1.36e-03	—
	1.49e-01	3.82e-01	0.84	4.86e-03	3.18	1.84e-03	3.12	2.48e-03	2.97	6.68e-05	4.44
	7.44e-02	1.99e-01	0.94	6.16e-04	2.96	2.35e-04	2.95	3.30e-04	2.89	4.43e-06	3.89
	3.71e-02	1.02e-01	0.96	7.75e-05	2.98	2.96e-05	2.97	4.29e-05	2.93	2.89e-07	3.92
3	2.94e-01	7.44e-01	—	2.76e-03	—	9.30e-04	—	1.09e-03	—	9.74e-05	—
	1.49e-01	4.18e-01	0.85	1.43e-04	4.36	5.16e-05	4.26	6.09e-05	4.25	1.84e-06	5.85
	7.44e-02	2.21e-01	0.91	8.51e-06	4.05	3.25e-06	3.97	3.70e-06	4.02	5.95e-08	4.92
	3.71e-02	1.14e-01	0.96	5.17e-07	4.02	2.02e-07	3.99	2.30e-07	4.00	1.92e-09	4.93
4	2.94e-01	7.79e-01	—	1.17e-04	—	3.88e-05	—	4.47e-05	—	7.06e-06	—
	1.49e-01	4.36e-01	0.85	2.91e-06	5.44	1.08e-06	5.27	1.28e-06	5.24	3.05e-08	8.02
	7.44e-02	2.33e-01	0.90	8.88e-08	5.01	3.42e-08	4.96	4.07e-08	4.95	5.20e-10	5.84
	3.71e-02	1.20e-01	0.96	2.80e-09	4.97	1.09e-09	4.96	1.31e-09	4.93	8.06e-12	5.99

(b) Embedded in a background grid.

k	h	$e(\nu)$	eoc	$e(\sigma)$	eoc	$e(u)$	eoc	$e(\gamma)$	eoc	$\hat{e}(u)$	eoc
1	1.00e-01	8.74e-01	—	5.00e-01	—	4.73e-02	—	1.41e-01	—	7.29e-02	—
	5.00e-02	6.88e-01	0.35	2.31e-01	1.11	1.38e-02	1.78	6.10e-02	1.20	2.34e-02	1.64
	2.50e-02	7.73e-01	-0.17	7.41e-02	1.64	2.94e-03	2.23	2.32e-02	1.39	4.37e-03	2.42
2	1.00e-01	8.90e-01	—	1.52e-01	—	7.32e-03	—	3.79e-02	—	1.55e-02	—
	5.00e-02	6.89e-01	0.37	9.36e-03	4.02	4.80e-04	3.93	2.52e-03	3.91	9.53e-04	4.03
	2.50e-02	7.79e-01	-0.18	1.34e-03	2.80	6.46e-05	2.89	3.02e-04	3.06	1.28e-04	2.90
3	1.00e-01	8.97e-01	—	3.24e-01	—	1.20e-02	—	5.21e-02	—	2.51e-02	—
	5.00e-02	6.90e-01	0.38	1.13e-03	8.16	3.58e-05	8.39	2.69e-04	7.60	7.47e-05	8.39
	2.50e-02	7.81e-01	-0.18	1.06e-04	3.42	2.93e-06	3.61	2.51e-05	3.42	5.94e-06	3.65
4	1.00e-01	9.01e-01	—	7.41e-04	—	2.99e-05	—	1.53e-04	—	6.36e-05	—
	5.00e-02	6.90e-01	0.39	5.44e-05	3.77	1.95e-06	3.94	7.88e-06	4.28	3.97e-06	4.00
	2.50e-02	7.83e-01	-0.18	8.21e-07	6.05	4.81e-08	5.34	2.11e-07	5.22	9.75e-08	5.35

6.3 Example 3: Benchmark experiment of fluid-structure interaction

We solve the steady-state obtained from the FSI1 benchmark [28].

In order to obtain the steady-state geometry, we solve FSI1 using an ALE formulation [24] with $\mathbf{P}_2$ Taylor-Hood elements for the fluid and standard H^1 -conforming $\mathbf{P}_2$ elements for the solid. The flag is then held fixed in its deformed configuration, with the interface described by spline interpolation of the ALE results, and we solve the coupled Navier–Stokes/elasticity problem on it with the inflow of the benchmark ($\text{Re} = 20$), linearizing the convective term by a Picard iteration in which each step is an Oseen problem [27]. The convective velocity $\mathbf{u}^{\text{OS}}$ for each of these steps is obtained by applying the divergence-free postprocessing detailed in [5] and the stability parameter is taken as

$$\tau^{\text{OS}} := \mu_f \tau + \frac{1}{2} \max_{e \in \mathcal{E}_h^f} \max_{x \in e} |\mathbf{u}^{\text{OS}} \cdot \mathbf{n}|.$$

As in Section 6.2, we compare two mesh strategies: the fluid mesh either interpolates the interface or is carved from a background grid aligned with the flag, while the flag itself is always meshed by interpolation; see Figure 4. In both cases the mesh is refined only near the cylinder and the flag, where h denotes the mesh size reported below, and coarsens in the rest of the channel.

All the results shown below are obtained using our method with $k = 2$. Table 3 reports the drag and lift on the cylinder and the flag, computed as volume integrals, and the displacement of the tip point A , together with the relative deviation from the reference values of [28]. On the interpolating meshes, the drag and lift are within 0.6% of the reference on every mesh, and $u_y(A)$ is within 1% for $h \leq 5 \cdot 10^{-3}$. The background grids are comparable, with deviations decreasing monotonically with h to 0.1% in the drag and 0.8% in $u_y(A)$. The displacement $u_x(A)$, which is 36 times smaller than $u_y(A)$, is underestimated by 5–9% on all meshes. This discrepancy is possibly explained by the fact that [28] assumes the solid is hyperelastic as opposed to linearly elastic as it is here. In every case the Picard iteration converges in 10 to

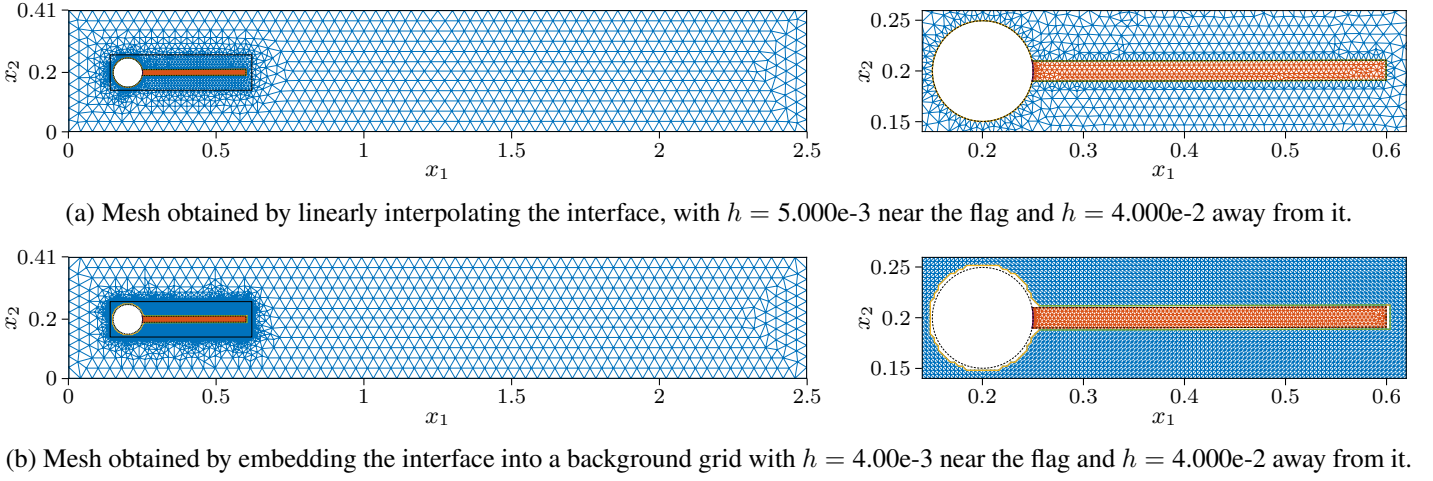


Figure 4: Meshes utilized in the FSI1 benchmark experiment.

11 steps, independently of h and of the mesh strategy. Figure 5 shows the fluid velocity magnitude and the displacement magnitude of the flag on the finest mesh of each strategy. The two solutions are visually indistinguishable.

Table 3: Results for the FSI1 benchmark using interpolating and background meshes, including the number of Picard iterations.

mesh	h	it.	drag	lift	$u_x(A) \cdot 10^5$	$u_y(A) \cdot 10^4$
reference			14.295	0.7638	2.270	8.209
interpolating	1.000e-2	10	14.291 (−0.0%)	0.7609 (−0.4%)	2.066 (−9.0%)	5.497 (−33.0%)
	5.000e-3	10	14.292 (−0.0%)	0.7596 (−0.6%)	2.158 (−4.9%)	8.198 (−0.1%)
	2.500e-3	11	14.293 (−0.0%)	0.7615 (−0.3%)	2.164 (−4.7%)	8.127 (−1.0%)
background	4.000e-3	11	14.461 (+1.2%)	0.7695 (+0.8%)	2.100 (−7.5%)	8.437 (+2.8%)
	2.857e-3	11	14.333 (+0.3%)	0.7664 (+0.3%)	2.115 (−6.8%)	8.367 (+1.9%)
	2.222e-3	11	14.327 (+0.2%)	0.7675 (+0.5%)	2.119 (−6.7%)	8.348 (+1.7%)
	1.539e-3	11	14.313 (+0.1%)	0.7664 (+0.3%)	2.125 (−6.4%)	8.274 (+0.8%)

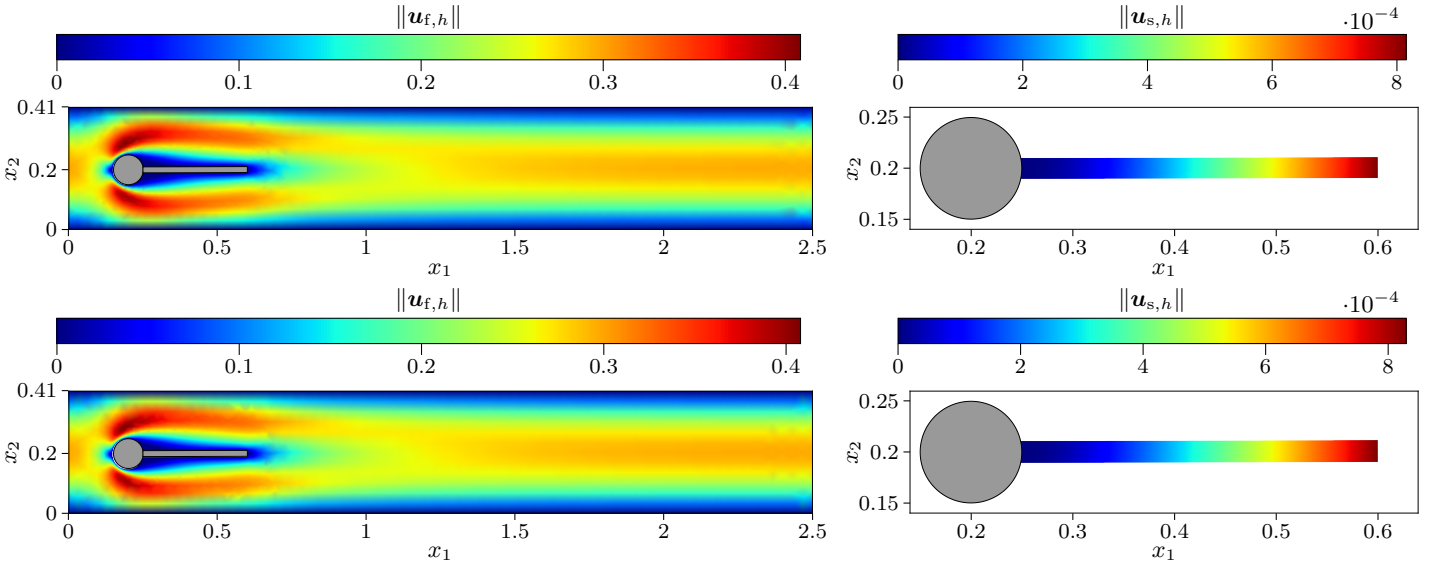


Figure 5: Fluid velocity and solid displacement magnitudes using an interpolating mesh ($h = 2.5000\text{e-}3$) and a mesh obtained by embedding into a background grid ($h = 1.539\text{e-}3$).

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