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SOLUCION NUMERICA DEL PROBLEMA DE CONFORMADO ELECTROMAGNETICO

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Resumen

En esta tesis se analizan algunos problemas de corrientes inducidas en dominios acotados con el fin de resolver el problema de conformado electromagnético tridimensional sin alguna simetría en particular. Se estudiará especialmente el caso en que las fuentes de corriente están proporcionadas en términos de intensidades y/o voltajes impuestos en partes de la frontera del dominio.

Inicialmente se demuestra la equivalencia entre dos formulaciones para el problema de corrientes inducidas en régimen armónico. La primera, es una formulación en términos del campo magnético en el conductor y un potencial escalar magnético en el dieléctrico. La segunda es una formulación en términos del campo magnético en todo el dominio y un multiplicador de Lagrange en el dieléctrico. La equivalencia se muestra a nivel discreto. Se muestran resultados numéricos que resaltan las ventajas e inconvenientes de cada una de las formulaciones.

A continuación, se analiza un método numérico para la formulación en términos del campo magnético del problema transitorio de corrientes inducidas con intensidades de corriente como dato. Se demuestra que la formulación débil tiene una única solución que satisface en cierto sentido el problema de partida. Se propone un espacio de elementos finitos para la discretización espacial basada en elementos de aristas de Nédélec. A continuación, se introduce un esquema de Euler implícito para la discretización en tiempo. Se demuestran estimaciones óptimas de error para los esquemas semidiscreto y totalmente discreto. Además, se introduce un potencial escalar magnético en el dieléctrico para imponer la condición de rotacional nulo, lo cual conduce a un ahorro computacional importante. Finalmente, el método se aplica para resolver dos problemas: un test con solución analítica conocida y una aplicación al conformado electromagnético.

La implementación de la formulación anterior, campo magnético/potencial escalar magnético, requiere la construcción de las llamadas *superficies de corte* en el dominio dieléctrico, cuando éste no es simplemente conexo. La construcción de estas superficies puede ser muy compleja en la práctica. Por ello, a continuación se aborda el problema de corrientes inducidas transitorio en términos de la primitiva del campo eléctrico. En este caso se estudia el problema con fuentes no locales en términos de intensidades y voltajes. En el dieléctrico es necesario introducir un multiplicador de Lagrange y el análisis conduce al estudio de un problema mixto parabólico degenerado. Se demuestran resultados de existencia y unicidad de solución así como resultados de convergencia para el método numérico propuesto. El método numérico es validado con un ejemplo con solución analítica conocida; se aplica además para calcular las corrientes inducidas en una topología donde la introducción de la superficie de corte no es trivial.

Finalmente, se aborda un problema transitorio de corrientes inducidas en un dominio acotado donde el dominio conductor cambia con el tiempo. En este caso, para simplificar el análisis, se plantea el problema con condiciones de contorno esenciales y una fuente volúmica conocida. Se propone una formulación en términos del campo magnético para la cual se demuestra que existe solución. Se propone también una técnica de penalización para imponer la restricción de rotacional nulo en el dieléctrico. Se demuestra que esta estrategia es eficaz para el problema con conductor fijo, tanto para el problema continuo como para su discretización con elementos finitos de Nédélec y un esquema de Euler implícito. Así mismo se demuestran estimaciones óptimas del error para este esquema numérico uniformes respecto al parámetro de penalización. Se presentan ensayos numéricos que confirman la convergencia del método de penalización propuesto. El método numérico con penalización se aplica a un caso donde el conductor se desplaza a lo largo del tiempo. Los resultados ponen de manifiesto que cabría esperar un orden de convergencia similar al caso de conductor fijo.

Contents

Introducción

0.1 Motivación

El conformado electromagnético es un proceso que consiste en la deformación de piezas metálicas sometidas a la acción de campos electromagnéticos de gran intensidad. Este proceso, conocido en inglés con las siglas EMF, *Electromagnetic Metal Forming*, se desarrolló alrededor de 1960 y se caracteriza porque la deformación de las piezas se produce en respuesta a la fuerza de Lorentz producida por una bobina próxima alimentada por una corriente eléctrica transitoria.

El proceso se inicia con un pulso de corriente muy intenso (ver Figura 1) que se hace pasar a través de una bobina por medio de la descarga rápida de un capacitor de alto voltaje. Esto crea un campo electromagnético variable muy potente que genera, a través de la inducción magnética, una corriente en la pieza metálica que se quiere deformar. De acuerdo a la ley de Lenz, los campos magnéticos dentro de la pieza y de la bobina se repelen fuertemente, de manera que se supera fácilmente la resistencia de la pieza causando una deformación permanente.

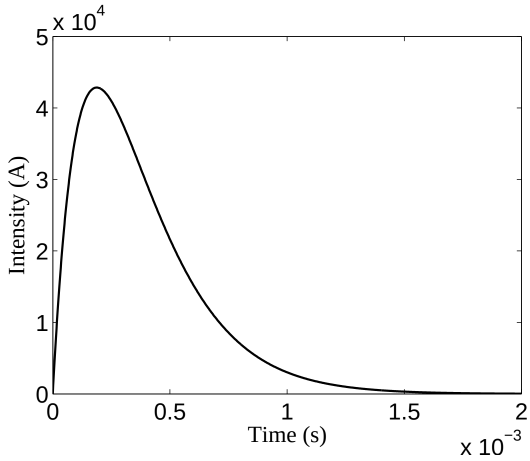

Figure 1: Intensidad de corriente (A) vs. tiempo (s) en un ejemplo típico de conformado electromagnético.

El proceso de conformado electromagnético ocurre de manera extremadamente rápida (en decenas de microsegundos) y, dada la magnitud de las fuerzas, partes de la pieza a deformar alcanzan aceleraciones importantes. Esta técnica forma parte de los denominados procesos de conformado a alta velocidad y es ampliamente utilizada en la industria para el tratamiento de materiales ligeros, como el aluminio o el magnesio, que responden peor a métodos tradicionales, frente a los cuales presenta numerosas ventajas como la reducción de arrugas, la alta repetibilidad de las piezas o la capacidad de reproducir pequeños micro-detalles. En el conformado electromagnético la pieza metálica puede ser deformada sin entrar en contacto con herramienta alguna, por ello este proceso es muy usado para encoger o expandir tubos cilíndricos, pero también para dar forma a hojas metálicas al hacerlas impactar sobre un molde a alta velocidad (Figura 2).

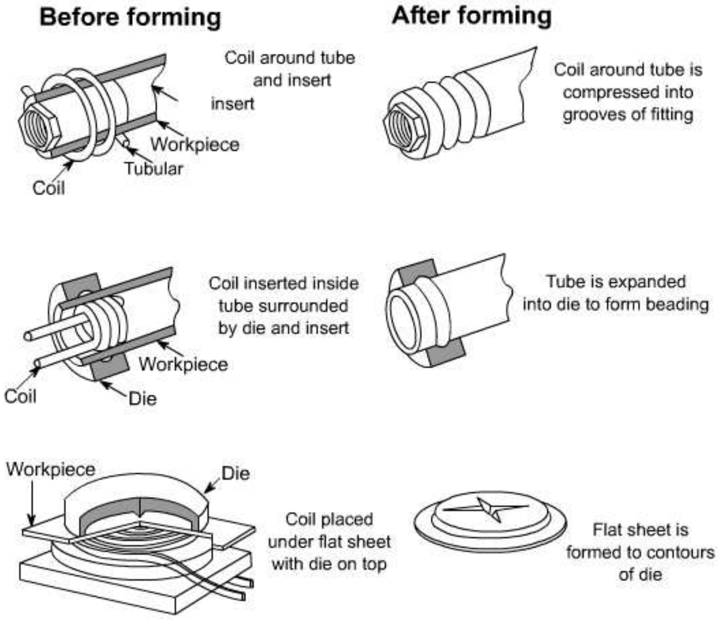

Figure 2: Ejemplos de aplicaciones de conformado electromagnético: expansión/compresión de tubos y estampado de láminas. Foto adquirida en http://nptel.iitm.ac.in/.

Figure 3: Ejemplos de productos obtenidos por conformado electromagnético. Fotos adquiridas en http://www.iap.com/magnetic-compaction.html y en http://www.amire.net/labeinweb.

El conformado electromagnético es una tecnología en desarrollo que necesita de la simulación numérica para diseñar sistemas de conformado eficientes. Se trata de un proceso multifísico, de modo que su modelado matemático debe tener en cuenta fenómenos electromagnéticos, térmicos y estructurales, todos ellos acoplados entre sí. Así, la simulación numérica de este proceso requiere resolver un modelo electromagnético basado en las ecuaciones de Maxwell para un cuerpo en movimiento, un modelo estructural no lineal con leyes constitutivas termo-viscoplásticas y un modelo térmico. En particular, destaca el acoplamiento magneto-mecánico debido a que la fuerza de Lorentz calculada en el modelo electromagnético es la fuerza volúmica que deforma la pieza y la deformación de ésta, modifica el dominio del submodelo electromagnético con el tiempo.

Una buena referencia para este proceso es el artículo [30] de El-Azab et al. Allí puede encontrarse una descripción del método de conformado electromagnético y sus aplicaciones, así como el marco matemático que debe tenerse en cuenta a la hora de desarrollar métodos numéricos para la simulación numérica del proceso; ver también [53, 54] para un estudio detallado del modelo magneto-mecánico en piezas que se derforman. Existe además una extensa lista de artículos que estudian distintos modelos para simular el conformado electromagnético. En general, estos trabajos se ocupan del modelado y/o resolución numérica del problema magneto-mecánico mediante métodos de elementos finitos [10, 35, 36, 50, 55, 61, 62, 64, 63, 66, 69], centrándose en muchos casos en sistemas de conformado con simetría cilíndrica [35, 62, 64]. Si además la pieza a deformar es muy delgada el modelo electromagnético podría abordarse mediante modelos de placas (ver, por ejemplo, [40]). Sin embargo, el análisis matemático y numérico de los modelos que surgen en este campo da lugar a problemas matemáticos que no han sido tratados en su mayor parte. Un objetivo de esta tesis es hacer una primera aproximación del análisis matemático y numérico del modelo electromagnético tridimensional que surge en el proceso de conformado electromagnético.

El modelo electromagnético está basado en el modelo de *corrientes inducidas o eddy currents*, que se obtiene a partir de las ecuaciones de Maxwell despreciando las corrientes de desplazamiento eléctrico en la ley de Ampére-Maxwell (ver seccion 0.2). Hay que señalar que el análisis de problemas tridimensionales de corrientes inducidas está fuertemente determinado por la formulación elegida (en términos de campos o potenciales) y por la forma de imponer las fuentes (fuentes volúmicas o fuentes dadas por un circuito externo, a saber, intensidades de corriente y/o caídas de voltaje). En [7] puede encontrarse, un estudio muy completo del análisis matemático y numérico de dichos problemas en régimen armónico. Dado que el proceso de conformado electromagnético se inicia con un pulso de corriente eléctrica transitoria, el modelo que mejor se ajusta a esta situación, es el modelo de corrientes inducidas en régimen transitorio con fuentes dadas por un circuito externo. En la literatura se encuentran bastantes trabajos que abordan el análisis numérico del problema transitorio de corrientes inducidas [1, 2, 26, 44, 45, 46, 48, 49, 72]. Sin embargo, todos ellos estudian el problema transitorio con fuente de corriente volúmica y, en caso de dominios acotados, con condiciones de contorno naturales y/o esenciales dependiendo de la variable principal. Por tal motivo, en los Capítulos 2 y 3 se estudia el problema transitorio de corrientes inducidas en un dominio acotado cuando la fuente de corriente se proporciona en términos de las intensidades de corriente y/o caídas de potencial. Así, el modelo electromagnético estudiado permite considerar el acoplamiento entre el circuito de descarga eléctrica y el modelo de corrientes inducidas. El análisis matemático y numérico desarrollado en dichos capítulos extiende los resultados ya obtenidos en régimen armónico en [8, 19]. Cabe señalar además que el Capítulo 1 estará también motivado por el problema de corrientes inducidas con datos intensidades, aportando contribuciones a su estudio en régimen armónico y considerando topologías generales.

Por otra parte, dado que en el proceso de conformado electromagnético la pieza a deformar puede desplazarse, el modelo electromagnético transitorio de corrientes inducidas debería tener en cuenta dicho movimiento a lo largo del tiempo. En particular, el modelo de corrientes inducidas con conductores móviles tiene un gran interés desde el punto de vista de las aplicaciones ya que surge en la simulación de otros dispositivos como máquinas eléctricas rotatorias [26], procesos de levitación magnética [47] o sistemas de inducción con inductores móviles [29]. Sin embargo, son pocos los trabajos de la literatura que se ocupan del análisis matemático y numérico de este problema tanto en dominios bidimensionales como tridimensionales. Así, en [26, 57, 58] se analiza el problema transitorio de corrientes inducidas que se obtiene del modelamiento de motores eléctricos. Aunque el modelo electromagnético estudiado en estos trabajos tiene en cuenta el movimiento del rotor, la geometría ocupada por el dominio en movimiento siempre es la misma, lo cual no ocurre en un proceso de conformado. Además, el movimiento de rotación es esencial en el análisis del problema. Por otro lado, recientemente en [16] se estudia el problema transitorio de corrientes inducidas con movimiento de la pieza en simetría axial en el marco de conformado electromagnético. En consecuencia, el estudio del problema en dominios genuinamente tridimensionales y con movimientos más generales de las piezas conductoras ofrece una línea de investigación abierta. El Capítulo 4 de la tesis pretende aportar contribuciones en esta línea. En particular, se estudia el problema transitorio de corrientes inducidas con conductores móviles en un dominio acotado. Como un paso inicial, se considera una fuente de corriente volúmica conocida soportada en un dominio fijo y con condiciones de contorno homogéneas.

0.2 Modelo de corrientes inducidas en un dominio acotado

Las ecuaciones de Maxwell son las ecuaciones que describen los fenómenos electromagnéticos. Reciben su nombre de James Clerk Maxwell quién recopiló la ley de Gauss para electricidad, la ley de Gauss para magnetismo, la ley de Faraday y la ley de Ampère.

Las ecuaciones de Maxwell en su forma diferencial son las siguientes:

$$\begin{aligned}
\partial_t \boldsymbol{D} - \mathbf{curl}\, \boldsymbol{H} &= -\boldsymbol{J} && \text{ley de Ampère-Maxwell,}\\
\partial_t \boldsymbol{B} + \mathbf{curl}\, \boldsymbol{E} &= \mathbf{0} && \text{ley de Faraday,}\\
\operatorname{div} \boldsymbol{D} &= \rho && \text{ley de Gauss,}\\
\operatorname{div} \boldsymbol{B} &= 0 && \text{ley de Gauss del magnetismo,}
\end{aligned}$$

donde

- $\boldsymbol{E}$ es la intensidad de campo eléctrico,
- $\boldsymbol{D}$ es el desplazamiento eléctrico,
- $\boldsymbol{H}$ es la intensidad de campo magnético,

- $\boldsymbol{B}$ es la inducción magnética,
- $\boldsymbol{J}$ es la densidad de corriente y
- ρ es la densidad de carga libre.

Todos los campos que aparecen en estas ecuaciones son funciones vectoriales que dependen de la variable espacial $\boldsymbol{x} \in \mathbb{R}^3$ y del tiempo t.

La ley de Ampère-Maxwell coincide con la ley de Ampère salvo por el término adicional $\partial_t \boldsymbol{D}$ introducido por Maxwell y conocido en la literatura como *corrientes de desplazamiento*.

La ley de Gauss y la ley de Gauss del magnetismo son consecuencia de la ley de Ampère-Maxwell y la ley de Faraday, bajo el supuesto de conservación de la carga. Formalmente, esto se demuestra tomando divergencia en la ley de Ampère-Maxwell y Faraday; así se obtiene

$$\operatorname{div}(\partial_t \boldsymbol{D}) = -\operatorname{div} \boldsymbol{J} \quad \text{y} \quad \operatorname{div}(\partial_t \boldsymbol{B}) = 0.$$

Por otra parte, si se conserva la carga, ρ y $\boldsymbol{J}$ estan relacionados mediante la expresión

$$\operatorname{div} \boldsymbol{J} + \partial_t \rho = 0,$$

y por lo tanto

$$\partial_t \operatorname{div} \boldsymbol{B} = \partial_t (\operatorname{div} \boldsymbol{D} - \rho) = 0.$$

Así, si la ley de Gauss y la ley de Gauss del magnetismo se cumplen en un tiempo inicial, se cumplen para todo instante de tiempo.

Los distintos campos $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$ y $\boldsymbol{H}$ están relacionados por medio de las leyes constitutivas, las cuales dependen de los materiales que forman el dominio de estudio. En la práctica, el dominio está compuesto por distintos materiales, es decir, es no homogéneo. Si las propiedades del material no dependen de la dirección del campo se dice que éste tiene un comportamiento isotrópico. Si además las propiedades físicas no dependen del campo aplicado, el material es lineal. En esta tesis se considerarán únicamente materiales lineales e isotrópicos; por tanto, las leyes constitutivas son

$$\begin{aligned} \boldsymbol{B} &= \mu \boldsymbol{H}, \\ \boldsymbol{D} &= \epsilon \boldsymbol{E}, \end{aligned}$$

donde ϵ es la *permitividad eléctrica* y μ es la *permeabilidad magnética*; ϵ y μ son coeficientes que solo dependen de los materiales.

El sistema se completa con la ley de Ohm que relaciona la densidad de corriente en el conductor con el campo eléctrico:

$$\boldsymbol{J} = \sigma \boldsymbol{E},$$

siendo σ es la *conductividad eléctrica*; σ es un coeficiente positivo en los materiales conductores y nulo en los no conductores o dieléctricos.

Para resolver estas ecuaciones deben considerarse así mismo términos fuente y condiciones iniciales. Respecto a los primeros, una forma usual de imponerlos es a través de una densidad de corriente conocida $\boldsymbol{J}_{\mathrm{S}}$ en un subdominio fijo. En tal caso, en lugar de la ley de Ohm se utiliza la así llamada *Ley de Ohm generalizada*

$$\boldsymbol{J} \;=\; \sigma\boldsymbol{E} + \boldsymbol{J}_{\mathrm{S}},$$

con $\operatorname{div}\boldsymbol{J}_{\mathrm{S}} = 0$.

Si las fuentes de corriente se imponen mediante el acoplamiento de las ecuaciones de Maxwell con un circuito externo, $\boldsymbol{J}_{\mathrm{S}} = \mathbf{0}$ y, en tal caso, los términos fuentes serán definidos mediante condiciones de contorno.

Utilizando las leyes constitutivas anteriores y la ley de Ohm generalizada, las ecuaciones de Maxwell pueden expresarse únicamente en términos de los campos de principal interés físico, $\boldsymbol{E}$ y $\boldsymbol{H}$, obteniéndose el sistema:

$$\begin{aligned} \partial_t(\epsilon\boldsymbol{E}) - \mathbf{curl}\,\boldsymbol{H} \;&=\; -\sigma\boldsymbol{E} - \boldsymbol{J}_{\mathrm{S}}, && (1)\\ \partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} \;&=\; \mathbf{0}, && (2)\\ \operatorname{div}(\epsilon\boldsymbol{E}) \;&=\; \rho, && (3)\\ \operatorname{div}(\mu\boldsymbol{H}) \;&=\; 0. && (4) \end{aligned}$$

El modelo de corrientes inducidas resulta de despreciar el término del desplazamiento eléctrico en la ley de Ampère-Maxwell (1). Eliminar este término es aceptable cuando la magnitud de las corrientes de desplazamiento es despreciable respecto al resto de términos de (1) (*hipótesis cuasi-estática*); ver por ejemplo, [22, Capítulo 15], o [7, Capítulo 1], para un análisis más detallado de esta hipótesis en el caso armónico. En tal caso, únicamente se considerará la ecuación (3) en el dieléctrico si se pretende determinar el campo eléctrico en los dominios dieléctricos.

En esta tesis se propondrán métodos de elementos finitos para resolver el modelo de corrientes inducidas, tratando así de determinar los campos $\boldsymbol{E}$ y $\boldsymbol{H}$ que verifiquen:

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{H} \;&=\; \sigma\boldsymbol{E} + \boldsymbol{J}_{\mathrm{S}}\,,\\ \partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} \;&=\; \mathbf{0}\,,\\ \operatorname{div}(\mu\boldsymbol{H}) \;&=\; 0. \end{aligned}$$

Dado que estas ecuaciones están definidas en $\mathbb{R}^3$, para resolver el problema mediante un método de elementos finitos es necesario restringir las ecuaciones a un dominio acotado Ω e imponer condiciones sobre su frontera de tal forma que el problema tenga una única solución.

En los Capítulos 1, 2 y 3, las fuentes de corriente se proporcionarán a través de un circuito eléctrico externo, mediante intensidades de corriente y/o voltajes impuestos en partes de la frontera por lo que $\boldsymbol{J}_{\mathrm{S}} = \mathbf{0}$. Para ello, es necesario introducir alguna notación que será utilizada en dichos capítulos y que permitirá explicar con mayor claridad la organización de la tesis.

Sea Ω un dominio acotado simplemente conexo, formado por dos partes, Ω_{C} y Ω_{D}, siendo Ω_{C} el dominio conductor y Ω_{D} el dominio dieléctrico. La frontera de Ω se supone conexa, Lipschitz continua y se descompone de la forma $\partial\Omega = \Gamma_{\mathrm{C}} \cup \Gamma_{\mathrm{D}}$, donde Γ_{C} y Γ_{D} son las fronteras exteriores de Ω_{C} y Ω_{D}, respectivamente. Sea $\boldsymbol{n}$ el vector normal unitario dirigido hacia el exterior de $\partial\Omega$.

Se supone que las componentes conexas del conductor $\Omega_{\mathrm{C}}^1, \dots, \Omega_{\mathrm{C}}^M$ son disjuntas y de dos tipos: *inductores* Ω_{C}^n, $n = 1, ..., N$, y *piezas inducidas* (*workpieces*) Ω_{C}^n, $n = N+1, ..., M$; los primeros intersecan la frontera de Ω y las segundos están totalmente contenidos en Ω (ver Figura 4). Además, se supone que la frontera de cada inductor tiene dos componentes conexas disjuntas $\Gamma_{\boldsymbol{J}}^n$ y $\Gamma_{\boldsymbol{E}}^n$; se denota $\Gamma_{\boldsymbol{E}} := \Gamma_{\boldsymbol{E}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{E}}^N$ y $\Gamma_{\boldsymbol{J}} := \Gamma_{\boldsymbol{J}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{J}}^N$. En un ejemplo de conformado electromagnético, la bobina sería un inductor y el objeto a deformar una pieza inducida.

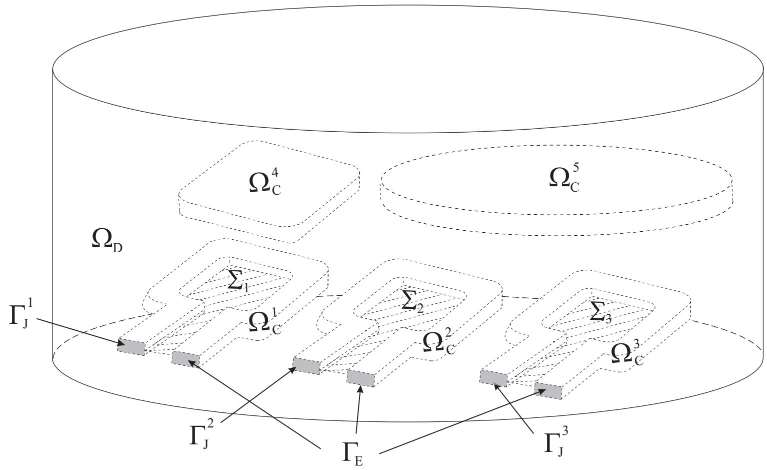

Figure 4: Esquema del dominio tridimensional Ω con fuentes intensidades de corriente y/o voltajes. Dominios conductores fijos.

Si se conocen las intensidades de corriente I_n a la entrada de cada inductor Ω_{C}^n, $n = 1, \dots, N$, el problema de corrientes inducidas se reduce a

$$\mathbf{curl}\,\boldsymbol{H} = \sigma \boldsymbol{E} \quad \text{en } [0,T] \times \Omega\,, \tag{5}$$

$$\partial_t(\mu \boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{en } [0,T] \times \Omega\,, \tag{6}$$

$$\operatorname{div}(\mu \boldsymbol{H}) = 0 \quad \text{en } [0,T] \times \Omega\,, \tag{7}$$

$$\int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\,\boldsymbol{H}(t) \cdot \boldsymbol{n} = I_n(t), \quad n = 1, \dots, N, \quad t \in [0,T]\,, \tag{8}$$

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{en } [0,T] \times \Gamma_{\boldsymbol{E}} \cup \Gamma_{\boldsymbol{J}}\,, \tag{9}$$

$$\mu \boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{en } [0,T] \times \partial\Omega\,, \tag{10}$$

con la condición inicial $\boldsymbol{H}(0) = \boldsymbol{H}_0$, donde $\boldsymbol{H}_0$ es el campo magnético en el instante inicial (T denota el tiempo final). El término I_n en (8) representa la intensidad de corriente a través de la superficie $\Gamma_{\boldsymbol{J}}^n$. Las condiciones (9)–(10) han sido introducidas en [22] en un contexto más general. La primera implica suponer que la corriente eléctrica es perpendicular a las superficies de entrada y salida de corriente, mientras que la última significa que el campo magnético es tangencial en la frontera. Estas condiciones son aproximaciones admisibles si los conductores que llevan la corriente eléctrica son suficientemente largos cerca de la frontera exterior del dominio y ortogonales a la misma.

Como consecuencia de (6) y (10), se demuestra que existe un potencial superficial $V(t)$ definido en $\partial\Omega$ tal que $\boldsymbol{n} \times \boldsymbol{E}(t) \times \boldsymbol{n} = -\,\mathbf{grad}_\tau V(t)$ en $\partial\Omega$, donde $\mathbf{grad}_\tau$ denota el gradiente superficial. Además, la condición (9) implica que $V(t)$ debe ser constante en cada componente conexa de $\Gamma_{\boldsymbol{J}}$ y $\Gamma_{\boldsymbol{E}}$. La diferencia entre estas constantes en cada conductor Ω_{C}^n, $V_n(t) := V|_{\Gamma_{\boldsymbol{E}}^n}(t) - V|_{\Gamma_{\boldsymbol{J}}^n}(t)$, es la caída de potencial (o voltaje) entre $\Gamma_{\boldsymbol{J}}^n$ y $\Gamma_{\boldsymbol{E}}^n$, $n = 1, \ldots, N$.

Por tanto, si en lugar de las intensidades de corriente se conocen las caídas de voltaje $V_n(t)$, $n = 1, \ldots, N$, el problema de corrientes inducidas se reduce a

$$\mathbf{curl}\,\boldsymbol{H} = \sigma\boldsymbol{E} \quad \text{en } [0,T] \times \Omega\,, \tag{11}$$

$$\partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{en } [0,T] \times \Omega\,, \tag{12}$$

$$\operatorname{div}(\mu\boldsymbol{H}) = 0 \quad \text{en } [0,T] \times \Omega\,, \tag{13}$$

$$\boldsymbol{n} \times \boldsymbol{E} \times \boldsymbol{n} = -\,\mathbf{grad}_\tau V \quad \text{en } \partial\Omega, \quad \text{con } V|_{\Gamma_{\boldsymbol{E}}^n} - V|_{\Gamma_{\boldsymbol{J}}^n} = V_n\,, \quad n = 1, \ldots, N, \quad t \in [0,T]\,, \tag{14}$$

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{en } [0,T] \times \Gamma_{\boldsymbol{E}} \cup \Gamma_{\boldsymbol{J}}\,, \tag{15}$$

$$\mu\boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{en } [0,T] \times \partial\Omega\,. \tag{16}$$

Por otra parte, en el Capítulo 4 se estudia un problema con conductores móviles, lo cual introduce dificultades importantes en el análisis teórico. Por esta razón la topología y las condiciones de contorno se simplificarán notablemente. Se considerará únicamente un dominio Ω conteniendo una bobina Ω_{S} con fuente de corriente conocida $\boldsymbol{J}_{\mathrm{S}}$, una pieza conductora Ω_{C}^t que se mueve a lo largo del tiempo y el aire alrededor. Se considerará un dominio suficientemente grande de modo que en la frontera del dominio se impondrá la condición de contorno $\boldsymbol{H} \times \boldsymbol{n} = \mathbf{0}$ en $\partial\Omega$.

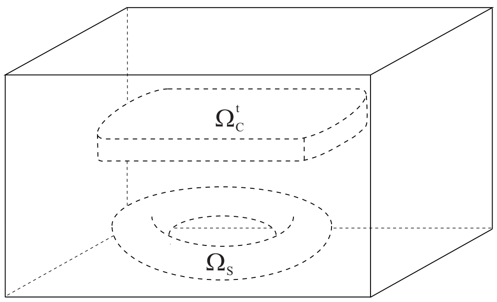

Figure 5: Esquema del dominio con fuente volúmica $\boldsymbol{J}_{\mathrm{S}}$ conocida en la bobina. Dominio conductor en movimiento.

Por tanto, el problema definido en el dominio Ω, puede resumirse en este caso mediante el conjunto de ecuaciones:

$$\mathbf{curl}\,\boldsymbol{H} = \sigma\boldsymbol{E} + \boldsymbol{J}_{\mathrm{S}} \quad \text{en } [0,T] \times \Omega, \tag{17}$$

$$\partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{en } [0,T] \times \Omega, \tag{18}$$

$$\boldsymbol{H} \times \boldsymbol{n} = \mathbf{0} \quad \text{en } [0,T] \times \partial\Omega, \tag{19}$$

donde $\boldsymbol{J}_{\mathrm{S}}$ tiene su soporte en $\bar{\Omega}_{\mathrm{S}} \subset\subset \Omega$, con condición inicial $\boldsymbol{H}(0) = \boldsymbol{H}_0$ como en el caso anterior.

0.3 Organización de la tesis

En esta sección se describen brevemente las aportaciones de cada uno de los capítulos de la tesis.

En el Capítulo 1 se estudia el problema de corrientes inducidas en régimen armónico en un dominio acotado proporcionando las fuentes de corriente en términos de intensidades. Como se ha mencionado previamente, el conformado electromagnético requiere un modelo de corrientes inducidas genuinamente transitorio. Sin embargo, en la primera etapa de la tesis al abordar el problema en dominios con topologías generales, se observó que la formulación introducida en [5] en régimen armónico podía extenderse al caso de fuentes no locales dadas en términos de intensidades de corriente en la frontera. Eso motivó la inclusión de este capítulo.

El modelo de corrientes inducidas en régimen armónico se obtiene al suponer que todos los campos que aparecen en las ecuaciones de Maxwell son sinusoidales en tiempo, es decir, tienen la forma

$$\boldsymbol{F}(\boldsymbol{x},t) \;=\; \mathrm{Re}\left[e^{\mathrm{i}\omega t}\boldsymbol{\mathcal{F}}(\boldsymbol{x})\right]$$

donde $\boldsymbol{\mathcal{F}}$ es la amplitud compleja del campo $\boldsymbol{F}$ y el parámetro $\omega \neq 0$ es la frecuencia angular, $\omega = 2\pi f$, siendo f la frecuencia de corriente. Esta hipótesis es adecuada por ejemplo en régimen de corriente alterna. Por tanto, el modelo de corrientes inducidas en régimen armónico puede escribirse en términos de las amplitudes complejas de los distintos campos. En este capítulo, se abordará dicho problema en una topología como la descrita en la Figura 4 y considerando conocidas las intensidades a la entrada de los inductores. En particular, si las intensidades de corriente son sinusoidales, $I_n(t) \;=\; \mathrm{Re}\left[e^{\mathrm{i}\omega t}\mathcal{I}_n(t)\right]$, se probará la equivalencia entre dos formulaciones distintas para resolver el problema

$$\mathbf{curl}\,\boldsymbol{\mathcal{H}} \;=\; \sigma\boldsymbol{\mathcal{E}} \quad \text{en } \Omega, \tag{20}$$

$$\mathrm{i}\omega\mu\boldsymbol{\mathcal{H}} + \mathbf{curl}\,\boldsymbol{\mathcal{E}} \;=\; \mathbf{0} \quad \text{en } \Omega, \tag{21}$$

$$\mathrm{div}\,(\mu\boldsymbol{\mathcal{H}}) \;=\; 0 \quad \text{en } \Omega, \tag{22}$$

$$\int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\,\boldsymbol{\mathcal{H}}\cdot\boldsymbol{n} = \mathcal{I}_n \quad \text{en } \Gamma_{\boldsymbol{J}}^n,\ n = 1,\ldots,N, \tag{23}$$

$$\boldsymbol{\mathcal{E}}\times\boldsymbol{n} = \mathbf{0} \quad \text{en } \Gamma_{\boldsymbol{E}}\cup\Gamma_{\boldsymbol{J}}, \tag{24}$$

$$\mu\boldsymbol{\mathcal{H}}\cdot\boldsymbol{n} = 0 \quad \text{en } \partial\Omega. \tag{25}$$

donde $\boldsymbol{\mathcal{E}}$ y $\boldsymbol{\mathcal{H}}$ son las amplitudes complejas asociadas al campo eléctrico y al campo magnético, respectivamente. Nótese que en régimen armónico, la ecuación (22) es una consecuencia de la ley de Faraday (21).

En [18] se ha analizado una formulación en términos del campo magnético para este problema. La restricción $\mathbf{curl}\,\boldsymbol{\mathcal{H}} \;=\; \mathbf{0}$ en el dominio dieléctrico, Ω_{D}, se impone en ese caso reemplazando el campo magnético en Ω_{D} por un potencial escalar magnético. Esta alternativa conduce a un ahorro computacional muy importante pero requiere construir ciertas *superficies de corte* si el dominio dieléctrico no es simplemente conexo. Estas superficies, denotadas por Σ_i en la Figura 4, pueden ser difíciles de construir en la práctica. La Figura 6 muestra un ejemplo donde el dieléctrico es el aire que rodea al conjunto hélice-cilindro y donde la construcción de dicha superficie no es trivial; esta configuración geométrica es clásica en problemas de calentamiento por inducción ([11]). Una alternativa que evita construir estas superficies es la propuesta en [5] en un caso con fuentes volúmicas y conductores totalmente contenidos en el dieléctrico. En este capítulo, inspirándose en tal formulación, se introduce un multiplicador de Lagrange para imponer la restricción de rotacional nulo en el dieléctrico y resolver el problema con datos intensidades (20)–(25). Se demostrará un resultado de equivalencia entre la formulación mixta y la formulación campo magnético/potencial escalar magnético a nivel discreto. De hecho, el campo magnético calculado en ambas formulaciones es exactamente el mismo. Esto hace innecesario obtener estimaciones del error para la segunda formulación ya que para la primera ya fueron demostradas en [18].

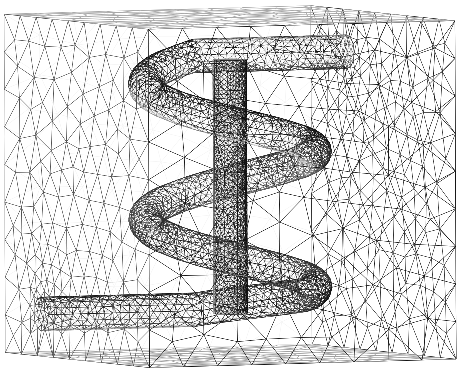

Figure 6: Ejemplo de configuración geométrica con dominio dieléctrico no simplemente conexo.

El esquema discreto propuesto para la formulación mixta está basado en elementos finitos Nédélec para el campo magnético en Ω y en el rotacional de los elementos finitos de Nédélec para el multiplicador de Lagrange en Ω_{D}. Las estimaciones de error para dicho esquema surgen del resultado de equivalencia demostrado. Dado que el esquema conduce a un sistema linear singular, se introduce un nuevo multiplicador de Lagrange en Ω_{D} que garantiza unicidad y que es aproximado mediante elementos finitos de Crouzeix-Raviart. Por tanto, el esquema numérico propuesto involucra un gran número de incógnitas, pero evita la construcción de las superficies de corte. En el capítulo, se muestran resultados numéricos que confirman los resultados de equivalencia demostrados y la aplicación a topologías generales como la presentada en la Figura 6.

Algunos de los resultados de este capítulo se recogen en la publicación [13]:

- Bermúdez, A., López-Rodríguez, B., Rodríguez, R., & Salgado, P. (2010) *Equivalence between two finite element methods for the eddy current problem,* Comptes Rendus de l'Académie des Sciences, **34**, 769–774.

En el Capítulo 2 se analiza un método numérico para resolver el problema transitorio de corrientes inducidas con intensidades de corriente $I_n(t)$, $n = 1, \dots, N$, como dato, es decir, el problema

(5)–(10). Se consideran topologías bastante generales y se propone una formulación en términos del campo magnético. Si las intensidades de corriente son suficientemente regulares, se prueba que la formulación débil tiene una única solución mediante un levantamiento adecuado de las condiciones (8). Además, se demuestran las propiedades que verifica dicha solución débil. En particular, la densidad de corriente y el campo eléctrico se determinan de forma única en el dominio conductor mediante la relación $\boldsymbol{J} = \mathbf{curl}\,\boldsymbol{H}$ y $\boldsymbol{E} := \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}$. En el dieléctrico no se determina el campo eléctrico por lo que la ecuación (6) solo se obtiene en el conductor.

Se propone un método de elementos finitos para la discretización espacial del problema basada en elementos finitos de Nédélec para aproximar el campo magnético. Se demuestra la existencia de solución del sistema semidiscreto y se obtienen estimaciones óptimas de error. El hecho de que la interpolada de Nédélec de la solución verifique exactamente la condición de intensidades (8), permite obtener dichas estimaciones. Para la discretización temporal se introduce un esquema de Euler implícito y se demuestran estimaciones de error para el esquema totalmente discreto. Por otra parte, la restricción $\mathbf{curl}\,\boldsymbol{H} = \mathbf{0}$ en el dieléctrico se resuelve introduciendo un potencial escalar magnético.

Aunque la parte central del capítulo aborda el problema con intensidades como dato, se realizan las observaciones necesarias para extender los resultados al caso en el que los datos son caídas de potencial. El esquema numérico se ha implementado y se presentan resultados numéricos para un test analítico con solución conocida y un ejemplo de aplicación al conformado electromagnético.

Los resultados de este capítulo dieron origen al artículo [14]:

- Bermúdez, A., López-Rodríguez, B., Rodríguez, R., & Salgado, P. *Numerical solution of transient eddy current problems with input current intensities as boundary data,* IMA Journal of Numerical Analysis (online; doi: 10.1093/imanum/drr028).

Tal y como se ha mencionado previamente, esta formulación campo magnético/potencial escalar magnético permite un ahorro computacional muy importante pero requiere la construcción de las superficies de corte. Para evitar esta dificultad en geometrías complejas, en el Capítulo 3 se propone una formulación del problema transitorio en términos de una nueva variable: una primitiva del campo eléctrico. Concretamente, la incógnita principal del problema será

$$\boldsymbol{u}(t,\boldsymbol{x}) := \int_0^t \boldsymbol{E}(s,\boldsymbol{x})\,ds.$$

Cabe señalar que en la literatura la incógnita "$\boldsymbol{u}$" ha sido introducida en el modelo de corrientes inducidas en [31] y se ha utilizado especialmente en dominios conductores (ver, por ejemplo, [42, 43]). En este capítulo $\boldsymbol{u}$ estará definida en todo el dominio.

Las fuentes de corriente serán las proporcionadas por un circuito externo y se analizará separadamente el problema con datos intensidades (5)–(10) y el problema con datos voltajes (11)–(16).

Con el objetivo de analizar el problema en términos de $\boldsymbol{u}$, es necesario establecer ecuaciones adicionales que permitan determinar esta variable de manera única. En principio, el campo eléctrico $\boldsymbol{E}$ en el dieléctrico queda únicamente determinado si se dispone de su componente normal $\epsilon\boldsymbol{E}(t)|_{\Omega_{\mathrm{D}}}\cdot$

$\boldsymbol{n}$ en la frontera exterior del dieléctrico Γ_{D}. En tal caso, deben añadirse a las ecuaciones (5)–(10) (o bien (11)–(16) en el caso de datos voltajes) las siguientes

$$\operatorname{div}(\epsilon \boldsymbol{E}) = 0 \quad \text{en } (0,T) \times \Omega_{\mathrm{D}}, \tag{26}$$

$$\epsilon \boldsymbol{E}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n} = g \quad \text{en } [0,T] \times \Gamma_{\mathrm{D}}, \tag{27}$$

$$\int_{\Gamma_{\mathrm{I}}^k} \epsilon \boldsymbol{E}(t)|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n} = 0, \quad k = 2, \ldots, M, \quad t \in [0,T], \tag{28}$$

donde, como ya se dijo, g es un dato adicional y $\Gamma_{\mathrm{I}}^k := \Omega_{\mathrm{D}} \cap \partial\Omega_{\mathrm{C}}^k$, $k = 1, \ldots, M$.

Nótese que el campo eléctrico verifica la ecuación (26) en ausencia de cargas en el dieléctrico. Para imponer estas restricciones se introduce un multiplicador de Lagrange en el dieléctrico obteniéndose como resultado una formulación mixta parabólica degenerada. Se demuestra la existencia y unicidad de solución tanto con datos intensidades como con datos voltajes; para este análisis, la formulación en campo magnético introducida en el Capítulo 2 y las propiedades de su solución débil serán esenciales.

Un inconveniente de esta formulación es que requiere un dato adicional: $g = \epsilon \boldsymbol{E}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}$ en $[0,T] \times \Gamma_{\mathrm{D}}$. Sin embargo demostraremos que las principales cantidades físicas, es decir, el campo magnético en todo el dominio y el campo eléctrico en el dominio conductor, son independientes del valor de g. Por esta razón, es posible considerar otro modelo que no requiere del dato adicional g. En efecto, si el campo eléctrico $\boldsymbol{E}$ en el dieléctrico no es una magnitud de interés, pueden resolverse las mismas ecuaciones pero con un dato g arbitrario (por ejemplo, $g = 0$). En este caso, aunque sigamos denotándola por $\boldsymbol{E}$, esta variable en el dieléctrico no será el verdadero campo eléctrico, sino simplemente una variable auxiliar que permite resolver el problema.

Para la discretización espacial se proponen elementos finitos de Nédélec para aproximar $\boldsymbol{u}$ y elementos finitos lineales a trozos y continuos para el multiplicador. Para demostrar que el esquema semidiscreto tiene una única solución se realiza un levantamiento del dato g y el diagrama de De Rham a nivel discreto ([9]). Para la discretización en tiempo, se introduce un esquema de Euler implícito y se demuestran estimaciones de error para los esquema semidiscreto y totalmente discreto. Esta misma formulación del problema de corrientes inducidas en términos de $\boldsymbol{u}$, pero con fuente volúmica de corriente y condiciones de contorno esenciales homogéneas, ha sido analizado previamente en [2]. Cabe resaltar que la condición de contorno que se utiliza para incorporar las fuentes de corriente genera cambios significativos en las demostraciones existentes.

El método numérico implementado se ha aplicado para resolver un problema con solución analítica conocida así como un problema de inducción en la configuración geométrica presentada en la Figura 6.

Los resultados de este capítulo se recogen en el siguiente artículo [12]:

- Bermúdez, A., López-Rodríguez, B., Rodríguez, R., & Salgado, P. *An eddy current problem in terms of a time-primitive of the electric field with non-local source conditions*, Preprint del Departamento de Ingeniería Matemática de la Universidad de Concepción, 2012.

Finalmente, el Capítulo 4 aborda el problema transitorio de corrientes inducidas teniendo en cuenta el movimiento de las partes conductoras para poder simular de un modo más realista un problema de conformado electromagnético. El hecho de que las ecuaciones sean de distinta naturaleza en conductor y dieléctrico introduce dificultades notables en el análisis del problema. Por ello, para centrarse en el movimiento, se simplifica el dominio, las fuentes y condiciones de contorno y se estudia el problema (17)–(19) suponiendo que la velocidad de la pieza es conocida. Para el análisis matemático del problema se ha valorado trabajar en términos de $\boldsymbol{H}$ o en términos de $\boldsymbol{u}$. En el primer caso, la dificultad mas relevante es la restricción $\mathbf{curl}\,\boldsymbol{H} = \mathbf{0}$ en el dieléctrico y en el segundo caso, restricciones del tipo (26)–(28) también definidas en el dieléctrico. Nótese que el movimiento de la pieza hace que estas restricciones estén definidas en un dominio que cambia con el tiempo. Hasta el momento sólo se ha conseguido demostrar un resultado de existencia de solución para una formulación débil en términos del campo magnético. En este capítulo se propone un método numérico que tenga en cuenta el movimiento de la pieza en dominios tridimensionales, evitando el remallado en cada paso de tiempo, y que al mismo tiempo no sea excesivamente costoso.

El método numérico propuesto se basa en un método de penalización (ver, por ejemplo, [38]) en términos del campo magnético, que consiste en reemplazar las ecuaciones

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{H} &= \mathbf{0} \quad \text{en el aire}, \\ \mathbf{curl}\,\boldsymbol{H} &= \boldsymbol{J}_{\mathrm{S}} \quad \text{en la bobina}, \end{aligned}$$

por

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{H} &= \varepsilon \boldsymbol{E} \quad \text{en el aire}, \\ \mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J}_{\mathrm{S}} &= \varepsilon \boldsymbol{E} \quad \text{en la bobina}, \end{aligned}$$

donde ε es un parámetro positivo suficientemente pequeño y destinado a tender a cero. Desde un punto de vista físico, en el modelo de corrientes inducidas, el método de penalización supone de algún modo reemplazar aire y bobina por un conductor muy pobre.

En primer lugar, se demuestra que esta estrategia de penalización conduce a un problema con solución única que converge al problema de partida cuando ε tiende a cero. Se propone una discretización espacial y temporal del problema parabólico penalizado y se demuestra asimismo la convergencia del esquema totalmente discreto. Los resultados numéricos obtenidos con dominio conductor fijo confirman los resultados de convergencia demostrados teóricamente. Posteriormente, se plantea el problema parabólico penalizado para el modelo de corrientes inducidas con dominio conductor en movimiento y el esquema totalmente discreto correspondiente. Se presentan los resultados numéricos obtenidos para un problema con solución analítica conocida y para un ejemplo axisimétrico que permite realizar comparaciones con el código desarrollado en [16]. Estos resultados ponen de manifiesto que el método de penalización ofrece una alternativa interesante para simular problemas tridimensionales de conformado electromagnético.

Con los resultados de este capítulo se espera redactar el siguiente artículo:

- Bermúdez, A., López-Rodríguez, B., Rodríguez, R., & Salgado, P. *Numerical solution of a transient 3D eddy current model with moving conductors* (en preparación).

Cabe señalar que cada capítulo de la tesis es autocontenido al corresponderse con trabajos publicados o pendientes de publicación. Por ello, la notación utilizada para los espacios funcionales y campos vectoriales será la misma que la utilizada en la publicación correspondiente e independiente en cada capítulo.

La tesis finaliza con las conclusiones y con una breve descripción de las líneas de investigación abiertas.

Chapter 1

Equivalence between two finite element methods for the harmonic eddy current problem

1.1 Introduction

In this chapter, we will deal with the time-harmonic eddy current model in a bounded domain which includes conducting and dielectric parts by imposing the current intensities as boundary data. This problem involves non-local source conditions and has been analyzed in terms of different unknowns (see, for instance, [7], [19]). In particular, we refer the reader to Chapter 8 of [7], where authors give an exhaustive description of different formulations used to solve the problem. The advantages and drawbacks of each formulation are usually related with the topology of the dielectric domain and the definition of the interface and boundary conditions.

We consider two different formulations of the problem. One of them, is in terms of magnetic field/magnetic scalar potential and was introduced in [17] and analyzed in [19]. The other one, is a mixed formulation in terms of the magnetic field and a Lagrange multiplier to impose the curl-free condition in the dielectric domain. This formulation was introduced and analyzed in [5] in the case of volumic source current as data. We adapt this formulation to our problem with current intensities as boundary data and prove that the corresponding finite element approximation is equivalent to that of the previous formulation. Therefore, there is no need of an additional convergence analysis.

Although the mixed formulation leads to an increasing of the number of unknowns, it avoids building cutting surface which is necessary for the application of the other formulation (see [19]). A reduced version of this chapter has been published in [13].

The outline of the chapter is the following: in Section 1.2 we describe the time-harmonic eddy current model with current intensities as boundary data and detail the topology of the bounded domain that will be considered. Section 1.3 is devoted to recall the magnetic field/scalar magnetic potential formulation, the well-posedness results and its discretization. In Section 1.4 we will prove equivalence results between this formulation and the mixed one proposed in the same section.

Finally, some numerical results are presented in Section 1.5 which validate the theoretical results and show the applicability of the numerical method proposed.

1.2 Statement of the problem

We start introducing the time-harmonic eddy current problem:

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}, \tag{1.1}$$

$$\mathrm{i}\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0}, \tag{1.2}$$

$$\boldsymbol{J} = \sigma\boldsymbol{E}, \tag{1.3}$$

where $\boldsymbol{E}$ is the complex amplitude of the electric field, $\boldsymbol{H}$ the complex amplitude of the magnetic field, $\boldsymbol{J}$ the complex amplitude of the current density, ω the angular frequency, μ the magnetic permeability and σ the electric conductivity.

We are interested in solving these equations in a simply connected three-dimensional bounded domain Ω, which consists of two parts, Ω_{C} and Ω_{D}, occupied by conductors and dielectrics, respectively. The electric conductivity σ vanishes in the dielectric domain. We denote $\Omega_{\mathrm{C}}^1, \dots, \Omega_{\mathrm{C}}^M$ the disjoint connected components of Ω_{C}, which are of two types: "inductors" which go through the boundary of Ω and represent the conducting parts directly connected to an external circuit, and "workpieces" which have their closure included in Ω, where only induced currents exist. We denote $\Omega_{\mathrm{C}}^1, \dots, \Omega_{\mathrm{C}}^N$ the former and $\Omega_{\mathrm{C}}^{N+1}, \dots, \Omega_{\mathrm{C}}^M$ the latter. Moreover, we assume that the latter, Ω_{C}^n, $n = N+1, \dots, M$, are connected and have a connected boundary $\partial\Omega_{\mathrm{C}}^n$. We also assume that Ω_{D} is connected.

The domain Ω is assumed to have a Lipschitz-continuous connected boundary $\partial\Omega$, which splits into two parts: $\partial\Omega = \Gamma_{\mathrm{C}} \cup \Gamma_{\mathrm{D}}$, with $\Gamma_{\mathrm{C}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega$ and $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \cap \partial\Omega$ being the outer boundaries of the conducting and dielectric domains, respectively. We denote $\Gamma_{\mathrm{I}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega_{\mathrm{D}}$, the interface between dielectrics and conductors. Note that $\Gamma_{\mathrm{I}} = \bigcup_{n=1}^{M} \Gamma_{\mathrm{I}}^n$, where $\Gamma_{\mathrm{I}}^n := \partial\Omega_{\mathrm{D}} \cap \partial\Omega_{\mathrm{C}}^n$, $n = 1, \dots, M$. Notice that, for $n = N+1, \dots, M$, $\Gamma_{\mathrm{I}}^n = \partial\Omega_{\mathrm{C}}^n$. If we denote $\Gamma_{\mathrm{D}}^\star := \Gamma_{\mathrm{D}} \cup \bigcup_{n=1}^{N} \Gamma_{\mathrm{I}}^n$ then the boundary of Ω_{D} has $M - N + 1$ disjoint connected components, namely, $\Gamma_{\mathrm{D}}^\star, \Gamma_{\mathrm{I}}^{N+1}, \dots, \Gamma_{\mathrm{I}}^M$. We also denote by $\boldsymbol{n}$ and $\boldsymbol{n}_{\mathrm{C}}$ the outer unit normal vectors to $\partial\Omega$ and $\partial\Omega_{\mathrm{C}}$, respectively. We assume that the outer boundary of each connected component, $\partial\Omega_{\mathrm{C}}^n \cap \partial\Omega$, $n = 1, \dots, N$, has two connected components, both with non-zero measure: the current entrance, $\Gamma_{\boldsymbol{J}}^n$, where it is connected to a wire supplying alternating electric current, and the current exit, $\Gamma_{\boldsymbol{E}}^n$. Finally, we denote $\Gamma_{\boldsymbol{J}} := \Gamma_{\boldsymbol{J}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{J}}^N$ and $\Gamma_{\boldsymbol{E}} := \Gamma_{\boldsymbol{E}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{E}}^N$, and we assume that $\Gamma_{\boldsymbol{J}} \cap \Gamma_{\boldsymbol{E}} = \emptyset$. (See a sketch of the domain in Figure 1.1.)

By following [22], we are interested in solving equations (1.1)–(1.3) with the following boundary conditions considered initially in [19]:

$$\int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\,\boldsymbol{H} \cdot \boldsymbol{n} = I_n \quad \text{on } \Gamma_{\boldsymbol{J}}^n,\ n = 1, \dots, N, \tag{1.4}$$

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\boldsymbol{E}} \cup \Gamma_{\boldsymbol{J}}, \tag{1.5}$$

$$\mu\boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

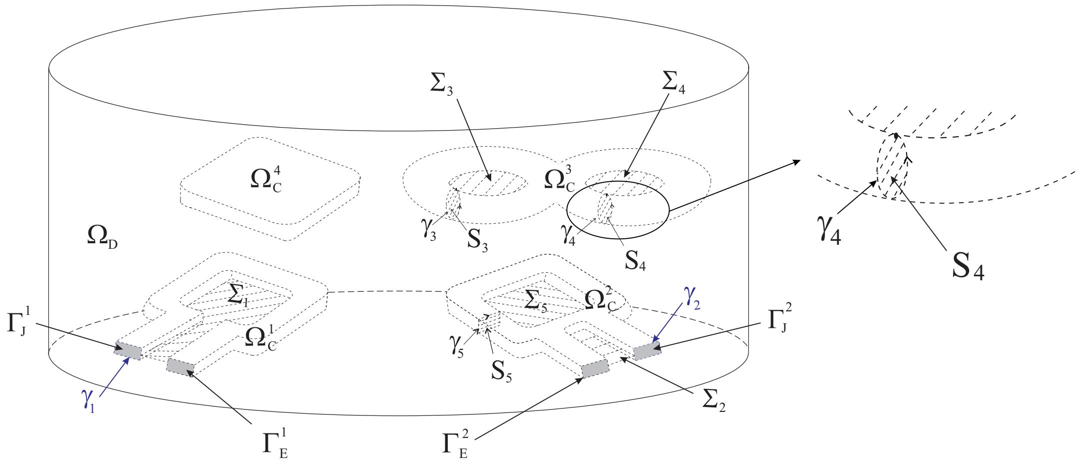

Figure 1.1: Sketch of the domain with a zoom around S_4

where the only data are the complex amplitudes of the current intensities I_n through each wire.

Conditions (1.4) account for the input current intensities through each Γ_J^n. Conditions (1.5)–(1.6) have been proposed in [22] in a more general setting. They will appear as natural boundary conditions of the weak formulation of the problem to be given below. The former implies the assumption that the electric current is normal to the current entrance and exit surfaces, whereas the latter means that the magnetic field is tangential to the boundary. (See [18] for further discussions on these boundary conditions and [19] for its application to the modeling of an electric furnace.)

1.3 The magnetic field/magnetic scalar potential formulation

In this section we recall the magnetic field/scalar magnetic potential formulation introduced in [19] for solving the above problem and its numerical solution by a finite element method.

Let

$$\boldsymbol{\mathcal{X}} := \left\{ \boldsymbol{G} \in \mathrm{H}(\mathbf{curl};\Omega) : \ \mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}} \right\}$$

and $a : \mathrm{H}(\mathbf{curl};\Omega) \times \mathrm{H}(\mathbf{curl};\Omega) \longrightarrow \mathbb{C}$ be the sesquilinear continuous form defined by

$$a(\boldsymbol{H},\boldsymbol{G}) := \mathrm{i}\omega \int_\Omega \mu \boldsymbol{H} \cdot \bar{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\bar{\boldsymbol{G}}.$$

Given $\boldsymbol{I} := (I_n) \in \mathbb{C}^N$, we introduce the closed linear manifold of $\boldsymbol{\mathcal{X}}$ defined as follows:

$$\boldsymbol{\mathcal{V}}(\boldsymbol{I}) := \left\{ \boldsymbol{G} \in \boldsymbol{\mathcal{X}} : \ \langle \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\boldsymbol{J}}^n} = I_n, \ n = 1, \ldots, N \right\}.$$

A weak formulation of the above eddy current problem in terms of the magnetic field has been obtained in [19] and reads as follows:

Problem 1.1 *Given $\boldsymbol{I} \in \mathbb{C}^N$, find $\boldsymbol{H} \in \boldsymbol{\mathcal{V}}(\boldsymbol{I})$ such that*

$$a(\boldsymbol{H},\boldsymbol{G}) = 0 \quad \forall \boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\boldsymbol{0}).$$

Notice that this formulation needs to impose the curl-free constraint in the space $\boldsymbol{\mathcal{V}}(\boldsymbol{I})$. An alternative to deal with this constraint consists in replacing the magnetic field by a scalar potential in the dielectric domain. Next, we summarize this procedure.

We assume there exist L connected "cut" surfaces $\Sigma_n \subset \Omega_{\mathrm{D}}$, $n = 1, \cdots, L$, such that $\partial\Sigma_n \subset \partial\Omega_{\mathrm{D}}$ and $\widetilde{\Omega}_{\mathrm{D}} := \Omega_{\mathrm{D}} \setminus \bigcup_{n=1}^{L} \Sigma_n$ is pseudo-Lipschitz and simply connected (see, for instance, [9]). We also assume that $\bar{\Sigma}_n \cap \bar{\Sigma}_m = \emptyset$ for $n \neq m$ (see Figure 1.1). For each inductor, Ω_{C}^n, $n = 1, \ldots, N$, there exists one cut surface Σ_n satisfying $\partial\Sigma_n \cap \Gamma_{\mathrm{D}} \neq \emptyset$ (see Figure 1.1). The remaining cut surfaces, $\Sigma_{N+1}, \ldots, \Sigma_L$, are assumed to be contained in the interior of Ω_{D} (see Figure 1.1, again).

For each cut surface Σ_n we assume that there exists a surface $S_n \subset \bar{\Omega}_{\mathrm{C}}^n$, with $\partial S_n \subset \partial\Omega_{\mathrm{C}}^n$ and such that its boundary γ_n is a simple closed curve which intersects $\bar{\Sigma}_n$ once and only once, and does not intersect $\bar{\Sigma}_m$, $m \neq n$. Note that, for $n = 1, \ldots, N$, we can take $S_n = \Gamma_J^n$. We denote the two faces of each Σ_n by Σ_n^- and Σ_n^+. We choose an orientation for each γ_n by taking its initial and end points on Σ_n^- and Σ_n^+, respectively. We denote by $\boldsymbol{t}_n$ the unit vector tangent to γ_n according to this orientation.

For any function $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$, we denote by

$$[\![\widetilde{\Psi}]\!]_{\Sigma_n} := \widetilde{\Psi}|_{\Sigma_n^-} - \widetilde{\Psi}|_{\Sigma_n^+}$$

the jump of $\widetilde{\Psi}$ through Σ_n along $\boldsymbol{n}_n$. The gradient of $\widetilde{\Psi}$ in $\mathcal{D}'(\widetilde{\Omega}_{\mathrm{D}})$ can be extended to $\mathrm{L}^2(\Omega_{\mathrm{D}})^3$ and will be denoted by $\widetilde{\mathbf{grad}}\,\widetilde{\Psi}$.

Let Θ be the linear subspace of $\mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$ defined by

$$\Theta := \left\{ \widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) : \ [\![\widetilde{\Psi}]\!]_{\Sigma_n} = \text{constant}, \ n = 1, \ldots, L \right\}.$$

Then, for $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$, we have that $\widetilde{\mathbf{grad}}\,\widetilde{\Psi} \in \mathrm{H}(\mathbf{curl}; \Omega_{\mathrm{D}})$ if and only if $\widetilde{\Psi} \in \Theta$, in which case $\mathbf{curl}(\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) = \mathbf{0}$ (see Lemma 3.11 in [9]).

We use the following notation: given $\boldsymbol{G}_{\mathrm{C}} \in \mathrm{L}^2(\Omega_{\mathrm{C}})^3$ and $\boldsymbol{G}_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})^3$, $(\boldsymbol{G}_{\mathrm{C}}|\boldsymbol{G}_{\mathrm{D}})$ denotes the field $\boldsymbol{G} \in \mathrm{L}^2(\Omega)^3$ defined by $\boldsymbol{G}|_{\Omega_{\mathrm{C}}} := \boldsymbol{G}_{\mathrm{C}}$ and $\boldsymbol{G}|_{\Omega_{\mathrm{D}}} := \boldsymbol{G}_{\mathrm{D}}$.

Let us denote by $\boldsymbol{\mathcal{Y}}$ the linear space given by

$$\boldsymbol{\mathcal{Y}} := \left\{ (\boldsymbol{G}, \widetilde{\Psi}) \in \mathrm{H}(\mathbf{curl}; \Omega_{\mathrm{C}}) \times (\Theta/\mathbb{C}) : \ (\boldsymbol{G}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) \in \mathrm{H}(\mathbf{curl}; \Omega) \right\}.$$

Then $(\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}}$ if and only if $(\boldsymbol{G}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) \in \boldsymbol{\mathcal{X}}$.

Given $\boldsymbol{I} \in \mathbb{C}^N$, it is natural to search the solution of our problem in the linear manifold

$$\boldsymbol{\mathcal{W}}(\boldsymbol{I}) := \left\{ (\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}} : \ [\![\widetilde{\Psi}]\!]_{\Sigma_n} = I_n, \ n = 1, \ldots, N \right\}$$

(see [19]).

Let $\widetilde{a} : \ \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Y}} \longrightarrow \mathbb{C}$ be the sesquilinear elliptic form defined by

$$\widetilde{a}\left((\boldsymbol{H}, \widetilde{\Phi}), (\boldsymbol{G}, \widetilde{\Psi})\right) := a\left((\boldsymbol{H}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Phi}), (\boldsymbol{G}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Psi})\right).$$

Then, Problem 1.1 can be written in terms of the magnetic field/magnetic scalar potential as follows.

Problem 1.2 *Given* $\boldsymbol{I} \in \mathbb{C}^N$, *find* $(\boldsymbol{H}, \widetilde{\Phi}) \in \boldsymbol{\mathcal{W}}(\boldsymbol{I})$ *such that*

$$\widetilde{a}\left((\boldsymbol{H}, \widetilde{\Phi}), (\boldsymbol{G}, \widetilde{\Psi})\right) = 0 \quad \forall (\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{W}}(\mathbf{0}).$$

For the numerical solution let us assume that Ω, Ω_{C} and Ω_{D} are Lipschitz polyhedra and consider regular tetrahedral meshes $\mathcal{T}_h$ of Ω, such that each element $K \in \mathcal{T}_h$ is contained either in Ω_{C} or in Ω_{D} (h stands as usual for the corresponding mesh-size). We employ "edge" finite elements to approximate the magnetic field. More precisely, the lowest-order Nédélec finite element space:

$$\boldsymbol{\mathcal{N}}_h(\Omega) := \{\boldsymbol{G}_h \in \mathrm{H}(\mathbf{curl}; \Omega) : \ \boldsymbol{G}_h|_K \in \boldsymbol{\mathcal{N}}(K) \ \forall K \in \mathcal{T}_h\},$$

where, for each tetrahedron K,

$$\boldsymbol{\mathcal{N}}(K) := \left\{\boldsymbol{G}_h \in \mathcal{P}_1(K)^3 : \ \boldsymbol{G}_h(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b}, \ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3, \ \boldsymbol{x} \in K\right\}.$$

We introduce the finite-dimensional space

$$\boldsymbol{\mathcal{X}}_h := \{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega) : \ \mathbf{curl}\, \boldsymbol{G}_h = \mathbf{0} \ \text{ in } \Omega_{\mathrm{D}}\} \subset \boldsymbol{\mathcal{X}}$$

and the linear manifolds $\boldsymbol{\mathcal{V}}_h(\boldsymbol{I}) := \boldsymbol{\mathcal{V}}(\boldsymbol{I}) \cap \boldsymbol{\mathcal{X}}_h$, $\boldsymbol{I} \in \mathbb{C}^N$.

Then, the discrete problem reads as follows.

Problem 1.3 *Given* $\boldsymbol{I} \in \mathbb{C}^N$, *find* $\boldsymbol{H}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})$ *such that*

$$a(\boldsymbol{H}_h, \boldsymbol{G}_h) = 0 \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0}).$$

In [19] it is proved that this problem has a unique solution which is an optimal order approximation of the solution to Problem 1.1. In that paper, a discrete multivalued magnetic scalar potential is introduced in the dielectric domain in order to approximate Problem 1.2. Next, we recall the corresponding discrete approximation.

Assume that the cut surfaces Σ_n are polyhedral and the meshes are compatible with them, in the sense that each Σ_n is a union of faces of tetrahedra $K \in \mathcal{T}_h$. Therefore, $\mathcal{T}_h^{\Omega_{\mathrm{D}}} := \left\{K \in \mathcal{T}_h : \ K \subset \Omega_{\mathrm{D}}\right\}$ can also be seen as a mesh of $\widetilde{\Omega}_{\mathrm{D}}$.

Let us introduce the following discrete spaces:

$$\begin{aligned}
\mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) &:= \left\{\widetilde{\Psi}_h \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) : \ \widetilde{\Psi}_h|_K \in \mathcal{P}_1(K) \ \forall K \in \mathcal{T}_h^{\Omega_{\mathrm{D}}}\right\}, \\
\Theta_h &:= \left\{\widetilde{\Psi}_h \in \mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) : \ [\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = \text{constant}, \ n = 1, \dots L\right\} \quad \text{and} \\
\boldsymbol{\mathcal{Y}}_h &:= \left\{(\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{C}}) \times (\Theta_h/\mathbb{C}) : \ \boldsymbol{G}_h \times \boldsymbol{n}_{\mathrm{C}} = \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h \times \boldsymbol{n}_{\mathrm{C}} \text{ on } \Gamma_{\mathrm{I}}\right\}.
\end{aligned}$$

Given $\boldsymbol{I} \in \mathbb{C}^N$, let

$$\boldsymbol{\mathcal{W}}_h(\boldsymbol{I}) := \left\{(\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{Y}}_h : \ [\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = I_n, \ n = 1, \dots, N\right\}.$$

Now, consider the following discrete problem.

Problem 1.4 *Given $\boldsymbol{I} \in \mathbb{C}^N$, find $(\boldsymbol{H}_h, \widetilde{\Phi}_h) \in \boldsymbol{\mathcal{W}}_h(\boldsymbol{I})$ such that*

$$\widetilde{a}\big((\boldsymbol{H}_h, \widetilde{\Phi}_h), (\boldsymbol{G}_h, \widetilde{\Psi}_h)\big) = 0 \quad \forall (\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{W}}_h(\boldsymbol{0}).$$

Next theorem, proved in [19], shows that Problems 1.3 and 1.4 are equivalent.

Theorem 1.5 *Given $\boldsymbol{I} \in \mathbb{C}^N$, $\boldsymbol{H}_h$ is a solution of Problem 1.3 if and only if there exists $\widetilde{\Phi}_h \in \Theta_h$ such that $\boldsymbol{H}_h|_{\Omega_{\mathrm{D}}} = \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h$ and $(\boldsymbol{H}_h|_{\Omega_{\mathrm{C}}}, \widetilde{\Phi}_h)$ is a solution of Problem 1.4.*

Problem 1.4 leads to an important saving in computational effort, since it involves a scalar field instead of a vector field in the dielectric. Notice that its implementation requires imposing the following constraints:

- $\boldsymbol{G}_h \times \boldsymbol{n}_{\mathrm{C}} = \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h \times \boldsymbol{n}_{\mathrm{C}}$ on Γ_{I}, which arises from definition of $\boldsymbol{\mathcal{Y}}_h$;
- $[\![\widetilde{\Psi}_h]\!]_{\Sigma_n} =$ constant, $n = 1, \dots, N$, which arise from definition of Θ_h.

A procedure to impose these constraints was proposed in [17], where numerical experiments exhibiting the performance of the method were also reported. In particular, the last condition involves the explicit building of cutting surfaces in the dielectric mesh.

1.4 Mixed formulations and equivalence results

As we have noticed above, Problem 1.4 represents a less expensive alternative to approximate the proposed eddy current model. The only drawback is that it needs finite element meshes involving "cuts", which sometimes can be difficult to build. In what follows we will introduce a mixed discrete formulation of the same eddy current model given above, which does not need any cut, and we will show that it is completely equivalent to Problem 1.3.

This mixed formulation has been previously analyzed in [5] for other boundary conditions, without establishing any relation with a magnetic field/magnetic scalar potential discretization as that of Problem 1.3 and, consequently, without taking advantage of the equivalence between such discrete problems. The formulation is based on using a Lagrange multiplier to impose the curl-free constraint in the dielectric instead of introducing the scalar potential in Ω_{D}, so that cuts are not required in the mesh.

Given $\boldsymbol{I} \in \mathbb{C}^N$, let

$$\boldsymbol{\mathcal{U}}_h(\boldsymbol{I}) := \left\{ \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega) : \int_{\Gamma_J^n} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{n} = I_n,\ n = 1, \dots, N \right\}.$$

Notice that

$$\mathbf{curl}\,\big(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\big) = \left\{ \begin{array}{c} \boldsymbol{F}_h \in \boldsymbol{\mathcal{RT}}_h(\Omega_{\mathrm{D}}) \,:\, \operatorname{div} \boldsymbol{F}_h = 0,\ \int_{\Gamma_{\mathrm{D}}^\star} \boldsymbol{F}_h \cdot \boldsymbol{n} = 0 \\ \text{and } \int_{\Gamma_{\mathrm{I}}^k} \boldsymbol{F}_h \cdot \boldsymbol{n} = 0,\ k = N+1, \dots, M \end{array} \right\},$$

where $\boldsymbol{\mathcal{RT}}_h(\Omega_{\mathrm{D}})$ is the space of lowest-order Raviart-Thomas elements (see, Lemma III.5.11 in [38]). Thus, the discrete mixed problem reads as follows.

Problem 1.6 *Given $\boldsymbol{I} \in \mathbb{C}^N$, find $\boldsymbol{H}_h \in \boldsymbol{\mathcal{U}}_h(\boldsymbol{I})$ and $\boldsymbol{Z}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$ such that*

$$a(\boldsymbol{H}_h, \boldsymbol{G}_h) + \int_{\Omega_{\mathrm{D}}} \boldsymbol{Z}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}_h = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0}), \tag{1.7}$$

$$\int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{H}_h \cdot \bar{\boldsymbol{F}}_h = 0 \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right). \tag{1.8}$$

Since $\left\{\boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0}) : \int_{\Omega_{\mathrm{D}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \bar{\boldsymbol{F}}_h = 0\ \forall \boldsymbol{F}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)\right\} = \boldsymbol{\mathcal{U}}_h(\mathbf{0}) \cap \boldsymbol{\mathcal{X}}$ and a is coercive on $\boldsymbol{\mathcal{X}}$, we only need the following *inf-sup* condition to conclude that Problem 1.6 has a unique solution.

Proposition 1.7 *There exist a constant $\beta_h > 0$ such that*

$$\sup_{\boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0}):\ \boldsymbol{G}_h \neq \mathbf{0}} \frac{\left|\int_{\Omega_{\mathrm{D}}} \boldsymbol{F}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}_h\right|}{\|\boldsymbol{G}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \geq \beta_h \|\boldsymbol{F}_h\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})} \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right).$$

Proof. Since $\dim \boldsymbol{\mathcal{U}}_h(\mathbf{0}) < \infty$, it is enough to prove that for all non-vanishing $\boldsymbol{F}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$ there exists $\boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0})$ such that $\int_{\Omega_{\mathrm{D}}} \boldsymbol{F}_h \cdot \mathbf{curl}\, \boldsymbol{G}_h \neq 0$. Let $\boldsymbol{U}_h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ be such that $\boldsymbol{F}_h = \mathbf{curl}\, \boldsymbol{U}_h$ in Ω_{D}. In general $\boldsymbol{U}_h \notin \boldsymbol{\mathcal{U}}_h(\mathbf{0})$, but if we are able to find $\boldsymbol{Y}_h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ satisfying $\mathbf{curl}\, \boldsymbol{Y}_h = \mathbf{0}$ in Ω_{D} and $\int_{\gamma_n} \boldsymbol{Y}_h \cdot \boldsymbol{t}_n = -\int_{\gamma_n} \boldsymbol{U}_h \cdot \boldsymbol{t}_n$, $n = 1, \ldots, N$, it is straightforward to check that $\boldsymbol{G}_h := \boldsymbol{U}_h + \boldsymbol{Y}_h$ satisfies the above requirements (recall $\gamma_n := \partial \Gamma_{\boldsymbol{J}}^n$). Such a $\boldsymbol{Y}_h$ can be defined as follows:

$$\boldsymbol{Y}_h := -\sum_{m=1}^{N} \left(\int_{\gamma_m} \boldsymbol{U}_h \cdot \boldsymbol{t}_m\right) \boldsymbol{Y}_h^m,$$

where $\boldsymbol{Y}_h^m \in \boldsymbol{\mathcal{N}}_h(\Omega)$ is such that $\boldsymbol{Y}_h^m|_{\Omega_{\mathrm{D}}} = \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h^m$, with $\widetilde{\Phi}_h^m \in \Theta_h$ satisfying $[\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n} = \delta_{nm}$, $n, m = 1, \ldots, N$. ∎

An *inf-sup* condition analogous to that in Proposition 1.7 has been proved to hold uniformly in h, in the proof of Theorem 5.2 from [5] for the problem considered in that paper. However, this is not necessary, at least in our case, since we will obtain error estimates for the component $\boldsymbol{H}_h$ of the solution to Problem 1.6 as a direct consequence of the following equivalence result.

Proposition 1.8 *Given $\boldsymbol{I} \in \mathbb{C}^N$, a discrete field $\boldsymbol{H}_h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ is solution of Problem 1.3 (equivalently, of Problem 1.4) if and only if there exists $\boldsymbol{Z}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$ such that $(\boldsymbol{H}_h, \boldsymbol{Z}_h)$ solves Problem 1.6.*

Proof. Since each problem has a unique solution, it is enough to prove that if $(\boldsymbol{H}_h, \boldsymbol{Z}_h)$ solves Problem 1.6, then $\boldsymbol{H}_h$ solves Problem 1.3. For this purpose, let us take $\boldsymbol{F}_h = \mathbf{curl}\, \boldsymbol{H}_h$ as test function in (1.8). We deduce $\mathbf{curl}\, \boldsymbol{H}_h = \mathbf{0}$ in Ω_{D} and then $\boldsymbol{H}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})$. Finally, we complete the proof by testing (1.7) with $\boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0})$. ∎

Although Problem 1.6 has a unique solution, its direct implementation leads to a singular linear system. Indeed, when functions $\boldsymbol{F}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$ are written as $\boldsymbol{F}_h = \mathbf{curl}\, \boldsymbol{U}_h$, with $\boldsymbol{U}_h \in \boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})$, such $\boldsymbol{U}_h$ is clearly not unique and this leads to a singular matrix. However, as stated in [5, Remark 5.1], since the kernel of this matrix is well separated from the rest of the spectrum, a conjugate gradient type method will work for its numerical solution.

An alternative leading to a system with a non-singular matrix was also proposed in [5]. Let $\mathcal{Q}_h$ be the space of piecewise constant functions in $\mathcal{T}_h^{\Omega_{\rm D}}$ and $\boldsymbol{\mathcal{CR}}_h(\Omega_{\rm D})$ the space of lowest-order 3D Crouzeix-Raviart elements (see [5], for instance), namely

$$\mathcal{Q}_h := \left\{ F_h \in \mathrm{L}^2(\Omega_{\rm D}) \ : \ F_h|_K \in \mathbb{P}_0 \ \forall K \in \mathcal{T}_h^{\Omega_{\rm D}} \right\},$$

$$\boldsymbol{\mathcal{CR}}_h(\Omega_{\rm D}) := \left\{ \begin{array}{l} q_h \in \mathrm{L}^2(\Omega_{\rm D}) \ : \ q_h|_K \in \mathbb{P}_1 \ \forall K \in \mathcal{T}_h^{\Omega_{\rm D}} \text{ and } q_h \text{ is continuous at} \\ \text{the centroid of any face } f \text{ common to two elements in } \mathcal{T}_h^{\Omega_{\rm D}} \end{array} \right\}.$$

We consider the subspace

$$\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D}) := \left\{ \begin{array}{l} q_h \in \boldsymbol{\mathcal{CR}}_h(\Omega_{\rm D}) \ : \ q_h(\boldsymbol{x}) = 0 \text{ for all midpoints } \boldsymbol{x} \text{ of faces of } \Gamma_{\rm D}^{\star}, \text{ and,} \\ \text{for } k = N+1, \dots, M, \ q_h(\boldsymbol{x}) = c_k \text{ (constant) for all midpoints } \boldsymbol{x} \text{ of faces of } \Gamma_{\rm I}^k \end{array} \right\}.$$

For $q_h \in \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})$, let $\mathbf{grad}_h \, q_h$ denote the vector field in $\mathcal{Q}_h^3$ defined by $(\mathbf{grad}_h \, q_h)|_K := \mathbf{grad}(q_h|_K)$, $K \in \mathcal{T}_h^{\Omega_{\rm D}}$. The following result has been proved in [51, Theorem 4.9] (see also [5, Lemma 5.4] for $\partial\Omega_{\rm D}$ non connected).

Lemma 1.9 *$\mathcal{Q}_h^3 = \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right) \oplus \mathbf{grad}_h\left(\boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})\right)$ and the decomposition is orthogonal in $L^2(\Omega_{\rm D})^3$.*

Taking into account this decomposition, we consider the following discrete problem.

Problem 1.10 *Given $\boldsymbol{I} \in \mathbb{C}^N$, find $\boldsymbol{H}_h \in \boldsymbol{\mathcal{U}}_h(\boldsymbol{I})$, $\boldsymbol{Z}_h \in \mathcal{Q}_h^3$ and $p_h \in \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D})$ such that*

$$a(\boldsymbol{H}_h, \boldsymbol{G}_h) + \int_{\Omega_{\rm D}} \boldsymbol{Z}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{G}}_h = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{U}}_h(\mathbf{0}), \tag{1.9}$$

$$\int_{\Omega_{\rm D}} \mathbf{curl}\, \boldsymbol{H}_h \cdot \bar{\boldsymbol{F}}_h + \int_{\Omega_{\rm D}} \mathbf{grad}_h \, p_h \cdot \bar{\boldsymbol{F}}_h = 0 \qquad \forall \boldsymbol{F}_h \in \mathcal{Q}_h^3, \tag{1.10}$$

$$\int_{\Omega_{\rm D}} \boldsymbol{Z}_h \cdot \mathbf{grad}_h \, \bar{q}_h = 0 \qquad \forall q_h \in \boldsymbol{\mathcal{CR}}_h^0(\Omega_{\rm D}). \tag{1.11}$$

Next result shows that Problem 1.10 is equivalent to Problems 1.6 and, hence, to Problem 1.3 and 1.4, too.

Proposition 1.11 *Let $\boldsymbol{I} \in \mathbb{C}^N$. If $(\boldsymbol{H}_h, \boldsymbol{Z}_h)$ is solution of Problem 1.6, then $(\boldsymbol{H}_h, \boldsymbol{Z}_h, 0)$ solves Problem 1.10. Conversely, if $(\boldsymbol{H}_h, \boldsymbol{Z}_h, p_h)$ solves Problem 1.10, then $p_h = 0$ and $(\boldsymbol{H}_h, \boldsymbol{Z}_h)$ is solution of Problem 1.6.*

Proof. Let $(\boldsymbol{H}_h, \boldsymbol{Z}_h)$ be solution of Problem 1.6. Then $(\boldsymbol{H}_h, \boldsymbol{Z}_h, 0)$ satisfies (1.9) and (1.10), the latter by virtue of Lemma 1.9. On the other hand, (1.11) follows from the fact that $\boldsymbol{Z}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})\right)$ and Lemma 1.9 again. Conversely, let $(\boldsymbol{H}_h, \boldsymbol{Z}_h, p_h)$ be solution of Problem 1.10. By testing (1.10) with $\boldsymbol{F}_h = \mathbf{grad}_h \, p_h$, it follows from Lemma 1.9 that $p_h = 0$. The same lemma and

(1.11) imply that $\boldsymbol{Z}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}})\right)$. Hence, for $p_h = 0$, (1.9) and (1.10) shows that $(\boldsymbol{H}_h, \boldsymbol{Z}_h)$ solves Problem 1.6. ■

As a consequence of the above proposition and the well-posedness of Problem 1.6, it follows that Problem 1.10 has also a unique solution. Thus, using standard basis for the finite element spaces leads to a linear system with a non-singular matrix. On the other hand, the approximation properties proved in [19] for Problem 1.3, automatically lead to optimal order error estimates for the component $\boldsymbol{H}_h$ of the solution to Problem 1.10.

1.5 Numerical experiments

In this section we report some numerical results obtained with a MATLAB code which implements the numerical methods described above.

First, we have solved an example to confirm the theoretical equivalence proved above. Figure 1.2 shows a sketch of the domain where the conducting part Ω_{C} and the whole domain Ω are coaxial cylinders.

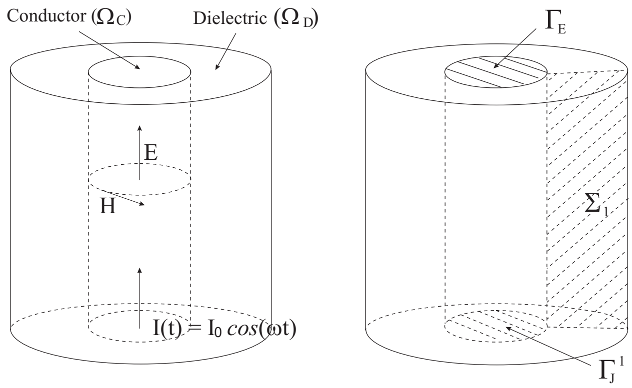

Figure 1.2: Sketch of the domain in the analytical example.

This example has been previously used in [19] to validate the computer code. In this section, we have compared the solution obtained with the formulation based on the magnetic field/magnetic scalar potential (Problem 1.6) and the code based on the mixed Problem 1.10. The numerical results provided by both techniques coincide up to rounding errors. Notice that in this simple geometry, the cutting surface can be placed in any plane θ-constant and the magnetic field/magnetic scalar potential formulation has clear advantages in terms of the number of unknowns; namely, in the finest mesh used for this geometry, Problem 1.6 involves $146,500$ unknowns while Problem 1.10 requires almost 9 times this number, $1,298,499$. At this point, we would like to remark that the mixed Problem 1.10 guarantees uniqueness of solution, but we could also avoid the cutting surfaces by solving the singular Problem 1.6 by means of a suitable iterative solver with an efficient preconditioner. Indeed, we have experimented with a preconditioned gradient method based on the preconditioner described in [67] and we have obtained the same results if the convergence tolerance is low enough; the number of unknowns is in this case $586,659$ (4 times those of Problem 1.6).

The advantage of the mixed formulation is clear in cases with complex geometries. For instance, we have computed the induced currents produced in a cylindrical workpiece which is surrounded by a helical coil carrying an alternating current. This configuration is typical, for instance, in induction heating furnaces (see [11]). The coil and the workpiece are shown in Figure 1.3, which also shows the box Ω surrounding the conducting domain.

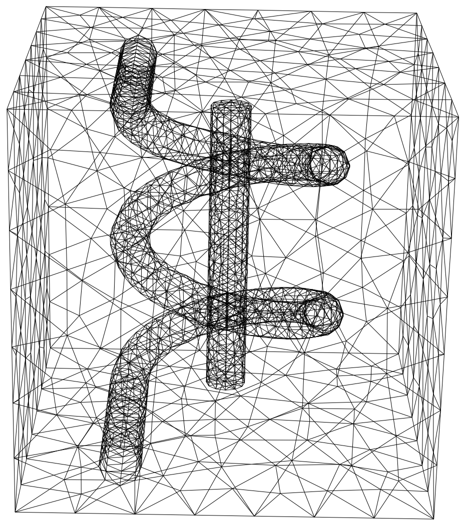

Figure 1.3: Domain composed by an helical coil, a cylindrical workpiece and air around.

Notice that the cutting surface which is needed in the dielectric mesh for Problem 1.4 is not easy to build in this case. To avoid it, we have solved the mixed discrete Problem 1.10. Concerning the physical data of the experiment, we have considered $\mu = 4\pi \times 10^{-7}\,\mathrm{Hm}^{-1}$, $\sigma = 11 \times 10^{6}\,(\Omega\mathrm{m})^{-1}$ in the workpiece and $58 \times 10^{6}\,(\Omega\mathrm{m})^{-1}$ in the coil. The amplitude of the alternating current intensity is taken equal to 40000 A and the angular frequency $2\pi \times 35000\,\mathrm{Hz}$. Figures 1.4 and 1.5 show the real part of the current density vector field in the conducting domain. Figure 1.6 shows the modulus of the current density in the conducting domain.

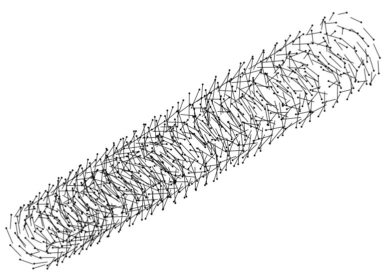

Figure 1.4: Distribution of the current density (real part) in the workpiece.

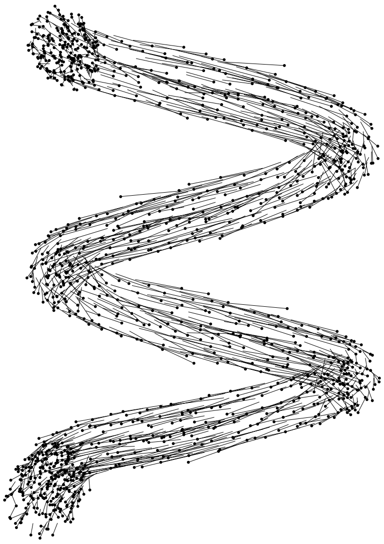

Figure 1.5: Distribution of the current density (real part) in the coil.

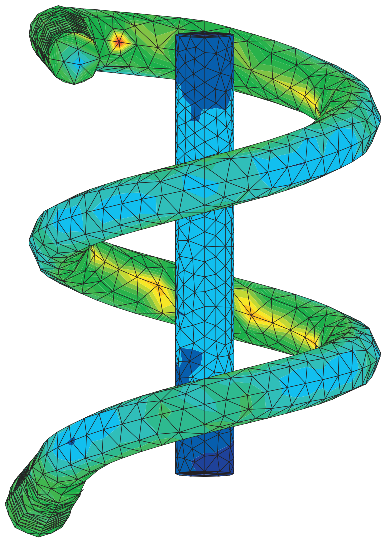

Figure 1.6: Modulus of the current density in the conductor.

Chapter 2

Numerical solution of transient eddy current problems with input current intensities as boundary data

2.1 Introduction

The objective of this work is to analyze a time-dependent eddy current problem defined in a three-dimensional domain including conducting and dielectric materials, when the source is given in terms of current intensities. This model arises in applications where the problem is posed in a bounded domain and it is necessary to link the electromagnetic fields with the sources provided by external circuits modeled by voltage drops and/or current intensities (see, for instance,[22]). In particular, we are interested in imposing the current intensities entering some conducting regions by means of boundary conditions. In this framework, we refer the reader to [6], where the authors give a systematic approach to eddy-current problems driven by voltage or current intensity in the harmonic regime. Numerical analysis of different finite element methods to solve this kind of models can be found in [19] and [8]; in both cases, the proposed numerical methods have been applied to simulate metallurgical furnaces by means of harmonic eddy current models subjected to boundary conditions proposed in [22]. However, if the exciting source is non-sinusoidal or if the materials have a non-linear behavior, a genuine transient eddy current problem must be solved. The present paper aims to extend the analysis of the model studied in [19] for the harmonic regime to the general transient situation.

Several papers devoted to the numerical analysis of the three-dimensional time-dependent eddy current model, both in bounded and unbounded domains, by using finite element and coupled boundary element - finite element methods, can be found in the literature: [1, 2, 45, 46, 48, 49, 72]. However, in all these works, the current source is given as a volume current in a conducting region. Moreover, the models proposed in bounded domains only deal with homogeneous essential and/or natural boundary conditions. To the authors' knowledge, the transient linear eddy current problem with source current intensities as boundary data has not been analyzed before, and this is the main objective of the present paper.

Following [19], we propose a formulation based on the magnetic field in the conductor regions and a scalar magnetic potential in the dielectric one. The scalar potential is defined from the curl-free condition of the magnetic field in dielectrics and can be multivalued on certain cut surfaces in order to consider general topologies. Note that the introduction of this potential has two main advantages: it leads to an important saving from a computational point of view and it allows direct imposing of the current intensities in terms of the jumps of the scalar potential. From a mathematical point of view, we will obtain a parabolic problem and, provided the input intensities are smooth enough, we prove its well-posedness by using a suitable lifting of the boundary conditions. Then, we show that the weak solution satisfies, in some sense, the eddy current model initially posed. We propose a finite element method combined with a backward Euler time discretization to numerically solve the problem. Concerning the space discretization, the magnetic field is approximated by the lowest-order Nédélec edge finite elements and the magnetic potential by standard piecewise linear continuous elements. The current intensities are imposed as jumps of the multivalued magnetic potential on some prescribed cut surfaces. We obtain convergence results for the main physical quantities, namely, the magnetic field and the current density.

The outline of the paper is as follows: In Section 2.2 we introduce the transient eddy current model and state the geometrical framework for the analysis. In Section 2.3 we obtain a weak formulation of the problem and prove that it is well-posed. In Section 2.4 we introduce a space discretization based on finite elements and prove error estimates. In Section 2.5 we propose a backward Euler scheme for time discretization and obtain error estimates for the fully discretized problem. In Section 2.6 we report some numerical results; firstly, we present the results obtained for a test problem with known analytical solution, which confirms the order of convergence predicted by the theory and allows us to assess the performance of the method; secondly, we simulate an application to electromagnetic forming, where the transient simulation in the time domain is mandatory. We end the paper with an appendix where we prove an additional regularity result for parabolic problems that we have used in Section 2.3.

Throughout the paper, we use standard notation for function spaces, norms, and duality pairings.

2.2 A time-dependent eddy current problem with input current intensities as boundary data

Three dimensional eddy current problems describe low-frequency electromagnetic phenomena. In this case, displacement currents may be neglected (see, for instance, [21, Chapter 8]), so that

Maxwell's equations become

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J} \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{2.1}$$

$$\partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{2.2}$$

$$\operatorname{div}(\mu\boldsymbol{H}) = 0 \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{2.3}$$

$$\boldsymbol{J} = \sigma\boldsymbol{E} \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{2.4}$$

where $\boldsymbol{E}(t,\boldsymbol{x})$ is the electric field, $\boldsymbol{H}(t,\boldsymbol{x})$ is the magnetic field, $\boldsymbol{J}(t,\boldsymbol{x})$ the current density, μ the magnetic permeability and σ the electric conductivity. Here and thereafter, we use boldface letters to denote vector fields and variables as well as vector-valued operators.

To solve these equations we restrict them to a simply connected three-dimensional bounded domain Ω, which consists of two parts, Ω_{C} and Ω_{D}, occupied by conductors and dielectrics, respectively. The mathematical framework we are going to analyze covers transient eddy current problems posed on different geometrical settings. We sketch in Figure 2.1 a particular case including several connected components of the conducting domain with different topological properties.

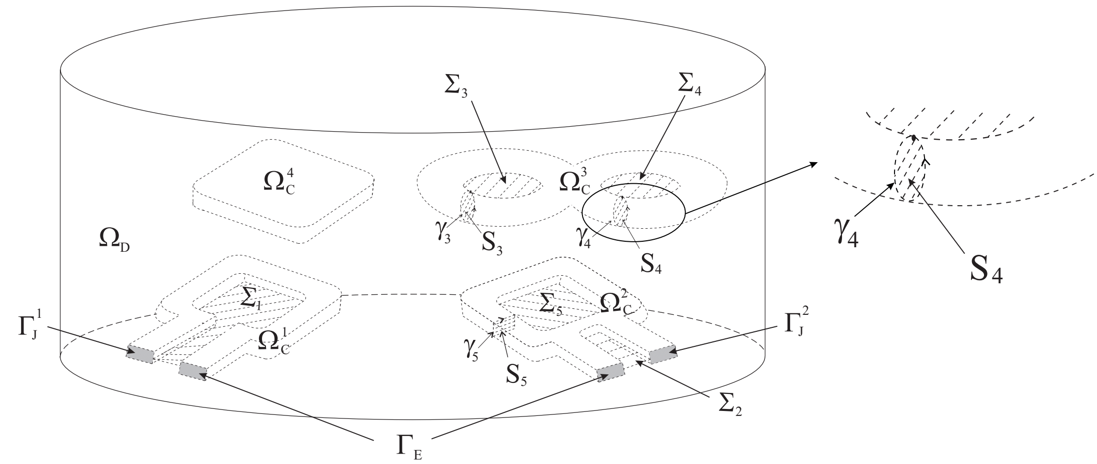

Figure 2.1: Sketch of the domain with a zoom around S_4.

The domain Ω is assumed to have a Lipschitz-continuous connected boundary $\partial\Omega$. We denote by Γ_{C}, Γ_{D} and Γ_{I} the open surfaces such that $\bar{\Gamma}_{\mathrm{C}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega$ is the outer boundary of the conductors domain, $\bar{\Gamma}_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \cap \partial\Omega$ that of the dielectrics domain and $\bar{\Gamma}_{\mathrm{I}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega_{\mathrm{D}}$ the interface between both domains. We also denote by $\boldsymbol{n}$, $\boldsymbol{n}_{\mathrm{C}}$ and $\boldsymbol{n}_{\mathrm{D}}$ the outer unit normal vectors to $\partial\Omega$, $\partial\Omega_{\mathrm{C}}$ and $\partial\Omega_{\mathrm{D}}$, respectively. Note that $\boldsymbol{n}_{\mathrm{C}} = \boldsymbol{n}$ on Γ_{C}, $\boldsymbol{n}_{\mathrm{D}} = \boldsymbol{n}$ on Γ_{D} and $\boldsymbol{n}_{\mathrm{C}} = -\boldsymbol{n}_{\mathrm{D}}$ on Γ_{I}.

As shown in Figure 2.1, the connected components of the conducting domain are of two types: "inductors" which go through the boundary of Ω, and "workpieces" which have their closure included in Ω. We denote $\Omega_{\mathrm{C}}^1, \ldots, \Omega_{\mathrm{C}}^N$ the former and $\Omega_{\mathrm{C}}^{N+1}, \ldots, \Omega_{\mathrm{C}}^M$ the latter.

We assume that the outer boundary of each inductor, $\partial\Omega_{\mathrm{C}}^n \cap \partial\Omega$ $(n = 1, \ldots, N)$, has two disjoint connected components, both being the closure of open surfaces: the current entrance $\Gamma_{\boldsymbol{J}}^n$, where the inductor is connected to a transient electric current source, and the current exit $\Gamma_{\boldsymbol{E}}^n$. We denote

$\Gamma_{\boldsymbol{J}} := \Gamma_{\boldsymbol{J}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{J}}^N$ and $\Gamma_{\boldsymbol{E}} := \Gamma_{\boldsymbol{E}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{E}}^N$. Furthermore, we assume that $\bar{\Gamma}_{\mathrm{J}}^n \cap \bar{\Gamma}_{\mathrm{J}}^m = \emptyset$, $\bar{\Gamma}_{\mathrm{E}}^n \cap \bar{\Gamma}_{\mathrm{E}}^m = \emptyset$, $1 \leq m, n \leq N$, $m \neq n$, and $\bar{\Gamma}_{\mathrm{J}} \cap \bar{\Gamma}_{\mathrm{E}} = \emptyset$.

We consider that μ and σ are time-independent and that there exist constants $\underline{\mu}$, $\overline{\mu}$, $\overline{\sigma}$ and $\underline{\sigma}$ such that

$$\begin{aligned} &0 < \underline{\mu} \leq \mu(\boldsymbol{x}) \leq \overline{\mu}, \quad \text{a.e. } \boldsymbol{x} \in \Omega, \\ &0 < \underline{\sigma} \leq \sigma(\boldsymbol{x}) \leq \overline{\sigma}, \quad \text{a.e. } \boldsymbol{x} \in \Omega_{\mathrm{C}} \quad \text{and} \quad \sigma \equiv 0 \text{ in } \Omega_{\mathrm{D}}. \end{aligned}$$

We have to complete the model with an initial condition, $\boldsymbol{H}(0) = \boldsymbol{H}_0$, and suitable boundary conditions. For the latter, we consider the following ones, which were proposed in [22] and were analyzed in [19] in the harmonic regime:

$$\int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\, \boldsymbol{H}(t) \cdot \boldsymbol{n} = I_n(t), \quad n = 1, \ldots, N, \quad t \in [0, T], \tag{2.5}$$

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } [0, T] \times \Gamma_{\boldsymbol{E}}, \tag{2.6}$$

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } [0, T] \times \Gamma_{\boldsymbol{J}}, \tag{2.7}$$

$$\mu \boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on } [0, T] \times \partial\Omega, \tag{2.8}$$

where the only data are the current intensities I_n through each surface $\Gamma_{\boldsymbol{J}}^n$.

Conditions (2.5) account for the input current intensities through each $\Gamma_{\boldsymbol{J}}^n$. Conditions (2.6)–(2.8) have been proposed in [22] in a more general setting. They will appear as natural boundary conditions of our weak formulation of the problem. The former implies the assumption that the electric current is normal to the current entrance and exit surfaces, whereas the latter means that the magnetic field is tangential to the boundary. (See [18] for further discussions on these boundary conditions and [19] for its application to the modeling of an electric furnace.)

2.3 Variational formulation. Existence and uniqueness of the solution

Our first goal is to give a variational formulation of the transient eddy current problem (2.1)–(2.8) in terms of the magnetic field. To do this, we follow the arguments from [19], which we include for the sake of completeness.

Let $\boldsymbol{G}$ be a smooth function such that

$$\mathbf{curl}\, \boldsymbol{G} = \mathbf{0} \quad \text{in } \Omega_{\mathrm{D}} \qquad \text{and} \qquad \int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n} = 0, \quad n = 1, \ldots, N. \tag{2.9}$$

From the first equation above, we have that these functions satisfy

$$\mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}_{\mathrm{D}} = 0 \quad \text{on } \partial\Omega_{\mathrm{D}}. \tag{2.10}$$

On the other hand, for each connected component Ω_{C}^n, there holds $\int_{\partial\Omega_{\mathrm{C}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}_{\mathrm{C}} = 0$. Therefore,

$$\int_{\Gamma_{\boldsymbol{E}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n} = -\int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, \qquad n = 1, \ldots, N. \tag{2.11}$$

We test (2.2) with a function $\boldsymbol{G}$ of this kind:

$$\int_\Omega \mu\partial_t \boldsymbol{H}\cdot\boldsymbol{G} + \int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{G} = 0. \tag{2.12}$$

Formal calculations allow us to show that boundary condition (2.8) implies that the tangential component of the electric field $\boldsymbol{E}$ is a gradient. Indeed, after integrating $\mu\partial_t\boldsymbol{H}\cdot\boldsymbol{n}$ on any surface S contained in $\partial\Omega$, by using (2.2) and Stokes' Theorem, we obtain

$$0 = \int_S \mu\partial_t\boldsymbol{H}\cdot\boldsymbol{n} = -\int_S \mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{n} = -\int_{\partial S}\boldsymbol{E}\cdot\boldsymbol{t} = -\int_{\partial S}\boldsymbol{n}\times(\boldsymbol{E}\times\boldsymbol{n})\cdot\boldsymbol{t},$$

with $\boldsymbol{t}$ being a unit vector tangent to ∂S. Therefore, since $\partial\Omega$ is simply connected, we can assert that there exists a sufficiently smooth function V defined in Ω up to a constant, such that $V|_{\partial\Omega}$ is a surface potential of the tangential component of $\boldsymbol{E}$; namely, $\boldsymbol{E}\times\boldsymbol{n} = -\,\mathbf{grad}\,V\times\boldsymbol{n}$ on $\partial\Omega$. On the other hand, (2.6) and (2.7) imply that V must be constant on each connected component of $\Gamma_{\boldsymbol{J}}$ and $\Gamma_{\boldsymbol{E}}$. Then, we can transform the second term of (2.12) by using a Green's formula as follows:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{G} = \int_\Omega \boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{G} - \int_{\partial\Omega}\boldsymbol{E}\times\boldsymbol{n}\cdot\boldsymbol{G} = \int_\Omega \boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{G}, \tag{2.13}$$

the last equality because

$$-\int_{\partial\Omega}\boldsymbol{E}\times\boldsymbol{n}\cdot\boldsymbol{G} = \int_{\partial\Omega}\mathbf{grad}\,V\times\boldsymbol{n}\cdot\boldsymbol{G} = \int_\Omega \mathbf{grad}\,V\cdot\mathbf{curl}\,\boldsymbol{G} = \int_{\partial\Omega} V\,\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n} = 0, \tag{2.14}$$

where, for the last equality, we have used that V is constant on each $\Gamma_{\boldsymbol{J}}^n$ and $\Gamma_{\boldsymbol{E}}^n$, (2.9), (2.10) and (2.11).

Now, by substituting (2.13) into (2.12), we obtain

$$\int_\Omega \mu\partial_t\boldsymbol{H}\cdot\boldsymbol{G} + \int_\Omega \boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{G} = 0.$$

Moreover, because of the first equation in (2.9), the second integral above reduces to the conducting domain Ω_{C}, whereas (2.1) and (2.4) lead to $\boldsymbol{E} = \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}$. Thus, we obtain

$$\int_\Omega \mu\partial_t\boldsymbol{H}\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{G} = 0.$$

Let

$$\boldsymbol{\mathcal{X}} := \left\{\boldsymbol{G}\in \mathrm{H}(\mathbf{curl};\Omega):\ \mathbf{curl}\,\boldsymbol{G} = \mathbf{0}\ \text{in}\ \Omega_{\mathrm{D}}\right\}.$$

We denote by $\langle\cdot,\cdot\rangle_{\partial\Omega}$ the duality pairing in $\mathrm{H}^{-1/2}(\partial\Omega)\times\mathrm{H}^{1/2}(\partial\Omega)$. For all $\boldsymbol{G}\in\boldsymbol{\mathcal{X}}$, we have that $\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}\in\mathrm{H}^{-1/2}(\partial\Omega)$ and $\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n} = 0$ on Γ_{D}, the latter in the following sense: $\langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},\zeta\rangle_{\partial\Omega} = 0$ for all $\zeta\in\mathrm{H}^{1/2}(\partial\Omega)$ that vanishes on Γ_{C}. In fact, for any such ζ, let $\hat{\zeta}\in\mathrm{H}^1(\Omega)$ be such that $\hat{\zeta}|_{\partial\Omega} = \zeta$ with $\hat{\zeta}|_{\Omega_{\mathrm{C}}} = 0$. Then, $\langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},\zeta\rangle_{\partial\Omega} = \int_\Omega \mathbf{curl}\,\boldsymbol{G}\cdot\mathbf{grad}\,\hat{\zeta} = 0$ because $\mathbf{curl}\,\boldsymbol{G} = 0$ in Ω_{D} and $\mathbf{grad}\,\hat{\zeta} = 0$ in Ω_{C}.

Moreover, for $\boldsymbol{G}\in\boldsymbol{\mathcal{X}}$, the expression $\langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},1\rangle_{\Gamma_{\boldsymbol{J}}^n}$ makes sense. Indeed, let ζ_n be any smooth function defined on $\partial\Omega$ such that $\zeta_n|_{\Gamma_{\boldsymbol{J}}^m} = \delta_{nm}$ and $\zeta_n = 0$ on $\Gamma_{\boldsymbol{E}}$ (such functions exist since

$\bar{\Gamma}^n_{\rm J} \cap \bar{\Gamma}^m_{\rm J} = \emptyset$, $m \neq n$, and $\bar{\Gamma}^n_{\rm J} \cap \bar{\Gamma}_{\rm E} = \emptyset$). Then, $\langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma^n_{\boldsymbol{J}}} := \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, \zeta_n \rangle_{\partial\Omega}$ is well defined, since its value does not depend on the particular choice of ζ_n, because $\mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n} = 0$ on $\Gamma_{\rm D}$. Hence, we define the following closed subspace of $\boldsymbol{\mathcal{X}}$:

$$\boldsymbol{\mathcal{V}} := \left\{ \boldsymbol{G} \in \boldsymbol{\mathcal{X}} : \ \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma^n_{\boldsymbol{J}}} = 0, \ n = 1, \dots, N \right\}.$$

Thus we arrive at the following problem: find $\boldsymbol{H} : [0, T] \to \boldsymbol{\mathcal{X}}$ such that

$$\int_{\Gamma^n_{\boldsymbol{J}}} \mathbf{curl}\, \boldsymbol{H}(t) \cdot \boldsymbol{n} = I_n(t), \quad n = 1, \dots, N, \tag{2.15}$$

$$\int_{\Omega} \mu \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{G} + \int_{\Omega_{\rm C}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}(t) \cdot \mathbf{curl}\, \boldsymbol{G} = 0 \quad \forall \boldsymbol{G} \in \boldsymbol{\mathcal{V}}, \tag{2.16}$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0. \tag{2.17}$$

Remark 2.1 *Many physical applications also involve voltage drops as boundary data. In such a case, the above approach also works and yields a very similar problem. For instance, let us suppose that the boundary data consist of the input current intensities I_n, for $n = 1, \dots, \widehat{N}$, and the voltage drops $V_n := V|_{\Gamma^n_{\boldsymbol{J}}} - V|_{\Gamma^n_{\boldsymbol{E}}}$, for $n = \widehat{N} + 1, \dots, N$. Then, we must use test functions $\boldsymbol{G}$ lying in the space $\widehat{\boldsymbol{\mathcal{V}}} := \left\{ \boldsymbol{G} \in \boldsymbol{\mathcal{X}} : \ \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma^n_{\boldsymbol{J}}} = 0, \ n = 1, \dots, \widehat{N} \right\}$. For such a function $\boldsymbol{G}$, by proceeding as to derive (2.14), we have that*

$$- \int_{\partial\Omega} \boldsymbol{E} \times \boldsymbol{n} \cdot \boldsymbol{G} = \int_{\partial\Omega} V\, \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n} = \sum_{n=\widehat{N}+1}^{N} V_n \int_{\Gamma^n_{\boldsymbol{J}}} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}.$$

Therefore, we are led to the following problem instead of (2.15)–(2.17): find $\boldsymbol{H} : [0, T] \to \boldsymbol{\mathcal{X}}$ such that

$$\int_{\Gamma^n_{\boldsymbol{J}}} \mathbf{curl}\, \boldsymbol{H}(t) \cdot \boldsymbol{n} = I_n(t), \quad n = 1, \dots, \widehat{N},$$

$$\int_{\Omega} \mu \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{G} + \int_{\Omega_{\rm C}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}(t) \cdot \mathbf{curl}\, \boldsymbol{G} = - \sum_{n=\widehat{N}+1}^{N} V_n(t) \int_{\Gamma^n_{\boldsymbol{J}}} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n} \quad \forall \boldsymbol{G} \in \widehat{\boldsymbol{\mathcal{V}}},$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0.$$

2.3.1 Introducing a magnetic potential

In what follows we show how problem (2.15)–(2.17) can be rewritten by replacing the magnetic field in the dielectric domain $\Omega_{\rm D}$ by a (scalar) magnetic potential.

We assume there exist L connected "cut" surfaces $\Sigma_n \subset \Omega_{\rm D}$, $n = 1, \cdots, L$, such that $\partial\Sigma_n \subset \partial\Omega_{\rm D}$ and $\widetilde{\Omega}_{\rm D} := \Omega_{\rm D} \setminus \bigcup_{n=1}^{L} \Sigma_n$ is pseudo-Lipschitz and simply connected (see, for instance, [9]). We also assume that $\bar{\Sigma}_n \cap \bar{\Sigma}_m = \emptyset$ for $n \neq m$ (see Figure 2.1). For each inductor, $\Omega^n_{\rm C}$, $n = 1, \dots, N$, there exists one cut surface Σ_n such that, necessarily, $\partial\Sigma_n \cap \Gamma_{\rm D} \neq \emptyset$ (see Figure 2.1). The remaining cut surfaces, $\Sigma_{N+1}, \dots, \Sigma_L$, are assumed to be contained in the interior of $\Omega_{\rm D}$ (see Figure 2.1, again).

For each cut surface Σ_n we assume that there exists a surface $S_n \subset \Omega_{\mathrm{C}}^n$, with $\partial S_n \subset \partial\Omega_{\mathrm{C}}^n$ and such that its boundary γ_n is a simple closed curve which intersects $\bar{\Sigma}_n$ once and only once, and does not intersect $\bar{\Sigma}_m$, $m \neq n$. Note that, for $n = 1, \dots, N$, we can take $S_n = \Gamma_{\boldsymbol{J}}^n$. We denote the two faces of each Σ_n by Σ_n^- and Σ_n^+. We choose an orientation for each γ_n by taking its initial and end points on Σ_n^- and Σ_n^+, respectively. We denote by $\boldsymbol{t}_n$ the unit vector tangent to γ_n according to this orientation.

Each function $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$ has in general different traces on each face of Σ_n and we denote by

$$[\![\widetilde{\Psi}]\!]_{\Sigma_n} := \widetilde{\Psi}|_{\Sigma_n^+} - \widetilde{\Psi}|_{\Sigma_n^-}$$

the jump of $\widetilde{\Psi}$ through Σ_n. The gradient of $\widetilde{\Psi}$ in $\mathcal{D}'(\widetilde{\Omega}_{\mathrm{D}})$ can be extended to $\mathrm{L}^2(\Omega_{\mathrm{D}})^3$ and will be denoted by $\widetilde{\mathbf{grad}}\,\widetilde{\Psi}$.

Let Θ be the linear subspace of $\mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$ defined by

$$\Theta := \left\{ \widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) : \ [\![\widetilde{\Psi}]\!]_{\Sigma_n} = \text{ constant}, \ n = 1, \dots, L \right\}.$$

Then, for $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$, we have that $\widetilde{\mathbf{grad}}\,\widetilde{\Psi} \in \mathrm{H}(\mathbf{curl}; \Omega_{\mathrm{D}})$ if and only if $\widetilde{\Psi} \in \Theta$, in which case $\mathbf{curl}(\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) = \mathbf{0}$ (see [9, Lemma 3.11]).

We use the following notation: given $\boldsymbol{G}_{\mathrm{C}} \in \mathrm{L}^2(\Omega_{\mathrm{C}})^3$ and $\boldsymbol{G}_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})^3$, $(\boldsymbol{G}_{\mathrm{C}}|\boldsymbol{G}_{\mathrm{D}})$ denotes the field $\boldsymbol{G} \in \mathrm{L}^2(\Omega)^3$ defined by $\boldsymbol{G}|_{\Omega_{\mathrm{C}}} := \boldsymbol{G}_{\mathrm{C}}$ and $\boldsymbol{G}|_{\Omega_{\mathrm{D}}} := \boldsymbol{G}_{\mathrm{D}}$.

Let us denote by $\boldsymbol{\mathcal{Y}}$ the linear space given by

$$\boldsymbol{\mathcal{Y}} := \left\{ (\boldsymbol{G}, \widetilde{\Psi}) \in \mathrm{H}(\mathbf{curl}; \Omega_{\mathrm{C}}) \times (\Theta/\mathbb{R}) : \ \left(\boldsymbol{G}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Psi}\right) \in \mathrm{H}(\mathbf{curl}; \Omega) \right\}.$$

Then $(\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}}$ if and only if $\left(\boldsymbol{G}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Psi}\right) \in \boldsymbol{\mathcal{X}}$.

When a magnetic potential $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$ is used, boundary condition (2.15) can be imposed by fixing its jumps on the cut surfaces. Indeed, if $(\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}}$ is smooth enough for the following integrals to make sense, we have that

$$\langle \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\boldsymbol{J}}^n} = \int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{n} = \int_{\gamma_n} \boldsymbol{G} \cdot \boldsymbol{t}_n = \int_{\gamma_n} \widetilde{\mathbf{grad}}\,\widetilde{\Psi} \cdot \boldsymbol{t}_n = [\![\widetilde{\Psi}]\!]_{\Sigma_n}, \tag{2.18}$$

where we have used Stokes' Theorem and the fact that $\boldsymbol{n}_{\mathrm{C}} \times (\boldsymbol{G} \times \boldsymbol{n}_{\mathrm{C}}) = \boldsymbol{n}_{\mathrm{D}} \times (\widetilde{\mathbf{grad}}\,\widetilde{\Psi} \times \boldsymbol{n}_{\mathrm{D}})$ on $\Gamma_{\mathrm{I}} \supset \gamma_n$.

Therefore, problem (2.15)–(2.17) reduces to find $(\boldsymbol{H}, \widetilde{\Phi}) : [0, T] \to \boldsymbol{\mathcal{Y}}$ such that

$$[\![\widetilde{\Phi}(t)]\!]_{\Sigma_n} = I_n(t), \quad n = 1, \dots, N, \tag{2.19}$$

$$\int_{\Omega_{\mathrm{C}}} \mu \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{G} + \int_{\Omega_{\mathrm{D}}} \mu \partial_t \widetilde{\mathbf{grad}}\,\widetilde{\Phi}(t) \cdot \widetilde{\mathbf{grad}}\,\widetilde{\Psi} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\,\boldsymbol{H}(t) \cdot \mathbf{curl}\,\boldsymbol{G} = 0 \quad \forall (\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}}^0, \tag{2.20}$$

$$\left(\boldsymbol{H}(0)|\,\widetilde{\mathbf{grad}}\,\widetilde{\Phi}(0)\right) = \boldsymbol{H}_0, \tag{2.21}$$

where

$$\boldsymbol{\mathcal{Y}}^0 := \left\{ (\boldsymbol{G}, \widetilde{\Psi}) \in \boldsymbol{\mathcal{Y}} \ : \ [\![\widetilde{\Psi}]\!]_{\Sigma_n} = 0, \ n = 1, \dots, N \right\}.$$

2.3.2 Existence and uniqueness of the solution

In this section we will prove the existence and uniqueness of the solution to the transient eddy current problem (2.15)–(2.17). With this aim, we recall the classical functional framework for functions defined on a bounded interval $[0,T]$ and with values in a separable Hilbert space X. We use the notation $\mathcal{C}([0,T];X)$ for the Banach space consisting of all continuous functions $f:[0,T]\to X$. We also consider the space $\mathrm{L}^2(0,T;X)$ of classes of functions $f:[0,T]\to X$ that are Böchner-measurable and such that $\|f\|_{\mathrm{L}^2(0,T;X)} := \left(\int_0^T \|f(t)\|_X^2\,dt\right)^{1/2} < \infty$. Furthermore, we will use the space $\mathrm{H}^1(0,T;X) := \left\{f\in \mathrm{L}^2(0,T;X):\ \partial_t f\in \mathrm{L}^2(0,T;X)\right\}$. (We will use indistinctly the notations $\partial_t f$ and $\dfrac{df}{dt}$ for the derivative with respect to the variable t.) Analogously, we define $\mathrm{H}^k(0,T;X)$ for all $k\in\mathbb{N}$.

We denote by $\boldsymbol{\mathcal{H}_\mathcal{V}}$ the closure of $\boldsymbol{\mathcal{V}}$ in $\mathrm{L}^2(\Omega)^3$ and by $\boldsymbol{\mathcal{V}}'$ the dual space of $\boldsymbol{\mathcal{V}}$ with respect to the pivot space $\boldsymbol{\mathcal{H}_\mathcal{V}}$ with measure $\mu(\boldsymbol{x})\,d\boldsymbol{x}$ (which is topologically equivalent to $\mathrm{L}^2(\Omega)^3$ with the standard Lebesgue measure). Hence, for $\boldsymbol{F}\in\boldsymbol{\mathcal{H}_\mathcal{V}}$ we have

$$\langle \boldsymbol{F},\boldsymbol{G}\rangle_{\boldsymbol{\mathcal{V}}'\times\boldsymbol{\mathcal{V}}} = \int_\Omega \mu\boldsymbol{F}\cdot\boldsymbol{G} \qquad \forall\boldsymbol{G}\in\boldsymbol{\mathcal{V}}.$$

We will also use the closure of $\boldsymbol{\mathcal{X}}$ in $\mathrm{L}^2(\Omega)^3$, which we denote by $\boldsymbol{\mathcal{H}_\mathcal{X}}$, and the dual space $\boldsymbol{\mathcal{X}}'$ of $\boldsymbol{\mathcal{X}}$ with respect to the pivot space $\boldsymbol{\mathcal{H}_\mathcal{X}}$ with measure $\mu(\boldsymbol{x})\,d\boldsymbol{x}$. We have the following characterization.

Lemma 2.2

$$\boldsymbol{\mathcal{H}_\mathcal{X}} = \left\{\boldsymbol{G}\in\mathrm{L}^2(\Omega)^3:\ \mathbf{curl}\,\boldsymbol{G}=\mathbf{0}\ \textit{in}\ \Omega_{\mathrm{D}}\right\}.$$

Proof. Given that $\left\{\boldsymbol{G}\in\mathrm{L}^2(\Omega)^3:\ \mathbf{curl}\,\boldsymbol{G}=\mathbf{0}\text{ in }\Omega_{\mathrm{D}}\right\}$ is a closed subspace of $\mathrm{L}^2(\Omega)^3$, it is enough to prove that $\boldsymbol{\mathcal{X}}$ is densely included in this subspace.

With this aim, let $\boldsymbol{G}\in\mathrm{L}^2(\Omega)^3$ be such that $\mathbf{curl}\,\boldsymbol{G}=\mathbf{0}$ in Ω_{D}. Let $\widetilde{\boldsymbol{G}}\in\mathrm{H}(\mathbf{curl};\Omega)$ be such that $\widetilde{\boldsymbol{G}}=\boldsymbol{G}$ in Ω_{D}. Since $\mathcal{D}(\Omega_{\mathrm{C}})$ is dense in $\mathrm{L}^2(\Omega_{\mathrm{C}})$, there exists a sequence $\{\boldsymbol{\Phi}_k\}_{k\in\mathbb{N}}\subset\mathcal{D}(\Omega_{\mathrm{C}})^3$ such that $\|\boldsymbol{\Phi}_k-(\boldsymbol{G}-\widetilde{\boldsymbol{G}})\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\to 0$ as $k\to\infty$. If we denote by $\widetilde{\boldsymbol{\Phi}}_k$ the extension by zero of $\boldsymbol{\Phi}_k$ to Ω, then $\widetilde{\boldsymbol{\Phi}}_k+\widetilde{\boldsymbol{G}}\in\boldsymbol{\mathcal{X}}$ for all $k\in\mathbb{N}$ and $\|(\widetilde{\boldsymbol{\Phi}}_k+\widetilde{\boldsymbol{G}})-\boldsymbol{G}\|_{\mathrm{L}^2(\Omega)^3}\to 0$ as $k\to\infty$. ■

Let a be defined over $\boldsymbol{\mathcal{X}}\times\boldsymbol{\mathcal{X}}$ by

$$a(\boldsymbol{K},\boldsymbol{G}) := \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{K}\cdot\mathbf{curl}\,\boldsymbol{G}.$$

This is a continuous bilinear form and it satisfies the following Gårding's inequality: for each $\lambda>0$ there exists $\alpha>0$ such that

$$a(\boldsymbol{G},\boldsymbol{G})+\lambda\|\boldsymbol{G}\|^2_{\mathrm{L}^2(\Omega)^3}\ge\alpha\|\boldsymbol{G}\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \qquad \forall\boldsymbol{G}\in\boldsymbol{\mathcal{X}}. \tag{2.22}$$

Therefore, problem (2.15)–(2.17) can be written as follows:

Problem 2.3 *Find* $\boldsymbol{H}\in\mathrm{L}^2(0,T;\boldsymbol{\mathcal{X}})\cap\mathrm{H}^1(0,T;\boldsymbol{\mathcal{H}_\mathcal{X}})$ *such that*

$$\langle\mathbf{curl}\,\boldsymbol{H}(t)\cdot\boldsymbol{n},1\rangle_{\Gamma_J^n} = I_n(t), \quad n=1,\dots,N, \tag{2.23}$$

$$\int_\Omega \mu\partial_t\boldsymbol{H}(t)\cdot\boldsymbol{G}+a(\boldsymbol{H}(t),\boldsymbol{G})=0 \quad \forall\boldsymbol{G}\in\boldsymbol{\mathcal{V}}, \tag{2.24}$$

$$\boldsymbol{H}(0)=\boldsymbol{H}_0. \tag{2.25}$$

For the data of this problem we make the following assumptions (the reason for them will be clear below):

$$I_n \in \mathrm{H}^2(0,T), \quad n = 1, \dots, N, \tag{2.26}$$

$$\boldsymbol{H}_0 \in \boldsymbol{\mathcal{X}} \quad \text{and} \quad \langle \mathbf{curl}\, \boldsymbol{H}_0 \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = I_n(0), \quad n = 1, \dots, N. \tag{2.27}$$

In order to show that Problem 2.3 has a unique solution, our first step is to build an auxiliary function $\widehat{\boldsymbol{H}} \in \mathrm{H}^2(0,T;\boldsymbol{\mathcal{X}})$ satisfying (2.23). With this aim, we will use the unique solutions $w_n \in \mathrm{H}^1(\Omega_{\mathrm{C}}^n)$, $n = 1, \dots, N$, of the following problems:

$$\begin{aligned} -\Delta w_n &= 0 \quad \text{in } \Omega_{\mathrm{C}}^n, \\ \frac{\partial w_n}{\partial n} &= \begin{cases} \dfrac{1}{\operatorname{area}(\Gamma_{\boldsymbol{J}}^n)} & \text{on } \Gamma_{\boldsymbol{J}}^n, \\ 0 & \text{on } \partial\Omega_{\mathrm{C}}^n \cap \Gamma_{\mathrm{I}}, \end{cases} \\ w_n &= 0 \quad \text{on } \Gamma_{\boldsymbol{E}}^n. \end{aligned}$$

Let $\boldsymbol{Q}_n \in \mathrm{L}^2(\Omega)^3$ be defined by

$$\boldsymbol{Q}_n := \begin{cases} \mathbf{grad}\, w_n & \text{in } \Omega_{\mathrm{C}}^n, \\ \mathbf{0} & \text{in } \Omega_{\mathrm{C}}^m, \quad 1 \le m \le M, \quad m \neq n, \\ \mathbf{0} & \text{in } \Omega_{\mathrm{D}}. \end{cases}$$

Since $\operatorname{div}(\mathbf{grad}\, w_n) = 0$ in Ω_{C}^n and $\mathbf{grad}\, w_n \cdot \boldsymbol{n}_{\mathrm{C}} = 0$ on $\partial\Omega_{\mathrm{C}}^n \cap \Gamma_{\mathrm{I}}$, we have that $\boldsymbol{Q}_n \in \mathrm{H}(\operatorname{div};\Omega)$ and $\operatorname{div} \boldsymbol{Q}_n = 0$ in Ω. Then, since $\partial\Omega$ is connected, there exists a vector potential $\widehat{\boldsymbol{H}}_n \in \mathrm{H}(\mathbf{curl};\Omega)$ such that

$$\mathbf{curl}\, \widehat{\boldsymbol{H}}_n = \boldsymbol{Q}_n,$$

which satisfies $\operatorname{div} \widehat{\boldsymbol{H}}_n = 0$ and $\widehat{\boldsymbol{H}}_n \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ (see [38, Theorem I.3.5]).

Therefore, we define

$$\widehat{\boldsymbol{H}}(t) := \sum_{n=1}^{N} I_n(t) \widehat{\boldsymbol{H}}_n. \tag{2.28}$$

Note that, by virtue of (2.26), $\widehat{\boldsymbol{H}} \in \mathrm{H}^2(0,T;\boldsymbol{\mathcal{X}})$ and, clearly,

$$\|\widehat{\boldsymbol{H}}\|_{H^2(0,T;\boldsymbol{\mathcal{X}})} \le C \sum_{n=1}^{N} \|I_n\|_{\mathrm{H}^2(0,T)}; \tag{2.29}$$

here and thereafter C denotes a generic constant not necessarily the same at each occurrence. Moreover,

$$\langle \mathbf{curl}\, \widehat{\boldsymbol{H}}(t) \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = \langle I_n(t) \boldsymbol{Q}_n \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = I_n(t) \int_{\Gamma_{\boldsymbol{J}}^n} \frac{\partial w_n}{\partial n} = I_n(t), \qquad n = 1, \dots, N, \tag{2.30}$$

so that $\widehat{\boldsymbol{H}}$ satisfies (2.23).

We propose to find a solution to Problem 2.3 of the form $\boldsymbol{H} = \widetilde{\boldsymbol{H}} + \widehat{\boldsymbol{H}}$. To this end, we define $f : [0, T] \to \boldsymbol{\mathcal{V}}'$ by

$$\langle f(t), \boldsymbol{G} \rangle_{\boldsymbol{\mathcal{V}}' \times \boldsymbol{\mathcal{V}}} := - \int_\Omega \mu \partial_t \widehat{\boldsymbol{H}}(t) \cdot \boldsymbol{G} - a(\widehat{\boldsymbol{H}}(t), \boldsymbol{G}), \qquad \boldsymbol{G} \in \boldsymbol{\mathcal{V}}. \tag{2.31}$$

Then, $\widetilde{\boldsymbol{H}}$ has to be a solution to the following problem:

Problem 2.4 *Find* $\widetilde{\boldsymbol{H}} \in \mathrm{L}^2(0, T; \boldsymbol{\mathcal{V}}) \cap \mathrm{H}^1(0, T; \boldsymbol{\mathcal{V}}')$ *such that*

$$\begin{aligned} \langle \partial_t \widetilde{\boldsymbol{H}}(t), \boldsymbol{G} \rangle_{\boldsymbol{\mathcal{V}}' \times \boldsymbol{\mathcal{V}}} + a(\widetilde{\boldsymbol{H}}(t), \boldsymbol{G}) = \langle f(t), \boldsymbol{G} \rangle_{\boldsymbol{\mathcal{V}}' \times \boldsymbol{\mathcal{V}}} \quad \forall \boldsymbol{G} \in \boldsymbol{\mathcal{V}}, \\ \widetilde{\boldsymbol{H}}(0) = \boldsymbol{H}_0 - \widehat{\boldsymbol{H}}(0). \end{aligned}$$

Regarding the data of this problem, from (2.28) and (2.31), we have that

$$\|f(t)\|^2_{\boldsymbol{\mathcal{V}}'} \le C \left\{ \sum_{n=1}^N |I'_n(t)|^2 + \sum_{n=1}^N |I_n(t)|^2 \right\}$$

and, hence, $f \in \mathrm{L}^2(0, T; \boldsymbol{\mathcal{V}}')$. On the other hand, as a consequence of (2.27) and (2.30) we have that $\boldsymbol{H}_0 - \widehat{\boldsymbol{H}}(0) \in \boldsymbol{\mathcal{V}} \subset \boldsymbol{\mathcal{H}}_{\boldsymbol{\mathcal{V}}}$.

Thus we are in a position to apply the classical theory for parabolic problems (see, for instance, [28, Chapter XVIII]) combined with an exponential shift (since a is not elliptic but satisfies a Gårding's inequality) to conclude that Problem 2.4 has a unique solution and there exists a constant $C > 0$, independent of I_n and $\boldsymbol{H}_0$, such that

$$\sup_{t \in [0,T]} \|\widetilde{\boldsymbol{H}}(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^T \|\widetilde{\boldsymbol{H}}(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \, dt \le C \left\{ \|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3} + \sum_{n=1}^N \|I_n\|^2_{\mathrm{H}^1(0,T)} \right\}.$$

However, the previous argument is not enough to derive that $\boldsymbol{H} = \widetilde{\boldsymbol{H}} + \widehat{\boldsymbol{H}}$ is a solution to Problem 2.3, since such a solution has to satisfy $\boldsymbol{H} \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{H}}_{\boldsymbol{\mathcal{X}}})$. To obtain this, we must prove additional regularity for the solution to Problem 2.4.

With this aim, note that $f \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{V}}')$. In fact, $\partial_t f : [0, T] \to \boldsymbol{\mathcal{V}}'$ is given by

$$\langle \partial_t f(t), \boldsymbol{G} \rangle_{\boldsymbol{\mathcal{V}}' \times \boldsymbol{\mathcal{V}}} := - \int_\Omega \mu \partial_{tt} \widehat{\boldsymbol{H}}(t) \cdot \boldsymbol{G} - a(\partial_t \widehat{\boldsymbol{H}}(t), \boldsymbol{G}), \qquad \boldsymbol{G} \in \boldsymbol{\mathcal{V}},$$

and hence

$$\|\partial_t f(t)\|^2_{\boldsymbol{\mathcal{V}}'} \le C \left\{ \sum_{n=1}^N |I''_n(t)|^2 + \sum_{n=1}^N |I'_n(t)|^2 \right\}.$$

Therefore, we have the following result, whose proof can be found in the appendix (cf. Corollary A.2).

Lemma 2.5 *The solution to Problem 2.4 satisfies* $\widetilde{\boldsymbol{H}} \in \mathrm{L}^\infty(0, T; \boldsymbol{\mathcal{V}})$ *and* $\partial_t \widetilde{\boldsymbol{H}} \in \mathrm{L}^2(0, T; \boldsymbol{\mathcal{H}}_{\boldsymbol{\mathcal{V}}})$. *Moreover, there exists a constant* $C > 0$, *independent of* I_n *and* $\boldsymbol{H}_0$, *such that*

$$\|\widetilde{\boldsymbol{H}}\|^2_{\mathrm{L}^\infty(0,T;\boldsymbol{\mathcal{V}})} + \|\partial_t \widetilde{\boldsymbol{H}}\|^2_{\mathrm{L}^2(0,T;\boldsymbol{\mathcal{H}}_{\boldsymbol{\mathcal{V}}})} \le C \left\{ \|\boldsymbol{H}_0\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \sum_{n=1}^N \|I_n\|^2_{\mathrm{H}^2(0,T)} \right\}. \tag{2.32}$$

Now, we are in a position to prove that Problem 2.3 is well-posed.

Theorem 2.6 *Let I_n, $n = 1, \dots, N$, and $\boldsymbol{H}_0$ satisfy (2.26) and (2.27). Then, Problem 2.3 has a unique solution $\boldsymbol{H}$. Furthermore, $\boldsymbol{H} \in \mathrm{L}^\infty(0, T; \boldsymbol{\mathcal{X}})$ and there exists a constant $C > 0$, independent of I_n and $\boldsymbol{H}_0$, such that*

$$\|\boldsymbol{H}\|^2_{\mathrm{L}^\infty(0,T;\boldsymbol{\mathcal{X}})} + \|\partial_t \boldsymbol{H}\|^2_{\mathrm{L}^2(0,T;\boldsymbol{\mathcal{H}_\mathcal{X}})} \le C \left\{ \|\boldsymbol{H}_0\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \sum_{n=1}^{N} \|I_n\|^2_{\mathrm{H}^2(0,T)} \right\}$$

Proof. Let $\widehat{\boldsymbol{H}}$ be the function defined by (2.28) and $\widetilde{\boldsymbol{H}}$ the unique solution to Problem 2.4. According to Lemma 2.5, $\partial_t \widetilde{\boldsymbol{H}} \in \mathrm{L}^2(0, T; \boldsymbol{\mathcal{H}_\mathcal{V}}) \subset \mathrm{L}^2(0, T; \boldsymbol{\mathcal{H}_\mathcal{X}})$, whereas $\widehat{\boldsymbol{H}} \in \mathrm{H}^2(0, T; \boldsymbol{\mathcal{X}})$. Therefore, $\boldsymbol{H} := \widetilde{\boldsymbol{H}} + \widehat{\boldsymbol{H}} \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{H}_\mathcal{X}})$ is a solution to Problem 2.3, which furthermore satisfies $\boldsymbol{H} \in \mathrm{L}^\infty(0, T; \boldsymbol{\mathcal{X}})$. Moreover, from (2.32) and (2.29), we immediately obtain the estimate of the theorem.

There only remains to prove that the problem has at most one solution. With this aim, let $\breve{\boldsymbol{H}}$ be a solution of the corresponding problem with vanishing data $I_n = 0$, $n = 1, \dots, N$, and $\boldsymbol{H}_0 = \mathbf{0}$. Then, by taking $\boldsymbol{G} = \breve{\boldsymbol{H}}(t)$ and integrating over $[0, t]$, we obtain

$$\frac{1}{2}\underline{\mu} \|\breve{\boldsymbol{H}}(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^t a(\breve{\boldsymbol{H}}(s), \breve{\boldsymbol{H}}(s))\, ds \le 0.$$

Hence, since $a(\breve{\boldsymbol{H}}(s), \breve{\boldsymbol{H}}(s)) \ge 0$, we have that $\breve{\boldsymbol{H}} \equiv \mathbf{0}$. Thus, we conclude the proof. ■

Remark 2.7 *The analysis above can be readily adapted to show that the problem introduced in Remark 2.1 also has a unique solution, provided $V_n \in \mathrm{H}^1(0, T)$, $n = \widehat{N} + 1, \dots, N$.*

Other quantities of physical interest can be computed from the solution $\boldsymbol{H}$ to Problem 2.3; in particular, $\boldsymbol{J}(t) := \mathbf{curl}\, \boldsymbol{H}(t)$ in Ω and $\boldsymbol{E}(t) := \frac{1}{\sigma} \boldsymbol{J}(t)$ in Ω_{C}. Our next goal is to prove that these quantities satisfy equations (2.1)–(2.8) in their respective domains of definition.

Theorem 2.8 *Let $\boldsymbol{H}$ be the solution to Problem 2.3, with data $\boldsymbol{H}_0$ satisfying*

$$\operatorname{div}(\mu \boldsymbol{H}_0) = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \mu \boldsymbol{H}_0 \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega.$$

Let $\boldsymbol{J}(t) := \mathbf{curl}\, \boldsymbol{H}(t)$ and $\boldsymbol{E}(t) := \left(\frac{1}{\sigma} \boldsymbol{J}(t)\right)\big|_{\Omega_{\mathrm{C}}}$. Then, the following properties hold true a.e. $t \in (0, T)$:

$$\operatorname{div}(\mu \boldsymbol{H}(t)) = 0 \quad \text{in } \Omega, \tag{2.33}$$

$$\mu \partial_t \boldsymbol{H}(t) + \mathbf{curl}\, \boldsymbol{E}(t) = \mathbf{0} \quad \text{in } \Omega_{\mathrm{C}}, \tag{2.34}$$

$$\boldsymbol{J}(t) = \mathbf{0} \quad \text{in } \Omega_{\mathrm{D}}, \tag{2.35}$$

$$\langle \mathbf{curl}\, \boldsymbol{H}(t) \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_J^n} = I_n(t), \quad n = 1, \dots, N, \tag{2.36}$$

$$\mu \boldsymbol{H}(t) \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{2.37}$$

$$\boldsymbol{E}(t) \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{C}}. \tag{2.38}$$

Proof. The proof follows the lines of that of Theorem 7 from [19]. Given $v \in \mathcal{D}(\Omega)$, it follows that $\mathbf{grad}\, v \in \boldsymbol{\mathcal{V}}$. Then (2.24) yields

$$\int_\Omega \mu \partial_t \boldsymbol{H}(t) \cdot \mathbf{grad}\, v = 0.$$

Hence, $\operatorname{div}(\mu\partial_t \boldsymbol{H}(t)) = 0$ and, consequently, $\partial_t \operatorname{div}(\mu \boldsymbol{H}(t)) = 0$ (see [71, Theorems 111 & 113]). Therefore, (2.33) follows from the fact that $\operatorname{div}(\mu \boldsymbol{H}(0)) = \operatorname{div}(\mu \boldsymbol{H}_0) = 0$.

Now, let $\boldsymbol{G} \in \mathcal{D}(\Omega)^3$ be such that $\operatorname{supp} \boldsymbol{G} \subset \Omega_{\mathrm{C}}$. Then $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}$ too and (2.24) yields

$$\int_{\Omega_{\mathrm{C}}} \mu \partial_t \boldsymbol{H}(t) \cdot \boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}(t) \cdot \mathbf{curl}\, \boldsymbol{G} = 0.$$

Hence, $\boldsymbol{E}(t) := \left(\frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}(t)\right)\big|_{\Omega_{\mathrm{C}}}$ satisfies (2.34).

Equation (2.35) follows from the definition of $\boldsymbol{J}(t)$ and the fact that $\boldsymbol{H}(t) \in \boldsymbol{\mathcal{X}}$, whereas equation (2.36) follows from (2.23).

To prove (2.37), note that $\mu\partial_t \boldsymbol{H}(t) \in \mathrm{H}(\operatorname{div};\Omega)$ because of (2.33). Then $\mu\partial_t \boldsymbol{H}(t)\cdot \boldsymbol{n} \in \mathrm{H}^{-1/2}(\partial\Omega)$. Moreover, given $v \in \mathrm{H}^1(\Omega)$, we have that

$$\langle \mu\partial_t \boldsymbol{H}(t)\cdot \boldsymbol{n}, v\rangle_{\partial\Omega} = \int_\Omega \operatorname{div}\left(\mu\partial_t \boldsymbol{H}(t)\right) v + \int_\Omega \mu\partial_t \boldsymbol{H}(t)\cdot \mathbf{grad}\, v = 0,$$

the last equality because of (2.33) and (2.24), since $\mathbf{grad}\, v \in \boldsymbol{\mathcal{V}}$. Therefore $\partial_t(\mu \boldsymbol{H}(t)) \cdot \boldsymbol{n} = 0$ in $\mathrm{H}^{-1/2}(\partial\Omega)$, which together with the fact that $\mu \boldsymbol{H}_0 \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ leads to (2.37).

Finally, to prove (2.38), let $\boldsymbol{v} \in \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{C}})^3$ be such that $\boldsymbol{v}|_{\Gamma_{\mathrm{I}}} = \boldsymbol{0}$. Let $\boldsymbol{G} \in \mathrm{H}^1(\Omega)^3$ vanishing in Ω_{D} and such that $\boldsymbol{G}|_{\partial\Omega_{\mathrm{C}}} = \boldsymbol{v}$. Clearly, $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$. In what follows, we prove that $\langle \mathbf{curl}\, \boldsymbol{G}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = 0$, $n = 1, \dots, N$. With this aim, let ζ_n be a smooth function defined in Ω such that $\zeta_n|_{\Gamma_{\boldsymbol{J}}^m} = \delta_{nm}$ and $\zeta_n = 0$ on $\Gamma_{\boldsymbol{E}}$. Then, using Green's formula twice,

$$\begin{aligned}
\langle \mathbf{curl}\, \boldsymbol{G}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} := \langle \mathbf{curl}\, \boldsymbol{G}\cdot\boldsymbol{n}, \zeta_n\rangle_{\partial\Omega} &= \int_\Omega \mathbf{curl}\, \boldsymbol{G}\cdot\mathbf{grad}\,\zeta_n \\
&= -\int_{\partial\Omega} \boldsymbol{G}\times\boldsymbol{n}\cdot\mathbf{grad}\,\zeta_n = -\int_{\Gamma_{\mathrm{C}}} \boldsymbol{G}\times\boldsymbol{n}\cdot\mathbf{grad}\,\zeta_n,
\end{aligned}$$

the last equality because $\boldsymbol{G}$ vanishes in Ω_{D} and, hence, $\boldsymbol{G}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ_{D}. Consequently,

$$\langle \mathbf{curl}\, \boldsymbol{G}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = -\int_{\Gamma_{\mathrm{C}}} \boldsymbol{G}\times\boldsymbol{n}\cdot\mathbf{grad}\,\zeta_n = \int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{grad}\,\zeta_n\times\boldsymbol{n}\cdot\boldsymbol{v} = 0,$$

the last equality because ζ_n is constant on $\Gamma_{\boldsymbol{J}}^n$. Therefore, $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}$ and we can use it to test (2.24). Using a Green's formula and (2.34), we obtain

$$\begin{aligned}
0 &= \int_\Omega \mu\partial_t \boldsymbol{H}(t)\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \boldsymbol{E}(t)\cdot\mathbf{curl}\,\boldsymbol{G} \\
&= \int_{\Omega_{\mathrm{C}}} \mu\partial_t \boldsymbol{H}(t)\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \mathbf{curl}\,\boldsymbol{E}(t)\cdot\boldsymbol{G} + \langle \boldsymbol{E}(t)\times\boldsymbol{n}_{\mathrm{C}}, \boldsymbol{G}\rangle_{\partial\Omega_{\mathrm{C}}} \\
&= \langle \boldsymbol{E}(t)\times\boldsymbol{n}_{\mathrm{C}}, \boldsymbol{v}\rangle_{\partial\Omega_{\mathrm{C}}}.
\end{aligned}$$

Since this holds for any $\boldsymbol{v} \in \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{C}})^3$ with $\boldsymbol{v}|_{\Gamma_{\mathrm{I}}} = \boldsymbol{0}$, we obtain (2.38) in a weak sense and conclude the proof. ∎

Remark 2.9 *The theorem above shows that Problem 2.3 allows us to determine the electric field* $\boldsymbol{E}$ *only in* Ω_{C}. *This is not a drawback in most applications, where the typical goal is to model the behavior of conductors. In the framework of the eddy current model with current intensities as data, we refer the reader to [7, Theorem 8.6], where the authors introduce a problem whose unique solution is an extension of the electric field, which satisfies equation (2.2) in the whole domain* Ω. *This theorem is proved for a particular topology and in the harmonic regime. The same arguments can be used to obtain an analogous result in the transient regime.*

2.4 Space discretization

From now on, we assume that Ω, Ω_{C}, and Ω_{D} are Lipschitz polyhedra and consider regular tetrahedral meshes $\mathcal{T}_h$ of Ω, such that each element $K \in \mathcal{T}_h$ is contained either in Ω_{C} or in Ω_{D} (h stands as usual for the corresponding mesh-size). We employ edge finite elements to approximate the magnetic field, more precisely, lowest-order Nédélec finite elements:

$$\boldsymbol{\mathcal{N}}_h(\Omega) := \{\boldsymbol{G}_h \in \mathrm{H}(\mathbf{curl};\Omega) : \ \boldsymbol{G}_h|_K \in \boldsymbol{\mathcal{N}}(K) \ \forall K \in \mathcal{T}_h\}.$$

The magnetic field is approximated in each tetrahedron K by a polynomial vector field in the space

$$\boldsymbol{\mathcal{N}}(K) := \left\{\boldsymbol{G}_h \in \mathbb{P}_1^3 : \ \boldsymbol{G}_h(\boldsymbol{x}) = \mathbf{a} \times \boldsymbol{x} + \mathbf{b}, \ \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \ \boldsymbol{x} \in K\right\}.$$

We introduce the following discrete spaces:

$$\begin{aligned}
\boldsymbol{\mathcal{X}}_h :=& \{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega) : \ \mathbf{curl}\,\boldsymbol{G}_h = \mathbf{0} \text{ in } \Omega_{\mathrm{D}}\} \subset \boldsymbol{\mathcal{X}}, \\
\boldsymbol{\mathcal{V}}_h :=& \left\{\boldsymbol{G}_h \in \boldsymbol{\mathcal{X}}_h : \ \int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\,\boldsymbol{G}_h \cdot \boldsymbol{n} = 0, \ n = 1, \dots, N\right\} \subset \boldsymbol{\mathcal{V}}.
\end{aligned}$$

Then, the space-discretization of Problem 2.3 reads as follows.

Problem 2.10 *Find* $\boldsymbol{H}_h : [0,T] \to \boldsymbol{\mathcal{X}}_h$ *such that*

$$\int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\,\boldsymbol{H}_h(t) \cdot \boldsymbol{n} = I_n(t), \quad n = 1, \dots, N, \tag{2.39}$$

$$\int_\Omega \mu \partial_t \boldsymbol{H}_h(t) \cdot \boldsymbol{G}_h + a(\boldsymbol{H}_h(t), \boldsymbol{G}_h) = 0 \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h, \tag{2.40}$$

$$\boldsymbol{H}_h(0) = \boldsymbol{H}_{0h}, \tag{2.41}$$

where $\boldsymbol{H}_{0h} \in \boldsymbol{\mathcal{X}}_h$ *is an approximation of* $\boldsymbol{H}_0$.

To prove that this problem is well-posed, we will use a function $\widehat{\boldsymbol{H}}_h \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{X}}_h)$ satisfying $\int_{\Gamma_{\boldsymbol{J}}^n} \mathbf{curl}\,\widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{n} = I_n(t)$, $n = 1, \dots, N$, $t \in [0,T]$. To define $\widehat{\boldsymbol{H}}_h$, first we choose functions $\widehat{\boldsymbol{H}}_h^j \in \boldsymbol{\mathcal{X}}_h$ such that $\int_{\gamma_n} \widehat{\boldsymbol{H}}_h^j \cdot \boldsymbol{t}_n = \delta_{nj}$, $j, n = 1, \dots, N$; such $\widehat{\boldsymbol{H}}_h^j$ are easy to construct, under our geometrical assumptions, once a basis of $\boldsymbol{\mathcal{X}}_h$ is given (see Remark 2.14). Then we define

$$\widehat{\boldsymbol{H}}_h(t) := \sum_{j=1}^N I_j(t) \widehat{\boldsymbol{H}}_h^j. \tag{2.42}$$

Hence,

$$\int_{\Gamma_J^n} \mathbf{curl}\, \widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{n} = \int_{\gamma_n} \widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{t}_n = \sum_{j=1}^{N} I_j(t) \int_{\gamma_n} \widehat{\boldsymbol{H}}_h^j \cdot \boldsymbol{t}_n = I_n(t).$$

Since $I_n \in \mathrm{H}^2(0,T)$, $n = 1, \dots, N$, we conclude that $\widehat{\boldsymbol{H}}_h$ actually lies in $\mathrm{H}^2(0,T;\boldsymbol{\mathcal{X}}_h)$.

Now, if we write $\boldsymbol{H}_h = \widetilde{\boldsymbol{H}}_h + \widehat{\boldsymbol{H}}_h$, Problem 2.10 is equivalent to finding $\widetilde{\boldsymbol{H}}_h \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{V}}_h)$ such that

$$\int_\Omega \mu \partial_t \widetilde{\boldsymbol{H}}_h(t) \cdot \boldsymbol{G}_h + a(\widetilde{\boldsymbol{H}}_h(t), \boldsymbol{G}_h) = -\int_\Omega \mu \partial_t \widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{G}_h - a(\widehat{\boldsymbol{H}}_h(t), \boldsymbol{G}_h) \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h, \tag{2.43}$$

$$\widetilde{\boldsymbol{H}}_h(0) = \boldsymbol{H}_{0h} - \widehat{\boldsymbol{H}}_h(0). \tag{2.44}$$

Next, let $\{\boldsymbol{\Phi}_i\}_{i=1}^K$ be a basis of $\boldsymbol{\mathcal{V}}_h$. We write

$$\widetilde{\boldsymbol{H}}_h(t, \boldsymbol{x}) = \sum_{i=1}^{K} \beta_i(t) \boldsymbol{\Phi}_i(\boldsymbol{x}).$$

Let $\boldsymbol{\beta}(t) := (\beta_i(t))_{1 \le i \le K}$ and $\boldsymbol{F}_h(t) := (f_{h_i}(t))_{1 \le i \le K}$ with

$$f_{h_i}(t) = -\int_\Omega \mu \partial_t \widehat{\boldsymbol{H}}_h(t) \cdot \boldsymbol{\Phi}_i - a(\widehat{\boldsymbol{H}}_h(t), \boldsymbol{\Phi}_i), \qquad 1 \le i \le K,$$

and let $\mathcal{K}$ and $\mathcal{M}$ be $\mathbb{R}^{K \times K}$ matrices given by

$$\mathcal{K}_{i,j} := a(\boldsymbol{\Phi}_i, \boldsymbol{\Phi}_j), \qquad \mathcal{M}_{i,j} := \int_\Omega \mu \boldsymbol{\Phi}_i \cdot \boldsymbol{\Phi}_j, \qquad 1 \le i, j \le K.$$

Then problem (2.43)–(2.44) reads as follows: find $\boldsymbol{\beta} : [0,T] \to \mathbb{R}^K$ such that

$$\mathcal{M}\boldsymbol{\beta}'(t) + \mathcal{K}\boldsymbol{\beta}(t) = \boldsymbol{F}_h(t),$$
$$\boldsymbol{\beta}(0) = \boldsymbol{\beta}_0.$$

Since $\mathcal{M}$ is symmetric positive definite, this linear system of differential equations has a unique solution. Thus, we conclude that Problem 2.10 admits a unique solution, too.

Our next goal is to obtain error estimates for this semi-discrete scheme. For $r \in \left(\frac{1}{2}, 1\right]$ let

$$\boldsymbol{\mathcal{X}}^r := \left\{ \boldsymbol{G} \in \boldsymbol{\mathcal{X}} : \ \boldsymbol{G}|_{\Omega_\mathrm{C}} \in \mathrm{H}^r(\mathbf{curl}; \Omega_\mathrm{C}) \text{ and } \boldsymbol{G}|_{\Omega_\mathrm{D}} \in \mathrm{H}^r(\Omega_\mathrm{D})^3 \right\},$$

where $\mathrm{H}^r(\mathbf{curl}; \Omega_\mathrm{C}) := \left\{ \boldsymbol{G} \in \mathrm{H}^r(\Omega_\mathrm{C})^3 : \ \mathbf{curl}\, \boldsymbol{G} \in \mathrm{H}^r(\Omega_\mathrm{C})^3 \right\}$. If $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}^r$, then its Nédélec interpolant $\mathcal{I}_h^{\mathcal{N}} \boldsymbol{G} \in \boldsymbol{\mathcal{N}}_h(\Omega)$ is well defined (see [17, Lemma 5.1] and [9]).

From now on, we assume that the solution to Problem 2.3 satisfies $\boldsymbol{H} \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{X}}^r)$, which in particular implies that the initial condition $\boldsymbol{H}_0 \in \boldsymbol{\mathcal{X}}^r$. Therefore, the Nédélec interpolant $\mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}(t)$ is well defined and satisfies

$$\int_{\Gamma_J^n} \mathbf{curl}\, \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}(t) \cdot \boldsymbol{n} = \int_{\gamma_n} \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}(t) \cdot \mathbf{t}_n = \int_{\gamma_n} \boldsymbol{H}(t) \cdot \mathbf{t}_n = \langle \mathbf{curl}\, \boldsymbol{H}(t) \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_J^n} = I_n(t).$$

Thus, we are allowed to use $\boldsymbol{H}_{0h} := \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0$ as initial data for Problem 2.10.

Let $\boldsymbol{\rho}_h(t) := \boldsymbol{H}(t) - \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t)$ and $\boldsymbol{\delta}_h(t) := \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t) - \boldsymbol{H}_h(t)$. Note that from the last equality we have that $\boldsymbol{\delta}_h(t) \in \boldsymbol{\mathcal{V}}_h$. A straightforward computation yields

$$\int_\Omega \mu\partial_t\boldsymbol{\delta}_h(t)\cdot\boldsymbol{G}_h + a(\boldsymbol{\delta}_h(t),\boldsymbol{G}_h) = -\int_\Omega \mu\partial_t\boldsymbol{\rho}_h(t)\cdot\boldsymbol{G}_h - a(\boldsymbol{\rho}_h(t),\boldsymbol{G}_h) \qquad \forall\boldsymbol{G}_h\in\boldsymbol{\mathcal{V}}_h. \tag{2.45}$$

By taking $\boldsymbol{G}_h := \boldsymbol{\delta}_h(t)$, using (2.22) and the Cauchy-Schwarz inequality, we obtain

$$\partial_t\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \le C\left\{\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \|\,\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\right\}. \tag{2.46}$$

Integrating in time, using the fact that $\boldsymbol{\delta}_h(0) = \mathbf{0}$ and Gronwall's inequality leads to

$$\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \le C\left\{\int_0^T \|\partial_t\boldsymbol{\rho}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds + \int_0^T \|\,\mathbf{curl}\,\boldsymbol{\rho}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds\right\}.$$

Using this estimate on the right-hand side of (2.46) and integrating in time again, we have that

$$\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^t \|\boldsymbol{\delta}_h(s)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,ds \le C\left\{\int_0^T \|\partial_t\boldsymbol{\rho}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds + \int_0^T \|\,\mathbf{curl}\,\boldsymbol{\rho}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds\right\}. \tag{2.47}$$

On the other hand, by taking $\boldsymbol{G}_h := \partial_t\boldsymbol{\delta}_h(t)$ in (2.45), we obtain that

$$\begin{aligned}&\underline{\mu}\,\|\partial_t\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \frac{1}{2}\frac{d}{dt}a(\boldsymbol{\delta}_h(t),\boldsymbol{\delta}_h(t))\\ &\qquad\le -\int_\Omega \mu\partial_t\boldsymbol{\rho}_h(t)\cdot\partial_t\boldsymbol{\delta}_h(t) + a(\partial_t\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t)) - \frac{d}{dt}a(\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t)).\end{aligned}$$

Integrating in time, since $\frac{1}{\overline{\sigma}}\|\,\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \le a(\boldsymbol{\delta}_h(t),\boldsymbol{\delta}_h(t))$, the Cauchy-Schwarz inequality yields

$$\begin{aligned}&\int_0^t \|\partial_t\boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds + \|\,\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\\ &\qquad\le C\left\{\int_0^t \|\,\mathbf{curl}\,\boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds + \int_0^T \|\partial_t\boldsymbol{\rho}_h(s)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,ds + \sup_{0\le s\le T}\|\,\mathbf{curl}\,\boldsymbol{\rho}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\right\}.\end{aligned}$$

Hence, using Gronwall's inequality to estimate $\|\,\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}$ and substituting this term by the resulting estimate on the right-hand side of the inequality above, we obtain

$$\begin{aligned}&\int_0^t \|\partial_t\boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds + \|\,\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\\ &\qquad\le C\left\{\int_0^T \|\partial_t\boldsymbol{\rho}_h(s)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,ds + \sup_{0\le s\le T}\|\,\mathbf{curl}\,\boldsymbol{\rho}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\right\}.\end{aligned}$$

Finally, combining this equation and (2.47) leads to

$$\begin{aligned}&\sup_{0\le t\le T}\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \int_0^T \|\partial_t\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\,dt\\ &\qquad\le C\left\{\int_0^T \|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,dt + \sup_{0\le t\le T}\|\,\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\right\}.\end{aligned} \tag{2.48}$$

Now, we are in a position to prove the following error estimate.

Theorem 2.11 *Let $\boldsymbol{H}$ be the solution to Problem 2.3 and $\boldsymbol{H}_h$ that to Problem 2.10. If $\boldsymbol{H} \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{X}}^r)$, with $r \in \left(\frac{1}{2}, 1\right]$, then there exists a constant $C > 0$, independent of h, such that*

$$\sup_{0\le t\le T} \|\boldsymbol{H}(t) - \boldsymbol{H}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \int_0^T \|\partial_t(\boldsymbol{H}(t) - \boldsymbol{H}_h(t))\|^2_{\mathrm{L}^2(\Omega)^3}\,dt$$
$$\le Ch^{2r}\left\{\int_0^T \left[\|\partial_t \boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega_{\mathrm{C}})} + \|\partial_t \boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3}\right]dt\right.$$
$$\left. + \sup_{0\le t\le T}\left[\|\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega_{\mathrm{C}})} + \|\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3}\right]\right\}.$$

Proof. Note that the regularity on $\boldsymbol{H}$ implies that $\partial_t(\mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t)) = \mathcal{I}_h^{\mathcal{N}}(\partial_t\boldsymbol{H}(t))$ a.e. $t \in [0,T]$ (see [71, Theorems 111 & 113]). Therefore (see [17]),

$$\|\boldsymbol{\rho}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le Ch^r\left\{\|\boldsymbol{H}(t)\|_{\mathrm{H}^r(\mathbf{curl};\Omega_{\mathrm{C}})} + \|\boldsymbol{H}(t)\|_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3}\right\},$$
$$\|\partial_t\boldsymbol{\rho}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le Ch^r\left\{\|\partial_t\boldsymbol{H}(t)\|_{\mathrm{H}^r(\mathbf{curl};\Omega_{\mathrm{C}})} + \|\partial_t\boldsymbol{H}(t)\|_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3}\right\}.$$

Thus, the result follows by writing $\boldsymbol{H}(t) - \boldsymbol{H}_h(t) = \boldsymbol{\rho}_h(t) + \boldsymbol{\delta}_h(t)$ and using estimate (2.48). ∎

For the implementation of Problem 2.10 we resort to its formulation in terms of a magnetic potential. With this aim, we assume that the cut surfaces Σ_n are polyhedral and the meshes are compatible with them, in the sense that each Σ_n is a union of faces of tetrahedra $K \in \mathcal{T}_h$. Therefore, $\mathcal{T}_h^{\Omega_{\mathrm{D}}} := \{K \in \mathcal{T}_h :\ K \subset \Omega_{\mathrm{D}}\}$ can also be seen as a mesh of $\widetilde{\Omega}_{\mathrm{D}}$.

We introduce an approximation of the space Θ. Let

$$\mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) := \left\{\widetilde{\Psi}_h \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) :\ \widetilde{\Psi}_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h^{\Omega_{\mathrm{D}}}\right\}$$

and consider the finite-dimensional subspace of Θ given by

$$\Theta_h := \left\{\widetilde{\Psi}_h \in \mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) :\ [\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = \text{constant},\ n = 1,\dots,L\right\}.$$

We introduce the following finite-dimensional subspaces of $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{Y}}^0$, respectively:

$$\boldsymbol{\mathcal{Y}}_h := \left\{(\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{C}}) \times (\Theta_h/\mathbb{R}) :\ (\boldsymbol{G}_h|\widetilde{\mathbf{grad}}\,\widetilde{\Psi}_h) \in \mathrm{H}(\mathbf{curl};\Omega)\right\},$$
$$\boldsymbol{\mathcal{Y}}_h^0 := \left\{(\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{Y}}_h :\ [\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = 0,\ n = 1,\dots,N\right\}.$$

Proceeding as in [19], it is immediate to show that Problem 2.10 is equivalent to finding $(\boldsymbol{H}_h, \widetilde{\Phi}_h) : [0,T] \to \boldsymbol{\mathcal{Y}}_h$ such that

$$[\![\widetilde{\Phi}_h(t)]\!]_{\Sigma_n} = I_n(t), \quad n = 1,\dots,N, \tag{2.49}$$
$$\int_{\Omega_{\mathrm{C}}} \mu\partial_t\boldsymbol{H}_h(t)\cdot\boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}_h(t)\cdot\mathbf{curl}\,\boldsymbol{G}_h + \int_{\Omega_{\mathrm{D}}} \mu\partial_t\widetilde{\mathbf{grad}}\,\widetilde{\Phi}_h(t)\cdot\widetilde{\mathbf{grad}}\,\widetilde{\Psi}_h = 0 \qquad \forall(\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{Y}}_h^0, \tag{2.50}$$
$$\left(\boldsymbol{H}_h(0)|\widetilde{\mathbf{grad}}\,\widetilde{\Phi}_h(0)\right) = \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}_0. \tag{2.51}$$

Let us remark that the first equation above is actually equivalent to (2.39) because $\boldsymbol{H}_h(t)$ and $\widetilde{\Phi}_h(t)$ are smooth enough for (2.18) to hold. The above problem can be seen as a discretization of the magnetic field - magnetic potential formulation (2.19)–(2.21).

2.5 Time discretization

We consider a uniform partition of $[0,T]$, $t_k := k\Delta t$, $k = 0,\dots,M$, with time step $\Delta t := \frac{T}{M}$. A fully discrete approximation of Problem 2.3 is defined as follows.

Problem 2.12 *Find* $\boldsymbol{H}_h^m \in \boldsymbol{\mathcal{X}}_h$, $m = 1,\dots,M$, *such that*

$$\int_{\Gamma_J^n} \mathbf{curl}\, \boldsymbol{H}_h^m \cdot \boldsymbol{n} = I_n(t_m), \quad n = 1,\dots,N,$$

$$\int_\Omega \mu \frac{\boldsymbol{H}_h^m - \boldsymbol{H}_h^{m-1}}{\Delta t} \cdot \boldsymbol{G}_h + a(\boldsymbol{H}_h^m, \boldsymbol{G}_h) = 0 \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h,$$

$$\boldsymbol{H}_h^0 = \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0.$$

Hence, at each iteration step we have to find $\boldsymbol{H}_h^m \in \boldsymbol{\mathcal{X}}_h$ satisfying

$$\int_{\Gamma_J^n} \mathbf{curl}\, \boldsymbol{H}_h^m \cdot \boldsymbol{n} = I_n(t_m), \quad n = 1,\dots,N,$$

$$\int_\Omega \mu \boldsymbol{H}_h^m \cdot \boldsymbol{G}_h + \Delta t\, a(\boldsymbol{H}_h^m, \boldsymbol{G}_h) = \int_\Omega \mu \boldsymbol{H}_h^{m-1} \cdot \boldsymbol{G}_h \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h.$$

The problem above has a unique solution. In fact, taking $\widehat{\boldsymbol{H}}_h^m := \widehat{\boldsymbol{H}}_h(t_m)$, with $\widehat{\boldsymbol{H}}_h$ as in (2.42), and writing $\boldsymbol{H}_h^m = \widetilde{\boldsymbol{H}}_h^m + \widehat{\boldsymbol{H}}_h^m$, this problem is equivalent to find $\widetilde{\boldsymbol{H}}_h^m \in \boldsymbol{\mathcal{V}}_h$ such that

$$\int_\Omega \mu \widetilde{\boldsymbol{H}}_h^m \cdot \boldsymbol{G}_h + \Delta t\, a(\widetilde{\boldsymbol{H}}_h^m, \boldsymbol{G}_h) = \int_\Omega \mu \widetilde{\boldsymbol{H}}_h^{m-1} \cdot \boldsymbol{G}_h + \int_\Omega \mu \widehat{\boldsymbol{H}}_h^{m-1} \cdot \boldsymbol{G}_h - \int_\Omega \mu \widehat{\boldsymbol{H}}_h^m \cdot \boldsymbol{G}_h - \Delta t\, a(\widehat{\boldsymbol{H}}_h^m, \boldsymbol{G}_h)$$

for all $\boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h$, which leads to a linear system of equations with a positive definite symmetric matrix.

Our next goal is to obtain error estimates for this fully discrete scheme. Let

$$\boldsymbol{\rho}_h^k := \boldsymbol{H}(t_k) - \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}(t_k), \quad \boldsymbol{\delta}_h^k := \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}(t_k) - \boldsymbol{H}_h^k \quad \text{and} \quad \boldsymbol{\tau}^k := \frac{\boldsymbol{H}(t_k) - \boldsymbol{H}(t_{k-1})}{\Delta t} - \partial_t \boldsymbol{H}(t_k).$$

A straightforward computation allows us to show that

$$\int_\Omega \mu \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h + a(\boldsymbol{\delta}_h^k, \boldsymbol{G}_h) = \int_\Omega \mu \boldsymbol{\tau}^k \cdot \boldsymbol{G}_h - \int_\Omega \mu \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h - a(\boldsymbol{\rho}_h^k, \boldsymbol{G}_h) \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h. \tag{2.52}$$

Choosing $\boldsymbol{G}_h := \boldsymbol{\delta}_h^k$ and using that

$$\int_\Omega \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \cdot \boldsymbol{\delta}_h^k \geq \frac{1}{2\Delta t} \left\{ \|\boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} - \|\boldsymbol{\delta}_h^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3} \right\}$$

and $a(\boldsymbol{\delta}_h^k, \boldsymbol{\delta}_h^k) \geq \frac{1}{\bar{\sigma}} \|\mathbf{curl}\, \boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega)^3}$, together with the Cauchy-Schwarz inequality, yield

$$\begin{aligned} &\|\boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} - \|\boldsymbol{\delta}_h^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \|\mathbf{curl}\, \boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} \\ &\qquad \leq \frac{\Delta t}{2T} \|\boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} + C \Delta t \left\{ \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} + \|\mathbf{curl}\, \boldsymbol{\rho}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} \right\}. \end{aligned}$$

Summing over k and using the discrete Gronwall's inequality and the fact that $\boldsymbol{\delta}_h^0 = \mathbf{0}$, we obtain for all $m = 1, \ldots, M$,

$$\|\boldsymbol{\delta}_h^m\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^{m} \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^k\|_{\mathrm{L}^2(\Omega)^3}^2 \leq C\Delta t \sum_{k=1}^{m} \left\{ \|\boldsymbol{\tau}^k\|_{\mathrm{L}^2(\Omega)^3}^2 + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^k\|_{\mathrm{L}^2(\Omega)^3}^2 \right\}. \tag{2.53}$$

On the other hand, by taking $\boldsymbol{G}_h := \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t}$ in (2.52) and using that

$$a\left(\boldsymbol{\delta}_h^k, \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t}\right) \geq \frac{1}{2\Delta t} \left\{ a(\boldsymbol{\delta}_h^k, \boldsymbol{\delta}_h^k) - a(\boldsymbol{\delta}_h^{k-1}, \boldsymbol{\delta}_h^{k-1}) \right\}$$

and

$$a\left(\boldsymbol{\rho}_h^k, \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t}\right) = \frac{1}{\Delta t} \left\{ a(\boldsymbol{\rho}_h^k, \boldsymbol{\delta}_h^k) - a(\boldsymbol{\rho}_h^{k-1}, \boldsymbol{\delta}_h^{k-1}) \right\} - a\left(\frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t}, \boldsymbol{\delta}_h^{k-1}\right),$$

together with the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \underline{\mu}\Delta t \left\| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 &+ a(\boldsymbol{\delta}_h^k, \boldsymbol{\delta}_h^k) - a(\boldsymbol{\delta}_h^{k-1}, \boldsymbol{\delta}_h^{k-1}) \\ &\leq C\Delta t \left\{ \|\boldsymbol{\tau}^k\|_{\mathrm{L}^2(\Omega)^3}^2 + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^{k-1}\|_{\mathrm{L}^2(\Omega)^3}^2 \right\} \\ &\quad - 2\left\{ a(\boldsymbol{\rho}_h^k, \boldsymbol{\delta}_h^k) - a(\boldsymbol{\rho}_h^{k-1}, \boldsymbol{\delta}_h^{k-1}) \right\}. \end{aligned}$$

Summing over k leads to

$$\begin{aligned} &\Delta t \sum_{k=1}^{m} \left\| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^m\|_{\mathrm{L}^2(\Omega)^3}^2 \\ &\leq C \left\{ \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^m\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^{m} \left[\|\boldsymbol{\tau}^k\|_{\mathrm{L}^2(\Omega)^3}^2 + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^{k-1}\|_{\mathrm{L}^2(\Omega)^3}^2 \right] \right\}. \end{aligned}$$

Adding this inequality to (2.53) and using again (2.53) to estimate $\Delta t \sum_{k=1}^{m} \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^{k-1}\|_{\mathrm{L}^2(\Omega)^3}^2$, we obtain

$$\begin{aligned} &\|\boldsymbol{\delta}_h^m\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{m} \left\| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 \\ &\leq C \left\{ \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^m\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^{m} \left[\|\boldsymbol{\tau}^k\|_{\mathrm{L}^2(\Omega)^3}^2 + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^k\|_{\mathrm{L}^2(\Omega)^3}^2 \right] \right\}. \end{aligned} \tag{2.54}$$

Therefore, we are in position to write the main result of this paper which provides error estimates for the physical quantities of interest, the magnetic field $\boldsymbol{H}$ and the current density $\boldsymbol{J} = \mathbf{curl}\,\boldsymbol{H}$.

Theorem 2.13 *Let $\boldsymbol{H}$ be the solution to Problem 2.3 and $\boldsymbol{H}_h^k$, $k = 1, \ldots, M$, that to Problem 2.12. If $\boldsymbol{H} \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{X}}^r) \cap \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)$, with $r \in \left(\frac{1}{2}, 1\right]$, then there exists a constant $C > 0$, independent of h and Δt, such that*

$$\max_{1\le k\le M} \|\boldsymbol{H}(t_k) - \boldsymbol{H}_h^k\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^{M} \left\| \partial_t \boldsymbol{H}(t_k) - \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3}$$
$$\le C \left\{ (\Delta t)^2 \|\boldsymbol{H}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)} + h^{2r} \sup_{0\le t\le T} \left[\|\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega_{\mathrm{C}})} + \|\boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3} \right] \right.$$
$$\left. + h^{2r} \int_0^T \left[\|\partial_t \boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega_{\mathrm{C}})} + \|\partial_t \boldsymbol{H}(t)\|^2_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3} \right] dt \right\}.$$

Proof. A Taylor expansion shows that

$$\sum_{k=1}^{M} \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} = \sum_{k=1}^{M} \left\| \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} (t_k - s) \partial_{tt} \boldsymbol{H}(s)\, ds \right\|^2_{\mathrm{L}^2(\Omega)^3} \le \Delta t \int_0^T \|\partial_{tt} \boldsymbol{H}(t)\|^2_{\mathrm{L}^2(\Omega)^3}\, dt.$$

Moreover, since $\boldsymbol{\rho}_h^k = \boldsymbol{\rho}_h(t_k)$, $k = 0, \ldots, M$, we have that

$$\sum_{k=1}^{M} \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \le \frac{1}{\Delta t} \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\, dt.$$

Using estimate (2.54) and the facts that

$$\boldsymbol{H}(t_k) - \boldsymbol{H}_h^k = \boldsymbol{\rho}_h^k + \boldsymbol{\delta}_h^k \qquad \text{and} \qquad \partial_t \boldsymbol{H}(t_k) - \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} = \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} + \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} - \boldsymbol{\tau}^k,$$

we obtain

$$\|\boldsymbol{H}(t_m) - \boldsymbol{H}_h^m\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^{m} \left\| \partial_t \boldsymbol{H}(t_k) - \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3}$$
$$\le C \left\{ \|\boldsymbol{\rho}_h^m\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^{m} \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{m} \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^{m} \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} \right\}$$
$$\le C \left\{ (\Delta t)^2 \int_0^T \|\partial_{tt} \boldsymbol{H}(t)\|^2_{\mathrm{L}^2(\Omega)^3}\, dt + \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\, dt + \max_{0\le m\le M} \|\boldsymbol{\rho}_h(t_m)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} \right\}.$$

Thus, since $\boldsymbol{\rho}_h := \boldsymbol{H} - \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}$, the result follows from the assumed regularity of $\boldsymbol{H}$ and standard error estimates for the Nédélec interpolant. ∎

For the actual computation of Problem 2.12 we proceed as for the semidiscrete problem and rewrite it in terms of a magnetic potential: Find $\left(\boldsymbol{H}_h^m, \widetilde{\Phi}_h^m\right) \in \boldsymbol{\mathcal{Y}}_h$, $m = 1, \ldots, M$, such that

$$[\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n} = I_n(t_m), \quad n = 1, \ldots, N,$$

$$\int_{\Omega_C} \mu \, \frac{\boldsymbol{H}_h^m - \boldsymbol{H}_h^{m-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_C} \frac{1}{\sigma} \, \mathbf{curl} \, \boldsymbol{H}_h^m \cdot \mathbf{curl} \, \boldsymbol{G}_h + \int_{\Omega_D} \mu \, \frac{\widetilde{\mathbf{grad}} \, \widetilde{\Phi}_h^m - \widetilde{\mathbf{grad}} \, \widetilde{\Phi}_h^{m-1}}{\Delta t} \cdot \widetilde{\mathbf{grad}} \, \widetilde{\Psi}_h = 0 \qquad \forall (\boldsymbol{G}_h, \widetilde{\Psi}_h) \in \boldsymbol{\mathcal{Y}}_h^0,$$

$$\left(\boldsymbol{H}_h^0 | \, \widetilde{\mathbf{grad}} \, \widetilde{\Phi}_h^0\right) = \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0.$$

This can be seen as a backward Euler time discretization of (2.49)–(2.51). This is the discrete problem that we have actually implemented in a computer code, because a less expensive scalar variable $\widetilde{\Phi}_h^m$ is used instead of a vector field $\boldsymbol{H}_h^m$ in the dielectric domain.

Note that at each time t_m the following constraints must be imposed:

- $\left(\boldsymbol{H}_h^m | \, \widetilde{\mathbf{grad}} \, \widetilde{\Phi}_h^m\right) \in \mathrm{H}(\mathbf{curl}; \Omega)$, which arises from the definition of $\boldsymbol{\mathcal{Y}}_h$;
- $[\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n} = \text{constant}$, $n = 1, \ldots, L$, which arise from the definition of Θ_h.

To deal with these conditions, we employ the following procedure (see [17] for more details).

For the first one we use that, for $\left(\boldsymbol{H}_h^m | \, \widetilde{\mathbf{grad}} \, \widetilde{\Phi}_h^m\right) \in \mathrm{H}(\mathbf{curl}; \Omega)$,

$$\int_\ell \boldsymbol{H}_h^m \cdot \boldsymbol{t}_\ell = \int_\ell \widetilde{\mathbf{grad}} \, \widetilde{\Phi}_h^m \cdot \boldsymbol{t}_\ell = \widetilde{\Phi}_h^m \left(P_\ell^+\right) - \widetilde{\Phi}_h^m \left(P_\ell^-\right) \qquad \forall \ell \text{ edge of } \mathcal{T}_h : \ \ell \subset \Gamma_{\mathrm{I}}, \tag{2.55}$$

where P_ℓ^- and P_ℓ^+ are the endpoints of ℓ and $\boldsymbol{t}_\ell$ the unit tangent vector pointing from P_ℓ^- to P_ℓ^+. Then the degrees of freedom (d.o.f.) of $\boldsymbol{H}_h^m$ associated with the edges $\ell \subset \Gamma_{\mathrm{I}}$ are eliminated by static condensation in terms of those of $\widetilde{\Phi}_h^m$ corresponding to the vertices of the mesh on Γ_{I}.

Regarding the second constraint, for each cut surface Σ_n, we in principle distinguish the d.o.f. of $\widetilde{\Phi}_h^m$ on Σ_n^+ from those on Σ_n^-. Then the former are eliminated by using

$$\widetilde{\Phi}_h^m|_{\Sigma_n^+} = \widetilde{\Phi}_h^m|_{\Sigma_n^-} + [\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n},$$

with $[\![\widetilde{\Psi}_h]\!]_{\Sigma_n} = 0$ for the test functions and $[\![\widetilde{\Phi}_h^m]\!]_{\Sigma_n} = I_n(t_m)$ for the trial functions, where $I_n(t_m)$, $n = 1, \cdots, N$, are the input current intensities and $I_n(t_m)$, $n = N+1, \cdots, L$, are unknowns of the problem.

Remark 2.14 *To prove that Problem 2.10 is well posed, we have used vector fields* $\widehat{\boldsymbol{H}}_h^j \in \boldsymbol{\mathcal{X}}_h$ *satisfying* $\int_{\gamma_n} \widehat{\boldsymbol{H}}_h^j \cdot \boldsymbol{t}_n = \delta_{nj}$, $j, n = 1, \ldots, N$. *Such* $\widehat{\boldsymbol{H}}_h^j$ *can be constructed as follows. Let* $\widetilde{\Psi}_h^j \in \Theta_h$ *be such that* $\widetilde{\Psi}_h^j(P) = 1$ *for all vertices* P *lying on* $\bar{\Sigma}_j^+$ *and* $\widetilde{\Psi}_h^j(P) = 0$ *for all other vertices* P *of the mesh in* $\widetilde{\Omega}_{\mathrm{D}}$. *Therefore* $\widetilde{\Psi}_h^j|_{\Sigma_j^+} = 1$ *and* $\widetilde{\Psi}_h^j|_{\Sigma_j^-} = 0$, *so that* $[\![\widetilde{\Psi}_h^j]\!]_{\Sigma_j} = 1$. *Moreover, clearly* $[\![\widetilde{\Psi}_h^j]\!]_{\Sigma_n} = 0$, $n \neq j$. *Let* $\boldsymbol{G}_h^j \in \boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{C}})$ *be such that its d.o.f. corresponding to the edges* $\ell \subset \bar{\Gamma}_{\mathrm{I}}$ *are defined as in (2.55):* $\int_\ell \boldsymbol{G}_h^j \cdot \boldsymbol{t}_\ell = \widetilde{\Psi}_h^j \left(P_\ell^+\right) - \widetilde{\Psi}_h^j \left(P_\ell^-\right)$. *The d.o.f. of* $\boldsymbol{G}_h^j$ *corresponding to edges not lying on* $\bar{\Gamma}_{\mathrm{I}}$ *can be chosen arbitrarily (for instance, equal to zero). Let* $\widehat{\boldsymbol{H}}_h^j := \left(\boldsymbol{G}_h^j | \, \widetilde{\mathbf{grad}} \, \widetilde{\Psi}_h^j\right) \in \boldsymbol{\mathcal{X}}_h$. *Then,* $\int_{\gamma_n} \widehat{\boldsymbol{H}}_h^j \cdot \boldsymbol{t}_n = [\![\widetilde{\Psi}_h^j]\!]_{\Sigma_n} = \delta_{nj}$, $j, n = 1, \ldots, N$.

2.6 Numerical experiments

In this section we report some numerical result obtained with a MATLAB code implementing the numerical method described above. First, we present a test with a known analytical solution to validate the computer code and to test the error estimates proved above. Finally, we will apply the method to a problem arising from an electromagnetic forming process.

2.6.1 A test with known analytical solution

The problem solved in this section has been already solved in [17] in harmonic regime. This is the reason why we only give here a brief description and refer the reader to this reference for further details.

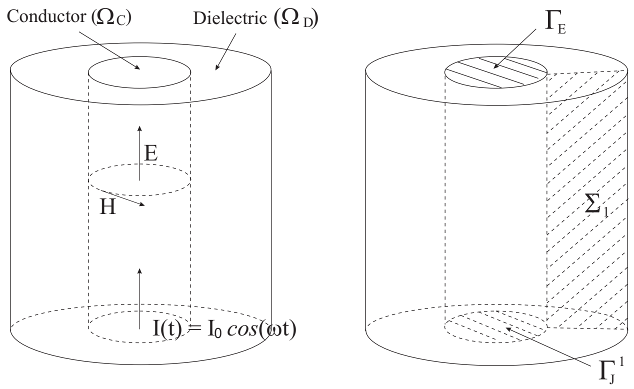

Figure 2.2: Sketch of the domain in the analytical example and cut surface.

Figure 2.2 shows a sketch of the domain, where the conducting part Ω_{C} and the whole domain Ω are coaxial cylinders. An alternating current of intensity $I(t) = I_0 \cos(\omega t)$ enters the conductor through Γ_J^1 and goes through Ω_{C} in the axial direction; I_0 denotes the amplitude of the intensity and ω the angular frequency. It is easy to obtain an analytical solution of the eddy current problem in Ω by writing all the fields in the form $\boldsymbol{F}(t, \boldsymbol{x}) = \mathrm{Re}(e^{\mathrm{i}\omega t}\boldsymbol{\mathcal{F}}(\boldsymbol{x}))$. In particular, the solution leads to a magnetic field that has only an azimuthal component and is defined by a scalar multivalued potential in the dielectric domain. Note that in this case we only need one cut surface in the dielectric domain (see Figure 2.2).

The numerical method has been used on several successively refined meshes and the time step has been accordingly reduced to determine the order of convergence with respect to both, the mesh size and the time step simultaneously. More precisely, we have used a sequence of six meshes with mesh size $h_n := \frac{h_1}{n}$, $n = 1, \dots, 6$, and, for each one, we have used a time step $\Delta t_n := \frac{\Delta t_1}{n}$. We have compared the obtained numerical solutions with the analytical one.

Figure 2.3 shows log-log plots of the relative error for the magnetic field (left) and its time derivative (right) in the discrete norms considered in Theorem 2.13 versus the number of d.o.f.

The slopes of the curves clearly show an order of convergence $\mathcal{O}(h + \Delta t)$ for both quantities, which agrees with the theoretical results, since the solution is smooth and hence the hypotheses of

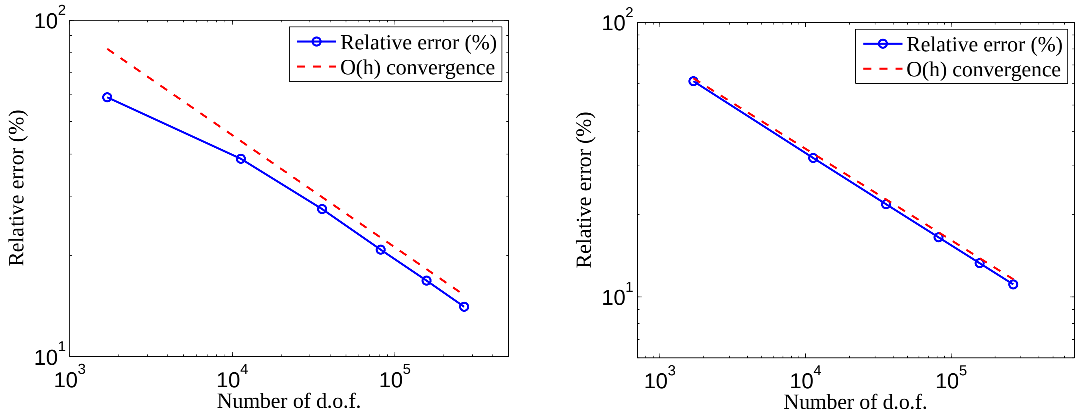

Figure 2.3: Relative errors $\dfrac{\max\limits_{1\le k\le M}\|\boldsymbol{H}(t_k)-\boldsymbol{H}_h^k\|_{\mathrm{H}(\mathbf{curl};\Omega)}}{\max\limits_{1\le k\le M}\|\boldsymbol{H}(t_k)\|_{\mathrm{H}(\mathbf{curl};\Omega)}}$ (left) and $\dfrac{\sqrt{\Delta t}\left\{\sum_{k=1}^{M}\left\|\partial_t\boldsymbol{H}(t_k)-\frac{\boldsymbol{H}_h^k-\boldsymbol{H}_h^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3}\right\}^{1/2}}{\sqrt{\Delta t}\left\{\sum_{k=1}^{M}\|\partial_t\boldsymbol{H}(t_k)\|^2_{\mathrm{L}^2(\Omega)^3}\right\}^{1/2}}$ (right) versus number of d.o.f. (log-log scale).

Theorem 2.13 are fulfilled for $r = 1$.

2.6.2 A problem arising from an electromagnetic forming process

Electromagnetic forming is a metal-working process that relies on the use of electromagnetic forces to deform metallic workpieces at high speeds. A transient electric current is induced in a coil that produces a magnetic field that penetrates a nearby conductive workpiece where an eddy current is generated. The magnetic field, together with the eddy current, induce Lorentz forces that drive the deformation of the workpiece (see, for instance, [30]). In this section we have simulated the electromagnetic behavior of a three-dimensional workpiece under the action of a coil. It corresponds to a similar configuration to the one presented in [65] but with simpler geometry and workpiece data. The coil and workpiece are presented in Figure 2.4, which also shows a typical mesh of the conducting domain. Domain Ω has been chosen as a box surrounding the conductor. Note that we only need one cut surface in the dielectric domain. The current intensity that enters the coil is shown in Figure 2.5; it corresponds to a typical curve in electromagnetic forming. With regard to the physical properties, the workpiece is a magnesium alloy and the coil is made with copper (see Table 2 of [65]).

Figure 2.6 shows the computed resultant of the Lorentz force versus time in the workpiece; the peak value corresponds to the time in which the input current intensity reaches its maximum (0.00018 s). All the other reported results correspond to this time. Figure 2.7 shows the modulus of the current density in the conducting domain. Figure 2.8 shows the current density vector field. Finally, Figure 2.9 shows the Lorentz force in the workpiece.

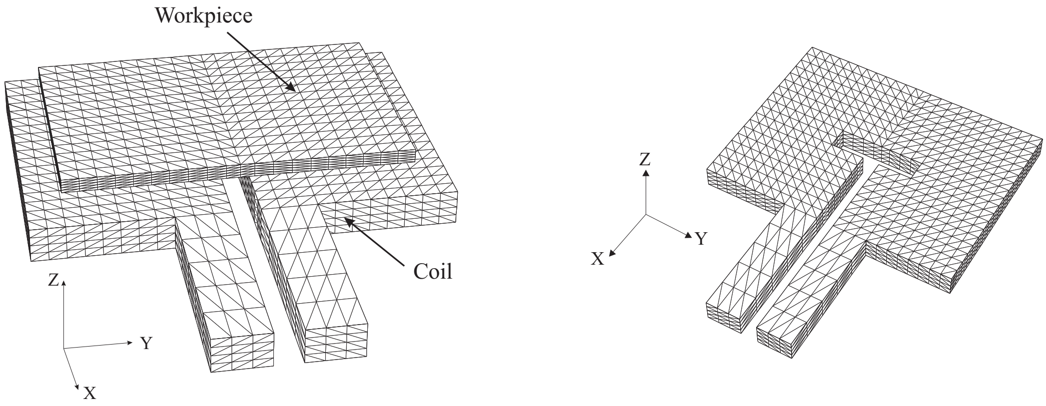

Figure 2.4: Mesh of the conducting domain (left). Detail of the coil mesh (right).

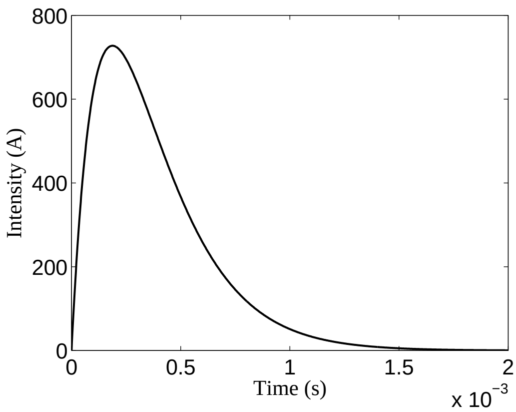

Figure 2.5: Current intensity (A) vs. time (s).

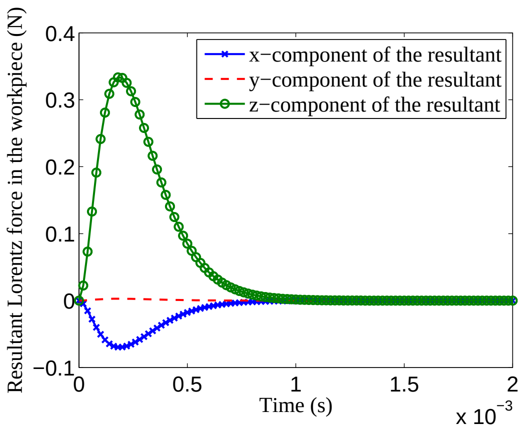

Figure 2.6: Resultant of the Lorentz force (N) in the workpiece vs. time (s).

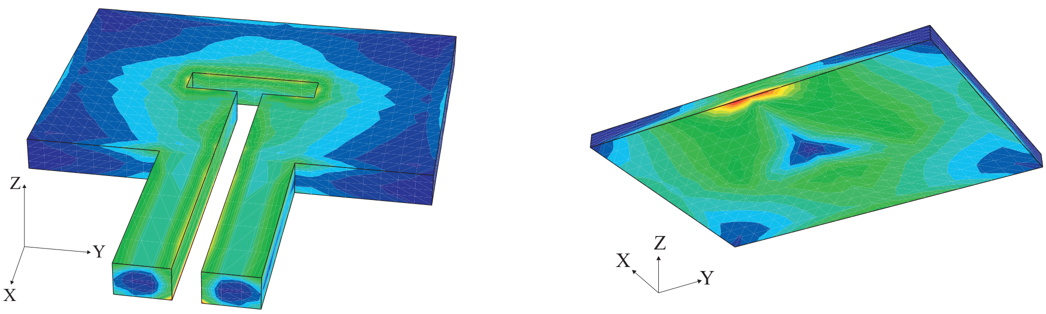

Figure 2.7: Modulus of the current density in coil and workpiece (underside) at time 0.00018 s.

Since this approach is also able to deal with non simply connected conductors we have solved another example in which the workpiece is as shown in Figure 2.10. In this case an additional cut surface contained in the interior of Ω_D is needed. As we have explained above, in this example,

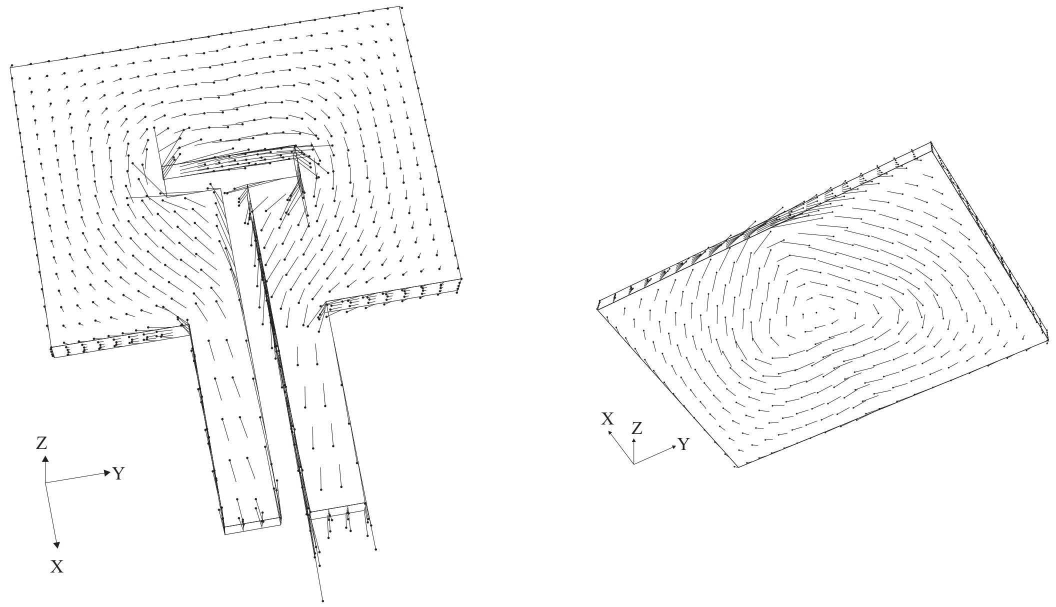

Figure 2.8: Distribution of the current density (vector field) in coil and workpiece (underside) at time 0.00018 s.

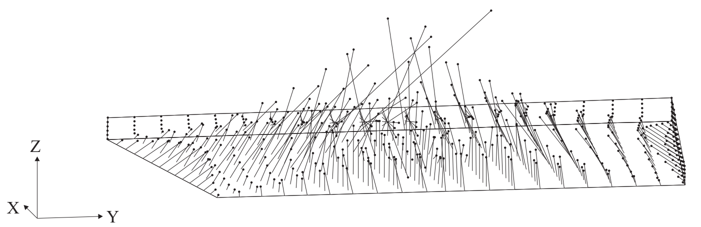

Figure 2.9: Lorentz force in the workpiece at time 0.00018 s.

the induced current intensity in the workpiece is an additional unknown which must be computed at each time step. Figure 2.11 shows the modulus of the induced current density in the workpiece at $t = 0.00018$ s. Figure 2.12 shows the additional unknown (induced current intensity in the workpiece) versus time.

A. Appendix

In this appendix, we will show an additional regularity result for evolution problems like Problem 2.4. We consider two real separable Hilbert spaces V and H. We suppose that V is densely and continuously included in H, so that, by identifying H with its dual H', we have that $V \hookrightarrow H \hookrightarrow V'$, both inclusions being dense and continuous.

Given $u_0 \in H$ and $g \in \mathrm{L}^2(0,T;V')$, we consider the following problem: find $u \in \mathrm{L}^2(0,T;V) \cap$

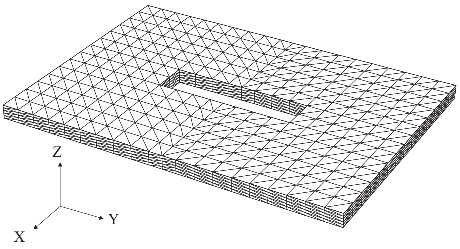

Figure 2.10: Mesh of the workpiece.

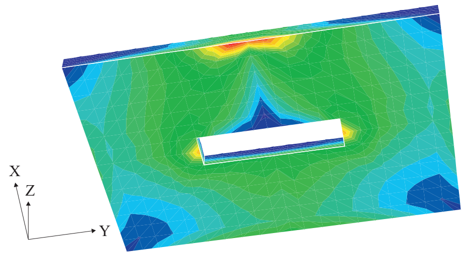

Figure 2.11: Modulus of the current density in workpiece at time 0.00018 s.

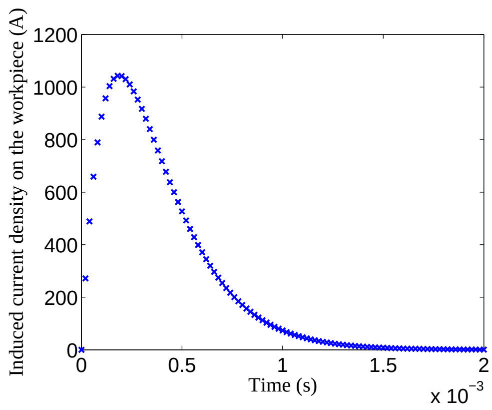

Figure 2.12: Induced current intensity (A) vs. time (s).

$\mathrm{H}^1(0,T;V')$ such that

$$\langle \partial_t u(t), v\rangle_{V'\times V} + c(u(t),v) = \langle g(t), v\rangle_{V'\times V} \quad \forall v\in V, \tag{A.1}$$

$$u(0) = u_0, \tag{A.2}$$

where $c : V \times V \to \mathbb{R}$ is a bounded bilinear form.

When c is elliptic, problem (A.1)–(A.2) has a unique solution and there exists $C > 0$ such that

$$\sup_{t\in[0,T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|_V^2\, dt \le C\left\{\|u_0\|_H^2 + \|g\|^2_{\mathrm{L}^2(0,T;V')}\right\}.$$

(See Theorems 1 and 2, Section 3, Chapter XVIII from [28].) Moreover, we have the following additional regularity result.

Theorem A.1 *Let c be elliptic, $u_0 \in V$ and $g \in \mathrm{H}^1(0,T;V')$. Then, the solution to problem (A.1)–(A.2) satisfies $u \in \mathrm{L}^\infty(0,T;V)$, $\partial_t u \in \mathrm{L}^2(0,T;H)$ and*

$$\|u\|^2_{\mathrm{L}^\infty(0,T;V)} + \|\partial_t u\|^2_{\mathrm{L}^2(0,T;H)} \le C\left\{\|u_0\|_V^2 + \|g\|^2_{\mathrm{H}^1(0,T;V')}\right\},$$

with C a positive constant independent of u_0 and g.

Proof. The proof, which follows the lines of that of Theorem 5, Chapter 7 from [32], is based on a Galerkin approximation technique (see Section 3, Chapter XVIII from [28]).

Let $\{w_j\}_{j\in\mathbb{N}}$ be a Hilbert basis of V. Let V_m be the finite-dimensional subspaces spanned by $\{w_j\}_{j=1}^m$, $m \in \mathbb{N}$. For $u_0 \in V$, let $u_{0m} \in V_m$ be the truncated expansion of u_0 in this basis, so that $u_{0m} \xrightarrow{m} u_0$ in V.

For fixed $m \in \mathbb{N}$, consider the following problem: find $u_m(t) := \sum_{j=1}^m \xi_j(t) w_j$ satisfying

$$(\partial_t u_m(t), w_i)_H + c(u_m(t), w_i) = \langle g(t), w_i\rangle_{V'\times V}, \quad 1 \le i \le m, \tag{A.3}$$

$$u_m(0) = u_{0m}. \tag{A.4}$$

This is a well-posed initial value problem for a system of ordinary differential equations. By proceeding as in the proof of Theorem 3, Chapter 7 from [32], we know that there exists a subsequence of $\{u_m\}_{m\in\mathbb{N}}$, that we also denote $\{u_m\}_{m\in\mathbb{N}}$, such that

$$u_m \to u \text{ weakly in } \mathrm{L}^2(0,T;V) \qquad \text{and} \qquad \partial_t u_m \to \partial_t u \text{ weakly in } \mathrm{L}^2(0,T;V'). \tag{A.5}$$

We multiply (A.3) by $\xi_j'(t)$ and sum from $j = 1$ to m, to obtain

$$\|\partial_t u_m(t)\|_H^2 + \frac{1}{2}\frac{d}{dt} c(u_m(t), u_m(t)) = \frac{d}{dt}\langle g(t), u_m(t)\rangle_{V'\times V} - \langle \partial_t g(t), u_m(t)\rangle_{V'\times V}.$$

Integrating over t, the ellipticity of c and Cauchy-Schwarz inequality yield

$$\begin{aligned}\int_0^t \|\partial_t u_m(s)\|_H^2\,ds + \|u_m(t)\|_V^2 &\le C\left\{\|u_{0m}\|_V^2 + \sup_{0\le s\le t}\|g(s)\|_{V'}^2 + \int_0^t \|\partial_t g(s)\|_{V'}^2\,ds + \int_0^t \|u_m(s)\|_V^2\,ds\right\}\\ &\le C\left\{\|u_0\|_V^2 + \|g\|_{\mathrm{H}^1(0,T;V')}^2 + \int_0^t \|u_m(s)\|_V^2\,ds\right\}.\end{aligned}$$

Hence, using Gronwall's inequality, we obtain

$$\|u_m(t)\|_V^2 \le C\left\{\|u_0\|_V^2 + \|g\|_{\mathrm{H}^1(0,T;V')}^2\right\}.$$

Therefore

$$\int_0^T \|\partial_t u_m(t)\|_H^2\,dt + \operatorname*{ess\,sup}_{0\le t\le T}\|u_m(t)\|_V^2 \le C\left\{\|u_0\|_V^2 + \|g\|_{\mathrm{H}^1(0,T;V')}^2\right\}.$$

Hence, passing to the limit as $m \to \infty$ and using (A.5), we conclude the proof by proceeding as in Step 2 of the proof of Theorem 5, Chapter 7 from [32]. ■

Corollary A.2 *Let us assume that the bilinear form c satisfies the following Gårding inequality: there exists λ and $\alpha > 0$, such that $c(v,v) + \lambda\|v\|_H^2 \ge \alpha\|v\|_V^2$ for all $v \in V$. Then, problem (A.1)–(A.2) has a unique solution and the previous theorem is also valid.*

Proof. By doing the change of variable $u = we^{\lambda t}$, we have that w satisfies

$$\langle \partial_t w(t), v\rangle_{V'\times V} + \widetilde{c}(w(t), v) = \langle e^{-\lambda t} g(t), v\rangle_{V'\times V} \quad \forall v \in V,$$

$$w(0) = u_0,$$

where $\widetilde{c}(w,v) := c(w,v) + \lambda(w,v)_H$ is bilinear, bounded and elliptic. Thus, we are in a position to apply the previous results to conclude the proof. ■

Chapter 3

An eddy current problem in terms of a time-primitive of the electric field with non-local source conditions

3.1 Introduction

The goal of this paper is to analyze a time-dependent eddy current problem defined in a three-dimensional bounded domain including conducting and dielectric materials, subject to source boundary conditions feasible from the physical point of view. This model arises in applications where the problem is reduced to a bounded domain and it is necessary to link the electromagnetic fields with sources provided by an external circuit by means of current intensities or voltage drops (see, for instance, [22, 41]). Both cases of source data will be separately analyzed in domains with a rather complex geometry, which allows modeling a great variety of applications.

In the literature, we can find some papers related to the numerical analysis of the three-dimensional time-dependent eddy current model in bounded domains containing conducting and dielectric materials [2, 26, 44, 48, 72, 14]. Most of these articles deal with the case where the conducting materials are strictly contained in the computational domain and the source current is imposed in an inner subdomain. These formulations involve natural and/or essential boundary conditions which differ depending on the primary unknown.

In order to consider sources provided by external circuits, the authors of this paper have recently analyzed in [14] (cf. Chapter 2, this thesis) a transient eddy current problem where the input current intensities are imposed in terms of source boundary conditions. The problem is written in terms of the magnetic field, which must satisfy the curl-free condition in the dielectric domain. At the discrete level, a magnetic scalar potential is introduced in the dielectric domain, which allows an important saving in computational effort. However, this formulation requires to build "cutting" surfaces to make the dielectric domain simply connected. These cutting surfaces can be difficult to build in complex geometries.

The present paper analyzes a formulation of the problem based on the time-primitive of the electric field and a Lagrange multiplier to impose the divergence-free constraint of the electric displacement. Although the computer cost is significantly more expensive than that of the method proposed in [14], it does not need of cutting surfaces, which is a significant advantage in case of complex geometries.

The time primitive of the electric field has been introduced in the literature of electrical engineering in [31] and it is usually known as *modified magnetic vector potential*. This potential has been used later by other authors (see, for instance, [42, 43, 68]) which usually couple this vector field with different unknowns in the dielectric part.

The same variable, the time-primitive of the electric field, has been used as the main unknown in the analysis of transient eddy current problems with inner current sources and standard essential and natural boundary conditions in [2, 26]. In the present paper, we obtain a degenerate parabolic problem as in these references; however, we cannot use the same arguments to prove the well-posedness of continuous and discrete problems due to the presence of the non-local source conditions. Thus, in order to analyze the resulting weak formulation, we resort to some results obtained in [14].

As in [2], the formulation analyzed in this paper need as a data the normal component of the electric displacement on the outer boundary. However, we prove that this boundary data has no effect on the value of the main physical quantities, namely, the magnetic field in the whole domain and the electric field in the conducting one. Thus, the data actually needed in practice for this formulation reduces to inputs current intensities or voltage drops.

To discretize the mixed problem we propose a finite element method on tetrahedral meshes based on Nédélec edge elements for the main variable and standard piecewise linear elements for the Lagrange multiplier. We prove that this leads to a degenerate algebraic-differential problem, which we prove is well-posed. Then, we obtain optimal order error estimates for this as well as for a fully discrete problem obtained by an implicit time discretization.

Let us remark that under the assumption of time-independent electromagnetic coefficients, similar arguments lead to a formulation in terms of the electric field, too. In principle the techniques in this paper could be tried to analyze such a formulation, provided further smoothness in time holds for the boundary data.

The outline of the paper is as follows. In Section 3.2 we introduce the transient eddy current model and state the geometrical framework for the analysis. In Section 3.3 we analyze the problem with input current intensities as boundary data. We obtain a time-dependent weak mixed formulation of the problem with input current intensities as boundary data and prove that it is well-posed. We introduce a space discretization based on finite elements and prove error estimates. We propose a backward Euler scheme for time discretization and obtain error estimates for the fully discretized problem. In Section 3.4 we perform a similar analysis for the transient eddy current problem, but now with voltage drops as boundary data. In Section 3.5, we report some numerical results. We present a test with known analytical solution which allows us to confirm the order of convergence predicted by the theory in both cases, namely, using intensities or voltage drops as source data. Finally, we apply the method to an application in non destructive testing which involves a more complex geometry.

3.2 Statement of the problem

Three dimensional eddy current problems describe low-frequency electromagnetic phenomena. In this case, displacement currents may be neglected (see, for instance, [21, Chapter 8]), so that Maxwell's equations restricted to a domain Ω become

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J} \quad \text{in } [0,T]\times\Omega, \tag{3.1}$$

$$\partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } [0,T]\times\Omega, \tag{3.2}$$

$$\operatorname{div}(\mu\boldsymbol{H}) = 0 \quad \text{in } [0,T]\times\Omega, \tag{3.3}$$

$$\boldsymbol{J} = \sigma\boldsymbol{E} \quad \text{in } [0,T]\times\Omega, \tag{3.4}$$

where $\boldsymbol{E}(t,\boldsymbol{x})$ is the electric field, $\boldsymbol{H}(t,\boldsymbol{x})$ the magnetic field, $\boldsymbol{J}(t,\boldsymbol{x})$ the current density, μ the magnetic permeability and σ the electric conductivity. Here and thereafter, we use boldface letters to denote vector fields and variables as well as vector-valued operators.

We assume that Ω is a simply connected three-dimensional bounded domain, which consists of two parts, Ω_{C} and Ω_{D}, occupied by conductors and dielectrics, respectively. We assume that Ω_{D} is connected. The domain Ω is assumed to have a Lipschitz-continuous connected boundary. We denote by Γ_{C}, Γ_{D} and Γ_{I} the open surfaces such that $\bar{\Gamma}_{\mathrm{C}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega$ is the outer boundary of the conductor domain, $\bar{\Gamma}_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \cap \partial\Omega$ that of the dielectric domain and $\bar{\Gamma}_{\mathrm{I}} := \partial\Omega_{\mathrm{C}} \cap \partial\Omega_{\mathrm{D}}$ the interface between both domains. We also denote by $\boldsymbol{n}$, $\boldsymbol{n}_{\mathrm{C}}$ and $\boldsymbol{n}_{\mathrm{D}}$ the outer unit normal vectors to $\partial\Omega$, $\partial\Omega_{\mathrm{C}}$ and $\partial\Omega_{\mathrm{D}}$, respectively. Notice that $\boldsymbol{n}_{\mathrm{C}} = \boldsymbol{n}$ on Γ_{C}, $\boldsymbol{n}_{\mathrm{D}} = \boldsymbol{n}$ on Γ_{D} and $\boldsymbol{n}_{\mathrm{C}} = -\boldsymbol{n}_{\mathrm{D}}$ on Γ_{I}.

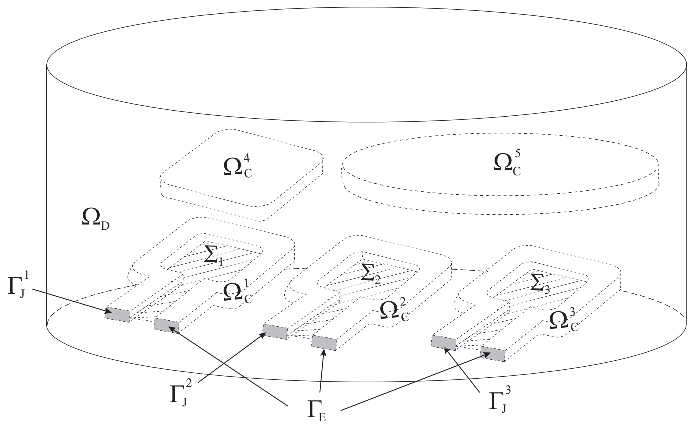

Figure 3.1: Sketch of the domain.

As shown in Figure 3.1, the disjoint connected components of the conducting domain are of two types: "inductors" which go through the boundary of Ω, and "workpieces" which have their closure included in Ω. We denote $\Omega_{\mathrm{C}}^1, \dots, \Omega_{\mathrm{C}}^N$ the former and $\Omega_{\mathrm{C}}^{N+1}, \dots, \Omega_{\mathrm{C}}^M$ the latter. Moreover, we assume that each Ω_{C}^n, $n = 1, \dots, M$, is simply connected with a connected boundary $\partial\Omega_{\mathrm{C}}^n$. We assume that Γ_{I} splits in connected components as follows: $\Gamma_{\mathrm{I}} = \bigcup_{n=1}^M \Gamma_{\mathrm{I}}^n$, where $\Gamma_{\mathrm{I}}^n := \Gamma_{\mathrm{I}} \cap \partial\Omega_{\mathrm{C}}^n$, $n = 1, \dots, M$.

We assume that the outer boundary of each inductor, $\partial\Omega_{\mathrm{C}}^n \cap \partial\Omega$, $n = 1, \ldots, N$, has two disjoint connected components, both being the closure of open simply connected surfaces: the current entrance $\Gamma_{\boldsymbol{J}}^n$, where the inductor is connected to a transient electric current source, and the current exit $\Gamma_{\boldsymbol{E}}^n$. We denote $\Gamma_{\boldsymbol{J}} := \Gamma_{\boldsymbol{J}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{J}}^N$ and $\Gamma_{\boldsymbol{E}} := \Gamma_{\boldsymbol{E}}^1 \cup \cdots \cup \Gamma_{\boldsymbol{E}}^N$. Furthermore, we assume that $\bar{\Gamma}_{\mathrm{J}}^n \cap \bar{\Gamma}_{\mathrm{J}}^m = \emptyset$, $\bar{\Gamma}_{\mathrm{E}}^n \cap \bar{\Gamma}_{\mathrm{E}}^m = \emptyset$, $1 \leq m, n \leq N$, $m \neq n$, and $\bar{\Gamma}_{\mathrm{J}} \cap \bar{\Gamma}_{\mathrm{E}} = \emptyset$.

We assume that for each inductor, Ω_{C}^n, $n = 1, \ldots, N$, there exists one connected "cut" surface $\Sigma_n \subset \Omega_{\mathrm{D}}$, with $\partial\Sigma_n \subset \partial\Omega_{\mathrm{C}}^n \cup \Gamma_{\mathrm{D}}$, such that $\widetilde{\Omega}_{\mathrm{D}} := \Omega_{\mathrm{D}} \setminus \bigcup_{n=1}^N \Sigma_n$ is pseudo-Lipschitz and simply connected (see, for instance, [9]). We also assume that $\bar{\Sigma}_n \cap \bar{\Sigma}_m = \emptyset$ for $n \neq m$ (see Figure 3.1). We denote $\Sigma := \bigcup_{n=1}^N \Sigma_n$ and assume that Γ_{D} and $\Gamma_{\mathrm{D}} \setminus \partial\Sigma$ are connected.

We suppose that μ and σ are time-independent and there exist positive constants $\underline{\mu}$, $\overline{\mu}$, $\overline{\sigma}$ and $\underline{\sigma}$ such that

$$\begin{aligned}
&0 < \underline{\mu} \leq \mu(\boldsymbol{x}) \leq \overline{\mu}, \quad \text{a.e. } \boldsymbol{x} \in \Omega,\\
&0 < \underline{\sigma} \leq \sigma(\boldsymbol{x}) \leq \overline{\sigma}, \quad \text{a.e. } \boldsymbol{x} \in \Omega_{\mathrm{C}} \quad \text{and} \quad \sigma \equiv 0 \text{ in } \Omega_{\mathrm{D}}.
\end{aligned}$$

We have to complete the model with an initial condition, $\boldsymbol{H}(0) = \boldsymbol{H}_0$, the source terms and suitable boundary conditions. For the latter, we consider the following ones:

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } [0,T] \times \Gamma_{\boldsymbol{E}}, \tag{3.5}$$

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } [0,T] \times \Gamma_{\boldsymbol{J}}, \tag{3.6}$$

$$\mu\boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on } [0,T] \times \partial\Omega. \tag{3.7}$$

Conditions (3.5), (3.6) and (3.7) have been proposed in [22] in a more general setting. They will appear as natural boundary conditions of our weak formulation of the problem. The former mean that the electric current density is normal to the current entrance and exit surfaces, whereas the latter means that the magnetic field is tangential to the boundary of the whole domain Ω.

To consider sources provided by external circuits we have two possibilities: either the intensities of the input current or the voltage drops most be given for each inductor Ω_{C}^n, $n = 1, \ldots, N$. In the first case, from (3.4), we have that

$$\int_{\Gamma_{\boldsymbol{J}}^n} \sigma\boldsymbol{E} \cdot \boldsymbol{n} = I_n \quad \text{in } [0,T], \tag{3.8}$$

where I_n is the current intensity through the surface $\Gamma_{\boldsymbol{J}}^n$.

To write down the equation corresponding to the second case, let V_n be the input voltage drop along the inductor Ω_{C}^n. It follows from (3.7) and (3.2) that $\mathbf{curl}\,\boldsymbol{E} \cdot \boldsymbol{n} = 0$ on $[0,T] \times \partial\Omega$. Hence, there exists a surface potential $V(t,\boldsymbol{x})$ defined on the boundary of the whole Ω and such that $\boldsymbol{n} \times \boldsymbol{E}(t,\boldsymbol{x}) \times \boldsymbol{n} = -\,\mathbf{grad}_\tau V(t,\boldsymbol{x})$ on $\partial\Omega$, where $\mathbf{grad}_\tau$ denotes the surface gradient (cf. [25]). Moreover, (3.5) and (3.6) imply that $V(t,\boldsymbol{x})$ must be constant on each connected component of $\Gamma_{\boldsymbol{J}}$ and $\Gamma_{\boldsymbol{E}}$. The difference $V_n(t) := V|_{\Gamma_{\boldsymbol{E}}^n}(t) - V|_{\Gamma_{\boldsymbol{J}}^n}(t)$ is the voltage drop along the conductor Ω_{C}^n. Thus, given V_n, the boundary condition reads

$$\boldsymbol{n} \times \boldsymbol{E} \times \boldsymbol{n} = -\,\mathbf{grad}_\tau V \text{ on } \partial\Omega, \text{ with } V|_{\Gamma_{\boldsymbol{E}}^n} - V|_{\Gamma_{\boldsymbol{J}}^n} = V_n \quad \text{in } [0,T]. \tag{3.9}$$

Although in a same problem, we could consider that voltage drops are known for some inductors and current intensities for the others, for simplicity we will study each case separately.

We have shown in [14] (cf. Chapter 2, this thesis) that equations (3.1)–(3.7) with boundary data (3.8) or (3.9) and initial condition $\boldsymbol{H}_0$ satisfying appropriate assumptions, lead to a well-posed problem. Note that, since the electric conductivity coefficient σ vanishes in Ω_{D}, we do not have uniqueness of the electric field $\boldsymbol{E}$ in Ω_{D}; in fact, if we add to a solution $\boldsymbol{E}$ the gradient of any function with compact support in Ω_{D}, the resulting field also solves (3.1)–(3.7).

Therefore, we must add equations so that $\boldsymbol{E}$ is uniquely determined. With this aim we introduce the following conditions as in [2, 7] which assumes absence of electric charge in the dielectric domain:

$$\operatorname{div}(\epsilon \boldsymbol{E}) = 0 \quad \text{in } (0,T)\times\Omega_{\mathrm{D}}, \tag{3.10}$$

$$\epsilon \boldsymbol{E}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n} = g \quad \text{on } [0,T]\times\Gamma_{\mathrm{D}}, \tag{3.11}$$

$$\int_{\Gamma_{\mathrm{I}}^k} \epsilon \boldsymbol{E}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n} = 0, \quad k = 2,\dots,M, \quad \text{in } [0,T], \tag{3.12}$$

where ϵ is the electric permittivity and g is an additional data. Notice that $\int_{\Gamma_{\mathrm{I}}^1} \epsilon \boldsymbol{E}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}$ is also fixed. In fact, from the equations above and Gauss Theorem, $\int_{\Gamma_{\mathrm{I}}^1} \epsilon \boldsymbol{E}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n} = -\int_{\Gamma_{\mathrm{D}}} g$.

Boundary condition (3.11) involves the knowledge of an additional boundary data, the normal trace of $\epsilon \boldsymbol{E}$ on Γ_{D}, which can be difficult to obtain in practice. However, we prove in the present paper that $\boldsymbol{E}|_{\Omega_{\mathrm{C}}}$ and $\boldsymbol{H}$ in the whole domain Ω are independent of the value of g. Hence, this allows us to choose, for instance, $g = 0$ in (3.11) if we are not interested in the electric field in Ω_{D} (see Remark 3.5). In such a case, $\boldsymbol{E}|_{\Omega_{\mathrm{D}}}$ is not the actual electric field but just an auxiliary variable which allows us to compute the typical quantities of interest: $\boldsymbol{E}|_{\Omega_{\mathrm{C}}}$ and $\boldsymbol{H}$.

Throughout this paper, we will use standard notation for Sobolev spaces and norms. We will also use the well known Hilbert spaces $\mathrm{H}(\mathbf{curl};\Omega)$, $\mathrm{H}(\operatorname{div};\Omega)$, $\mathrm{H}_0(\operatorname{div}^0;\Omega)$, etc. (see, for instance, [9]).

Let us remark that, given η and $\varsigma \in \mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{D}})$, we say that $\eta = \varsigma$ on Γ, where Γ is an open surface contained in $\partial\Omega_{\mathrm{D}}$, if $\eta = \varsigma$ on $\mathrm{H}_{00}^{-1/2}(\Gamma)$; namely, if $\langle \eta, \phi\rangle_{\partial\Omega_{\mathrm{D}}} = \langle \varsigma, \phi\rangle_{\partial\Omega_{\mathrm{D}}}$ $\forall \phi \in \mathrm{H}_{00}^{1/2}(\Gamma)$, where $\langle\cdot,\cdot\rangle_{\partial\Omega_{\mathrm{D}}}$ denotes the duality pairing in $\mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{D}}) \times \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{D}})$. In particular, equation (3.11) must be understood in this sense. Similarly, equation (3.8) has to be understood as the a duality paring $\langle \sigma \boldsymbol{E}(t)\cdot\boldsymbol{n}, 1\rangle_{\Gamma_J^n}$. This paring is well defined because $\sigma\boldsymbol{E}(t)\cdot\boldsymbol{n} = 0$ on Γ_{D} (see, [33, Proposition 3.3]). The same happens with equation (3.12).

3.3 Eddy current problem with input current intensities as source data

The aim of this section is to analyze a formulation in terms of a time-primitive of the electric field of the transient eddy current problem given by equations (3.1)–(3.8), the latter for $n = 1,\dots,N$, and (3.10)–(3.12) with an adequate initial condition $\boldsymbol{H}_0$, under appropriate assumptions of the data. In particular, we assume that $g \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma_{\mathrm{D}}))$, $I_n \in \mathrm{H}^2(0,T)$, $n = 1,\dots,N$, and the initial data $\boldsymbol{H}_0$ satisfies

$$\boldsymbol{H}_0 \in \boldsymbol{\mathcal{X}}, \qquad \langle \mathbf{curl}\, \boldsymbol{H}_0 \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = I_n(0),\ n = 1, \ldots, N, \qquad \text{and} \qquad \mu \boldsymbol{H}_0 \in \mathrm{H}_0(\mathrm{div}^0; \Omega), \tag{3.13}$$

where

$$\boldsymbol{\mathcal{X}} := \left\{ \boldsymbol{G} \in \mathrm{H}(\mathbf{curl}; \Omega) : \ \mathbf{curl}\, \boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}} \right\}.$$

Let us introduce the time-primitive of the electric field

$$\boldsymbol{u}(t, \boldsymbol{x}) := \int_0^t \boldsymbol{E}(s, \boldsymbol{x})\, ds.$$

Integrating (3.2) over $[0, t]$ we obtain

$$\mu(\boldsymbol{x}) \boldsymbol{H}(t, \boldsymbol{x}) - \mu(\boldsymbol{x}) \boldsymbol{H}_0(\boldsymbol{x}) + \mathbf{curl}\, \boldsymbol{u}(t, \boldsymbol{x}) = \mathbf{0}. \tag{3.14}$$

Using that $\boldsymbol{u}(0, \boldsymbol{x}) = \mathbf{0}$, it is easy to write the transient eddy current equations in terms of $\boldsymbol{u}$ as follows:

$$\sigma \partial_t \boldsymbol{u} + \mathbf{curl}\left(\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{u} \right) = \mathbf{curl}\, \boldsymbol{H}_0 \quad \text{in } [0, T] \times \Omega, \tag{3.15}$$

$$\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } [0, T] \times \partial\Omega, \tag{3.16}$$

$$\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } [0, T] \times \Gamma_{\mathrm{C}}, \tag{3.17}$$

$$\mathrm{div}(\epsilon \boldsymbol{u}) = 0 \quad \text{in } [0, T] \times \Omega_{\mathrm{D}}, \tag{3.18}$$

$$\epsilon \boldsymbol{u}(t) \cdot \boldsymbol{n} = \int_0^t g(s)\, ds \quad \text{on } \Gamma_{\mathrm{D}}, \quad t \in [0, T], \tag{3.19}$$

$$\langle \epsilon \boldsymbol{u}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\mathrm{I}}^k} = 0, \quad k = 2, \ldots, M, \quad \text{in } [0, T], \tag{3.20}$$

$$\langle \sigma \boldsymbol{u}(t) \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = \int_0^t I_n(s)\, ds, \quad n = 1, \ldots, N, \quad t \in [0, T], \tag{3.21}$$

$$\boldsymbol{u}(0) = \mathbf{0} \quad \text{in } \Omega. \tag{3.22}$$

Our next goal is to obtain a weak formulation of this problem. With this end, we introduce the following space:

$$\boldsymbol{\mathcal{U}} := \{ \boldsymbol{w} \in \mathrm{H}(\mathbf{curl}; \Omega) \ : \ \boldsymbol{w} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma_{\mathrm{C}} \text{ and } \mathbf{curl}\, \boldsymbol{w} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \}.$$

Notice that, according to (3.16)–(3.17), we have that $\boldsymbol{u}(t) \in \boldsymbol{\mathcal{U}}$ at each $t \in [0, T]$. On the other hand, for all $\boldsymbol{w} \in \boldsymbol{\mathcal{U}}$ there exists a unique $W \in \mathcal{W}$ such that $\boldsymbol{n} \times \boldsymbol{w} \times \boldsymbol{n} = -\,\mathbf{grad}_\tau W$ on $\partial\Omega$, where $\mathcal{W}$ is defined by

$$\mathcal{W} := \left\{ W \in \mathrm{H}^{1/2}(\partial\Omega)/\mathbb{R} \ : \ W|_{\Gamma_{\boldsymbol{J}}^n} \text{ and } W|_{\Gamma_{\boldsymbol{E}}^n} \text{ constant, } n = 1, \ldots, N \right\}$$

(see Lemma 2.1 in [18]).

Let $L_n : \boldsymbol{\mathcal{U}} \to \mathbb{R}$, $n = 1, \ldots, N$, be defined by

$$L_n(\boldsymbol{w}) := W|_{\Gamma_{\boldsymbol{E}}^n} - W|_{\Gamma_{\boldsymbol{J}}^n}, \tag{3.23}$$

where $W \in \mathcal{W}$ is the only function in this space such that $\boldsymbol{n} \times \boldsymbol{w} \times \boldsymbol{n} = -\,\mathbf{grad}_\tau W$ on $\partial\Omega$. We have that L_n are bounded linear functionals. In fact,

$$|L_n(\boldsymbol{w})| \leq \frac{1}{|\Gamma_{\boldsymbol{E}}|^{1/2}} \|\boldsymbol{w}\|_{\mathrm{L}^2(\Gamma_{\boldsymbol{E}})} + \frac{1}{|\Gamma_{\boldsymbol{J}}|^{1/2}} \|\boldsymbol{w}\|_{\mathrm{L}^2(\Gamma_{\boldsymbol{J}})} \leq C \,\|W\|_{\mathrm{H}^{1/2}(\partial\Omega)} \leq C \,\|\boldsymbol{w}\|_{\mathrm{H}(\mathbf{curl};\Omega)},$$

where, for the last inequality, we have used results from [25, Remark 5.2]. Here and thereafter C denotes a generic constant not necessarily the same at each occurrence.

The following lemma will be used to impose the boundary conditions involving the input current intensities. Here and thereafter $\langle\cdot,\cdot\rangle$ denotes the duality pairing $\langle\cdot,\cdot\rangle_{\mathrm{H}^{-1/2}_{\partial\Omega}(\mathrm{div}_\tau;\partial\Omega)\times\mathrm{H}^{-1/2}_{\partial\Omega}(\mathbf{curl}_\tau;\partial\Omega)}$ as defined in Section 5 from [25].

Lemma 3.1 *For all $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$ and $W \in \mathcal{W}$ we have*

$$\langle \boldsymbol{G} \times \boldsymbol{n}, \mathbf{grad}_\tau W\rangle = \sum_{n=1}^{N} \left(W|_{\Gamma_{\boldsymbol{E}}^n} - W|_{\Gamma_{\boldsymbol{J}}^n} \right) \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n}.$$

Proof. Let Φ_n be any smooth function defined in Ω and such that $\Phi_n|_{\Omega_{\mathrm{C}}^m} = \delta_{nm}$, $n = 1, \ldots, N$, $m = 1, \ldots, M$. Then, for $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$,

$$\begin{aligned}
\langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} + \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{E}}^n} &= \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, \Phi_n\rangle_{\partial\Omega} \\
&= \int_\Omega \mathbf{curl}\, \boldsymbol{G} \cdot \mathbf{grad}\, \Phi_n \\
&= \int_{\Omega_{\mathrm{C}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \mathbf{grad}\, \Phi_n = 0.
\end{aligned}$$

Hence, $\langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = -\langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{E}}^n}$, $n = 1, \ldots, N$.

Therefore, for $W \in \mathcal{W}$, if $\Psi \in \mathrm{H}^1(\Omega)$ is such that $\Psi|_{\partial\Omega} = W$

$$\begin{aligned}
\langle \boldsymbol{G} \times \boldsymbol{n}, \mathbf{grad}_\tau W\rangle &= -\int_\Omega \mathbf{curl}\, \boldsymbol{G} \cdot \mathbf{grad}\, \Psi = -\langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, W\rangle_{\partial\Omega} \\
&= -\sum_{n=1}^{N} W|_{\Gamma_{\boldsymbol{J}}^n} \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} - \sum_{n=1}^{N} W|_{\Gamma_{\boldsymbol{E}}^n} \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{E}}^n} \\
&= \sum_{n=1}^{N} \left(W|_{\Gamma_{\boldsymbol{E}}^n} - W|_{\Gamma_{\boldsymbol{J}}^n} \right) \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n}.
\end{aligned}$$

Thus we conclude the proof. ∎

Now, we are in a position to obtain a weak formulation of (3.15)–(3.22). By testing (3.15) with $\boldsymbol{w} \in \boldsymbol{\mathcal{U}}$ we have

$$\int_\Omega \sigma \partial_t \boldsymbol{u} \cdot \boldsymbol{w} + \int_\Omega \frac{1}{\mu} \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{w} - \left\langle \frac{1}{\mu} \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{w} \right\rangle = \int_\Omega \mathbf{curl}\, \boldsymbol{H}_0 \cdot \boldsymbol{w}.$$

Provided $\boldsymbol{H} \in \boldsymbol{\mathcal{X}}$, according to (3.14), $\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{u}(t) \in \boldsymbol{\mathcal{X}}$. Then, since $W \in \mathcal{W}$, applying Lemma 3.1

we have that

$$\begin{aligned}\left\langle \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}(t)\times\boldsymbol{n},\boldsymbol{w}\right\rangle &= -\left\langle \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}(t)\times\boldsymbol{n},\mathbf{grad}_\tau W\right\rangle\\
&= -\sum_{n=1}^{N}\left(W|_{\Gamma_{\boldsymbol{J}}^n} - W|_{\Gamma_{\boldsymbol{E}}^n}\right)\left\langle \mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}(t)\right)\cdot\boldsymbol{n},1\right\rangle_{\Gamma_{\boldsymbol{J}}^n}\\
&= -\sum_{n=1}^{N} L_n(\boldsymbol{w})\langle(\sigma\partial_t\boldsymbol{u}(t)-\mathbf{curl}\,\boldsymbol{H}_0)\cdot\boldsymbol{n},1\rangle_{\Gamma_{\boldsymbol{J}}^n}\\
&= \sum_{n=1}^{N} L_n(\boldsymbol{w})(I_n(t)-I_n(0)),\end{aligned}$$

the last equality because of (3.15), (3.13) and (3.21).

Therefore

$$\int_{\Omega_{\mathrm{C}}}\sigma\partial_t\boldsymbol{u}(t)\cdot\boldsymbol{w}+\int_\Omega\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}(t)\cdot\mathbf{curl}\,\boldsymbol{w}=\sum_{n=1}^{N}L_n(\boldsymbol{w})(I_n(t)-I_n(0))+\int_\Omega\mathbf{curl}\,\boldsymbol{H}_0\cdot\boldsymbol{w}\qquad\forall\boldsymbol{w}\in\boldsymbol{\mathcal{U}}. \tag{3.24}$$

On the other hand, we introduce the following space to impose (3.18)–(3.20) by means of a Lagrange multiplier:

$$\boldsymbol{\mathcal{M}}(\Omega_{\mathrm{D}}) := \left\{\varphi\in\mathrm{H}^1(\Omega_{\mathrm{D}})\,:\,\varphi|_{\Gamma_{\mathrm{I}}^1}=0,\ \varphi|_{\Gamma_{\mathrm{I}}^k}=\text{constant},\ k=2,\dots,M\right\}.$$

It is easy to show that, for $\boldsymbol{u}(t)\in\boldsymbol{\mathcal{U}}$,

$$\int_{\Omega_{\mathrm{D}}}\epsilon\boldsymbol{u}(t)\cdot\mathbf{grad}\,\varphi=\int_{\Gamma_{\mathrm{D}}}\left(\int_0^t g(s)\,ds\right)\varphi\quad\forall\varphi\in\boldsymbol{\mathcal{M}}(\Omega_{\mathrm{D}})\quad\Leftrightarrow\quad\begin{cases}\operatorname{div}(\epsilon\boldsymbol{u}(t))=0 & \text{in } \Omega_{\mathrm{D}},\\ \epsilon\boldsymbol{u}(t)\cdot\boldsymbol{n}=\int_0^t g(s)\,ds & \text{on } \Gamma_{\mathrm{D}},\\ \langle\epsilon\boldsymbol{u}(t)\cdot\boldsymbol{n},1\rangle_{\Gamma_{\mathrm{I}}^k}=0, & k=2,\dots,M.\end{cases} \tag{3.25}$$

Thus, we are led to the following problem:

Problem 3.2 *Given $g\in\mathrm{L}^2(0,T;L^2(\Gamma_{\mathrm{D}}))$, $I_n\in\mathrm{H}^2(0,T)$, $n=1,\dots,N$, and $\boldsymbol{H}_0$ satisfying (3.13), find $\boldsymbol{u}\in\mathrm{L}^2(0,T;\boldsymbol{\mathcal{U}})$ with $\boldsymbol{u}|_{\Omega_{\mathrm{C}}}\in\mathrm{H}^1(0,T;\mathrm{H}_{\Gamma_{\mathrm{C}}}(\mathbf{curl};\Omega_{\mathrm{C}}))$ and $\xi\in\mathrm{L}^2(0,T;\boldsymbol{\mathcal{M}}(\Omega_{\mathrm{D}}))$ such that*

$$\begin{aligned}&\int_{\Omega_{\mathrm{C}}}\sigma\partial_t\boldsymbol{u}(t)\cdot\boldsymbol{w}+\int_\Omega\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}(t)\cdot\mathbf{curl}\,\boldsymbol{w}+\int_{\Omega_{\mathrm{D}}}\epsilon\boldsymbol{w}\cdot\mathbf{grad}\,\xi(t)\\
&\qquad\qquad=\sum_{n=1}^{N}L_n(\boldsymbol{w})(I_n(t)-I_n(0))+\int_\Omega\mathbf{curl}\,\boldsymbol{H}_0\cdot\boldsymbol{w}\quad\forall\boldsymbol{w}\in\boldsymbol{\mathcal{U}},\\
&\int_{\Omega_{\mathrm{D}}}\epsilon\boldsymbol{u}(t)\cdot\mathbf{grad}\,\varphi=\int_{\Gamma_{\mathrm{D}}}\left(\int_0^t g(s)\,ds\right)\varphi\quad\forall\varphi\in\boldsymbol{\mathcal{M}}(\Omega_{\mathrm{D}}),\\
&\boldsymbol{u}(0)=\boldsymbol{0}\quad\textit{in }\Omega.\end{aligned}$$

As stated above, an alternative weak formulation of (3.1)–(3.8) in terms of the magnetic field was analyzed in [14] (cf. Chapter 2, this thesis). In this reference it was shown (cf. [14, Theorem 3.6] or Theorem 2.6 – this thesis) that there exists a unique $\boldsymbol{H} \in \mathrm{L}^2(0,T;\boldsymbol{\mathcal{X}}) \cap \mathrm{H}^1(0,T;\boldsymbol{\mathcal{H}}_{\boldsymbol{\mathcal{X}}})$ such that

$$\langle \mathbf{curl}\,\boldsymbol{H}(t)\cdot\boldsymbol{n},1\rangle_{\Gamma_{\boldsymbol{J}}^n} = I_n(t), \quad n=1,\dots,N, \tag{3.26}$$

$$\int_\Omega \mu\partial_t\boldsymbol{H}(t)\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}(t)\cdot\mathbf{curl}\,\boldsymbol{G} = 0 \quad \forall \boldsymbol{G}\in\boldsymbol{\mathcal{V}}, \tag{3.27}$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0, \tag{3.28}$$

where

$$\boldsymbol{\mathcal{H}}_{\boldsymbol{\mathcal{X}}} := \left\{\boldsymbol{G}\in\mathrm{L}^2(\Omega)^3 : \mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega_{\mathrm{D}}\right\} \text{ and } \boldsymbol{\mathcal{V}} := \left\{\boldsymbol{G}\in\boldsymbol{\mathcal{X}} : \langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},1\rangle_{\Gamma_{\boldsymbol{J}}^n} = 0,\ n=1,\dots,N\right\}.$$

Moreover, it was shown in Theorem 3.8 of the same reference that defining $\boldsymbol{E}_{\mathrm{C}}(t) := \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}(t)$ in Ω_{C}, the following properties hold true a.e. $t\in(0,T)$:

$$\operatorname{div}(\mu\boldsymbol{H}(t)) = 0 \quad \text{in } \Omega, \tag{3.29}$$

$$\mu\partial_t\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}_{\mathrm{C}}(t) = \mathbf{0} \quad \text{in } \Omega_{\mathrm{C}}, \tag{3.30}$$

$$\boldsymbol{J}(t) = \mathbf{0} \quad \text{in } \Omega_{\mathrm{D}}, \tag{3.31}$$

$$\mu\boldsymbol{H}(t)\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{3.32}$$

$$\boldsymbol{E}_{\mathrm{C}}(t)\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{C}}, \tag{3.33}$$

$$\langle \mathbf{curl}\,\boldsymbol{H}(t)\cdot\boldsymbol{n},1\rangle_{\Gamma_{\boldsymbol{J}}^n} = I_n(t), \quad n=1,\dots,N. \tag{3.34}$$

We will use these results to prove that Problem 3.2 also has a unique solution. With this aim, we need to extend $\boldsymbol{E}_{\mathrm{C}}$ to the dielectric domain in order to define $\boldsymbol{u}$ as its primitive. Next result shows how this can be done, taking into account that $\partial_t\boldsymbol{H} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)$, $\boldsymbol{E}_{\mathrm{C}} \in \mathrm{L}^2(0,T;\mathrm{H}_{\Gamma_{\mathrm{C}}}(\mathbf{curl};\Omega_{\mathrm{C}}))$ and $g\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma_{\mathrm{D}}))$.

Lemma 3.3 *There exists a unique* $\boldsymbol{E}_{\mathrm{D}}\in\mathrm{L}^2(0,T;\mathrm{H}(\mathbf{curl};\Omega_{\mathrm{D}}))$ *which satisfies a.e.* $t\in[0,T]$*:*

$$\mathbf{curl}\,\boldsymbol{E}_{\mathrm{D}}(t) = -\mu\partial_t\boldsymbol{H}(t) \quad \textit{in } \Omega_{\mathrm{D}}, \tag{3.35}$$

$$\boldsymbol{E}_{\mathrm{D}}(t)\times\boldsymbol{n}_{\mathrm{D}} = -\boldsymbol{E}_{\mathrm{C}}(t)\times\boldsymbol{n}_{\mathrm{C}} \quad \textit{on } \Gamma_{\mathrm{I}}, \tag{3.36}$$

$$\operatorname{div}(\epsilon\boldsymbol{E}_{\mathrm{D}}(t)) = 0 \quad \textit{in } \Omega_{\mathrm{D}}, \tag{3.37}$$

$$\epsilon\boldsymbol{E}_{\mathrm{D}}(t)\cdot\boldsymbol{n} = g(t) \quad \textit{on } \Gamma_{\mathrm{D}}, \tag{3.38}$$

$$\langle\epsilon\boldsymbol{E}_{\mathrm{D}}(t)\cdot\boldsymbol{n},1\rangle_{\Gamma_{\mathrm{I}}^k} = 0, \quad k=2,\dots,M. \tag{3.39}$$

Proof. To prove that this problem is well-posed, let us write $\boldsymbol{E}_{\mathrm{D}}(t) := \widetilde{\boldsymbol{E}}_{\mathrm{D}}(t) + \widehat{\boldsymbol{E}}_{\mathrm{D}}(t)$ a.e. $t\in[0,T]$, where $\widetilde{\boldsymbol{E}}_{\mathrm{D}}(t)$, $\widehat{\boldsymbol{E}}_{\mathrm{D}}(t)\in\mathrm{H}(\mathbf{curl};\Omega_{\mathrm{D}})$ are respective solutions to the following problems:

$$\begin{aligned}
\mathbf{curl}\,\widetilde{\boldsymbol{E}}_{\mathrm{D}}(t) &= \mathbf{0} \quad \text{in } \Omega_{\mathrm{D}},\\
\widetilde{\boldsymbol{E}}_{\mathrm{D}}(t)\times\boldsymbol{n}_{\mathrm{D}} &= \mathbf{0} \quad \text{on } \Gamma_{\mathrm{I}},\\
\operatorname{div}(\epsilon\widetilde{\boldsymbol{E}}_{\mathrm{D}}(t)) &= 0 \quad \text{in } \Omega_{\mathrm{D}},\\
\epsilon\widetilde{\boldsymbol{E}}_{\mathrm{D}}(t)\cdot\boldsymbol{n} &= g(t) \quad \text{on } \Gamma_{\mathrm{D}},\\
\langle\epsilon\widetilde{\boldsymbol{E}}_{\mathrm{D}}(t)\cdot\boldsymbol{n},1\rangle_{\Gamma_{\mathrm{I}}^k} &= 0, \quad k=2,\dots,M,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{curl}\,\widehat{\boldsymbol{E}}_{\mathrm{D}}(t) &= -\mu\partial_t\boldsymbol{H}(t) && \text{in } \Omega_{\mathrm{D}},\\
\widehat{\boldsymbol{E}}_{\mathrm{D}}(t)\times\boldsymbol{n}_{\mathrm{D}} &= -\boldsymbol{E}_{\mathrm{C}}(t)\times\boldsymbol{n}_{\mathrm{C}} && \text{on } \Gamma_{\mathrm{I}},\\
\operatorname{div}(\epsilon\widehat{\boldsymbol{E}}_{\mathrm{D}}(t)) &= 0 && \text{in } \Omega_{\mathrm{D}},\\
\epsilon\widehat{\boldsymbol{E}}_{\mathrm{D}}(t)\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma_{\mathrm{D}},\\
\langle\epsilon\widehat{\boldsymbol{E}}_{\mathrm{D}}(t)\cdot\boldsymbol{n},1\rangle_{\Gamma_{\mathrm{I}}^k} &= 0, && k=2,\dots,M.
\end{aligned}$$

It was proved in [33, Theorem 8.4] (see also [34, Lemma 3.2]) that for $g(t)\in \mathrm{L}^2(\Gamma_{\mathrm{D}})$, the first problem has a unique solution $\widetilde{\boldsymbol{E}}_{\mathrm{D}}(t)$ which satisfies

$$\|\widetilde{\boldsymbol{E}}_{\mathrm{D}}(t)\|_{\mathrm{H}(\mathbf{curl};\Omega_{\mathrm{D}})} \leq C\|g(t)\|_{\mathrm{L}^2(\Gamma_{\mathrm{D}})}.$$

To prove that the second problem is also well-posed, we follow the steps of the proof of [7, Theorem 8.6], where a similar result was obtained in the harmonic case and for a particular topology. The key point of this proof is that the term $\mu\partial_t\boldsymbol{H}(t,\cdot)\in\mathrm{L}^2(\Omega_{\mathrm{D}})^3$. Moreover, we also obtain

$$\|\widehat{\boldsymbol{E}}_{\mathrm{D}}(t)\|_{\mathrm{H}(\mathbf{curl};\Omega_{\mathrm{D}})} \leq C\left\{\|\partial_t\boldsymbol{H}(t)\|_{\mathrm{L}^2(\Omega_{\mathrm{D}})^3} + \|\boldsymbol{E}_{\mathrm{C}}(t)\|_{\mathrm{H}(\mathbf{curl};\Omega_{\mathrm{C}})}\right\}.$$

Thus, we have that $\boldsymbol{E}_{\mathrm{D}}(t) := \widetilde{\boldsymbol{E}}_{\mathrm{D}}(t)+\widehat{\boldsymbol{E}}_{\mathrm{D}}(t)$ is a solution to problem (3.35)–(3.39). Furthermore, $\boldsymbol{E}_{\mathrm{D}}\in\mathrm{L}^2(0,T;\mathrm{H}(\mathbf{curl};\Omega_{\mathrm{D}}))$ because of the above estimates for $\widetilde{\boldsymbol{E}}_{\mathrm{D}}$ and $\widehat{\boldsymbol{E}}_{\mathrm{D}}$. Moreover, this problem has at most one solution as a consequence of [33, Proposition 6.3]. Thus, we conclude the proof. ■

Now, we are in a position to conclude the following result.

Theorem 3.4 *Problem 3.2 has a unique solution $(\boldsymbol{u},\xi)$, with Lagrange multiplier $\xi\equiv 0$.*

Proof. To prove existence, let $\boldsymbol{H}\in\mathrm{L}^2(0,T;\boldsymbol{\mathcal{X}})\cap\mathrm{H}^1(0,T;\boldsymbol{\mathcal{H}}_{\boldsymbol{\mathcal{X}}})$ be the solution of (3.26)–(3.28) and let $\boldsymbol{E}_{\mathrm{C}} := \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}|_{\Omega_{\mathrm{C}}}\in\mathrm{L}^2(0,T;\mathrm{H}_{\Gamma_{\mathrm{C}}}(\mathbf{curl};\Omega_{\mathrm{C}}))$, so that (3.29)–(3.34) hold true a.e. $t\in(0,T)$. Let

$$\boldsymbol{E}(t) := \begin{cases} \boldsymbol{E}_{\mathrm{C}}(t) & \text{in } \Omega_{\mathrm{C}},\\ \boldsymbol{E}_{\mathrm{D}}(t) & \text{in } \Omega_{\mathrm{D}},\end{cases} \tag{3.40}$$

where $\boldsymbol{E}_{\mathrm{D}}\in\mathrm{L}^2(0,T;\mathrm{H}(\mathbf{curl};\Omega_{\mathrm{D}}))$ is the solution to (3.35)–(3.39) a.e. $t\in(0,T)$. As a consequence of (3.36), $\boldsymbol{E}(t)\in\mathrm{H}(\mathbf{curl};\Omega)$ a.e. $t\in(0,T)$ and, hence $\boldsymbol{E}\in\mathrm{L}^2(0,T;\mathrm{H}_{\Gamma_{\mathrm{C}}}(\mathbf{curl};\Omega))$. Thus, defining

$$\boldsymbol{u}(t,\boldsymbol{x}) := \int_0^t \boldsymbol{E}(s,\boldsymbol{x})\,ds, \qquad t\in[0,T], \quad \boldsymbol{x}\in\Omega, \tag{3.41}$$

$\boldsymbol{u}\in\mathrm{L}^2(0,T;\mathrm{H}_{\Gamma_{\mathrm{C}}}(\mathbf{curl};\Omega))$, too. Moreover, from (3.30) and (3.35) we have that $\mathbf{curl}\,\boldsymbol{E} = -\mu\partial_t\boldsymbol{H}$ in Ω and integrating in time

$$\mathbf{curl}\,\boldsymbol{u} = \mu\boldsymbol{H}_0 - \mu\boldsymbol{H} \qquad \text{in } [0,T]\times\Omega. \tag{3.42}$$

Therefore, from (3.13) and (3.32) we conclude that $\boldsymbol{u}\in\mathrm{L}^2(0,T;\boldsymbol{\mathcal{U}})$ and, since $\partial_t\boldsymbol{u}=\boldsymbol{E}$, we have that $\boldsymbol{u}|_{\Omega_{\mathrm{C}}}\in\mathrm{H}^1(0,T;\mathrm{H}_{\Gamma_{\mathrm{C}}}(\mathbf{curl};\Omega_{\mathrm{C}}))$.

Our next step is to prove that $(\boldsymbol{u}, 0)$ is a solution to Problem 3.2. With this aim, first we notice that by virtue of (3.29)–(3.34), the definition of $\boldsymbol{E}_{\mathrm{C}}(t)$ and (3.35)–(3.39), it is straightforward to show that $\boldsymbol{u}(t, \boldsymbol{x})$ satisfies (3.15)–(3.22). Then, the same steps that lead to (3.24) allow us to prove this expression in our case, which means that $(\boldsymbol{u}, 0)$ satisfies the first equation of Problem 3.2.

On the other hand, we integrate in time (3.37)–(3.39) and use the fact that $\boldsymbol{u}(0) = \mathbf{0}$ in Ω, to conclude that $\boldsymbol{u}(t)$ satisfies the conditions on the right hand side of (3.25), which was shown to be equivalent to the second equation from Problem 3.2.

Thus, we have proved that $(\boldsymbol{u}, 0)$ is a solution of Problem 3.2. There only remains to prove that this problem has a unique solution. With this aim let $(\overline{\boldsymbol{u}}, \overline{\xi})$ be a solution of Problem 3.2 with vanishing data $g = 0$, $I_n = 0$, $n = 1, \ldots, N$, and $\boldsymbol{H}_0 = \mathbf{0}$, namely,

$$\int_{\Omega_{\mathrm{C}}} \sigma \partial_t \overline{\boldsymbol{u}}(t) \cdot \boldsymbol{w} + \int_{\Omega} \frac{1}{\mu} \operatorname{\mathbf{curl}} \overline{\boldsymbol{u}}(t) \cdot \operatorname{\mathbf{curl}} \boldsymbol{w} + \int_{\Omega_{\mathrm{D}}} \epsilon \boldsymbol{w} \cdot \operatorname{\mathbf{grad}} \overline{\xi}(t) = 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{U}}, \tag{3.43}$$

$$\int_{\Omega_{\mathrm{D}}} \epsilon \overline{\boldsymbol{u}}(t) \cdot \operatorname{\mathbf{grad}} \varphi = 0 \qquad \forall \varphi \in \boldsymbol{\mathcal{M}}(\Omega_{\mathrm{D}}), \tag{3.44}$$

$$\overline{\boldsymbol{u}}(0) = \mathbf{0} \qquad \text{in } \Omega. \tag{3.45}$$

By taking $\boldsymbol{w} = \overline{\boldsymbol{u}}(t)$ and $\varphi = \overline{\xi}(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt} \int_{\Omega_{\mathrm{C}}} \sigma |\overline{\boldsymbol{u}}(t)|^2 + \int_{\Omega} \frac{1}{\mu} |\operatorname{\mathbf{curl}} \overline{\boldsymbol{u}}(t)|^2 = 0 \quad \text{a.e. } t \in [0, T]$$

and integrating in time

$$\frac{1}{2} \underline{\sigma} \|\overline{\boldsymbol{u}}(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \int_0^t \int_{\Omega} \frac{1}{\mu} |\operatorname{\mathbf{curl}} \overline{\boldsymbol{u}}(s)|^2 \, ds \leq 0,$$

which implies that $\overline{\boldsymbol{u}}(t) = \mathbf{0}$ in Ω_{C} and $\operatorname{\mathbf{curl}} \overline{\boldsymbol{u}}(t) = \mathbf{0}$ in Ω. From this (3.44) and (3.25), we deduce that $\overline{\boldsymbol{u}}(t)$ is a solution of (3.35)–(3.39) with vanishing right hand sides. Hence $\overline{\boldsymbol{u}}(t) \equiv \mathbf{0}$ in Ω_{D} (see Proposition 6.3 in [33]) and we conclude that $\overline{\boldsymbol{u}}(t)$ vanishes in the whole domain.

On the other hand, let $\widetilde{\xi}(t)$ be the extension of $\overline{\xi}(t)$ defined by: $\widetilde{\xi}(t)|_{\Omega^k_{\mathrm{C}}} = \overline{\xi}(t)|_{\Gamma^k_{\mathrm{I}}}$, $k = 1, \ldots, M$. Then $\operatorname{\mathbf{grad}} \widetilde{\xi}(t) \in \boldsymbol{\mathcal{U}}$ and taking $\boldsymbol{w} = \operatorname{\mathbf{grad}} \widetilde{\xi}(t)$ in (3.43) we obtain $\operatorname{\mathbf{grad}} \overline{\xi}(t) = \mathbf{0}$ in Ω_{D}. Hence, $\overline{\xi}(t)$ vanishes because Ω_{D} is connected and $\overline{\xi}(t)|_{\Gamma^1_{\mathrm{I}}} = 0$. ■

Remark 3.5 *As was shown in the proof of the previous theorem, actually* $\boldsymbol{u} \in \mathrm{H}^1(0, T; \mathrm{H}_{\Gamma_{\mathrm{C}}}(\mathbf{curl}; \Omega))$. *Then, the physical quantities can be recovered from (3.41) and (3.42) as follows:*

$$\widetilde{\boldsymbol{E}} := \partial_t \boldsymbol{u} \quad \textit{and} \quad \widetilde{\boldsymbol{H}} := \boldsymbol{H}_0 - \frac{1}{\mu} \operatorname{\mathbf{curl}} \boldsymbol{u}.$$

Different choices of the data g *lead to different solutions* $\boldsymbol{u}$ *to Problem 3.2. However only* $\widetilde{\boldsymbol{E}}|_{\Omega_{\mathrm{D}}}$ *actually depends on* g. *In fact, we have shown in the proof of the theorem above that* $\widetilde{\boldsymbol{H}}$ *as defined above is the solution* $\boldsymbol{H}$ *to problem (3.26)–(3.28) (which does not depend on* g*) and* $\widetilde{\boldsymbol{E}}|_{\Omega_{\mathrm{C}}} = \boldsymbol{E}|_{\Omega_{\mathrm{C}}} = \frac{1}{\sigma} \operatorname{\mathbf{curl}} \boldsymbol{H}|_{\Omega_{\mathrm{C}}}$. *This is an important fact because, if we do not know the values of* $\epsilon \boldsymbol{E} \cdot \boldsymbol{n}$ *on* Γ_{D} *and we are not interested in computing* $\boldsymbol{E}$ *in the dielectric, then we can simply choose* $g = 0$ *and compute the magnetic field* $\boldsymbol{H}$ *in* Ω *and the electric field* $\boldsymbol{E}$ *in* Ω_{C}, *which are typically the most relevant quantities in physical applications.*

3.3.1 Space discretization

From now on, we assume that Ω, Ω_{C} and Ω_{D} are Lipschitz polyhedra and consider regular tetrahedral meshes $\mathcal{T}_h$ of Ω, such that each element $K \in \mathcal{T}_h$ is contained either in Ω_{C} or in Ω_{D} (h stands as usual for the corresponding mesh-size). Therefore, $\mathcal{T}_h^{\Omega_{\mathrm{D}}} := \{K \in \mathcal{T}_h : \ K \subset \Omega_{\mathrm{D}}\}$ is a mesh of Ω_{D}. We employ edge finite elements to approximate $\boldsymbol{u}$, more precisely, lowest-order Nédélec finite elements:

$$\boldsymbol{\mathcal{N}}_h(\Omega) := \{\boldsymbol{w}_h \in \mathrm{H}(\mathbf{curl}; \Omega) : \ \boldsymbol{w}_h|_K \in \boldsymbol{\mathcal{N}}(K) \ \forall K \in \mathcal{T}_h\},$$

where, for each tetrahedron K,

$$\boldsymbol{\mathcal{N}}(K) := \left\{\boldsymbol{w}_h \in \mathbb{P}_1^3 : \ \boldsymbol{w}_h(\boldsymbol{x}) = \mathbf{a} \times \boldsymbol{x} + \mathbf{b}, \ \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \ \boldsymbol{x} \in K\right\}.$$

On the other hand, we use standard finite elements for the Lagrange multiplier ξ:

$$\mathcal{L}_h(\Omega_{\mathrm{D}}) := \left\{\varphi_h \in \mathrm{H}^1(\Omega_{\mathrm{D}}) : \varphi_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h^{\Omega_{\mathrm{D}}}\right\}.$$

We introduce the following discrete spaces:

$$\begin{aligned}
\boldsymbol{\mathcal{U}}_h &:= \{\boldsymbol{w}_h \in \boldsymbol{\mathcal{N}}_h(\Omega) : \boldsymbol{w}_h \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma_{\mathrm{C}} \text{ and } \mathbf{curl}\, \boldsymbol{w}_h \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\
\mathcal{Q}_h &:= \left\{\varphi_h \in \mathcal{L}_h(\Omega_{\mathrm{D}}) : \varphi_h|_{\Gamma_{\mathrm{I}}^1} = 0, \, \varphi_h|_{\Gamma_{\mathrm{I}}^k} = \text{constant}, \, k = 2, \dots, M\right\}.
\end{aligned}$$

To discretize Problem 3.2, we consider a convenient way to compute the right hand side for the discrete test functions. Let

$$\widetilde{L}_n(\boldsymbol{w}_h) := \int_{C_n} \boldsymbol{w}_h \cdot \boldsymbol{t} \qquad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h, \tag{3.46}$$

where C_n is a simple curve on $\partial\Omega$ joining $\Gamma_{\boldsymbol{E}}^n$ with $\Gamma_{\boldsymbol{J}}^n$, $n = 1, \dots, N$, and $\boldsymbol{t}$ being a unit vector tangent to C_n. It is easy to see that $L_n(\boldsymbol{w}_h) = \widetilde{L}_n(\boldsymbol{w}_h)$ for all $\boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h$ (cf. [18, Lemma 2.4]).

Then, the space discretization of Problem 3.2 reads as follows:

Problem 3.6 *Given $g \in \mathrm{L}^2(0, T; L^2(\Gamma_{\mathrm{D}}))$, $I_n \in \mathrm{H}^2(0, T)$, $n = 1, \dots, N$, and the initial condition $\boldsymbol{H}_0$ satisfying (3.13), find $\boldsymbol{u}_h : [0, T] \to \boldsymbol{\mathcal{U}}_h$ and $\xi_h : [0, T] \to \mathcal{Q}_h$ such that*

$$\begin{aligned}
&\int_{\Omega_{\mathrm{C}}} \sigma \partial_t \boldsymbol{u}_h(t) \cdot \boldsymbol{w}_h + \int_\Omega \frac{1}{\mu} \mathbf{curl}\, \boldsymbol{u}_h(t) \cdot \mathbf{curl}\, \boldsymbol{w}_h + \int_{\Omega_{\mathrm{D}}} \epsilon \boldsymbol{w}_h \cdot \mathbf{grad}\, \xi_h(t) \\
&\qquad\qquad = \sum_{n=1}^N \widetilde{L}_n(\boldsymbol{w}_h)(I_n(t) - I_n(0)) + \int_\Omega \mathbf{curl}\, \boldsymbol{H}_0 \cdot \boldsymbol{w}_h \qquad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h,
\end{aligned} \tag{3.47}$$

$$\int_{\Omega_{\mathrm{D}}} \epsilon \boldsymbol{u}_h(t) \cdot \mathbf{grad}\, \varphi_h = \int_{\Gamma_{\mathrm{D}}} \left(\int_0^t g(s)\, ds\right) \varphi_h \qquad \forall \varphi_h \in \mathcal{Q}_h, \tag{3.48}$$

$$\boldsymbol{u}_h(0) = \mathbf{0} \quad \textit{in } \Omega. \tag{3.49}$$

To prove that this problem is well-posed we will use the discrete kernel

$$\mathcal{K}_h := \left\{\boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h : \int_{\Omega_{\mathrm{D}}} \epsilon \boldsymbol{w}_h \cdot \mathbf{grad}\, \varphi_h = 0 \quad \forall \varphi_h \in \mathcal{Q}_h\right\}$$

and the following *inf-sup* condition.

Lemma 3.7 *There exists $\beta > 0$ (independent to h) such that*

$$\sup_{\substack{\boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h \\ \boldsymbol{w}_h \neq \boldsymbol{0}}} \frac{\int_{\Omega_\mathrm{D}} \epsilon \boldsymbol{w}_h \cdot \mathbf{grad}\, \varphi_h}{\|\boldsymbol{w}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \geq \beta \|\varphi_h\|_{\mathrm{H}^1(\Omega_\mathrm{D})^3} \qquad \forall \varphi_h \in \mathcal{Q}_h. \tag{3.50}$$

Proof. For $\varphi_h \in \mathcal{Q}_h$ let $\tilde{\varphi}_h$ be its extension to Ω_C defined by $\tilde{\varphi}_h|_{\Omega_\mathrm{C}^k} = \varphi_h|_{\Gamma_\mathrm{I}^k}$ (constant), $k = 1, \ldots, M$. Then, $\mathbf{grad}\, \tilde{\varphi}_h \in \boldsymbol{\mathcal{U}}_h$ and

$$\sup_{\substack{\boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h \\ \boldsymbol{w}_h \neq \boldsymbol{0}}} \frac{\int_{\Omega_\mathrm{D}} \epsilon \boldsymbol{w}_h \cdot \mathbf{grad}\, \varphi_h}{\|\boldsymbol{w}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \geq \frac{\int_{\Omega_\mathrm{D}} \epsilon\, \mathbf{grad}\, \tilde{\varphi}_h \cdot \mathbf{grad}\, \varphi_h}{\|\, \mathbf{grad}\, \tilde{\varphi}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \geq \frac{\underline{\epsilon} \|\, \mathbf{grad}\, \varphi_h\|^2_{\mathrm{L}^2(\Omega_\mathrm{D})^3}}{\|\, \mathbf{grad}\, \varphi_h\|_{\mathrm{L}^2(\Omega_\mathrm{D})^3}} \geq \beta \|\varphi_h\|_{\mathrm{H}^1(\Omega_\mathrm{D})^3},$$

where we have used Poincaré inequality since, for all $\varphi_h \in \mathcal{Q}_h$, $\varphi_h|_{\Gamma_\mathrm{I}^1} = 0$. ■

Next step is to prove that there exist a particular solution to equation (3.48).

Lemma 3.8 *Given $g \in \mathrm{L}^2(0, T; L^2(\Gamma_\mathrm{D}))$, there exists $\widehat{\boldsymbol{u}}_h \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{U}}_h)$ such that*

$$\int_{\Omega_\mathrm{D}} \epsilon \widehat{\boldsymbol{u}}_h(t) \cdot \mathbf{grad}\, \varphi_h = \int_{\Gamma_\mathrm{D}} \left(\int_0^t g(s)\, ds \right) \varphi_h \quad \forall \varphi_h \in \mathcal{Q}_h. \tag{3.51}$$

Proof. Consider the following auxiliary problem: for each $t \in [0, T]$, find $\widehat{\boldsymbol{v}}_h(t) \in \boldsymbol{\mathcal{K}}_h^{\perp \boldsymbol{\mathcal{U}}_h}$ such that

$$\int_{\Omega_\mathrm{D}} \epsilon \widehat{\boldsymbol{v}}_h(t) \cdot \mathbf{grad}\, \varphi_h = \int_{\Gamma_\mathrm{D}} g(t)\, \varphi_h \quad \forall \varphi_h \in \mathcal{Q}_h.$$

According to [38, Lemma I.4.1(*iii*)] because of the *inf-sup* condition (3.50), this problem has a unique solution and the following estimate holds true:

$$\|\widehat{\boldsymbol{v}}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C\, \|g(t)\|_{\mathrm{L}^2(\Gamma_\mathrm{D})}, \quad t \in [0, T].$$

Now, let $\widehat{\boldsymbol{u}}_h(t) := \int_0^t \widehat{\boldsymbol{v}}_h(s)\, ds$. From the above inequality it is immediate to show that $\widehat{\boldsymbol{u}}_h \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{U}}_h)$ and that it satisfies (3.51). ■

Now, if we write $\boldsymbol{u}_h = \widetilde{\boldsymbol{u}}_h + \widehat{\boldsymbol{u}}_h$, Problem 3.6 is equivalent to finding $\widetilde{\boldsymbol{u}}_h : [0, T] \to \boldsymbol{\mathcal{K}}_h$ such that

$$\begin{aligned} &\int_{\Omega_\mathrm{C}} \sigma \partial_t \widetilde{\boldsymbol{u}}_h(t) \cdot \boldsymbol{w}_h + \int_\Omega \frac{1}{\mu} \mathbf{curl}\, \widetilde{\boldsymbol{u}}_h(t) \cdot \mathbf{curl}\, \boldsymbol{w}_h \\ &= \sum_{n=1}^N \widetilde{L}_n(\boldsymbol{w}_h)(I_n(t) - I_n(0)) + \int_\Omega \mathbf{curl}\, \boldsymbol{H}_0 \cdot \boldsymbol{w}_h - \int_{\Omega_\mathrm{C}} \sigma \partial_t \widehat{\boldsymbol{u}}_h(t) \cdot \boldsymbol{w}_h - \int_\Omega \frac{1}{\mu} \mathbf{curl}\, \widehat{\boldsymbol{u}}_h(t) \cdot \mathbf{curl}\, \boldsymbol{w}_h \\ &\hspace{30em} \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{K}}_h, \end{aligned} \tag{3.52}$$

$$\widetilde{\boldsymbol{u}}_h(0) = \boldsymbol{0} \quad \text{in } \Omega. \tag{3.53}$$

In what follows we prove that this problem has a unique solution.

Lemma 3.9 *There exists a unique $\widetilde{\boldsymbol{u}}_h \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{K}}_h)$ solution of (3.52)–(3.53).*

Proof. Let $\{\boldsymbol{\Phi}_i\}_{i=1}^K$ be a basis of $\boldsymbol{\mathcal{K}}_h$ such that the last functions furnish a basis $\{\boldsymbol{\Phi}_i\}_{i=K_1+1}^K$ of the subspace $\{\boldsymbol{w}_h \in \boldsymbol{\mathcal{K}}_h \,:\, \boldsymbol{w}_h = \mathbf{0} \text{ in } \Omega_{\mathrm{D}}\}$. We write

$$\widetilde{\boldsymbol{u}}_h(t,\boldsymbol{x}) = \sum_{i=1}^K \alpha_i(t)\boldsymbol{\Phi}_i(\boldsymbol{x}).$$

Let $\boldsymbol{\alpha}(t) := (\alpha_i(t))_{1\le i\le K}$ and $\boldsymbol{b}(t) := (b_i(t))_{1\le i\le K}$, with

$$b_i(t) := \sum_{n=1}^N \widetilde{L}_n(\boldsymbol{\Phi}_i)(I_n(t)-I_n(0)) + \int_\Omega \mathbf{curl}\,\boldsymbol{H}_0\cdot\boldsymbol{\Phi}_i - \int_{\Omega_{\mathrm{C}}} \sigma\partial_t\widehat{\boldsymbol{u}}_h(t)\cdot\boldsymbol{\Phi}_i - \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\widehat{\boldsymbol{u}}_h(t)\cdot\mathbf{curl}\,\boldsymbol{\Phi}_i.$$

We consider $\boldsymbol{\mathcal{M}} := (M_{ij})_{1\le i,j\le K}$ and $\boldsymbol{\mathcal{K}} := (K_{ij})_{1\le i,j\le K}$ given by

$$M_{ij} := \int_{\Omega_{\mathrm{C}}} \sigma\boldsymbol{\Phi}_i\cdot\boldsymbol{\Phi}_j, \quad K_{ij} := \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{\Phi}_i\cdot\mathbf{curl}\,\boldsymbol{\Phi}_j, \qquad 1\le i,j\le K. \tag{3.54}$$

Then, (3.52)–(3.53) reads as follows: Find $\boldsymbol{\alpha}:[0,T]\to\mathbb{R}^K$ such that

$$\begin{aligned}&\boldsymbol{\mathcal{M}}\boldsymbol{\alpha}'(t) + \boldsymbol{\mathcal{K}}\boldsymbol{\alpha}(t) = \boldsymbol{b}(t),\\ &\boldsymbol{\alpha}(0) = \mathbf{0}.\end{aligned} \tag{3.55}$$

Because of the degenerate character of the problem, we decompose $\boldsymbol{\alpha}(t)$ as follows:

$$\boldsymbol{\alpha}(t) = \begin{bmatrix}\boldsymbol{\alpha}_1(t)\\ \boldsymbol{\alpha}_2(t)\end{bmatrix},$$

with $\boldsymbol{\alpha}_1(t) := (\alpha_i(t))_{1\le i\le K_1}$. We use a similar decomposition for $\boldsymbol{b}(t)$ and matrices $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{K}}$ to write

$$\boldsymbol{b}(t) = \begin{bmatrix}\boldsymbol{b}_1(t)\\ \boldsymbol{b}_2(t)\end{bmatrix}, \qquad \boldsymbol{\mathcal{M}} = \begin{bmatrix}\boldsymbol{\mathcal{M}}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \qquad \boldsymbol{\mathcal{K}} = \begin{bmatrix}\boldsymbol{\mathcal{K}}_{11} & \boldsymbol{\mathcal{K}}_{12}\\ \boldsymbol{\mathcal{K}}_{12}^T & \boldsymbol{\mathcal{K}}_{22}\end{bmatrix}. \tag{3.56}$$

Provided $\boldsymbol{\mathcal{K}}_{22}$ is invertible, (3.55) is equivalent to

$$\begin{aligned}&\boldsymbol{\mathcal{M}}_{11}\boldsymbol{\alpha}_1'(t) = \boldsymbol{b}_1(t) + [\boldsymbol{\mathcal{K}}_{12}\boldsymbol{\mathcal{K}}_{22}^{-1}\boldsymbol{\mathcal{K}}_{12}^T - \boldsymbol{\mathcal{K}}_{11}]\boldsymbol{\alpha}_1(t) - \boldsymbol{\mathcal{K}}_{12}\boldsymbol{\mathcal{K}}_{22}^{-1}\boldsymbol{b}_2(t),\\ &\boldsymbol{\alpha}_1(0) = \mathbf{0}.\end{aligned}$$

Therefore, in such a case, the existence and uniqueness of solution of (3.55) follows from the fact that $\boldsymbol{\mathcal{M}}_{11}$ is positive definite.

Thus, to conclude that (3.52)–(3.53) has a unique solution, we are going to check that $\boldsymbol{\mathcal{K}}_{22}$ is positive definite. First notice that

$$\boldsymbol{\beta}^T\boldsymbol{\mathcal{K}}_{22}\boldsymbol{\beta} = \int_{\Omega_{\mathrm{D}}} \frac{1}{\mu}\left|\mathbf{curl}\left(\sum_{i=K_1+1}^K \beta_i\boldsymbol{\Phi}_i\right)\right|^2 \ge 0.$$

Let us assume that the expression above vanishes. Then, $\boldsymbol{w}_h := \sum_{i=K_1+1}^{K} \beta_i \boldsymbol{\Phi}_i$ satisfies

$$\boldsymbol{w}_h \in \boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{D}}), \tag{3.57}$$

$$\mathbf{curl}\, \boldsymbol{w}_h = \mathbf{0} \quad \text{in } \Omega_{\mathrm{D}}, \tag{3.58}$$

$$\boldsymbol{w}_h \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{I}}, \tag{3.59}$$

$$\int_{\Omega_{\mathrm{D}}} \epsilon \boldsymbol{w}_h \cdot \mathbf{grad}\, \varphi_h = 0 \quad \forall \varphi_h \in \mathcal{Q}_h. \tag{3.60}$$

Since $\Omega_{\mathrm{D}} \setminus \Sigma$ is pseudo Lipschitz and simply connected, as a consequence of (3.57) and (3.58) there exists $\vartheta_h \in \mathcal{L}_h(\Omega_{\mathrm{D}} \setminus \Sigma)/\mathbb{R}$ such that $\boldsymbol{w}_h = \widetilde{\mathbf{grad}} \vartheta_h$, where

$$\mathcal{L}_h(\Omega_{\mathrm{D}} \setminus \Sigma) := \left\{ \varrho_h \in \mathcal{C}(\Omega_{\mathrm{D}} \setminus \Sigma) : \varrho_h|_K \in \mathbb{P}_1(K) \, \forall K \in \mathcal{T}_h^{\Omega_{\mathrm{D}}} \text{ with } [\![\varrho_h]\!]_{\Sigma_n} = \text{constant}, \ n = 1, \dots, N \right\},$$

with $[\![\cdot]\!]_{\Sigma_n}$ denoting the jump across Σ_n. From (3.59) we obtain $\mathbf{grad}_\tau \vartheta_h = \mathbf{0}$ on $\Gamma_{\mathrm{I}} \setminus \Sigma$. Thus, ϑ_h is constant on $\Gamma_{\mathrm{I}}^n \setminus \Sigma_n$, which implies that $[\![\vartheta_h]\!]_{\Sigma_n} = 0$, $n = 1, \dots, N$, and, whence, ϑ_h can be extended to a continuous function in Ω_{D}. By setting $\vartheta_h|_{\Gamma_{\mathrm{I}}^1} = 0$ we obtain $\vartheta_h \in \mathcal{Q}_h$ and, from (3.60), ϑ_h is a constant in Ω_{D} and then $\widehat{\boldsymbol{w}}_h = \mathbf{0}$ in Ω_{D}. Therefore, we conclude that $\boldsymbol{\mathcal{K}}_{22}$ is positive definite.

Thus, we have shown that (3.55) has a unique solution $\boldsymbol{\alpha} \in \mathrm{H}^1(0, T; \mathbb{R}^K)$ and, consequently, (3.52)–(3.53) also has a unique solution $\widetilde{\boldsymbol{u}}_h \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{K}}_h)$. ■

Finally, notice that for any solution to Problem 3.6, the Lagrange multiplier ξ_h necessarily vanishes, as in the continuous case.

Lemma 3.10 *If $(\boldsymbol{u}_h, \xi_h)$ is a solution to Problem 3.6, then $\xi_h \equiv 0$.*

Proof. Let $\tilde{\xi}_h$ be extension to Ω_{C} of ξ_h defined by $\tilde{\xi}_h|_{\Omega_{\mathrm{C}}^k} = \xi_h|_{\Gamma_{\mathrm{I}}^k}$, $k = 1, \dots, M$. Then, $\mathbf{grad}\, \tilde{\xi}_h \in \boldsymbol{\mathcal{U}}_h$ and $\widetilde{L}_n(\mathbf{grad}\, \tilde{\xi}_h) = 0$, $n = 1, \dots, N$. Furthermore, $\int_{\Omega_{\mathrm{C}}} \sigma \partial_t \boldsymbol{u}_h \cdot \mathbf{grad}\, \tilde{\xi}_h = 0$ and $\int_{\Omega} \mathbf{curl}\, \boldsymbol{H}_0 \cdot \mathbf{grad}\, \tilde{\xi}_h = 0$, because $\mathbf{grad}\, \tilde{\xi}_h$ vanishes in Ω_{C} and $\mathbf{curl}\, \boldsymbol{H}_0$ vanishes in Ω_{D}. Hence, taking $\boldsymbol{w}_h = \mathbf{grad}\, \tilde{\xi}_h$ in (3.47), we obtain that $\int_{\Omega_{\mathrm{D}}} \epsilon |\, \mathbf{grad}\, \xi_h|^2 = 0$. Therefore, ξ_h is a constant in Ω_{D} and, since $\xi_h|_{\Gamma_{\mathrm{I}}^1} = 0$, we have $\xi_h \equiv 0$. ■

Now, we are in a position to conclude the following result.

Theorem 3.11 *Problem 3.6 has a unique solution $(\boldsymbol{u}_h, \xi_h)$, with $\boldsymbol{u}_h \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{U}}_h)$ and $\xi_h = 0$.*

Proof. Let $\boldsymbol{u}_h := \widehat{\boldsymbol{u}}_h + \widetilde{\boldsymbol{u}}_h$, with $\widehat{\boldsymbol{u}}_h$ and $\widetilde{\boldsymbol{u}}_h$ being respective solutions of (3.51) and (3.52)–(3.53), then, for any $\xi_h : [0, T] \to \mathcal{Q}_h$, $(\boldsymbol{u}_h, \xi_h)$ satisfies (3.47)–(3.49), the first equation only for $\boldsymbol{w}_h \in \boldsymbol{\mathcal{K}}_h$. Hence, because of [38, Lemma I.4.1(*ii*)] and the *inf-sup* condition (3.50), for each $t \in [0, T]$ there exists a unique $\xi_h(t) \in \mathcal{Q}_h$ such that (3.47) holds for all $\boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h$. Moreover, according to Lemma 3.10, $\xi_h = 0$. Finally, the uniqueness follows from the fact that $(\boldsymbol{u}_h, \xi_h)$ is a solution to Problem 3.6 if and only if $\widetilde{\boldsymbol{u}}_h = \boldsymbol{u} - \widehat{\boldsymbol{u}}_h$ is the unique solution to (3.52)–(3.53) (cf. Lemma 3.9) and $\xi_h = 0$. ■

Our next goal is to obtain error estimates for this semi-discrete scheme. With this aim, from now on, we assume that the solution to Problem 3.2 satisfies $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ for $r \in \left(\frac{1}{2},1\right]$, where $\mathrm{H}^r(\mathbf{curl};\Omega) := \left\{\boldsymbol{G} \in \mathrm{H}^r(\Omega)^3 : \mathbf{curl}\,\boldsymbol{G} \in \mathrm{H}^r(\Omega)^3\right\}$. Let $\mathcal{I}_h^{\mathcal{N}}$ denote the Nédélec interpolant operator. According to [18, Lemma 2.2], we have that if $\boldsymbol{w} \in \mathrm{H}^r(\mathbf{curl};\Omega) \cap \boldsymbol{\mathcal{U}}$, then $\mathcal{I}_h^{\mathcal{N}}\boldsymbol{w} \in \boldsymbol{\mathcal{U}}_h$. We decompose the error of $\boldsymbol{u}$ as follows

$$\boldsymbol{u}(t) - \boldsymbol{u}_h(t) = \boldsymbol{\rho}_h(t) - \boldsymbol{\delta}_h(t), \tag{3.61}$$

with

$$\boldsymbol{\rho}_h(t) := \boldsymbol{u}(t) - \mathcal{I}_h^{\mathcal{N}}\boldsymbol{u}(t) \qquad \text{and} \qquad \boldsymbol{\delta}_h(t) := \mathcal{I}_h^{\mathcal{N}}\boldsymbol{u}(t) - \boldsymbol{u}_h(t).$$

First, we prove the following auxiliary error estimate.

Lemma 3.12 *Let $\boldsymbol{u}$ be the solution to Problem 3.2 and $\boldsymbol{u}_h$ that to Problem 3.6. If $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ with $r \in \left(\frac{1}{2},1\right]$, then there exists a constant $C > 0$, independent of h, such that*

$$\begin{aligned}\sup_{0\le t\le T}\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \sup_{0\le t\le T}\|\,\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^T \|\partial_t\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\,dt \\ \le C\left\{\sup_{0\le t\le T}\|\,\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^T \|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,dt\right\}.\end{aligned}$$

Proof. Since $\xi = 0$ and $\xi_h = 0$ (cf. Theorem 3.4 and 3.11), subtracting the first equation in Problem 3.6 from that in Problem 3.2, we have

$$\int_{\Omega_{\mathrm{C}}} \sigma\partial_t(\boldsymbol{u}(t) - \boldsymbol{u}_h(t))\cdot\boldsymbol{w}_h + \int_\Omega \frac{1}{\mu}\mathbf{curl}(\boldsymbol{u}(t) - \boldsymbol{u}_h(t))\cdot\mathbf{curl}\,\boldsymbol{w}_h = 0 \quad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h.$$

A straightforward computation yields

$$\begin{aligned}\int_{\Omega_{\mathrm{C}}} \sigma\partial_t\boldsymbol{\delta}_h(t)\cdot\boldsymbol{w}_h + \int_\Omega \frac{1}{\mu}\mathbf{curl}\,\boldsymbol{\delta}_h(t)\cdot\mathbf{curl}\,\boldsymbol{w}_h \\ = -\int_{\Omega_{\mathrm{C}}} \sigma\partial_t\boldsymbol{\rho}_h(t)\cdot\boldsymbol{w}_h - \int_\Omega \frac{1}{\mu}\mathbf{curl}\,\boldsymbol{\rho}_h(t)\cdot\mathbf{curl}\,\boldsymbol{w}_h \qquad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h.\end{aligned} \tag{3.62}$$

By taking $\boldsymbol{w}_h = \boldsymbol{\delta}_h(t)$, the hypotheses over σ and μ and Young's inequality lead to

$$\frac{d}{dt}\int_{\Omega_{\mathrm{C}}} \sigma|\boldsymbol{\delta}_h(t)|^2 + \frac{1}{\overline{\mu}}\|\,\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \le \int_{\Omega_{\mathrm{C}}} \sigma|\boldsymbol{\delta}_h(t)|^2 + \overline{\sigma}\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \frac{\overline{\mu}}{\underline{\mu}^2}\|\,\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}. \tag{3.63}$$

Using Gronwall's inequality together with the fact that $\boldsymbol{\delta}_h(0) = \mathbf{0}$ and integrating in time lead to

$$\begin{aligned}\|\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \int_0^t \|\,\mathbf{curl}\,\boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds \\ \le C\left\{\int_0^T \|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\,dt + \int_0^T \|\,\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\,dt\right\}.\end{aligned} \tag{3.64}$$

On the other hand, the assumed regularity of $\boldsymbol{u}$ implies that $\partial_t\left(\mathcal{I}_h^{\mathcal{N}}\boldsymbol{u}(t)\right)=\mathcal{I}_h^{\mathcal{N}}\left(\partial_t\boldsymbol{u}(t)\right)$ a.e. $t\in[0,T]$ (see Theorems 111 and 113 from [71]). Thus, by taking $\boldsymbol{w}_h=\partial_t\boldsymbol{\delta}_h(t)$ in (3.62), from the hypotheses over σ and μ, straightforward computations lead to

$$\begin{aligned}
&\frac{\underline{\sigma}}{2}\|\partial_t\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}+\frac{1}{2\overline{\mu}}\frac{d}{dt}\|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\\
&\qquad\le\frac{d}{dt}\int_\Omega\frac{1}{\mu}\operatorname{\mathbf{curl}}\boldsymbol{\rho}_h(t)\cdot\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)+\frac{1}{2\underline{\mu}}\|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}+C\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}. \qquad (3.65)
\end{aligned}$$

Integrating in time, using again the fact that $\boldsymbol{\delta}_h(0)=\mathbf{0}$ together with Young's inequality we obtain

$$\begin{aligned}
&\frac{\underline{\sigma}}{2}\int_0^t\|\partial_t\boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\,ds+\frac{1}{4\overline{\mu}}\|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\\
&\qquad\le\frac{\overline{\mu}}{\underline{\mu}^2}\|\operatorname{\mathbf{curl}}\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}+\frac{1}{2\underline{\mu}}\int_0^t\|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega)^3}\,ds+C\int_0^t\|\partial_t\boldsymbol{\rho}_h(s)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,ds. \qquad (3.66)
\end{aligned}$$

Using Gronwall's inequality and the fact that $\boldsymbol{\delta}_h(0)=\mathbf{0}$ we have that

$$\|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\le C\left\{\sup_{0\le t\le T}\|\operatorname{\mathbf{curl}}\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}+\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,dt\right\}.$$

Substituting this term in (3.66), we obtain

$$\begin{aligned}
&\int_0^t\|\partial_t\boldsymbol{\delta}_h(s)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\,ds+\|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}\\
&\qquad\qquad\le C\left\{\sup_{0\le t\le T}\|\operatorname{\mathbf{curl}}\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}+\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{H}(\mathbf{curl};\Omega)}\,dt\right\}.
\end{aligned}$$

Thus, the result follows from (3.64) and the above inequality. ■

Now, we are in a position to prove the following error estimates.

Theorem 3.13 *Let* $\boldsymbol{u}$ *be the solution to Problem 3.2 and* $\boldsymbol{u}_h$ *that to Problem 3.6. If* $\boldsymbol{u}\in\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ *with* $r\in\left(\frac{1}{2},1\right]$*, then there exists a constant* $C>0$*, independent of* h*, such that*

$$\begin{aligned}
&\sup_{0\le t\le T}\|\boldsymbol{u}(t)-\boldsymbol{u}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}+\sup_{0\le t\le T}\|\operatorname{\mathbf{curl}}\boldsymbol{u}(t)-\operatorname{\mathbf{curl}}\boldsymbol{u}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3}+\int_0^T\|\partial_t(\boldsymbol{u}(t)-\boldsymbol{u}_h(t))\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\,dt\\
&\qquad\le C\,h^{2r}\left\{\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)}+\int_0^T\|\partial_t\boldsymbol{u}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)}\,dt\right\}\\
&\qquad\le C\,h^{2r}\|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}.
\end{aligned}$$

Proof. Classical estimates for the Nédélec interpolant lead to

$$\|\boldsymbol{\rho}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)}\le C\,h^r\|\boldsymbol{u}(t)\|_{\mathrm{H}^r(\mathbf{curl};\Omega)},\quad\|\partial_t\boldsymbol{\rho}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)}\le C\,h^r\|\partial_t\boldsymbol{u}(t)\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}. \qquad (3.67)$$

Thus, the result follows from the decomposition (3.61) by using these estimates and the previous lemma. ■

Remark 3.14 *This theorem allows us to obtain error estimates for the physical variables of interest in most applications,* $\boldsymbol{E}|_{\Omega_{\mathrm{C}}}$ *and* $\boldsymbol{H}$*. For the first one, we define* $\boldsymbol{E}_h(t,\boldsymbol{x}) := \partial_t \boldsymbol{u}_h(t,\boldsymbol{x})$ *and we have the following error estimate:*

$$\int_0^T \|\boldsymbol{E}(t) - \boldsymbol{E}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\, dt \leq C\, h^{2r} \|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}.$$

To approximate $\boldsymbol{H}$*, we make use of (3.14) and define* $\boldsymbol{H}_h(t,\boldsymbol{x}) := \boldsymbol{H}_0(\boldsymbol{x}) - \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}_h(t,\boldsymbol{x})$*. Then, we have*

$$\sup_{0\leq t\leq T} \|\boldsymbol{H}(t) - \boldsymbol{H}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} \leq C\, h^{2r} \|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}.$$

Remark 3.15 *The assumption* $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ *does not seem realistic when the magnetic permeability of conductor and dielectric are not the same (cf. (3.15)). However, Theorem 3.13 holds true if this assumption is substituted by*

$$\boldsymbol{u}|_{\Omega_{\mathrm{C}}} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega_{\mathrm{C}})) \quad \textit{and} \quad \boldsymbol{u}|_{\Omega_{\mathrm{D}}} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega_{\mathrm{D}})).$$

3.3.2 Time discretization

We consider a uniform partition of $[0,T]$, $t_k := k\Delta t$, $k = 0,\ldots,M$, with time step $\Delta t := \frac{T}{M}$. A fully discrete approximation of Problem 3.2 by means of a backward Euler scheme reads as follows:

Problem 3.16 *Find* $\boldsymbol{u}_h^m \in \boldsymbol{\mathcal{U}}_h$ *and* $\xi_h^m \in \mathcal{Q}_h$*,* $m = 1,\ldots,M$*, such that*

$$\int_{\Omega_{\mathrm{C}}} \sigma \frac{\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}}{\Delta t} \cdot \boldsymbol{w}_h + \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}_h^m \cdot \mathbf{curl}\,\boldsymbol{w}_h + \int_{\Omega_{\mathrm{D}}} \epsilon \boldsymbol{w}_h \cdot \mathbf{grad}\,\xi_h^m$$
$$= \sum_{n=1}^N \widetilde{L}_n(\boldsymbol{w}_h)(I_n(t_m) - I_n(0)) + \int_\Omega \mathbf{curl}\,\boldsymbol{H}_0 \cdot \boldsymbol{w}_h \quad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h,$$
$$\int_{\Omega_{\mathrm{D}}} \epsilon \boldsymbol{u}_h^m \cdot \mathbf{grad}\,\varphi_h = \int_{\Gamma_{\mathrm{D}}} \left(\int_0^{t_m} g(s)\, ds\right) \varphi_h \quad \forall \varphi_h \in \mathcal{Q}_h,$$
$$\boldsymbol{u}_h^0 = \mathbf{0} \quad \textit{in } \Omega.$$

We proceed as for the semi-discrete scheme. First, the same arguments allows us to show that any solution of Problem 3.16 satisfies $\xi_h^m = 0$, $m = 1,\ldots,M$. Secondly, let $\widehat{\boldsymbol{u}}_h \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{U}}_h)$ be as above so that it satisfies (3.51). Let $\widehat{\boldsymbol{u}}_h^m := \widehat{\boldsymbol{u}}_h(t^m)$ and $\boldsymbol{u}_h^m = \widetilde{\boldsymbol{u}}_h^m + \widehat{\boldsymbol{u}}_h^m$. Then, it is clear that Problem 3.16 has a unique solution if only if there exist unique $\widetilde{\boldsymbol{u}}_h^m \in \boldsymbol{\mathcal{K}}_h$, $m = 1,\ldots,M$, such that

$$\int_{\Omega_{\mathrm{C}}} \sigma \widetilde{\boldsymbol{u}}_h^m \cdot \boldsymbol{w}_h + \Delta t \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\widetilde{\boldsymbol{u}}_h^m \cdot \mathbf{curl}\,\boldsymbol{w}_h = \int_{\Omega_{\mathrm{C}}} \sigma \widetilde{\boldsymbol{u}}_h^{m-1} \cdot \boldsymbol{w}_h - \int_{\Omega_{\mathrm{C}}} \sigma(\widehat{\boldsymbol{u}}_h^m - \widehat{\boldsymbol{u}}_h^{m-1}) \cdot \boldsymbol{w}_h$$
$$-\Delta t \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\widehat{\boldsymbol{u}}_h^m \cdot \mathbf{curl}\,\boldsymbol{w}_h + \Delta t \sum_{n=1}^N \widetilde{L}_n(\boldsymbol{w}_h)(I_n(t_m) - I_n(0)) + \Delta t \int_\Omega \mathbf{curl}\,\boldsymbol{H}_0 \cdot \boldsymbol{w}_h \qquad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{K}}_h,$$

with $\widetilde{\boldsymbol{u}}_h^0 = \mathbf{0}$.

To prove that this problem has a unique solution, we proceed as in the proof of Lemma 3.9. We write $\widetilde{\boldsymbol{u}}_h^m$ in the basis $\{\boldsymbol{\Phi}_i\}_{i=1}^K$ of $\boldsymbol{\mathcal{K}}_h$, $\widetilde{\boldsymbol{u}}_h^m = \sum_{i=1}^K \alpha_i^m \boldsymbol{\Phi}_i$, and obtain the following matrix form of the problem above:

$$\widetilde{\boldsymbol{\mathcal{M}}}\boldsymbol{\alpha}^m = \boldsymbol{\mathcal{M}}\boldsymbol{\alpha}^{m-1} + \Delta t \boldsymbol{b}^m,$$

with $\boldsymbol{b}^m \in \mathbb{R}^K$ beging the vector arising from the right hand side of the problem, $\boldsymbol{\mathcal{M}}$ as in (3.54) and

$$\widetilde{\boldsymbol{\mathcal{M}}} := \boldsymbol{\mathcal{M}} + \Delta t\, \boldsymbol{\mathcal{K}} = \begin{bmatrix} \boldsymbol{\mathcal{M}}_{11} + \Delta t\, \boldsymbol{\mathcal{K}}_{11} & \Delta t\, \boldsymbol{\mathcal{K}}_{12} \\ \Delta t\, \boldsymbol{\mathcal{K}}_{12}^T & \Delta t\, \boldsymbol{\mathcal{K}}_{22} \end{bmatrix},$$

where we have used the block matrices from (3.56). Since $\boldsymbol{\mathcal{K}}$ is semi-positive definite and $\boldsymbol{\mathcal{M}}_{11}$ and $\boldsymbol{\mathcal{K}}_{22}$ are positive definite, it is easy to check that $\widetilde{\boldsymbol{\mathcal{M}}}$ is also positive definite. Thus, we conclude that Problem 3.16 has a unique solution.

Our next goal is to obtain error estimates for this fully discrete scheme. With this aim, we write

$$\partial_t \boldsymbol{u}(t_k) - \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t} = \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} + \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} - \boldsymbol{\tau}^k, \tag{3.68}$$

where

$$\boldsymbol{\rho}_h^k := \boldsymbol{u}(t_k) - \mathcal{I}_h^{\mathcal{N}} \boldsymbol{u}(t_k), \qquad \boldsymbol{\delta}_h^k := \mathcal{I}_h^{\mathcal{N}} \boldsymbol{u}(t_k) - \boldsymbol{u}_h^k \quad \text{and} \qquad \boldsymbol{\tau}^k := \frac{\boldsymbol{u}(t_k) - \boldsymbol{u}(t_{k-1})}{\Delta t} - \partial_t \boldsymbol{u}(t_k).$$

Lemma 3.17 *Let $\boldsymbol{u}$ be the solution to Problem 3.2 and $\boldsymbol{u}_h^k$, $k = 1, \dots, M$, that to Problem 3.16. If $\boldsymbol{u} \in \mathrm{H}^1(0, T; \mathrm{H}^r(\mathbf{curl}; \Omega))$ with $r \in \left(\frac{1}{2}, 1\right]$, then there exists a constant $C > 0$, independent of h and Δt, such that*

$$\max_{1\le k\le M} \|\boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \max_{1\le k\le M} \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^M \left\| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}$$
$$\le C \left(\max_{1\le k\le M} \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^M \left\{ \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \right\} \right).$$

Proof. Proceeding as in the proof of Lemma 3.12 and using (3.68), a straightforward computation allows us to show that

$$\int_{\Omega_{\mathrm{C}}} \sigma \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \cdot \boldsymbol{w}_h + \int_\Omega \frac{1}{\mu}\, \mathbf{curl}\, \boldsymbol{\delta}_h^k \cdot \mathbf{curl}\, \boldsymbol{w}_h$$
$$= \int_{\Omega_{\mathrm{C}}} \sigma \boldsymbol{\tau}^k \cdot \boldsymbol{w}_h - \int_{\Omega_{\mathrm{C}}} \sigma \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \cdot \boldsymbol{w}_h - \int_\Omega \frac{1}{\mu}\, \mathbf{curl}\, \boldsymbol{\rho}_h^k \cdot \mathbf{curl}\, \boldsymbol{w}_h \quad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h. \tag{3.69}$$

By taking $\boldsymbol{w}_h = \boldsymbol{\delta}_h^k$, using the inequality $2(p-q)p \ge p^2 - q^2$ and Young's inequality, yield

$$\int_{\Omega_{\mathrm{C}}} \sigma |\boldsymbol{\delta}_h^k|^2 - \int_{\Omega_{\mathrm{C}}} \sigma |\boldsymbol{\delta}_h^{k-1}|^2 + \frac{\Delta t}{\overline{\mu}} \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega)^3}$$
$$\le \frac{\Delta t}{2T} \int_{\Omega_{\mathrm{C}}} \sigma |\boldsymbol{\delta}_h^k|^2 + \frac{\overline{\mu}\Delta t}{\underline{\mu}^2} \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} + 4\Delta t\, T \int_{\Omega_{\mathrm{C}}} \sigma |\boldsymbol{\tau}^k|^2 + 4\Delta t\, T \int_{\Omega_{\mathrm{C}}} \sigma \left| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right|^2.$$

Summing from $k=1$ to m ($m \leq M$) and using the discrete Gronwall's inequality and the fact that $\boldsymbol{\delta}_h^0 = \mathbf{0}$, we obtain

$$\|\boldsymbol{\delta}_h^m\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \Delta t \sum_{k=1}^{m} \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega)^3}$$
$$\leq C\,\Delta t \sum_{k=1}^{m} \left\{ \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \right\}. \quad (3.70)$$

On the other hand, by taking $\boldsymbol{w}_h = \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t}$ in (3.69), similar arguments and the equation

$$\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{\rho}_h^k \cdot \mathbf{curl}\left(\frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t}\right)$$
$$= -\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\left(\frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t}\right) \cdot \mathbf{curl}\,\boldsymbol{\delta}_h^{k-1} + \frac{1}{\Delta t}\left\{\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{\rho}_h^k \cdot \mathbf{curl}\,\boldsymbol{\delta}_h^k - \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{\rho}_h^{k-1} \cdot \mathbf{curl}\,\boldsymbol{\delta}_h^{k-1}\right\}$$

lead to

$$\Delta t \int_{\Omega_{\mathrm{C}}} \sigma \left| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right|^2 + \int_\Omega \frac{1}{\mu} |\,\mathbf{curl}\,\boldsymbol{\delta}_h^k|^2 - \int_\Omega \frac{1}{\mu} |\,\mathbf{curl}\,\boldsymbol{\delta}_h^{k-1}|^2$$
$$\leq -2\left\{\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{\rho}_h^k \cdot \mathbf{curl}\,\boldsymbol{\delta}_h^k - \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{\rho}_h^{k-1} \cdot \mathbf{curl}\,\boldsymbol{\delta}_h^{k-1}\right\}$$
$$+ \Delta t\, C \left\{ \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \left\| \mathbf{curl}\left(\frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t}\right) \right\|^2_{\mathrm{L}^2(\Omega)^3} + \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3} \right\}.$$

Summing from $k=1$ to m ($m \leq M$) and using to (3.70) to estimate $\Delta t \sum_{k=1}^{m} \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3}$, we obtain

$$\Delta t \sum_{k=1}^{m} \left\| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \|\,\mathbf{curl}\,\boldsymbol{\delta}_h^m\|^2_{\mathrm{L}^2(\Omega)^3}$$
$$\leq C \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^m\|^2_{\mathrm{L}^2(\Omega)^3} + C\,\Delta t \sum_{k=1}^{m} \left\{ \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{\rho}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} \right\}.$$

Finally, the result follows from the above inequality and (3.70). ■

Now, we are in a position to write one of the main results of this paper.

Theorem 3.18 *Let $\boldsymbol{u}$ be the solution to Problem 3.2 and $\boldsymbol{u}_h^k$, $k = 1, \ldots, M$, that to Problem 3.16. If $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ for $r \in \left(\frac{1}{2}, 1\right]$, and $\boldsymbol{u}|_{\Omega_{\mathrm{C}}} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{C}})^3)$, then there exists a constant*

$C > 0$, *independent of* h *and* Δt, *such that*

$$\max_{1\le k\le M}\|\boldsymbol{u}(t_k)-\boldsymbol{u}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}+\max_{1\le k\le M}\|\,\mathbf{curl}(\boldsymbol{u}(t_k)-\boldsymbol{u}_h^k)\|^2_{\mathrm{L}^2(\Omega)^3}+\Delta t\sum_{k=1}^{M}\left\|\partial_t\boldsymbol{u}(t_k)-\frac{\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}$$
$$\le C\left\{(\Delta t)^2\int_0^T\|\partial_{tt}\boldsymbol{u}(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\,dt+h^{2r}\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)}+h^{2r}\int_0^T\|\partial_t\boldsymbol{u}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)}\,dt\right\}$$
$$\le C\left\{(\Delta t)^2\|\boldsymbol{u}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{C}})^3)}+h^{2r}\|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}\right\}.$$

Proof. A Taylor expansion shows that

$$\sum_{k=1}^{M}\|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}=\sum_{k=1}^{M}\left\|\frac{1}{\Delta t}\int_{t_{k-1}}^{t_k}(t_k-s)\partial_{tt}\boldsymbol{u}(s)\,ds\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\le\Delta t\int_0^T\|\partial_{tt}\boldsymbol{u}(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\,dt.$$

Moreover,

$$\sum_{k=1}^{M}\left\|\frac{\boldsymbol{\rho}_h^k-\boldsymbol{\rho}_h^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\le\frac{1}{\Delta t}\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\,dt.$$

Since $\boldsymbol{u}(t_k)-\boldsymbol{u}_h^k=\boldsymbol{\delta}_h^k+\boldsymbol{\rho}_h^k$, the result follows from (3.67), (3.68) and the previous lemma. ∎

Remark 3.19 *As in the semi-discrete scheme, if we approximate the electric field* $\boldsymbol{E}$ *and the magnetic field* $\boldsymbol{H}$ *at each time* t_k, $k=1,\dots,M$, *by taking* $\boldsymbol{E}_h^k:=\frac{\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1}}{\Delta t}$ *and* $\boldsymbol{H}_h^k:=\boldsymbol{H}_0-\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}_h^k$, *respectively, then*

$$\Delta t\sum_{k=1}^{M}\left\|\boldsymbol{E}(t_k)-\boldsymbol{E}_h^k\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\le C\left\{(\Delta t)^2\|\boldsymbol{u}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{C}})^3)}+h^{2r}\|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}\right\},$$
$$\max_{1\le k\le M}\|\boldsymbol{H}(t_k)-\boldsymbol{H}_h^k)\|^2_{\mathrm{L}^2(\Omega)^3}\le C\left\{(\Delta t)^2\|\boldsymbol{u}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{C}})^3)}+h^{2r}\|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}\right\}.$$

Remark 3.20 *The same observation made in Remark 3.15 holds in this case.*

3.4 Eddy current problem with voltage drops as boundary data

The goal of this section is to analyze the transient eddy current problem with voltage drops as boundary data. We consider equations (3.1)–(3.7) together with (3.9), for $n=1,\dots,N$, and (3.10)–(3.12). Notice that the only difference with respect the problem studied in the previous section is that (3.9) replaces (3.8). As in the previous section, we assume that the initial data $\boldsymbol{H}_0$ satisfies (3.13) and that $g\in\mathrm{L}^2(0,T;L^2(\Gamma_{\mathrm{D}}))$. Furthermore, we assume that $V_n\in\mathrm{H}^1(0,T)$, $n=1,\dots,N$.

Let $\boldsymbol{u}(t):=\int_0^t\boldsymbol{E}(s)\,ds$ as above. Integrating in time (3.9), we have

$$\boldsymbol{n}\times\boldsymbol{u}(t)\times\boldsymbol{n}=-\int_0^t\mathbf{grad}_\tau V(s)\,ds=-\,\mathbf{grad}_\tau\left(\int_0^t V(s)\,ds\right)\qquad\text{on }\partial\Omega.$$

Thus, according to (3.23), we have that $L_n(\boldsymbol{u}(t)) = \int_0^t V_n(s)\,ds$, $n = 1,\dots,N$, $t \in [0,T]$.

Therefore, the transient eddy current problem with voltage drops as boundary data written in terms of $\boldsymbol{u}$ is given by equations (3.15)–(3.20) and

$$L_n(\boldsymbol{u}(t)) = \int_0^t V_n(s)\,ds, \quad n = 1,\dots,N, \quad t \in [0,T]$$

(the latter instead of (3.21)), with the initial condition (3.22).

Similar arguments to those used in Section 3.3 allow us to obtain the following problem:

Problem 3.21 *Given $g \in \mathrm{L}^2(0,T;L^2(\Gamma_{\mathrm{D}}))$, $V_n \in \mathrm{H}^1(0,T)$, $n = 1,\dots,N$, and an initial condition $\boldsymbol{H}_0$ satisfying (3.13), find $\boldsymbol{u} \in \mathrm{L}^2(0,T;\boldsymbol{\mathcal{U}})$ with $\boldsymbol{u}|_{\Omega_{\mathrm{C}}} \in \mathrm{H}^1(0,T;\mathrm{H}_{\Gamma_{\mathrm{C}}}(\mathbf{curl};\Omega_{\mathrm{C}}))$ and $\xi \in \mathrm{L}^2(0,T;\boldsymbol{\mathcal{M}}(\Omega_{\mathrm{D}}))$ such that*

$$L_n(\boldsymbol{u}(t)) = \int_0^t V_n(s)\,ds, \qquad n = 1,\dots,N, \quad a.e.\ t \in [0,T], \tag{3.71}$$

$$\int_{\Omega_{\mathrm{C}}} \sigma\partial_t\boldsymbol{u}(t)\cdot\boldsymbol{w} + \int_\Omega \frac{1}{\mu}\mathbf{curl}\,\boldsymbol{u}(t)\cdot\mathbf{curl}\,\boldsymbol{w} + \int_{\Omega_{\mathrm{D}}} \epsilon\boldsymbol{w}\cdot\mathbf{grad}\,\xi(t) = \int_\Omega \mathbf{curl}\,\boldsymbol{H}_0\cdot\boldsymbol{w} \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{U}}_0, \tag{3.72}$$

$$\int_{\Omega_{\mathrm{D}}} \epsilon\boldsymbol{u}(t)\cdot\mathbf{grad}\,\varphi = \int_{\Gamma_{\mathrm{D}}} \left(\int_0^t g(s)\,ds\right)\varphi \qquad \forall\varphi \in \boldsymbol{\mathcal{M}}(\Omega_{\mathrm{D}}), \tag{3.73}$$

$$\boldsymbol{u}(0) = \mathbf{0} \quad in\ \Omega, \tag{3.74}$$

where $\boldsymbol{\mathcal{U}}^0 = \{\boldsymbol{w} \in \boldsymbol{\mathcal{U}} : L_n(\boldsymbol{w}) = 0,\ n = 1,\dots,N\}$.

A formulation of the same problem in terms of the magnetic field $\boldsymbol{H}$ was analyzed in [14] (cf. Chapter 2, this thesis). In particular, it was shown in this reference that equations (3.1)–(3.7) with voltage drops $V_n(t)$, $n = 1,\dots,N$, as boundary data lead to a well-posed problem (cf. [14, Remark 3.7] or Remark 2.7 – this thesis) which consists of finding $\boldsymbol{H} \in \mathrm{L}^2(0,T;\boldsymbol{\mathcal{X}}) \cap \mathrm{H}^1(0,T;\boldsymbol{\mathcal{H}}_{\boldsymbol{\mathcal{X}}})$ such that

$$\int_\Omega \mu\partial_t\boldsymbol{H}(t)\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\mathbf{curl}\,\boldsymbol{H}(t)\cdot\mathbf{curl}\,\boldsymbol{G} = -\sum_{n=1}^N V_n(t)\langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},1\rangle_{\Gamma_J^n} \qquad \forall\boldsymbol{G} \in \boldsymbol{\mathcal{X}}, \tag{3.75}$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0. \tag{3.76}$$

Defining $\boldsymbol{E}_{\mathrm{C}}(t) := \frac{1}{\sigma}\mathbf{curl}\,\boldsymbol{H}(t)$ in Ω_{C}, the arguments from [14, Theorem 3.8] can be repeated to prove that $\boldsymbol{H}(t)$ and $\boldsymbol{E}_{\mathrm{C}}(t)$ satisfy (3.29)–(3.33), a.e. $t \in (0,T)$.

Theorem 3.22 *Problem 3.21 has a unique solution $(\boldsymbol{u},\xi)$ and the Lagrange multiplier ξ vanishes.*

Proof. The existence of solution follows by repeating the arguments of the proof of Theorem 3.4. In fact, now we begin with the solution $\boldsymbol{H}$ of (3.75)–(3.76) (instead of that of (3.26)–(3.28)). Repeating the steps of the proof of Theorem 3.4, we define $\boldsymbol{E}_{\mathrm{C}}(t) := \frac{1}{\sigma}\mathbf{curl}\,\boldsymbol{H}(t)$ in Ω_{C} and show that $\boldsymbol{H}(t)$ and $\boldsymbol{E}_{\mathrm{C}}(t)$ satisfy (3.29)–(3.33) a.e. $t \in (0,T)$. Next, we define $\boldsymbol{E}_{\mathrm{D}}(t)$, $t \in [0,T]$, as the solution of (3.35)–(3.39), $\boldsymbol{E}(t)$ as in (3.40) and $\boldsymbol{u}$ as in (3.41). Proceeding as in the proof of Theorem 3.4 and using the fact that $L_n(\boldsymbol{w}) = 0$ for $\boldsymbol{w} \in \boldsymbol{\mathcal{U}}^0$, we prove that $(\boldsymbol{u}, 0)$ satisfies (3.72)–(3.74). Thus, to conclude the existence of solution, there only remains to prove that $\boldsymbol{u}$ satisfies (3.71).

To prove this, note that as a consequence of (3.30), (3.35), (3.36) and (3.32), there exists a function $\widetilde{V}$ defined in Ω up to a constant, such that $\widetilde{V}|_{\partial\Omega}$ is a surface potential of the tangential component of $\boldsymbol{E}$; namely, $\boldsymbol{n} \times \boldsymbol{E} \times \boldsymbol{n} = -\,\mathbf{grad}_\tau \widetilde{V}$ on $\partial\Omega$. On the other hand, (3.33) implies that $\widetilde{V}$ is constant on each connected component of $\Gamma_{\boldsymbol{J}}$ and $\Gamma_{\boldsymbol{E}}$.

From (3.75), using successively, the definition of $\boldsymbol{E}$, a Green's formula, (3.32), (3.2) and Lemma 3.1, we have

$$\begin{aligned}
-\sum_{n=1}^{N} V_n(t)\langle \mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} &= \int_\Omega \mu\partial_t\boldsymbol{H}(t)\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}(t)\cdot\mathbf{curl}\,\boldsymbol{G} \\
&= \int_\Omega \mu\partial_t\boldsymbol{H}(t)\cdot\boldsymbol{G} + \int_\Omega \boldsymbol{E}(t)\cdot\mathbf{curl}\,\boldsymbol{G} \\
&= \langle \boldsymbol{E}(t)\times\boldsymbol{n}, \boldsymbol{G}\rangle = -\langle \mathbf{grad}_\tau \widetilde{V}(t)\times\boldsymbol{n}, \boldsymbol{G}\rangle \\
&= -\sum_{n=1}^{N}\left(\widetilde{V}(t)|_{\Gamma_{\boldsymbol{J}}^n} - \widetilde{V}(t)|_{\Gamma_{\boldsymbol{E}}^n}\right)\langle \mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} \qquad \forall \boldsymbol{G}\in\boldsymbol{\mathcal{X}}.
\end{aligned}$$

Next, we take as test function $\boldsymbol{G}^m \in \boldsymbol{\mathcal{X}}$ satisfying $\langle \mathbf{curl}\,\boldsymbol{G}^m\cdot\boldsymbol{n}, 1\rangle_{\Gamma_{\boldsymbol{J}}^n} = \delta_{mn}$, $m, n = 1, \ldots, N$ (see [14, Remark 5.3] or Remark 2.14 – this thesis for the existence of such $\boldsymbol{G}^m$). By so doing, we obtain $L_n(\boldsymbol{E}(t)) = \widetilde{V}(t)|_{\Gamma_{\boldsymbol{J}}^n} - \widetilde{V}(t)|_{\Gamma_{\boldsymbol{E}}^n} = V_n(t)$, $n = 1, \ldots, N$, from which it follows (3.71).

Finally, the proof of uniqueness of solution is identical to that in Theorem 3.4. ■

Remark 3.23 *As in Section 3.3, we conclude that we can use the simplest choice of data $g = 0$ on Γ_{D} without affecting the quantities of main interest, namely, $\boldsymbol{H}$ in the whole domain Ω and $\boldsymbol{E}$ in the conducting domain Ω_{C}.*

Next step is the space discretization of Problem 3.21. Let $\boldsymbol{\mathcal{U}}_h$ and $\mathcal{Q}_h$ be as in Subsection 3.3.1. Let $\boldsymbol{\mathcal{U}}_h^0 := \{\boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h : \widetilde{L}_n(\boldsymbol{w}_h) = 0, n = 1, \ldots, N\}$ with $\widetilde{L}_n$ as defined in (3.46). The space-discretization reads as follows:

Problem 3.24 *Given $g \in \mathrm{L}^2(0, T; L^2(\Gamma_{\mathrm{D}}))$, $V_n \in \mathrm{H}^1(0, T)$, $n = 1, \ldots, N$, and $\boldsymbol{H}_0$ satisfying (3.13), find $\boldsymbol{u}_h : [0, T] \to \boldsymbol{\mathcal{U}}_h$ and $\xi_h : [0, T] \to \mathcal{Q}_h$ such that*

$$\widetilde{L}_n(\boldsymbol{u}_h(t)) = \int_0^t V_n(s)\,ds, \quad n = 1, \ldots, N,$$

$$\int_{\Omega_{\mathrm{C}}} \sigma\partial_t\boldsymbol{u}_h(t)\cdot\boldsymbol{w}_h + \int_\Omega \frac{1}{\mu}\mathbf{curl}\,\boldsymbol{u}_h(t)\cdot\mathbf{curl}\,\boldsymbol{w}_h + \int_{\Omega_{\mathrm{D}}} \epsilon\boldsymbol{w}_h\cdot\mathbf{grad}\,\xi_h(t) = \int_\Omega \mathbf{curl}\,\boldsymbol{H}_0\cdot\boldsymbol{w}_h \quad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h^0,$$

$$\int_{\Omega_{\mathrm{D}}} \epsilon\boldsymbol{u}_h(t)\cdot\mathbf{grad}\,\varphi_h = \int_{\Gamma_{\mathrm{D}}} \left(\int_0^t g(s)\,ds\right)\varphi_h \quad \forall \varphi_h \in \mathcal{Q}_h,$$

$$\boldsymbol{u}_h(0) = \mathbf{0} \quad \text{in } \Omega.$$

To prove that this problem is well-posed, our first step is to build an auxiliary function $\check{\boldsymbol{u}}_h \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{U}}_h)$ satisfying $\widetilde{L}_n(\check{\boldsymbol{u}}_h(t)) = \int_0^t V_n(s)\,ds$, $n = 1, \ldots, N$, $t \in [0, T]$. To define $\check{\boldsymbol{u}}_h$, first we choose functions $\mathbf{\Phi}_m \in \boldsymbol{\mathcal{U}}_h$ such that $\widetilde{L}_n(\mathbf{\Phi}_m) = \delta_{mn}$, $m, n = 1, \ldots, N$; such $\mathbf{\Phi}_m$ are easy to construct once a basis of $\boldsymbol{\mathcal{U}}_h$ is given (see Remark 3.33 below). Then, we define

$$\check{\boldsymbol{u}}_h(t) := \sum_{m=1}^{N} \int_0^t V_m(s)\, ds\, \mathbf{\Phi}_m. \tag{3.77}$$

Hence,

$$\widetilde{L}_n(\check{\boldsymbol{u}}_h(t)) = \sum_{m=1}^{N} \int_0^t V_m(s)\, ds\, \widetilde{L}_n(\mathbf{\Phi}_m) = \int_0^t V_n(s)\, ds.$$

Moreover, since $V_n \in \mathrm{H}^1(0,T)$, $n = 1, \cdots, N$, we conclude that $\check{\boldsymbol{u}}_h \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{U}}_h)$.

Now, if we write $\boldsymbol{u}_h = \bar{\boldsymbol{u}}_h + \check{\boldsymbol{u}}_h$, Problem 3.24 is equivalent to finding $\bar{\boldsymbol{u}}_h : [0,T] \to \boldsymbol{\mathcal{U}}_h^0$ and $\xi_h : [0,T] \to \mathcal{Q}_h$ such that

$$\begin{aligned}
&\int_{\Omega_{\mathrm{C}}} \sigma \partial_t \bar{\boldsymbol{u}}_h(t) \cdot \boldsymbol{w}_h + \int_\Omega \frac{1}{\mu} \mathbf{curl}\, \bar{\boldsymbol{u}}_h(t) \cdot \mathbf{curl}\, \boldsymbol{w}_h + \int_{\Omega_{\mathrm{D}}} \epsilon \boldsymbol{w}_h \cdot \mathbf{grad}\, \xi_h(t) \\
&\qquad = -\int_{\Omega_{\mathrm{C}}} \sigma \partial_t \check{\boldsymbol{u}}_h(t) \cdot \boldsymbol{w}_h - \int_\Omega \frac{1}{\mu} \mathbf{curl}\, \check{\boldsymbol{u}}_h(t) \cdot \mathbf{curl}\, \boldsymbol{w}_h + \int_\Omega \mathbf{curl}\, \boldsymbol{H}_0 \cdot \boldsymbol{w}_h \quad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h^0, \\
&\int_{\Omega_{\mathrm{D}}} \epsilon \bar{\boldsymbol{u}}_h(t) \cdot \mathbf{grad}\, \varphi_h = \int_{\Gamma_{\mathrm{D}}} \left(\int_0^t g(s)\, ds \right) \varphi_h - \int_{\Omega_{\mathrm{D}}} \epsilon \check{\boldsymbol{u}}_h(t) \cdot \mathbf{grad}\, \varphi_h \quad \forall \varphi_h \in \mathcal{Q}_h, \\
&\bar{\boldsymbol{u}}_h(0) = \mathbf{0} \quad \text{in } \Omega.
\end{aligned}$$

The well-posedness of this problem is obtained by following the same arguments used for Problem 3.6 in Section 3.3.1. The main difference is that now we need a discrete *inf-sup* condition similar to (3.50), but taking supremum in $\boldsymbol{\mathcal{U}}_h^0$ instead of $\boldsymbol{\mathcal{U}}_h$. However, for the proof of (3.50) it was used a function $\boldsymbol{w}_h = \mathbf{grad}\, \widetilde{\varphi}_h$ which actually lies in $\boldsymbol{\mathcal{U}}_h^0$. Altogether, we conclude that Problem 3.24 has a unique solution.

Next, the arguments in Section 3.3.1 can be readily adapted to obtain the error estimate. With this aim, we need the following result.

Lemma 3.25 *If $\boldsymbol{w} \in \mathrm{H}^r(\mathbf{curl};\Omega) \cap \boldsymbol{\mathcal{U}}$ then $\widetilde{L}_n(\mathcal{I}_h^{\mathcal{N}} \boldsymbol{w}) = L_n(\boldsymbol{w})$, $n = 1, \ldots, N$.*

Proof. We recall that for $\boldsymbol{w} \in \boldsymbol{\mathcal{U}}$ there exists a unique $W \in \mathcal{W}$ such that $\boldsymbol{n} \times \boldsymbol{w} \times \boldsymbol{n} = -\,\mathbf{grad}_\tau W$ on $\partial\Omega$ and $L_n(\boldsymbol{w}) = W|_{\Gamma_{\boldsymbol{E}}^n} - W|_{\Gamma_{\boldsymbol{J}}^n}$, $n = 1, \ldots, N$ (cf. (3.23)). On the other hand, for $\boldsymbol{w} \in \mathrm{H}^r(\mathbf{curl};\Omega)$ we have that $\boldsymbol{w}|_{\partial\Omega} \in \mathrm{H}^{r-1/2}(\partial\Omega)^3$ and, hence, $W \in \mathrm{H}^{r+1/2}(\partial\Omega)$. Thus, $\boldsymbol{w}$ and W are smooth enough to write

$$\boldsymbol{n} \times \mathcal{I}_h^{\mathcal{N}} \boldsymbol{w} \times \boldsymbol{n} = \mathcal{I}_h^{\mathcal{N}_{2D}} (\boldsymbol{n} \times \boldsymbol{w} \times \boldsymbol{n}) = -\mathcal{I}_h^{\mathcal{N}_{2D}} (\mathbf{grad}_\tau W) = -\,\mathbf{grad}_\tau (\mathcal{I}_h^{\mathcal{L}} W) \quad \text{on } \partial\Omega,$$

where $\mathcal{I}_h^{\mathcal{N}_{2D}}$ and $\mathcal{I}_h^{\mathcal{L}}$ denote the two-dimensional Nédélec and Lagrange interpolant operators, respectively. Then,

$$\widetilde{L}_n(\mathcal{I}_h^{\mathcal{N}} \boldsymbol{w}) = \int_{C_n} \mathcal{I}_h^{\mathcal{N}} \boldsymbol{w} \cdot \boldsymbol{t} = -\int_{C_n} \mathbf{grad}_\tau (\mathcal{I}_h^{\mathcal{L}} W) \cdot \boldsymbol{t} = \mathcal{I}_h^{\mathcal{L}} W|_{\Gamma_{\boldsymbol{E}}^n} - \mathcal{I}_h^{\mathcal{L}} W|_{\Gamma_{\boldsymbol{J}}^n} = W|_{\Gamma_{\boldsymbol{E}}^n} - W|_{\Gamma_{\boldsymbol{J}}^n} = L_n(\boldsymbol{w}).$$

Thus we conclude the proof. ∎

Now we are in a position to prove the following error estimate.

Theorem 3.26 *Let $\boldsymbol{u}$ be the solution to Problem 3.21 and $\boldsymbol{u}_h$ that to Problem 3.24. If $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ with $r \in \left(\frac{1}{2},1\right]$, then there exists a constant $C>0$, independent of h, such that*

$$\sup_{0\le t\le T} \|\boldsymbol{u}(t)-\boldsymbol{u}_h(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \sup_{0\le t\le T} \|\mathbf{curl}\,\boldsymbol{u}(t)-\mathbf{curl}\,\boldsymbol{u}_h(t)\|^2_{\mathrm{L}^2(\Omega)^3} + \int_0^T \|\partial_t(\boldsymbol{u}(t)-\boldsymbol{u}_h(t))\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\,dt$$
$$\le C\,h^{2r}\left\{\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)} + \int_0^T \|\partial_t\boldsymbol{u}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)}\,dt\right\}$$
$$\le C\,h^{2r}\|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}.$$

Proof. As a first step, we need to prove the analogue to Lemma 3.12 for $\boldsymbol{u}$ and $\boldsymbol{u}_h$ being solution to Problem 3.21 and 3.24, respectively. The only difference in this proof is that, now, the test functions $\boldsymbol{w}_h$ in (3.62) lie in $\boldsymbol{\mathcal{U}}_h^0$ instead of $\boldsymbol{\mathcal{U}}_h$. Therefore, we need to ensure that $\boldsymbol{\delta}_h(t) := \mathcal{I}_h^{\mathcal{N}}(\boldsymbol{u}(t)) - \boldsymbol{u}_h(t)$ and $\partial_t\boldsymbol{\delta}_h(t)$ belong to $\boldsymbol{\mathcal{U}}_h^0$; namely, $\widetilde{L}_n(\boldsymbol{\delta}_h(t)) = \widetilde{L}_n(\partial_t\boldsymbol{\delta}_h(t)) = 0$. The former follows from Lemma 3.25 and the fact that $L_n(\boldsymbol{u}(t)) = \widetilde{L}_n(\boldsymbol{u}_h(t))$ (cf. the first equations in Problem 3.21 and 3.24). For the latter we use the same arguments and the assumption $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$. The rest of the proof follows identically as that of Theorem 3.13. ■

Finally, we introduce the fully discrete approximation of Problem 3.21 defined as follows:

Problem 3.27 *Given $V_n \in \mathrm{H}^1(0,T)$, $n=1,\dots,N$, and $\boldsymbol{H}_0$ satisfying (3.13), find $\boldsymbol{u}_h^m \in \boldsymbol{\mathcal{U}}_h$ and $\xi_h^m \in \mathcal{Q}_h$, $m=1,\dots,M$, such that*

$$\widetilde{L}_n(\boldsymbol{u}_h(t_m)) = \int_0^{t_m} V_n(s)\,ds, \quad n=1,\dots,N,$$
$$\int_{\Omega_{\mathrm{C}}} \sigma\frac{\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}}{\Delta t}\cdot\boldsymbol{w}_h + \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}_h^m\cdot\mathbf{curl}\,\boldsymbol{w}_h + \int_{\Omega_{\mathrm{D}}} \epsilon\boldsymbol{w}_h\cdot\mathbf{grad}\,\xi_h^m = \int_\Omega \mathbf{curl}\,\boldsymbol{H}_0\cdot\boldsymbol{w}_h \quad \forall\boldsymbol{w}_h\in\boldsymbol{\mathcal{U}}_h^0,$$
$$\int_{\Omega_{\mathrm{D}}} \epsilon\boldsymbol{u}_h^m\cdot\mathbf{grad}\,\varphi_h = \int_{\Gamma_{\mathrm{D}}}\left(\int_0^{t_m} g(s)\,ds\right)\varphi_h \quad \forall\varphi_h\in\mathcal{Q}_h,$$
$$\boldsymbol{u}_h^0 = \mathbf{0} \quad \text{in } \Omega.$$

This problem has a unique solution. In fact, taking $\check{\boldsymbol{u}}_h^m := \check{\boldsymbol{u}}_h(t^m)$, with $\check{\boldsymbol{u}}_h$ as in (3.77), and writing $\boldsymbol{u}_h^m = \bar{\boldsymbol{u}}_h^m + \check{\boldsymbol{u}}_h^m$, the m-th step of Problem 3.27 is equivalent to find $\bar{\boldsymbol{u}}_h^m \in \boldsymbol{\mathcal{U}}_h^0$ and $\xi_h^m \in \mathcal{Q}_h$ such that

$$\int_{\Omega_{\mathrm{C}}} \sigma\bar{\boldsymbol{u}}_h^m\cdot\boldsymbol{w}_h + \Delta t\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\bar{\boldsymbol{u}}_h^m\cdot\mathbf{curl}\,\boldsymbol{w}_h + \Delta t\int_{\Omega_{\mathrm{D}}} \epsilon\boldsymbol{w}_h\cdot\mathbf{grad}\,\xi_h^m$$
$$= \int_{\Omega_{\mathrm{C}}} \sigma\bar{\boldsymbol{u}}_h^{m-1}\cdot\boldsymbol{w}_h + \Delta t\int_\Omega \mathbf{curl}\,\boldsymbol{H}_0\cdot\boldsymbol{w}_h - \int_{\Omega_{\mathrm{C}}} \sigma(\check{\boldsymbol{u}}_h^m - \check{\boldsymbol{u}}_h^{m-1})\cdot\boldsymbol{w}_h - \Delta t\int_\Omega\frac{1}{\mu}\,\mathbf{curl}\,\check{\boldsymbol{u}}_h^m\cdot\mathbf{curl}\,\boldsymbol{w}_h \quad \forall\boldsymbol{w}_h\in\boldsymbol{\mathcal{U}}_h^0,$$
$$\Delta t\int_{\Omega_{\mathrm{D}}} \epsilon\bar{\boldsymbol{u}}_h^m\cdot\mathbf{grad}\,\varphi_h = \Delta t\int_{\Gamma_{\mathrm{D}}}\left(\int_0^{t_m} g(s)\,ds\right)\varphi_h + \Delta t\int_{\Omega_{\mathrm{D}}}\epsilon\check{\boldsymbol{u}}_h^m\cdot\mathbf{grad}\,\varphi_h \quad \forall\varphi_h\in\mathcal{Q}_h.$$

The well-posedness of this problem follows identically as that of Problem 3.16. The same happens with the error estimates analogous to those in Theorem 3.18. Therefore, we conclude the following result.

Theorem 3.28 *Let $\boldsymbol{u}$ be the solution to Problem 3.21 and $\boldsymbol{u}_h^k$, $k = 1, \ldots, M$, that to Problem 3.27. If $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ for $r \in \left(\frac{1}{2}, 1\right]$ and $\boldsymbol{u}|_{\Omega_\mathrm{C}} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_\mathrm{C})^3)$, then there exists a constant $C > 0$, independent of h and Δt, such that*

$$\max_{1\le k\le M} \|\boldsymbol{u}(t_k) - \boldsymbol{u}_h^k\|^2_{\mathrm{L}^2(\Omega_\mathrm{C})^3} + \max_{1\le k\le M} \|\,\mathbf{curl}(\boldsymbol{u}(t_k) - \boldsymbol{u}_h^k)\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{M} \left\| \partial_t \boldsymbol{u}(t_k) - \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega_\mathrm{C})^3}$$
$$\le C \left\{ (\Delta t)^2 \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|^2_{\mathrm{L}^2(\Omega_\mathrm{C})^3}\, dt + h^{2r} \sup_{0\le t\le T} \|\boldsymbol{u}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)} + h^{2r} \int_0^T \|\partial_t \boldsymbol{u}(t)\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)}\, dt \right\}$$
$$\le C \left\{ (\Delta t)^2 \|\boldsymbol{u}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_\mathrm{C})^3)} + h^{2r} \|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))} \right\}.$$

Remark 3.29 *As in the case of Problem 3.16, we approximate the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ at each time t_k, $k = 1, \ldots, M$, by means of $\boldsymbol{E}_h^k := \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t}$ and $\boldsymbol{H}_h^k := \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}_h^k - \boldsymbol{H}_0$, respectively. Then, Theorem 3.28 yields the following error estimates:*

$$\Delta t \sum_{k=1}^{M} \left\| \boldsymbol{E}(t_k) - \boldsymbol{E}_h^k \right\|^2_{\mathrm{L}^2(\Omega_\mathrm{C})^3} \le C \left\{ (\Delta t)^2 \|\boldsymbol{u}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_\mathrm{C})^3)} + h^{2r} \|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))} \right\},$$
$$\max_{1\le k\le M} \|\boldsymbol{H}(t_k) - \boldsymbol{H}_h^k)\|^2_{\mathrm{L}^2(\Omega)^3} \le C \left\{ (\Delta t)^2 \|\boldsymbol{u}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_\mathrm{C})^3)} + h^{2r} \|\boldsymbol{u}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))} \right\}.$$

Remark 3.30 *The same observation made in Remark 3.15 holds in this case.*

Let us remark that the constraints $\widetilde{L}_n(\boldsymbol{u}_h(t_m)) = \int_0^{t_m} V_n(s)\, ds$, $n = 1, \ldots, N$, can be imposed by means of a Lagrange multiplier. In such a case, we are led to the following problem:

Problem 3.31 *Given $V_n \in \mathrm{H}^1(0,T)$, $n = 1, \ldots, N$, and $\boldsymbol{H}_0$ satisfying (3.13), find $\boldsymbol{u}_h^m \in \boldsymbol{\mathcal{U}}_h$, $\xi_h^m \in \mathcal{Q}_h$ and $\mathbb{I}^m = (\mathbb{I}_1^m, \ldots, \mathbb{I}_N^m) \in \mathbb{R}^N$, $m = 1, \ldots, M$, such that*

$$\int_{\Omega_\mathrm{C}} \sigma \frac{\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1}}{\Delta t} \cdot \boldsymbol{w}_h + \int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}_h^m \cdot \mathbf{curl}\,\boldsymbol{w}_h + \int_{\Omega_\mathrm{D}} \epsilon \boldsymbol{w}_h \cdot \mathbf{grad}\,\xi_h^m + \sum_{n=1}^{N} \mathbb{I}_n^m \widetilde{L}_n(\boldsymbol{w}_h)$$
$$= \int_\Omega \mathbf{curl}\,\boldsymbol{H}_0 \cdot \boldsymbol{w}_h \quad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{U}}_h,$$
$$\int_{\Omega_\mathrm{D}} \epsilon \boldsymbol{u}_h^m \cdot \mathbf{grad}\,\varphi_h = \int_{\Gamma_\mathrm{D}} \left(\int_0^{t_m} g(s)\, ds \right) \varphi \quad \forall \varphi_h \in \mathcal{Q}_h,$$
$$\sum_{n=1}^{N} \widetilde{L}_n(\boldsymbol{u}_h^m)\, \mathbb{J}_n = \sum_{n=1}^{N} \int_0^t V_n(s)\, ds\, \mathbb{J}_n \quad \forall \mathbb{J} = (\mathbb{J}_1, \ldots, \mathbb{J}_N) \in \mathbb{R}^N,$$
$$\boldsymbol{u}_h^0 = \mathbf{0} \quad \text{in } \Omega.$$

The following lemma shows that this and Problem 3.27 are actually equivalent:

Lemma 3.32 *Given $V_n \in \mathrm{H}^1(0,T)$, $n = 1, \ldots, N$, and $\boldsymbol{H}_0$ satisfying (3.13), $(\boldsymbol{u}_h^m, \xi_h^m)$, $m = 1, \ldots, M$, is the solution to Problem 3.24 if and only if there exist $\mathbb{I}^m \in \mathbb{R}^N$ such that $(\boldsymbol{u}_h^m, \xi_h^m, \mathbb{I}^m)$, $m = 1, \ldots, M$, is the unique solution to Problem 3.31.*

Proof. The result is a consequence of the existence and uniqueness of the solution to Problem 3.27 and the fact that the bilinear form $c : \boldsymbol{\mathcal{U}}_h \times \mathbb{R}^N \to \mathbb{R}$ defined by $c(\boldsymbol{w}_h, \mathbb{J}) := \sum_{n=1}^N \widetilde{L}_n(\boldsymbol{w}_h)\,\mathbb{J}_n$ satisfies a discrete *inf-sup* condition, see [18, Lemma 3.3]. ■

The Lagrange multipliers $\mathbb{I}_n^m$ have a physical meaning. In fact, the equations of Problem 3.31 are exactly the same as those of Problem 3.16, with $\mathbb{I}_n^m$ instead of $I_n(t_m) - I_n(0)$. Therefore by solving Problem 3.31, we can compute the input currents on each conductor Ω_{C}^n by means of $I_n(t_m) = I_n(0) + \mathbb{I}_n^m$ (provided $I_n(0)$ is known).

3.5 Numerical experiments

In this section we present some numerical results obtained with a Matlab code implementing the numerical method described above. First, we give some details about the computer implementation. Then, we present a test with a known analytical solution which we use to validate the computer code and to check the error estimates proved above. Finally, we apply the method to a problem in a more realistic geometry.

3.5.1 Implementation issues

We have implemented in our codes matrix forms of Problem 3.6 and 3.31. In both cases we need a basis of $\boldsymbol{\mathcal{U}}_h$. We have used the following one taken from [18, Section 3]

$$\left\{\boldsymbol{\Phi}_e \,:\, e \in \mathring{\boldsymbol{\mathcal{E}}}_h\right\} \cup \left\{\mathbf{grad}\,\varphi_v \,:\, v \in \boldsymbol{\mathcal{V}}_h^{\Gamma_{\mathrm{D}}}\right\} \cup \left\{\mathbf{grad}\,\varphi_n^{\mathrm{J}} \,:\, n = 1,\dots,N\right\} \cup \left\{\mathbf{grad}\,\varphi_n^{\mathrm{E}} \,:\, n = 1,\dots,N\right\},$$

where

- $\mathring{\boldsymbol{\mathcal{E}}}_h$ is the set of inner edges of the mesh $\mathcal{T}_h$ (i.e., edges $e \not\subseteq \partial\Omega$) and, for each $e \in \mathring{\boldsymbol{\mathcal{E}}}_h$, $\boldsymbol{\Phi}_e \in \boldsymbol{\mathcal{N}}_h(\Omega)$ is the Nedéléc basis function associated to e;
- $\boldsymbol{\mathcal{V}}_h^{\Gamma_{\mathrm{D}}}$ is the set of vertices of the mesh $\mathcal{T}_h$ lying on the open surface Γ_{D} and, for all vertices $v \in \bar{\Omega}_{\mathrm{D}}$, $\varphi_v \in \mathcal{L}_h(\Omega_{\mathrm{D}})$ is the piecewise linear function associated to v;
- φ_n^{J} is the piecewise linear function such that $\varphi_n^{\mathrm{J}} = 1$ for all vertices of the mesh $\mathcal{T}_h$ lying on the closed surface $\bar{\Gamma}_{\mathrm{J}}^n$ and $\varphi_n^{\mathrm{J}} = 0$ otherwise;
- φ_n^{E} is the piecewise linear function such that $\varphi_n^{\mathrm{E}} = 1$ for all vertices of the mesh $\mathcal{T}_h$ lying on the closed surface $\bar{\Gamma}_{\mathrm{E}}^n$ and $\varphi_n^{\mathrm{E}} = 0$ otherwise.

In spite of the fact that $\widetilde{L}_n$ is defined by means of an integral on a particular curve C_n (cf. (3.46)), in practice, there is no need to construct such curves. In fact, to impose the constraint $\widetilde{L}_n(\boldsymbol{u}_h(t_k)) = \int_0^{t_k} V_n(s)\,ds$, it is enough to evaluate $\widetilde{L}_n$ for the basis functions of $\boldsymbol{\mathcal{U}}_h$ by means of (3.46). Thus, we obtain

$$\widetilde{L}_n(\boldsymbol{w}_h) = \begin{cases} 0, & \text{if } \boldsymbol{w}_h = \boldsymbol{\Phi}_e, \\ 0, & \text{if } \boldsymbol{w}_h = \mathbf{grad}\,\varphi_v \text{ for } v \in \boldsymbol{\mathcal{V}}_h^{\Gamma_{\mathrm{D}}}, \\ \delta_{mn}, & \text{if } \boldsymbol{w}_h = \mathbf{grad}\,\varphi_m^{\mathrm{J}}, \\ -\delta_{mn}, & \text{if } \boldsymbol{w}_h = \mathbf{grad}\,\varphi_m^{\mathrm{E}}. \end{cases}$$

On the other hand, a basis of $\mathcal{Q}_h$ is given by

$$\left\{\varphi_v \,:\, v \in \boldsymbol{\mathcal{V}}_h^{\Gamma_{\mathrm{D}}}\right\} \cup \{\varphi_v \,:\, v \in \Omega_{\mathrm{D}}\} \cup \{\varphi_k \,:\, k = 2, \ldots, M\},$$

where φ_v are as defined above and φ_k is the piecewise linear function such that $\varphi_k = 1$ for all vertices of the mesh $\mathcal{T}_h$ lying on the closed surface $\bar{\Gamma}_{\mathrm{I}}^k$, $k = 2, \ldots, M$, and vanishing at all the other vertices.

Remark 3.33 *Let us recall that to prove that Problem 3.24 is well-posed, we have used functions* $\boldsymbol{\Phi}_m$ *satisfying* $\widetilde{L}_n(\boldsymbol{\Phi}_m) = \delta_{mn}$, $m, n = 1, \ldots, N$. *An example of one such* $\boldsymbol{\Phi}_m$ *is defined by* $\boldsymbol{\Phi}_m := \mathbf{grad}\, \varphi_m^{\mathrm{J}}$, *where* φ_m^{J} *is as above.*

3.5.2 A test with known analytical solution

To test our codes, we applied the proposed method to the same problem solved in [17] in harmonic regime. This is the reason why we only give here a brief description and refer the reader to the quoted paper for further details. Figure 3.2 shows a sketch of the domain where the conducting

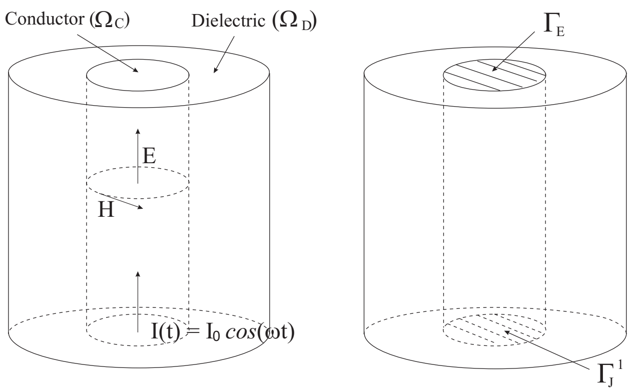

Figure 3.2: Sketch of the domain in the analytical example.

part Ω_{C} and the whole domain Ω are coaxial cylinders of respective radius $R_{\mathrm{C}} = 0.25\,\mathrm{m}$ and $R_{\mathrm{D}} = 0.5\,\mathrm{m}$ and height $A = 0.5\,\mathrm{m}$. First, we solve the problem with input intensity as boundary data. An alternating current of intensity $I(t) = I_0 \cos(\omega t)$ enters the conductor through Γ_J^1 and crosses Ω_{C} in the axial direction; I_0 denotes the amplitude of the intensity and ω the angular frequency. Under these assumptions, by using a cylindrical coordinate system, it is easy to obtain an analytical solution of the eddy current problem in Ω by writing all the fields in the form $\boldsymbol{F}(t, \boldsymbol{x}) = \mathrm{Re}(e^{\mathrm{i}\omega t} \boldsymbol{\mathcal{F}}(\boldsymbol{x}))$.

To solve Problem 3.16 we also need the data $g \in \mathrm{L}^2(0, T; \mathrm{L}^2(\Gamma_{\mathrm{D}}))$. However, as stated above, the most relevant physical quantities $\boldsymbol{H}$ and $\boldsymbol{E}|_{\Omega_{\mathrm{C}}}$ are independent of the chosen g. Because of this, we have solved Problem 3.16 by means of the easiest choice: $g = 0$.

The numerical method has been used on several successively refined meshes and the time-step has been conveniently reduced to analyze the convergence with respect to both, the mesh-size

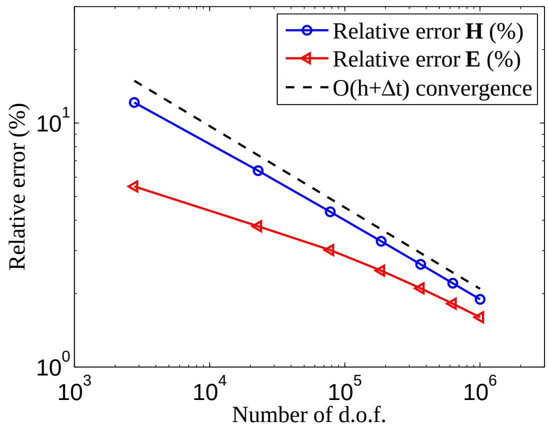

Figure 3.3: $\frac{\max_{1\le k\le M}\|\boldsymbol{H}(t_k)-\boldsymbol{H}_h^k\|_{\mathrm{L}^2(\Omega)^3}}{\max_{1\le k\le M}\|\boldsymbol{H}(t_k)\|_{\mathrm{L}^2(\Omega)^3}}$ and $\frac{\sqrt{\Delta t}\left\{\sum_{k=1}^M\left\|\boldsymbol{E}(t_k)-\boldsymbol{E}_h^k\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\right\}^{1/2}}{\sqrt{\Delta t}\left\{\sum_{k=1}^M\|\boldsymbol{E}(t_k)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\right\}^{1/2}}$ versus number of d.o.f. (log-log scale).

and the time-step simultaneously. We have compared the obtained numerical solutions with the analytical one.

In order to show the linear convergence with respect to the mesh-size and the time-step, we have computed the relative errors of the different fields corresponding to $\frac{h}{n}$, $\frac{\Delta t}{n}$, $n = 1, \ldots, 7$. Figure 3.3 shows log-log plots of the relative error for the physical variables of interest, the magnetic field and the electric field in the conductor domain, in the discrete norms considered in Remark 3.19 versus the number of degrees of freedom (d.o.f.).

Secondly we consider voltage drops as boundary data for the same problem. In this case it is easy to show that the corresponding voltage drop is given by $V(t) = \mathrm{Re}(e^{\mathrm{i}\omega t}\mathcal{V})$ (see [7, Section 8.1.5]), where

$$\mathcal{V} = \frac{\gamma A I_0}{2\pi\sigma R_C}\frac{\mathcal{I}_0(\gamma R_C)}{\mathcal{I}_1(\gamma R_C)} + \mathrm{i}\omega\mu\frac{AI_0}{2\pi}\log\left(\frac{R_D}{R_C}\right),$$

with $\gamma = \sqrt{\mathrm{i}\omega\mu\sigma}$ and $\mathcal{I}_0$, $\mathcal{I}_1$ the Bessel's function of order 0 and 1, respectively.

We have compared the obtained numerical solutions with the analytical one. As in the previous case, we have chosen $g = 0$. Figure 3.4 shows log-log plots of the relative errors for the magnetic field and for the electric field in Ω_{C} in the discrete norms considered in Remark 3.29 versus the number of degrees of freedom.

In both tests, with intensities or voltage drops as boundary data, the error curves show a very good agreement with the theoretically predicted order of convergence. In fact, the relative error of $\boldsymbol{H}$ behaves always very close to $\mathcal{O}(h + \Delta t)$. The order of convergence of $\boldsymbol{E}$ is initially worse (although the relative errors are smaller than those of $\boldsymbol{H}$) but finally it is also almost $\mathcal{O}(h + \Delta t)$. Moreover, these results are actually independent of the choice of g. In fact, we have also solved Problems 3.6 and 3.24 with two other choices of g: a random one and the exact value of $\epsilon\boldsymbol{E}|_{\Gamma_{\mathrm{D}}}$

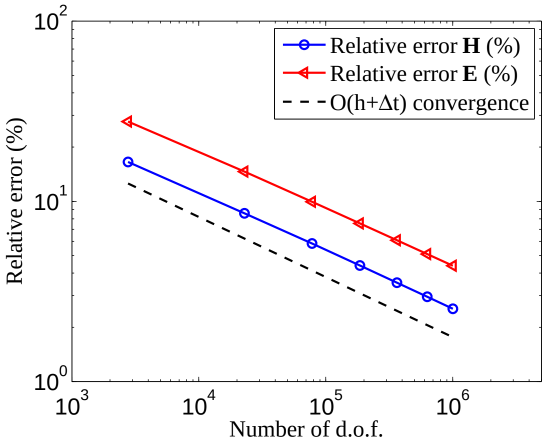

Figure 3.4: $\frac{\max_{1\le k\le M}\|\boldsymbol{H}(t_k)-\boldsymbol{H}_h^k\|_{\mathrm{L}^2(\Omega)^3}}{\max_{1\le k\le M}\|\boldsymbol{H}(t_k)\|_{\mathrm{L}^2(\Omega)^3}}$ and $\frac{\sqrt{\Delta t}\left\{\sum_{k=1}^M \|\boldsymbol{E}(t_k)-\boldsymbol{E}_h^k\|^2_{\mathrm{L}^2(\Omega_\mathrm{C})^3}\right\}^{1/2}}{\sqrt{\Delta t}\left\{\sum_{k=1}^M \|\boldsymbol{E}(t_k)\|^2_{\mathrm{L}^2(\Omega_\mathrm{C})^3}\right\}^{1/2}}$ versus number of d.o.f. (log-log scale).

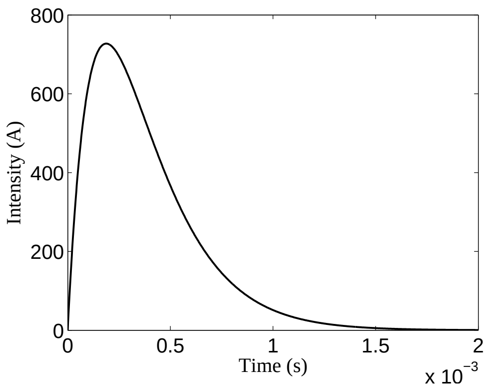

Figure 3.5: Imposed current intensity (A) vs. time (s).

(which was obtained by analytical computations similar to those in [7, Section 8.1.5]). In all cases the computed values of $\boldsymbol{H}_h^k$ and $\boldsymbol{E}_h^k$, the latter only in Ω_C, coincide up to rounding errors.

Additionally, when the exact value of $g = \epsilon \boldsymbol{E}|_{\Gamma_\mathrm{D}}$ was used, we have tested whether the computed values $\frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t}$ approximate the exact electric fields $\boldsymbol{E}_\mathrm{D}$ in Ω_D. In this case, although the theoretical results only guarantees such a convergence in Ω_C, we checked an $\mathcal{O}(h + \Delta t)$ convergence, too.

3.5.3 A problem in a more complex geometry

In this section we have computed the eddy currents induced by a coil in a metallic plate. The coil and the plate are shown in Figure 3.6, which also shows a typical mesh of the conducting domain. Such configuration is usually found, for instance, in problems related to non destructive

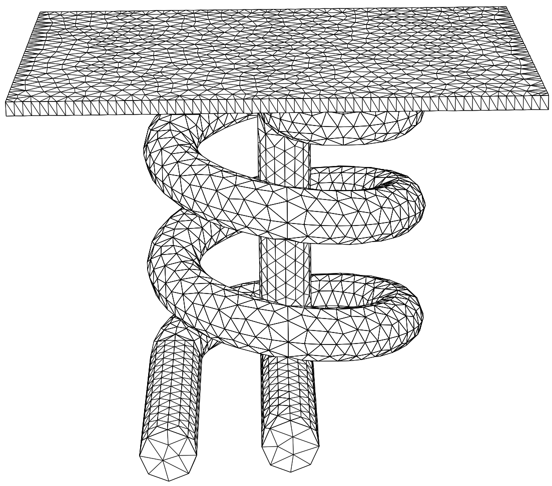

Figure 3.6: Mesh of the conducting domain (left). Detail of the coil mesh (right).

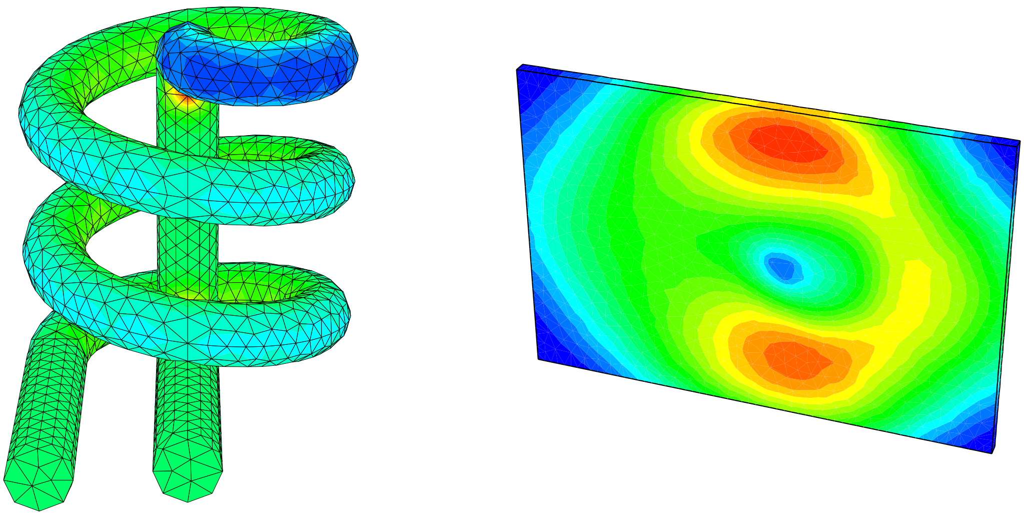

Figure 3.7: Modulus of the current density in coil and plate at time 0.00018 s (different scales).

testing or electromagnetic forming (see, e.g., [56]).

Domain Ω has been chosen as a sufficiently large box surrounding the conductor. Notice that in order to introduce a scalar potential in the dielectric domain to use the formulation proposed in [14] (cf. Chapter 2, this thesis), we would need to build a cutting surface in this domain, what would not be easy in this case.

The current intensity which enters the coil is shown in Figure 3.5. Here, we have used $g \equiv 0$ on $[0,T] \times \Gamma_{\mathrm{D}}$, too.

In this test, the eddy currents induced in the plate are in the range $2.7{\times}10^4 - 1.8{\times}10^3$ A/m^2. They are significantly smaller than those in the coil (range $2.9{\times}10^8 - 8.1{\times}10^8$ A/m^2). This is the reason why we show coil and plate on separate figures. Figure 3.7 shows the modulus of the current density in the conducting domain. Figures 3.8 and 3.9 show the current density vector field. All the

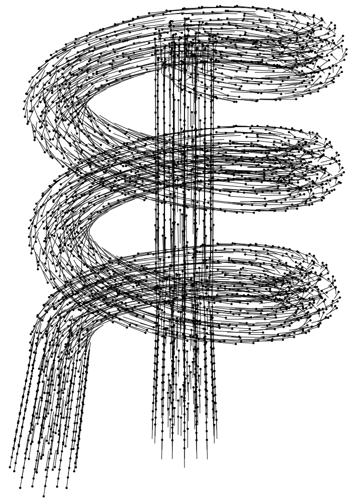

Figure 3.8: Distribution of the current density (vector field) in coil at time 0.00018 s.

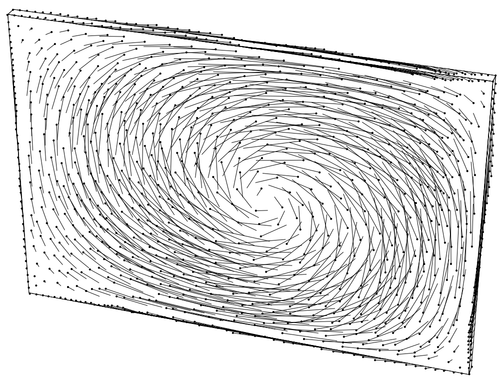

Figure 3.9: Distribution of the current density (vector field) in the plate at time 0.00018 s.

reported results correspond to the time at which the input current intensity reaches its maximum (0.00018 s).

Chapter 4

Numerical solution of a transient 3D eddy current model with moving conductors

Introduction

In this chapter we propose a finite element method to solve a time-dependent eddy current model in a three-dimensional bounded domain which contains moving conductors. As we have detailed in the introduction of this thesis, this model arises in different physical applications and, in particular, our work is motivated by the electromagnetic forming process (EMF) [30]. Thus, we will focus on solving the transient eddy current model in the case where the conducting part is a workpiece which moves along the time, while a coil which provides the current source is placed in a fixed position.

As a first step in the modeling of the EMF, we will assume that the velocity of the workpiece is known and our objective will consist in computing the electromagnetic fields in the workpiece along the time.

The motion of the workpiece introduces serious difficulties in the mathematical analysis of the eddy current model, mainly due to the different nature of the equations in dielectric and conducting parts. In fact, we have not obtained until now a result which guarantees thoroughly the well-posedness of the continuous problem. We emphasize that in the literature we can find only few papers which deal with the analysis of the eddy current model by considering moving conductors in either a two-dimensional or a three-dimensional framework.

In particular, in [24, 27], the authors analyze a two-dimensional transient eddy current problem which arises in the modeling of electrical engines by considering the motion of the rotor. The three-dimensional case is studied in [26], where a time-primitive of the electric field is used as the main unknown, leading to a degenerate parabolic problem. We notice that while the electromagnetic model used in these papers takes into account the motion of the rotor, the geometry occupied by the moving part is always the same which does not happen in our case. Moreover, the fact that the motion is a rotation is essential in the theoretical proofs of existence and uniqueness of solution in these problems

On the other hand, an axisymmetric eddy current model with workpiece motion has been recently studied in [16], also in the framework of EMF. The main unknown in this case is the magnetic vector potential and the resulting problem is also parabolic and degenerate; the authors prove the well-posedness of the problem by means of a regularization argument.

The extension of the two previous formulations to our problem does not seem to lead to convenient numerical methods. Regarding the first one, for using a time primitive of the electric field, we would need a Lagrange multiplier defined in the dielectric domain (see, [2, 12]). Regarding the second one, in three-dimensional problems, the use of a vector potential as in [16] requires an additional scalar potential defined in the conducting domain [3, 20]. Since neither the dielectric nor the conducting domain are fixed in time, the discretization of these formulations would need of moving meshes or a fictitious domain approach (see for instance [39]).

Instead, we have chosen a formulation in terms of the magnetic field. This leads to a parabolic problem for which, following ideas from [59], we obtained an existence result for its weak formulation; however, the uniqueness is still an open question.

Next, we introduce a numerical method to solve this parabolic problem. It is based on Nédélec finite element for the space discretization and a backward Euler scheme for time discretization; the curl-free constraint in the dielectric domain is imposed by means of a penalty strategy (see, for instance [38, Section I.4.3]). This approach applied to the eddy current model corresponds in some sense to replace the dielectric domain by a very poor conductor ([37]). Let us emphasize that, in spite of the fact that the dielectric domain change along time, the proposed numerical method does not need of moving meshes; namely, the mesh remains the same at all time steps.

In the case of fixed domains we have proved convergence of the penalized continuous problem as the penalization parameter goes to zero. We also proved this result for the discrete problem uniformly in the discretization parameters. Moreover, we obtain error estimates for the convergence of the discrete penalized problem with respect to the penalization and the discretization parameters. We also report some numerical test which allows us to assess the performance of this approach.

We do not have a convergence analysis for the numerical method applied to a problem with a moving workpiece. However, we report very promissory numerical results obtained in some test problems. This includes an EMF problem on a cylindrical geometry, which allows us to compare our results with those obtained with the axisymmetric code from [16].

The outline of the chapter is as follows: in Section 4.1 we introduce the transient eddy current model and state the geometrical framework for the analysis. In Section 4.2 we apply the abstract results from [59] to prove an existence result. In Section 4.3 we introduce a penalty method to approximate the $\boldsymbol{H}$-formulation of the eddy current model in the case of fixed conducting parts. We prove convergence results for the continuous penalty method and propose a fully discrete scheme to approximate the corresponding problem. Convergence results are also provided in this discrete case. Moreover, we report in this section the results of a numerical test which confirms the convergence of the penalized approach. In Section 4.4 we describe the penalty technique to approximate the $\boldsymbol{H}$-formulation with moving conductors. We detail some implementation issues to consider a fixed mesh along the time and we report the results of a couple of tests. The numerical results predict that we may expect a similar order of convergence as that proved for fixed domains.

4.1 Statement of the problem

Let us consider a workpiece whose position is changing along the time and a coil which carries a given transient current density $\boldsymbol{J}_{\mathrm{S}}$. Let Ω be a simply connected bounded three-dimensional domain with a Lipschitz continuous connected boundary which contains the coil, the workpiece and the air around. We assume that $\boldsymbol{J}_{\mathrm{S}}$ is supported in $\bar{\Omega}_{\mathrm{S}}$ where $\bar{\Omega}_{\mathrm{S}} \subset \Omega$. We are interested in computing the induced currents in the workpiece along the time. For each time t, we denote by Ω^t_{C} the domain of the workpiece at time t and we assume that $\Omega^t_{\mathrm{C}} \cap \Omega_{\mathrm{S}} = \emptyset$. We assume that Ω^t_{C} is connected and simply connected with connected boundary. We denote $\Omega^t_{\mathrm{A}} := \Omega \setminus (\bar{\Omega}^t_{\mathrm{C}} \cup \bar{\Omega}_{\mathrm{S}})$ the domain occupied by air and $\Omega^t_{\mathrm{NC}} := \Omega \setminus \bar{\Omega}^t_{\mathrm{C}}$. We notice that $\bar{\Omega}_{\mathrm{S}} \subset \Omega^t_{\mathrm{NC}}$, so that $\boldsymbol{J}_{\mathrm{S}}|_{\Omega^t_{\mathrm{C}}} = \boldsymbol{0}$ for all time t (see Figure 4.1).

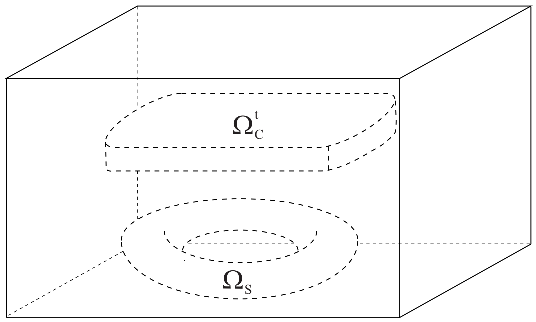

Figure 4.1: Sketch of the domain.

The problem to be solved is

$$\partial_t(\mu \boldsymbol{H}) + \mathbf{curl}\, \boldsymbol{E} = \boldsymbol{0} \quad \text{in } [0,T] \times \Omega, \tag{4.1}$$

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} = \sigma \boldsymbol{E} + \boldsymbol{J}_{\mathrm{S}} \quad \text{in } [0,T] \times \Omega, \tag{4.2}$$

where $\boldsymbol{E}(t,\boldsymbol{x})$ is the electric field, $\boldsymbol{H}(t,\boldsymbol{x})$ the magnetic field, $\boldsymbol{J}(t,\boldsymbol{x})$ the current density, μ the magnetic permeability and σ the electric conductivity. Here and thereafter, we use boldface letters to denote vector fields and variables as well as vector-valued operators.

We assume that $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega))$. Since $\boldsymbol{J}_{\mathrm{S}}$ is supported in $\bar{\Omega}_{\mathrm{S}}$, this implies that $\boldsymbol{J}_{\mathrm{S}}|_{\Omega_{\mathrm{S}}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$, too. On the other hand, we consider that there exist constants $\underline{\mu}$, $\overline{\mu}$, $\overline{\sigma}$ and $\underline{\sigma}$ such that

$$\begin{aligned} &0 < \underline{\mu} \le \mu(t,\boldsymbol{x}) \le \overline{\mu}, \quad \text{a.e. } \boldsymbol{x} \in \Omega, \quad \text{a.e. } t \in [0,T], \\ &0 < \underline{\sigma} \le \sigma(t,\boldsymbol{x}) \le \overline{\sigma}, \quad \text{a.e. } \boldsymbol{x} \in \Omega^t_{\mathrm{C}} \quad \text{and} \quad \sigma \equiv 0 \text{ in } \Omega^t_{\mathrm{NC}}, \quad \text{a.e. } t \in [0,T]. \end{aligned}$$

Moreover, we denote by μ_0 the constant magnetic permeability of vacuum.

We add the initial condition

$$\boldsymbol{H}(0,\boldsymbol{x}) = \boldsymbol{H}_0(\boldsymbol{x}) \quad \text{in } \Omega,$$

where $\boldsymbol{H}_0 \in \mathrm{H}(\mathbf{curl};\Omega)$ satisfies $\mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{J}_{\mathrm{S}}(0)$ in Ω^0_{NC}.

The model (4.1)–(4.2) is completed in the next section with suitable boundary conditions. In particular, we will work with boundary conditions which guarantee the well-posedness of the problem in the case of fixed conductors.

4.2 $\boldsymbol{H}$-formulation

Our goal is to give a variational formulation, based on the magnetic field, of the transient eddy current model (4.1)–(4.2) with appropriate boundary conditions and in the presence of moving conductors.

The transient eddy current model (4.1)–(4.2) defined in the whole space $\mathbb{R}^3$ with fixed conductors has been studied in [49] by using the magnetic field as the main unknown. The results of this reference can be easily adapted to bounded domains by using as boundary condition $\boldsymbol{H}\times\boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$ (see Subsection 4.3.1). These are the conditions that we are going to consider in our problem with moving conductors, too; namely,

$$\begin{aligned} \partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} &= \mathbf{0} \quad \text{in } [0,T]\times\Omega,\\ \mathbf{curl}\,\boldsymbol{H} &= \sigma\boldsymbol{E} + \boldsymbol{J}_{\mathrm{S}} \quad \text{in } [0,T]\times\Omega,\\ \boldsymbol{H}\times\boldsymbol{n} &= \mathbf{0} \quad \text{on } [0,T]\times\partial\Omega. \end{aligned}$$

Since $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$, there exists $\widehat{\boldsymbol{H}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ such that $\mathbf{curl}\,\widehat{\boldsymbol{H}}(t) = \boldsymbol{J}_{\mathrm{S}}(t)$ in Ω, $t\in[0,T]$ and there exists $C>0$ such that $\|\widehat{\boldsymbol{H}}\|_{\mathrm{H}^1(0,T;\mathrm{H}(\mathbf{curl};\Omega))} \le C\|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}$ ([38, Theorem I.3.6]). Notice that, in particular, $\widehat{\boldsymbol{H}}(0)$ satisfies $\mathbf{curl}\,\widehat{\boldsymbol{H}}(0) = \boldsymbol{J}_{\mathrm{S}}(0)$ in Ω^0_{NC}.

Now, if we write $\boldsymbol{H} = \widetilde{\boldsymbol{H}} + \widehat{\boldsymbol{H}}$, we are led to the equations

$$\partial_t(\mu\widetilde{\boldsymbol{H}}) + \mathbf{curl}\,\boldsymbol{E} = -\partial_t(\mu\widehat{\boldsymbol{H}}) \quad \text{in } [0,T]\times\Omega, \tag{4.3}$$

$$\mathbf{curl}\,\widetilde{\boldsymbol{H}} = \sigma\boldsymbol{E} \quad \text{in } [0,T]\times\Omega, \tag{4.4}$$

$$\widetilde{\boldsymbol{H}}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } [0,T]\times\partial\Omega, \tag{4.5}$$

with initial condition $\widetilde{\boldsymbol{H}}(0) := \boldsymbol{H}_0 - \widehat{\boldsymbol{H}}(0)$.

For each $t\in[0,T]$, let

$$\boldsymbol{\mathcal{Y}}^t := \left\{\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega)\,:\,\mathbf{curl}\,\boldsymbol{G} = \mathbf{0} \text{ in } \Omega^t_{\mathrm{NC}}\right\}$$

and let

$$\mathrm{L}^2\left(0,T;\boldsymbol{\mathcal{Y}}^t\right) := \left\{\boldsymbol{G}\in\mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))\,:\,\boldsymbol{G}(t)\in\boldsymbol{\mathcal{Y}}^t\right\}.$$

The latter is a closed subspace of $\mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ and hence a Hilbert space (cf. [59]).

Notice that because of (4.4), $\widetilde{\boldsymbol{H}}\in\mathrm{L}^2\left(0,T;\boldsymbol{\mathcal{Y}}^t\right)$. By testing (4.3) with a $\boldsymbol{G}\in\mathrm{L}^2\left(0,T;\boldsymbol{\mathcal{Y}}^t\right)$, we have that

$$\int_\Omega \partial_t(\mu\widetilde{\boldsymbol{H}})\cdot\boldsymbol{G} + \int_{\Omega^t_{\mathrm{C}}} \boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{G} = -\int_\Omega \partial_t(\mu\widehat{\boldsymbol{H}})\cdot\boldsymbol{G} \quad \text{a.e. } t\in[0,T].$$

Next, using (4.4) we have that

$$\int_\Omega \partial_t(\mu\widetilde{\boldsymbol{H}})\cdot\boldsymbol{G} + \int_{\Omega^t_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}\cdot\mathbf{curl}\,\boldsymbol{G} = -\int_\Omega \partial_t(\mu\widehat{\boldsymbol{H}})\cdot\boldsymbol{G} \quad \text{a.e. } t\in[0,T]. \tag{4.6}$$

Let $f\in \mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega)')$ be defined by

$$\langle f(t),\boldsymbol{G}\rangle := -\int_\Omega \partial_t(\mu\widehat{\boldsymbol{H}})\cdot\boldsymbol{G} \qquad \forall\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathrm{H}_0(\mathbf{curl};\Omega)$ and $\mathrm{H}_0(\mathbf{curl};\Omega)'$.

Let $a(t;\cdot,\cdot)$ be the continuous bilinear form defined in $\mathrm{H}_0(\mathbf{curl};\Omega)\times\mathrm{H}_0(\mathbf{curl};\Omega)$ by

$$a(t;\widetilde{\boldsymbol{G}},\boldsymbol{G}) := \int_{\Omega^t_{\mathrm{C}}} \frac{1}{\sigma(t)}\,\mathbf{curl}\,\widetilde{\boldsymbol{G}}\cdot\mathbf{curl}\,\boldsymbol{G}$$

and $c(t,\cdot,\cdot)$ be the continuous bilinear form defined in $\mathrm{L}^2(\Omega)^3\times\mathrm{L}^2(\Omega)^3$ by

$$c(t;\widetilde{\boldsymbol{G}},\boldsymbol{G}) := \int_\Omega \mu(t)\,\widetilde{\boldsymbol{G}}\cdot\boldsymbol{G}.$$

In case that $\mu(t,\boldsymbol{x})=\mu_0$ in the whole domain Ω, integrating by parts in time the first term in (4.6) lead us to the following problem:

Problem 4.1 *Find* $\widetilde{\boldsymbol{H}}\in\mathrm{L}^2\left(0,T;\boldsymbol{\mathcal{Y}}^t\right)$ *such that*

$$\int_0^T a(t;\widetilde{\boldsymbol{H}}(t),\boldsymbol{G}(t))\,dt - \int_0^T c(t;\widetilde{\boldsymbol{H}}(t),\partial_t\boldsymbol{G}(t))\,dt = \int_0^T \langle f(t),\boldsymbol{G}(t)\rangle\,dt + c(0;\widetilde{\boldsymbol{H}}(0),\boldsymbol{G}(0))$$

for all $\boldsymbol{G}\in\mathrm{L}^2\left(0,T;\boldsymbol{\mathcal{Y}}^t\right)\cap\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ *with* $\boldsymbol{G}(T)=0$.

Theorem 4.2 *If* $\mu(t,\boldsymbol{x})=\mu_0$, $t\in[0,T]$, $x\in\Omega$, *then Problem 4.1 has a solution.*

Proof. The result is an immediate application of Theorem 1 from [59]. In what follows we check the hypothesis of this theorem:

1. $\boldsymbol{\mathcal{Y}}^t\subset\mathrm{H}_0(\mathbf{curl};\Omega)\subset\mathrm{L}^2(\Omega)^3$.

2. $\{c(t;\cdot,\cdot)\ :\ t\in[0,T]\}$ is a regular family on $\mathrm{L}^2(\Omega)^3$; namely, for all $\widetilde{\boldsymbol{G}},\,\boldsymbol{G}\in\mathrm{L}^2(\Omega)^3$, the function $t\to c(t;\widetilde{\boldsymbol{G}},\boldsymbol{G})$ is absolutely continuous in $[0,T]$ and there exists $M\in\mathrm{L}^1(0,T)$ such that for all $\widetilde{\boldsymbol{G}},\,\boldsymbol{G}\in\mathrm{H}(\mathbf{curl};\Omega)$ we have

$$|c'(t;\widetilde{\boldsymbol{G}},\boldsymbol{G})|\le M(t)\|\widetilde{\boldsymbol{G}}\|_{\mathrm{H}(\mathbf{curl};\Omega)}\,\|\boldsymbol{G}\|_{\mathrm{H}(\mathbf{curl};\Omega)}, \quad \text{a.e. } t\in[0,T].$$

   Both properties clearly hold in our case under the assumption $\mu(t,\boldsymbol{x})=\mu_0$.

3. $c(0;\boldsymbol{G},\boldsymbol{G})\ge 0\ \forall\boldsymbol{G}\in\mathrm{L}^2(\Omega)^3$.

4. $\forall\lambda\in\mathbb{R}$, $\lambda>0$ there exists $\alpha>0$, such that

$$\lambda\,c(t;\boldsymbol{G},\boldsymbol{G})+c'(t;\boldsymbol{G},\boldsymbol{G})+a(t;\boldsymbol{G},\boldsymbol{G})\ge\alpha\|\boldsymbol{G}\|_{\mathrm{H}(\mathbf{curl};\Omega)} \quad \forall\boldsymbol{G}\in\boldsymbol{\mathcal{Y}}^t, \quad \text{a.e. } t\in[0,T],$$

   which clearly holds in our case.

Thus, we conclude the proof. ■

However, we cannot apply Theorem 2 from [59] to conclude that Problem 4.1 has a unique solution, because two of its hypothesis are not fulfilled. On one side, $\{a(t;\cdot,\cdot)\,:\,t\in[0,T]\}$ should be a regular family on $\boldsymbol{\mathcal{Y}}^t$. For this property to hold, apparently we would need a Reynold's transport formula for $|\operatorname{\mathbf{curl}}\boldsymbol{G}|^2$ with $\boldsymbol{G}\in\boldsymbol{\mathcal{Y}}^t\subset \mathrm{H}_0(\operatorname{\mathbf{curl}};\Omega)$. Recently, a version of the Reynold's transport formula for functions with reduced smoothness was proved in [15]. However, it seems mandatory that the integrand (in our case $|\operatorname{\mathbf{curl}}\boldsymbol{G}|^2$) be in $\mathrm{W}^{1,1}$, which is not our case. On the other hand, another hypothesis of Theorem 2 from [59] is that the family of spaces $\boldsymbol{\mathcal{Y}}^t$ has to be decreasing in the sense that

$$t>s,\ t,s\in[0,T]\Rightarrow \boldsymbol{\mathcal{Y}}^t\subset\boldsymbol{\mathcal{Y}}^s.$$

This would hold in our case only if the workpiece shrinked without other motion, which would be a rather particular case. Thus, the uniqueness of the solution to Problem 4.1 remains an open question.

In the following sections, we will focus on the numerical solution of the problem and we will propose a penalty approach. With this aim, we first analyze this technique in fixed domains.

4.3 A penalized $\boldsymbol{H}$-formulation in fixed domains

In this section we will introduce a penalty method to approximate the $\boldsymbol{H}$-formulation of the eddy current model in the case of fixed conducting parts. The main goal is to show that the approximation by penalty works in this case to exploit the idea of its easy implementation in the case of moving conductors. Thus, we proceed as follows; first, we describe the penalty method and the convergence results for the continuous problem; next, we prove convergence results for the fully discrete problem and finally, we illustrate the convergence by reporting some numerical results.

4.3.1 Continuous penalty formulation

In this section we suppose that the conducting workpiece occupies a fixed position which is denoted by Ω_{C}. The complementary set of Ω_{C} in Ω is denoted by Ω_{NC}, that is, $\Omega_{\mathrm{NC}}:=\Omega\setminus\Omega_{\mathrm{C}}$. We also assume that μ, σ and ϵ are time-independent. Let us introduce the following space

$$\boldsymbol{\mathcal{Y}}:=\left\{\boldsymbol{G}\in\mathrm{H}_0(\operatorname{\mathbf{curl}};\Omega)\,:\,\operatorname{\mathbf{curl}}\boldsymbol{G}=\boldsymbol{0}\text{ in }\Omega_{\mathrm{NC}}\right\}.$$

Thus, the problem to be solved is

Problem 4.3 *Find* $\boldsymbol{H}\in\mathrm{L}^2(0,T;\mathrm{H}_0(\operatorname{\mathbf{curl}};\Omega))\cap\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ *such that*

$$\begin{aligned}
\operatorname{\mathbf{curl}}\boldsymbol{H}&=\boldsymbol{J}_{\mathrm{S}} \quad \text{in } \Omega_{\mathrm{NC}},\\
\int_\Omega\mu\partial_t\boldsymbol{H}\cdot\boldsymbol{G}+\int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\operatorname{\mathbf{curl}}\boldsymbol{H}\cdot\operatorname{\mathbf{curl}}\boldsymbol{G}&=0 \quad \forall\boldsymbol{G}\in\boldsymbol{\mathcal{Y}},\\
\boldsymbol{H}(0)&=\boldsymbol{H}_0.
\end{aligned}$$

Theorem 4.4 *Let* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$ *and* $\boldsymbol{H}_0 \in \mathrm{H}_0(\mathbf{curl};\Omega)$ *satisfying* $\mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{J}_{\mathrm{S}}(0)$ *in* Ω_{NC}. *Then, Problem 4.3 has a unique solution. Furthermore,* $\boldsymbol{H} \in \mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ *and there exist* $C>0$ *independent* $\boldsymbol{H}_0$ *and* $\boldsymbol{J}_{\mathrm{S}}$ *such that*

$$\|\boldsymbol{H}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))} + \|\partial_t \boldsymbol{H}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)} \le C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}. \tag{4.7}$$

Proof. The proof follows lines similar to those developed in [14], Section 3.2 (or Section 2.3, this thesis). Let $\mathcal{H}_{\boldsymbol{\mathcal{Y}}}$ be the closure of $\boldsymbol{\mathcal{Y}}$ in $\mathrm{L}^2(\Omega)^3$ characterized by $\mathcal{H}_{\boldsymbol{\mathcal{Y}}} := \left\{\boldsymbol{G} \in \mathrm{L}^2(\Omega)^3 : \mathbf{curl}\,\boldsymbol{G} = \boldsymbol{0} \text{ in } \Omega_{\mathrm{NC}}\right\}$ (see [14, Lemma 3.2] or Lemma 2.2, this thesis). Since $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$ there exists $\widehat{\boldsymbol{H}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ such that $\mathbf{curl}\,\widehat{\boldsymbol{H}} = \boldsymbol{J}_{\mathrm{S}}$ in Ω and there exists $C>0$ independent of $\boldsymbol{J}_{\mathrm{S}}$ such that

$$\|\widehat{\boldsymbol{H}}\|_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))} \le C\|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}. \tag{4.8}$$

We consider the following problem: find $\widetilde{\boldsymbol{H}} \in \mathrm{L}^2(0,T;\boldsymbol{\mathcal{Y}}) \cap \mathrm{H}^1(0,T;\mathcal{H}_{\boldsymbol{\mathcal{Y}}})$ such that

$$\int_\Omega \mu\partial_t\widetilde{\boldsymbol{H}}\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\widetilde{\boldsymbol{H}}\cdot\mathbf{curl}\,\boldsymbol{G} = -\int_\Omega \mu\partial_t\widehat{\boldsymbol{H}}\cdot\boldsymbol{G} \qquad \forall \boldsymbol{G}\in\boldsymbol{\mathcal{Y}},$$

$$\widetilde{\boldsymbol{H}}(0) = \boldsymbol{H}_0 - \widehat{\boldsymbol{H}}(0).$$

As a consequence of [32, Theorem 5, Chapter 7] this problem has a unique solution $\widetilde{\boldsymbol{H}}$ that satisfies $\widetilde{\boldsymbol{H}} \in \mathrm{L}^\infty(0,T;\boldsymbol{\mathcal{Y}})$ and there exists $C>0$ independent of $\boldsymbol{H}_0$ and $\boldsymbol{J}_{\mathrm{S}}$ such that

$$\|\widetilde{\boldsymbol{H}}\|^2_{\mathrm{L}^\infty(0,T;\boldsymbol{\mathcal{Y}})} + \|\partial_t\widetilde{\boldsymbol{H}}\|^2_{\mathrm{L}^2(0,T;\mathcal{H}_{\boldsymbol{\mathcal{Y}}})} \le C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}.$$

Now, if we write $\boldsymbol{H} := \widetilde{\boldsymbol{H}} + \widehat{\boldsymbol{H}}$ then $\boldsymbol{H} \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$, because $\partial_t\widetilde{\boldsymbol{H}} \in \mathrm{L}^2(0,T;\mathcal{H}_{\boldsymbol{\mathcal{Y}}}) \subset \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)$ and $\widehat{\boldsymbol{H}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$. Thus, $\boldsymbol{H}$ is a solution to Problem 4.3. Moreover $\boldsymbol{H} \in \mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ and as a consequence of the previous inequality and (4.8) we obtain the estimate of the theorem. The uniqueness of the solution is an immediate consequence of this estimate. ■

The constraint $\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}_{\mathrm{S}}$ in Ω_{NC} can be imposed in a weak form as in the following problem.

Problem 4.5 *Find* $\boldsymbol{H} \in \mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ *and* $\boldsymbol{K} \in \mathrm{L}^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))$ *such that*

$$\int_\Omega \mu\partial_t\boldsymbol{H}\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{G} + \int_{\Omega_{\mathrm{NC}}}\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{K} = 0 \qquad \forall\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega),$$

$$\int_{\Omega_{\mathrm{NC}}}\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{F} = \int_{\Omega_{\mathrm{NC}}}\boldsymbol{J}_{\mathrm{S}}\cdot\boldsymbol{F} \qquad \forall\boldsymbol{F}\in\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}),$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0.$$

Next, we prove that Problem 4.5 is actually equivalent to Problem 4.3.

Lemma 4.6 *Given* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$ *and* $\boldsymbol{H}_0 \in \mathrm{H}_0(\mathbf{curl};\Omega)$ *satisfying* $\mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{J}_{\mathrm{S}}(0)$ *in* Ω_{NC}, *let* $\boldsymbol{H}$ *be the solution of Problem 4.3. Then there exists* $\boldsymbol{K} \in \mathrm{L}^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))$ *such that* $(\boldsymbol{H},\boldsymbol{K})$ *is the unique solution of Problem 4.5. Moreover, there exists* $C>0$ *independent of* $\boldsymbol{H}_0$ *and* $\boldsymbol{J}_{\mathrm{S}}$, *such that*

$$\|\boldsymbol{K}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))} \le C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}. \tag{4.9}$$

Proof. Let $\boldsymbol{H}$ be the solution of Problem 4.3. Then we define $h : [0,T] \to \mathrm{H}_0(\mathbf{curl};\Omega)'$ by

$$\langle h(t), \boldsymbol{G}\rangle := -\int_\Omega \mu\partial_t \boldsymbol{H}\cdot\boldsymbol{G} - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{G}$$

and we have that $h \in \mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega)')$ and

$$\langle h(t), \boldsymbol{G}\rangle = 0 \quad \forall \boldsymbol{G} \in \mathrm{H}_0(\mathbf{curl};\Omega). \tag{4.10}$$

On the other hand, it is easy to show that the bilinear form $b : \mathrm{H}_0(\mathbf{curl};\Omega)\times\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}) \to \mathbb{R}$ defined by $b(\boldsymbol{G},\boldsymbol{F}) = \int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{F}$ satisfies the following *inf-sup* condition (see, for instance [5, Lemma 5.3])

$$\sup_{\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega)} \frac{b(\boldsymbol{G},\boldsymbol{F})}{\|\boldsymbol{G}\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \geq \beta\|\boldsymbol{F}\|_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3} \qquad \forall \boldsymbol{F} \in \mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}).$$

Consequently, for each $t \in [0,T]$, because of (4.10) there exists a unique $\boldsymbol{K}(t) \in \mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}})$ such that $b(\boldsymbol{G},\boldsymbol{K}(t)) = \langle h(t),\boldsymbol{G}\rangle$ $\forall \boldsymbol{G} \in \mathrm{H}_0(\mathbf{curl};\Omega)$ (see [38, Lemma I.4.1]). Hence, $(\boldsymbol{H},\boldsymbol{K})$ is a solution of Problem 4.5. There only remains to prove that this problem has at most one solution. With this aim consider $(\breve{\boldsymbol{H}},\breve{\boldsymbol{K}})$ a solution of Problem 4.5 with data $\boldsymbol{J}_{\mathrm{S}} = \boldsymbol{0}$ and $\boldsymbol{H}_0 = \boldsymbol{0}$. For each $t \in [0,T]$ we take $\boldsymbol{G} := \breve{\boldsymbol{H}}(t)$ and $\boldsymbol{F} := \breve{\boldsymbol{K}}(t)$, subtracting the resulting equations we obtain

$$\int_\Omega \mu\partial_t\breve{\boldsymbol{H}}(t)\cdot\breve{\boldsymbol{H}}(t) + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}|\,\mathbf{curl}\,\breve{\boldsymbol{H}}(t)|^2 = 0.$$

Since $\int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}|\,\mathbf{curl}\,\breve{\boldsymbol{H}}(t)|^2 \geq 0$, $\int_\Omega \mu\partial_t\breve{\boldsymbol{H}}(t)\cdot\breve{\boldsymbol{H}}(t) = \frac{1}{2}\frac{d}{dt}\|\breve{\boldsymbol{H}}(t)\|^2_{\mathrm{L}^2(\Omega)^3}$ and $\breve{\boldsymbol{H}}(0) = \boldsymbol{0}$, it follows that $\breve{\boldsymbol{H}}(t) = \boldsymbol{0}$. Finally, the *inf-sup* condition for b guarantees that $\breve{\boldsymbol{K}}(t) = \boldsymbol{0}$ and the estimate (4.9). ■

Our next step is to introduce a penalization to impose the constraint

$$\mathbf{curl}\,\boldsymbol{H}(t) = \boldsymbol{J}_{\mathrm{S}}(t) \quad \text{in } \Omega_{\mathrm{NC}} \quad \text{a.e. } t \in [0,T]$$

as follows:

$$\mathbf{curl}\,\boldsymbol{H}(t) - \boldsymbol{J}_{\mathrm{S}}(t) = \varepsilon\boldsymbol{E}(t) \quad \text{in } \Omega_{\mathrm{NC}} \quad \text{a.e. } t \in [0,T],$$

where ε is a positive parameter which will tend to 0. Therefore, for $0 < \varepsilon \leq 1$, we introduce the following penalized problem to approximate Problem 4.3.

Problem 4.7 *Find* $\boldsymbol{H}_\varepsilon \in \mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ *such that*

$$\int_\Omega \mu\partial_t\boldsymbol{H}_\varepsilon\cdot\boldsymbol{G} + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}_\varepsilon\cdot\mathbf{curl}\,\boldsymbol{G} + \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{NC}}}\mathbf{curl}\,\boldsymbol{H}_\varepsilon\cdot\mathbf{curl}\,\boldsymbol{G} = \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{NC}}}\boldsymbol{J}_{\mathrm{S}}\cdot\mathbf{curl}\,\boldsymbol{G} \qquad \forall \boldsymbol{G} \in \mathrm{H}_0(\mathbf{curl};\Omega),$$

$$\boldsymbol{H}_\varepsilon(0) = \boldsymbol{H}_0.$$

By applying Corollary A.2 from [14] (cf. Appendix–Chapter 2, this thesis), we conclude that Problem 4.7 has a unique solution that satisfies $\boldsymbol{H}_\varepsilon \in \mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))$ and there exists a constant $C>0$, independent of ε, $\boldsymbol{J}_{\mathrm{S}}$ and $\boldsymbol{H}_0$ such that

$$\|\boldsymbol{H}_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))} + \|\partial_t \boldsymbol{H}_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)} \leq C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}.$$

To prove that $\boldsymbol{H}_\varepsilon \to \boldsymbol{H}$ when $\varepsilon \to 0$ we also introduce the equivalent mixed variational formulation of Problem 4.7.

Problem 4.8 *Find* $\boldsymbol{H}_\varepsilon \in \mathrm{L}^2(0,T;\mathrm{H}_0(\mathbf{curl};\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)^3)$ *and* $\boldsymbol{K}_\varepsilon \in \mathrm{L}^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))$ *such that*

$$\begin{aligned}
&\int_\Omega \mu \partial_t \boldsymbol{H}_\varepsilon \cdot \boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_\varepsilon \cdot \mathbf{curl}\, \boldsymbol{G} + \int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{K}_\varepsilon = 0 && \forall \boldsymbol{G} \in \mathrm{H}_0(\mathbf{curl};\Omega),\\
&\int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\, \boldsymbol{H}_\varepsilon \cdot \boldsymbol{F} - \varepsilon \int_{\Omega_{\mathrm{NC}}} \boldsymbol{K}_\varepsilon \cdot \boldsymbol{F} = \int_{\Omega_{\mathrm{NC}}} \boldsymbol{J}_{\mathrm{S}} \cdot \boldsymbol{F} && \forall \boldsymbol{F} \in \mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}),\\
&\boldsymbol{H}_\varepsilon(0) = \boldsymbol{H}_0.
\end{aligned}$$

Theorem 4.9 *Given* $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$ *and* $\boldsymbol{H}_0 \in \mathrm{H}_0(\mathbf{curl};\Omega)$ *satisfying* $\mathbf{curl}\, \boldsymbol{H}_0 = \boldsymbol{J}_{\mathrm{S}}(0)$ *in* Ω_{NC} *and* $0 < \varepsilon \leq 1$. *Problem 4.8 has a unique solution and there exists* $C>0$ *independent of* ε, $\boldsymbol{H}_0$ *and* $\boldsymbol{J}_{\mathrm{S}}$, *such that*

$$\begin{aligned}
&\|\boldsymbol{H}_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}_0(\mathbf{curl};\Omega))} + \|\partial_t \boldsymbol{H}_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)}\\
&\qquad + \|\boldsymbol{K}_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))} \leq C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}.
\end{aligned} \tag{4.11}$$

Moreover, the following error estimate holds for $\varepsilon \leq \varepsilon_0$ *small enough*

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|\boldsymbol{H}(t) - \boldsymbol{H}_\varepsilon(t)\|_{\mathrm{L}^2(\Omega)^3} + \|\,\mathbf{curl}\, \boldsymbol{H} - \mathbf{curl}\, \boldsymbol{H}_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)}\\
&\qquad + \sqrt{\varepsilon}\|\boldsymbol{K} - \boldsymbol{K}_\varepsilon\|_{L^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))} \leq \sqrt{\varepsilon}\, C\left\{\|\boldsymbol{H}_0\|_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}.
\end{aligned} \tag{4.12}$$

Proof. The proof follows the lines of that of Theorem I.4.3 from [38]. First, we show that Problem 4.8 has a solution; namely, for $0 < \varepsilon \leq 1$ let $\boldsymbol{H}_\varepsilon$ be a solution to Problem 4.7. By taking $\boldsymbol{K}_\varepsilon := \frac{1}{\varepsilon}\big(\mathbf{curl}\, \boldsymbol{H}_\varepsilon - \boldsymbol{J}_{\mathrm{S}}\big)$, $(\boldsymbol{H}_\varepsilon, \boldsymbol{K}_\varepsilon)$ is solution to Problem 4.8. Now, by taking $\boldsymbol{G} := \boldsymbol{H}_\varepsilon(t)$ and $\boldsymbol{F} := \boldsymbol{K}_\varepsilon(t)$ in Problem 4.8 we obtain the estimate (4.11) which also leads to the uniqueness of the solution. Secondly, we prove the error estimate (4.12). To attain this goal, we subtract Problem 4.8 and Problem 4.5 to obtain

$$\int_\Omega \mu \partial_t (\boldsymbol{H}_\varepsilon - \boldsymbol{H}) \cdot \boldsymbol{G} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}(\boldsymbol{H}_\varepsilon - \boldsymbol{H}) \cdot \mathbf{curl}\, \boldsymbol{G} + \int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\, \boldsymbol{G} \cdot (\boldsymbol{K}_\varepsilon - \boldsymbol{K}) = 0 \qquad \forall \boldsymbol{G} \in \mathrm{H}_0(\mathbf{curl};\Omega), \tag{4.13}$$

$$\int_{\Omega_{\mathrm{NC}}} \mathbf{curl}(\boldsymbol{H}_\varepsilon - \boldsymbol{H}) \cdot \boldsymbol{F} - \varepsilon \int_{\Omega_{\mathrm{NC}}} \boldsymbol{K}_\varepsilon \cdot \boldsymbol{F} = 0 \qquad \forall \boldsymbol{F} \in \mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}), \tag{4.14}$$

$$\boldsymbol{H}_\varepsilon(0) - \boldsymbol{H}(0) = \mathbf{0}. \tag{4.15}$$

As a consequence of the *inf-sup* condition proved above for b, we have

$$\beta\|\boldsymbol{K}_\varepsilon(t)-\boldsymbol{K}(t)\|_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3} \leq \sup_{\boldsymbol{G}\in\mathrm{H}_0(\mathbf{curl};\Omega)} \frac{b(\boldsymbol{G},\boldsymbol{K}_\varepsilon(t)-\boldsymbol{K}(t))}{\|\boldsymbol{G}\|_{\mathrm{H}(\mathbf{curl};\Omega)}}$$
$$\leq C\left\{\|\partial_t(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))\|^2_{\mathrm{L}^2(\Omega)^3}+\|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\right\}^{1/2}$$

thus

$$\|\boldsymbol{K}_\varepsilon-\boldsymbol{K}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))} \leq C\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3}+\|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}. \tag{4.16}$$

On the other hand, by taking $\boldsymbol{G}:=\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t)$ and $\boldsymbol{F}:=\boldsymbol{K}_\varepsilon(t)-\boldsymbol{K}(t)$ in (4.13) and (4.14), and subtracting the resulting expression we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mu|\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t)|^2+\int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))|^2+\varepsilon\int_{\Omega_{\mathrm{NC}}}\boldsymbol{K}_\varepsilon(t)\cdot(\boldsymbol{K}_\varepsilon(t)-\boldsymbol{K}(t))=0.$$

By adding and subtracting the quantity $\varepsilon\int_{\Omega_{\mathrm{NC}}}\boldsymbol{K}(t)\cdot(\boldsymbol{K}_\varepsilon(t)-\boldsymbol{K}(t))$ and since $\varepsilon\int_{\Omega_{\mathrm{NC}}}|\boldsymbol{K}_\varepsilon(t)-\boldsymbol{K}(t)|^2>0$ for all ε we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mu|\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t)|^2+\int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))|^2$$
$$\leq\frac{\varepsilon}{2}\left(\|\boldsymbol{K}(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}+\|\boldsymbol{K}_\varepsilon(t)-\boldsymbol{K}(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}\right).$$

By taking $\boldsymbol{F}:=\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))-\varepsilon\boldsymbol{K}_\varepsilon(t)$ in (4.14) we obtain that $\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))=\varepsilon\boldsymbol{K}_\varepsilon(t)$ in Ω_{NC} and hence $\|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}=\varepsilon^2\|\boldsymbol{K}_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}$. Furthermore, we know that there exists $\alpha>0$ such that

$$\int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))|^2+\int_{\Omega_{\mathrm{NC}}}|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))|^2\geq\alpha\|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))\|^2_{\mathrm{L}^2(\Omega)^3},$$

therefore

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mu|\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t)|^2+\int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))|^2+\int_{\Omega_{\mathrm{NC}}}|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))|^2$$
$$\geq\frac{1}{2}\underline{\mu}\frac{d}{dt}\|\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t)\|^2_{\mathrm{L}^2(\Omega)^3}+\alpha\|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))\|^2_{\mathrm{L}^2(\Omega)^3}$$

and we conclude that

$$\frac{1}{2}\underline{\mu}\frac{d}{dt}\|\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t)\|^2_{\mathrm{L}^2(\Omega)^3}+\alpha\|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t))\|^2_{\mathrm{L}^2(\Omega)^3}$$
$$\leq\frac{\varepsilon}{2}\left(\|\boldsymbol{K}(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}+\|\boldsymbol{K}_\varepsilon(t)-\boldsymbol{K}(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}\right)+\alpha\varepsilon^2\|\boldsymbol{K}_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}.$$

Integrating in time we obtain

$$\frac{1}{2}\underline{\mu}\|\boldsymbol{H}_\varepsilon(t)-\boldsymbol{H}(t)\|^2_{\mathrm{L}^2(\Omega)^3}+\alpha\int_0^t\|\,\mathbf{curl}(\boldsymbol{H}_\varepsilon(s)-\boldsymbol{H}(s))\|^2_{\mathrm{L}^2(\Omega)^3}\,ds$$
$$\leq\frac{\varepsilon}{2}\left(\|\boldsymbol{K}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))}+\|\boldsymbol{K}_\varepsilon-\boldsymbol{K}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))}\right)+\alpha\varepsilon^2\|\boldsymbol{K}_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}_{\partial\Omega}(\mathrm{div}^0;\Omega_{\mathrm{NC}}))}.$$

Using (4.9), (4.16), (4.11) and the fact that when ε is sufficiently small $\varepsilon^2 < \varepsilon$, we conclude that there exists $C > 0$, independent of ε, $\boldsymbol{J}_{\mathrm{S}}$ and $\boldsymbol{H}_0$, such that

$$\sup_{0\le t\le T} \|\boldsymbol{H}(t) - \boldsymbol{H}_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega)^3} \le \varepsilon C \left\{ \|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))} \right\},$$
$$\|\operatorname{\mathbf{curl}}\boldsymbol{H} - \operatorname{\mathbf{curl}}\boldsymbol{H}_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^3)} \le \varepsilon C \left\{ \|\boldsymbol{H}_0\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))} \right\}.$$

Thus, the last estimate of the theorem follows from these inequalities and (4.16). ∎

4.3.2 Fully discrete penalty formulation

From now on, we assume that Ω, Ω_{S}, Ω_{C}, and Ω_{NC} are Lipschitz polyhedra and consider regular tetrahedral meshes $\mathcal{T}_h$ of Ω, such that each element $K \in \mathcal{T}_h$ is contained either in Ω_{S}, or in Ω_{C} or in $\Omega \setminus (\Omega_{\mathrm{S}} \cup \Omega_{\mathrm{C}})$ (h stands as usual for the corresponding mesh-size). We employ edge finite elements to approximate the magnetic field, more precisely, lowest-order Nédélec finite elements:

$$\boldsymbol{\mathcal{N}}_h(\Omega) := \{\boldsymbol{G}_h \in \mathrm{H}(\operatorname{\mathbf{curl}};\Omega) : \ \boldsymbol{G}_h|_K \in \boldsymbol{\mathcal{N}}(K) \ \forall K \in \mathcal{T}_h\}.$$

The magnetic field is approximated in each tetrahedron K by a polynomial vector field in the space

$$\boldsymbol{\mathcal{N}}(K) := \left\{\boldsymbol{G}_h \in \mathbb{P}_1^3 : \ \boldsymbol{G}_h(\boldsymbol{x}) = \mathbf{a}\times\boldsymbol{x} + \mathbf{b}, \ \mathbf{a},\mathbf{b}\in\mathbb{R}^3, \ \boldsymbol{x}\in K\right\}.$$

We introduce the following discrete spaces

$$\boldsymbol{\mathcal{N}}_h^0(\Omega) := \{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega) : \ \boldsymbol{G}_h \times \boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\} \subset \mathrm{H}_0(\operatorname{\mathbf{curl}};\Omega),$$
$$\boldsymbol{\mathcal{Y}}_h := \left\{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega) : \operatorname{\mathbf{curl}}\boldsymbol{G}_h = \mathbf{0} \text{ in } \Omega_{\mathrm{NC}}\right\} \subset \boldsymbol{\mathcal{Y}}.$$

We consider a uniform partition of $[0,T]$, $t_k := k\Delta t$, $k = 0,\dots,M$, with time step $\Delta t := \frac{T}{M}$. A fully discrete approximation based on an backward Euler scheme of Problem 4.5 reads as follows:

Problem 4.10 *Find $\boldsymbol{H}_h^m \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and $\boldsymbol{K}_h^m \in \operatorname{\mathbf{curl}}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right)$, $m = 1,\dots,M$, such that*

$$\int_\Omega \mu \frac{\boldsymbol{H}_h^m - \boldsymbol{H}_h^{m-1}}{\Delta t}\cdot\boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\operatorname{\mathbf{curl}}\boldsymbol{H}_h^m\cdot\operatorname{\mathbf{curl}}\boldsymbol{G}_h + \int_{\Omega_{\mathrm{NC}}}\operatorname{\mathbf{curl}}\boldsymbol{G}_h\cdot\boldsymbol{K}_h^m = 0 \qquad \forall\boldsymbol{G}_h\in\boldsymbol{\mathcal{N}}_h^0(\Omega),$$
$$\int_{\Omega_{\mathrm{NC}}}\operatorname{\mathbf{curl}}\boldsymbol{H}_h^m\cdot\boldsymbol{F}_h = \int_{\Omega_{\mathrm{NC}}}\boldsymbol{J}_{\mathrm{S}}(t_m)\cdot\boldsymbol{F}_h \qquad \forall\boldsymbol{F}_h\in\operatorname{\mathbf{curl}}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right),$$
$$\boldsymbol{H}_h^0 = \boldsymbol{H}_{0h},$$

where $\boldsymbol{H}_{0h} \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ is an approximation of $\boldsymbol{H}_0$.

Theorem 4.11 *For $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$, Problem 4.10 admits a unique solution $(\boldsymbol{H}_h^k,\boldsymbol{K}_h^k)$, $k = 1,\dots,M$. Moreover, there exists a constant $C > 0$, independent of h and Δt, such that*

$$\max_{1\le k\le M}\|\boldsymbol{H}_h^k\|^2_{\mathrm{H}(\operatorname{\mathbf{curl}};\Omega)} + \Delta t\sum_{k=1}^M\left\|\frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t\sum_{k=1}^M\left\|\boldsymbol{K}_h^k\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}$$
$$\le C\left\{\|\boldsymbol{H}_{0h}\|^2_{\mathrm{H}(\operatorname{\mathbf{curl}};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}.$$

Proof. Since the bilinear form b satisfies a discrete *inf-sup* condition (see [5, Lemma 5.3]), the theory of discrete mixed problems ensures the existence and uniqueness of the solution for all $k = 1, \dots, M$.

To prove that a priori estimate, let us define $h_h : [0,T] \to \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right)'$ by

$$\langle h_h(t), \boldsymbol{G}_h\rangle := \int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{J}_{\mathrm{S}}(t), \qquad \boldsymbol{G}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right),$$

then $h_h \in \mathrm{H}^1\left(0,T; \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right)'\right)$ and as a consequence of the discrete *inf-sup* condition, for each $t \in [0,T]$ there exists a unique $\widehat{\boldsymbol{H}}_h(t) \in \boldsymbol{\mathcal{Y}}_h^{\perp_{\boldsymbol{\mathcal{N}}_h^0(\Omega)}}$, $\widehat{\boldsymbol{H}}_h(t) \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ such that $b(\boldsymbol{G}_h, \widehat{\boldsymbol{H}}_h(t)) = \langle h_h(t), \boldsymbol{G}_h\rangle$ $\forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and a constant $\beta > 0$ independent of h such that $\beta\|\widehat{\boldsymbol{H}}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le \|\boldsymbol{J}_{\mathrm{S}}(t)\|_{\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}})}$. Therefore,

$$\|\widehat{\boldsymbol{H}}_h\|_{\mathrm{H}^1(0,T;\mathrm{H}(\mathbf{curl};\Omega))} \le C\|\boldsymbol{J}_{\mathrm{S}}\|_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}. \tag{4.17}$$

Now, if we write $\boldsymbol{H}_h^k = \widetilde{\boldsymbol{H}}_h^k + \widehat{\boldsymbol{H}}_h^k$ where $\widehat{\boldsymbol{H}}_h^k := \widehat{\boldsymbol{H}}_h(t_k)$, Problem 4.10 is equivalent to finding $\widetilde{\boldsymbol{H}}_h^k \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and $\boldsymbol{K}_h^k \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right)$, $k = 1, \dots, M$, such that

$$\begin{aligned}
&\int_\Omega \mu \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t}\cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^k \cdot \mathbf{curl}\,\boldsymbol{G}_h + \int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\,\boldsymbol{G}_h\cdot \boldsymbol{K}_h^k = \\
&\qquad -\int_\Omega \mu \frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t}\cdot \boldsymbol{G}_h - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\mathbf{curl}\,\widehat{\boldsymbol{H}}_h^k \cdot \mathbf{curl}\,\boldsymbol{G}_h \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega), &(4.18)\\
&\int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^k \cdot \boldsymbol{F}_h = 0 \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right), &(4.19)\\
&\widetilde{\boldsymbol{H}}_h^0 = \boldsymbol{H}_{0h} - \widehat{\boldsymbol{H}}_h(0). &(4.20)
\end{aligned}$$

By taking $\boldsymbol{G}_h := \widetilde{\boldsymbol{H}}_h^k$ in (4.18), using (4.19), the inequality $2(p-q)p \ge p^2 - q^2$ and Young's inequality, we obtain

$$\begin{aligned}
&\int_\Omega \mu|\widetilde{\boldsymbol{H}}_h^k|^2 - \int_\Omega \mu|\widetilde{\boldsymbol{H}}_h^{k-1}|^2 + \frac{\Delta t}{\overline{\sigma}}\|\mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}\\
&\qquad \le \frac{\Delta t}{2T}\int_\Omega \mu|\widetilde{\boldsymbol{H}}_h^k|^2 + C\,\Delta t\left(\|\mathbf{curl}\,\widehat{\boldsymbol{H}}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \left\|\frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3}\right).
\end{aligned}$$

Summing up from $k = 1$ to m ($m \le M$), using the estimate $\Delta t\sum_{k=1}^M \left\|\frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t}\right\|^2_{\mathrm{L}^2(\Omega)^3} \le C\|\widehat{\boldsymbol{H}}_h\|^2_{\mathrm{H}^1(0,T;\mathrm{H}(\mathbf{curl};\Omega))}$, the bound (4.17) and the discrete Gronwall's inequality, we obtain

$$\|\widetilde{\boldsymbol{H}}_h^m\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t\sum_{k=1}^m \|\mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} \le C\left\{\|\boldsymbol{H}_{0h}\|^2_{\mathrm{L}^2(\Omega)^3} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}. \tag{4.21}$$

On the other hand, by taking $\boldsymbol{G}_h := \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t}$ in (4.18) and using that

$$\int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k \cdot \mathbf{curl} \left(\frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right) \geq \frac{1}{2\Delta t} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} |\,\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k|^2 - \frac{1}{2\Delta t} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} |\,\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1}|^2,$$

$$\begin{aligned} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^k \cdot \mathbf{curl} \left(\frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right) &= \frac{1}{\Delta t} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^k \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k \\ &\quad - \frac{1}{\Delta t} \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^{k-1} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} \\ &\quad - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl} \left(\frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right) \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1}, \end{aligned}$$

together with (4.18) and the Young's inequality, we obtain

$$\begin{aligned} \Delta t \int_{\Omega} \mu \left| \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right|^2 &+ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \left| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k \right|^2 - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \left| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} \right|^2 \\ &\leq -2 \left\{ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^k \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^k - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^{k-1} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} \right\} \\ &\quad + \Delta t\, C \left\{ \left\| \frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \left\| \mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1} \right\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \right\}. \end{aligned}$$

Summing up from $k = 1$ to m ($m \leq M$) leads to

$$\begin{aligned} \Delta t \sum_{k=1}^{m} \left\| \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 &+ \|\,\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^m\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \\ &\leq C \Bigg(\|\,\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^0\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 + \|\,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^0\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 + \|\,\mathbf{curl}\, \widehat{\boldsymbol{H}}_h^m\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \\ &\quad + \Delta t \sum_{k=1}^{m} \left\{ \left\| \frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\,\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1}\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \right\} \Bigg). \end{aligned}$$

Using the estimate $\Delta t \sum_{k=1}^{M} \left\| \frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 \leq C \|\widehat{\boldsymbol{H}}_h\|_{\mathrm{H}^1(0,T;\mathrm{H}(\mathbf{curl};\Omega))}^2$, the bound (4.17) and, using (4.21) to estimate $\Delta t \sum_{k=2}^{m+1} \|\,\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^{k-1}\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2$, we obtain

$$\Delta t \sum_{k=1}^{m} \left\| \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \|\,\mathbf{curl}\, \widetilde{\boldsymbol{H}}_h^m\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \leq C \left\{ \|\boldsymbol{H}_{0h}\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}^2 \right\}.$$

Note that, from (4.19), we have that $\mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^m = \mathbf{0}$ in Ω_{NC}, therefore we have

$$\Delta t \sum_{k=1}^{m} \left\| \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \| \mathbf{curl}\,\widetilde{\boldsymbol{H}}_h^m \|_{\mathrm{L}^2(\Omega)^3}^2 \leq C \left\{ \|\boldsymbol{H}_{0h}\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}^2 \right\}.$$

Adding this inequality to (4.21) we obtain

$$\|\widetilde{\boldsymbol{H}}_h^m\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{m} \left\| \frac{\widetilde{\boldsymbol{H}}_h^k - \widetilde{\boldsymbol{H}}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 \leq C \left\{ \|\boldsymbol{H}_{0h}\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}^2 \right\}.$$

Finally, since $\boldsymbol{H}_h^k = \widetilde{\boldsymbol{H}}_h^k + \widehat{\boldsymbol{H}}_h^k$ and thanks to the boundedness (4.17), we conclude that

$$\max_{1\leq m\leq M} \|\boldsymbol{H}_h^m\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{M} \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 \leq C \left\{ \|\boldsymbol{H}_{0h}\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}^2 \right\}.$$

The last step is to estimate the terms involving the Lagrange multipliers $\boldsymbol{K}_h^k$. To do this, note that as a consequence of the *inf-sup* condition we have

$$\beta \|\boldsymbol{K}_h^k\|_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3} \leq \sup_{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega)} \frac{b(\boldsymbol{G}_h, \boldsymbol{K}_h^k)}{\|\boldsymbol{G}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)}} \leq C \left\{ \left\| \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \| \mathbf{curl}\,\boldsymbol{H}_h^k \|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \right\}^{1/2}.$$

Therefore

$$\Delta t \sum_{k=1}^{M} \|\boldsymbol{K}_h^k\|_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}^2 \leq C \left\{ \|\boldsymbol{H}_{0h}\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_{\mathrm{S}}\|_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}^2 \right\},$$

which allows us to conclude the proof. ∎

Now, we are interested in obtaining an error estimate for the fully discrete scheme. With this aim, we introduce for $r \in \left(\frac{1}{2}, 1\right]$ the space $\mathrm{H}^r(\mathbf{curl};\Omega) := \{\boldsymbol{G} \in \mathrm{H}^r(\Omega) : \mathbf{curl}\,\boldsymbol{G} \in \mathrm{H}^r(\Omega)\}$. If $\boldsymbol{G} \in \mathrm{H}^r(\mathbf{curl};\Omega) \cap \mathrm{H}_0(\mathbf{curl};\Omega)$, then its Nédélec interpolant $\mathcal{I}_h^{\mathcal{N}}\boldsymbol{G} \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ is well defined (see [9]).

From now on, we assume that the solution of Problem 4.3 satisfies $\boldsymbol{H} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$, which in particular implies that the initial condition $\boldsymbol{H}_0 \in \mathrm{H}^r(\mathbf{curl};\Omega)$. Therefore, the Nédélec interpolant $\mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t)$ for all $t \in [0,T]$ is well defined. Thus, we are allowed to use $\boldsymbol{H}_{0h} := \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}_0$.

Next, let us write the error as follows

$$\boldsymbol{H}(t_k) - \boldsymbol{H}_h^k := (\boldsymbol{H}(t_k) - \mathring{\boldsymbol{H}}_h^k) + (\mathring{\boldsymbol{H}}_h^k - \boldsymbol{H}_h^k)$$

where $\mathring{\boldsymbol{H}}_h^k \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ is such that $\mathring{\boldsymbol{H}}_h^k - \boldsymbol{H}_h^k \in \boldsymbol{\mathcal{Y}}_h$, $k = 1, \ldots, M$. To obtain such $\mathring{\boldsymbol{H}}_h^k$, we proceed as in the proof of [38, Theorem II.1.1]; namely, let $\widehat{b}$ be the bilinear form defined in $\mathrm{H}_0(\mathbf{curl};\Omega) \times \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right)$ by

$$\widehat{b}(\boldsymbol{G}, \boldsymbol{F}_h) := \int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\,\boldsymbol{G} \cdot \boldsymbol{F}_h.$$

Let us consider its associated operator $\mathbb{B} : \mathrm{H}_0(\mathbf{curl};\Omega) \to \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right)'$ defined by

$$\langle \mathbb{B}(\boldsymbol{G}), \boldsymbol{G}_h \rangle := \widehat{b}(\boldsymbol{G}, \boldsymbol{G}_h), \quad \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega).$$

For $t \in [0,T]$, $\boldsymbol{H}(t) - \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t) \in \mathrm{H}_0(\mathbf{curl};\Omega)$ then $\mathbb{B}(\boldsymbol{H}(t) - \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t)) \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right)'$; thus, as a consequence of *inf-sup* condition, there exists $\boldsymbol{Z}_h(t) \in \boldsymbol{\mathcal{Y}}_h^{\perp_{\boldsymbol{\mathcal{N}}_h^0(\Omega)}}$ such that

$$\mathbb{B}(\boldsymbol{Z}_h(t)) = \mathbb{B}(\boldsymbol{H}(t) - \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t))$$

and

$$\beta \|\boldsymbol{Z}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq \|\mathbb{B}\| \, \|\boldsymbol{H}(t) - \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)},$$

where $\|\mathbb{B}\|$ and $\beta > 0$ are independent of h. Moreover, if we denote $\mathring{\boldsymbol{H}}_h(t) := \boldsymbol{Z}_h(t) + \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t)$, then

$$\int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\, \mathring{\boldsymbol{H}}_h(t) \cdot \boldsymbol{F}_h = \int_{\Omega_{\mathrm{NC}}} \boldsymbol{J}_{\mathrm{S}}(t) \cdot \boldsymbol{F}_h \quad \forall \boldsymbol{F}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_{\mathrm{NC}}}\right)$$

and

$$\|\boldsymbol{H}(t) - \mathring{\boldsymbol{H}}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq \left(1 + \frac{\|\mathbb{B}\|}{\beta}\right) \|\boldsymbol{H}(t) - \mathcal{I}_h^{\mathcal{N}}\boldsymbol{H}(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)}.$$

By using classical estimates for the Nédélec interpolant (see, for instance, [52, Theorem 5.41]) we have that $\|\boldsymbol{G} - \mathcal{I}_h^{\mathcal{N}}\boldsymbol{G}\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C\, h^r \|\boldsymbol{G}\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}$, $\boldsymbol{G} \in \mathrm{H}^r(\mathbf{curl};\Omega)$. Therefore,

$$\|\boldsymbol{H}(t) - \mathring{\boldsymbol{H}}_h(t)\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C\, h^r \|\boldsymbol{H}(t)\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}, \tag{4.22}$$

$$\|\partial_t(\boldsymbol{H}(t) - \mathring{\boldsymbol{H}}_h(t))\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C\, h^r \|\partial_t \boldsymbol{H}(t)\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}. \tag{4.23}$$

Next, in order to obtain the error estimate we write

$$\partial_t \boldsymbol{H}(t_k) - \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} = \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} + \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} - \boldsymbol{\tau}^k, \tag{4.24}$$

where

$$\boldsymbol{\rho}_h^k := \boldsymbol{H}(t_k) - \mathring{\boldsymbol{H}}_h(t_k), \qquad \boldsymbol{\delta}_h^k := \mathring{\boldsymbol{H}}_h(t_k) - \boldsymbol{H}_h^k \quad \text{and} \qquad \boldsymbol{\tau}^k := \frac{\boldsymbol{H}(t_k) - \boldsymbol{H}(t_{k-1})}{\Delta t} - \partial_t \boldsymbol{H}(t_k).$$

Lemma 4.12 *Let* $\boldsymbol{H}$ *be the first component of the solution to Problem 4.5 and* $\boldsymbol{H}_h^k$*,* $k = 1, \ldots, M$*, that to Problem 4.10. If* $\boldsymbol{H} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ *for* $r \in \left(\frac{1}{2}, 1\right]$*, then there exists a constant* $C > 0$*, independent of* h *and* Δt*, such that*

$$\max_{1\leq k\leq M} \|\boldsymbol{\delta}_h^k\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{M} \left\| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2$$

$$\leq C \left(\|\boldsymbol{\delta}_h^0\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \max_{1\leq k\leq M} \|\,\mathbf{curl}\, \boldsymbol{\rho}_h^k\|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 + \Delta t \sum_{k=1}^{M} \|\boldsymbol{\tau}^k\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^{M} \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 \right).$$

Proof. By testing with $\boldsymbol{G}_h \in \boldsymbol{\mathcal{Y}}_h$ in the first equation to Problem 4.5 and in the first equation to Problem 4.10, a straightforward computation allows us to show that

$$\int_\Omega \mu \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \boldsymbol{\delta}_h^k \cdot \mathbf{curl}\, \boldsymbol{G}_h$$
$$= \int_\Omega \mu \boldsymbol{\tau}^k \cdot \boldsymbol{G}_h - \int_\Omega \mu \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \boldsymbol{\rho}_h^k \cdot \mathbf{curl}\, \boldsymbol{G}_h \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{Y}}_h. \quad (4.25)$$

By taking $\boldsymbol{G}_h := \boldsymbol{\delta}_h^k$, using the inequality $2(p-q)p \geq p^2 - q^2$ and Young's inequality, we obtain

$$\int_\Omega \mu |\boldsymbol{\delta}_h^k|^2 - \int_\Omega \mu |\boldsymbol{\delta}_h^{k-1}|^2 + \frac{\Delta t}{\overline{\sigma}} \|\,\mathbf{curl}\, \boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}$$
$$\leq \frac{\Delta t}{2T} \int_\Omega \mu |\boldsymbol{\delta}_h^k|^2 + C \left(\|\,\mathbf{curl}\, \boldsymbol{\rho}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \int_\Omega \mu |\boldsymbol{\tau}^k|^2 + \int_\Omega \mu \left| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right|^2 \right).$$

Summing up from $k = 1$ to m $(m \leq M)$ and using the discrete Gronwall's inequality, we obtain

$$\|\boldsymbol{\delta}_h^m\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^m \|\,\mathbf{curl}\, \boldsymbol{\delta}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}$$
$$\leq C \left(\|\boldsymbol{\delta}_h^0\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^m \left\{ \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} + \|\,\mathbf{curl}\, \boldsymbol{\rho}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{L}^2(\Omega)^3} \right\} \right). \quad (4.26)$$

On the other hand, by taking $\boldsymbol{G}_h := \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t}$ in (4.25), similar arguments and the fact that

$$\int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \boldsymbol{\rho}_h^k \cdot \mathbf{curl}\, \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} = - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl} \left(\frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right) \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^{k-1}$$
$$+ \frac{1}{\Delta t} \left\{ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \boldsymbol{\rho}_h^k \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^k - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \boldsymbol{\rho}_h^{k-1} \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^{k-1} \right\}$$

lead to

$$\Delta t \int_\Omega \mu \left| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right|^2 + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} |\,\mathbf{curl}\, \boldsymbol{\delta}_h^k|^2 - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} |\,\mathbf{curl}\, \boldsymbol{\delta}_h^{k-1}|^2$$
$$\leq -2 \left\{ \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \boldsymbol{\rho}_h^k \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^k - \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \,\mathbf{curl}\, \boldsymbol{\rho}_h^{k-1} \cdot \mathbf{curl}\, \boldsymbol{\delta}_h^{k-1} \right\}$$
$$+ \Delta t\, C \left\{ \|\boldsymbol{\tau}^k\|^2_{\mathrm{L}^2(\Omega)^3} + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|^2_{\mathrm{H}(\mathbf{curl};\Omega)} + \|\,\mathbf{curl}\, \boldsymbol{\delta}_h^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3} \right\}.$$

Summing up from $k = 1$ to m $(m \leq M)$ and using to (4.26) to estimate $\Delta t \sum_{k=2}^{m+1} \|\,\mathbf{curl}\, \boldsymbol{\delta}_h^{k-1}\|^2_{\mathrm{L}^2(\Omega)^3}$, we obtain

$$\Delta t \sum_{k=1}^{m} \left\| \frac{\boldsymbol{\delta}_h^k - \boldsymbol{\delta}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2 + \| \mathbf{curl}\, \boldsymbol{\delta}_h^m \|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2$$
$$\leq C \left(\Delta t \sum_{k=1}^{m} \left\{ \|\boldsymbol{\tau}^k\|_{\mathrm{L}^2(\Omega)^3}^2 + \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \| \mathbf{curl}\, \boldsymbol{\rho}_h^k \|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \right\} \right.$$
$$\left. + \| \mathbf{curl}\, \boldsymbol{\delta}_h^0 \|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 + \| \mathbf{curl}\, \boldsymbol{\rho}_h^m \|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 + \| \mathbf{curl}\, \boldsymbol{\rho}_h^0 \|_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3}^2 \right).$$

Thus, the result follows by combining the above inequality with (4.26) and the fact that $\boldsymbol{\delta}_h^k \in \boldsymbol{\mathcal{Y}}_h$. ■

Now, we are in a position to write an optimal error estimate for this fully discrete scheme.

Theorem 4.13 *Let $\boldsymbol{H}$ be the first component of the solution to Problem 4.5 and $\boldsymbol{H}_h^k$, $k = 1, \ldots, M$, that to Problem 4.10. If $\boldsymbol{H} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ for $r \in \left(\frac{1}{2}, 1\right]$, and $\boldsymbol{H} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)$, then there exists a constant $C > 0$, independent of h and Δt, such that*

$$\max_{1\leq k\leq M} \|\boldsymbol{H}(t_k) - \boldsymbol{H}_h^k\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{M} \left\| \partial_t \boldsymbol{H}(t_k) - \frac{\boldsymbol{H}_h^k - \boldsymbol{H}_h^{k-1}}{\Delta t} \right\|_{\mathrm{L}^2(\Omega)^3}^2$$
$$\leq C \left\{ (\Delta t)^2 \int_0^T \|\partial_{tt} \boldsymbol{H}(t)\|_{\mathrm{L}^2(\Omega)^3}^2 \, dt + h^{2r} \sup_{0\leq t\leq T} \|\boldsymbol{H}(t)\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}^2 + h^{2r} \int_0^T \|\partial_t \boldsymbol{H}(t)\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}^2 \, dt \right\}$$
$$\leq C \left\{ (\Delta t)^2 \|\boldsymbol{H}\|_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)}^2 + h^{2r} \|\boldsymbol{H}\|_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))}^2 \right\}.$$

Proof. A Taylor expansion shows that

$$\sum_{k=1}^{M} \|\boldsymbol{\tau}^k\|_{\mathrm{L}^2(\Omega)^3}^2 = \sum_{k=1}^{M} \left\| \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} (t_k - s) \partial_{tt} \boldsymbol{H}(s) \, ds \right\|_{\mathrm{L}^2(\Omega)^3}^2 \leq \Delta t \int_0^T \|\partial_{tt} \boldsymbol{H}(t)\|_{\mathrm{L}^2(\Omega)^3}^2 \, dt.$$

Moreover,

$$\sum_{k=1}^{M} \left\| \frac{\boldsymbol{\rho}_h^k - \boldsymbol{\rho}_h^{k-1}}{\Delta t} \right\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 \leq \frac{1}{\Delta t} \int_0^T \|\partial_t (\boldsymbol{H}(t) - \mathring{\boldsymbol{H}}_h(t))\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 \, dt,$$

and

$$\|\boldsymbol{\delta}_h^0\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 \leq 2\|\boldsymbol{\rho}_h^0\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 + 2\|\boldsymbol{H}_0 - \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2 \leq C\|\boldsymbol{H}_0 - \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0\|_{\mathrm{H}(\mathbf{curl};\Omega)}^2.$$

Since $\boldsymbol{H}(t_k) - \boldsymbol{H}_h^k = \boldsymbol{\delta}_h^k + \boldsymbol{\rho}_h^k$, the result follows from (4.22), (4.23), (4.24) and the previous lemma. ■

If our goal were to solve Problem 4.3 with a fixed conductor Ω_{C}, the natural procedure would be to implement Problem 4.10, for which the convergence results of the previous theorem hold. However, when the conductors move, it would not be easy to introduce the space where the Lagrange multiplier $\boldsymbol{K}_h^k$ lies, unless a different mesh were used at each time step. Taking this in mind, in what follows we consider a penalization approach to overcome this drawback. Thus, to compute the solution of Problem 4.3, we propose the following regularized problem.

Problem 4.14 *Find $\boldsymbol{H}_{h,\varepsilon}^m \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$, $m = 1, \ldots, M$, such that*

$$\int_\Omega \mu \frac{\boldsymbol{H}_{h,\varepsilon}^m - \boldsymbol{H}_{h,\varepsilon}^{m-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_\mathrm{C}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^m \cdot \mathbf{curl}\, \boldsymbol{G}_h + \frac{1}{\varepsilon} \int_{\Omega_\mathrm{NC}} \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^m \cdot \mathbf{curl}\, \boldsymbol{G}_h$$
$$= \frac{1}{\varepsilon} \int_{\Omega_\mathrm{NC}} \boldsymbol{J}_\mathrm{S}(t_m) \cdot \mathbf{curl}\, \boldsymbol{G}_h \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h^0(\Omega),$$
$$\boldsymbol{H}_{h,\varepsilon}^0 = \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0.$$

Note that, for $0 < \varepsilon \le 1$ the Lax-Milgram theorem ensures the existence and uniqueness of the solution for all $m = 1, \ldots, M$. By using similar ideas as those developed in the continuous case, we introduce the following equivalent mixed problem to analyze the convergence of Problem 4.14 to Problem 4.3.

Problem 4.15 *Find $\boldsymbol{H}_{h,\varepsilon}^m \in \boldsymbol{\mathcal{N}}_h^0(\Omega)$ and $\boldsymbol{K}_{h,\varepsilon}^m \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_\mathrm{NC}}\right)$, $m = 1, \ldots, M$, such that*

$$\int_\Omega \mu \frac{\boldsymbol{H}_{h,\varepsilon}^m - \boldsymbol{H}_{h,\varepsilon}^{m-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_\mathrm{C}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^m \cdot \mathbf{curl}\, \boldsymbol{G}_h + \int_{\Omega_\mathrm{NC}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{K}_{h,\varepsilon}^m = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{Y}}_h,$$
$$\int_{\Omega_\mathrm{NC}} \mathbf{curl}\, \boldsymbol{H}_{h,\varepsilon}^m \cdot \boldsymbol{F}_h - \varepsilon \int_{\Omega_\mathrm{NC}} \boldsymbol{K}_{h,\varepsilon}^m \cdot \boldsymbol{F}_h = \int_{\Omega_\mathrm{NC}} \boldsymbol{J}_\mathrm{S}(t_m) \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_\mathrm{NC}}\right),$$
$$\boldsymbol{H}_{h,\varepsilon}^0 = \mathcal{I}_h^{\mathcal{N}} \boldsymbol{H}_0.$$

Now, we will show that this regularized fully discrete scheme converges with ε to the fully discrete scheme, as in the continuous case.

Theorem 4.16 *Given $\boldsymbol{J}_\mathrm{S} \in \mathrm{H}^1(0, T; \mathrm{H}_0(\mathrm{div}^0; \Omega_\mathrm{S}))$, let $\boldsymbol{H}_h^k$, $\boldsymbol{K}_h^k$, $k = 1, \ldots, M$ be the solution to Problem 4.10 and $\boldsymbol{H}_{h,\varepsilon}^k$, $\boldsymbol{K}_{h,\varepsilon}^k$, $k = 1, \ldots, M$, be the solution to Problem 4.15. There exists $C > 0$, independent of ε, h and Δt, such that*

$$\max_{1 \le k \le M} \|\boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^M \|\mathbf{curl}(\boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k)\|_{\mathrm{L}^2(\Omega)^3}^2 + \Delta t \sum_{k=1}^M \|\boldsymbol{K}_{h,\varepsilon}^k - \boldsymbol{K}_h^k\|_{\mathrm{L}^2(\Omega_\mathrm{NC})^3}^2$$
$$\le C\varepsilon^2 \left\{ \|\boldsymbol{H}_0\|_{\mathrm{H}^r(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{J}_\mathrm{S}\|_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_\mathrm{S}))}^2 \right\},$$

Proof. If we denote $\boldsymbol{u}_h^k := \boldsymbol{H}_{h,\varepsilon}^k - \boldsymbol{H}_h^k$ and $\boldsymbol{P}_h^k := \boldsymbol{K}_{h,\varepsilon}^k - \boldsymbol{K}_h^k$, subtracting Problem 4.10 and Problem 4.15 we obtain

$$\int_\Omega \mu \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega_\mathrm{C}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{u}_h^k \cdot \mathbf{curl}\, \boldsymbol{G}_h + \int_{\Omega_\mathrm{NC}} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{P}_h^k = 0 \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{Y}}_h,$$
$$\int_{\Omega_\mathrm{NC}} \mathbf{curl}\, \boldsymbol{u}_h^k \cdot \boldsymbol{F}_h - \varepsilon \int_{\Omega_\mathrm{NC}} \boldsymbol{P}_h^k \cdot \boldsymbol{F}_h = \varepsilon \int_{\Omega_\mathrm{NC}} \boldsymbol{K}_h^k \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathbf{curl}\left(\boldsymbol{\mathcal{N}}_h^0(\Omega)|_{\Omega_\mathrm{NC}}\right),$$
$$\boldsymbol{u}_h^0 = \mathbf{0}.$$

By taking $\boldsymbol{G}_h := \boldsymbol{u}_h^k$, $\boldsymbol{F}_h := \boldsymbol{P}_h^k$, using the inequality $2(p-q)p \geq p^2 - q^2$ and Young's inequality, we get

$$\int_\Omega \mu|\boldsymbol{u}_h^k|^2 - \int_\Omega \mu|\boldsymbol{u}_h^{k-1}|^2 + \frac{\Delta t}{\underline{\sigma}}\|\,\mathbf{curl}\,\boldsymbol{u}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \Delta t\|\boldsymbol{P}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3} \leq \Delta t\varepsilon^2\|\boldsymbol{K}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}.$$

Summing from $k=1$ to m $(m \leq M)$, using the discrete Gronwall's inequality and the fact that $\boldsymbol{u}_h^0 = \mathbf{0}$, we obtain

$$\begin{aligned}
\|\boldsymbol{u}_h^m\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t\sum_{k=1}^m \|\,\mathbf{curl}\,\boldsymbol{u}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{C}})^3} + \Delta t\sum_{k=1}^m \|\boldsymbol{P}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}& \\
\leq C\varepsilon^2\Delta t\sum_{k=1}^m \|\boldsymbol{K}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}& \\
\leq C\varepsilon^2\Delta t\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}&,
\end{aligned}$$

where the last inequality is a consequence of the estimate in Theorem 4.11. There only remains to estimate $\Delta t\sum_{k=1}^m \|\,\mathbf{curl}\,\boldsymbol{u}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}$. With this aim, notice that we have $\mathbf{curl}\,\boldsymbol{u}_h^k = \varepsilon(\boldsymbol{P}_h^k + \boldsymbol{K}_h^k)$ in Ω_{NC}. Thus,

$$\|\,\mathbf{curl}\,\boldsymbol{u}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3} \leq 2\varepsilon^2\left(\|\boldsymbol{P}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3} + \|\boldsymbol{K}_h^k\|^2_{\mathrm{L}^2(\Omega_{\mathrm{NC}})^3}\right),$$

Hence from the two previous inequalities, we have

$$\Delta t\sum_{k=1}^m \|\,\mathbf{curl}\,\boldsymbol{u}_h^k\|^2_{\mathrm{L}^2(\Omega)^3} \leq C\left(\varepsilon^2+\varepsilon^4\right)\left\{\|\boldsymbol{H}_0\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{H^1(0,T,\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\right\}.$$

Thus, we conclude the proof. ■

Finally, we are in a position to write the main result of this section.

Theorem 4.17 *Given $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$, let $\boldsymbol{H}$ be the first component of the solution to Problem 4.5 and $\boldsymbol{H}^k_{h,\varepsilon}$, $k=1,\dots,M$, the solution to Problem 4.15. If $\boldsymbol{H} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ for some $r \in \left(\frac{1}{2},1\right]$, and $\boldsymbol{H} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)$, then there exists a constant $C>0$, independent of ε, h and Δt, such that*

$$\begin{aligned}
&\max_{1\leq k\leq M}\|\boldsymbol{H}(t_k) - \boldsymbol{H}^k_{h,\varepsilon}\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t\sum_{k=1}^M \|\,\mathbf{curl}(\boldsymbol{H}(t_k) - \boldsymbol{H}^k_{h,\varepsilon})\|^2_{\mathrm{L}^2(\Omega)^3} \\
&\leq C\Big\{(\Delta t)^2\|\boldsymbol{H}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{C}})^3)} + h^{2r}\|\boldsymbol{H}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))} + \varepsilon^2\Big(\|\boldsymbol{H}_0\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{H^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\Big)\Big\}.
\end{aligned}$$

From the equivalence between primal and mixed formulations of the problem (cf. Lemma 4.6) and the analogous result for the corresponding penalized problems, we have the following corollary.

Corollary 4.18 *Given $\boldsymbol{J}_{\mathrm{S}} \in \mathrm{H}^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))$, let $\boldsymbol{H}$ be the solution to Problem 4.3 and $\boldsymbol{H}^k_{h,\varepsilon}$, $k=1,\dots,M$, that to Problem 4.14. If $\boldsymbol{H} \in \mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))$ for $r \in \left(\frac{1}{2},1\right]$ and*

$\boldsymbol{H} \in \mathrm{H}^2(0,T;\mathrm{L}^2(\Omega)^3)$, *then there exists a constant* $C > 0$, *independent of* ε, h *and* Δt, *such that*

$$\max_{1\leq k\leq M} \|\boldsymbol{H}(t_k) - \boldsymbol{H}^k_{h,\varepsilon}\|^2_{\mathrm{L}^2(\Omega)^3} + \Delta t \sum_{k=1}^{M} \|\,\mathbf{curl}(\boldsymbol{H}(t_k) - \boldsymbol{H}^k_{h,\varepsilon})\|^2_{\mathrm{L}^2(\Omega)^3}$$
$$\leq C\Big\{(\Delta t)^2 \|\boldsymbol{H}\|^2_{\mathrm{H}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{C}})^3)} + h^{2r}\|\boldsymbol{H}\|^2_{\mathrm{H}^1(0,T;\mathrm{H}^r(\mathbf{curl};\Omega))} + \varepsilon^2\Big(\|\boldsymbol{H}_0\|^2_{\mathrm{H}^r(\mathbf{curl};\Omega)} + \|\boldsymbol{J}_{\mathrm{S}}\|^2_{H^1(0,T;\mathrm{H}_0(\mathrm{div}^0;\Omega_{\mathrm{S}}))}\Big)\Big\}.$$

4.3.3 Numerical experiments with fixed conductors

In this section we report some numerical results obtained with a MATLAB code which implements the penalty method described above, to illustrate the convergence with respect to the parameter ε. In particular, we have implemented Problem 4.14 and an equivalent variant of Problem 4.10 and compared the results at each time step for different values of the parameter ε.

The implementation of Problem 4.14 is straightforward, but we give some details concerning that of Problem 4.10. At each time step, this problem leads to a singular system, if the multiplier is written as the curl of edge basis functions. To avoid this singularity, we implement a double mixed problem by following the ideas from [5] which have also been developed in Chapter 1 of this thesis in the harmonic case. More precisely, let $\mathcal{Q}_h$ be the space of piecewise constant functions in $\mathcal{T}_h^{\Omega_{\mathrm{NC}}}$, where $\mathcal{T}_h^{\Omega_{\mathrm{NC}}} := \{K \in \mathcal{T}_h : \ K \subset \Omega_{\mathrm{NC}}\}$, and let $\mathcal{CR}_h(\Omega_{\mathrm{NC}})$ be the space of lowest-order 3D Crouzeix-Raviart elements; namely,

$$\mathcal{Q}_h := \left\{F_h \in \mathrm{L}^2(\Omega_{\mathrm{NC}}) \ : \ F_h|_K \in \mathbb{P}_0 \ \forall K \in \mathcal{T}_h\right\},$$
$$\mathcal{CR}_h(\Omega_{\mathrm{NC}}) := \left\{\begin{array}{l} q_h \in \mathrm{L}^2(\Omega_{\mathrm{NC}}) \ : \ q_h|_K \in \mathbb{P}_1 \ \forall K \in \mathcal{T}_h \text{ and } q_h \text{ is continuous at} \\ \text{the midpoints of any face } f \text{ common to two elements in } \mathcal{T}_h \end{array}\right\}.$$

We consider the subspace:

$$\mathcal{CR}^0_h(\Omega_{\mathrm{NC}}) := \{q_h \in \mathcal{CR}_h(\Omega_{\mathrm{NC}}) \ : \ q_h(\boldsymbol{x}) = 0 \text{ for all midpoints } \boldsymbol{x} \text{ of faces of } \partial\Omega_{\mathrm{C}}\},$$

and, for $q_h \in \mathcal{CR}^0_h(\Omega_{\mathrm{D}})$, $\mathbf{grad}_h\, q_h$ denote the vector field in $\mathcal{Q}_h^3$ defined by

$$(\mathbf{grad}_h\, q_h)|_K := \mathbf{grad}(q_h|_K) \quad \forall K \in \mathcal{T}_h^{\Omega_{\mathrm{NC}}}.$$

Thus, the problem to solve is the following:

Problem 4.19 *Find* $\boldsymbol{H}^m_h \in \boldsymbol{\mathcal{N}}^0_h(\Omega)$, $\boldsymbol{Z}^m_h \in \mathcal{Q}^3_h$ *and* $p^m_h \in \mathcal{CR}^0_h(\Omega_{\mathrm{NC}})$, $m = 1, \ldots, M$, *such that*

$$\int_\Omega \mu \boldsymbol{H}^m_h \cdot \boldsymbol{G}_h + \Delta t \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\, \boldsymbol{H}^m_h \cdot \mathbf{curl}\, \boldsymbol{G}_h + \Delta t \int_{\Omega_{\mathrm{NC}}} \boldsymbol{Z}^m_h \cdot \mathbf{curl}\, \boldsymbol{G}_h = \int_\Omega \mu \boldsymbol{H}^{m-1}_h \cdot \boldsymbol{G}_h \qquad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}^0_h(\Omega),$$

$$\Delta t \int_{\Omega_{\mathrm{NC}}} \mathbf{curl}\, \boldsymbol{H}^m_h \cdot \boldsymbol{F}_h + \Delta t \int_{\Omega_{\mathrm{NC}}} \mathbf{grad}_h\, p^m_h \cdot \boldsymbol{F}_h = \Delta t \int_{\Omega_{\mathrm{NC}}} \boldsymbol{J}_{\mathrm{S}}(t_m) \cdot \boldsymbol{F}_h \qquad \forall \boldsymbol{F}_h \in \mathcal{Q}^3_h,$$

$$\Delta t \int_{\Omega_{\mathrm{NC}}} \boldsymbol{Z}^m_h \cdot \mathbf{grad}_h\, q_h = 0 \qquad \forall q_h \in \mathcal{CR}^0_h(\Omega_{\mathrm{NC}}),$$

$$\boldsymbol{H}^0_h = \mathcal{I}^{\mathcal{N}}_h \boldsymbol{H}_0.$$

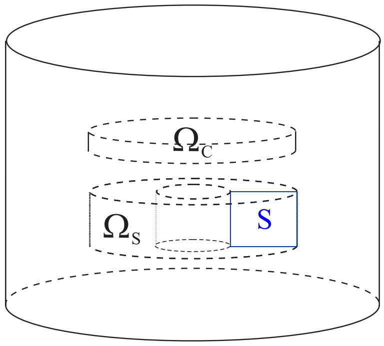

Figure 4.2: Sketch of the domain with fixed conductor.

The results from [5] applied to the problem above at each time step show that Problem 4.19 is equivalent to Problem 4.10.

Let us consider the geometry sketched in Figure 4.2. The workpiece and the whole domain are cylinders of respective radius $R_{\mathrm{P}} = 1\,\mathrm{m}$ and $R_{\Omega} = 2\,\mathrm{m}$, and heights $A_{\mathrm{P}} = 0.2\,\mathrm{m}$ and $A_{\Omega} = 1.9\,\mathrm{m}$. The coil Ω_{S} is a toroidal core of rectangular cross section S, with inner radius equal to $R_{\mathrm{c}} = 0.5\,\mathrm{m}$, outer radius $R_{\mathrm{C}} = 1\,\mathrm{m}$ and height $A = 0.5\,\mathrm{m}$. The source current density is supported in Ω_{S} and is given by

$$\boldsymbol{J}_{\mathrm{S}}(t, \boldsymbol{x}) = \frac{I(t)}{\mathrm{meas}(S)} \begin{bmatrix} -\frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ 0 \end{bmatrix} \quad \text{in } \Omega_{\mathrm{S}}, \tag{4.27}$$

where the current intensity $I(t)$ in the coil is shown in Figure 4.3. The shape of the curve $I(t)$ is similar to a typical one used in EMF benchmark. Concerning the physical properties, we have considered $\mu = \mu_0 = 4\pi \times 10^{-7}\,\mathrm{Hm}^{-1}$ and $\sigma = 1 \times 10^{6}\,(\Omega\mathrm{m})^{-1}$ in the workpiece.

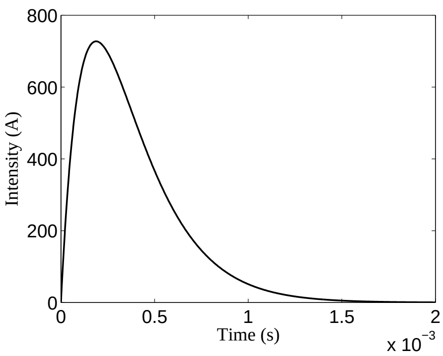

Figure 4.3: Current intensity (A) vs. time (s).

The problem fits in the framework of Problem 4.3, but an analytical solution is not available. However, this is not a problem since we will focus on the convergence with respect to ε of the solution of Problem 4.14 to that of Problem 4.19. More precisely, let $\boldsymbol{H}^m_{h,\varepsilon}$ and $\boldsymbol{H}^m_h$ be the respective solutions of these problems. We compare both solutions; namely, we fixed the mesh-size, the time-step, and varied ε from 10^0 to 10^{-4}.

ε	$\dfrac{\max_{1\le k\le M}\lVert\boldsymbol{H}^m_{h,\varepsilon}-\boldsymbol{H}^m_h\rVert_{\mathrm{L}^2(\Omega)^3}}{\max_{1\le k\le M}\lVert\boldsymbol{H}^m_h\rVert_{\mathrm{L}^2(\Omega)^3}}$	$\dfrac{\sqrt{\Delta t}\left\{\sum_{k=1}^m \lVert\,\mathbf{curl}\,\boldsymbol{H}^m_{h,\varepsilon}-\mathbf{curl}\,\boldsymbol{H}^m_h\rVert^2_{\mathrm{L}^2(\Omega)^3}\right\}^{1/2}}{\sqrt{\Delta t}\left\{\sum_{k=1}^m \lVert\,\mathbf{curl}\,\boldsymbol{H}^m_h\rVert^2_{\mathrm{L}^2(\Omega)^3}\right\}^{1/2}}$
1	0.4013258	0.1234511
10^{-1}	0.0412257	0.0124055
10^{-2}	0.0041338	0.0012411
10^{-3}	0.0004134	0.0001241
10^{-4}	0.0000413	0.0000190

Table 4.1: Relative percentual errors in the norms considered in Theorem 4.16.

The numerical results show a clear linear convergence with respect to the parameter ε until this parameter becomes too small, which confirm the theoretical results proved in Theorem 4.16. This convergence is shown for the magnetic field and the current density in Figure 4.4. Table 4.1 shows the exact values of the relative percentual errors for each value of ε. The relative errors are actually very small; however, for values of $\varepsilon \le 10^{-4}$, the convergence deteriorates due to the poor conditioning of the linear system.

The reported results do not change significantly by repeating the experiments with different time-step and mesh-size, which confirms the theoretical results (cf. Theorem 4.17).

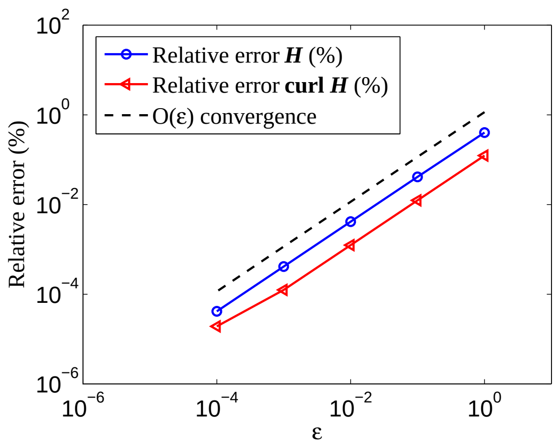

Figure 4.4: $\dfrac{\max_{1\le k\le M}\lVert\boldsymbol{H}^m_{h,\varepsilon}-\boldsymbol{H}^m_h\rVert_{\mathrm{L}^2(\Omega)^3}}{\max_{1\le k\le M}\lVert\boldsymbol{H}^m_h\rVert_{\mathrm{L}^2(\Omega)^3}}$ and $\dfrac{\sqrt{\Delta t}\left\{\sum_{k=1}^m \lVert\,\mathbf{curl}\,\boldsymbol{H}^m_{h,\varepsilon}-\mathbf{curl}\,\boldsymbol{H}^m_h\rVert^2_{\mathrm{L}^2(\Omega)^3}\right\}^{1/2}}{\sqrt{\Delta t}\left\{\sum_{k=1}^m \lVert\,\mathbf{curl}\,\boldsymbol{H}^m_h\rVert^2_{\mathrm{L}^2(\Omega)^3}\right\}^{1/2}}$ versus ε (log-log scale).

4.4 A penalized $\boldsymbol{H}$-formulation to deal with moving conductors

Penalization seems to be an interesting alternative to solve the eddy current model with moving conductors, because it allows to work on a fixed mesh with a reasonable number of unknowns. This technique avoids introducing additional variables as in Problem 4.19, which makes the latter significantly more expensive. On the other hand, in the case of moving domains, there are not yet theoretical results which guarantee the convergence as it happens on fixed domains. However, the numerical results from this section show that the penalty approach seems to be a good option.

Our goal is to approximate the solution of Problem 4.1 by means of a penalization to impose the condition $\mathbf{curl}\,\boldsymbol{H}(t) = \boldsymbol{J}_{\mathrm{S}}(t)$ in Ω^t_{NC} a.e. $t \in [0, T]$. Namely, for $\varepsilon > 0$ we want to find $\boldsymbol{H}_\varepsilon(t)$ such that $\mathbf{curl}\,\boldsymbol{H}_\varepsilon(t) - \boldsymbol{J}_{\mathrm{S}}(t) = \varepsilon \boldsymbol{E}(t)$ in Ω^t_{NC} a.e. $t \in [0, T]$ which leads to the following problem:

Problem 4.20 *Find* $\boldsymbol{H}_\varepsilon \in \mathrm{L}^2(0, T; \mathrm{H}_0(\mathbf{curl}; \Omega)) \cap \mathrm{H}^1(0, T; \mathrm{H}_0(\mathbf{curl}; \Omega)')$ *such that*

$$\frac{d}{dt}\int_\Omega \mu \boldsymbol{H}_\varepsilon \cdot \boldsymbol{G} + \int_{\Omega^t_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}_\varepsilon \cdot \mathbf{curl}\,\boldsymbol{G} + \frac{1}{\varepsilon}\int_{\Omega^t_{\mathrm{NC}}} \mathbf{curl}\,\boldsymbol{H}_\varepsilon \cdot \mathbf{curl}\,\boldsymbol{G} = \frac{1}{\varepsilon}\int_{\Omega^t_{\mathrm{NC}}} \boldsymbol{J}_{\mathrm{S}} \cdot \mathbf{curl}\,\boldsymbol{G} \quad \forall \boldsymbol{G} \in \mathrm{H}_0(\mathbf{curl}; \Omega),$$

$$\boldsymbol{H}_\varepsilon(0) = \boldsymbol{H}_0.$$

From now on, we assume $\mu(t, \boldsymbol{x}) = \mu_0$ (constant). Then for each $\varepsilon > 0$, the problem above lies in the framework of [60]. In fact, by applying Proposition III.2.2 and Proposition III.2.3 we show that Problem 4.20 has as unique solution.

A fully discrete approximation of Problem 4.20 reads as follows:

Problem 4.21 *Find* $\boldsymbol{H}^m_{h,\varepsilon} \in \boldsymbol{\mathcal{N}}^0_h(\Omega)$, $m = 1, \ldots, M$, *such that*

$$\int_\Omega \mu_0 \frac{\boldsymbol{H}^m_{h,\varepsilon} - \boldsymbol{H}^{m-1}_{h,\varepsilon}}{\Delta t} \cdot \boldsymbol{G}_h + \int_{\Omega^{t_m}_{\mathrm{C}}} \frac{1}{\sigma(t_m)}\,\mathbf{curl}\,\boldsymbol{H}^m_{h,\varepsilon} \cdot \mathbf{curl}\,\boldsymbol{G}_h + \frac{1}{\varepsilon}\int_{\Omega^{t_m}_{\mathrm{NC}}} \mathbf{curl}\,\boldsymbol{H}^m_{h,\varepsilon} \cdot \mathbf{curl}\,\boldsymbol{G}_h = \frac{1}{\varepsilon}\int_{\Omega^{t_m}_{\mathrm{NC}}} \boldsymbol{J}_{\mathrm{S}}(t_m) \cdot \mathbf{curl}\,\boldsymbol{G}_h \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}^0_h(\Omega),$$

$$\boldsymbol{H}^0_{h,\varepsilon} = \boldsymbol{H}_{0h}.$$

Notice that a fixed mesh is used along the time. To deal with the motion of the workpiece we will compute the integrals by using low order quadrature rules with a large number of integration points.

The motion of the workpiece only affects the domains of the second and the third integral, which can we written as a unique term $\int_\Omega \frac{1}{\sigma_\varepsilon(t_m)}\,\mathbf{curl}\,\boldsymbol{H}^m_{h,\varepsilon} \cdot \mathbf{curl}\,\boldsymbol{G}_h$ with

$$\sigma_\varepsilon(t_m, \boldsymbol{x}) := \begin{cases} \sigma(t_m, \boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega^{t_m}_{\mathrm{C}}, \\ \varepsilon & \text{if } \boldsymbol{x} \in \Omega^{t_m}_{\mathrm{NC}}. \end{cases}$$

These integrals involve a discontinuous coefficient on those tetrahedra which do not lie entirely in $\Omega^{t_m}_{\mathrm{C}}$ or $\Omega^{t_m}_{\mathrm{NC}}$. To compute them, we use a low order quadrature rule with a large number of integration points.

4.4.1 Numerical experiments with moving conductors

We present two numerical examples which allow us to assess the performance of the penalty approach. First, we describe the results obtained for a test with known analytical solution. Secondly, we solve a problem with cylindrical symmetry and compare the results with those obtained with an axisymmetric code introduced in [16].

Problem with known analytical solution

We will approximate the solution of the source problem

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{H} &= \sigma\boldsymbol{E} \quad \text{in } (0,T)\times\Omega, \\ \mu_0\partial_t\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} &= \boldsymbol{f} \quad \text{in } (0,T)\times\Omega, \end{aligned}$$

where $\boldsymbol{f}$ is a data.

We consider the domain $\Omega := (0,1)\times(0,1)\times(0,3)$ and we assume that the workpiece moves as a rigid body, with velocity $v = \boldsymbol{e}_z$, so that $\Omega_{\mathrm{C}}^t = (0,1)\times(0,1)\times(1+t,2+t)$ (see Figure 4.5).

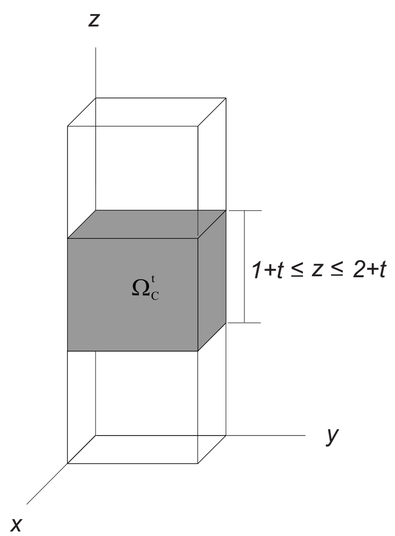

Figure 4.5: Sketch of the domain.

We must solve the following problem: find $\boldsymbol{H}(t)\in\boldsymbol{\mathcal{Y}}^t$ such that

$$\int_\Omega \mu_0\partial_t\boldsymbol{H}(t)\cdot\boldsymbol{G}(t) + \int_{\Omega_{\mathrm{C}}^t}\frac{1}{\sigma(t)}\,\mathbf{curl}\,\boldsymbol{H}(t)\cdot\mathbf{curl}\,\boldsymbol{G}(t) = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{G}(t) + \int_{\partial\Omega}\boldsymbol{E}(t)\times\boldsymbol{n}\cdot\boldsymbol{G}(t) \qquad \forall\boldsymbol{G}(t)\in\boldsymbol{\mathcal{Y}}^t,$$

$$\boldsymbol{H}(0) = \boldsymbol{H}_0.$$

We consider $\boldsymbol{f}(t,\boldsymbol{x})$ so that the analytical solution is

$$\boldsymbol{H}(t,\boldsymbol{x}) := \begin{bmatrix}\varphi(t,z)\\ \varphi(t,z)\\ 0\end{bmatrix}, \quad \boldsymbol{E}(t,\boldsymbol{x}) := \begin{cases}\frac{1}{\sigma(t)}\,\mathbf{curl}\,\boldsymbol{H}(t,\boldsymbol{x}) & \text{in } \Omega_{\mathrm{C}}^t,\\ \boldsymbol{0} & \text{in } \Omega_{NC}^t,\end{cases}$$

with

$$\varphi(t,z) := \begin{cases} 100(z-1-t)^2(z-2-t)^2 & z \in [1+t, 2+t], \\ 0 & z \notin [1+t, 2+t]. \end{cases}$$

Notice that $\mathbf{curl}\,\boldsymbol{H}(t) = \mathbf{0}$ in Ω^t_{NC} for all $t \in [0, T]$.

To approximate this solution, we fix $\varepsilon > 0$ and solve the following problem: find $\boldsymbol{H}_\varepsilon(t) \in \mathrm{H}(\mathbf{curl}; \Omega)$ such that

$$\int_\Omega \mu_0 \partial_t \big(\boldsymbol{H}_\varepsilon(t)\big) \cdot \boldsymbol{G} + \int_{\Omega^t_{\mathrm{C}}} \frac{1}{\sigma(t)} \mathbf{curl}\,\boldsymbol{H}_\varepsilon(t) \cdot \mathbf{curl}\,\boldsymbol{G} + \frac{1}{\varepsilon} \int_{\Omega^t_{\mathrm{NC}}} \mathbf{curl}\,\boldsymbol{H}_\varepsilon(t) \cdot \mathbf{curl}\,\boldsymbol{G}$$
$$= \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{G} + \int_{\partial\Omega} \boldsymbol{E}(t) \times \boldsymbol{n} \cdot \boldsymbol{G} \quad \forall \boldsymbol{G} \in \mathrm{H}(\mathbf{curl}; \Omega),$$
$$\boldsymbol{H}_\varepsilon(0) = \boldsymbol{H}_0$$

We have solved the problem for a fixed $\varepsilon = 10^{-2}$ with several successively refined uniform meshes. The time step has been conveniently reduced to analyze the convergence with respect to both, the mesh-size and the time-step simultaneously. We have compared the obtained numerical solution with the analytical one. Figure 4.6 shows a log-log plot of the relative error in the discrete norms given in Theorem 4.17, versus the number of degrees of freedom (d.o.f.).

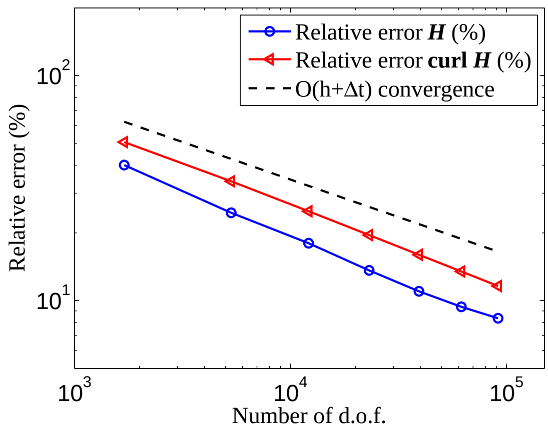

Figure 4.6: $\dfrac{\max_{1\le k\le M} \|\boldsymbol{H}(t_k) - \boldsymbol{H}^k_{h,\varepsilon}\|_{\mathrm{L}^2(\Omega)^3}}{\max_{1\le k\le M} \|\boldsymbol{H}(t_k)\|_{\mathrm{L}^2(\Omega)^3}}$ and $\dfrac{\sqrt{\Delta t}\left\{\sum_{k=1}^M \|\,\mathbf{curl}\,\boldsymbol{H}(t_k) - \mathbf{curl}\,\boldsymbol{H}^k_{h,\varepsilon})\|^2_{\mathrm{L}^2(\Omega)^3}\right\}^{1/2}}{\sqrt{\Delta t}\left\{\sum_{k=1}^M \|\mathbf{curl}\,\boldsymbol{H}(t_k)\|^2_{\mathrm{L}^2(\Omega)^3}\right\}^{1/2}}$ versus number of d.o.f. (log-log scale).

These curves show that, in the case of moving conductors, we can expect a similar order of convergence as that obtained for fixed conductors.

We also checked that the penalization parameter was correctly chosen. With this aim, we checked that the numerical results for $\varepsilon = 10^{-4}$ were the same up to rounding errors.

Problem with cylindrical symmetry. Comparison with an axisymmetric code

We consider the domain Ω with cylindrical symmetry sketched in Figure 4.7 (left) and we assume that the workpiece moves as a rigid body, with velocity $\boldsymbol{v} = 50\boldsymbol{e}_z$. Thus Ω^t_{C} is the cylinder of radius $R_{\mathrm{P}} = 1\,\mathrm{m}$ and its z-coordinate varies between $(1.2 + 50\,t, 1.4 + 50\,t)$. The coil Ω_{S} is a toroidal core of rectangular cross section S, with inner radius equal to $R_{\mathrm{s}} = 0.5\,\mathrm{m}$, outer radius $R_{\mathrm{S}} = 1\,\mathrm{m}$ and height $A_{\mathrm{s}} = 0.5\,\mathrm{m}$. The source current density is supported in Ω_{S} and given by (4.27). Note that, since the source current density field has only azimuthal non-zero component, the solution will be axisymmetric. In particular, we can solve the problem in the meridional section depicted in Figure 4.7 (right). Concerning the physical properties, we have taken $\mu = \mu_0 = 4\pi \times 10^{-7}\,\mathrm{Hm}^{-1}$ and $\sigma = 1 \times 10^4\,(\Omega\mathrm{m})^{-1}$ in the workpiece.

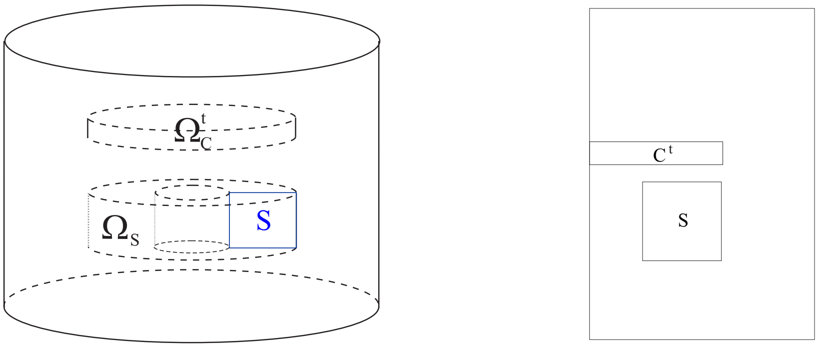

Figure 4.7: Sketch of the domain Ω (left) and its meridian section (right).

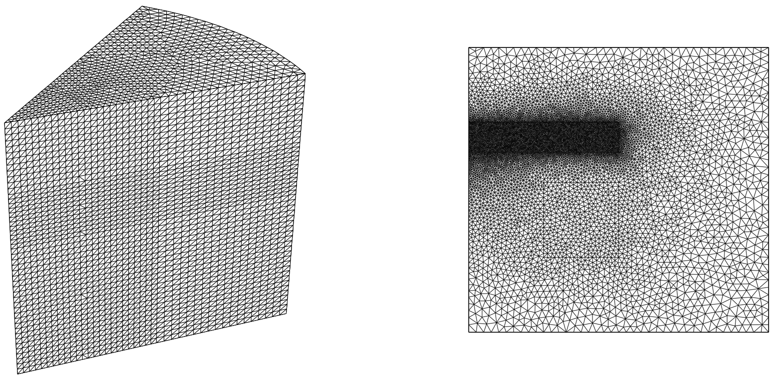

Figure 4.8: Dielectric mesh for the 3D code (left) and mesh used by the axisymmetric code (right).

In this case, there is no analytical solution, so we will asses the behavior of the method by comparing the computed results with those obtained with an axisymmetric code on the very fine mesh shown in Figure 4.8 (right) and with a very small tiem step, which will be taken as 'exact' solution.

The axisymmetric problem has been solved by using a scalar formulation written in terms of the azimuthal component of a magnetic vector potential A_θ. The corresponding weak formulation, although with boundary conditions different to those of our case, has been analyzed in [16] with moving domains. In particular, the method was proved to converge with optimal order error estimates in terms of h and Δt under appropriate assumptions. From the numerical point of view, to apply this method it is needed to approximate integrals on triangles of piecewise discontinuous functions; low quadrature rules with a large number of points are also used to do this.

We have used the axisymmetric code with natural homogeneous boundary condition, which correspond to $\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{n}=\mathbf{0}$ on $\partial\Omega$ as in our problem.

Figure 4.8 (right) shows the mesh used in the axisymmetric code. Concerning the 3D mesh, we have exploited the symmetry of the problem and solve it in 1/8 of the whole domain to reduce the number of degrees of freedom. The used mesh is shown in Figure 4.8 (left).

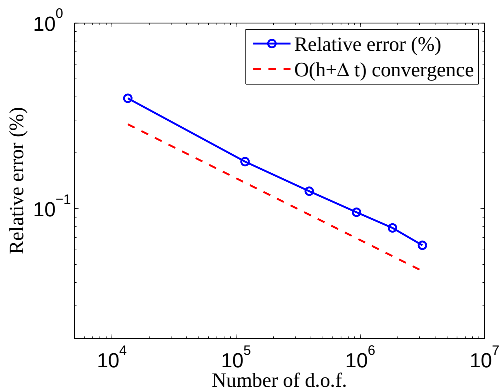

Figure 4.9: $\dfrac{\max\limits_{1\le k\le M}\|\boldsymbol{H}(t_k)-\boldsymbol{H}^k_{h,\varepsilon}\|_{\mathrm{L}^2(\Omega)^3}}{\max\limits_{1\le k\le M}\|\boldsymbol{H}(t_k)\|_{\mathrm{L}^2(\Omega)^3}}$ versus number of d.o.f. (log-log scale).

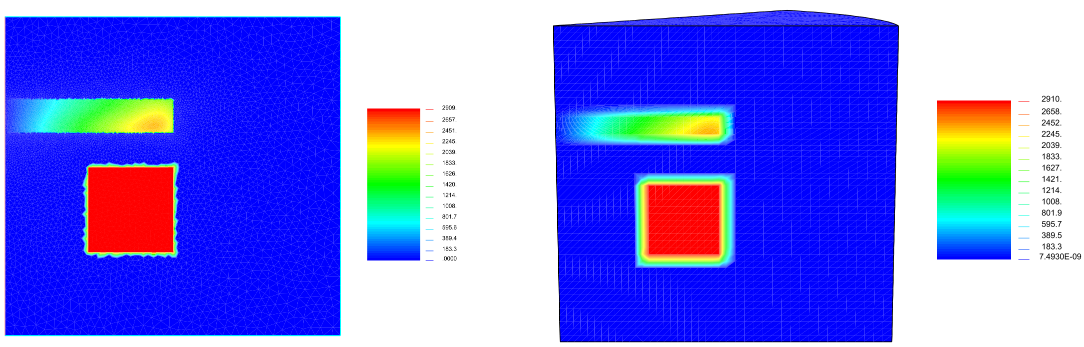

Figure 4.10: Modulus of the current density at time 0.00018 s, axisymmetric (left) and the 3D (right).

As in the previous subsection, for $\varepsilon=10^{-6}$ fixed, we have solved the problem with several successively refined meshes and a time-step conveniently reduced to analyze the convergence with respect to both, the mesh-size and the time-step simultaneously. Figure 4.9 shows a log-log plot of the relative error for the magnetic field as in Theorem 4.17, versus the number of degrees of freedom (d.o.f.). The curve shows that in the case of moving conductors the results obtained with the penalization technique converge to the 'exact' ones as h and Δt go to zero.

Finally, Figure 4.10 shows the modulus of the current density obtained with the axisymemtric and the 3D code at the time in which the input current intensity reaches its maximum (0.00018 s). We should notice that, the graph on the left shows computed values only in the workpiece, because the current density is imposed in the coil and vanishes identically in the dielectric (cf. [16, pag. 22]). In the workpiece no postprocess is needed since the computed current density is continuous and piecewise linear. The graph on the right, instead, is a continuous piecewise linear postprocess obtained from the piecewise constant values of the computed current density $\boldsymbol{J}^k_{h,\varepsilon} = \mathbf{curl}\,\boldsymbol{H}^k_{h,\varepsilon}$. This graph shows more diffusion due to this postprocess.

Chapter 5

Conclusiones y trabajo futuro

En este capítulo presentamos un resumen de las principales aportaciones de esta tesis y una descripción del trabajo futuro a desarrollar.

5.1 Conclusiones

La tesis recoge el análisis matemático y numérico de distintos modelos electromagnéticos de corrientes inducidas en dominios acotados. El estudio de estos problemas ha estado motivado por el proceso físico de conformado electromagnético, el cual requiere resolver un modelo electromagnético genuinamente transitorio con movimiento de las piezas conductoras. En este proceso, las fuentes de corriente son proporcionadas a través de un circuito eléctrico cuyos datos son intensidades de corriente o caídas de voltaje. Por ello, con el fin de tener en cuenta las distintas características del proceso de conformado, en la tesis se han estudiado varios modelos que abordan de manera gradual las dificultades del problema. En particular, el movimiento de las piezas conductoras solo ha sido considerado en el Capítulo 4.

A continuación se enumeran los resultados más importantes divididos en dos grandes grupos

1. **Dominio conductor fijo. Fuentes no locales: condiciones de contorno en términos de intensidades de corriente y/o voltajes.**

   Los resultados obtenidos para este problema son fundamentalmente en régimen transitorio. Sin embargo, paralelamente se ha obtenido un resultado relevante para el problema de corrientes inducidas en régimen armónico que se recoge en el Capítulo 1.

   - Se demostró un resultado de equivalencia entre los problemas discretos que surgen de dos formulaciones que permiten resolver el problema de corrientes inducidas en régimen armónico y con fuentes no locales en términos de intensidades de corriente. El resultado muestra a nivel discreto que la formulación clásica campo magnético/potencial escalar magnético puede reemplazarse por una formulación que combina el campo magnético con varios multiplicadores escalares y vectoriales definidos en el dieléctrico. De este modo, es posible aproximar el problema con la formulación que mayores ventajas ofrezca teniendo en cuenta la topología del dominio dieléctrico.

- Se propuso una formulación en términos del campo magnético para resolver el problema evolutivo de corrientes inducidas con fuentes no locales dadas en términos de intensidades de corriente. Para tratar la restricción $\mathbf{curl}\, \boldsymbol{H} = \mathbf{0}$ en el dieléctrico, se introdujo un potencial escalar magnético obteniéndose un importante ahorro computacional. Se demostraron resultados de existencia y unicidad de solución para el problema continuo y resultados de convergencia óptima para los esquemas discretos propuestos.
- Se propuso una formulación en términos de una primitiva del campo eléctrico para resolver el problema evolutivo de corrientes inducidas, con fuentes no locales dadas en términos de intensidades de corriente y caídas de voltaje. En este caso se imponen restricciones adicionales sobre el campo eléctrico en el dominio dieléctrico a fin de tener unívocamente determinado dicho campo en todo el dominio. Si bien en principio esta formulación requiere un dato adicional en la frontera exterior del dominio dieléctrico, se demuestra que las cantidades físicas de más relevancia son independientes de este dato adicional. Esto permite tomar por ejemplo ese dato homogéneo sin afectar la aproximación de estas cantidades.

  Se demostraron resultados de existencia y unicidad de solución para el problema continuo y resultados de convergencia óptima para los esquemas discretos propuestos. Desde el punto de vista computacional, aunque el número de incógnitas aumenta, permite resolver el problema en geometrías un poco más generales, en particular, aquellas en las que las superficies de corte no son fáciles de construir.
- Se desarrollaron códigos propios escritos en MATLAB que permiten resolver los esquemas numéricos propuestos previamente. La convergencia de los métodos numéricos ha sido validada mediante ejemplos académicos con solución analítica conocida. Además, se muestra la versatilidad de los distintos métodos para simular distintas aplicaciones, resaltando las ventajas e inconvenientes de cada formulación.

2. **Dominio conductor móvil. Fuente de corriente volúmica conocida en una región totalmente incluida en el dominio.**

- Se propuso una formulación en términos del campo magnético para resolver el problema evolutivo de corrientes inducidas con fuente volúmica dada y condiciones de contorno esenciales homogéneas. Se demostró existencia de solución para el problema continuo. En el caso de conductores fijos la solución es única. Demostrar la unicidad en el caso de conductores móviles es aún un problema abierto.
- Se propuso un método numérico basado en una técnica de penalización para resolver el problema con dominio móvil. Para evaluar la viabilidad de este método desde el punto de vista matemático y numérico se demostraron resultados de convergencia con conductores fijos. En particular, se demostró que esta técnica es viable tanto para el

problema continuo como para su discretización. Se demostraron estimaciones óptimas del error para este esquema numérico en el caso de conductores fijos.

- Se implementó el método de penalización para el caso de conductores móviles y se validó con un ejemplo analítico. Los resultados muestran que cabe esperar un orden de convergencia similar al que se demuestra para conductores fijos. Luego se aplicó el método a un problema con simetría cilíndrica y se compararon los resultados con los que se obtienen mediante un modelo axisimétrico de otra formulación (en términos del potencial vectorial magnético). Los resultados muestran que ambos métodos convergen a la misma solución.

5.2 Trabajo futuro

1. El resultado de existencia demostrado para la formulación débil planteada en el Capítulo 4 es válido si todos los materiales tienen la misma permeabilidad magnética y en consecuencia esta propiedad no cambia con el tiempo.

Se estudiará si este resultado puede extenderse a un caso más general, tratando además de avanzar en el estudio de la unicidad de dicho problema. En particular, hay que evaluar si pueden relajarse las hipótesis del teorema de unicidad presentado en [59].

2. Se estudiará si el análisis de convergencia del problema penalizado realizado para conductores fijos puede extenderse al caso de conductores en movimiento, ya que los resultados numéricos muestran convergencia.

3. Como se mencionó antes, al considerar como dato la corriente en la bobina se está despreciando el acoplamiento del sistema de conformado con el circuito eléctrico de descarga. Se estudiará si el resultado obtenido en el Capítulo 4 se extiende al problema evolutivo de corrientes inducidas con fuentes no locales dadas en términos de intensidades de corriente y caídas de voltaje.

4. El acoplamiento magneto-mecánico para abordar el problema real del conformado electromagnético solo podrá llevarse a cabo desde un punto de vista teórico si se consiguen resultados para el problema electromagnético. Desde el punto de vista numérico se tratará de explotar la técnica de penalización para simular al menos algún ejemplo acoplado con movimiento rígido (como el acoplamiento presentado en [16]).

5. No parece claro que otras formulaciones diferenciales ofrezcan ventajas para abordar el análisis del problema evolutivo de corrientes inducidas con dominio conductor que cambia con el tiempo. La dificultad fundamental de todas las formulaciones es la naturaleza distinta de las ecuaciones en conductor y dieléctrico. Sin embargo, habría que explotar las llamadas formulaciones *integro-diferenciales*, que permiten trabajar únicamente en conductores siempre que estos tengan un comportamiento lineal no magnético (ver, por ejemplo, [4, 23]). En la literatura puede encontrarse algún trabajo donde se aplican estas formulaciones para simular problemas de inducción con movimiento de alguna de sus partes [29].

6. En el desarrollo de la tesis, se ha visto que la formulación en términos de la primitiva del campo eléctrico representa una opción muy interesante para resolver el problema de corrientes inducidas. Puede aplicarse fácilmente a topologías complejas y computacionalmente no es excesivamente costosa. Sin embargo, en la literatura, las formulaciones con potenciales escalares y vectoriales $(\boldsymbol{A}, V)$ tienen una gran popularidad (ver, por ejemplo, [20]) y están implementadas en la mayoría de los códigos comerciales. Si los conductores están totalmente contenidos en el dominio, las formulaciones $(\boldsymbol{A}, V)$ y las formulaciones en $\boldsymbol{u}$ podrían relacionarse con condiciones de *gauge* adecuadas. Por ello, se tratará de establecer formalmente esta relación tanto a nivel continuo como discreto ([43]). Además, aprovechando los resultados del Capítulo 3, esta relación tratará de establecerse también con fuentes no locales (intensidades y potenciales).

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