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MIXED FINITE ELEMENT METHODS FOR TIME-DEPENDENT WAVE PROPAGATION PROBLEMS: ELASTODYNAMICS AND ELASTOACOUSTICS

(MÉTODOS DE ELEMENTOS FINITOS MIXTOS PARA PROBLEMAS DE PROPAGACIÓN DE ONDAS DEPENDIENTES DEL TIEMPO: ELASTODINÁMICA Y ELASTOACÚSTICA)

Tesis para optar al grado de Doctor en Ciencias Aplicadas con Mención en Ingeniería Matemática

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Mixed Finite Element Methods for Time-dependent Wave Propagation Problems: Elastodynamic and Elastoacoustic

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Abstract

This thesis aims at the formulation, analysis and implementation of mixed finite element methods for the time-dependent wave propagation models arising in elastodynamic and elastoacoustic. The analysis of the dependence on time of the formulations of these problems is one of the main goals of this thesis. This motivates us to follow the traditional approach for this kind of evolutive models, which consists of splitting the analysis into two steps: the first one deals with the semidiscrete scheme, in which only the space is discretized, whereas the second one refers to the fully-discrete scheme, where both space and time are discretized by using finite elements and finite differences, respectively.

For the elastodynamic problem, the unknowns of our formulation are the stress and rotation tensors, where the latter is introduced as a Lagrange multiplier taking care of the symmetry of the stress tensor in a weak sense. Once the well-posedness of the formulation is established, we introduce a semidiscretization of the problem using a family of mixed finite elements satisfying standard hypotheses for the elasticity problem with reduced symmetry, and then prove the abstract error estimates. Finally, we use the Newmark method to obtain a fully-discrete version of our formulation and derive the corresponding error estimates as well. An interesting feature of our formulation is the fact that it does not involve the displacement since this is replaced in terms of the stress thanks to the momentum equation. Nevertheless, we show that this unknown can be easily recovered later on by means of a post-processing formula, thus leading to a suitable approximation of it. Moreover, a similar post-processing technique allows to obtain a numerical approximation of the acceleration.

On the other hand, the unknowns of the formulation for the fluid-structure interaction problem are given by the stress tensor in the solid and the pressure in the fluid. Once the wellâposedness of the formulation is proved, we proceed to build an appropriate auxiliary operator whose properties are useful for the analysis of the semidiscrete scheme, and then we obtain the corresponding error estimates using the Arnold-Falk-Whinter mixed finite elements in the solid and the classic Lagrange elements in the fluid domain. The analysis of the corresponding fully-discrete scheme for this problem is performed in a separate chapter.

Finally, we remark that for both problems analyzed in this thesis, we have shown that the discrete schemes employed are uniformly stable with respect to the discretization parameters, and have derived the respective asymptotic error estimates. In particular, we prove that they are immune to locking phenomenon in the nearly incompressible case. In addition, numerical experiments supporting the theoretical results and illustrating the robustness of each method are reported.

Resumen

Esta tesis apunta a la formulación, análisis e implementación de métodos de elementos finitos mixtos para los modelos de propagación de ondas dependientes del tiempo que surgen en elastodinámica y elastoacústica. El análisis de la dependencia del tiempo de las formulaciones de estos problemas es uno de los objetivos principales de esta tesis. Ello nos motiva a seguir el enfoque tradicional para este tipo de modelos evolutivos, el cual consiste en dividir el análisis en dos pasos: el primero tiene que ver con el esquema semidiscreto, en el que sólo el espacio es discretizado, mientras que el segundo se refiere al esquema completamente discreto, donde tanto el espacio como el tiempo son discretizados usando elementos finitos y diferencias finitas, respectivamente.

Para el problema elastodinámico, las incógnitas de nuestra formulación son los tensores de esfuerzo y rotación, donde este último se introduce como un multiplicador de Lagrange que se encarga de la simetría del esfuerzo en un sentido débil. Una vez demostrado que la formulación es bien propuesta, introducimos una semidiscretización del problema utilizando una familia de elementos finitos mixtos que satisface las hipótesis usuales para el problema de elasticidad con simetría reducida, y luego probamos estimaciones de error abstractas. Por último, utilizamos el método de Newmark para obtener una versión totalmente discreta de nuestra formulación y derivar también las estimaciones de error correspondientes. Una característica interesante de nuestra formulación es el hecho que ella no involucra el desplazamiento, ya que éste se sustituye en términos del esfuerzo gracias a la ecuación de momento. Sin embargo, mostramos que esta incógnita puede ser recuperada fácilmente más tarde por medio de una fórmula de post-procesamiento, generando así una aproximación apropiada de ella. Además, una técnica de post-procesamiento similar permite obtener una aproximación numérica de la aceleración.

Por otro lado, las incógnitas de la formulación para el problema de interacción fluido-estructura están dadas por el tensor de esfuerzo en el sólido y la presión en el fluido. Una vez probado que la formulación está bien propuesta, procedemos a construir un operador auxiliar apropiado cuyas propiedades son útiles para el análisis del esquema semidiscreto, y luego obtenemos las estimaciones de error correspondientes utilizando los elementos finitos mixtos de Arnold-Falk-Whinter en el sólido y los clásicos elementos de Lagrange en el fluido. El análisis del esquema totalmente discreto correspondiente para este problema se realiza en un capítulo aparte.

Finalmente, observamos que para ambos problemas analizados en esta tesis hemos probado que los esquemas discretos son uniformemente estables con respecto a los parámetros de discretización, y hemos deducido las estimaciones de error asintóticas respectivas. En particular, probamos que son inmunes al fenómeno de bloqueo en el caso casi incompresible. Además, presentamos experimentos numéricos que apoyan los resultados teóricos e ilustran la robustez de cada método.

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Introduction

Elastic materials constitute a very useful model for many important problems in engineering sciences and physics, particularly solid mechanics, and are used for a variety of practical applications. For example, when designing a bridge, we need a good mathematical model for computing the kinematic features accurately. There are many other important areas in which elastic materials are related to, such as seismology in geophysics, and hence the study of elastic materials has been and is of great interest.

The kinematic behaviors of elastic materials are formulated mathematically as partial differential equations (PDEs). However, a partial differential equation in material science is complicated in general even in simple physical models. Not surprisingly, solving the PDE analytically is very difficult, impossible in most cases. The above motivates to introduce numerical methods that allow to obtain an approximation of the solution of an PDE, with an acceptable approximation error. The finite element method is among the most important approaches in the numerical study of solutions of PDEs, and specially, the mixed finite element method. In this thesis, we study numerical methods for time dependent problems of linear elastic solids using mixed finite element methods.

In the classical energy minimization form of linear elasticity problems, displacement is the only unknown of the equation and the numerical solution for stress is obtained by numerical differentiation using the approximate solution for displacement. In turn, in mixed methods for linear elasticity based on stress and displacement, there are two unknowns, stress and displacement. At first glance, this approach increases the number of unknowns and leads to a larger system of equations, but there are other benefits that make mixed methods attractive. A key advantage of mixed methods for linear elasticity is that they directly deliver the numerical solution for stress. Since stress is directly linked to destruction of materials, it is of great interest in engineering applications. Another advantage of mixed methods for elasticity is the robustness for nearly incompressible materials. In the displacement based approach, although the error for stress converges to zero as mesh size converges to zero, the error bound often contains a constant which is very large when a material is nearly incompressible, so we need a very small mesh size to get a sufficiently small error. This phenomenon is known in the literature as “ locking ”. However, the mixed methods in general, and in particular those developed in this thesis, do not present this deficiency.

Since there are two unknowns, we need a pair of finite element spaces for mixed methods. One subtlety in mixed methods is to find a pair of finite elements which guarantee existence of numerical solutions with good approximation properties. A choice of mixed finite element spaces is called stable if it guarantees existence of numerical solutions. Necessary and sufficient conditions for stable mixed finite elements are known thanks to the theory of Babuska and Brezzi. However, finding stable mixed finite elements for elasticity has long been a difficult problem, being the symmetry of the stress the major obstacle. Since stress is symmetric, it is natural that the finite elements for stress be symmetric as well, but it is very difficult to find stable finite element spaces that satisfy this constraint. One way of solving this problem is to impose the symmetry of stress weakly, by requiring it to be orthogonal to the space of skew-symmetric tensors. From another point of view, we introduce a skew-symmetric tensor, which is the Lagrange multiplier for the symmetry of stress, and rewrite the original elasticity problems with the Lagrange multiplier. The Lagrange multiplier is often called the rotation because it is the skew-symmetric part of the gradient of displacement. Therefore, in this approach, we have three unknowns, the stress tensor, the displacement vector, and the rotation. In [4], Arnold, Falk, and Winther introduced an exterior calculus framework for the study of mixed finite elements for elasticity. The Arnold-Falk-Winther elements are defined in two and three dimensions and for higher orders with simple descriptions and have small numbers of degrees of freedom. Later on, other elements were developed following the analysis of the exterior calculus framework. For example, in [13] Cockburn, Gopalakrishnan, and Guzmán constructed a family of elements such that the finite element spaces for stress are based on the Raviart-Thomas-Nédélec elements with additional terms using bubble functions. More recently, another family of elements was developed by Gopalakrishnan and Guzmán [28], which have fewer degrees of freedom than their previous elements with same accuracy of errors. Finally, Stenberg in [40] developed a similar finite element family.

Mixed finite element methods for elastodynamics

As we have seen in the previous section, mixed finite elements for elasticity with weak symmetry have relatively few degrees of freedom and are relatively easy to implement in both two and three dimensions. Therefore, in this thesis, we will use these elements to address the problem of elastodynamics.

In this way, in **Chapter 1**, we analyze a mixed finite element approximation of the linear elastodynamic problem with reduced symmetry. There are several families of mixed finite elements with weak symmetry for the steady elasticity problem, [4, 13, 28, 40]. Our aim here is to prove the stability of the corresponding Galerkin schemes for the elastodynamic problem in a bounded domain Ω, which, given a body force $\boldsymbol{f}$ and a finite time T, reads: find the displacement vector $\boldsymbol{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) && \text{in } \Omega\times(0,T]\,, \\ \rho\,\ddot{\boldsymbol{u}} - \mathbf{div}\,\boldsymbol{\sigma}(t) &= \boldsymbol{f}(t) && \text{in } \Omega\times(0,T]\,, \end{aligned} \tag{0.0.1}$$

in addition to appropriate boundary and initial conditions to be specified later on (cf. (1.2.1)), where $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\Big\{\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}$ is the linearized strain tensor and $\mathcal{C}$ is the elasticity stiffness tensor.

Mixed methods for (0.0.1) have already been studied in [5, 7, 11, 16, 31]. In contrast to the strong symmetry approach considered in [7, 16, 31] for the stress tensor, we are interested here in a weak imposition of this restriction, as in [5, 11]. The displacement-stress formulation method presented in [11] relies on a dual hybrid method introduced in [19] for a two-dimensional problem. Here, we follow [5] and carry out a multi-dimensional error analysis for a class of mixed finite elements satisfying conditions that are known to hold true for the mixed families introduced in [4, 13, 28, 40]. More precisely, our present approach can be formally considered as the second order version of the first order hyperbolic system studied in [5], whose main variables are the stress tensor and the velocity. As a consequence, we only maintain the stress tensor as primary unknown (besides the rotation) and end up with a classical wave equation for a tensorial grad-div operator. The advantage of our formulation is that it naturally provides an a priori error bound for the stress variable in the H(div)-norm, which improves the L^2-error estimate obtained for this variable in [5]. Moreover, our error estimates are shown to be uniform with respect to compressibility at the semi-discrete and fully discrete levels. Finally, it is worthwhile to mention that, while the displacement is not explicitly involved in our formulation, it can be numerically post-processed by integrating twice the linear momentum equation. This first contribution originally was published in the paper:

- C. García, G.N. Gatica, and S. Meddahi, *A new mixed finite element analysis of the elastodynamic equations.* Appl. Math. Lett. 59 (2016), 48–55.
- C. García, G.N. Gatica, and S. Meddahi, *A new mixed finite element method for elastodynamics with weak symmetry.* J. Sci. Comput., to appear.

Mixed finite element methods for a elastoacoustic problem

Straight away in **Chapter 2**, we aim to compute the vibrations of an elastic structure enclosing in its interior an inviscid compressible fluid. The model problem consists in a scalar-valued equation describing the propagation of acoustic pressure waves and a vector-valued equation modeling the propagation of elastic waves. The two systems are coupled through adequate transmission conditions on the contact boundary. This problem is known as elastoacoustic problem.

Traditionally, a displacement formulation in the solid is combined with a formulation using either the acoustic pressure (as in [18]) or the fluid displacement (as in [8]) as main variables in the fluid domain. The displacement-pressure formulation studied in [18] leads to a non-symmetric weak formulation involving time derivatives on boundary terms. However, the convergence analysis in [18] becomes a bit cumbersome since it requires complicated energy spaces. In addition, computational effort is needed to solve the non-symmetric linear systems of equations resulting from any implicit time-discretization of the semi-discrete formulation. On the other hand, the displacement-displacement formulation introduced in [8] is symmetric but it is not adapted to deal with nearly incompressible elastic materials. Similar drawbacks hold with the approaches developed in [38], which is based on the velocity in the solid and the pressure in the fluid. Nevertheless, an interesting feature of the method in [38] is its ability to deal with meshes that are non-conforming at the contact boundary. This permits to mesh independently the domains corresponding to each media in order to handle efficiently situations in which big disparities between the fluid and solid waves lengths occur.

More recently, dual-mixed formulations have been considered in the solid for the static elastoacoustic source problem (see, e.g., [24] and [25]). This approach may be considered as the dual procedure to the one proposed in [8]. In such a case, the Cauchy stress tensor is used as a main variable in the solid structure, in combination with the pressure in the fluid domain. The resulting formulation is symmetric and delivers direct finite element approximations of the stresses. In addition, it has been shown that an approximation scheme based on the Lagrange and Arnold-Falk-Winther (AFW) [4] finite elements in the fluid and solid domains, respectively, provides a stable Galerkin method in the nearly incompressible case. Moreover, it has been proved in [35] that the former mixed finite element method provides a spectrally correct approximation of the corresponding eigenproblem.

In the **Chapter 2**, we use the same Galerkin method for a space discretization of the elastoacoustic problem in the time domain transient version and conclude that it provides the same convergence and stability performances shown in [35] and [25] for the spectral and the static source problems, respectively. However, the technical tools used in the present analysis, being based on a new suitable projector and its discrete approximation, differ from the ones used in [25].

More precisely, we prove the stability of the AFW/Lagrange finite element scheme when the Lamé coefficient λ tends to infinity and when the mesh size h goes to 0, and then we establish asymptotic error estimates. In other words, the mixed scheme that we propose in this chapter is free of the locking phenomenon. In addition, the above mentioned symmetry yields discrete linear systems with the same property which facilitates their resolution. Moreover, we emphasize that the method proposed here provides direct approximations of the stress, thus avoiding the numerical differentiation and the corresponding loss of accuracy arising from the use of a displacement-based formulation to approximate that tensor. This feature of our mixed method is very important for many applications, among which the simulation of earthquakes becomes one of the most significant ones. The importance of approximating the stress by means of a mixed formulation has also been emphasized in several other places (see, e.g. [5] and the references therein), in which the authors highlight that this kind of approach can be easily extended to materials with more complex constitutive equations, such as viscoelasticity, plasticity and poroelasticity. Finally, while as said before our unknowns are only the stress tensor in the solid and the pressure in the fluid, we are also able to provide approximations of the displacement and the velocity through suitable postprocessing formulae, and to derive the associated rates of convergence. The contents presented in this chapter originally were published in the paper:

- C. García, G.N. Gatica, and S. Meddahi, *Finite element analysis of a pressure-stress formulation for the time-domain fluid-structure interaction problem.* IMA J. Numer. Anal., to appear.

Finally, in **Chapter 3**, we complete the study given in the chapter 2 by carrying out the convergence analysis of an implicit time integration based on the Newmark trapezoidal rule. Following the steps given in [21, Section 6], we establish the unconditional stability of the resulting fully discrete method when the mesh parameters h and Δt go to 0 and when the Lamé coefficient λ tends to infinity. Finally, we prove that if the $k^{\text{th}}-$order Arnold-Falk-Winther element and the $k^{\text{th}}-$order Lagrange element ($k \geq 1$) are used in the solid and the fluid domains, respectively, then the error exhibits a combined space-time asymptotic behaviour given by $O(h^k) + O((\Delta t)^2)$. The contents of this chapter appears in the following preprint:

- C. García, G.N. Gatica, A. Márquez and S. Meddahi , *A fully discrete scheme for the pressure-stress formulation of the time-domain fluid-structure interaction problem.* Preprint 2017-03, Centro de Investigacion en Ingenieria Matematica, Universidad de Concepcion, (2017).
  Available at: `http://www.ci2ma.udec.cl/publicaciones/prepublicaciones/`

Notations and preliminary results

We denote by $\boldsymbol{I}$ the identity matrix of $\mathbb{R}^{d\times d}$ $(d = 2, 3)$, and $\mathbf{0}$ represents the null vector in $\mathbb{R}^d$ or the null tensor in $\mathbb{R}^{d\times d}$. Given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbb{R}^{d\times d}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, the trace $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^d \tau_{ii}$, the deviatoric tensor $\boldsymbol{\tau}^{\mathtt{D}} := \boldsymbol{\tau} - \frac{1}{d}(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I}$, and the tensor inner product $\boldsymbol{\tau} : \boldsymbol{\sigma} := \sum_{i,j=1}^d \tau_{ij}\sigma_{ij}$. Let Ω be a polyhedral Lipschitz bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$, with boundary $\partial\Omega$. We denote by $\mathcal{D}(\Omega)$ the space of indefinitely differentiable functions with compact support in Ω. For $s \in \mathbb{R}$, $\|\cdot\|_{s,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^s(\Omega)$, $\mathbf{H}^s(\Omega) := \mathrm{H}^s(\Omega)^d$ or $\mathbb{H}^s(\Omega) := \mathrm{H}^s(\Omega)^{d\times d}$, with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also denote by $(\cdot,\cdot)$ the inner product in $\mathrm{L}^2(\Omega)$, $\boldsymbol{L}^2(\Omega) := \mathrm{L}^2(\Omega)^d$ or $\mathbb{L}^2(\Omega) := \mathrm{L}^2(\Omega)^{d\times d}$. We introduce the Hilbert space

$$\mathbb{H}(\mathbf{div},\Omega) := \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \ \mathbf{div}\,\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)\right\},$$

whose norm is given by $\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega)} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega}$. Since we will deal with a time-domain problem, besides the Sobolev spaces defined above, we need to introduce spaces of functions defined on a bounded time interval $(0,T)$ and with values in a separable Hilbert space V, whose norm is denoted here by $\|\cdot\|_V$. For $1 \le p \le \infty$, $\mathrm{L}^p(V)$ is the space of classes of functions $f : \ (0,T) \to V$ that are Böchner-measurable and such that $\|f\|_{\mathrm{L}^p(V)} < \infty$, with

$$\|f\|^p_{\mathrm{L}^p(V)} := \int_0^T \|f(t)\|^p_V \,\mathrm{d}t \quad (1 \le p < \infty), \quad \|f\|_{\mathrm{L}^\infty(V)} := \operatorname*{ess\,sup}_{[0,T]} \|f(t)\|_V .$$

We use the notation $\mathcal{C}^0(V)$ for the Banach space consisting of all continuous functions $f : \ [0,T] \to V$. More generally, for any $k \in \mathbb{N}$, $\mathcal{C}^k(V)$ denotes the subspace of $\mathcal{C}^0(V)$ of all functions f with (strong) derivatives $\dfrac{\mathrm{d}^j f}{\mathrm{d}t^j}$ in $\mathcal{C}^0(V)$ for all $1 \le j \le k$. In what follows, we will use indistinctly the notations

$$\dot{f} := \frac{\mathrm{d}f}{\mathrm{d}t} \quad \text{and} \quad \ddot{f} := \frac{\mathrm{d}^2 f}{\mathrm{d}t^2}$$

to express the first and second derivatives with respect to the variable t. Furthermore, we will use the Sobolev space

$$\mathrm{W}^{1,p}(V) := \left\{ f : \ \exists g \in \mathrm{L}^p(V) \text{ and } \exists f_0 \in V \text{ such that} \right.$$
$$\left. f(t) = f_0 + \int_0^t g(s)\,\mathrm{d}s \quad \forall t \in [0,T] \right\},$$

and denote $\mathrm{H}^1(V) := \mathrm{W}^{1,2}(V)$. The space $\mathrm{W}^{k,p}(V)$ is defined recursively for all $k \in \mathbb{N}$.

Finally, we need to recall a classical result that will be recurrently used in the following and that concerns the well-posedness of a variational problem defined in terms of a bilinear form satisfying the inf-sup condition. Indeed, given two Hilbert spaces $\big(S, \langle\cdot,\cdot\rangle_S\big)$ and $\big(Q, \langle\cdot,\cdot\rangle_Q\big)$ and a bounded bilinear form $\mathcal{A} : S \times Q \to \mathbb{R}$, we let $\mathbf{A} : S \to Q$ be the bounded linear operator induced by $\mathcal{A}$, that is $\langle \mathbf{A}(s), q\rangle_Q = \mathcal{A}(s,q) \quad \forall\, (s,q) \in S \times Q$, and introduce the null space

$$N(\mathbf{A}) := \Big\{ s \in S : \ \mathbf{A}(s) = 0 \Big\} \ = \ \Big\{ s \in S : \ \mathcal{A}(s,q) = 0 \quad \forall\, q \in Q \Big\}$$

and its polar

$$N(\mathbf{A})^\circ \ := \ \Big\{ \chi \in S' ; \quad \chi(s) = 0 \quad \forall\, s \in N(\mathbf{A}) \Big\} \,.$$

In addition, we let $\mathcal{R}_S : S' \to S$ be the corresponding Riesz operator. Then, we have the following theorem (cf. [12]).

Theorem 0.0.1. *Assume that there exists $\kappa > 0$ such that*

$$\|\mathbf{A}^*(q)\| \ := \sup_{0 \neq s \in S} \frac{\mathcal{A}(s,q)}{\|s\|_S} \geq \kappa \, \|q\|_Q \qquad \forall\, q \in Q \,. \tag{0.0.2}$$

Then, for each $\ell \in N(\mathbf{A})^\circ$ there exists a unique $q \in Q$ such that $\mathbf{A}^*(q) = \mathcal{R}_S(\ell)$, that is

$$\mathcal{A}(s,q) = \ell(s) \quad \forall\, s \in S \,.$$

Proof. It suffices to see that (0.0.2) establishes, equivalently, that $\mathbf{A}^*$ is injective and has closed range $R(\mathbf{A}^*)$, whence $R(\mathbf{A}^*) \ = \ N(\mathbf{A})^\perp \ = \ \mathcal{R}_S\big(N(\mathbf{A})^\circ\big)$. □

Throughout this paper we use C (with or without subscripts) to denote generic constants independent of the parameters indicated at each instance. We point out that these constants may take different values at different places.

Introducción

Los materiales elásticos constituyen un modelo muy útil para muchos problemas importantes en ciencias de la ingeniería y en física, particularmente en la mecánica de sólidos, y se utilizan para una variedad de aplicaciones prácticas. Por ejemplo, al diseñar un puente, necesitamos un buen modelo matemático para calcular las características cinemáticas con precisión. Hay muchas otras áreas importantes en las que se relacionan materiales elásticos, como la sismología en geofísica, por lo que el estudio de materiales elásticos ha sido y sigue siendo de gran interés.

Los comportamientos cinemáticos de los materiales elásticos se formulan matemáticamente como ecuaciones diferenciales parciales(EDPs). Sin embargo, una ecuación diferencial parcial en ciencia de materiales es, en general, difícil de resolver, incluso en modelos físicos simples. No es sorprendente el hecho de que, resolver una EDP es analíticamente muy difícil, imposible en la mayoría de los casos. Lo anterior motiva introducir métodos numéricos que permitan obtener una aproximación de la solución de una EDP, con un error de aproximación aceptable. El método de los elementos finitos está entre una de los herramientas más importantes en el estudio numérico de soluciones de EDPs, y especialmente el método de los elementos finitos mixtos. En esta tesis, estudiamos métodos numéricos para problemas dependientes del tiempo de sólidos linealmente elásticos usando elementos finitos mixtos.

Al usar la estrategia clásica de minimizar un funcional de energía en problemas de elasticidad lineal, el desplazamiento es la único incógnita de la ecuación y la solución numérica para el tensor de esfuerzos se obtiene por medio de diferenciación numérica a partir de la solución aproximada del desplazamiento. Por otro lado, en los métodos mixtos para elasticidad lineal hay dos incógnitas, el tensor de esfuerzos y el desplazamiento. A primera vista, este enfoque aumenta el número de incógnitas y conduce a un sistema de ecuaciones más grande, pero existen otros beneficios que hacen que los métodos mixtos sean atractivos. Una ventaja clave de los métodos mixtos para elasticidad lineal es que entregan directamente la solución numérica para el tensor de esfuerzos. Dado que el tensor de esfuerzos está directamente relacionado con el colapso de materiales, es de gran interés el cálculo de este tensor en aplicaciones de ingeniería. Otra ventaja de los métodos mixtos para elasticidad es su robustez para materiales casi incompresibles. En el enfoque basado en el desplazamiento, aunque el error de aproximación para el esfuerzo converge a cero a medida que el tamaño de malla disminuye, la constante que acota este error es muy grande en el caso de un material casi incompresible, por lo que necesitamos un tamaño de malla mucho más pequeño para obtener una error lo suficientemente aceptable. Este fenómeno se conoce en la literatura como “bloqueo numérico”. Sin embargo, los métodos mixtos en general, y en particular los que desarrollamos en esta tesis, no presentan esta deficiencia.

Como hay dos incógnitas en los métodos mixtos, necesitamos dos espacios de elementos finitos para estos métodos. Un aspecto clave en los métodos mixtos es encontrar espacios de elementos finitos que garanticen la existencia de soluciones numéricas con buenas propiedades de aproximación. Una elección de espacios de elementos finitos mixtos se llama estable si garantiza la existencia de soluciones numéricas. Se conocen condiciones necesarias y suficientes para espacios de elementos finitos estables gracias a la teoría de Babuska y Brezzi. Sin embargo, la búsqueda de espacios de elementos finitos mixtos estables para la elasticidad ha sido durante mucho tiempo un problema difícil, siendo la simetría del tensor de esfuerzos el principal obstáculo. Dado que el esfuerzo es simétrico, es natural pedir que las aproximaciones por elementos finitos para el esfuerzo también sean simétricos, pero es muy difícil encontrar espacios de elementos finitos estables que satisfagan esa restricción. Una manera de resolver este problema es imponer la simetría del tensor de esfuerzos débilmente, requiriendo que éste sea ortogonal al espacio de los tensores antisimétricos. Desde otro punto de vista, introducimos un tensor asimétrico, el cual corresponde al multiplicador de Lagrange para la simetría del esfuerzo, y reescribimos los problemas de elasticidad originales con el multiplicador de Lagrange. El multiplicador de Lagrange es, en este contexto, llamado rotación porque corresponde a la parte asimétrica del gradiente del desplazamiento. Por lo tanto, en este enfoque, tenemos tres incógnitas, el tensor de esfuerzos, el vector de desplazamiento y la rotación. En [4] Arnold, Falk y Winther introdujeron la teoría del cálculo exterior para el estudio de elementos finitos mixtos para la elasticidad. Los elementos de Arnold-Falk-Winther se definen en dos y tres dimensiones, incluyendo órdenes superiores con descripciones simples y pocos grados de libertad. Después de este trabajo, se desarrollaron otros elementos finitos siguiendo el análisis hecho con la teoría del cálculo exterior. Por ejemplo, en [13] Cockburn, Gopalakrishnan, y Guzmán construyeron una familia de elementos finitos tales que el espacio de aproximación para el esfuerzoi está basado los elementos de Raviart-Thomas-Nédélec con términos adicionales usando funciones burbuja. Más recientemente, otra familia de elementos fue desarrollado por Gopalakrishnan y Guzmán [28], los cuales tienen menos grados de libertad que los elementos previos pero mantienen la misma precisión para los errores. Finalmente, Stenberg en [40] desarrolló una familia de elementos finitos similar.

Métodos de elementos finitos mixtos para elastodinámica

Como hemos visto en la sección anterior, los elementos finitos mixtos para la elasticidad con simetría débil tienen relativamente pocos grados de libertad y son relativamente fáciles de implementar tanto en dos como en tres dimensiones. Por lo tanto, en esta tesis, usaremos esos elementos para abordar el problema de la elastodinámica.

De este modo, en el **Capítulo 1**, analizamos un método de elementos finitos mixtos para el problema elastodinámico con simetría reducida. Nuestro objetivo aqui es probar la estabilidad del correspondiente esquema de Galerkin para el problema elastodinámico en un dominio acotado Ω, el cual, dada una fuerza $\boldsymbol{f}$ y un tiempo finito T, se lee como: Encontrar el vector desplazamiento the $\boldsymbol{u}$ y el tensor de esfuerzos de Cauchy $\boldsymbol{\sigma}$ tal que

$$\begin{aligned} \boldsymbol{\sigma}(t) \,&=\, \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) && \text{in } \Omega\times(0,T]\,, \\ \rho\,\ddot{\boldsymbol{u}} \,-\, \mathbf{div}\,\boldsymbol{\sigma}(t) \,&=\, \boldsymbol{f}(t) && \text{in } \Omega\times(0,T]\,, \end{aligned} \tag{0.0.3}$$

además de apropiadas condiciones iniciales y de borde que especificaremos posteriormente (cf. (1.2.1)), aqui $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\Big\{\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}$ es el tensor de pequeñas deformaciones y $\mathcal{C}$ es el tensor de rígidez.

Los métodos mixtos para (0.0.3) ya han sido estudiados en [5, 7, 11, 16, 31]. En contraste con el enfoque de imponer de manera fuerte la simetría del tensor de esfuerzos considerado en [7, 16, 31], aquí estamos interesado en imponer de manera débil esta restricción, de manera similar a lo hecho en [5, 11]. La formulación desplazamiento-esfuerzos presentada en [11] se basa en un método dual hı́brido introducido en [19] para un problema en dos dimensiones. Aquí, seguimos lo hecho en [5] y llevamos a cabo un análisis de error multi-dimensional para una clase de elementos finitos mixtos que satisfacen condiciones, las que verifican las familias de elementos finitos mixtos introducidas en [4, 13, 28, 40]. Más precisamente, nuestro enfoque del problema puede ser considerao formalmente como la versión de segundo orden del sistema hiperbólico de primer orden estudiado en [5], cuyas principales incógnitas son el tensor de esfuerzos y la velocidad. En consecuencia, sólo mantenemos el tensor de esfuerzos como incógnita principal(junto con la rotación) y obtenemos la clásica ecuación de ondas en términos de los operadores gradiente y divergente. La ventaja de nuestra formulación es que proporciona de manera natural una estimación de error a priori para el tensor de esfuerzos en la norma H(div), lo cual mejora la estimación de error en norma L^2 obtenida para esta variable en [5]. Además, nuestras estimaciones de error resultan uniformes con respecto al caso casi incompresible tanto a nivel semidiscreto como a nivel completamente discreto. Finalmente, vale la pena mencionar que, si bien el desplazamiento no está explícitamente incorporado en nuestra formulación, éste puede ser recuperado numéricamente por medio de un post-proceso el cual se basa en integrar dos veces respecto al tiempo la ecuación de momentum. Esta primera contribución se publicó en:

- C. García, G.N. Gatica, and S. Meddahi, *A new mixed finite element analysis of the elastodynamic equations*. Appl. Math. Lett. 59 (2016), 48–55.
- C. García, G.N. Gatica, and S. Meddahi, *A new mixed finite element method for elastodynamics with weak symmetry*. J. Sci. Comput., to appear.

Métodos de elementos finitos mixtos para elastoacústica

En el **Capítulo 2**, nuestro objetivo es calcular las vibraciones de una estructura sólida elástica que encierra en su interior un fluido compresible no viscoso. El problema del modelo consiste en una ecuación a valores escalares que describe la propagación de ondas de presión acústica(en el medio acústico) y una ecuación a valores vectoriales que modela la propagación de ondas elásticas (en el medio elástico). Los dos sistemas se acoplan a través de apropiadas condiciones de transmisión en la superfice de contacto. Este problema es conocido como problema de la elastoacústica.

Tradicionalmente, una formulación con el vector desplazamiento en el sólido se combina con una formulación con la presión (como en [18]) o bien con una formulación con el desplazamiento (como en [8]) en el fluido como incógnitas principales en el fluido. La formulación desplazamiento-presión estudiada en [18] conduce a una formulación débil no simétrica que contiene derivadas temporales sobre los términos del borde. Sin embargo, el análisis de la convergencia hecho en [18] se vuelve algo engorroso ya que requiere espacios de energía más complejos. Además, un mayor costo computacional es necesario para resolver los sistemas lineales no simétricos resultantes de cualquier discretización completamente discreta.

Por otro lado, la formulación desplazamiento-desplazamiento introcida en [8] es simétrica pero no está adaptada para tratar el caso casi-incompresible. Inconvenientes similares se encuentran en el enfoque del problema desarrollado en [38], el cual se basa en una formulación con la velocidad en el sólido y la presión en el fluido. No obstante, una característica interesante del método desarrollado en [38] es su habilidad para tratar con mallas no-conformes en el superficie de contacto. Esto permite usar mallas independientes en cada dominio (sólido y fluido) con el objeto de tratar eficientemente situaciones en que aparezcan grandes disparidades entre las longitudes de las ondas en el fluido y en el sólido.

Más recientemente, formulaciones duales-mixtas han sido consideradas en el sólido para el problema estacionario de la elastoacústica con término fuente (ver por ejemplo, [24] y [25]). Este enfoque puede ser visto como el procedimiento dual del propuesto en [8]. En tal caso, el tensor de esfuerzos de Cauchy es usado como variable principal en la estructura sólida, en combinación con la presión en el fluido. La formulación resultante es simétrica y proporciona directamente aproximaciones por elementos finitos para el esfuerzo. Además, se ha demostrado que un esquema de aproximación basado en los elementos de Lagrange y los elementos de Arnold-Falk-Winther(AFW) [4], en los dominios fluido y sólido respectivamente, constituyen un esquema de Galerkin estable en el caso casi incompresible. Incluso, se ha demostrado en [35] que tal método de elementos finitos mixtos proporciona una aproximación correcta del correspondiente problema de valores propios.

En el **Capítulo 2**, usamos el mismo esquema de Galerkin para una discretización espacial para una versión transiente (dependiente del tiempo) del problema elastoacústico y concluimos que podemos obtener los mismos resultados de convergencia y estabilidad que en [35] y [25] en donde se analizaron las versiones espectrales y estáticas, respectivamente. Sin embargo, las herramientas usadas en nuestro análisis, que se basan en definir apropidamente un nuevo proyector junto con su versión discreta, difieren de las técnicas usadas en [25].

Más precisamente, probamos la estabilidad del esquema de elementos finitos basado en elementos AFW/Lagrange cuando el coeficiente de Lamé λ tiende a infinito y cuando el tamaño de malla h convrege a 0, y entonces establecemos las correspondientes estimaciones asintóticas del error. En otras palabras, el esquema mixto que proponemos en este capítulo es inmune al fenómenos del "bloque numérico". Además, la simetría del problema anteriormente mencionada, se traduce en que los correspondientes sistemas lineales sean también simétricos, lo cual facilita su resolución. Más aún, mostramos que el método propuesto aquí proporciona una aproximación directa para el tensor de esfuerzos, esto evita hacer uso de la diferenciación numérica que trae la correspondiente pérdida de precisión, técnica usada para aproximar el tensor de esfuerzos a partir de una formulación basada en el desplazamiento. Ésta característica de nuestro método mixto es muy importante para muchas aplicaciones, como la simulación de terremotos por mencionar una de las más significativas. La importancia de aproximar el esfuerzo por medio de una formulación mixta ha sido señalada en muchas publicaciones (ver [5] y sus referencias), en donde los autores destacan que este tipo de enfoque puede ser fácilmente extendido a materiales con ecuaciones constitutivas más complejas, tales como viscoelasticidad, plasticidad y poroelasticidad. Finalmente, si bien es cierto como se dijo anteriormente aunque nuestras incógnitas son sólo el tensor de esfuerzos en el sólido y la presión en el fluido, también estamos en posición de proporcionar aproximaciones para el desplazamiento y la velocidad a a través de una apropiada fórmula de post-proceso y obtener las correspondientes razones de convergencia. Los contenidos presentados en este capítulo se pueden encontrar en la publicación:

- C. García, G.N. Gatica, and S. Meddahi, *Finite element analysis of a pressure-stress formulation for the time-domain fluid-structure interaction problem.* IMA J. Numer. Anal., to appear.

Finalmente, en el **capítulo 3**, completamos el estudio dado en el capítulo 2 realizando el análisis de la convergencia de esquema completamente discreto basado en el método de Newmark implícito. Siguiendo los pasos dados en [21, Section 6], establecemos la que el método completamente discreto es incondicionalmente estable cuando los parámetros de tamaño de malla h y el paso de tiempo Δt convergen a 0 y cuando el coeficiente de Lamé λ tiende a infinito. Finalmente, probamos que al usar los elementos de Arnold-Falk-Winther de orden $k^{\text{th}}-$ y los elementos de Lagrange de orden $k^{\text{th}}-$ (con $k \geq 1$) en los dominios del sólido y del fluido respectivamente, entonces el error muestra un comportamiento espacio-tiempo dado por $O(h^k) + O((\Delta t)^2)$. Los contenidos de este capítulo se encuentran en el siguiente preprint:

- C. García, G.N. Gatica, A. Márquez and S. Meddahi , *A fully discrete scheme for the pressure-stress formulation of the time-domain fluid-structure interaction problem.* Preprint 2017-03, Centro de Investigacion en Ingenieria Matematica, Universidad de Concepcion, (2017).
  Available at: `http://www.ci2ma.udec.cl/publicaciones/prepublicaciones/`

Chapter 1

A new mixed finite element method for elastodynamics with weak symmetry

1.1 A wave equation in $\mathbb{H}(\mathbf{div}, \Omega)$

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open bounded Lipschitz polygonal/polyhedral domain. We denote by $\boldsymbol{n}$ the outward unit normal vector to $\partial\Omega$. We consider a subset $\emptyset \neq \Gamma \subset \partial\Omega$ and denote its complement by $\Sigma := \partial\Omega \setminus \Gamma$. We consider the closed subspace of $\mathbb{H}(\mathbf{div}, \Omega)$ given by

$$\mathcal{W} := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega) : \quad \langle \boldsymbol{\tau n}, \boldsymbol{v} \rangle_{\partial\Omega} = 0 \quad \forall \boldsymbol{v} \in \mathbf{H}^{1/2}(\partial\Omega),\ \boldsymbol{v}|_\Gamma = \mathbf{0} \right\},$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ stands for the duality pairing between $\mathbf{H}^{-1/2}(\partial\Omega)$ and $\mathbf{H}^{1/2}(\partial\Omega)$ with respect to the $\boldsymbol{L}^2(\partial\Omega)$-inner product. Alternatively, recalling that the restriction of $\boldsymbol{\tau n}$ to Σ belongs to $\mathbf{H}_{00}^{-1/2}(\Sigma) := \mathbf{H}_{00}^{1/2}(\Sigma)'$, where $\mathbf{H}_{00}^{1/2}(\Sigma)$ is the subspace of functions in $\mathbf{H}^{1/2}(\Sigma)$ whose extensions by zero on Γ are in $\mathbf{H}^{1/2}(\partial\Omega)$, we can also set

$$\mathcal{W} := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega) : \quad \boldsymbol{\tau n} = 0 \quad \text{on} \quad \Sigma \right\}.$$

Next, we assume that $\{\Omega_j, \quad j = 1 \cdots, J\}$ is a set of polygonal/polyhedral disjoint partition of $\bar{\Omega}$, i.e.,

$$\Omega_j \cap \Omega_i = \emptyset \quad \text{for all } 1 \leq i \neq j \leq J \quad \text{and} \quad \bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j.$$

Then we consider piecewise constant functions $\mu(\boldsymbol{x})$, $\lambda(\boldsymbol{x})$, and $\rho(\boldsymbol{x})$ defined, for $j = 1, \ldots, J$, by $\mu|_{\Omega_j} := \mu_j > 0$, $\lambda|_{\Omega_j} := \lambda_j > 0$, and $\rho|_{\Omega_j} := \rho_j > 0$, and assume that there exist positive constants $\underline{\mu}$, $\bar{\mu}$, $\underline{\lambda}$, $\bar{\lambda}$, $\underline{\rho}$, and $\bar{\rho}$, such that

$$\underline{\mu} \leq \mu_j \leq \bar{\mu}, \quad \underline{\lambda} \leq \lambda_j \leq \bar{\lambda} \quad \text{and} \quad \underline{\rho} \leq \rho_j \leq \bar{\rho} \quad (1 \leq j \leq J).$$

In turn, we introduce on $\boldsymbol{L}^2(\Omega)$ the inner product $(\boldsymbol{u}, \boldsymbol{v})_\rho := (\rho^{-1}\boldsymbol{u}, \boldsymbol{v})$ and denote the corresponding norm

$$\|\boldsymbol{v}\|_\rho := \sqrt{(\boldsymbol{v}, \boldsymbol{v})_\rho}.$$

In addition, we consider the elasticity stiffness tensor $\mathcal{C}$ defined by $\mathcal{C}\boldsymbol{\tau} := \lambda\,(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I} + 2\mu\boldsymbol{\tau}$ and recall that its inverse (the compliance tensor) is given by

$$\mathcal{C}^{-1}\boldsymbol{\tau} = \frac{1}{2\mu}\left\{\boldsymbol{\tau} - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}\right\}.$$

We endow $\mathbb{L}^2(\Omega)$ with the norm

$$\|\boldsymbol{\tau}\|^2_{\mathcal{C}} := (\boldsymbol{\tau},\boldsymbol{\tau})_{\mathcal{C}} := (\mathcal{C}^{-1}\boldsymbol{\tau},\boldsymbol{\tau}) = \int_\Omega \frac{1}{2\mu}\boldsymbol{\tau}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \frac{1}{d(d\lambda + 2\mu)}\operatorname{tr}(\boldsymbol{\tau})^2. \tag{1.1.1}$$

The following result proves that

$$\|\boldsymbol{\tau}\|^2_{\mathcal{C},\mathrm{div}} := \|\boldsymbol{\tau}\|^2_{\mathcal{C}} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{\rho}$$

is a Hilbertian norm on $\boldsymbol{\mathcal{W}}$ that is equivalent to the $\mathbb{H}(\mathbf{div},\Omega)$-norm uniformly in λ.

Lemma 1.1.1. *There exists a constant $\alpha > 0$, independent of λ, such that*

$$\alpha\,\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega)} \le \|\boldsymbol{\tau}\|^2_{\mathcal{C},\mathrm{div}} \le \max(\frac{1}{2\underline{\mu}},\frac{1}{\underline{\rho}})\,\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega)} \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}. \tag{1.1.2}$$

Proof. We deduce from (1.1.1) that

$$\frac{1}{2\bar{\mu}}\left\|\boldsymbol{\tau}^{\mathtt{D}}\right\|^2_{0,\Omega} \le \|\boldsymbol{\tau}\|^2_{\mathcal{C}} \le \frac{1}{2\underline{\mu}}\|\boldsymbol{\tau}\|^2_{0,\Omega} \quad \forall \boldsymbol{\tau} \in \mathbb{L}^2(\Omega), \tag{1.1.3}$$

which gives the upper bound of (1.1.2). Next, given $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div},\Omega)$, we let $\boldsymbol{\tau}_0 := \boldsymbol{\tau} - \frac{1}{d|\Omega|}\left(\int_\Omega \operatorname{tr}\boldsymbol{\tau}\right)\boldsymbol{I}$. It is proved in [12, Proposition 9.1.1] that there exists $C_0 > 0$, depending only on Ω, such that

$$\|\boldsymbol{\tau}_0\|^2_{0,\Omega} \le C_0\left(\left\|\boldsymbol{\tau}^{\mathtt{D}}\right\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega}\right) \qquad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div},\Omega).$$

On the other hand, it is shown in [26, Lemma 2.5] (see also [23, Lemma 2.2]) that there exists $C_1 > 0$, depending only on Ω, such that

$$\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega)} \le C_1\,\|\boldsymbol{\tau}_0\|^2_{\mathrm{H}(\mathbf{div},\Omega)} \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

The lower bound of (1.1.2) follows now directly from (1.1.3), the last two inequalities, and the fact that $\mathbf{div}\,\boldsymbol{\tau}_0 = \mathbf{div}\,\boldsymbol{\tau}$ in Ω. $\square$

We now introduce the space of skew symmetric tensors

$$\boldsymbol{\mathcal{Q}} := \left\{\boldsymbol{s} \in \mathbb{L}^2(\Omega); \quad \boldsymbol{s} = -\boldsymbol{s}^{\mathtt{t}}\right\},$$

and observe that the subspace $\boldsymbol{\mathcal{S}}$ of symmetric tensors in $\boldsymbol{\mathcal{W}}$ can be written, equivalently, as

$$\boldsymbol{\mathcal{S}} := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}; \quad (\boldsymbol{\tau},\boldsymbol{s}) = 0 \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}\}.$$

In addition, we notice that $\boldsymbol{\mathcal{S}}$ is closed in $\mathbb{H}(\mathbf{div},\Omega)$, and hence, the density of $\mathbb{H}(\mathbf{div},\Omega)$ in $\mathbb{L}^2(\Omega)$ proves that $\boldsymbol{\mathcal{S}}$ is also densely embedded in $\mathbb{L}^2_{\mathrm{sym}}(\Omega) := \left\{\boldsymbol{s} \in \mathbb{L}^2(\Omega); \quad \boldsymbol{s} = \boldsymbol{s}^{\mathtt{t}}\right\}$. We may then identify $\mathbb{L}^2_{\mathrm{sym}}(\Omega)$ with its dual space and consider the Gelfand triple

$$\boldsymbol{\mathcal{S}} \hookrightarrow \mathbb{L}^2_{\mathrm{sym}}(\Omega) \hookrightarrow \boldsymbol{\mathcal{S}}',$$

where $\boldsymbol{\mathcal{S}}'$ is the dual space of $\boldsymbol{\mathcal{S}}$. The following inf-sup condition ([1, 9]) is essential in the forthcoming analysis: there exists $\beta > 0$ such that

$$\sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}} \frac{(\boldsymbol{\tau},\boldsymbol{s}) + (\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{v})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \geq \beta(\|\boldsymbol{s}\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega}), \tag{1.1.4}$$

for all $(\boldsymbol{s},\boldsymbol{v}) \in \boldsymbol{\mathcal{Q}} \times \boldsymbol{L}^2(\Omega)$.

Having established the above notations and preliminary results, we now introduce the wave equation in $\mathbb{H}(\mathbf{div},\Omega)$. Indeed, given $\boldsymbol{f} \in \mathrm{L}^1(\boldsymbol{L}^2(\Omega))$, $\boldsymbol{\sigma}_0 \in \boldsymbol{\mathcal{S}}$, $\boldsymbol{\sigma}_1 \in \mathbb{L}^2_{\mathrm{sym}}(\Omega)$, and $\boldsymbol{r}_0, \boldsymbol{r}_1 \in \boldsymbol{\mathcal{Q}}$, we consider the problem:

$$\begin{aligned}
&\text{Find } \boldsymbol{\sigma} \in \mathrm{L}^\infty(\boldsymbol{\mathcal{W}}) \cap \mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega)) \text{ and } \boldsymbol{r} \in \mathrm{W}^{1,\infty}(\boldsymbol{\mathcal{Q}}) \text{ such that}\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathcal{C}^{-1}\boldsymbol{\sigma}(t) + \boldsymbol{r}(t),\boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\sigma}(t),\mathbf{div}\,\boldsymbol{\tau})_\rho \;=\; -(\boldsymbol{f}(t),\mathbf{div}\,\boldsymbol{\tau})_\rho\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\boldsymbol{\sigma}(t),\boldsymbol{s}) \;=\; 0\,,
\end{aligned} \tag{1.1.5}$$

for all $(\boldsymbol{\tau},\boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$, and such that the following initial conditions are satisfied:

$$\begin{aligned}
\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0\,, &\qquad \dot{\boldsymbol{\sigma}}(0) = \boldsymbol{\sigma}_1,\\
\boldsymbol{r}(0) = \boldsymbol{r}_0\,, &\qquad \dot{\boldsymbol{r}}(0) = \boldsymbol{r}_1.
\end{aligned} \tag{1.1.6}$$

We notice here that the second equation of (1.1.5) is the weak imposition of the symmetry of $\boldsymbol{\sigma}$, where $\boldsymbol{r}$ is the corresponding Lagrange multiplier. In this way, testing in particular the first equation of (1.1.5) with $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}$, we arrive at the following reduced form of (1.1.5):

$$\begin{aligned}
&\text{Find } \boldsymbol{\sigma} \in \mathrm{L}^\infty(\boldsymbol{\mathcal{S}})) \cap \mathrm{W}^{1,\infty}(\mathbb{L}^2_{\mathrm{sym}}(\Omega)) \text{ such that}\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathcal{C}^{-1}\boldsymbol{\sigma}(t),\boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\sigma}(t),\mathbf{div}\,\boldsymbol{\tau})_\rho = -(\boldsymbol{f}(t),\mathbf{div}\,\boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}\,,\\
&\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \dot{\boldsymbol{\sigma}}(0) = \boldsymbol{\sigma}_1.
\end{aligned} \tag{1.1.7}$$

In what follows it will be useful to consider the energy functional $\mathcal{E} : W^{1,\infty}(\mathbb{H}(\mathbf{div},\Omega)) \to \mathrm{L}^\infty((0,T))$ defined by

$$\mathcal{E}(\boldsymbol{\tau})(t) := \frac{1}{2}\,\|\dot{\boldsymbol{\tau}}(t)\|^2_{\mathcal{C}} + \frac{1}{2}\,\|\mathbf{div}\,\boldsymbol{\tau}(t)\|^2_\rho \quad \forall \boldsymbol{\tau} \in W^{1,\infty}(\mathbb{H}(\mathbf{div},\Omega))\,, \quad \forall t \in [0,T]\,. \tag{1.1.8}$$

Lemma 1.1.2. *Assume that* $\boldsymbol{f} \in \mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega))$. *Then, problem* (1.1.7) *admits at least a solution and there exists a constant* $C > 0$ *such that*

$$\operatorname*{ess\,sup}_{[0,T]} \mathcal{E}(\boldsymbol{\sigma})^{1/2}(t) \leq C\Big\{\,\|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{\sigma}_0\|_{\mathcal{C},\mathrm{div}} + \|\boldsymbol{\sigma}_1\|_{\mathcal{C}}\,\Big\}\,. \tag{1.1.9}$$

Proof. We only give a sketch of the proof since it follows by the classical Galerkin procedure (cf. [17, 37]). In fact, we first consider a family of finite dimensional subspaces $\{\boldsymbol{\mathcal{S}}_n\}$ of $\boldsymbol{\mathcal{S}}$ such that, for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}$,

$$\lim_{n\to\infty} \inf_{\boldsymbol{\tau}_n\in\boldsymbol{\mathcal{S}}_n} \|\boldsymbol{\tau} - \boldsymbol{\tau}_n\|_{\mathrm{H}(\mathbf{div},\Omega)} = 0.$$

Then we denote by $\boldsymbol{\sigma}_{0,n}$ the $(\boldsymbol{\mathcal{S}},\ \|\cdot\|_{\mathcal{C},\mathrm{div}})$-orthogonal projection of $\boldsymbol{\sigma}_0$ onto $\boldsymbol{\mathcal{S}}_n$ and by $\boldsymbol{\sigma}_{1,n}$ the $(\mathbb{L}^2_{\mathrm{sym}}(\Omega), \|\cdot\|_{\mathcal{C}})$-orthogonal projection of $\boldsymbol{\sigma}_1$ onto $\boldsymbol{\mathcal{S}}_n$. It is easy to show, by using the classical ODE theory, that the problem:

$$\begin{aligned}
&\text{Find } \boldsymbol{\sigma}_n \in \mathcal{C}^1(\boldsymbol{\mathcal{S}}_n) \text{ such that,}\\
&(\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}_n(t), \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\sigma}_n(t), \mathbf{div}\, \boldsymbol{\tau})_\rho = -(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}_n\,,\\
&\boldsymbol{\sigma}_n(0) = \boldsymbol{\sigma}_{0,n}\,, \quad \dot{\boldsymbol{\sigma}}_n(0) = \boldsymbol{\sigma}_{1,n}\,,
\end{aligned} \tag{1.1.10}$$

admits a unique solution. The first step of the proof reduces to deriving energy estimates for $\boldsymbol{\sigma}_n(t)$. To this end, we take $\boldsymbol{\tau} = \dot{\boldsymbol{\sigma}}_n(t)$ in (1.1.10) and integrate the resulting identity over $(0, t)$, which gives

$$\mathcal{E}(\boldsymbol{\sigma}_n)(t) - \mathcal{E}(\boldsymbol{\sigma}_n)(0) = -\int_0^t (\boldsymbol{f}(s), \mathbf{div}\, \dot{\boldsymbol{\sigma}}_n(s))_\rho \, \mathrm{d}s.$$

Next, integrating by parts the right-hand side yields

$$\mathcal{E}(\boldsymbol{\sigma}_n)(t) \;=\; \int_0^t (\dot{\boldsymbol{f}}(s), \mathbf{div}\, \boldsymbol{\sigma}_n(s))_\rho \, \mathrm{d}s - (\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\sigma}_n(t))_\rho \;+\; (\boldsymbol{f}(0), \mathbf{div}\, \boldsymbol{\sigma}_{0,n})_\rho + \mathcal{E}(\boldsymbol{\sigma}_n)(0)\,. \tag{1.1.11}$$

We now notice, according to the definition of $\boldsymbol{\sigma}_{0,n}$ and $\boldsymbol{\sigma}_{1,n}$, that

$$\mathcal{E}(\boldsymbol{\sigma}_n)(0) \le \frac{1}{2}\, \|\boldsymbol{\sigma}_0\|^2_{\mathcal{C},\mathrm{div}} + \frac{1}{2}\, \|\boldsymbol{\sigma}_1\|^2_{\mathcal{C}}\;.$$

In turn, using the Sobolev embedding $\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega)) \hookrightarrow \mathcal{C}^0(\boldsymbol{L}^2(\Omega))$ (see [39, Lemma 7.1]) and the Cauchy-Schwarz inequality, we deduce easily from (1.1.11) that there exists a constant $C > 0$ such that

$$\max_{[0,T]} \mathcal{E}(\boldsymbol{\sigma}_n)^{1/2}(t) \le C \Big\{ \, \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{\sigma}_0\|_{\mathcal{C},\mathrm{div}} + \|\boldsymbol{\sigma}_1\|_{\mathcal{C}} \, \Big\}\,. \tag{1.1.12}$$

It follows from (1.1.12) that $(\dot{\boldsymbol{\sigma}}_n)_n$ is uniformly bounded in $\mathrm{L}^\infty(\mathbb{L}^2_{\mathrm{sym}}(\Omega))$ and $(\boldsymbol{\sigma}_n)_n$ is uniformly bounded in $\mathrm{L}^\infty(\boldsymbol{\mathcal{S}})$. We can then extract a weak-$*$ convergent subsequence (also denoted $(\boldsymbol{\sigma}_n)_n$) satisfying

$$\begin{aligned}
&\int_0^T -(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}_n(t), \boldsymbol{\tau})\dot{\psi}(t) \;+\; (\mathbf{div}\, \boldsymbol{\sigma}_n(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \psi(t)\, \mathrm{d}t\\
&\quad = -\int_0^T (\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \psi(t)\, \mathrm{d}t \;+\; \psi(0)\, (\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}_n(0), \boldsymbol{\tau})
\end{aligned} \tag{1.1.13}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}_n$ and for all $\psi \in \mathcal{C}^1([0,T])$ such that $\psi(T) = 0$. A classical procedure shows that the limit $\boldsymbol{\sigma} \in \mathrm{L}^\infty(\boldsymbol{\mathcal{S}}) \cap \mathrm{W}^{1,\infty}(\mathbb{L}^2_{\mathrm{sym}}(\Omega))$ of the subsequence $(\boldsymbol{\sigma}_n)_n$ satisfies

$$\begin{aligned}
&\int_0^T -(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t), \boldsymbol{\tau})\dot{\psi}(t) \;+\; (\mathbf{div}\, \boldsymbol{\sigma}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \psi(t)\, \mathrm{d}t\\
&\quad = -\int_0^T (\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \psi(t)\, \mathrm{d}t \;+\; \psi(0)\, (\mathcal{C}^{-1}\boldsymbol{\sigma}_1, \boldsymbol{\tau})
\end{aligned} \tag{1.1.14}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}$ and for all $\psi \in \mathcal{C}^1([0,T])$ such that $\psi(T) = 0$. This proves that $\boldsymbol{\sigma}$ solves (1.1.7) if the time derivative is interpreted in the sense of distributions. In addition, we notice that $\boldsymbol{\sigma}_n$ also converges weakly to $\boldsymbol{\sigma}$ in $\mathrm{H}^1(\mathbb{L}^2_{\mathrm{sym}}(\Omega))$ and hence $\boldsymbol{\sigma}_n(0)$ converges weakly to $\boldsymbol{\sigma}(0)$ in $\mathbb{L}^2_{\mathrm{sym}}(\Omega)$. Moreover, since $\boldsymbol{\sigma}_n(0) = \boldsymbol{\sigma}_{0,n}$ converges to $\boldsymbol{\sigma}_0$ in $\mathbb{L}^2(\Omega)$ as well, we conclude that the initial condition $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ is meaningful. Furthermore, it is clear from (1.1.14) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t), \boldsymbol{\tau}) = -(\mathbf{div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}\,, \tag{1.1.15}$$

from which it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t)$ belongs to $\mathrm{L}^1(\boldsymbol{\mathcal{S}}')$, and thus $\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t) \in \mathrm{W}^{1,1}(\boldsymbol{\mathcal{S}}') \hookrightarrow \mathcal{C}^0(\boldsymbol{\mathcal{S}}')$. Next, testing (1.1.15) with $\psi \in \mathcal{C}^1([0,T])$ such that $\psi(T) = 0$ yields

$$\begin{aligned} &\int_0^T -(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(t), \boldsymbol{\tau})\dot{\psi}(t) \,+\, (\mathbf{div}\,\boldsymbol{\sigma}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \psi(t)\,\mathrm{d}t \\ &= -\int_0^T (\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \psi(t)\,\mathrm{d}t \,+\, \psi(0)\,\left\langle \mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}(0), \boldsymbol{\tau}\right\rangle_{\boldsymbol{\mathcal{S}}} \end{aligned} \tag{1.1.16}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}$, where $\langle \cdot, \cdot \rangle_{\boldsymbol{\mathcal{S}}}$ stands for the duality bracket between $\boldsymbol{\mathcal{S}}'$ and $\boldsymbol{\mathcal{S}}$ pivotal $\mathbb{L}^2_{\mathrm{sym}}(\Omega)$, and hence, comparing (1.1.14) with (1.1.16) we deduce that $\dot{\boldsymbol{\sigma}}(0) = \boldsymbol{\sigma}_1$ in $\mathbb{L}^2_{\mathrm{sym}}(\Omega)$. Finally, the stability estimate (1.1.9) is obtained by taking the limit in (1.1.12). □

Lemma 1.1.3. *The solution of problem* (1.1.7) *is unique.*

Proof. Assume that $\boldsymbol{\sigma}$ is a solution of (1.1.7) with homogeneous data $\boldsymbol{f}(t) = \mathbf{0}$ and $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_1 = \mathbf{0}$. We proceed as in [17, 37] and consider with $s \in (0,T)$ fixed

$$\boldsymbol{w}(t) = \begin{cases} -\int_t^s \boldsymbol{\sigma}(z)\,\mathrm{d}z & t < s \\ 0 & t \geq s \end{cases} \;\in\; \mathrm{W}^{1,2}(\boldsymbol{\mathcal{S}})\,.$$

Then, testing (1.1.7) with $\boldsymbol{w}(t)$ and integrating by parts in the time variable, we obtain

$$\int_0^T (\mathbf{div}\,\boldsymbol{\sigma}(t), \mathbf{div}\,\boldsymbol{w}(t))_\rho - (\dot{\boldsymbol{\sigma}}(t), \dot{\boldsymbol{w}}(t))_{\mathcal{C}}\,\mathrm{d}t \,=\, 0\,,$$

which can be rewritten as

$$\frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{div}\,\boldsymbol{w}(t)\|^2_\rho - \|\boldsymbol{\sigma}(t)\|^2_{\mathcal{C}} \Big)\mathrm{d}t = 0\,.$$

It follows that $\|\mathbf{div}\,\boldsymbol{w}(0)\|^2_\rho + \|\boldsymbol{\sigma}(s)\|^2_{\mathcal{C}} = 0$, and the proof is finished. □

It is important to remark that, following [37, Section 11.2.4], one can also show that the solution $\boldsymbol{\sigma}$ to problem (1.1.7) is actually in $\mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{\mathrm{sym}}(\Omega))$.

Theorem 1.1.1. *Assume that* $\boldsymbol{f} \in \mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega))$. *Then problem* (1.1.5) *admits a unique solution. Moreover, there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\max_{[0,T]} \|\boldsymbol{\sigma}(t)\|_{\mathrm{H}(\mathbf{div},\Omega)} \,+\, \max_{[0,T]} \|\dot{\boldsymbol{\sigma}}(t)\|_{0,\Omega} \,+\, \|\boldsymbol{r}(t)\|_{\mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega))} \\ &\quad\leq\; C\left\{ \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega))} \,+\, \|\boldsymbol{\sigma}_0\|_{\mathrm{H}(\mathbf{div},\Omega)} + \|\boldsymbol{\sigma}_1\|_{0,\Omega} \,+\, \|\boldsymbol{r}_0\|_{0,\Omega} + \|\boldsymbol{r}_1\|_{0,\Omega} \right\}. \end{aligned} \tag{1.1.17}$$

Proof. We only have to prove the existence and uniqueness of the Lagrange multiplier $\boldsymbol{r}$. To this end, we consider $\mathcal{G} \in \mathcal{C}^1(\boldsymbol{\mathcal{W}}')$ given by

$$\begin{aligned} \Big\langle \mathcal{G}(t), \boldsymbol{\tau} \Big\rangle_{\boldsymbol{\mathcal{W}}} \;:=\; &(\mathcal{C}^{-1}\boldsymbol{\sigma}(t), \boldsymbol{\tau}) + \int_0^t \Big(\int_0^s (\mathbf{div}\,\boldsymbol{\sigma}(z) + \boldsymbol{f}(z), \mathbf{div}\,\boldsymbol{\tau})_\rho\,\mathrm{d}z \Big)\mathrm{d}s \\ &- t(\mathcal{C}^{-1}\boldsymbol{\sigma}_1 + \boldsymbol{r}_1, \boldsymbol{\tau}) - (\mathcal{C}^{-1}\boldsymbol{\sigma}_0 + \boldsymbol{r}_0, \boldsymbol{\tau})\,, \end{aligned} \tag{1.1.18}$$

where $\left\langle \cdot,\cdot \right\rangle_{\boldsymbol{\mathcal{W}}}$ denotes the duality bracket between $\boldsymbol{\mathcal{W}}'$ and $\boldsymbol{\mathcal{W}}$ pivotal $\mathbb{L}^2(\Omega)$. Then, taking $\boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}$ in (1.1.18), using that $\boldsymbol{r}_0, \boldsymbol{r}_1 \in \boldsymbol{\mathcal{Q}}$, and integrating (1.1.7) twice with respect to time, we deduce that

$$\left\langle \mathcal{G}(t), \boldsymbol{\tau} \right\rangle_{\boldsymbol{\mathcal{W}}} = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}\,,$$

which means that $\mathcal{G}(t)$ belongs to the polar set of $\boldsymbol{\mathcal{S}}$ in $\boldsymbol{\mathcal{W}}'$. Moreover, $\boldsymbol{\mathcal{S}}$ is the kernel of $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \ni (\boldsymbol{\tau}, \boldsymbol{r}) \mapsto \int_\Omega \boldsymbol{\tau} : \boldsymbol{r}$ and (1.1.4) implies that this bilinear form satisfies the inf-sup condition for the pair $\{\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{Q}}\}$, which guarantees the existence of $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ such that

$$(\boldsymbol{r}(t), \boldsymbol{\tau}) = -\left\langle \mathcal{G}(t), \boldsymbol{\tau} \right\rangle_{\boldsymbol{\mathcal{W}}} \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}. \tag{1.1.19}$$

We conclude that the pair $\{\boldsymbol{\sigma}, \boldsymbol{r}\}$ solves the first equation of (1.1.5) by differentiating twice the last identity in the sense of distributions with respect to t. Moreover, evaluating (1.1.19) and its time derivative at $t = 0$, we deduce that $\boldsymbol{r}(0) = \boldsymbol{r}_0$ and $\dot{\boldsymbol{r}}(0) = \boldsymbol{r}_1$. Finally, using the inf-sup condition (1.1.4), the Cauchy-Schwarz inequality and (1.1.2), we deduce that there exists $C_1 > 0$ such that

$$\begin{aligned} \beta \, \|\dot{\boldsymbol{r}}(t)\|_{0,\Omega} &\leq \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}} \frac{\int_\Omega \dot{\boldsymbol{r}}(t) : \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} = \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}} \frac{\left\langle \dot{\mathcal{G}}(t), \boldsymbol{\tau} \right\rangle_{\boldsymbol{\mathcal{W}}}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\ &\leq \; C_1 \left\{ \|\boldsymbol{\sigma}_1\|_{0,\Omega} + \|\boldsymbol{r}_1\|_{0,\Omega} + \max_{[0,T]} \mathcal{E}(\boldsymbol{\sigma})^{1/2}(t) + \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\mathrm{L}^2(\Omega)^n)} \right\}. \end{aligned} \tag{1.1.20}$$

Finally, we deduce from the fundamental theorem of calculus that

$$\|\boldsymbol{\sigma}(t)\|_{\mathcal{C}} \leq T \max_{[0,T]} \|\dot{\boldsymbol{\sigma}}(t)\|_{\mathcal{C}} + \|\boldsymbol{\sigma}_0\|_{\mathcal{C}} \quad \text{and} \quad \|\boldsymbol{r}(t)\|_{0,\Omega} \leq T \max_{[0,T]} \|\dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \|\boldsymbol{r}_0\|_{0,\Omega}\,, \tag{1.1.21}$$

so that (1.1.17) is obtained by combining (1.1.9), (1.1.20), and (1.1.21). □

1.2 Relationship with the elastodynamic problem

We assume that Ω represents an isotropic and linearly elastic body with mass density ρ and Lamé coefficients μ and λ. The solid is assumed to be fixed at Γ and free of stresses on Σ. The elastodynamic equations with body force $\boldsymbol{f} : \Omega \times [0,T] \to \mathbb{R}^d$ and initial data $\boldsymbol{u}_0, \boldsymbol{u}_1 : \Omega \to \mathbb{R}$ are given by

$$\begin{aligned} \rho \ddot{\boldsymbol{u}} - \mathbf{div}\, \mathcal{C} \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \boldsymbol{f}(t) \qquad & \text{in } \Omega \times (0,T], \\ \boldsymbol{u}(t) = \mathbf{0} \qquad & \text{on } \Gamma \times (0,T], \\ \mathcal{C} \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \boldsymbol{n} = \mathbf{0} \qquad & \text{on } \Sigma \times (0,T], \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad & \text{in } \Omega, \\ \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 \qquad & \text{in } \Omega, \end{aligned} \tag{1.2.1}$$

where $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is the displacement field and $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\left\{ \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathtt{t}} \right\}$ is the linearized strain tensor. In order to establish a relationship between problems (1.2.1) and (1.1.5) we need to introduce the subspace

$$\mathbb{V} := \left\{ (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}; \quad (\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}, \boldsymbol{\tau}) + (\boldsymbol{\sigma}, \boldsymbol{s}) = 0 \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}} \right\}, \tag{1.2.2}$$

where

$$\boldsymbol{\mathcal{K}} := \left\{ \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}; \quad \mathbf{div}\, \boldsymbol{\tau} = \mathbf{0} \right\}.$$

Lemma 1.2.1. *The linear operator* $D : \mathbb{V} \to \boldsymbol{L}^2(\Omega)$ *uniquely characterized by*

$$(\mathbf{div}\,\boldsymbol{\tau}, D(\boldsymbol{\sigma},\boldsymbol{r})) = -(\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}, \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \tag{1.2.3}$$

is well-defined and bounded.

Proof. We deduce from (1.1.4) that the bilinear form $(\boldsymbol{\tau},\boldsymbol{v}) \mapsto \int_\Omega \mathbf{div}\,\boldsymbol{\tau}\cdot\boldsymbol{v}$ satisfies the inf-sup condition for the pair $\{\boldsymbol{\mathcal{W}}, \boldsymbol{L}^2(\Omega)\}$. Moreover, by definition of $\mathbb{V}$, the linear form $\boldsymbol{\tau} \mapsto (\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}, \boldsymbol{\tau})$ vanishes identically on the kernel $\boldsymbol{\mathcal{K}}$ of this bilinear form. This proves the existence of a unique $D(\boldsymbol{\sigma},\boldsymbol{r}) \in \boldsymbol{L}^2(\Omega)$ satisfying (1.2.3). $\square$

We are now ready to give the main result of this section.

Theorem 1.2.1. *We consider the same right hand side* $\boldsymbol{f}$ *in* (1.1.5) *and* (1.2.1)*, and assume that the initial data of these problems satisfy*

$$(\boldsymbol{\sigma}_0,\boldsymbol{r}_0)\,,\ (\boldsymbol{\sigma}_1,\boldsymbol{r}_1) \in \mathbb{V}\,, \quad \boldsymbol{u}_0 := D(\boldsymbol{\sigma}_0,\boldsymbol{r}_0)\,, \quad \textit{and} \quad \boldsymbol{u}_1 := D(\boldsymbol{\sigma}_1,\boldsymbol{r}_1)\,. \tag{1.2.4}$$

Then

$$\boldsymbol{u}(t) := \int_0^t \left\{ \int_0^s \rho^{-1}\Big(\mathbf{div}\,\boldsymbol{\sigma}(z) + \boldsymbol{f}(z)\Big)\, dz \right\} ds + \boldsymbol{u}_0 + t\boldsymbol{u}_1 \tag{1.2.5}$$

solves the (primal) weak formulation of problem (1.2.1)*. Moreover, the solution* $(\boldsymbol{\sigma}(t),\boldsymbol{r}(t))$ *of* (1.1.5) *coincides with the stress and rotation tensors associated with* $\boldsymbol{u}(t)$*, that is*

$$\boldsymbol{\sigma}(t) = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad \textit{and} \quad \boldsymbol{r}(t) = \frac{1}{2}\left[\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\right]. \tag{1.2.6}$$

Proof. We first notice that, testing (1.1.5) with $(\boldsymbol{\tau},\boldsymbol{s}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ and taking into account (1.2.4), we deduce that $(\boldsymbol{\sigma}(t),\boldsymbol{r}(t)) \in \mathbb{V}$ for all $t \in [0,T]$. Hence, Lemma 1.2.1 ensures that there exists a unique $\boldsymbol{u}(t) := D(\boldsymbol{\sigma}(t),\boldsymbol{r}(t)) \in \boldsymbol{L}^2(\Omega)$ satisfying

$$(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}(t)) = -(\mathcal{C}^{-1}\boldsymbol{\sigma}(t) + \boldsymbol{r}(t), \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \quad \forall t \in [0,T]. \tag{1.2.7}$$

On the other hand, integrating the first equation of (1.1.5) twice with respect to time yields,

$$(\mathcal{C}^{-1}\boldsymbol{\sigma}(t) + \boldsymbol{r}(t), \boldsymbol{\tau}) = (\mathcal{C}^{-1}\boldsymbol{\sigma}_0 + \boldsymbol{r}_0, \boldsymbol{\tau}) + t(\mathcal{C}^{-1}\boldsymbol{\sigma}_1 + \boldsymbol{r}_1, \boldsymbol{\tau}) - \int_0^t \Big(\int_0^s (\mathbf{div}\,\boldsymbol{\sigma}(z) + \boldsymbol{f}(z), \mathbf{div}\,\boldsymbol{\tau})_\rho\, \mathrm{d}z \Big) \mathrm{d}s.$$

Comparing the last identity with (1.2.7) we deduce that $\boldsymbol{u}$ is given by (1.2.5). On the other hand, recalling that $\mathcal{D}(\Omega)$ stands for the space of indefinitely differentiable functions with compact support in Ω, and testing (1.2.7) with a tensor $\boldsymbol{\tau} \in \mathcal{D}(\Omega)^{d\times d}$, yields

$$\nabla\boldsymbol{u}(t) = \mathcal{C}^{-1}\boldsymbol{\sigma}(t) + \boldsymbol{r}(t) \in \mathbb{L}^2(\Omega)\,, \tag{1.2.8}$$

so that considering the symmetric and skew symmetric parts of this identity gives (1.2.6). Then, integrating by parts the left-hand side of (1.2.7) and using the last identity yields

$$\Big\langle \boldsymbol{\tau}\cdot\boldsymbol{n}, \boldsymbol{u}(t) \Big\rangle_{\partial\Omega} = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

Consequently, $\boldsymbol{u} \in \mathcal{C}^1(\mathbf{H}^1_\Gamma(\Omega))$ where $\mathbf{H}^1_\Gamma(\Omega) := \left\{\boldsymbol{v} \in \mathbf{H}^1(\Omega); \quad \boldsymbol{v}|_\Gamma = \mathbf{0}\right\}$. Multiplying (1.2.5) by ρ, testing with $\boldsymbol{v} \in \mathbf{H}^1_\Gamma(\Omega)$, integrating by parts in space and differentiating twice in time we find that $\boldsymbol{u} \in \mathcal{C}^1(\mathbf{H}^1_\Gamma(\Omega))$ satisfies the weak displacement-based variational formulation of (1.2.1), namely,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho \boldsymbol{u}(t), \boldsymbol{v}) + (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})(t), \boldsymbol{\varepsilon}(\boldsymbol{v})) = (\boldsymbol{f}(t), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\Gamma(\Omega)\,,$$

and the result follows. □

Henceforth, we assume that condition (1.2.4) is satisfied, which will permit us to interpret the solution pair $(\boldsymbol{\sigma}, \boldsymbol{r})$ of (1.1.5) as the stress tensor and the rotation associated to the solution $\boldsymbol{u}$ of (1.2.1).

1.3 Semi-discretization in space

1.3.1 Finite element subspaces

We consider finite dimensional families of subspaces

$$\boldsymbol{\mathcal{W}}_h \subset \boldsymbol{\mathcal{W}} \qquad \boldsymbol{\mathcal{Q}}_h \subset \boldsymbol{\mathcal{Q}} \qquad \mathcal{U}_h \subset \boldsymbol{L}^2(\Omega)$$

indexed with a parameter $h \to 0$, and assume that there holds

$$\lim_{h\to 0} \left\{ \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div},\Omega)} + \inf_{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{r} - \boldsymbol{s}_h\|_{0,\Omega} + \inf_{\boldsymbol{v}_h \in \mathcal{U}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{0,\Omega} \right\} = 0 \tag{1.3.1}$$

for all $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$, $\boldsymbol{r} \in \boldsymbol{\mathcal{Q}}$ and $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$. Besides the approximation property (1.3.1) we need to impose conditions ensuring that the triple of spaces $\{\boldsymbol{\mathcal{W}}_h, \mathcal{U}_h, \boldsymbol{\mathcal{Q}}_h\}$ provides a stable Galerkin approximation method for the dual-mixed formulation of the (steady state) elasticity problem with weak symmetry. By virtue of the Babuška-Brezzi theory, such a stability is guaranteed by the following two hypotheses and Lemma 1.1.1 (see [4]).

Hypothesis 1. *There exists $\beta^* > 0$, independent of h, such that*

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{(\boldsymbol{\tau}, \boldsymbol{s}) + (\mathbf{div}\, \tau, \boldsymbol{v})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \geq \beta^*(\|\boldsymbol{s}\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega}), \tag{1.3.2}$$

for all $(\boldsymbol{s}, \boldsymbol{v}) \in \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h$.

Hypothesis 2. $\mathbf{div}(\boldsymbol{\mathcal{W}}_h) = \mathcal{U}_h$ *and* $\rho^{-1}\,\mathbf{div}(\boldsymbol{\mathcal{W}}_h) = \mathcal{U}_h$.

We point out that in practice, as ρ is assumed to be a piecewise constant function, we will be able to choose the triangulations upon which the finite element spaces $\boldsymbol{\mathcal{W}}_h$ and $\mathcal{U}_h$ are constructed in such a way that the two conditions of Hypothesis 2 are equivalent.

Finally, we assume the existence of an operator satisfying the following stability and commuting diagram properties.

Hypothesis 3. *There exists a linear operator* $\Pi_h : \mathbb{H}(\mathbf{div}, \Omega) \cap \mathbb{H}^\epsilon(\Omega) \to \boldsymbol{\mathcal{W}}_h$*, with* $\epsilon > 0$*, such that*

$$\|\Pi_h \boldsymbol{\tau}\|_{0,\Omega} \leq C \Big\{ \|\boldsymbol{\tau}\|_{\epsilon,\Omega} + \|\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega} \Big\} \quad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega) \cap \mathbb{H}^\epsilon(\Omega) \tag{1.3.3}$$

for a constant $C > 0$ *independent of* h *and*

$$\mathbf{div}\, \Pi_h \boldsymbol{\tau} = U_h\, \mathbf{div}\, \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\, \Omega) \cap \mathbb{H}^\epsilon(\Omega)\,, \tag{1.3.4}$$

where U_h *is the orthogonal projection from* $(\boldsymbol{L}^2(\Omega), \|\cdot\|_{0,\Omega})$ *onto* $\mathcal{U}_h$*.*

Remark 1.3.1. *Regarding Hypothesis 3, we recall from [34, Lemmas 3.2 and 3.4] that when the Lamé coefficients* λ *and* μ *are constant in* Ω*, the space* $\mathbb{V}$ *(cf.* (1.2.2)*) is indeed contained in* $\mathbb{H}^\epsilon(\Omega) \times \mathbb{H}^\epsilon(\Omega)$ *for some* $\epsilon \in (0, 1]$*. However, since in the present case these coefficients are assumed to be piecewise constant only, we need to explicitly require the* $\mathbb{H}^\epsilon$*-regularity.*

We now introduce the discrete analogue of $\mathbb{V}$ (cf. (1.2.2)), that is

$$\mathbb{V}_h := \Big\{ (\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h; \quad (\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{r}_h, \boldsymbol{\tau}) + (\boldsymbol{\sigma}_h, \boldsymbol{s}) = 0 \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h \Big\},$$

where

$$\boldsymbol{\mathcal{K}}_h := \Big\{ \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h; \quad \mathbf{div}\, \boldsymbol{\tau} = \mathbf{0} \Big\}.$$

Then, the discrete version of Lemma 1.2.1 reads as follows.

Lemma 1.3.1. *The linear operator* $D_h : \mathbb{V}_h \to \mathcal{U}_h$ *uniquely characterized by*

$$(\mathbf{div}\, \boldsymbol{\tau}, D_h(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)) = -(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{r}_h, \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h. \tag{1.3.5}$$

is well-defined and uniformly bounded.

Proof. The result is obtained by following the same steps given in the proof of its continuous counterpart and by using the discrete inf-sup condition (1.3.2). □

1.3.2 An auxiliary operator

In order to facilitate our analysis we now introduce an auxiliary operator $\varXi$ and its discrete counterpart $\varXi_h$. More precisely, we define

$$\begin{array}{rcl} \varXi : \boldsymbol{\mathcal{W}} & \to & \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \times \boldsymbol{L}^2(\Omega) \\ \boldsymbol{\sigma} & \mapsto & \varXi \boldsymbol{\sigma} := (\boldsymbol{\sigma}^*, \boldsymbol{r}^*, \boldsymbol{u}^*), \end{array}$$

where $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*, \boldsymbol{u}^*) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \times \boldsymbol{L}^2(\Omega)$ is the solution of

$$\begin{array}{rcll} (\mathcal{C}^{-1}\boldsymbol{\sigma}^* + \boldsymbol{r}^*, \boldsymbol{\tau}) + (\boldsymbol{u}^*, \mathbf{div}\, \boldsymbol{\tau}) & = & 0 & \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \\ (\mathbf{div}\, \boldsymbol{\sigma}^*, \boldsymbol{v}) & = & (\mathbf{div}\, \boldsymbol{\sigma}, \boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \\ (\boldsymbol{\sigma}^*, \boldsymbol{s}) & = & 0, & \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \end{array} \tag{1.3.6}$$

It is easy to prove, using the continuous inf-sup condition (1.1.4), Lemma 1.1.1, and the Babuška-Brezzi theory, that $\Xi : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \times \mathbf{L}^2(\Omega)$ is well-defined and uniformly bounded in λ. In addition, we notice that $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) \in \mathbb{V}$ for all $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$ and $\boldsymbol{u}^* = D(\boldsymbol{\sigma}^*, \boldsymbol{r}^*)$. Moreover, it is crucial for the forthcoming analysis to observe that

$$(\boldsymbol{\sigma}^*, \boldsymbol{r}^*, \boldsymbol{u}^*) := \Xi\boldsymbol{\sigma} = (\boldsymbol{\sigma}, \boldsymbol{r}, D(\boldsymbol{\sigma}, \boldsymbol{r})) \qquad \forall (\boldsymbol{\sigma}, \boldsymbol{r}) \in \mathbb{V}. \tag{1.3.7}$$

Indeed, by virtue of Lemma 1.2.1, given $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \mathbb{V}$ there exists a unique $\boldsymbol{u} := D(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{L}^2(\Omega)$ such that

$$(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}) = -(\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}, \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$

from which it follows that $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{\mathcal{Q}}$ is the unique solution to problem (1.3.6) with datum $\mathbf{div}\,\boldsymbol{\sigma}$.

In turn, the discrete counterpart of Ξ is given by

$$\begin{aligned} \Xi_h : \boldsymbol{\mathcal{W}} &\to \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \\ \boldsymbol{\sigma} &\mapsto \Xi_h\boldsymbol{\sigma} := (\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*, \boldsymbol{u}_h^*) \end{aligned}$$

where $(\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*, \boldsymbol{u}_h^*) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h$ is the solution of

$$\begin{aligned} (\mathcal{C}^{-1}\boldsymbol{\sigma}_h^* + \boldsymbol{r}_h^*, \boldsymbol{\tau}) + (\boldsymbol{u}_h^*, \mathbf{div}\,\boldsymbol{\tau}) &= 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \\ (\mathbf{div}\,\boldsymbol{\sigma}_h^*, \boldsymbol{v}) &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{U}_h, \\ (\boldsymbol{\sigma}_h^*, \boldsymbol{s}) &= 0, \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h. \end{aligned} \tag{1.3.8}$$

Similarly to the continuous case, the discrete inf-sup condition given by Hypothesis 1, Lemma 1.1.1, the first condition of Hypothesis 2, and the Babuška-Brezzi theory imply that $\Xi_h : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h$ is well-defined and uniformly bounded in h and λ. In addition, there holds $(\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*) \in \mathbb{V}_h$ for all $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$ and $\boldsymbol{u}_h^* = D_h(\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*)$. Furthermore, we have the Céa estimate

$$\begin{aligned} &\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{H}(\mathbf{div},\Omega)} + \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{0,\Omega} + \|\boldsymbol{u}^* - \boldsymbol{u}_h^*\|_{0,\Omega} \\ &\qquad \leq C\Big\{ \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{\sigma}^* - \boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div},\Omega)} + \inf_{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{r}^* - \boldsymbol{s}_h\|_{0,\Omega} + \inf_{\boldsymbol{v}_h \in \mathcal{U}_h} \|\boldsymbol{u}^* - \boldsymbol{v}_h\|_{0,\Omega} \Big\} \end{aligned} \tag{1.3.9}$$

with $C > 0$ independent of h and λ.

1.3.3 The semi-discrete problem

From now on, we assume that the discrete initial data are given by

$$(\boldsymbol{\sigma}_{0,h}, \boldsymbol{r}_{0,h}, \boldsymbol{u}_{0,h}) := \Xi_h\boldsymbol{\sigma}_0 \qquad \text{and} \qquad (\boldsymbol{\sigma}_{1,h}, \boldsymbol{r}_{1,h}, \boldsymbol{u}_{1,h}) := \Xi_h\boldsymbol{\sigma}_1, \tag{1.3.10}$$

which, according to a previous observation, yields $\boldsymbol{u}_{0,h} := D_h(\boldsymbol{\sigma}_{0,h}, \boldsymbol{r}_{0,h})$ and $\boldsymbol{u}_{1,h} := D_h(\boldsymbol{\sigma}_{1,h}, \boldsymbol{r}_{1,h})$. Then, we consider the following semi-discrete counterpart of (1.1.5):

Find $\boldsymbol{\sigma}_h \in \mathcal{C}^1(\boldsymbol{\mathcal{W}}_h)$ and $\boldsymbol{r}_h \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}}_h)$ such that

$$\begin{aligned} (\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}_h(t) + \ddot{\boldsymbol{r}}_h(t), \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\sigma}_h(t), \mathbf{div}\,\boldsymbol{\tau})_\rho &= -(\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \\ (\boldsymbol{\sigma}_h(t), \boldsymbol{s}) &= \mathbf{0} \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h, \end{aligned} \tag{1.3.11}$$

and

$$\begin{aligned} \boldsymbol{\sigma}_h(0) &= \boldsymbol{\sigma}_{0,h}\,, \qquad \dot{\boldsymbol{\sigma}}_h(0) = \boldsymbol{\sigma}_{1,h}, \\ \boldsymbol{r}_h(0) &= \boldsymbol{r}_{0,h}\,, \qquad \dot{\boldsymbol{r}}_h(0) = \boldsymbol{r}_{1,h}\,. \end{aligned} \tag{1.3.12}$$

The kernel of the bilinear form $\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \ni (\boldsymbol{\tau}, \boldsymbol{s}) \mapsto \int_\Omega \boldsymbol{\tau} : \boldsymbol{s}$ is defined by

$$\boldsymbol{\mathcal{S}}_h := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h; \quad (\boldsymbol{\tau}, \boldsymbol{s}) = 0 \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h\}\ ,$$

which, being the subspace of $\boldsymbol{\mathcal{W}}_h$ whose elements are symmetric only in a discrete sense, is generally not contained in $\boldsymbol{\mathcal{S}}$. Then, as in the continuous case, we now introduce a reduced version of problem (1.3.11):

$$\begin{aligned} &\text{Find } \boldsymbol{\sigma}_h \in \mathcal{C}^1(\boldsymbol{\mathcal{S}}_h) \text{ such that} \\ &(\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}_h(t), \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\sigma}_h(t), \mathbf{div}\,\boldsymbol{\tau})_\rho = -(\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{S}}_h, \\ &\boldsymbol{\sigma}_h(0) = \boldsymbol{\sigma}_{0,h}, \quad \dot{\boldsymbol{\sigma}}_h(0) = \boldsymbol{\sigma}_{1,h}\,, \end{aligned} \tag{1.3.13}$$

whose unique solvability is ensured by classical ODE theory.

Next, we prove the existence of the Lagrange multiplier $\boldsymbol{r}_h(t)$ by proceeding as in the continuous case. To this end, we let $\mathcal{G}_h(t) \in \mathcal{C}^1(\boldsymbol{\mathcal{W}}_h')$ be given by

$$\begin{aligned} \Big\langle \mathcal{G}_h(t), \boldsymbol{\tau} \Big\rangle \ :=\ & (\mathcal{C}^{-1}\boldsymbol{\sigma}_h(t), \boldsymbol{\tau}) + \int_0^t \Big\{ \int_0^s (\mathbf{div}\,\boldsymbol{\sigma}_h(z) + \boldsymbol{f}(z), \mathbf{div}\,\boldsymbol{\tau})_\rho \,\mathrm{d}z \Big\}\,\mathrm{d}s \\ & -(\mathcal{C}^{-1}\boldsymbol{\sigma}_{0,h} + \boldsymbol{r}_{0,h}, \boldsymbol{\tau}) - t(\mathcal{C}^{-1}\boldsymbol{\sigma}_{1,h} + \boldsymbol{r}_{1,h}, \boldsymbol{\tau})\,. \end{aligned}$$

Using the fact that $\boldsymbol{\sigma}_h(t)$ solves (1.3.13), we deduce that $\mathcal{G}_h(t)$ belongs to the polar set of $\boldsymbol{\mathcal{S}}_h$ in $\boldsymbol{\mathcal{W}}_h'$, and hence, by virtue of the discrete inf-sup condition (1.3.2), we deduce that there exists a unique $\boldsymbol{r}_h \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}}_h)$ such that

$$(\boldsymbol{r}_h(t), \boldsymbol{\tau}) = -\Big\langle \mathcal{G}_h(t), \boldsymbol{\tau} \Big\rangle \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \tag{1.3.14}$$

which proves that $(\boldsymbol{\sigma}_h(t), \boldsymbol{r}_h(t))$ is the unique solution to (1.3.11). In turn, since by construction (see (1.3.10)) $(\boldsymbol{\sigma}_{0,h}, \boldsymbol{r}_{0,h}) \in \mathbb{V}_h$ and $(\boldsymbol{\sigma}_{1,h}, \boldsymbol{r}_{1,h}) \in \mathbb{V}_h$, we find from (1.3.11) that $(\boldsymbol{\sigma}_h(t), \boldsymbol{r}_h(t)) \in \mathbb{V}_h$ for all $t \in [0, T]$. Consequently, we can propose the function $\boldsymbol{u}_h(t) := D_h(\boldsymbol{\sigma}_h(t), \boldsymbol{r}_h(t))$ as a semi-discrete approximation of the displacement $\boldsymbol{u}(t)$, which can be computed by solving a saddle point problem of the form (1.3.8) with datum $\mathbf{div}\,\boldsymbol{\sigma}_h(t)$. However, comparing (1.3.5) with the first equation of (1.3.11), we easily obtain the following explicit expression for the semi-discrete displacement field:

$$\boldsymbol{u}_h(t) = \int_0^t \Big\{ \int_0^s \rho^{-1}\Big(\ \mathbf{div}\,\boldsymbol{\sigma}_h(z) + U_h\boldsymbol{f}(z)\Big)\,\mathrm{d}z\Big\}\,\mathrm{d}s + \boldsymbol{u}_{0,h} + t\boldsymbol{u}_{1,h}\,. \tag{1.3.15}$$

1.3.4 Convergence analysis

We begin by recalling from Section 1.3.2 (cf. (1.3.8)) that $(\boldsymbol{\sigma}_h^*(t), \boldsymbol{r}_h^*(t), \boldsymbol{u}_h^*(t)) = \Xi_h\,\boldsymbol{\sigma}(t)$. Then, we introduce

$$\boldsymbol{e}_{\sigma,h}(t) := \boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}_h(t) \quad \text{and} \quad \boldsymbol{e}_{r,h}(t) := \boldsymbol{r}_h^*(t) - \boldsymbol{r}_h(t)\,,$$

and deduce from (1.3.10) that

$$\boldsymbol{e}_{\sigma,h}(0) = \boldsymbol{e}_{r,h}(0) = \mathbf{0} \qquad \text{and} \qquad \dot{\boldsymbol{e}}_{\sigma,h}(0) = \dot{\boldsymbol{e}}_{r,h}(0) = \mathbf{0}. \tag{1.3.16}$$

Lemma 1.3.2. *Assume that the solutions $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{sym}(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ to problem* (1.1.5) *satisfy the regularity assumptions $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div},\Omega) \cap \mathbb{H}^\epsilon(\Omega))$ for some $\epsilon > 0$ and $\boldsymbol{r} \in \mathcal{C}^2(\mathbb{L}^2(\Omega))$. Then, there exists a constant $C > 0$ independent of λ and h such that*

$$\begin{aligned}&\max_{[0,T]} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{\mathcal{C},\mathrm{div}} + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h)(t)\|_{\mathcal{C}} \\ &\quad \leq\ C\left\{ \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r} - Q_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \right\}.\end{aligned}$$

Proof. Let us first notice that, as $(\boldsymbol{\sigma}(t), \boldsymbol{r}(t)) \in \mathbb{V}$ for all $t \in [0,T]$ (see the proof of Theorem 1.2.1), we deduce from (1.3.7) that

$$(\boldsymbol{\sigma}^*(t), \boldsymbol{r}^*(t), \boldsymbol{u}^*(t)) := \varXi \boldsymbol{\sigma}(t) = (\boldsymbol{\sigma}(t), \boldsymbol{r}(t), D(\boldsymbol{\sigma}(t), \boldsymbol{r}(t))) \qquad \forall\, t \in [0,T]\,, \tag{1.3.17}$$

and because of the regularity assumptions we also have

$$\begin{aligned}\Big(\frac{\mathrm{d}^i \boldsymbol{\sigma}^*}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i \boldsymbol{r}^*}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i \boldsymbol{u}^*}{\mathrm{d}t^i}(t)\Big) &:= \frac{\mathrm{d}^i \varXi \boldsymbol{\sigma}(t)}{\mathrm{d}t^i} = \varXi \frac{\mathrm{d}^i \boldsymbol{\sigma}}{\mathrm{d}t^i}(t) \\ &= \Big(\frac{\mathrm{d}^i \boldsymbol{\sigma}}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i \boldsymbol{r}}{\mathrm{d}t^i}(t), D\big(\frac{\mathrm{d}^i \boldsymbol{\sigma}}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i \boldsymbol{r}}{\mathrm{d}t^i}(t)\big)\Big) \qquad \forall\, i \in \{1,2\}\,, \quad \forall\, t \in [0,T]\,.\end{aligned} \tag{1.3.18}$$

Moreover, by virtue of (1.3.9), (1.3.18) and Hypothesis 3, there holds

$$\begin{aligned}&\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{u}^* - \boldsymbol{u}_h^*\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \\ &\quad \leq C_0 \left\{ \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r} - Q_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \right\},\end{aligned} \tag{1.3.19}$$

with $C_0 > 0$ independent of h and λ. Next, it is straightforward to see that

$$\begin{aligned}&(\mathcal{C}^{-1} \ddot{\mathbf{e}}_{\sigma,h}(t) + \ddot{\mathbf{e}}_{r,h}(t), \boldsymbol{\tau}) + (\mathbf{div}\, \mathbf{e}_{\sigma,h}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \\ &\quad = (\mathcal{C}^{-1}(\ddot{\boldsymbol{\sigma}}_h^*(t) - \ddot{\boldsymbol{\sigma}}(t)), \boldsymbol{\tau}) + (\ddot{\boldsymbol{r}}_h^*(t) - \ddot{\boldsymbol{r}}(t), \boldsymbol{\tau}) + (\mathbf{div}(\boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}(t)), \mathbf{div}\, \boldsymbol{\tau})_\rho\end{aligned} \tag{1.3.20}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h$, and, as a consequence of (1.3.18),

$$\begin{aligned}&(\mathcal{C}^{-1} \ddot{\mathbf{e}}_{\sigma,h}(t) + \ddot{\mathbf{e}}_{r,h}(t), \boldsymbol{\tau}) + (\mathbf{div}\, \mathbf{e}_{\sigma,h}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \\ &\quad = (\mathcal{C}^{-1}(\ddot{\boldsymbol{\sigma}}_h^*(t) - \ddot{\boldsymbol{\sigma}}^*(t)), \boldsymbol{\tau}) + (\ddot{\boldsymbol{r}}_h^*(t) - \ddot{\boldsymbol{r}}^*(t), \boldsymbol{\tau}) + (\mathbf{div}(\boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}^*(t)), \mathbf{div}\, \boldsymbol{\tau})_\rho\end{aligned} \tag{1.3.21}$$

for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h$. Now, by definition of $\varXi$ and $\varXi_h$, we have that

$$\mathbf{div}\, \boldsymbol{\sigma}_h^*(t) = U_h\, \mathbf{div}\, \boldsymbol{\sigma}^*(t) \qquad \forall\, t \in [0,T]\,, \tag{1.3.22}$$

and the second condition of Hypothesis 2 implies that

$$(\mathbf{div}(\boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}^*(t)), \mathbf{div}\, \boldsymbol{\tau})_\rho = 0 \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,.$$

Consequently, $\boldsymbol{e}_{\sigma,h}(t) \in \boldsymbol{\mathcal{S}}_h$ and $\boldsymbol{e}_{r,h}(t) \in \boldsymbol{\mathcal{Q}}_h$ satisfy

$$\begin{aligned}(\mathcal{C}^{-1} \ddot{\mathbf{e}}_{\sigma,h}(t) + \ddot{\mathbf{e}}_{r,h}(t), \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{e}_{\sigma,h}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho &= F(\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,, \\ (\boldsymbol{e}_{\sigma,h}(t), \boldsymbol{s}) &= \mathbf{0} \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h\,,\end{aligned} \tag{1.3.23}$$

with

$$F(\boldsymbol{\tau}) := \Big(\mathcal{C}^{-1}\big(\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}^*\big)(t) + \big(\ddot{\boldsymbol{r}}_h^* - \ddot{\boldsymbol{r}}^*\big)(t), \boldsymbol{\tau}\Big).$$

Taking $\boldsymbol{\tau} = \dot{\boldsymbol{e}}_{\sigma,h}(t)$ in the first equation of (1.3.23) and using the Cauchy-Schwarz inequality yields

$$\frac{\dot{\mathcal{E}}(\boldsymbol{e}_{\sigma,h})(t)}{2\sqrt{\mathcal{E}(\boldsymbol{e}_{\sigma,h})(t)}} \leq \frac{1}{\sqrt{2}}\Big\{\big(\mathcal{C}(\ddot{\boldsymbol{r}}^* - \ddot{\boldsymbol{r}}_h^*)(t), (\ddot{\boldsymbol{r}}^* - \ddot{\boldsymbol{r}}_h^*)(t)\big)^{1/2} + \|(\ddot{\boldsymbol{\sigma}}^* - \ddot{\boldsymbol{\sigma}}_h^*)(t)\|_{\mathcal{C}}\Big\},$$

which, using (1.1.3) and the fact that

$$\mathcal{C}\boldsymbol{s} = 2\mu\boldsymbol{s} \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}, \tag{1.3.24}$$

implies that

$$\frac{\dot{\mathcal{E}}(\boldsymbol{e}_{\sigma,h})(t)}{2\sqrt{\mathcal{E}(\boldsymbol{e}_{\sigma,h})(t)}} \leq \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{2\underline{\mu}}}\|(\ddot{\boldsymbol{\sigma}}^* - \ddot{\boldsymbol{\sigma}}_h^*)(t)\|_{0,\Omega} + \sqrt{2\bar{\mu}}\,\|(\ddot{\boldsymbol{r}}^* - \ddot{\boldsymbol{r}}_h^*)(t)\|_{0,\Omega}\right\}.$$

Integrating with respect to time gives

$$\max_{[0,T]}\mathcal{E}(\boldsymbol{e}_{\sigma,h})^{1/2}(t) \leq \int_0^T\left\{\frac{1}{2\sqrt{\underline{\mu}}}\|(\ddot{\boldsymbol{\sigma}}^* - \ddot{\boldsymbol{\sigma}}_h^*)(t)\|_{0,\Omega} + \sqrt{\bar{\mu}}\,\|(\ddot{\boldsymbol{r}}^* - \ddot{\boldsymbol{r}}_h^*)(t)\|_{0,\Omega}\right\}\mathrm{d}t. \tag{1.3.25}$$

On the other hand, we deduce easily from the identity $\boldsymbol{e}_{\sigma,h}(t) = \int_0^t \dot{\boldsymbol{e}}_{\sigma,h}(s)$ and (1.1.8) that there exists a constant $C_0 > 0$, independent of λ and h, such that

$$\|\boldsymbol{e}_{\sigma,h}(t)\|_{\mathcal{C},\mathrm{div}} + \|\dot{\boldsymbol{e}}_{\sigma,h}(t)\|_{\mathcal{C}} \leq C_0\max_{[0,T]}\mathcal{E}(\boldsymbol{e}_{\sigma,h})^{1/2}(t) \qquad \forall\, t \in [0,T]. \tag{1.3.26}$$

In this way, combining (1.3.25) and (1.3.26) with the triangle inequality we arrive at

$$\begin{aligned}&\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{\mathcal{C},\mathrm{div}} + \|(\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h)(t)\|_{\mathcal{C}}\\ &\quad\leq \|\boldsymbol{e}_{\sigma,h}(t)\|_{\mathcal{C},\mathrm{div}} + \|\dot{\boldsymbol{e}}_{\sigma,h}(t)\|_{\mathcal{C}} + \|(\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*)(t)\|_{\mathcal{C},\mathrm{div}} + \|(\dot{\boldsymbol{\sigma}}^* - \dot{\boldsymbol{\sigma}}_h^*)(t)\|_{\mathcal{C}},\end{aligned} \tag{1.3.27}$$

which, together with (1.1.3) and (1.3.19), imply the existence of a constant $C_1 > 0$, independent of h and λ, such that

$$\begin{aligned}&\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{\mathcal{C},\mathrm{div}} + \|(\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h)(t)\|_{\mathcal{C}}\\ &\quad\leq C_1\Big\{\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r} - Q_h\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))}\Big\}\end{aligned}$$

for all $t \in [0,T]$ and the result follows. □

Lemma 1.3.3. *Under the hypotheses of Lemma 1.3.2 there exists a constant $C > 0$, independent of λ and h, such that*

$$\begin{aligned}\|\boldsymbol{r} - \boldsymbol{r}_h\|_{\mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega))} \leq C\Big\{&\max_{[0,T]}\|(\dot{\boldsymbol{r}}(t) - Q_h\dot{\boldsymbol{r}})(t)\|_{0,\Omega} + \max_{[0,T]}\|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{0,\Omega}\\ &+ \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_1 - \boldsymbol{r}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_0 - \boldsymbol{r}_{0,h}\|_{0,\Omega}\Big\}.\end{aligned} \tag{1.3.28}$$

Proof. By virtue of the inf-sup condition (1.3.2), and the identities provided by (1.1.19) and (1.3.14), we find that

$$\begin{aligned}\beta^* \left\|\dot{\boldsymbol{r}}_h(t) - Q_h\dot{\boldsymbol{r}}(t)\right\|_{0,\Omega} &\le \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{\int_\Omega(\dot{\boldsymbol{r}}_h(t) - Q_h\dot{\boldsymbol{r}}(t)) : \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\ &\le \ \|\dot{\boldsymbol{r}}(t) - Q_h\dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{|\int_\Omega(\dot{\boldsymbol{r}}(t) - \dot{\boldsymbol{r}}_h(t)) : \boldsymbol{\tau}|}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\ &= \ \|\dot{\boldsymbol{r}}(t) - Q_h\dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{|\left\langle \dot{\mathcal{G}}(t) - \dot{\mathcal{G}}_h(t), \boldsymbol{\tau}\right\rangle_{\boldsymbol{\mathcal{W}}}|}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}\,.\end{aligned} \tag{1.3.29}$$

In turn, using the Cauchy-Schwarz inequality and (1.1.3) we have that

$$\begin{aligned}&|\left\langle \dot{\mathcal{G}}(t) - \dot{\mathcal{G}}_h(t), \boldsymbol{\tau}\right\rangle_{\boldsymbol{\mathcal{W}}}| \le T \max_{[0,T]} \|\mathbf{div}\,\boldsymbol{\sigma}(t) - \mathbf{div}\,\boldsymbol{\sigma}_h(t)\|_\rho \|\mathbf{div}\,\boldsymbol{\tau}\|_\rho \\ &\quad + \left\{ \frac{1}{\sqrt{2\underline{\mu}}} \|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h(t)\|_{\mathcal{C}} + \frac{1}{2\underline{\mu}} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_1 - \boldsymbol{r}_{1,h}\|_{0,\Omega} \right\} \|\boldsymbol{\tau}\|_{0,\Omega} \\ &\quad \le \left\{ \frac{T}{\sqrt{\underline{\rho}}} \max_{[0,T]} \|\mathbf{div}\,\boldsymbol{\sigma}(t) - \mathbf{div}\,\boldsymbol{\sigma}_h(t)\|_\rho + \frac{1}{\sqrt{2\underline{\mu}}} \|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h(t)\|_{\mathcal{C}} \right. \\ &\quad \left. + \frac{1}{2\underline{\mu}} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_1 - \boldsymbol{r}_{1,h}\|_{0,\Omega} \right\} \|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}\,.\end{aligned} \tag{1.3.30}$$

In this way, combining (1.3.29) and (1.3.30) we deduce that

$$\begin{aligned}&\|\dot{\boldsymbol{r}}(t) - \dot{\boldsymbol{r}}_h(t)\|_{0,\Omega} \le \|\dot{\boldsymbol{r}}(t) - Q_h\dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \|Q_h\dot{\boldsymbol{r}}(t) - \dot{\boldsymbol{r}}_h(t)\|_{0,\Omega} \\ &\quad \le \left(1 + \frac{1}{\beta^*}\right) \|\dot{\boldsymbol{r}}(t) - Q_h\dot{\boldsymbol{r}}(t)\|_{0,\Omega} + \frac{1}{\beta^*}\left\{ \frac{T}{\sqrt{\underline{\rho}}} \max_{[0,T]} \|\mathbf{div}\,\boldsymbol{\sigma}(t) - \mathbf{div}\,\boldsymbol{\sigma}_h(t)\|_\rho \right. \\ &\quad \left. + \frac{1}{\sqrt{2\underline{\mu}}} \|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_h(t)\|_{\mathcal{C}} + \frac{1}{2\underline{\mu}} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|_{0,\Omega} + \|\boldsymbol{r}_1 - \boldsymbol{r}_{1,h}\|_{0,\Omega} \right\}.\end{aligned}$$

Finally, the bound for $\|\boldsymbol{r}(t) - \boldsymbol{r}_h(t)\|_{0,\Omega}$ is obtained from the foregoing estimate and the identity $\boldsymbol{r}(t) - \boldsymbol{r}_h(t) = \boldsymbol{r}_0 - \boldsymbol{r}_{0,h} + \int_0^t(\dot{\boldsymbol{r}}(s) - \dot{\boldsymbol{r}}_h(s))\,\mathrm{d}s$, which completes the proof. □

Lemma 1.3.4. *Under the hypotheses of Lemma 1.3.2, there exists a constant $C > 0$, independent of λ and h, such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{W}^{2,\infty}(\boldsymbol{L}^2(\Omega))} \le C\left\{ \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\boldsymbol{L}^2(\Omega))} + \max_{[0,T]} \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{0,\Omega} \right\}.$$

Proof. Using (1.2.5) and (1.3.15) we have that $\ddot{\boldsymbol{u}}(t) := D(\ddot{\boldsymbol{\sigma}}(t), \ddot{\boldsymbol{r}}(t))$ and $\ddot{\boldsymbol{u}}_h(t) := D_h(\ddot{\boldsymbol{\sigma}}_h(t), \ddot{\boldsymbol{r}}_h(t))$, which satisfy

$$(\mathbf{div}\,\boldsymbol{\tau}, \ddot{\boldsymbol{u}}(t) - \ddot{\boldsymbol{u}}_h(t)) = (\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t), \mathbf{div}\,\boldsymbol{\tau})_\rho \qquad \forall\,\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \quad \forall\, t \in [0,T]\,. \tag{1.3.31}$$

Then, it follows from (1.3.2), (1.3.31), (1.2.7), and Hypothesis 2, that

$$\begin{aligned}\beta^* \|\ddot{\boldsymbol{u}}_h(t) - U_h\ddot{\boldsymbol{u}}(t)\|_{0,\Omega} &\le \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{(\ddot{\boldsymbol{u}}_h(t) - U_h\ddot{\boldsymbol{u}}(t), \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\ &= \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{(\ddot{\boldsymbol{u}}_h(t) - \ddot{\boldsymbol{u}}(t), \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \le \frac{1}{\underline{\rho}} \|\mathbf{div}\,\boldsymbol{\sigma}(t) - \mathbf{div}\,\boldsymbol{\sigma}_h(t)\|_{0,\Omega}\,,\end{aligned}$$

which, thanks to the triangle inequality, gives the estimate

$$\max_{[0,T]} \|\ddot{\boldsymbol{u}}(t) - \ddot{\boldsymbol{u}}_h(t)\|_{0,\Omega} \leq \frac{1}{\beta^*} \max_{[0,T]} \|\ddot{\boldsymbol{u}}(t) - U_h \ddot{\boldsymbol{u}}(t)\|_{0,\Omega} + \frac{1}{\beta^* \underline{\rho}} \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|_{0,\Omega} .$$

The same estimates for $\|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_h(t)\|_{0,\Omega}$ and $\|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_{0,\Omega}$ are obtained after integrating, and the result follows. □

We conclude by providing the following convergence result.

Theorem 1.3.1. *Assume that the solutions* $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{sym}(\Omega))$ *and* $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ *to problem* (1.1.5) *satisfy* $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div}, \Omega) \cap \mathbb{H}^\epsilon(\Omega))$ *for some* $\epsilon > 0$ *and* $\boldsymbol{r} \in \mathcal{C}^2(\mathbb{L}^2(\Omega))$. *Then, there exists a constant* $C > 0$, *independent of* λ *and* h, *such that*

$$\begin{aligned} &\max_{[0,T]} \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t)\|_{\mathrm{H}(\mathbf{div},\Omega)} + \|\boldsymbol{r} - \boldsymbol{r}_h\|_{\mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \\ &\quad \leq C \left\{ \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathbb{H}(\mathbf{div},\Omega))} + \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} + \|\boldsymbol{r} - Q_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} \right\}. \end{aligned}$$

Proof. The required error estimate is a direct consequence of Lemmas 1.3.2, 1.3.3 and 1.3.4, and the norm equivalence provided by Lemma 1.1.1. □

Remark 1.3.2. *We notice that the uniformity of the error estimate provided by Theorem 1.3.1 with respect to the coefficient* λ *shows that the semi-discrete Galerkin scheme* (1.3.11) *is immune to locking phenomenon in the nearly incompressible case.*

Remark 1.3.3. *As already observed in Remark 1.3.1, we emphasize here that if the Lamé coefficients* λ *and* μ *were constant in* Ω, *we would only need to assume in Theorem 1.3.1 that the solution* $(\boldsymbol{\sigma}, \boldsymbol{r})$ *to problem* (1.1.5) *satisfies* $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div}, \Omega))$. *Indeed, in such a case, the regularity* $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div}, \Omega) \cap \mathbb{H}^\epsilon(\Omega))$ *for some* $\epsilon > 0$ *is guaranteed by the fact that* $(\boldsymbol{\sigma}(t), \boldsymbol{r}(t)) \in \mathbb{V}$, $\forall t \in [0, T]$.

1.4 Comparison with Arnold & Lee's method

The purpose of this section is to describe the differences and advantages of our approach when compared to [5]. In particular, we show that the present method is indeed different from the one given in [5] and that it leads to better convergence results. In addition, our stability proof in the nearly incompressible case is valid for general polygonal/polyhedral domains, whereas only convex domains can be considered in [5].

Let us first show that the two semi-discrete methods are not equivalent. With our notations, the semidiscrete problem of [5] consists of the first order system: Find $\boldsymbol{\sigma}_h^{[5]} \in \mathcal{C}^1(\boldsymbol{\mathcal{W}}_h)$, $\boldsymbol{v}_h^{[5]} \in \mathcal{C}^1(\mathcal{U}_h)$ and $\boldsymbol{r}_h^{[5]} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}}_h)$ such that

$$\begin{aligned} (\mathcal{C}^{-1} \dot{\boldsymbol{\sigma}}_h^{[5]}(t), \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\tau}, \boldsymbol{v}_h^{[5]}(t)) + (\dot{\boldsymbol{r}}_h^{[5]}(t), \boldsymbol{\tau}) &= 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h \\ (\rho \dot{\boldsymbol{v}}_h^{[5]}(t), \boldsymbol{w}) - (\mathbf{div}\, \boldsymbol{\sigma}_h^{[5]}(t), \boldsymbol{w}) &= (\boldsymbol{f}, \boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathcal{U}_h \\ (\dot{\boldsymbol{\sigma}}_h^{[5]}(t), \boldsymbol{s}) &= 0 \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h. \end{aligned} \tag{1.4.1}$$

As initial data we have, on one hand (see equation (4.5) of [5])

$$\boldsymbol{v}_h^{[5]}(0) = U_h \boldsymbol{u}_1 \,, \tag{1.4.2}$$

and on the other hand

$$\boldsymbol{\sigma}_h^{[5]}(0) := \boldsymbol{\sigma}_{0,h} \qquad \text{and} \qquad \boldsymbol{r}_h^{[5]}(0) := \boldsymbol{r}_{0,h}, \tag{1.4.3}$$

where $(\boldsymbol{\sigma}_{0,h}, \boldsymbol{r}_{0,h}, \boldsymbol{u}_{0,h}) := \Xi_h \boldsymbol{\sigma}_0$.

Next, it is easy to see that, taking $\boldsymbol{\tau} = \boldsymbol{\sigma}_h^{[5]}$ and $\boldsymbol{w} = \boldsymbol{v}_h$ in the first and second equations of (1.4.1), respectively, we obtain

$$\max_{[0,T]} \mathcal{E}_{[5]}\big((\boldsymbol{\sigma}_h^{[5]}, \boldsymbol{v}_h^{[5]})\big)^{1/2}(t) \leq C\Big(\|\boldsymbol{f}\|_{\mathrm{L}^2(\mathbf{L}^2(\Omega))} + \|\boldsymbol{\sigma}_{0,h}\|_{0,\Omega} + \|\boldsymbol{v}_{0,h}\|_{0,\Omega} \Big) \tag{1.4.4}$$

with a constant $C > 0$ that is independent of h and with an energy functional

$$\mathcal{E}_{[5]}\big((\boldsymbol{\sigma}_h^{[5]}, \boldsymbol{v}_h^{[5]})\big)(t) := \frac{1}{2} \left\|\boldsymbol{\sigma}_h^{[5]}(t)\right\|_{\mathcal{C}}^2 + \frac{1}{2} \left\|\boldsymbol{v}_h^{[5]}(t)\right\|_{\rho}^2 .$$

We now notice, thanks to Hypothesis 2 (condition A0 in [5]), that the second equation of (1.4.1) becomes

$$\dot{\boldsymbol{v}}_h^{[5]} = \rho^{-1}\big(\mathbf{div}\, \boldsymbol{\sigma}_h^{[5]} + U_h \boldsymbol{f} \big) \,, \tag{1.4.5}$$

and hence, differentiating the first equation of (1.4.1) with respect to time, and substituting back there the foregoing expression for $\dot{\boldsymbol{v}}_h^{[5]}$, we find that

$$\begin{aligned} (\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}_h^{[5]}(t) + \ddot{\boldsymbol{r}}_h^{[5]}(t), \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\sigma}_h^{[5]}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho &= -(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau})_\rho \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \\ (\boldsymbol{\sigma}_h^{[5]}(t), \boldsymbol{s}) &= \mathbf{0} \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h. \end{aligned} \tag{1.4.6}$$

In order to conclude from this that $\boldsymbol{\sigma}_h^{[5]}(t) = \boldsymbol{\sigma}_h(t)$ and $\boldsymbol{r}_h^{[5]}(t) = \boldsymbol{r}_h(t)$, we need to have $\dot{\boldsymbol{\sigma}}_h^{[5]}(0) = \boldsymbol{\sigma}_{1,h}$ and $\dot{\boldsymbol{r}}_h^{[5]}(0) = \boldsymbol{r}_{1,h}$. However, taking $t = 0$ in the first equation of (1.4.1) yields

$$(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}_h^{[5]}(0), \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\tau}, U_h \boldsymbol{u}_1) + (\dot{\boldsymbol{r}}_h^{[5]}(0), \boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h \,,$$

and, if the identities $\dot{\boldsymbol{\sigma}}_h^{[5]}(0) = \boldsymbol{\sigma}_{1,h}$ and $\dot{\boldsymbol{r}}_h^{[5]}(0) = \boldsymbol{r}_{1,h}$ were true, the comparison of the second one with (see the definition of Ξ_h in (1.3.8))

$$(\mathcal{C}^{-1}\boldsymbol{\sigma}_{1,h}, \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\tau}, \boldsymbol{u}_{1,h}) + (\boldsymbol{r}_{1,h}, \boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h \,,$$

would imply

$$(\mathbf{div}\, \boldsymbol{\tau}, U_h \boldsymbol{u}_1 - \boldsymbol{u}_{1,h}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h \,,$$

that is (by virtue of Hypothesis 2) $U_h \boldsymbol{u}_1 = \boldsymbol{u}_{1,h}$, which is obviously not true in general. Consequently, our initial assumptions $\dot{\boldsymbol{\sigma}}_h^{[5]}(0) = \boldsymbol{\sigma}_{1,h}$ and $\dot{\boldsymbol{r}}_h^{[5]}(0) = \boldsymbol{r}_{1,h}$ are not true either.

If the starting point is now the stress formulation (1.3.11), we can introduce the acceleration field

$$\dot{\boldsymbol{v}}_h = \rho^{-1}\big(\mathbf{div}\, \boldsymbol{\sigma}_h + U_h \boldsymbol{f} \big), \tag{1.4.7}$$

and rewrite (1.3.11) in the form

$$\begin{aligned}(\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}_h(t),\boldsymbol{\tau})_{\mathcal{C}} + (\mathbf{div}\,\boldsymbol{\tau},\dot{\boldsymbol{v}}_h) + (\ddot{\boldsymbol{r}}_h(t),\boldsymbol{\tau}) &= 0 \quad \forall\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h\\ (\dot{\boldsymbol{v}}_h(t),\boldsymbol{w})_\rho - (\mathbf{div}\,\boldsymbol{\sigma}_h(t),\boldsymbol{w}) &= (\boldsymbol{f},\boldsymbol{w}) \quad \forall\boldsymbol{w}\in\mathcal{U}_h\\ (\boldsymbol{\sigma}_h(t),\boldsymbol{s}) &= 0 \quad \forall\boldsymbol{s}\in\boldsymbol{\mathcal{Q}}_h.\end{aligned} \tag{1.4.8}$$

It follows that we can only retrieve the first equation of (1.4.1) by one time integration in the first equation of (1.4.8) if the initial value $\boldsymbol{v}_h(0)$ is such that

$$(\mathcal{C}^{-1}\dot{\boldsymbol{\sigma}}_h(0),\boldsymbol{\tau})_{\mathcal{C}} + (\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{v}_h(0)) + (\dot{\boldsymbol{r}}_h(0),\boldsymbol{\tau}) = (\mathcal{C}^{-1}\boldsymbol{\sigma}_{1,h},\boldsymbol{\tau})_{\mathcal{C}} + (\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{v}_h(0)) + (\boldsymbol{r}_{1,h},\boldsymbol{\tau}) = 0$$

for all $\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h$, which, however, does not hold true with the required choice $\boldsymbol{v}_h(0) = U_h\boldsymbol{u}_1$ given in (1.4.2).

We end this section with several other important remarks. First of all, it is important to notice here that in our formulation we can post process the acceleration and velocity fields directly by (1.4.7) and

$$\boldsymbol{v}_h(t) = \int_0^t \rho^{-1}(\mathbf{div}\,\boldsymbol{\sigma}_h(s) + U_h\boldsymbol{f}(s))\,\mathrm{d}s + \boldsymbol{u}_{1,h} \neq \boldsymbol{v}_h^{[5]}(t)\,,$$

respectively. In turn, we point out that defining the initial data for our scheme needs the solution of two mixed elastostatic problems of type (1.3.8) (with right hand-sides $\mathbf{div}\,\boldsymbol{\sigma}_0$ and $\mathbf{div}\,\boldsymbol{\sigma}_1$ respectively), while the method of [5] only requires the solution of one of them. Nevertheless, this negligible initial computational effort is clearly worthwhile since, as it can be seen from the energy estimate (1.4.4), and on the contrary to the implications arising from (1.1.12), the method from [5] does not provide any control on the L^2-norm of $\mathbf{div}\,\boldsymbol{\sigma}_h^{[5]}(t)$. Consequently, the error estimates provided by Theorem 4.2 and Theorem 6.1 of [5] for the variable $\boldsymbol{\sigma}\in\mathcal{C}^0([0,T],\mathbf{H}(\mathbf{div},\Omega))$ (at the semi and fully discrete levels) are given in the L^2-norm only, whereas our Theorems 1.3.1 and 1.5.1 (cf. next section) provide error estimates for the same variable in the full $\mathbf{H}(\mathbf{div},\Omega)$-norm. Moreover, the loss of control on the divergence of the stress tensor in [5] has also the following consequences:

- On one hand, it complicates the proof of robustness of the scheme in the nearly incompressible case. Such a proof is only provided in the semidiscrete case in Section 4.5 of [5]. It is based on Lemma 4.5 of [5] and it requires the introduction of the H^{-1}-norm and a duality argument. The proof of that Lemma 4.5 is given in reference [20] of [5] in the two-dimensional case. It requires full regularity for the Laplacian in Ω, which limits the result to convex domains Ω. It is also necessary to assume pure displacement boundary conditions and to consider constant Lamé coefficients. In our case, having control on the stress tensor in the $\mathbf{H}(\mathbf{div},\Omega)$-norm, our stability proof in the nearly incompressible case simply relies on our Lemma 1.1.1, which is valid for general polygonal/polyhedral domains, general boundary conditions and for piecewise constant Lamé coefficients.

- On the other hand, the aforementioned loss impedes one to take advantage of the motion equation (1.4.5) to obtain a direct and good approximation of the acceleration field. With our technique, and as already mentioned, we can easily post-process the velocity, the acceleration and the displacement fields.

1.5 Time-space discretization

1.5.1 The fully discrete scheme

Given $L \in \mathbb{N}$, we consider a uniform partition of the time interval $[0,T]$ with step size $\Delta t := T/L$. Then, for any continuous function $\phi : [0,T] \to \mathbb{R}$ and for each $k \in \{0,1,\dots,L\}$ we denote $\phi^k := \phi(t_k)$, where $t_k := k\,\Delta t$. In addition, we adopt the same notation for vector/tensor valued functions and consider $t_{k+\frac{1}{2}} := \frac{t_{k+1}+t_k}{2}$, $\phi^{k+\frac{1}{2}} := \frac{\phi^{k+1}+\phi^k}{2}$, $\phi^{k-\frac{1}{2}} := \frac{\phi^k+\phi^{k-1}}{2}$, and the discrete time derivatives

$$\partial_t \phi^k := \frac{\phi^{k+1}-\phi^k}{\Delta t} \quad \text{and} \quad \bar{\partial}_t \phi^k := \frac{\phi^k - \phi^{k-1}}{\Delta t}\,,$$

from which we notice that

$$\partial_t \bar{\partial}_t \phi^k = \frac{\bar{\partial}_t\phi^{k+1} - \bar{\partial}_t\phi^k}{\Delta t} = \frac{\partial_t\phi^k - \partial_t\phi^{k-1}}{\Delta t}\,.$$

In what follows we utilize the Newmark trapezoidal rule for the time discretization of (1.3.11): For $k = 1,\dots,L-1$, we look for $(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{r}_h^{k+1}) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ solution of

$$\begin{aligned} \Big(\partial_t\bar{\partial}_t(\mathcal{C}^{-1}\boldsymbol{\sigma}_h^k + \boldsymbol{r}_h^k), \boldsymbol{\tau}\Big) + \Big(\mathbf{div}\,\frac{\boldsymbol{\sigma}_h^{k+\frac{1}{2}} + \boldsymbol{\sigma}_h^{k-\frac{1}{2}}}{2}, \mathbf{div}\,\boldsymbol{\tau}\Big)_\rho &= -\Big(\boldsymbol{f}(t_k), \mathbf{div}\,\boldsymbol{\tau}\Big)_\rho \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,, \\ (\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{s}) &= 0 \qquad\qquad\qquad \forall\, \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h\,, \end{aligned} \tag{1.5.1}$$

where, for the sake of simplicity, we assume that the scheme (1.5.1) is started up with

$$(\boldsymbol{\sigma}_h^0, \boldsymbol{r}_h^0) := \Xi_h \boldsymbol{\sigma}_0, \quad \text{and} \quad (\boldsymbol{\sigma}_h^1, \boldsymbol{r}_h^1) := \Xi_h\boldsymbol{\sigma}(t_1). \tag{1.5.2}$$

Then, we introduce the functions

$$\boldsymbol{e}_{\sigma,h}^k := \boldsymbol{\sigma}_h^*(t_k) - \boldsymbol{\sigma}_h^k \in \boldsymbol{\mathcal{S}}_h \quad \text{and} \quad \boldsymbol{e}_{r,h}^k := \boldsymbol{r}_h^*(t_k) - \boldsymbol{r}_h^k \in \boldsymbol{\mathcal{Q}}_h,$$

where, as usual, $(\boldsymbol{\sigma}_h^*(t_k), \boldsymbol{r}_h^*(t_k)) := \Xi_h\boldsymbol{\sigma}(t_k)$. We note here that (1.5.2) permits us to ignore the error at the first two initial steps since $\boldsymbol{e}_{\sigma,h}^0 = \boldsymbol{e}_{\sigma,h}^1 = \mathbf{0}$ and $\boldsymbol{e}_{\boldsymbol{r},h}^0 = \boldsymbol{e}_{\boldsymbol{r},h}^1 = \mathbf{0}$. Next, it is straightforward to see that

$$\begin{aligned} &\Big(\partial_t\bar{\partial}_t(\mathcal{C}^{-1}\boldsymbol{e}_{\sigma,h}^k + \boldsymbol{e}_{r,h}^k), \boldsymbol{\tau}\Big) + \Big(\mathbf{div}\,\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} + \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{2}, \mathbf{div}\,\boldsymbol{\tau}\Big)_\rho \\ &\quad = (\boldsymbol{\chi}_1^k, \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\chi}_2^k, \mathbf{div}\,\boldsymbol{\tau})_\rho \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,, \end{aligned} \tag{1.5.3}$$

where

$$\boldsymbol{\chi}_1^k := \partial_t\bar{\partial}_t(\mathcal{C}^{-1}\boldsymbol{\sigma}_h^*(t_k) + \boldsymbol{r}_h^*(t_k)) \;-\; (\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}}^*(t_k) + \ddot{\boldsymbol{r}}^*(t_k))$$

and

$$\boldsymbol{\chi}_2^k := \frac{\boldsymbol{\sigma}_h^*(t_{k+1}) + 2\boldsymbol{\sigma}_h^*(t_k) + \boldsymbol{\sigma}_h^*(t_{k-1})}{4} - \boldsymbol{\sigma}^*(t_k)\,.$$

Moreover, thanks to the second condition of Hypothesis 2 there holds

$$(\mathbf{div}(\boldsymbol{\sigma}_h^*(t_k) - \boldsymbol{\sigma}^*(t_k)), \mathbf{div}\,\boldsymbol{\tau})_\rho \;=\; 0 \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h\,,$$

and hence the consistency term $\boldsymbol{\chi}_2^k$ can be substituted in the error equation (1.5.3) by

$$\bar{\boldsymbol{\chi}}_2^k \;=\; \boldsymbol{\chi}_2^k \;-\; (\boldsymbol{\sigma}_h^*(t_k) - \boldsymbol{\sigma}^*(t_k)) \;=\; \frac{\boldsymbol{\sigma}_h^*(t_{k+1}) - 2\boldsymbol{\sigma}_h^*(t_k) + \boldsymbol{\sigma}_h^*(t_{k-1})}{4}\,.$$

1.5.2 Convergence results

We begin the analysis with the following stability result for the main variable $\boldsymbol{\sigma}$.

Lemma 1.5.1. *There exists a constant $C > 0$, independent of λ, h and Δt, such that for each n there holds*

$$\begin{aligned}&\max_n \|\partial_t \boldsymbol{e}_{\sigma,h}^n\|_{\mathcal{C}} + \max_n \left\|\mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}}\right\|_\rho \\ &\quad\leq C\left\{\max_n \|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} + \max_n \|\mathbf{div}\, \partial_t \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} + \max_n \|\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega}\right\}.\end{aligned} \tag{1.5.4}$$

Proof. Taking $\boldsymbol{\tau} = \dfrac{\boldsymbol{e}_{\sigma,h}^{k+1} - \boldsymbol{e}_{\sigma,h}^{k-1}}{2\Delta t}$ in (1.5.3) and using that

$$\frac{\boldsymbol{e}_{\sigma,h}^{k+1} - \boldsymbol{e}_{\sigma,h}^{k-1}}{2\Delta t} = \frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} - \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{\Delta t} = \frac{\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2},$$

we find that

$$\begin{aligned}&\frac{1}{2\Delta t}\Big(\mathcal{C}^{-1}(\partial_t \boldsymbol{e}_{\sigma,h}^k - \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}), (\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1})\Big) + \frac{1}{2\Delta t}\Big(\mathbf{div}(\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} + \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}), \mathbf{div}(\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} - \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}})\Big)_\rho \\ &\quad= \left(\boldsymbol{\chi}_1^k, \frac{\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}\right) + \left(\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\, \frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} - \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{\Delta t}\right)_\rho,\end{aligned}$$

which can also be written as

$$\begin{aligned}&\frac{1}{2\Delta t}\left(\left\|\partial_t \boldsymbol{e}_{\sigma,h}^k\right\|_{\mathcal{C}}^2 - \left\|\partial_t \boldsymbol{e}_{\sigma,h}^{k-1}\right\|_{\mathcal{C}}^2\right) + \frac{1}{2\Delta t}\left(\left\|\mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}}\right\|_\rho^2 - \left\|\mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}\right\|_\rho^2\right) \\ &\quad= \left(\boldsymbol{\chi}_1^k, \frac{\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}\right) + \left(\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\, \frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} - \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{\Delta t}\right)_\rho.\end{aligned}$$

In this way, summing up the foregoing identity over $k = 1, \ldots, n$, gives

$$\begin{aligned}\left\|\partial_t \boldsymbol{e}_{\sigma,h}^n\right\|_{\mathcal{C}}^2 + \left\|\mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}}\right\|_\rho^2 &= 2\Delta t \sum_{k=1}^n \left(\boldsymbol{\chi}_1^k, \frac{\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}\right) + 2\Delta t \sum_{k=1}^n \left(\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\, \frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} - \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{\Delta t}\right)_\rho \\ &= 2\Delta t \sum_{k=1}^n \left(\boldsymbol{\chi}_1^k, \frac{\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}\right) - 2\,\Delta t \sum_{k=1}^{n-1} (\mathbf{div}\, \partial_t \bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}})_\rho + 2(\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^n, \mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}})_\rho.\end{aligned}$$

It is now straightforward to deduce from the last identity, the Cauchy-Schwarz inequality, and (1.1.3) that there exists a constant $C_0 > 0$, independent of λ, h and Δt, such that

$$\begin{aligned}&\max_n \left\|\partial_t \boldsymbol{e}_{\sigma,h}^n\right\|_{\mathcal{C}} + \max_n \left\|\mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}}\right\|_\rho \\ &\quad\leq C_0 \left\{\Delta t \sum_{k=1}^L \left\|\mathcal{C}\boldsymbol{\chi}_1^k\right\|_{\mathcal{C}} + \frac{\Delta t}{\sqrt{\underline{\rho}}} \sum_{k=1}^L \left\|\mathbf{div}\, \partial_t \bar{\boldsymbol{\chi}}_2^k\right\|_{0,\Omega} + \frac{1}{\sqrt{\underline{\rho}}} \max_n \|\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega}\right\},\end{aligned} \tag{1.5.5}$$

and the result follows. ◻

We now turn to prove stability estimates for the Lagrange multiplier $\boldsymbol{r}$.

Lemma 1.5.2. *There exists a constant $C > 0$, independent of h, such that for each n there holds*

$$\max_n \left\|\partial_t \boldsymbol{e}_{r,h}^n\right\|_{0,\Omega} \leq C \left\{ \max_n \|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} + \max_n \|\mathbf{div}\, \partial_t \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} + \max_n \|\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \right\}. \tag{1.5.6}$$

Proof. Given $k \geq 1$ we deduce from the error equation (1.5.3) that

$$(\bar{\partial}_t \boldsymbol{e}_{r,h}^{k+1} - \bar{\partial}_t \boldsymbol{e}_{r,h}^k, \boldsymbol{\tau}) = -(\mathcal{C}^{-1}(\bar{\partial}_t \boldsymbol{e}_{\sigma,h}^{k+1} - \bar{\partial}_t \boldsymbol{e}_{\sigma,h}^k), \boldsymbol{\tau}) - \Delta t \left(\mathbf{div}\, \frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} + \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{2}, \mathbf{div}\, \boldsymbol{\tau} \right)_\rho$$
$$+ \Delta t (\boldsymbol{\chi}_1^k, \boldsymbol{\tau}) + \Delta t (\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\, \boldsymbol{\tau})_\rho ,$$

which, summing over $k = 1, \ldots, n$, yields

$$(\partial_t \boldsymbol{e}_{r,h}^{n+1}, \boldsymbol{\tau}) = -(\mathcal{C}^{-1} \partial_t \boldsymbol{e}_{\sigma,h}^{n+1}, \boldsymbol{\tau}) - \Delta t \sum_{k=1}^n \left(\mathbf{div}\, \frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} + \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{2}, \mathbf{div}\, \boldsymbol{\tau} \right)_\rho$$
$$+ \Delta t \sum_{k=1}^n (\boldsymbol{\chi}_1^k, \boldsymbol{\tau}) + \Delta t \sum_{k=1}^n (\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^k, \mathbf{div}\, \boldsymbol{\tau})_\rho .$$

It follows from the inf-sup condition (1.3.2), the Cauchy-Schwarz inequality, and (1.1.3) that there exists a constant $C_1 > 0$, independent of λ, h and Δt, such that

$$\beta^* \left\|\partial_t \boldsymbol{e}_{r,h}^{n+1}\right\|_{0,\Omega} \leq \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{(\partial_t \boldsymbol{e}_{r,h}^{n+1}, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}$$
$$\leq C_1 \left\{ \max_n (\left\|\partial_t \boldsymbol{e}_{\sigma,h}^{n+1}\right\|_{\mathcal{C}} + \frac{1}{\sqrt{\underline{\rho}}} \max_n \left\|\mathbf{div}\, \boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}}\right\|_\rho + \max_n \|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} + \frac{1}{\underline{\rho}} \max_n \|\mathbf{div}\, \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \right\} ,$$

and the result follows from Lemma 1.5.1. ◻

Lemma 1.5.3. *Assume that the solutions $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{sym}(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ to problem (1.1.5) satisfy $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div},\Omega) \cap \mathbb{H}^\epsilon(\Omega)) \cap \mathcal{C}^4(\mathbb{H}(\mathbf{div},\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^4(\mathbb{L}^2(\Omega))$. Then, there exists a constant $C > 0$, independent of λ, h and Δt, such that*

$$\max_n \left\|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^n\right\|_{\mathcal{C}} + \max_n \left\|\mathbf{div}(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}})\right\|_\rho + \max_n \left\|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^n\right\|_{0,\Omega}$$
$$\leq C \Big\{ \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r} - Q_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} \tag{1.5.7}$$
$$+ \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} + (\Delta t)^2 \|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \Big\}.$$

Proof. It follows from the triangle inequality and the stability estimates (1.5.4) and (1.5.6) that

$$
\begin{aligned}
&\max_n \left\|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^n\right\|_{\mathcal{C}} + \max_n \left\|\mathbf{div}\,\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \mathbf{div}\,\boldsymbol{\sigma}_h^{n+\frac{1}{2}}\right\|_{\rho} + \max_n \left\|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^n\right\|_{0,\Omega} \\
&\quad\leq \max_n \left\|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^*(t_n)\right\|_{\mathcal{C}} + \max_n \left\|\,\mathbf{div}\left(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \frac{\boldsymbol{\sigma}_h^*(t_{n+1}) + \boldsymbol{\sigma}_h^*(t_n)}{2}\right)\right\|_{\rho} \\
&\quad+ \max_n \left\|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^*(t_n)\right\|_{0,\Omega} + \max_n \left\|\partial_t \boldsymbol{e}_{\sigma,h}^n\right\|_{\mathcal{C}} + \max_n \left\|\mathbf{div}\,\boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}}\right\|_{\rho} + \max_n \left\|\partial_t \boldsymbol{e}_{r,h}^n\right\|_{\mathcal{C}} \\
&\quad\leq \max_n \left\|\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^*(t_n)\right\|_{\mathcal{C}} + \max_n \left\|\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^*(t_n)\right\|_{0,\Omega} \\
&\quad+ \max_n \left\|\,\mathbf{div}\left(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \frac{\boldsymbol{\sigma}_h^*(t_{n+1}) + \boldsymbol{\sigma}_h^*(t_n)}{2}\right)\right\|_{\rho} \\
&\quad+ C\left\{ \max_n \|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} + \max_n \|\mathbf{div}\,\partial_t \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} + \max_n \|\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \right\}.
\end{aligned}
\tag{1.5.8}
$$

Then, using Taylor expansions centered at $t = t_n$ with integral remainder and keeping in mind (1.3.24) we have that

$$
\begin{aligned}
\mathcal{C}\boldsymbol{\chi}_1^n &= \ddot{\boldsymbol{\sigma}}_h^*(t_n) - \ddot{\boldsymbol{\sigma}}^*(t_n) + 2\mu(\ddot{\boldsymbol{r}}_h^*(t_n) - \ddot{\boldsymbol{r}}^*(t_n)) \\
&+ \frac{1}{6(\Delta t)^2}\int_{t_{n-1}}^{t_{n+1}} \left(\frac{\mathrm{d}^4 \boldsymbol{\sigma}_h^*(t)}{\mathrm{d}t^4} + 2\mu \frac{\mathrm{d}^4 \boldsymbol{r}_h^*(t)}{\mathrm{d}t^4}\right)(\Delta t - |t - t_n|)^3\,\mathrm{d}t\,,
\end{aligned}
\tag{1.5.9}
$$

$$
\bar{\boldsymbol{\chi}}_2^n = \frac{1}{4}\int_{t_{n-1}}^{t_{n+1}} \ddot{\boldsymbol{\sigma}}_h^*(t)(\Delta t - |t - t_n|)\,\mathrm{d}t\,,
\tag{1.5.10}
$$

and

$$
\begin{aligned}
\partial_t \bar{\boldsymbol{\chi}}_2^k &= \frac{\boldsymbol{\sigma}_h^*(t_{n+2}) - 3\boldsymbol{\sigma}_h^*(t_{n+1}) + 3\boldsymbol{\sigma}_h^*(t_n) - \boldsymbol{\sigma}_h^*(t_{n-1})}{4\Delta t} = \frac{1}{8\Delta t}\left\{\int_{t_n}^{t_{n+2}} \frac{\mathrm{d}^3 \boldsymbol{\sigma}_h^*}{\mathrm{d}t^3}(t)(t_{n+2} - t)^2\,\mathrm{d}t \right.\\
&\left. - 3\int_{t_n}^{t_{n+1}} \frac{\mathrm{d}^3 \boldsymbol{\sigma}_h^*}{\mathrm{d}t^3}(t)(t_{n+1} - t)^2\,\mathrm{d}t + \int_{t_{n-1}}^{t_n} \frac{\mathrm{d}^3 \boldsymbol{\sigma}_h^*(t)}{\mathrm{d}t^3}(t_{n-1} - t)^2\,\mathrm{d}t \right\}.
\end{aligned}
\tag{1.5.11}
$$

In turn, Taylor expansions centered this time at $t = t_{n+\frac{1}{2}}$ give

$$
\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \frac{\boldsymbol{\sigma}_h^*(t_{n+1}) + \boldsymbol{\sigma}_h^*(t_n)}{2} = \boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^*(t_{n+\frac{1}{2}}) - \frac{1}{2}\int_{t_n}^{t_{n+1}} \ddot{\boldsymbol{\sigma}}_h^*(t)\left(\frac{\Delta t}{2} - |t - t_{n+\frac{1}{2}}|\right)\mathrm{d}t,
\tag{1.5.12}
$$

$$
\begin{aligned}
\dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^*(t_n) &= \dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \dot{\boldsymbol{\sigma}}_h^*(t_{n+\frac{1}{2}}) - \frac{1}{2\Delta t}\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \frac{\mathrm{d}^3 \boldsymbol{\sigma}_h^*(t)}{\mathrm{d}t^3}(t_{n+1} - t)^2\,\mathrm{d}t \\
&- \frac{1}{2\Delta t}\int_{t_n}^{t_{n+\frac{1}{2}}} \frac{\mathrm{d}^3 \boldsymbol{\sigma}_h^*(t)}{\mathrm{d}t^3}(t_n - t)^2\,\mathrm{d}t\,.
\end{aligned}
\tag{1.5.13}
$$

and

$$
\begin{aligned}
\dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^*(t_n) &= \dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \dot{\boldsymbol{r}}_h^*(t_{n+\frac{1}{2}}) - \frac{1}{2\Delta t}\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \frac{\mathrm{d}^3 \boldsymbol{r}_h^*(t)}{\mathrm{d}t^3}(t_{n+1} - t)^2\,\mathrm{d}t \\
&- \frac{1}{2\Delta t}\int_{t_n}^{t_{n+\frac{1}{2}}} \frac{\mathrm{d}^3 \boldsymbol{r}_h^*(t)}{\mathrm{d}t^3}(t_n - t)^2\,\mathrm{d}t\,.
\end{aligned}
\tag{1.5.14}
$$

Having established the above estimates, we now deduce from (1.5.9), (1.5.10) and (1.5.11) that there exists a constant $C_1 > 0$, independent of λ, h and Δt, such that

$$\begin{aligned}\max_n \|\mathcal{C}\boldsymbol{\chi}_1^n\|_{\mathcal{C}} \,+\, \max_n \|\mathbf{div}\,\partial_t \bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \,+\, \max_n \|\mathbf{div}\,\bar{\boldsymbol{\chi}}_2^n\|_{0,\Omega} \,\le\, C_1 \Big\{ \|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \\ +\, \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} \,+\, (\Delta t)^2 \big(\|\boldsymbol{r}_h^*\|_{\mathrm{W}^{4,\infty}(\mathbb{L}^2(\Omega))} \,+\, \|\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \big) \Big\},\end{aligned} \tag{1.5.15}$$

whereas (1.5.12), (1.5.13) and (1.5.14) yield the existence of a constant $C_2 > 0$, independent of λ, h and Δt, such that

$$\begin{aligned}&\max_n \Big\|\dot{\boldsymbol{\sigma}}(t_{n+\frac12}) - \partial_t \boldsymbol{\sigma}_h^*(t_n)\Big\|_{\mathcal{C}} + \max_n \Big\|\dot{\boldsymbol{r}}(t_{n+\frac12}) - \partial_t \boldsymbol{r}_h^*(t_n)\Big\|_{0,\Omega} \\ &+\, \max_n \Big\|\, \mathbf{div}\, \Big(\boldsymbol{\sigma}(t_{n+\frac12}) - \frac{\boldsymbol{\sigma}_h^*(t_{n+1}) + \boldsymbol{\sigma}_h^*(t_n)}{2}\Big)\Big\|_{\rho} \,\le\, C_2 \Big\{ \|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{1,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \\ &+\, \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{\mathrm{W}^{1,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \,+\, (\Delta t)^2 \big(\|\boldsymbol{r}_h^*\|_{\mathrm{W}^{3,\infty}(\mathbb{L}^2(\Omega))} \,+\, \|\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{3,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \big) \Big\}.\end{aligned} \tag{1.5.16}$$

Finally, we deduce from the uniform boundedness of $\Xi_h : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h$ with respect to h and λ, and from our regularity assumptions, that there exists a constant $C_3 > 0$, independent of h and λ, such that

$$\|\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r}_h^*\|_{\mathrm{W}^{4,\infty}(\mathbb{L}^2(\Omega))} \,\le\, C_3\, \|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))}\,, \tag{1.5.17}$$

and thus, combining (1.5.15), (1.5.16), and (1.5.17) with (1.5.8), we conclude that

$$\begin{aligned}&\max_n \Big\|\dot{\boldsymbol{\sigma}}(t_{n+\frac12}) - \partial_t \boldsymbol{\sigma}_h^n\Big\|_{\mathcal{C}} \,+\, \max_n \Big\|\mathbf{div}\,\boldsymbol{\sigma}(t_{n+\frac12}) - \mathbf{div}\,\boldsymbol{\sigma}_h^{n+\frac12}\Big\|_{\rho} \,+\, \max_n \Big\|\dot{\boldsymbol{r}}(t_{n+\frac12}) - \partial_t \boldsymbol{r}_h^n\Big\|_{0,\Omega} \\ &\le\, C_4 \Big\{ \|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \,+\, \|\boldsymbol{r}^* - \boldsymbol{r}_h^*\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} \,+\, (\Delta t)^2\, \|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \Big\},\end{aligned} \tag{1.5.18}$$

and the result follows from (1.3.18) and (1.3.19). □

Lemma 1.5.4. *Under the hypotheses of Lemma 1.5.3 there exists a constant $C > 0$, independent of λ, h and Δt, such that*

$$\begin{aligned}\max_n \Big\|\boldsymbol{\sigma}(t_{n+\frac12}) - \boldsymbol{\sigma}_h^{n+\frac12}\Big\|_{\mathcal{C}} \,+\, \max_n \Big\|\boldsymbol{r}(t_{n+\frac12}) - \boldsymbol{r}_h^{n+\frac12}\Big\|_{0,\Omega} \,\le\, C \Big\{ \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \\ +\, \|\boldsymbol{r} - Q_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} \,+\, \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \\ +\, (\Delta t)^2 \left(\|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} \,+\, \|\boldsymbol{r}\|_{\mathrm{W}^{4,\infty}(\mathbb{L}^2(\Omega))} \right) \Big\}.\end{aligned} \tag{1.5.19}$$

Proof. We first notice that

$$\begin{aligned}(\boldsymbol{\sigma}(t_{k+\frac12}) - \boldsymbol{\sigma}_h^{k+\frac12}) - (\boldsymbol{\sigma}(t_{k-\frac12}) - \boldsymbol{\sigma}_h^{k-\frac12}) = \boldsymbol{\sigma}(t_{k+\frac12}) - \boldsymbol{\sigma}(t_{k-\frac12}) \\ -\, \frac{\Delta t}{2} (\dot{\boldsymbol{\sigma}}(t_{k+\frac12}) + \dot{\boldsymbol{\sigma}}(t_{k-\frac12})) + \frac{\Delta t}{2} (\dot{\boldsymbol{\sigma}}(t_{k+\frac12}) - \partial_t \boldsymbol{\sigma}_h^k) + \frac{\Delta t}{2} (\dot{\boldsymbol{\sigma}}(t_{k-\frac12}) - \partial_t \boldsymbol{\sigma}_h^{k-1})\end{aligned} \tag{1.5.20}$$

and

$$\begin{aligned}(\boldsymbol{r}(t_{k+\frac12}) - \boldsymbol{r}_h^{k+\frac12}) - (\boldsymbol{r}(t_{k-\frac12}) - \boldsymbol{r}_h^{k-\frac12}) = \boldsymbol{r}(t_{k+\frac12}) - \boldsymbol{r}(t_{k-\frac12}) \\ -\, \frac{\Delta t}{2} (\dot{\boldsymbol{r}}(t_{k+\frac12}) + \dot{\boldsymbol{r}}(t_{k-\frac12})) + \frac{\Delta t}{2} (\dot{\boldsymbol{r}}(t_{k+\frac12}) - \partial_t \boldsymbol{r}_h^k) + \frac{\Delta t}{2} (\dot{\boldsymbol{r}}(t_{k-\frac12}) - \partial_t \boldsymbol{r}_h^{k-1})\,.\end{aligned} \tag{1.5.21}$$

Then, using a Taylor expansion centered at $t = t_k$, we find that

$$\begin{aligned} \boldsymbol{\sigma}(t_{k+\frac{1}{2}}) - \boldsymbol{\sigma}(t_{k-\frac{1}{2}}) - \frac{\Delta t}{2}(\dot{\boldsymbol{\sigma}}(t_{k+\frac{1}{2}}) + \dot{\boldsymbol{\sigma}}(t_{k-\frac{1}{2}})) &= \frac{1}{2}\int_{t_k}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{\sigma}}{\mathrm{d}t^3}(t)(t_{k+\frac{1}{2}} - t)^2\,\mathrm{d}t \\ + \frac{1}{2}\int_{t_{k-\frac{1}{2}}}^{t_k} \frac{\mathrm{d}^3\boldsymbol{\sigma}}{\mathrm{d}t^3}(t)(t_{k-\frac{1}{2}} - t)^2\,\mathrm{d}t \; - \; \frac{\Delta t}{2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{\sigma}}{\mathrm{d}t^3}(t)(\frac{\Delta t}{2} - |t-t_k|)\,\mathrm{d}t & \end{aligned} \tag{1.5.22}$$

and

$$\begin{aligned} \boldsymbol{r}(t_{k+\frac{1}{2}}) - \boldsymbol{r}(t_{k-\frac{1}{2}}) - \frac{\Delta t}{2}(\dot{\boldsymbol{r}}(t_{k+\frac{1}{2}}) + \dot{\boldsymbol{r}}(t_{k-\frac{1}{2}})) &= \frac{1}{2}\int_{t_k}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}t^3}(t)(t_{k+\frac{1}{2}} - t)^2\,\mathrm{d}t \\ + \frac{1}{2}\int_{t_{k-\frac{1}{2}}}^{t_k} \frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}t^3}(t)(t_{k-\frac{1}{2}} - t)^2\,\mathrm{d}t - \frac{\Delta t}{2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}t^3}(t)(\frac{\Delta t}{2} - |t-t_k|)\,\mathrm{d}t\,. & \end{aligned} \tag{1.5.23}$$

In this way, substituting (1.5.22) in (1.5.20) and (1.5.23) in (1.5.21), and summing the resulting identities over $k = 1, \ldots, n$, we deduce that there exists a constant $C_0 > 0$, independent of λ, h and Δt, such that

$$\begin{aligned} \max_n \left\| \boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}} \right\|_{\mathcal{C}} + \max_n \left\| \boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}} \right\|_{0,\Omega} \;\leq\; C_0 \Big\{ (\Delta t)^2 \big(\|\boldsymbol{\sigma}\|_{\mathrm{W}^{3,\infty}(\mathbb{L}^2(\Omega))} \\ + \; \|\boldsymbol{r}\|_{\mathrm{W}^{3,\infty}(\mathbb{L}^2(\Omega))} \big) + \max_n \left\| \dot{\boldsymbol{\sigma}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{\sigma}_h^n \right\|_{\mathcal{C}} + \max_n \left\| \dot{\boldsymbol{r}}(t_{n+\frac{1}{2}}) - \partial_t \boldsymbol{r}_h^n \right\|_{0,\Omega} \Big\}. \end{aligned}$$

Finally, (1.5.19) is a direct consequence of the foregoing estimate and Lemma 1.5.3. □

It follows from (1.5.1) and the fact that $(\boldsymbol{\sigma}_h^0, \boldsymbol{r}_h^0)$ and $(\boldsymbol{\sigma}_h^1, \boldsymbol{r}_h^1)$ belong to $\mathbb{V}_h$ that for each $n \in \{2, \ldots, L\}$, $(\boldsymbol{\sigma}_h^n, \boldsymbol{r}_h^n)$ belongs to $\mathbb{V}_h$ as well. Hence, we may define $\boldsymbol{u}_h^n := D_h(\boldsymbol{\sigma}_h^n, \boldsymbol{r}_h^n) \in \mathcal{U}_h$, which is characterized by

$$(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}_h^n) \;=\; -(\mathcal{C}^{-1}\boldsymbol{\sigma}_h^n + \boldsymbol{r}_h^n, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \quad \forall\, n \in \{0, \ldots, L\}\,. \tag{1.5.24}$$

Moreover, we propose $\boldsymbol{u}_h^{n+\frac{1}{2}}$ and $\boldsymbol{a}_h^n := \bar{\partial}_t \partial_t \boldsymbol{u}^n$ as suitable approximations of the displacement field $u(t_{n+\frac{1}{2}})$ and the acceleration $\ddot{\boldsymbol{u}}(t_n)$, respectively. To this regard, we remark again that, one can compute $\boldsymbol{u}_h^{n+\frac{1}{2}}$ by solving a saddle point problem of the form (1.3.8) with right-hand side $\mathbf{div}\,\boldsymbol{\sigma}_h^{n+\frac{1}{2}}$. However, a better option consists in using an explicit representation of the fully discrete displacement field $\boldsymbol{u}_h^{n+\frac{1}{2}}$, which is obtained as follows. We first notice from the characterization (1.5.24) of the operator D_h that

$$\Big(\mathbf{div}\,\boldsymbol{\tau}, \bar{\partial}_t \partial_t \boldsymbol{u}_h^n \Big) = -\Big(\partial_t \bar{\partial}_t (\mathcal{C}^{-1}\boldsymbol{\sigma}_h^n + \boldsymbol{r}_h^n), \boldsymbol{\tau} \Big) \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \quad \forall\, n \in \{1, \ldots, L-1\}\,,$$

whereas from the first equation of (1.5.1) we have that

$$\Big(\mathbf{div}\,\boldsymbol{\tau}, \bar{\partial}_t \partial_t \boldsymbol{u}_h^n \Big) = \Big(\mathbf{div}\,\frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}} + \boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2} + \boldsymbol{f}(t_n), \mathbf{div}\,\boldsymbol{\tau} \Big)_\rho \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h, \quad \forall\, n \in \{1, \ldots, L-1\}\,.$$

It follows from the foregoing equation that

$$\boldsymbol{a}_h^n := \bar{\partial}_t \partial_t \boldsymbol{u}^n = \rho^{-1} \left\{ \mathbf{div}\,\frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}} + \boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2} + U_h \boldsymbol{f}(t_n) \right\}, \tag{1.5.25}$$

and summing twice the last identity we obtain

$$\boldsymbol{u}_h^n = (\Delta t)^2 \sum_{l=1}^{n-1} \sum_{k=1}^{l} \rho^{-1} \left\{ \mathbf{div}\, \frac{\boldsymbol{\sigma}_h^{k+\frac{1}{2}} + \boldsymbol{\sigma}_h^{k-\frac{1}{2}}}{2} + U_h \boldsymbol{f}(t_k) \right\} + \boldsymbol{u}_h^0 + t_n \bar{\partial}_t \boldsymbol{u}_h^1 \quad \forall\, n \in \{2, \ldots, L\}\,, \quad (1.5.26)$$

with $\boldsymbol{u}_h^0 := D_h(\boldsymbol{\sigma}_h^0, \boldsymbol{r}_h^0)$ and $\boldsymbol{u}_h^1 := D_h(\boldsymbol{\sigma}_h^1, \boldsymbol{r}_h^1)$.

Lemma 1.5.5. *Under the hypotheses of Lemma 1.5.3 there exists a constant $C > 0$, independent of λ, h and Δt, such that*

$$\max_n \|\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} + \max_n \left\| \boldsymbol{u}(t_{n+\frac{1}{2}}) - \boldsymbol{u}_h^{n+\frac{1}{2}} \right\|_{0,\Omega} \leq C \Big\{ (\Delta t)^2 \, \|\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))}$$
$$+ \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\boldsymbol{L}^2(\Omega))} + \max_n \left\| \boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}} \right\|_{\mathrm{H}(\mathbf{div},\Omega)} + \max_n \left\| \boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}} \right\|_{0,\Omega} \Big\}.$$

Proof. We begin by observing, thanks to the inf-sup condition (1.3.2) and Hypothesis 2, that

$$\beta^* \, \|U_h \ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} \quad \leq \quad \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{(U_h \ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n, \mathbf{div}\, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \quad = \quad \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{(\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n, \mathbf{div}\, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}}. \quad (1.5.27)$$

Next, we notice that by definition of $\ddot{\boldsymbol{u}}(t) = D(\ddot{\boldsymbol{\sigma}}(t), \ddot{\boldsymbol{r}}(t))$ and $\boldsymbol{a}_h^n$, it holds that

$$(\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n, \mathbf{div}\, \boldsymbol{\tau}) = \Big(\mathbf{div}\, \big(\boldsymbol{\sigma}(t_n) - \frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}} + \boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2}\big), \mathbf{div}\, \boldsymbol{\tau} \Big)_\rho. \quad (1.5.28)$$

In this way, writing

$$\boldsymbol{\sigma}(t_n) - \frac{\boldsymbol{\sigma}_h^{n+\frac{1}{2}} + \boldsymbol{\sigma}_h^{n-\frac{1}{2}}}{2} = -\frac{1}{2} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \ddot{\boldsymbol{\sigma}}(t) \Big(\frac{\Delta t}{2} - |t - t_n|\Big) \, \mathrm{d}t + \frac{1}{2}\big(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}\big) + \frac{1}{2}\big(\boldsymbol{\sigma}(t_{n-\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n-\frac{1}{2}}\big)\,,$$

we deduce from (1.5.27) and (1.5.28) that

$$\beta^* \max_n \|U_h \ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} \leq \max_n \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h} \frac{(\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n, \mathbf{div}\, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \leq \frac{(\Delta t)^2}{4\underline{\rho}} \, \|\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))}$$
$$+ \frac{1}{2\underline{\rho}} \max_n \left\| \mathbf{div}(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}) \right\|_{0,\Omega} + \frac{1}{2\underline{\rho}} \max_n \left\| \mathbf{div}(\boldsymbol{\sigma}(t_{n-\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n-\frac{1}{2}}) \right\|_{0,\Omega},$$

which, combined with the triangle inequality, gives

$$\max_n \|\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} \leq \max_n \|\ddot{\boldsymbol{u}}(t_n) - U_h \ddot{\boldsymbol{u}}(t_n)\|_{0,\Omega} + \max_n \|U_h \ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega}$$
$$\leq \max_n \|\ddot{\boldsymbol{u}}(t_n) - U_h \ddot{\boldsymbol{u}}(t_n)\|_{0,\Omega} + \frac{(\Delta t)^2}{4\underline{\rho}\beta^*} \, \|\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \frac{1}{2\underline{\rho}\beta^*} \max_n \left\| \mathbf{div}(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}) \right\|_{0,\Omega}.$$

On the other hand, in order prove error estimates for the displacement, we use again the inf-sup condition (1.3.2) and the identities (1.2.7) and (1.5.24) to obtain

$$\begin{aligned}
\beta^* \left\| \boldsymbol{u}_h^{n+\frac{1}{2}} - U_h \boldsymbol{u}(t_{n+\frac{1}{2}}) \right\|_{0,\Omega} &\leq \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{(\boldsymbol{u}_h^{n+\frac{1}{2}} - U_h \boldsymbol{u}(t_{n+\frac{1}{2}}), \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\
&= \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{(\boldsymbol{u}_h^{n+\frac{1}{2}} - \boldsymbol{u}(t_{n+\frac{1}{2}}), \mathbf{div}\,\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\
&= \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h} \frac{(\mathcal{C}^{-1}\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) + \boldsymbol{r}(t_{n+\frac{1}{2}}) - \mathcal{C}^{-1}\boldsymbol{\sigma}_h^{n+\frac{1}{2}} - \boldsymbol{r}_h^{n+\frac{1}{2}}, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \\
&\leq \left\| \boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}} \right\|_{0,\Omega} + \frac{1}{2\mu} \left\| \boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}} \right\|_{0,\Omega}.
\end{aligned}$$

Finally, the triangle inequality gives the estimate

$$\begin{aligned}
\max_n \left\| \boldsymbol{u}(t_{n+\frac{1}{2}}) - \boldsymbol{u}_h^{n+\frac{1}{2}} \right\|_{0,\Omega} &\leq \max_n \left\| \boldsymbol{u}(t_{n+\frac{1}{2}}) - U_h \boldsymbol{u}(t_{n+\frac{1}{2}}) \right\|_{0,\Omega} \\
+ \quad \tfrac{1}{\beta^*} \max\nolimits_{[0,T]} \left\| \boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}} \right\|_{0,\Omega} &+ \tfrac{1}{2\mu\beta^*} \max\nolimits_n \left\| \boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}} \right\|_{0,\Omega},
\end{aligned}$$

and the result follows. □

Theorem 1.5.1. *Assume that the solutions* $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{sym}(\Omega))$ *and* $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ *to problem* (1.1.5) *satisfy* $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}(\mathbf{div},\Omega) \cap \mathbb{H}^\epsilon(\Omega)) \cap \mathcal{C}^4(\mathbb{H}(\mathbf{div},\Omega))$ *and* $\boldsymbol{r} \in \mathcal{C}^4(\mathbb{L}^2(\Omega))$*. Then, there exists a constant* $C > 0$ *independent of* λ*,* h *and* Δt *such that*

$$\begin{aligned}
\max_n \left\| \boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}} \right\|_{\mathrm{H}(\mathbf{div},\Omega)} &+ \max_n \left\| \boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}} \right\|_{0,\Omega} + \max_n \|\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} \\
+ \max_n \left\| \boldsymbol{u}(t_{n+\frac{1}{2}}) - \boldsymbol{u}_h^{n+\frac{1}{2}} \right\|_{0,\Omega} &\leq C \Big\{ \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r} - Q_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}(\mathbb{L}^2(\Omega))} \\
&+ \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} + (\Delta t)^2 \big(\|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}(\mathrm{H}(\mathbf{div},\Omega))} + \|\boldsymbol{r}\|_{\mathrm{W}^{4,\infty}(\mathbb{L}^2(\Omega))} \big) \Big\}.
\end{aligned}$$

Proof. The result is a direct consequence of Lemmas 1.5.3, 1.5.4 and 1.5.5 and the norm equivalence provided by Lemma 1.1.1. □

Remark 1.5.1. *We end this section by remarking, as shown by Theorem 1.5.1, that the fully discrete scheme maintains the convergence properties obtained in Theorem 1.3.1 for the semidiscrete Galerkin scheme as discussed at the end of Section 1.3. Indeed,* (1.5.7) *shows that the fully discrete scheme can deal safely with nearly incompressible materials. Finally, we notice from* (1.5.26) *that the displacement field can also be post-processed at the fully discrete level.*

1.6 Asymptotic error estimates for the AFW element

It is important to notice that Hypotheses 1, 2 and 3 are satisfied for most known mixed finite elements [4, 13, 28, 40] for the steady elasticity problem with reduced symmetry (see [5] for more details).

However, for the sake of brevity we restrict our choice of finite element examples to the Arnold-Falk-Winther (AFW) family [4]. We consider shape regular affine meshes $\mathcal{T}_h$ that subdivide the domain $\bar{\Omega}$ into triangles/tetrahedra K of diameter h_K. The parameter $h := \max_{K\in\mathcal{T}_h}\{h_K\}$ represents the mesh size of $\mathcal{T}_h$. In what follows, we assume that $\mathcal{T}_h$ is compatible with the partition $\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j$, i.e.,

$$\left\{K \in \mathcal{T}_h, \quad K \subset \bar{\Omega}_j\right\} = \bar{\Omega}_j \quad \forall j = 1, \cdots, J.$$

Hereafter, given an integer $m \geq 0$ and a domain $D \subset \mathbb{R}^d$, $\mathcal{P}_m(D)$ denotes the space of polynomials of degree at most m on D. The space of piecewise polynomial functions of degree at most m relatively to $\mathcal{T}_h$ is denoted by

$$\mathcal{P}_m(\mathcal{T}_h) := \left\{v \in L^2(\Omega); \quad v|_K \in \mathcal{P}_m(K), \quad \forall K \in \mathcal{T}_h\right\}.$$

For $k \geq 1$, the finite element spaces

$$\boldsymbol{\mathcal{W}}_h := \mathcal{P}_k(\mathcal{T}_h)^{d\times d} \cap \boldsymbol{\mathcal{W}}, \qquad \boldsymbol{\mathcal{Q}}_h := \mathcal{P}_{k-1}(\mathcal{T}_h)^{d\times d} \cap \boldsymbol{\mathcal{Q}} \qquad \text{and} \qquad \mathcal{U}_h := \mathcal{P}_{k-1}(\mathcal{T}_h)^d$$

correspond to the Arnold-Falk-Winther (AFW) family introduced in [4] for the steady elasticity problem. It is shown in [4] that Hypothesis 1 and the first condition of Hypothesis 2 hold true. Moreover, the fact that $\mathcal{T}_h$ is compatible with the partition $\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j$ implies that the second condition of Hypothesis 2 follows from the first one.

We also let $\Pi_h : \mathbb{H}^1(\Omega) \to \boldsymbol{\mathcal{W}}_h$ be the tensorial version of the BDM-interpolation operator and recall the following classical error estimate, see [12, Proposition 2.5.4],

$$\|\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}\|_{0,\Omega} \leq Ch^m \|\boldsymbol{\tau}\|_{m,\Omega} \qquad \forall \boldsymbol{\tau} \in \mathbb{H}^m(\Omega) \quad \text{with } 1 \leq m \leq k+1. \tag{1.6.1}$$

Moreover, thanks to the commutativity property, if $\operatorname{div}\boldsymbol{\tau} \in \mathbf{H}^k(\Omega)$, then

$$\|\operatorname{div}(\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})\|_{0,\Omega} = \|\operatorname{div}\boldsymbol{\tau} - U_h\,\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \leq Ch^m \|\mathbf{div}\,\boldsymbol{\tau}\|_{m,\Omega} \quad \text{for } 0 \leq m \leq k. \tag{1.6.2}$$

In addition, it is well known (see, e.g. [29, Theorem 3.16]) that Π_h is defined on $\mathbb{H}^\epsilon(\Omega) \cap \mathbb{H}(\mathbf{div}, \Omega)$ for any $\epsilon > 0$ and there exists $C > 0$, independent of h, such that

$$\|\Pi_h\boldsymbol{\tau}\|_{0,\Omega} \leq C\left\{\|\boldsymbol{\tau}\|_{\epsilon,\Omega} + \|\,\mathbf{div}\ \boldsymbol{\tau}\|_{0,\Omega}\right\}, \tag{1.6.3}$$

which proves that Hypothesis 3 is satisfied.

We deduce from (1.6.1), (1.6.2) and Theorem 1.3.1 that if the solutions $\boldsymbol{\sigma} \in \mathcal{C}^0(\boldsymbol{\mathcal{S}}) \cap \mathcal{C}^1(\mathbb{L}^2_{\text{sym}}(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^1(\boldsymbol{\mathcal{Q}})$ to problem (1.1.5) satisfy $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{H}^k(\Omega))$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbf{H}^k(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^2(\mathbb{H}^k(\Omega))$, then there exists a constant $C > 0$ independent of h such that

$$\max_{[0,T]} \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t)\|_{\mathrm{H}(\mathbf{div},\Omega)} + \|\boldsymbol{r} - \boldsymbol{r}_h\|_{\mathrm{W}^{1,\infty}(\mathbb{L}^2(\Omega))} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{W}^{2,\infty}(\mathbf{L}^2(\Omega))} \leq Ch^k. \tag{1.6.4}$$

Similarly, it follows from Theorem 1.5.1 that if the solutions to problem (1.1.5) satisfy $\boldsymbol{\sigma} \in \mathcal{C}^4(\mathbb{H}(\mathbf{div}, \Omega)) \cap \mathcal{C}^2(\mathbb{H}^k(\Omega))$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbf{H}^k(\Omega))$ and $\boldsymbol{r} \in \mathcal{C}^4(\mathbb{L}^2(\Omega)) \cap \mathcal{C}^2(\mathbb{H}^k(\Omega))$ then, there exists a constant $C > 0$ independent of h, Δt and λ such that

$$\begin{aligned} &\max_n \left\|\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}\right\|_{\mathrm{H}(\mathbf{div},\Omega)} + \max_n \left\|\boldsymbol{r}(t_{n+\frac{1}{2}}) - \boldsymbol{r}_h^{n+\frac{1}{2}}\right\|_{0,\Omega} \\ &+ \max_n \|\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\|_{0,\Omega} + \max_n \left\|\boldsymbol{u}(t_{n+\frac{1}{2}}) - \boldsymbol{u}_h^{n+\frac{1}{2}}\right\|_{0,\Omega} \leq C\left\{h^k + (\Delta t)^2\right\}. \end{aligned} \tag{1.6.5}$$

$h = \Delta t$	$\mathtt{e}_h(\boldsymbol{\sigma})$	$\mathtt{r}_h(\boldsymbol{\sigma})$	$\mathtt{e}_h(\mathbf{r})$	$\mathtt{r}_h(\mathbf{r})$	$\ddot{\mathtt{e}}_h(\mathbf{u})$	$\ddot{\mathtt{r}}_h(\mathbf{u})$	$\mathtt{e}_h(\mathbf{u})$	$\mathtt{r}_h(\mathbf{u})$
1/8	4.65e−01	–	3.18e−02	–	9.53e−00	–	1.23e−01	–
1/16	1.08e−01	2.11	9.03e−03	1.82	2.27e−00	2.07	3.05e−02	2.01
1/32	2.65e−02	2.02	2.47e−03	1.87	5.59e−01	2.02	7.56e−03	2.01
1/64	6.63e−03	2.00	6.47e−04	1.93	1.39e−01	2.01	1.89e−03	2.00
1/128	1.65e−03	2.01	1.66e−04	1.96	3.47e−02	2.00	4.72e−04	2.00
1/256	4.10e−04	2.01	4.19e−05	1.99	8.67e−03	2.00	1.18e−04	2.00

Table 1.7.1: Convergence history in the case $\lambda = \mu = \omega = 1$ and $k = 2$.

1.7 A mixed FEM example and numerical results

We present a series of numerical experiments confirming the good performance of the fully discrete Galerkin scheme (1.5.1). For simplicity we consider a two-dimensional model problem and the AFW element element for the spatial discretization. All the numerical results have been obtained by using FEniCS [30].

We choose $\Omega = (0,1) \times (0,1)$, $T = 1$, $\rho = 1$ and select the data $\boldsymbol{f}$ so that the exact solution is given by

$$\boldsymbol{u}(x_1, x_2) = \sin(2\pi\omega x_1)\sin(2\pi\omega x_2)\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}. \tag{1.7.1}$$

We also assume that the body is fixed on the whole boundary, i.e., we take $\Gamma = \partial\Omega$. The numerical results have been obtained by considering nested sequences of uniform triangular meshes $\mathcal{T}_h$ of the unit square Ω. For each mesh size h, we take $\Delta t = h$ and the individual relative errors produced by the fully discrete Galerkin method (1.5.1) are measured at the final time step as follows:

$$\mathtt{e}_h(\boldsymbol{\sigma}) := \frac{\|\boldsymbol{\sigma}(t_{L-\frac{1}{2}}) - \boldsymbol{\sigma}_h^{L-\frac{1}{2}}\|_{\mathrm{H}(\mathbf{div},\Omega)}}{\|\boldsymbol{\sigma}(t_{L-\frac{1}{2}})\|_{\mathrm{H}(\mathbf{div},\Omega)}}, \qquad \mathtt{e}_h(\boldsymbol{r}) := \frac{\|\boldsymbol{r}(t_{L-\frac{1}{2}}) - \boldsymbol{r}_h^{L-\frac{1}{2}}\|_{0,\Omega}}{\|\boldsymbol{r}(t_{L-\frac{1}{2}})\|_{0,\Omega}},$$

$$\mathtt{e}_h(\boldsymbol{u}) := \frac{\|\boldsymbol{u}(t_{L-\frac{1}{2}}) - \boldsymbol{u}_h^{L-\frac{1}{2}}\|_{0,\Omega}}{\|\boldsymbol{u}(t_{L-\frac{1}{2}})\|_{0,\Omega}} \qquad \ddot{\mathtt{e}}_h(\boldsymbol{u}) := \frac{\|\ddot{\boldsymbol{u}}(t_{L-1}) - \boldsymbol{a}_h^{L-1}\|_{0,\Omega}}{\|\ddot{\boldsymbol{u}}(t_{L-1})\|_{0,\Omega}},$$

where $(\boldsymbol{\sigma}, \boldsymbol{r})$ and $\{(\boldsymbol{\sigma}_h^n, \boldsymbol{r}_h^n),\ n = 0, \ldots, L\}$ are the solutions of (1.1.5) and (1.5.1) respectively and $\boldsymbol{a}_h^{L-1}$ is obtained from (1.5.25). We introduce the experimental rates of convergence

$$\mathtt{r}_h(\boldsymbol{\sigma}) := \frac{\log(\mathtt{e}_h(\boldsymbol{\sigma})/\mathtt{e}_{\hat{h}}(\boldsymbol{\sigma}))}{\log(h/\hat{h})}, \qquad \mathtt{r}_h(\boldsymbol{r}) := \frac{\log(\mathtt{e}_h(\boldsymbol{r})/\mathtt{e}_{\hat{h}}(\boldsymbol{r}))}{\log(h/\hat{h})},$$

$$\mathtt{r}_h(\boldsymbol{u}) := \frac{\log(\mathtt{e}_h(\boldsymbol{u})/\mathtt{e}_{\hat{h}}(\boldsymbol{u}))}{\log(h/\hat{h})}, \qquad \ddot{\mathtt{r}}_h(\boldsymbol{u}) := \frac{\log(\ddot{\mathtt{e}}_h(\boldsymbol{u})/\ddot{\mathtt{e}}_{\hat{h}}(\boldsymbol{u}))}{\log(h/\hat{h})},$$

where $\mathtt{e}_h$ and $\hat{\mathtt{e}}_{\hat{h}}$ are the errors corresponding to two consecutive triangulations with mesh sizes h and $\hat{h}$, respectively.

We report in Table 1.7.1 the relative errors and the convergence orders obtained for the AFW element of order $k = 2$ (AFW(2)) and with an exact solution defined as in (1.7.1) with $\lambda = \mu = \omega = 1$. It is clear that the correct quadratic convergence rate of the error (see (1.6.5)) is attained in each variable.

$h=\Delta t$	$\mathtt{e}_h(\boldsymbol{\sigma})$	$\mathtt{r}_h(\boldsymbol{\sigma})$	$\mathtt{e}_h(\mathbf{r})$	$\mathtt{r}_h(\mathbf{r})$	$\ddot{\mathtt{e}}_h(\mathbf{u})$	$\ddot{\mathtt{r}}_h(\mathbf{u})$	$\mathtt{e}_h(\mathbf{u})$	$\mathtt{r}_h(\mathbf{u})$
1/8	4.18e−01	–	4.43e−00	–	5.28e+03	–	3.53e−01	–
1/16	9.70e−02	2.11	5.69e−01	2.96	1.25e+03	2.07	3.67e−02	3.26
1/32	2.23e−02	2.12	7.80e−02	2.87	3.09e+02	2.02	7.73e−03	2.25
1/64	6.12e−03	1.87	9.50e−03	3.04	7.67e+01	2.01	1.88e−03	2.04
1/128	1.48e−03	2.05	1.14e−03	3.06	1.91e+01	2.00	4.72e−04	2.00
1/256	3.43e−04	2.11	1.53e−04	2.90	4.79e+00	2.00	1.18e−04	2.00

Table 1.7.2: Convergence history in a nearly incompressible case: $\nu = 0.499$, $\omega = 1$, $k = 2$.

$h=\Delta t$	$\mathtt{e}_h(\boldsymbol{\sigma})$	$\mathtt{r}_h(\boldsymbol{\sigma})$	$\mathtt{e}_h(\mathbf{r})$	$\mathtt{r}_h(\mathbf{r})$	$\ddot{\mathtt{e}}_h(\mathbf{u})$	$\ddot{\mathtt{r}}_h(\mathbf{u})$	$\mathtt{e}_h(\mathbf{u})$	$\mathtt{r}_h(\mathbf{u})$
1/8	2.25e+02	–	1.22e+01	–	3.16e+04	–	4.39e+02	–
1/16	1.59e+02	0.50	1.68e−00	2.86	3.08e+04	0.04	2.29e+01	4.26
1/32	2.44e+01	2.70	1.78e−01	3.24	5.18e+03	2.57	1.21e−01	7.56
1/64	1.28e+00	4.25	4.88e−03	5.19	3.35e+02	3.95	7.75e−03	3.97
1/128	7.82e−02	4.03	2.29e−04	4.41	2.24e+01	3.90	5.31e−04	3.87
1/256	5.54e−03	3.82	1.25e−05	4.19	1.43e+00	3.98	3.37e−05	3.98

Table 1.7.3: Convergence history in the case $\lambda = \mu = 1$, $\omega = 16$ and $k = 4$.

To test the locking-free character of the method in the nearly incompressible case, we consider now Lamé coefficients λ and μ corresponding to a Poisson ratio $\nu = 0.499$ and a Young modulus $E = 10$. We fix the polynomial degree to $k = 2$, take $\omega = 1$ and report in Table 1.7.2 the experimental rates of convergence. We observe that the method is thoroughly robust for nearly incompressible materials. Finally, we notice that the higher ω is in (1.7.1), the smaller is the mesh size h needed to reduce the predominance of the spatial component of the error. In such a case, it is meaningful to use a polynomial degree $k > 2$ in order to make the error reach its asymptotic behavior without using too small mesh sizes h. This is illustrated in Table 1.7.3 where AFW(4) is used with the choice $\omega = 16$.

Chapter 2

Finite element analysis of a pressure-stress formulation for the time-domain fluid-structure interaction problem

2.1 The model problem

We aim to compute the linear oscillations of a structure $\Omega := \Omega_S \cup \Sigma \cup \Omega_F$, with $\Omega \subset \mathbb{R}^n (n = 2, 3)$ consisting of a solid body, represented by a polyhedral Lipschitz domain Ω_S, and a cavity Ω_F completely filled with a homogeneous, inviscid and compressible fluid, see Figure 2.1.1. The fluid-structure interface is given by $\Sigma := \partial\Omega_F$ and the external boundary $\Gamma := \partial\Omega$ of the solid consists of a part $\Gamma_D \neq \emptyset$ where the structure is fixed and a part Γ_N on which it is free from tractions. We impose on Σ the orientation given by the unit normal vector $\boldsymbol{n}$ pointing outward to Ω_F. The outward unit normal vector to Γ is also denoted by $\boldsymbol{n}$, as shown in Figure 2.1.1. We assume that the fluid-structure system is subject to a volume load $\boldsymbol{f} : (0, T] \times \Omega_S \to \mathbb{R}^n$ acting on the solid. We can combine the constitutive law

$$\mathcal{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega_S, \tag{2.1.1}$$

and the equation of motion

$$\rho_S \ddot{\boldsymbol{u}} = \mathbf{div}\, \boldsymbol{\sigma} + \boldsymbol{f} \qquad \text{in } \Omega_S, \tag{2.1.2}$$

to eliminate either the displacement field $\boldsymbol{u}$ in the solid or the Cauchy stress tensor $\boldsymbol{\sigma}$ from the global formulation of the fluid-structure problem. Here, $\rho_S > 0$ is a constant representing the solid density, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\Big\{\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}$ is the linearized strain tensor, and $\mathcal{C} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is the Hooke operator, which is given in terms of the Lamé coefficients λ and μ by

$$\mathcal{C}\boldsymbol{\tau} := \lambda\, (\operatorname{tr} \boldsymbol{\tau})\, \boldsymbol{I} + 2\mu\boldsymbol{\tau} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{R}^{n\times n}.$$

In what follows we eliminate the displacement $\boldsymbol{u}$ and maintain the stress tensor $\boldsymbol{\sigma}$ as a main variable, which leads to the following dual mixed formulation in the solid,

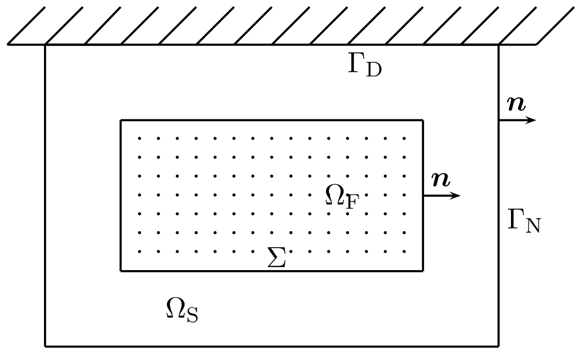

Figure 2.1.1: Fluid and solid domains

$$\mathcal{C}^{-1}\ddot{\boldsymbol{\sigma}} - {\rho_{\mathrm{S}}}^{-1}\boldsymbol{\varepsilon}\,(\mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f}) = \mathbf{0} \qquad \text{in } \Omega_{\mathrm{S}} \times (0,T], \tag{2.1.3}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}} \qquad \text{in } \Omega_{\mathrm{S}} \times (0,T], \tag{2.1.4}$$

$${\rho_{\mathrm{S}}}^{-1}(\mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f}) = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{D}} \times (0,T], \tag{2.1.5}$$

$$\boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{N}} \times (0,T] \tag{2.1.6}$$

$$\boldsymbol{\sigma}\boldsymbol{n} + p\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Sigma \times (0,T]. \tag{2.1.7}$$

We notice that the transmission condition (2.1.7) represents an equilibrium of forces on the contact boundary Σ where the fluid pressure p is acting here as a prescribed normal stress. The model problem is described in the fluid domain Ω_{F} in terms of the pressure,

$$c^{-2}\ddot{p} - \Delta p = 0 \qquad \text{in } \Omega_{\mathrm{F}} \times (0,T], \tag{2.1.8}$$

$$\frac{\partial p}{\partial \boldsymbol{n}} + \frac{\rho_{\mathrm{F}}}{\rho_{\mathrm{S}}}(\mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f}) \cdot \boldsymbol{n} = 0 \qquad \text{on } \Sigma \times (0,T]. \tag{2.1.9}$$

Here, $c > 0$ is the acoustic speed and ρ_{F} stands for the (constant) fluid density. Equation (2.1.9) corresponds to the so-called wall slipping condition, which expresses the matching of the normal components of the fluid and solid displacements on the transmission boundary Σ. Summing up, our model problem is given by the system (2.1.3)-(2.1.9) and the initial conditions

$$(\boldsymbol{\sigma}(0), p(0)) = (\boldsymbol{\sigma}_0, p_0) \quad \text{and} \quad (\dot{\boldsymbol{\sigma}}(0), \dot{p}(0)) = (\boldsymbol{\sigma}_1, p_1). \tag{2.1.10}$$

Now, we consider the orthogonal decomposition $[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n} = [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{sym}} \oplus [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}}$, where

$$[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{sym}} := \left\{ \boldsymbol{\tau} \in [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}; \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \right\}$$

and

$$[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}} := \left\{ \boldsymbol{\tau} \in [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}; \quad \boldsymbol{\tau} = -\boldsymbol{\tau}^{\mathtt{t}} \right\},$$

and introduce the closed subspaces of $\mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}})$ given by

$$\boldsymbol{\mathcal{W}} := \left\{ \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}}); \quad \boldsymbol{\tau}\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{N}} \right\}$$

and its symmetric counterpart

$$\boldsymbol{\mathcal{W}}^{\mathrm{sym}} := \boldsymbol{\mathcal{W}} \cap [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{sym}}\,.$$

On the other hand, since the equation (2.1.7) is an essential transmission condition that must be explicitly satisfied by the solution pair $(\boldsymbol{\sigma}, p)$, we need to consider the energy space

$$\mathbb{X} := \Big\{(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{W}} \times \mathrm{H}^1(\Omega_{\mathrm{F}}); \quad \boldsymbol{\tau}\boldsymbol{n} + q\boldsymbol{n} = 0 \qquad \text{on } \Sigma\Big\}, \tag{2.1.11}$$

which is a closed subspace of $\mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}}) \times \mathrm{H}^1(\Omega_{\mathrm{F}})$ when endowed with the Hilbertian norm

$$\|(\boldsymbol{\tau}, q)\|^2 := \|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})} + \|q\|^2_{1,\Omega_{\mathrm{F}}}\,.$$

We notice that the density of $\mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}}) \times \mathrm{H}^1(\Omega_{\mathrm{F}})$ in $\mathbb{H} := [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n} \times \mathrm{L}^2(\Omega_{\mathrm{F}})$ proves that the space $\mathbb{X}^{\mathrm{sym}} := \big\{(\boldsymbol{\tau}, q) \in \mathbb{X}; \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}}\big\}$ is also densely embedded in $\mathbb{H}^{\mathrm{sym}} := [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{sym}} \times \mathrm{L}^2(\Omega_{\mathrm{F}})$. We may then construct the dual $(\mathbb{X}^{\mathrm{sym}})'$ of $\mathbb{X}^{\mathrm{sym}}$ pivotal to $\mathbb{H}^{\mathrm{sym}}$, in such a way that the identification

$$\Big\langle (\boldsymbol{f}, g), (\boldsymbol{\tau}, q)\Big\rangle_{(\mathbb{X}^{\mathrm{sym}})',\mathbb{X}^{\mathrm{sym}}} = \Big((\boldsymbol{f}, g), (\boldsymbol{\tau}, q)\Big)_{\mathbb{H}} \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}^{\mathrm{sym}}$$

holds true for all $(\boldsymbol{f}, g) \in \mathbb{H}^{\mathrm{sym}} \hookrightarrow (\mathbb{X}^{\mathrm{sym}})'$. Here, $\langle\cdot,\cdot\rangle_{(\mathbb{X}^{\mathrm{sym}})',\mathbb{X}^{\mathrm{sym}}}$ represents the duality pairing between $(\mathbb{X}^{\mathrm{sym}})'$ and $\mathbb{X}^{\mathrm{sym}}$ and $(\cdot,\cdot)_{\mathbb{H}}$ is the natural inner product in $\mathbb{H}$ whose norm is given by

$$\|(\boldsymbol{\tau}, q)\|_0^2 := \|\boldsymbol{\tau}\|^2_{0,\Omega_{\mathrm{S}}} + \|q\|^2_{0,\Omega_{\mathrm{F}}}\,.$$

Next, given $\boldsymbol{f} \in \mathrm{L}^1((0, T); \mathrm{L}^2(\Omega_{\mathrm{S}})^n)$, $(\boldsymbol{\sigma}_0, p_0) \in \mathbb{X}^{\mathrm{sym}}$ and $(\boldsymbol{\sigma}_1, p_1) \in \mathbb{H}^{\mathrm{sym}}$, it is straightforward to show that the variational formulation of (2.1.3)-(2.1.10) is given by:

Find $(\boldsymbol{\sigma}, p) \in \mathrm{L}^\infty((0, T); \mathbb{X}^{\mathrm{sym}}) \cap \mathrm{W}^{1,\infty}((0, T); \mathbb{H}^{\mathrm{sym}})$ such that

$$\begin{aligned} &\Big((\ddot{\boldsymbol{\sigma}}, \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{\sigma}, p)(t), (\boldsymbol{\tau}, q)\Big) = -\rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}} \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}^{\mathrm{sym}} \\ &\big(\boldsymbol{\sigma}(0), p(0)\big) = (\boldsymbol{\sigma}_0, p_0), \qquad \big(\dot{\boldsymbol{\sigma}}(0), \dot{p}(0)\big) = (\boldsymbol{\sigma}_1, p_1), \end{aligned} \tag{2.1.12}$$

where

$$\Big((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} := (\mathcal{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + \frac{1}{\rho_{\mathrm{F}} c^2}(p, q)_{0,\Omega_{\mathrm{F}}} \tag{2.1.13}$$

and

$$A\Big((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)\Big) := \rho_{\mathrm{S}}^{-1}(\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + \rho_{\mathrm{F}}^{-1}(\nabla p, \nabla q)_{0,\Omega_{\mathrm{F}}}.$$

In the forthcoming analysis, we need to keep track of the parameter λ. For this reason, it is important to notice that

$$\begin{aligned} \|(\boldsymbol{\tau}, q)\|^2_{0,\mathcal{C}} := \Big((\boldsymbol{\tau}, q), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} &= \frac{1}{2\mu}\left\|\boldsymbol{\tau}^{\mathtt{D}}\right\|^2_{0,\Omega_{\mathrm{S}}} + \frac{1}{n(n\lambda + 2\mu)}\|\mathrm{tr}(\boldsymbol{\tau})\|^2_{0,\Omega_{\mathrm{S}}} + \frac{1}{\rho_{\mathrm{F}} c^2}\|q\|^2_{0,\Omega_{\mathrm{F}}} \\ &\le \max\left\{\frac{1}{2\mu}, \frac{1}{\rho_{\mathrm{F}} c^2}\right\}\|(\boldsymbol{\tau}, q)\|_0^2 \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{H}\,, \end{aligned} \tag{2.1.14}$$

where the deviatoric tensor $\boldsymbol{\tau}^{\mathtt{D}}$ has been defined at the beginning of Section 2. In addition, the following result proves that $\|(\boldsymbol{\tau}, q)\|^2_{0,\mathcal{C}} + A\Big((\boldsymbol{\tau}, q), (\boldsymbol{\tau}, q)\Big)$ is a Hilbertian norm on $\mathbb{X}$ that is equivalent to the $\mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}}) \times \mathrm{H}^1(\Omega_{\mathrm{F}})$-norm uniformly in λ.

Lemma 2.1.1. *There exists a constant $\alpha > 0$, independent of λ, such that*

$$\alpha\,\|(\boldsymbol{\tau},q)\|^2 \le \|(\boldsymbol{\tau},q)\|_{0,\mathcal{C}}^2 + A\Big((\boldsymbol{\tau},q),(\boldsymbol{\tau},q)\Big) \le C\,\|(\boldsymbol{\tau},q)\|^2 \qquad \forall (\boldsymbol{\tau},q)\in\mathbb{X}. \tag{2.1.15}$$

with $C = \max\left\{\frac{1}{2\mu},\frac{1}{\rho_{\rm S}},\frac{1}{\rho_{\rm F}},\frac{1}{\rho_{\rm F}c^2}\right\}$.

Proof. See [35, Lemma 2.1]. ◻

Theorem 2.1.1. *Assume that $\boldsymbol{f} \in \mathrm{W}^{1,1}((0,T);\mathrm{L}^2(\Omega_{\rm S})^n)$. Then, problem (2.1.12) admits a unique solution $(\boldsymbol{\sigma},p)\in\mathcal{C}^0((0,T);\mathbb{X}^{sym})\cap\mathcal{C}^1((0,T);\mathbb{H}^{sym})$. Moreover, there exists a constant $C>0$, independent of λ and T, such that*

$$\begin{aligned}&\operatorname*{ess\,sup}_{[0,T]}\|(\boldsymbol{\sigma},p)(t)\| + \operatorname*{ess\,sup}_{[0,T]}\|(\dot{\boldsymbol{\sigma}},\dot{p})(t)\|_{0,\mathcal{C}}\\ &\quad\le CT\left\{\|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega_{\rm S}))} + \|(\boldsymbol{\sigma}_0,p_0)\| + \|(\boldsymbol{\sigma}_1,p_1)\|_0\right\}.\end{aligned} \tag{2.1.16}$$

Proof. We only provide the main ideas of the proof, which makes use of the classical Galerkin procedure (cf. [17, 37]). More precisely, following the same steps adopted in [21, Lemma 3.2], we first consider a family of finite dimensional subspaces $\{\mathbb{X}_n^{\rm sym}\}_{n\in\mathbb{N}}$ of $\mathbb{X}^{\rm sym}$ such that

$$\lim_{n\to\infty}\ \inf_{(\boldsymbol{\tau}_n,q_n)\in\mathbb{X}_n^{\rm sym}}\|(\boldsymbol{\tau},q)-(\boldsymbol{\tau}_n,q_n)\| = 0 \qquad \forall(\boldsymbol{\tau},q)\in\mathbb{X}^{\rm sym}.$$

Next, we denote by $(\boldsymbol{\sigma}_{0,n},p_{0,n})$ the $(\mathbb{X}^{\rm sym},\|\cdot\|)$-orthogonal projection of $(\boldsymbol{\sigma}_0,p_0)$ onto $\mathbb{X}_n^{\rm sym}$ and by $(\boldsymbol{\sigma}_{1,n},p_{1,n})$ the $(\mathbb{H}^{\rm sym},\|\cdot\|_0)$-orthogonal projection of $(\boldsymbol{\sigma}_1,p_1)$ onto $\mathbb{X}_n^{\rm sym}$. Then, it is easy to show, by using the classical ODE theory, that the problem:

Find $(\boldsymbol{\sigma}_n,p_n)\in\mathcal{C}^1(\mathbb{X}^{\rm sym})$ such that

$$\begin{aligned}&\Big((\ddot{\boldsymbol{\sigma}}_n,\ddot{p}_n)(t),(\boldsymbol{\tau},q)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{\sigma}_n,p_n)(t),(\boldsymbol{\tau},q)\Big) = -\rho_{\rm S}^{-1}\big(\boldsymbol{f}(t),\mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\rm S}} \quad \forall(\boldsymbol{\tau},q)\in\mathbb{X}^{\rm sym},\\ &\big(\boldsymbol{\sigma}_n(0),p_n(0)\big) = (\boldsymbol{\sigma}_{0,n},p_{0,n}), \qquad \big(\dot{\boldsymbol{\sigma}}_n(0),\dot{p}_n(0)\big) = (\boldsymbol{\sigma}_{1,n},p_{1,n}),\end{aligned} \tag{2.1.17}$$

admits a unique solution. Furthermore, since $(\cdot,\cdot)_{\mathcal{C}}$ and $A(\cdot,\cdot)$ are symmetric bilinear forms, taking formally $(\boldsymbol{\tau},q) = (\dot{\boldsymbol{\sigma}}_n(t),\dot{p}_n(t))$ in (2.1.17) gives

$$\dot{\mathcal{E}}\big((\boldsymbol{\sigma}_n,p_n)\big)(t) \;=\; -\,\rho_{\rm S}^{-1}\big(\boldsymbol{f}(t),\mathbf{div}\,\dot{\boldsymbol{\sigma}}_n(t)\big)_{0,\Omega_{\rm S}}, \tag{2.1.18}$$

where the energy functional $\mathcal{E}$ is defined by

$$\mathcal{E}\big((\boldsymbol{\tau},q)\big)(t) := \frac{1}{2}\Big((\dot{\boldsymbol{\tau}},\dot{q})(t),(\dot{\boldsymbol{\tau}},\dot{q})(t)\Big)_{\mathcal{C}} + \frac{1}{2}A\Big((\boldsymbol{\tau},q)(t),(\boldsymbol{\tau},q)(t)\Big) \quad \forall\,(\boldsymbol{\tau},q)\in W^{1,\infty}(\mathbb{X}). \tag{2.1.19}$$

In this way, integrating (2.1.18) on $[0,t]$ and using the time regularity assumption on $\boldsymbol{f}$ to perform an integration by parts, we find that

$$\begin{aligned}\mathcal{E}\big((\boldsymbol{\sigma}_n,p_n)\big)(t) \;=\; &\mathcal{E}\big((\boldsymbol{\sigma}_n,p_n)\big)(0) + \int_0^t\big(\dot{\boldsymbol{f}}(s),\mathbf{div}\,\boldsymbol{\sigma}_n(s)\big)_{0,\Omega_{\rm S}}\,\mathrm{d}s\\ &-\big(\boldsymbol{f}(t),\mathbf{div}\,\boldsymbol{\sigma}_n(t)\big)_{0,\Omega_{\rm S}} + \big(\boldsymbol{f}(0),\mathbf{div}\,\boldsymbol{\sigma}_{0,n}\big)_{0,\Omega_{\rm S}},\end{aligned}$$

from which, employing the Cauchy-Schwarz inequality, the Sobolev embedding $\mathrm{W}^{1,1}((0,T);\mathrm{L}^2(\Omega)^n) \hookrightarrow \mathcal{C}^0((0,T);\mathrm{L}^2(\Omega)^n)$ (see [39, Lemma 7.1]), and the continuous dependence result for (2.1.17), we deduce that

$$\operatorname*{ess\,sup}_{[0,T]} \mathcal{E}\big((\boldsymbol{\sigma}_n, p_n)\big)^{1/2}(t) \;\leq\; C_1 \Big\{ \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega_{\mathrm{S}}))} + \|(\boldsymbol{\sigma}_{0,n}, p_{0,n})\| + \|(\boldsymbol{\sigma}_{1,n}, p_{1,n})\|_0 \Big\}. \tag{2.1.20}$$

with constant $C_1 > 0$, independent of λ and T. It follows now easily from the last estimate, (2.1.19) and fundamental theorem of calculus that

$$\begin{aligned} \operatorname*{ess\,sup}_{[0,T]} \|(\dot{\boldsymbol{\sigma}}_n, \dot{p}_n)(t)\|_{0,\mathcal{C}} &+ \operatorname*{ess\,sup}_{[0,T]} \Big(\|(\boldsymbol{\sigma}_n, p_n)(t)\|_{0,\mathcal{C}}^2 + A\Big((\boldsymbol{\sigma}_n, p_n)(t), (\boldsymbol{\sigma}_n, p_n)(t)\Big)\Big)^{1/2} \\ &\leq C_2\, T \Big\{ \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega_{\mathrm{S}}))} + \|(\boldsymbol{\sigma}_{0,n}, p_{0,n})\| + \|(\boldsymbol{\sigma}_{1,n}, p_{1,n})\|_0 \Big\}. \end{aligned} \tag{2.1.21}$$

Finally, using (2.1.15) and the fact that $\|(\boldsymbol{\sigma}_{0,n}, p_{0,n})\|$ and $\|(\boldsymbol{\sigma}_{1,n}, p_{1,n})\|_0$ are bounded by $\|(\boldsymbol{\sigma}_0, p_0)\|$ and $\|(\boldsymbol{\sigma}_1, p_1)\|_0$, respectively, yield

$$\begin{aligned} \operatorname*{ess\,sup}_{[0,T]} \|(\boldsymbol{\sigma}_n, p_n)(t)\| &+ \operatorname*{ess\,sup}_{[0,T]} \|(\dot{\boldsymbol{\sigma}}_n, \dot{p}_n)(t)\|_{0,\mathcal{C}} \\ &\leq C_3\, T \Big\{ \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega_{\mathrm{S}}))} + \|(\boldsymbol{\sigma}_0, p_0)\| + \|(\boldsymbol{\sigma}_1, p_1)\|_0 \Big\}. \end{aligned} \tag{2.1.22}$$

where the constants C_2 and C_3 are independent of λ and T. It is clear from (2.1.22) that $(\dot{\boldsymbol{\sigma}}_n, \dot{p}_n)_n$ and $(\boldsymbol{\sigma}_n, p_n)_n$ are uniformly bounded in the spaces $\mathrm{L}^\infty((0,T);\mathbb{H}^{\mathrm{sym}})$ and $\mathrm{L}^\infty((0,T);\mathbb{X}^{\mathrm{sym}})$, respectively, and hence, a classical procedure (cf. [21, Lemma 3.2]) shows that the sequence $\Big\{(\boldsymbol{\sigma}_n, p_n)\Big\}_{n\in\mathbb{N}}$ converges to a solution $(\boldsymbol{\sigma}, p) \in \mathrm{L}^\infty((0,T);\mathbb{X}^{\mathrm{sym}}) \cap \mathrm{W}^{1,\infty}((0,T);\mathbb{H}^{\mathrm{sym}})$ of (2.1.12). Then, taking the limit in (2.1.22) we arrive at the required estimate (2.1.16), whereas the uniqueness of solution follows from a standard procedure (cf. [17, 37] or [21, Lemma 3.3]). Finally, we remark that, following [37, Section 11.2.4], it can be shown that the solution $(\boldsymbol{\sigma}, p)$ to problem (2.1.12) is actually in $\mathcal{C}^0((0,T);\mathbb{X}^{\mathrm{sym}}) \cap \mathcal{C}^1((0,T);\mathbb{H}^{\mathrm{sym}})$. $\square$

At this point we find it important to notice that the kernel of the seminorm $A\Big((\boldsymbol{\tau}, q), (\boldsymbol{\tau}, q)\Big)^{1/2}$ is given by

$$\mathbb{K} := \Big\{ (\boldsymbol{\tau}, \xi) \in \mathbb{X}_c^{\mathrm{sym}}; \quad \mathbf{div}\, \boldsymbol{\tau} = 0 \Big\},$$

where $\mathbb{X}_c^{\mathrm{sym}} := \mathbb{X}_c \cap \mathbb{X}^{\mathrm{sym}}$ with

$$\mathbb{X}_c := \{ (\boldsymbol{\tau}, \xi) \in \mathbb{X}; \quad \xi = \text{constant} \}. \tag{2.1.23}$$

Finally, the orthogonal of $\mathbb{K}$ in $\mathbb{X}^{\mathrm{sym}}$ with respect to the inner product $(\cdot,\cdot)_{\mathcal{C}}$ is denoted

$$\mathbb{K}^\perp := \Big\{ (\boldsymbol{\sigma}, p) \in \mathbb{X}^{\mathrm{sym}}; \quad \Big((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, \xi)\Big)_{\mathcal{C}} = 0 \quad \forall (\boldsymbol{\tau}, \xi) \in \mathbb{K} \Big\}.$$

On the other hand, the existence of a constant $\beta_0 > 0$ such that

$$\sup_{\substack{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}} \\ \boldsymbol{\tau n} = 0 \text{ on } \Sigma}} \frac{(\boldsymbol{s}, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{v}, \mathbf{div}\, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})}} \;\geq\; \beta_0 \Big\{ \|\boldsymbol{v}\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}} \Big\} \tag{2.1.24}$$

for all $(\boldsymbol{v}, \boldsymbol{s}) \in \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}}$, constitutes a crucial inf-sup condition in the analysis of the mixed formulation of the elastostatic problem with reduced symmetry (cf. [4, 9]). Indeed, as we show next, it plays an essential role in the recovery of the displacement field $\boldsymbol{u}$ from $(\boldsymbol{\sigma}, p)$. To this end, we now define a suitable operator.

Definition 2.1.1. *We introduce the linear operator* $\mathbf{D} : \mathbb{K}^\perp \to \mathrm{L}^2(\Omega_\mathrm{S})^n \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{skew}$ *defined, for any* $(\boldsymbol{\sigma}, p) \in \mathbb{K}^\perp$, *by the unique solution* $(\boldsymbol{u}, \boldsymbol{r}) := \mathbf{D}(\boldsymbol{\sigma}, p) \in \mathrm{L}^2(\Omega_\mathrm{S})^n \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{skew}$ *of*

$$(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u})_{0,\Omega_\mathrm{S}} + (\boldsymbol{r}, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} = -\Big((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, \xi)\Big)_{\mathcal{C}} \quad \forall (\boldsymbol{\tau}, \xi) \in \mathbb{X}_c. \tag{2.1.25}$$

It is important to clarify here that, due to the definitions of $\mathbb{X}$ and $\mathbb{X}_c^{\mathrm{sym}}$ (cf. (2.1.11) and (2.1.23)), the second component $\xi \in \mathbb{R}$ on the right hand side of (2.1.25) is fixed by $\boldsymbol{\tau}$ according to the identity $\boldsymbol{\tau n} = -\,\xi\,\boldsymbol{n}$ on Σ. Moreover, the operator $\mathbf{D}$ is well-defined by virtue of Theorem 0.0.1 and (2.1.24). In fact, the functional on the right hand side of (2.1.25), that is $\mathbb{X}_c \ni (\boldsymbol{\tau}, \xi) \mapsto -\Big((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, \xi)\Big)_{\mathcal{C}}$, belongs to the polar of $\mathbb{K}$ in $(\mathbb{X}_c)'$, and the inf-sup condition

$$\sup_{(\boldsymbol{\tau},\xi)\in\mathbb{X}_c} \frac{(\boldsymbol{s}, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{v})_{0,\Omega_\mathrm{S}}}{\|(\boldsymbol{\tau}, \xi)\|} \geq \beta_0 \Big\{ \|\boldsymbol{v}\|_{0,\Omega_\mathrm{S}} + \|\boldsymbol{s}\|_{0,\Omega_\mathrm{S}} \Big\} \tag{2.1.26}$$

for all $(\boldsymbol{v}, \boldsymbol{s}) \in \mathrm{L}^2(\Omega_\mathrm{S})^n \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}}$, is a direct consequence of (2.1.24). Further properties concerning the range of $\mathbf{D}$ in $\mathrm{L}^2(\Omega_\mathrm{S})^n \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}}$ are provided by the following Lemma.

Lemma 2.1.2. *Given* $(\boldsymbol{\sigma}, p) \in \mathbb{X}^{sym}$, *the following two statements are equivalent:*

i) $(\boldsymbol{\sigma}, p) \in \mathbb{K}^\perp$

ii) *There exists a unique* $\boldsymbol{u} \in \mathrm{H}^1(\Omega_\mathrm{S})^n$ *with* $\boldsymbol{u}|_{\Gamma_\mathrm{D}} = \mathbf{0}$, *such that* $\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$,

$$\int_\Sigma \boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{\rho_\mathrm{F} c^2}\int_{\Omega_\mathrm{F}} p = 0, \tag{2.1.27}$$

and $\mathbf{D}(\boldsymbol{\sigma}, p) = (\boldsymbol{u}, \boldsymbol{r})$, *where* $\boldsymbol{r} = \Big\{\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}/2$.

Proof. Given $(\boldsymbol{\sigma}, p) \in \mathbb{K}^\perp$, we first let $(\boldsymbol{u}, \boldsymbol{r}) := \mathbf{D}(\boldsymbol{\sigma}, p)$ according to (2.1.25). Then, for each $\boldsymbol{\tau} \in [\mathcal{D}(\Omega_\mathrm{S})]^{n\times n}$ we have that $(\boldsymbol{\tau}, 0) \in \mathbb{X}_c$, which replaced into (2.1.25) yields $\nabla\boldsymbol{u} = \mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r} \in \mathrm{L}^2(\Omega_\mathrm{S})^{n\times n}$. From this identity and the fact that $\mathcal{C}^{-1}\boldsymbol{\sigma}$ is symmetric (because $\boldsymbol{\sigma}$ is), we readily deduce that there hold $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathcal{C}^{-1}\boldsymbol{\sigma}$ and $\boldsymbol{r} = \Big\{\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}/2$. In turn, testing now (2.1.25) with $(\boldsymbol{\tau}, 0) \in \mathbb{X}_c$ and $(\boldsymbol{\tau}, 1) \in \mathbb{X}_c^{\mathrm{sym}}$, and integrating by parts in both cases, we obtain the boundary condition $\boldsymbol{u}|_{\Gamma_\mathrm{D}} = \mathbf{0}$ and (2.1.27), respectively. Conversely, given $(\boldsymbol{\sigma}, p) \in \mathbb{X}^{\mathrm{sym}}$ such that ii) holds true, we set the tensor $\boldsymbol{r} = \Big\{\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}/2 \in [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}}$ and observe that $\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r} - \nabla\boldsymbol{u} = \mathbf{0}$. Hence, given $(\boldsymbol{\tau}, \xi) \in \mathbb{X}_c$, we test the foregoing equation with $\boldsymbol{\tau}$, integrate by parts in Ω_S, and use (2.1.27), to find

$$(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u})_{0,\Omega_\mathrm{S}} + (\boldsymbol{r}, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} = -(\mathcal{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} - \frac{1}{\rho_\mathrm{F} c^2}(p, \xi)_{0,\Omega_\mathrm{F}}\,. \tag{2.1.28}$$

Finally, since the left hand side of (2.1.28) vanishes for $(\boldsymbol{\tau}, \xi) \in \mathbb{K}$, we conclude from there that $(\boldsymbol{\sigma}, p) \in \mathbb{K}^\perp$ and $(\boldsymbol{u}, \boldsymbol{r}) = \mathbf{D}(\boldsymbol{\sigma}, p)$. □

The following result establishes the relation between the solution $(\boldsymbol{\sigma}, p)$ of (2.1.12) and the solution of the displacement-pressure formulation of the fluid-structure interaction problem.

Theorem 2.1.2. *Assume that the initial data of problem* (2.1.12) *are such that* $(\boldsymbol{\sigma}_0, p_0),\ (\boldsymbol{\sigma}_1, p_1) \in \mathbb{K}^\perp$, *and let* $(\boldsymbol{u}_0, \boldsymbol{r}_0) := \mathbf{D}(\boldsymbol{\sigma}_0, p_0)$ *and* $(\boldsymbol{u}_1, \boldsymbol{r}_1) := \mathbf{D}(\boldsymbol{\sigma}_1, p_1)$. *If* $(\boldsymbol{\sigma}, p)$ *is the solution of* (2.1.12) *then the pair* $(\boldsymbol{u}, p)$, *with*

$$\boldsymbol{u}(t) := \int_0^t \Big\{ \int_0^s \rho^{-1}\Big(\mathbf{div}\,\boldsymbol{\sigma}(z) + \boldsymbol{f}(z)\Big)\, dz \Big\}\, ds + \boldsymbol{u}_0 + t\boldsymbol{u}_1, \tag{2.1.29}$$

solves the displacement-pressure formulation of the fluid-structure interaction problem,

$$\rho_{\mathrm{S}}\ddot{\boldsymbol{u}} - \mathbf{div}\,\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{f} \qquad \text{in } \Omega_{\mathrm{S}} \times (0,T] \tag{2.1.30}$$

$$c^{-2}\ddot{p} - \Delta p = 0 \qquad \text{in } \Omega_{\mathrm{F}} \times (0,T] \tag{2.1.31}$$

$$\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n} + p\boldsymbol{n} = \mathbf{0} \qquad \text{in } \Sigma \times (0,T] \tag{2.1.32}$$

$$\frac{\partial p}{\partial \boldsymbol{n}} + \rho_{\mathrm{F}}\ddot{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \qquad \text{in } \Sigma \times (0,T] \tag{2.1.33}$$

$$\boldsymbol{u} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{D}} \times (0,T], \tag{2.1.34}$$

$$\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{N}} \times (0,T] \tag{2.1.35}$$

subject to the initial conditions $(\boldsymbol{u}(0), p(0)) = (\boldsymbol{u}_0, p_0)$ *and* $(\dot{\boldsymbol{u}}(0), \dot{p}(0)) = (\boldsymbol{u}_1, p_1)$.

Proof. Integrating the first equation of (2.1.12) twice with respect to time we deduce that

$$\begin{aligned}\Big((\boldsymbol{\sigma}(t), p(t)), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} &= \Big((\boldsymbol{\sigma}_0, p_0), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + t\Big((\boldsymbol{\sigma}_1, p_1), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} \\ &- \int_0^t \left(\int_0^s A\Big((\boldsymbol{\sigma}, p)(z), (\boldsymbol{\tau}, q)\Big) + \rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(z), \mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}}\, \mathrm{d}z \right) \mathrm{d}s\,,\end{aligned} \tag{2.1.36}$$

for all $(\boldsymbol{\tau}, q) \in \mathbb{X}^{\mathrm{sym}}$. It follows that $(\boldsymbol{\sigma}(t), p(t)) \in \mathbb{K}^\perp$ for all $t \in [0,T]$, and hence Lemma 2.1.2 ensures the existence of a unique pair $(\boldsymbol{u}(t), \boldsymbol{r}(t)) = \mathbf{D}(\boldsymbol{\sigma}(t), p(t)) \in \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}}$ satisfying $\boldsymbol{u}(t) \in \mathrm{H}^1(\Omega_{\mathrm{S}})^n$ $\boldsymbol{\sigma}(t) = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})(t)$, $\boldsymbol{u}|_{\Gamma_{\mathrm{D}}} = \mathbf{0}$ and

$$(\boldsymbol{r}(t), \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}(t))_{0,\Omega_{\mathrm{S}}} = -\Big((\boldsymbol{\sigma}(t), p(t)), (\boldsymbol{\tau}, \xi)\Big)_{\mathcal{C}} \quad \forall (\boldsymbol{\tau}, \xi) \in \mathbb{X}_c. \tag{2.1.37}$$

On the other hand, we readily obtain from (2.1.24) the inf-sup condition

$$\sup_{(\boldsymbol{\tau},q)\in\mathbb{X}} \frac{(\boldsymbol{s}, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}}}{\|(\boldsymbol{\tau}, q)\|} \geq \beta_0 \|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}} \quad \forall \boldsymbol{s} \in [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}}.$$

In this way, applying Theorem 0.0.1, we deduce from (2.1.36) the existence of a unique $\bar{\boldsymbol{r}}(t) \in [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}}$ satisfying

$$\begin{aligned}(\bar{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} &= -\Big((\boldsymbol{\sigma}(t), p(t)), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + \Big((\boldsymbol{\sigma}_0, p_0), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + t\Big((\boldsymbol{\sigma}_1, p_1), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} \\ &- \int_0^t \left(\int_0^s A\Big((\boldsymbol{\sigma}, p)(z), (\boldsymbol{\tau}, q)\Big) + \rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(z), \mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}}\, \mathrm{d}z \right) \mathrm{d}s\,,\end{aligned} \tag{2.1.38}$$

for all $(\boldsymbol{\tau}, q) \in \mathbb{X}$. Then, replacing (2.1.37) in (2.1.38), yields

$$\begin{aligned}(\bar{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} &= (\boldsymbol{r}(t), \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}(t))_{0,\Omega_{\mathrm{S}}} + \Big((\boldsymbol{\sigma}_0, p_0), (\boldsymbol{\tau}, \xi)\Big)_{\mathcal{C}} \\ &+ t\Big((\boldsymbol{\sigma}_1, p_1), (\boldsymbol{\tau}, \xi)\Big)_{\mathcal{C}} - \int_0^t \left(\int_0^s A\Big((\boldsymbol{\sigma}, p)(z), (\boldsymbol{\tau}, \xi)\Big) + \rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(z), \mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}}\, \mathrm{d}z \right) \mathrm{d}s \\ &= \big(\boldsymbol{r}(t) - \boldsymbol{r}_0 - t\boldsymbol{r}_1, \boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}} + \big(\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u}(t) - \boldsymbol{u}_0 - t\boldsymbol{u}_1\big)_{0,\Omega_{\mathrm{S}}} \\ &- \int_0^t \left(\int_0^s \big(\rho_{\mathrm{S}}^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}(z) + \rho_{\mathrm{S}}^{-1}\boldsymbol{f}(z), \mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}}\, \mathrm{d}z \right) \mathrm{d}s\end{aligned} \tag{2.1.39}$$

for all $(\boldsymbol{\tau}, \xi) \in \mathbb{X}_c$, from which it follows that

$$\Big(\boldsymbol{u}(t) - \int_0^t \Big\{ \int_0^s \rho^{-1}\Big(\mathbf{div}\,\boldsymbol{\sigma}(z) + \boldsymbol{f}(z)\Big)\,\mathrm{d}z\Big\}\,\mathrm{d}s \,-\, \boldsymbol{u}_0 \,-\, t\boldsymbol{u}_1,\, \mathbf{div}\,\boldsymbol{\tau}\Big)_{0,\Omega_\mathrm{S}} \\ + \quad \big(\boldsymbol{r}(t) - \boldsymbol{r}_0 - t\boldsymbol{r}_1 - \bar{\boldsymbol{r}}(t),\, \boldsymbol{\tau}\big)_{0,\Omega_\mathrm{S}} = 0 \qquad \forall \boldsymbol{\tau} \in \mathbb{X}_c\,.$$

Thus, the foregoing equation and the inf-sup condition (2.1.26) imply (2.1.29) and

$$\bar{\boldsymbol{r}}(t) = \boldsymbol{r}(t) - \boldsymbol{r}_0 \,-\, t\,\boldsymbol{r}_1 \qquad \forall\, t \in [0,T]\,. \tag{2.1.40}$$

Finally, differentiating (2.1.29) twice with respect to time we obtain the motion equation (2.1.30), whereas substituting (2.1.30) back into (2.1.9) yields (2.1.33), which completes the proof. □

We end this section remarking that, after differentiating (2.1.38) twice with respect to time and using (2.1.40), we find that

$$\Big((\ddot{\boldsymbol{\sigma}}, \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + \big(\ddot{\boldsymbol{r}}(t), \boldsymbol{\tau}\big)_{0,\Omega_\mathrm{S}} + A\Big((\boldsymbol{\sigma}, p)(t), (\boldsymbol{\tau}, q)\Big) = -\rho_\mathrm{S}^{-1}\big(\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_\mathrm{S}} \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}\,. \tag{2.1.41}$$

This identity is employed later on in Section 2.4.

2.2 The discrete problem

We consider shape regular affine meshes $\mathcal{T}_h$ that subdivide the domain $\bar{\Omega} = \bar{\Omega}_\mathrm{S} \cup \bar{\Omega}_\mathrm{F}$, into triangles/tetrahedra K of diameter h_K. The parameter $h := \max_{K\in\mathcal{T}_h}\{h_K\}$ represents the mesh size of $\mathcal{T}_h$. In what follows, we assume that each triange/tetrahedron of $\mathcal{T}_h$ is contained either in $\bar{\Omega}_\mathrm{S}$ or in $\bar{\Omega}_\mathrm{F}$, and denote

$$\mathcal{T}_h^\mathrm{S} := \big\{K \in \mathcal{T}_h;\quad K \subset \bar{\Omega}_\mathrm{S}\big\} \quad \text{and} \quad \mathcal{T}_h^\mathrm{F} := \big\{K \in \mathcal{T}_h;\quad K \subset \bar{\Omega}_\mathrm{F}\big\}\,.$$

Moreover, we let Σ_h be the triangulation induced by $\mathcal{T}_h$ on Σ. Next, given an integer $m \geq 0$ and a domain $D \subset \mathbb{R}^d$, $\mathcal{P}_m(D)$ denotes the space of polynomials of degree at most m on D. The space of piecewise polynomial functions of degree at most m associated with $\mathcal{T}_h^*$, $* \in \{\mathrm{S}, \mathrm{F}\}$, is denoted by

$$\mathcal{P}_m(\mathcal{T}_h^*) := \big\{v \in L^2(\Omega_*);\quad v|_K \in \mathcal{P}_m(K), \quad \forall K \in \mathcal{T}_h^*\big\}\,.$$

Similarly, $\mathcal{P}_m(\Sigma_h) := \big\{\phi \in L^2(\Sigma);\quad \phi|_T \in \mathcal{P}_m(T), \quad \forall T \in \Sigma_h\big\}$. In addition, for $k \geq 1$, the finite element spaces

$$\boldsymbol{\mathcal{W}}_h := \mathcal{P}_k(\mathcal{T}_h^\mathrm{S})^{n\times n} \cap \boldsymbol{\mathcal{W}}\,, \quad \boldsymbol{\mathcal{Q}}_h := \mathcal{P}_{k-1}(\mathcal{T}_h^\mathrm{S})^{n\times n} \cap [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}}\,, \quad \text{and} \quad \mathcal{U}_h := \mathcal{P}_{k-1}(\mathcal{T}_h^\mathrm{S})^n\,,$$

correspond to the k^{th}-order element of the Arnold-Falk-Winther (AFW) family introduced for the mixed formulation of elastostatic problem with reduced symmetry. It is shown in [3, Theorem 11.9] that the discrete inf-sup condition

$$\sup_{\substack{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h \\ \boldsymbol{\tau n}\,=\,0 \text{ on } \Sigma}} \frac{(\boldsymbol{\tau}, \boldsymbol{s})_{0,\Omega_\mathrm{S}} + (\mathbf{div}\,\tau, \boldsymbol{v})_{0,\Omega_\mathrm{S}}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega)}} \geq \beta_0^*\Big\{\, \|\boldsymbol{s}\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega}\,\Big\}, \quad \forall (\boldsymbol{s}, \boldsymbol{v}) \in \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \tag{2.2.1}$$

holds true for a constant $\beta_0^* > 0$ independent of h. It is important to notice that the weakly symmetric version

$$\boldsymbol{\mathcal{W}}_h^{\text{sym}} = \left\{ \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h; \quad \int_{\Omega_{\mathrm{S}}} \boldsymbol{\tau}_h : \boldsymbol{s} = 0 \quad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h \right\}$$

of $\boldsymbol{\mathcal{W}}_h$ is not a subspace of $\boldsymbol{\mathcal{W}}^{\text{sym}}$. Moreover, it is generally not possible to construct a basis for the finite element space $\boldsymbol{\mathcal{W}}_h^{\text{sym}}$. Hence, in all what follows, we implicitly assume that a Lagrange multiplier is needed in order to deal, from the practical point of view, with the weak symmetry constraint defining $\boldsymbol{\mathcal{W}}_h^{\text{sym}}$. We deliberately have chosen here to hide this additional variable (which is none other than the discrete counterpart of the rotation $\boldsymbol{r}$) for economy in notations.

We approximate the pressure in the usual Lagrange finite element space $V_h := \mathcal{P}_k(\mathcal{T}_h^{\mathrm{F}}) \cap \mathrm{H}^1(\Omega_{\mathrm{F}})$. We recall some well-known approximation properties of the finite element spaces introduced above. Given $s > 0$, it is well-known that the usual k^{th}-order Brezzi-Douglas-Marini (BDM) interpolation operator (see [12]) $\boldsymbol{\Pi}_h : [\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n} \cap \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h$ satisfies for $0 < s \leq 1/2$ the error estimate

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h\boldsymbol{\tau}\|_{0,\Omega_{\mathrm{S}}} \leq Ch^s \Big\{ \|\boldsymbol{\tau}\|_{s,\Omega_{\mathrm{S}}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega_{\mathrm{S}}} \Big\} \qquad \forall \boldsymbol{\tau} \in [\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n} \cap \boldsymbol{\mathcal{W}}. \tag{2.2.2}$$

For more regular functions $\boldsymbol{\tau} \in [\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n}$ with $s > 1/2$, it holds

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h\boldsymbol{\tau}\|_{0,\Omega_{\mathrm{S}}} \leq Ch^{\min\{s,k+1\}} \|\boldsymbol{\tau}\|_{s,\Omega_{\mathrm{S}}}, \qquad \forall \boldsymbol{\tau} \in [\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n}. \tag{2.2.3}$$

Moreover, we have the commuting diagram properties

$$\mathbf{div}(\boldsymbol{\Pi}_h\boldsymbol{\tau}) = U_h(\mathbf{div}\,\boldsymbol{\tau}) \quad \text{and} \quad (\boldsymbol{\Pi}_h\boldsymbol{\sigma})\boldsymbol{n} = \boldsymbol{\pi}_h(\boldsymbol{\sigma}\boldsymbol{n}) \tag{2.2.4}$$

for all $\boldsymbol{\tau} \in \mathrm{H}^s(\Omega_{\mathrm{S}})^{n\times n} \cap \mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}})$, $s > 0$, where $U_h :\ \mathrm{L}^2(\Omega_{\mathrm{S}})^n \to \mathcal{U}_h$ is the $\mathrm{L}^2(\Omega_{\mathrm{S}})^n$-orthogonal projector, π_h is the $\mathrm{L}^2(\Sigma)$-orthogonal projector onto $\mathcal{P}_k(\Sigma_h)$, and $\boldsymbol{\pi}_h$ is the vectorial version of π_h. In addition, we denote by $\boldsymbol{R}_h :\ [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\text{skew}} \to \boldsymbol{\mathcal{Q}}_h$ the orthogonal projector with respect to the $[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}$-norm, and let $\Pi_h : \mathrm{H}^1(\Omega_{\mathrm{F}}) \to V_h$ be the operator that, given $p \in \mathrm{H}^1(\Omega_{\mathrm{F}})$, is uniquely characterized by

$$(\nabla\Pi_h p, \nabla q)_{0,\Omega_{\mathrm{F}}} = (\nabla p, \nabla q)_{0,\Omega_{\mathrm{F}}} \quad \forall q \in V_h \quad \text{and} \quad \int_{\Omega_{\mathrm{F}}} \Pi_h p = 0\,. \tag{2.2.5}$$

Then, there hold

$$\|\boldsymbol{r} - \boldsymbol{R}_h\boldsymbol{r}\|_{0,\Omega_{\mathrm{S}}} \leq Ch^{\min\{s,k\}} \|\boldsymbol{r}\|_{s,\Omega_{\mathrm{S}}} \qquad \forall \boldsymbol{r} \in [\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n} \cap [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\text{skew}}, \tag{2.2.6}$$

$$\|\boldsymbol{v} - U_h\boldsymbol{v}\|_{0,\Omega_{\mathrm{S}}} \leq Ch^{\min\{s,k\}} \|\boldsymbol{v}\|_{s,\Omega_{\mathrm{S}}} \qquad \forall \boldsymbol{v} \in \mathrm{H}^s(\Omega_{\mathrm{S}})^n, \tag{2.2.7}$$

$$|p - \Pi_h p|_{1,\Omega_{\mathrm{F}}} \leq Ch^{\min\{s,k\}} \|p\|_{1+s,\Omega_{\mathrm{F}}} \qquad \forall p \in \mathrm{H}^{1+s}(\Omega_{\mathrm{F}}), \tag{2.2.8}$$

$$\|\boldsymbol{\varphi} - \pi_h\boldsymbol{\varphi}\|_{0,\Sigma} \leq Ch^{\min\{s,k+1\}} \Big(\sum_{e\in\Sigma_h} \|\boldsymbol{\varphi}\|^2_{s,e}\Big)^{1/2} \qquad \forall \boldsymbol{\varphi} \in \prod_{e\in\Sigma_h} \mathrm{H}^s(e)^n\,. \tag{2.2.9}$$

Furthermore, we introduce the discrete energy space

$$\mathbb{X}_h := \{(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{W}}_h \times V_h : \quad \boldsymbol{\tau}\boldsymbol{n} + p\boldsymbol{n} = 0 \quad \text{on} \quad \Sigma\}\,, \tag{2.2.10}$$

and its subspace $\mathbb{X}_{h,c} = \{(\boldsymbol{\tau}, \xi) \in \mathbb{X}_h; \quad \xi = \text{constant}\}$. We also consider their weakly symmetric versions

$$\mathbb{X}_h^{\text{sym}} := \big\{(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{W}}_h^{\text{sym}} \times V_h; \quad \boldsymbol{\tau}\boldsymbol{n} + p\boldsymbol{n} = 0 \qquad \text{on}\ \Sigma\big\}\,,$$

and $\mathbb{X}_{h,c}^{\rm sym} := \mathbb{X}_{h,c} \cap \mathbb{X}_h^{\rm sym}$, respectively. The kernel $\mathbb{K}_h$ of the bilinear form A in $\mathbb{X}_h^{\rm sym}$ is given by

$$\mathbb{K}_h := \Big\{ (\boldsymbol{\tau}, \xi) \in \mathbb{X}_{h,c}^{\rm sym}; \quad \mathbf{div}\, \boldsymbol{\tau} = 0 \Big\} .$$

In turn, we set

$$\mathbb{K}_h^{\perp} := \Big\{ (\boldsymbol{\sigma}_h, p_h) \in \mathbb{X}_h^{\rm sym}; \quad \Big((\boldsymbol{\sigma}_h, p_h), (\boldsymbol{\tau}, \xi) \Big)_{\mathcal{C}} = 0 \quad \forall (\boldsymbol{\tau}, \xi) \in \mathbb{K}_h \Big\} ,$$

and notice that, in general, neither $\mathbb{K}_h \subseteq \mathbb{K}$ nor $\mathbb{K}_h^{\perp} \subseteq \mathbb{K}^{\perp}$.

According to the above discussions and notations, we consider in what follows the following semi-discrete Galerkin discretization of (2.1.12):

$$\begin{aligned}
&\text{Find } (\boldsymbol{\sigma}_h, p_h) \in \mathcal{C}^1((0,T); \mathbb{X}_h^{\rm sym}) \text{ such that} \\
&\Big((\ddot{\boldsymbol{\sigma}}_h, \ddot{p}_h)(t), (\boldsymbol{\tau}, q) \Big)_{\mathcal{C}} + A\Big((\boldsymbol{\sigma}_h, p_h)(t), (\boldsymbol{\tau}, q) \Big) = -\rho_{\rm S}^{-1} \big(\boldsymbol{f}(t), \mathbf{div}\, \boldsymbol{\tau} \big)_{0,\Omega_{\rm S}} \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\rm sym} \\
&(\boldsymbol{\sigma}_h(0), p_h(0)) = (\boldsymbol{\sigma}_{0,h}, p_{0,h}), \qquad (\dot{\boldsymbol{\sigma}}_h(0), \dot{p}_h(0)) = (\boldsymbol{\sigma}_{1,h}, p_{1,h}),
\end{aligned} \tag{2.2.11}$$

where the discrete initial data $(\boldsymbol{\sigma}_{0,h}, p_{0,h}) \in \mathbb{K}_h^{\perp}$ and $(\boldsymbol{\sigma}_{1,h}, p_{1,h}) \in \mathbb{K}_h^{\perp}$ are given approximations of $(\boldsymbol{\sigma}_0, p_0)$ and $(\boldsymbol{\sigma}_1, p_1)$, respectively.

We end this section by remarking that, exactly as in [21, Section 5.3], [20], and the proof of Theorem 2.1.1, the well-posedness of (2.2.11) also follows from classical ODE theory. We omit further details and refer to those works or to Theorem 2.1.1. In turn, similarly as in [21, Section 5], the corresponding convergence analysis is carried out later on in Section 2.4 by applying the properties of the continuous and discrete versions of the auxiliary operator to be introduced in the following section.

Remark 2.2.1. *We point out that the analysis presented in this paper can be adapted to deal with independent meshes in $\Omega_{\rm S}$ and $\Omega_{\rm F}$ with mesh sizes $h_{\rm S}$ and $h_{\rm F}$ respectively. Indeed, following [32] (see also [15]), in that case we can redefine $\mathbb{X}_h$ (given above in (2.2.10)) as*

$$\mathbb{X}_{h_{\rm S}, h_{\rm F}} := \big\{ (\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{W}}_{h_{\rm S}} \times V_{h_{\rm F}} : \quad \boldsymbol{\tau n} + \pi_{h_{\rm S}}(p\boldsymbol{n}) = 0 \quad on \quad \Sigma \big\} ,$$

where $\pi_{h_{\rm S}}$ is the $[\mathrm{L}^2(\Sigma)]^n$-projection onto $\mathcal{P}_m(\Sigma_{h_{\rm S}})$, where $\Sigma_{h_{\rm S}}$ is the partition induced on Σ by the triangulation performed in $\Omega_{\rm S}$ with mesh size $h_{\rm S}$. We plan to address this issue, within the context of a fully-discrete scheme, in a separate work.

2.3 An auxiliary operator

As already announced, and in order to facilitate the convergence analysis of the Galerkin scheme (2.2.11), in this section we first introduce a suitable auxiliary operator and a discrete approximation of it, and then we derive the corresponding error estimate between them.

2.3.1 The continuous version

In what follows we define an operator $\Xi : \mathbb{X} \to \mathbb{X}^{\rm sym}$ whose restriction to $\mathbb{X}^{\rm sym}$ coincides with the $(\cdot,\cdot)_{\mathcal{C}}$-orthogonal projection of $\mathbb{X}^{\rm sym}$ onto $\mathbb{K}^{\perp}$. More precisely, given $(\boldsymbol{\sigma}, p) \in \mathbb{X}$, we first let

$$\bar{p} = p - \frac{1}{\Omega_{\rm F}} \int_{\Omega_{\rm F}} p \quad \text{in} \quad \Omega_{\rm F} , \tag{2.3.1}$$

and then define $\Xi(\boldsymbol{\sigma}, p) := (\boldsymbol{\sigma}^*, p^*)$, where

$$p^* := \bar{p} - \frac{\rho_F c^2}{|\Omega_F|} \int_\Sigma \boldsymbol{u}^* \cdot \boldsymbol{n} \quad \text{in} \quad \Omega_F \tag{2.3.2}$$

and the pair $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*)$ is characterized by the set of equations,

$$\begin{gathered} \mathcal{C}^{-1}\boldsymbol{\sigma}^* = \boldsymbol{\varepsilon}(\boldsymbol{u}^*) \quad \text{in} \quad \Omega_S\,, \quad \boldsymbol{\sigma}^* = (\boldsymbol{\sigma}^*)^{\mathtt{t}} \quad \text{in} \quad \Omega_S\,, \quad \mathbf{div}\,\boldsymbol{\sigma}^* = \mathbf{div}\,\boldsymbol{\sigma} \quad \text{in} \quad \Omega_S \\ \boldsymbol{\sigma}^*\boldsymbol{n} = -p^*\boldsymbol{n} \quad \text{on} \quad \Sigma\,, \quad \boldsymbol{\sigma}^*\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_N\,, \quad \boldsymbol{u}^* = \mathbf{0} \quad \text{on} \quad \Gamma_D\,. \end{gathered} \tag{2.3.3}$$

Note from (2.3.1) and (2.3.2) that there holds

$$\int_\Sigma \boldsymbol{u}^* \cdot \boldsymbol{n} + \frac{1}{\rho_F c^2} \int_{\Omega_F} p^* = 0\,. \tag{2.3.4}$$

Actually, the constant value given by the second term on the right hand side of (2.3.2) has been chosen so that (2.3.4) holds. Then, motivated by the Neumann boundary condition on Σ, we now consider the spaces

$$\boldsymbol{\mathcal{Y}} := \left\{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \quad \boldsymbol{\tau}\boldsymbol{n} \in \mathrm{L}^2(\Sigma)^n\right\} \quad \text{and} \quad \boldsymbol{\mathcal{Y}}^{\text{sym}} := \boldsymbol{\mathcal{Y}} \cap \boldsymbol{\mathcal{W}}^{\text{sym}}\,,$$

both endowed with the graph norm

$$\|\boldsymbol{\tau}\|^2_{\boldsymbol{\mathcal{Y}}} := \|\boldsymbol{\tau}\|^2_{\mathbf{H}(\mathbf{div},\Omega_S)} + \|\boldsymbol{\tau}\boldsymbol{n}\|^2_{0,\Sigma}\,. \tag{2.3.5}$$

Hence, with these notations at hand, and realizing that the auxiliary unknown $\boldsymbol{\psi}^* := \boldsymbol{u}^*|_\Sigma$ becomes the Lagrange multiplier corresponding to the weak imposition of the aforementioned condition on Σ, we arrive at the following dual-mixed variational formulation of problem (2.3.3)

Find $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*) \in \boldsymbol{\mathcal{Y}}^{\text{sym}} \times \mathrm{L}^2(\Omega_S)^n \times \mathrm{L}^2(\Sigma)^n$ such that

$$\begin{aligned} (\mathcal{C}^{-1}\boldsymbol{\sigma}^*, \boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{u}^*, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{\psi}^*, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}^{\text{sym}} \\ (\mathbf{div}\,\boldsymbol{\sigma}^*, \boldsymbol{v})_{0,\Omega_S} &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{0,\Omega_S} && \forall \boldsymbol{v} \in \mathrm{L}^2(\Omega_S)^n, \\ (\boldsymbol{\sigma}^*\boldsymbol{n}, \boldsymbol{\varphi})_{0,\Sigma} - \frac{c^2\rho_F}{|\Omega_F|}\left\{\int_\Sigma \boldsymbol{\psi}^* \cdot \boldsymbol{n}\right\}\left\{\int_\Sigma \boldsymbol{\varphi} \cdot \boldsymbol{n}\right\} &= -(\bar{p}, \boldsymbol{\varphi} \cdot \boldsymbol{n})_{0,\Sigma} && \forall \boldsymbol{\varphi} \in \mathrm{L}^2(\Sigma)^n\,. \end{aligned} \tag{2.3.6}$$

A further simplification is obtained by taking $\boldsymbol{\varphi} = \boldsymbol{n}$ in the last equation of (2.3.6), which yields

$$\int_\Sigma \boldsymbol{\psi}^* \cdot \boldsymbol{n} = \frac{|\Omega_F|}{\rho_F c^2 |\Sigma|}\left\{(\boldsymbol{\sigma}^*\boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} + \int_\Sigma \bar{p}\right\}\,. \tag{2.3.7}$$

As a consequence of the foregoing identity, and defining the space

$$\Psi := \left\{\boldsymbol{\varphi} \in \mathrm{L}^2(\Sigma)^n; \quad \int_\Sigma \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0\right\},$$

we can reformulate (2.3.6) as follows:

Find $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*) \in \boldsymbol{\mathcal{Y}}^{\text{sym}} \times \mathrm{L}^2(\Omega_S)^n \times \Psi$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) + (\boldsymbol{u}^*, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{\psi}_0^*, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} &= -\frac{|\Omega_F|}{\rho_F c^2 |\Sigma|^2} \int_\Sigma \bar{p}\,(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} && \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}^{\text{sym}}\,, \\ (\mathbf{div}\,\boldsymbol{\sigma}^*, \boldsymbol{v})_{0,\Omega_S} &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{0,\Omega_S} && \forall \boldsymbol{v} \in \mathrm{L}^2(\Omega_S)^n\,, \\ (\boldsymbol{\sigma}^*\boldsymbol{n}, \boldsymbol{\varphi})_{0,\Sigma} &= -(\bar{p}, \boldsymbol{\varphi} \cdot \boldsymbol{n})_{0,\Sigma} && \forall \boldsymbol{\varphi} \in \Psi\,, \end{aligned} \tag{2.3.8}$$

where

$$a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) \,:=\, (\mathcal{C}^{-1}\boldsymbol{\sigma}^*, \boldsymbol{\tau})_{0,\Omega_S} \,+\, \frac{|\Omega_F|}{\rho_F c^2 |\Sigma|^2}\,(\boldsymbol{\sigma}^*\boldsymbol{n}, \boldsymbol{n})_{0,\Sigma}\,(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{n})_{0,\Sigma}\,.$$

More precisely, the following lemma establishes the equivalence between (2.3.6) and (2.3.8).

Lemma 2.3.1. *Let* $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*)$ *be a solution of* (2.3.6), *and let*

$$\boldsymbol{\psi}_0^* := \boldsymbol{\psi}^* - \frac{1}{|\Sigma|}\left\{\int_\Sigma \boldsymbol{\psi}^* \cdot \boldsymbol{n}\right\}\boldsymbol{n}\,. \tag{2.3.9}$$

Then $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*)$ *is a solution of* (2.3.8). *Conversely, let* $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*)$ *be a solution of* (2.3.8) *and let*

$$\boldsymbol{\psi}^* := \boldsymbol{\psi}_0^* + \frac{|\Omega_F|}{\rho_F c^2 |\Sigma|^2}\left\{(\boldsymbol{\sigma}^*\boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} \,+\, \int_\Sigma \bar{p}\right\}\boldsymbol{n}\,. \tag{2.3.10}$$

Then $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*)$ *is solution of* (2.3.6).

Proof. Let $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*) \in \boldsymbol{\mathcal{Y}}^{\rm sym} \times \mathrm{L}^2(\Omega_S)^n \times \mathrm{L}^2(\Sigma)^n$ be a solution of (2.3.6), and define $\boldsymbol{\psi}_0^*$ by (2.3.9), which belongs to Ψ. Then, replacing $\boldsymbol{\psi}^*$ by $\boldsymbol{\psi}_0^* + \frac{1}{|\Sigma|}\left\{\int_\Sigma \boldsymbol{\psi}^* \cdot \boldsymbol{n}\right\}\boldsymbol{n}$ in the first equation of (2.3.6) and using (2.3.7), we deduce the first equation of (2.3.8). In turn, testing the third equation of (2.3.6) with $\boldsymbol{\varphi} \in \Psi$, we obtain the third equation of (2.3.8), and hence $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*)$ becomes a solution of (2.3.8). Conversely, let $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}_0^*) \in \boldsymbol{\mathcal{Y}}^{\rm sym} \times \mathrm{L}^2(\Omega_S)^n \times \Psi$ be a solution of (2.3.8), and define $\boldsymbol{\psi}^*$ as indicated in (2.3.10) (which is suggested by (2.3.7) and (2.3.9)). Then, replacing the resulting expression for $\boldsymbol{\psi}_0^*$ in the first equation of (2.3.8), we arrive at the first equation of (2.3.6). On the other hand, given $\boldsymbol{\varphi} \in \mathrm{L}^2(\Sigma)^n$, we certainly have that $\boldsymbol{\varphi}_0 \,:=\, \boldsymbol{\varphi} \,-\, \frac{1}{|\Sigma|}(\boldsymbol{\varphi}, \boldsymbol{n})_{0,\Sigma}\,\boldsymbol{n}$ belongs to Ψ. Thus, employing $\boldsymbol{\varphi}_0$ in the third equation of (2.3.8), and using from (2.3.10) that there holds (2.3.7), we obtain the third equation of (2.3.6), from which we conclude that $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*)$ is solution of (2.3.6). □

Next, in order to prove that problem (2.3.8) (or equivalently (2.3.6)) is well-posed, we need to establish the following inf-sup condition.

Lemma 2.3.2. *There exists a constant* $\beta_1 >$ *such that*

$$S(\boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \,:=\, \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{Y}}} \frac{(\boldsymbol{s}, \boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{v}, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{\varphi}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \geq \beta_1\Big\{\,\|\boldsymbol{s}\|_{0,\Omega_S} + \|\boldsymbol{v}\|_{0,\Omega_S} + \|\boldsymbol{\varphi}\|_{0,\Sigma}\,\Big\} \tag{2.3.11}$$

for all $(\boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \in [\mathrm{L}^2(\Omega_S)]^{n\times n}_{skew} \times \mathrm{L}^2(\Omega_S)^n \times \Psi$.

Proof. Given $(\boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \in [\mathrm{L}^2(\Omega_S)]^{n\times n}_{\rm skew} \times \mathrm{L}^2(\Omega_S)^n \times \Psi$, we first observe, thanks to (2.1.24), that there holds

$$S(\boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \,\geq\, \sup_{\substack{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}} \\ \boldsymbol{\tau}\boldsymbol{n}\,=\,0\text{ on }\Sigma}} \frac{(\boldsymbol{s}, \boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{v}, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_S}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega_S)}} \,\geq\, C_0\left\{\,\|\boldsymbol{s}\|_{0,\Omega_S} + \|\boldsymbol{v}\|_{0,\Omega_S}\,\right\}. \tag{2.3.12}$$

Next, we let $\boldsymbol{w} \in \mathrm{H}^1(\Omega_S)^n$ be the unique solution of the (vectorial) Laplace problem

$$\begin{array}{rcll} \mathbf{div}(\nabla\boldsymbol{w}) & = & \mathbf{0} & \text{in } \Omega_S\,, \\ \boldsymbol{w} & = & \mathbf{0} & \text{on } \Gamma_D\,, \\ (\nabla\boldsymbol{w})\boldsymbol{n} & = & \mathbf{0} & \text{on } \Gamma_N\,, \\ (\nabla\boldsymbol{w})\boldsymbol{n} & = & \boldsymbol{\varphi} & \text{on } \Sigma\,, \end{array} \tag{2.3.13}$$

and define $\bar{\boldsymbol{\sigma}} := \nabla \boldsymbol{w} \in \boldsymbol{\mathcal{Y}}$. It is clear from (2.3.13) and its associated continuous dependence result that $\mathbf{div}\,\bar{\boldsymbol{\sigma}} = \mathbf{0}$ in Ω_{S}, $\bar{\boldsymbol{\sigma}}\boldsymbol{n} = \boldsymbol{\varphi}$ on Σ, and that there exists $C_1 > 0$, independent of $\boldsymbol{\varphi}$, such that

$$\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}} \leq C_1 \|\boldsymbol{\varphi}\|_{0,\Sigma}\,.$$

It follows that

$$\begin{aligned} S(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) \,\geq\, & \frac{(\boldsymbol{s},\bar{\boldsymbol{\sigma}})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{v},\mathbf{div}\,\bar{\boldsymbol{\sigma}})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\varphi},\bar{\boldsymbol{\sigma}}\boldsymbol{n})_{0,\Sigma}}{\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} \,=\, \frac{(\boldsymbol{s},\bar{\boldsymbol{\sigma}})_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\varphi}\|_{0,\Sigma}^2}{\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} \\ \geq\, & \frac{\|\boldsymbol{\varphi}\|_{0,\Sigma}^2}{\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} - \|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}} \,\geq\, \frac{1}{C_1}\,\|\boldsymbol{\varphi}\|_{0,\Sigma} - \|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}}\,. \end{aligned} \tag{2.3.14}$$

In this way, the inf-sup condition (2.3.11) is now obtained by multiplying (2.3.14) by $\frac{C_0}{2}$ and adding the resulting estimate to (2.3.12). □

Next, defining the norm in $\boldsymbol{\mathcal{Y}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \Psi$ as

$$\|(\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\psi})\|^2 \,:=\, \|\boldsymbol{\sigma}\|_{\boldsymbol{\mathcal{Y}}}^2 + \|\boldsymbol{u}\|_{0,\Omega_{\mathrm{S}}}^2 + \|\boldsymbol{\psi}\|_{0,\Sigma}^2 \quad \forall (\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\psi}) \in \boldsymbol{\mathcal{Y}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \Psi\,,$$

we are now in a position to show that (2.3.8) is well posed.

Lemma 2.3.3. *There exists a unique* $(\boldsymbol{\sigma}^*,\boldsymbol{u}^*,\boldsymbol{\psi}_0^*) \in \boldsymbol{\mathcal{Y}}^{sym} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \Psi$ *solution of* (2.3.8)*, and there exists* $C > 0$*, independent of* λ *and the given* $(\boldsymbol{\sigma},p) \in \mathbb{X}$*, such that*

$$\|(\boldsymbol{\sigma}^*,\boldsymbol{u}^*,\boldsymbol{\psi}_0^*)\| \,\leq\, C\,\|(\boldsymbol{\sigma},p)\|\,. \tag{2.3.15}$$

Proof. We begin by introducing

$$\mathbf{K} := \Big\{\boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}^{\mathrm{sym}}; \quad (\boldsymbol{v},\mathbf{div},\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\varphi},\boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} = 0 \quad \forall (\boldsymbol{v},\boldsymbol{\varphi}) \in \mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \Psi\Big\}\,,$$

which reduces to $\mathbf{K} = \Big\{\boldsymbol{\tau}; \quad (\boldsymbol{\tau},\xi) \in \mathbb{K}\Big\}$. Then, using the estimate (2.1.15) and the fact that $\boldsymbol{\tau}\boldsymbol{n} = -\xi\boldsymbol{n}$ on Σ, we find that

$$\begin{aligned} a(\boldsymbol{\tau},\boldsymbol{\tau}) &= (\mathcal{C}^{-1}\boldsymbol{\tau},\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + \frac{|\Omega_{\mathrm{F}}|}{\rho_{\mathrm{F}}c^2}\xi^2 = \Big((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi)\Big) \\ &\geq\, \alpha\,\|(\boldsymbol{\tau},\xi)\|^2 \,=\, \alpha\left(\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})}^2 + \xi^2|\Omega_{\mathrm{F}}|\right) \,\geq\, \alpha\min\left\{1,\frac{|\Omega_{\mathrm{F}}|}{|\Sigma|}\right\}\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}^2 \end{aligned} \tag{2.3.16}$$

for all $\boldsymbol{\tau} \in \mathbf{K}$. In this way, thanks to the ellipticity property (2.3.16) and the inf-sup condition (2.3.11), a straightforward application of the well-known Babuška-Brezzi theory implies the well-posedness of the saddle point problem (2.3.8) and the continuous dependence estimate (2.3.15). □

As a consequence of the foregoing lemma, the auxiliary operator Ξ given originally by (2.3.1), (2.3.2), and (2.3.3), is now well defined. Actually, due to the equivalence between (2.3.6) and (2.3.8), and the identity (2.3.7), we can redefine Ξ as

$$\Xi(\boldsymbol{\sigma},p) := (\boldsymbol{\sigma}^*,p^*) \qquad \forall\,(\boldsymbol{\sigma},p) \in \mathbb{X}\,, \tag{2.3.17}$$

where

$$p^* := \bar{p} - \frac{1}{|\Sigma|}\left\{(\boldsymbol{\sigma}^*\boldsymbol{n},\boldsymbol{n})_{0,\Sigma} + \int_\Sigma \bar{p}\right\}, \tag{2.3.18}$$

$\bar{p}$ is given by (2.3.1), and $\boldsymbol{\sigma}^*$ is the first component of $(\boldsymbol{\sigma}^*,\boldsymbol{u}^*,\boldsymbol{\psi}_0^*) \in \boldsymbol{\mathcal{Y}}^{\text{sym}} \times \mathrm{L}^2(\Omega_\mathrm{S})^n \times \Psi$, the unique solution of (2.3.8). Moreover, we have the following result.

Lemma 2.3.4. *There exists a constant $C > 0$, independent of λ, such that*

$$\|\Xi(\boldsymbol{\sigma},p)\| \leq C\,\|(\boldsymbol{\sigma},p)\| \quad \forall (\boldsymbol{\sigma},p) \in \mathbb{X}\,.$$

In addition, the operator $\widetilde{\Xi} := \Xi|_{\mathbb{X}^{sym}}$ is the $(\cdot,\cdot)_{\mathcal{C}}$-orthogonal projection in $\mathbb{X}^{sym}$ onto $\mathbb{K}^\perp$.

Proof. The uniform boundedness of Ξ with respect to λ follows directly from the definition of this operator and (2.3.15). Furthermore, it is straightforward to see that $\widetilde{\Xi}^2 = \widetilde{\Xi}$, and hence we have the stable and direct splitting,

$$\mathbb{X}^{\text{sym}} = \widetilde{\Xi}(\mathbb{X}) + N(\widetilde{\Xi})\,. \tag{2.3.19}$$

On the other hand, it is clear that $N(\widetilde{\Xi}) = \mathbb{K}$, and the first equation of (2.3.8) shows that, for any $(\boldsymbol{\sigma}^*,p^*) = \widetilde{\Xi}(\boldsymbol{\sigma},p)$, with $(\boldsymbol{\sigma},p)$ arbitrary in $\mathbb{X}$, there holds

$$\left(\widetilde{\Xi}(\boldsymbol{\sigma},p),(\boldsymbol{\tau},\xi)\right)_{\mathcal{C}} = 0 \quad \forall (\boldsymbol{\tau},\xi) \in \mathbb{K}\,,$$

which proves that the decomposition (2.3.19) is $(\cdot,\cdot)_{\mathcal{C}}$-orthogonal and $\widetilde{\Xi}(\mathbb{X}) = \mathbb{K}^\perp$. □

We now notice from (2.3.11), taking in particular $(\boldsymbol{v},\boldsymbol{\varphi}) = (\mathbf{0},\mathbf{0})$, that there also holds

$$\sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{Y}}} \frac{(\boldsymbol{r},\boldsymbol{\tau})_{0,\Omega_\mathrm{S}}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \geq \beta_1\,\|\boldsymbol{r}\|_{0,\Omega_\mathrm{S}} \quad \forall\, \boldsymbol{r} \in [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\text{skew}}\,.$$

Hence, bearing in mind that the first equation of (2.3.8) can be rewritten as

$$a(\boldsymbol{\sigma}^*,\boldsymbol{\tau}) + (\boldsymbol{u}^*,\mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{\psi}_0^*,\boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} + \frac{|\Omega_\mathrm{F}|}{\rho_\mathrm{F} c^2|\Sigma|^2}\int_\Sigma \bar{p}(\boldsymbol{n},\boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} = 0 \quad \forall\,\boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}^{\text{sym}}\,,$$

and that $\boldsymbol{\mathcal{Y}}^{\text{sym}}$ is the kernel of the operator induced by the bilinear form $\mathcal{A} : \boldsymbol{\mathcal{Y}} \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\text{skew}} \to \mathbb{R}$ defined by $\mathcal{A}(\boldsymbol{\tau},\boldsymbol{r}) = (\boldsymbol{r},\boldsymbol{\tau})_{0,\Omega_\mathrm{S}}\ \forall\,(\boldsymbol{\tau},\boldsymbol{r}) \in \boldsymbol{\mathcal{Y}} \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\text{skew}}$, we can apply Theorem 0.0.1 to conclude that there exists a unique $\boldsymbol{r}^* \in [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\text{skew}}$ such that

$$a(\boldsymbol{\sigma}^*,\boldsymbol{\tau}) + (\boldsymbol{u}^*,\mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{\psi}_0^*,\boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} + \frac{|\Omega_\mathrm{F}|}{\rho_\mathrm{F} c^2|\Sigma|^2}\int_\Sigma \bar{p}(\boldsymbol{n},\boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} = -(\boldsymbol{r}^*,\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\tau} \in \boldsymbol{\mathcal{Y}}\,. \tag{2.3.20}$$

The following result provides an explicit connection between the operators $\mathbf{D}$ and Ξ.

Lemma 2.3.5. *Given $(\boldsymbol{\sigma},p) \in \mathbb{X}$, we let $(\boldsymbol{\sigma}^*,\boldsymbol{u}^*,\boldsymbol{\psi}^*) \in \boldsymbol{\mathcal{Y}} \times \mathrm{L}^2(\Omega_\mathrm{S})^n \times \mathrm{L}^2(\Sigma)^n$ and $\boldsymbol{r}^* \in [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{skew}$ be the unique solutions of* (2.3.6) *and* (2.3.20)*, respectively, and set $(\boldsymbol{\sigma}^*,p^*) := \Xi(\boldsymbol{\sigma},p)$ (*cf. (2.3.17)*). Then there hold*

$$(\boldsymbol{u}^*,\boldsymbol{r}^*) = \mathbf{D}(\boldsymbol{\sigma}^*,p^*) \qquad \text{and} \qquad \boldsymbol{\psi}^* = \boldsymbol{u}^*|_\Sigma\,.$$

Proof. Given $(\boldsymbol{\tau},\xi)\in\mathbb{X}_c$, that is $\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}$, $\xi\in\mathbb{R}$, and $\boldsymbol{\tau n} = -\xi\boldsymbol{n}$ on Σ, it is clear that $\boldsymbol{\tau}\in\boldsymbol{\mathcal{Y}}$. Then, recalling the definition of $\boldsymbol{\psi}_0^*$ (cf. (2.3.9)), we deduce from (2.3.20) that

$$(\boldsymbol{r}^*,\boldsymbol{\tau})_{0,\Omega_S} + (\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{u}^*)_{0,\Omega_S} = -(\mathcal{C}^{-1}\boldsymbol{\sigma}^*,\boldsymbol{\tau})_{0,\Omega_S} - (\boldsymbol{\psi}^*,\boldsymbol{\tau n})_{0,\Sigma} \quad \forall\,(\boldsymbol{\tau},\xi)\in\mathbb{X}_c\,.$$

Next, from the definition of p^* (cf. (2.3.18)), the identity (2.3.7), and the fact that $\boldsymbol{\tau n} = -\xi\boldsymbol{n}$ on Σ, we find after minor algebraic manipulations that

$$(\boldsymbol{r}^*,\boldsymbol{\tau})_{0,\Omega_S} + (\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{u}^*)_{0,\Omega_S} = -\Big((\boldsymbol{\sigma}^*,p^*),(\boldsymbol{\tau},\xi)\Big)_{\mathcal{C}} \quad \forall\,(\boldsymbol{\tau},\xi)\in\mathbb{X}_c\,,$$

which, according to the characterization (2.1.25) of the operator $\mathbf{D}$, yields $(\boldsymbol{u}^*,\boldsymbol{r}^*) = \mathbf{D}(\boldsymbol{\sigma}^*,p^*)$. Finally, the interpretation $\boldsymbol{\psi}^* = \boldsymbol{u}^*|_\Sigma$ is obtained by integrating by parts in the first equation of (2.3.6). □

2.3.2 The discrete version

We now aim to define a discrete version of the operator Ξ. To this end, we introduce the subspace of Ψ given by $\Psi_h := \mathcal{P}_k(\Sigma_h)^n\cap\Psi$, recall the definition of the operator Π_h (cf. (2.2.5)), and consider the following Galerkin approximation of problem (2.3.8):

Find $(\boldsymbol{\sigma}_h^*,\boldsymbol{u}_h^*,\boldsymbol{\psi}_{0,h}^*)\in\boldsymbol{\mathcal{W}}_h^{\rm sym}\times\mathcal{U}_h\times\Psi_h$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h^*,\boldsymbol{\tau}) + (\boldsymbol{u}_h^*,\mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{\psi}_{0,h}^*,\boldsymbol{\tau n})_{0,\Sigma} &= -\frac{|\Omega_F|}{\rho_F c^2|\Sigma|^2}\int_\Sigma \Pi_h p(\boldsymbol{n},\boldsymbol{\tau n})_{0,\Sigma} && \forall\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h^{\rm sym}\,,\\ (\mathbf{div}\,\boldsymbol{\sigma}_h^*,\boldsymbol{v})_{0,\Omega_S} &= (\mathbf{div}\,\boldsymbol{\sigma},\boldsymbol{v})_{0,\Omega_S} && \forall\boldsymbol{v}\in\mathcal{U}_h\,,\\ (\boldsymbol{\sigma}_h^*\boldsymbol{n},\boldsymbol{\varphi})_{0,\Sigma} &= -(\Pi_h p,\boldsymbol{\varphi}\cdot\boldsymbol{n})_{0,\Sigma} && \forall\boldsymbol{\varphi}\in\Psi_h\,. \end{aligned} \tag{2.3.21}$$

We begin the analysis of (2.3.21) with the following discrete inf-sup condition.

Lemma 2.3.6. *There exists a constant $\beta_1^*>0$, independent of h, such that*

$$S_h(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) := \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h}\frac{(\boldsymbol{s},\boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{v},\mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{\varphi},\boldsymbol{\tau n})_{0,\Sigma}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \geq \beta_1^*\Big\{\|\boldsymbol{s}\|_{0,\Omega_S}+\|\boldsymbol{v}\|_{0,\Omega_S}+\|\boldsymbol{\varphi}\|_{0,\Sigma}\Big\} \tag{2.3.22}$$

for all $(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi})\in\boldsymbol{\mathcal{Q}}_h\times\mathcal{U}_h\times\Psi_h$.

Proof. We proceed very similarly to the proof of Lemma 2.3.2. In fact, given $(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi})\in\boldsymbol{\mathcal{Q}}_h\times\mathcal{U}_h\times\Psi_h$, we first realize, thanks to (2.2.1), that there exists a constant $\beta_0^*>0$, independent of h, such that

$$S_h(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) \geq \sup_{\substack{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_h\\ \boldsymbol{\tau n}=0 \text{ on } \Sigma}}\frac{(\boldsymbol{s},\boldsymbol{\tau})_{0,\Omega_S} + (\boldsymbol{v},\mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_S}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega_S)}} \geq \beta_0^*\Big\{\|\boldsymbol{s}\|_{0,\Omega_S}+\|\boldsymbol{v}\|_{0,\Omega_S}\Big\}. \tag{2.3.23}$$

Now, we consider again the solution $\boldsymbol{w}$ of problem (2.3.13), but with a Neumann data $\boldsymbol{\varphi}\in\Psi_h\in \mathrm{L}^2(\Sigma)^n$. Then, classical regularity results for the Poisson problem in polyhedral (polygonal) domains (cf. [14]) ensure the existence of $\varepsilon\in(0,1)$, depending on the geometry of Ω_S, such that $\boldsymbol{w}\in\mathrm{H}^{1+\varepsilon}(\Omega_S)^n$ and

$$\|\boldsymbol{w}\|_{1+\varepsilon,\Omega_S} \leq C_1\,\|\boldsymbol{\varphi}\|_{0,\Sigma}\,. \tag{2.3.24}$$

It follows that $\bar{\boldsymbol{\sigma}} := \nabla \boldsymbol{w}$ belongs to $\boldsymbol{\mathcal{Y}} \cap \mathrm{H}^{\varepsilon}(\Omega_{\mathrm{S}})^n$, and hence $\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}$ is meaningful. In addition, by virtue of (2.2.4) we have that $\mathbf{div}\, \boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}} = \mathbf{0}$ in Ω_{S}, $(\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})\boldsymbol{n} = \mathbf{0}$ on Γ_{N}, and $(\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})\boldsymbol{n} = \boldsymbol{\varphi}$ on Σ, whereas the approximation property of $\boldsymbol{\Pi}_h$ (cf. (2.2.2)) yields the existence of a constant $C_2 > 0$, independent of h, such that

$$\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})} = \|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{0,\Omega_{\mathrm{S}}} \le C_2 \, \|\bar{\boldsymbol{\sigma}}\|_{\varepsilon,\Omega_{\mathrm{S}}} \,.$$

Combining the foregoing inequality with (2.3.24) gives

$$\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}} \le C_3 \, \|\boldsymbol{\varphi}\|_{0,\Sigma} \,,$$

and therefore, noting that $\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}} \in \boldsymbol{\mathcal{W}}_h$, we find that

$$\begin{aligned} S_h(\boldsymbol{s},\boldsymbol{v},\boldsymbol{\varphi}) \;&\ge\; \frac{(\boldsymbol{s},\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{v},\mathbf{div}\, \boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\varphi},\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\boldsymbol{n})_{0,\Sigma}}{\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} \\ = \frac{(\boldsymbol{s},\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}})_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\varphi}\|_{0,\Sigma}^2}{\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} \;&\ge\; \frac{\|\boldsymbol{\varphi}\|_{0,\Sigma}^2}{\|\boldsymbol{\Pi}_h \bar{\boldsymbol{\sigma}}\|_{\boldsymbol{\mathcal{Y}}}} - \|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}} \;\ge\; \frac{1}{C_3}\, \|\boldsymbol{\varphi}\|_{0,\Sigma} - \|\boldsymbol{s}\|_{0,\Omega_{\mathrm{S}}} \,. \end{aligned} \tag{2.3.25}$$

Finally, an adequate combination of (2.3.23) and (2.3.25) implies (2.3.22) and finishes the proof. □

The well-posedness of (2.3.21) is provided next.

Lemma 2.3.7. *There exists a unique* $(\boldsymbol{\sigma}_h^*, \boldsymbol{u}_h^*, \boldsymbol{\psi}_{0,h}^*) \in \boldsymbol{\mathcal{W}}_h^{\mathrm{sym}} \times \mathcal{U}_h \times \Psi_h$ *solution of* (2.3.21)*, and there exists* $C > 0$*, independent of* λ*,* h*, and the given* $(\boldsymbol{\sigma}, p) \in \mathbb{X}$*, such that*

$$\|(\boldsymbol{\sigma}_h^*, \boldsymbol{u}_h^*, \boldsymbol{\psi}_{0,h}^*))\| \;\le\; C \, \|(\boldsymbol{\sigma}, p)\| \,. \tag{2.3.26}$$

Proof. Proceeding analogously as in the proof of Lemma 2.3.4, we begin by introducing the discrete null space

$$\mathbf{K}_h := \Big\{ \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h^{\mathrm{sym}}; \quad (\boldsymbol{v}, \mathbf{div}\, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{\varphi}, \boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma} = 0 \quad \forall (\boldsymbol{v}, \boldsymbol{\varphi}) \in \mathcal{U}_h \times \Psi_h \Big\} \,,$$

which becomes $\mathbf{K}_h = \Big\{ \boldsymbol{\tau}; \quad (\boldsymbol{\tau}, \xi) \in \mathbb{K}_h \Big\}$. Then, it is readily seen that (2.1.15) implies the $\boldsymbol{\mathcal{Y}}$-ellipticity of $a(\cdot,\cdot)$ on $\mathbf{K}_h$, with a constant independent of h and λ. In this way, thanks to this result and the inf-sup condition (2.3.6), a direct application of the discrete Babuška-Brezzi theory completes the proof of the lemma. □

As a consequence of the foregoing lemma, we now introduce the operator $\Xi_h : \mathbb{X} \to \mathbb{X}_h^{\mathrm{sym}}$, which is defined as the discrete analogue of (2.3.17), that is

$$\Xi_h(\boldsymbol{\sigma}, p) \;:=\; (\boldsymbol{\sigma}_h^*, p_h^*) \qquad \forall\, (\boldsymbol{\sigma}, p) \,\in\, \mathbb{X} \,, \tag{2.3.27}$$

where

$$p_h^* := \Pi_h p - \frac{1}{|\Sigma|} \left\{ (\boldsymbol{\sigma}_h^* \boldsymbol{n}, \boldsymbol{n})_{0,\Sigma} + \int_{\Sigma} \Pi_h p \right\} \,, \tag{2.3.28}$$

and $\boldsymbol{\sigma}_h^*$ is the first component of the solution $(\boldsymbol{\sigma}_h^*, \boldsymbol{u}_h^*, \boldsymbol{\psi}_{0,h}^*) \in \boldsymbol{\mathcal{W}}_h^{\mathrm{sym}} \times \mathcal{U}_h \times \Psi_h$ of problem (2.3.21). In addition, there exists $C > 0$, independent of λ, h, and the given $(\boldsymbol{\sigma}, p) \in \mathbb{X}$, such that

$$\|\Xi_h(\boldsymbol{\sigma}, p)\| \;\le\; C \, \|(\boldsymbol{\sigma}, p)\| \qquad \forall\, (\boldsymbol{\sigma}, p) \,\in\, \mathbb{X} \,. \tag{2.3.29}$$

In the following section we deal with the a priori error estimate for $\Xi - \Xi_h$.

2.3.3 The associated error estimate

We begin by remarking that our analysis up to now has not required any formulation involving explicitly the rotation $\boldsymbol{r}^*$ nor its corresponding discrete version $\boldsymbol{r}_h^*$. Actually, the main novelty of our approach has been precisely the fact that these complementary unknowns have remained some how hiden. Nevertheless, for the derivation of the aforementioned error estimate, we now need to introduce the extended versions of (2.3.8) and (2.3.21), which are given, respectively, as follows:

Find $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*, \boldsymbol{u}^*, \psi_0^*) \in \boldsymbol{\mathcal{Y}} \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}} \times \mathrm{L}^2(\Omega_\mathrm{S})^n \times \Psi$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) + (\boldsymbol{r}^*, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{u}^*, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{\psi}_0^*, \boldsymbol{\tau n})_{0,\Sigma} &= -\frac{|\Omega_\mathrm{F}|}{\rho_\mathrm{F} c^2 |\Sigma|^2} \int_\Sigma \bar{p} (\boldsymbol{n}, \boldsymbol{\tau n})_{0,\Sigma}\,, \\ (\boldsymbol{\sigma}^*, \boldsymbol{s})_{0,\Omega_\mathrm{S}} &= 0\,, \\ (\mathbf{div}\,\boldsymbol{\sigma}^*, \boldsymbol{v})_{0,\Omega_\mathrm{S}} &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{0,\Omega_\mathrm{S}}\,, \\ (\boldsymbol{\sigma}^* \boldsymbol{n}, \boldsymbol{\varphi})_{0,\Sigma} &= -(\bar{p}, \boldsymbol{\varphi}\cdot\boldsymbol{n})_{0,\Sigma}\,, \end{aligned} \tag{2.3.30}$$

for all $(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \in \boldsymbol{\mathcal{Y}} \times [\mathrm{L}^2(\Omega_\mathrm{S})]^{n\times n}_{\mathrm{skew}} \times \mathrm{L}^2(\Omega_\mathrm{S})^n \times \Psi$, and

Find $(\boldsymbol{\sigma}_h^*, \boldsymbol{r}_h^*, \boldsymbol{u}_h^*, \psi_{0,h}^*) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \times \Psi_h$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h^*, \boldsymbol{\tau}) + (\boldsymbol{r}_h^*, \boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{u}_h^*, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_\mathrm{S}} + (\boldsymbol{\psi}_{0,h}^*, \boldsymbol{\tau n})_{0,\Sigma} &= -\frac{|\Omega_\mathrm{F}|}{\rho_\mathrm{F} c^2 |\Sigma|^2} \int_\Sigma \Pi_h p (\boldsymbol{n}, \boldsymbol{\tau n})_{0,\Sigma}\,, \\ (\boldsymbol{\sigma}_h^*, \boldsymbol{s})_{0,\Omega_\mathrm{S}} &= 0\,, \\ (\mathbf{div}\,\boldsymbol{\sigma}_h^*, \boldsymbol{v})_{0,\Omega_\mathrm{S}} &= (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{0,\Omega_\mathrm{S}}\,, \\ (\boldsymbol{\sigma}_h^* \boldsymbol{n}, \boldsymbol{\varphi})_{0,\Sigma} &= -(\Pi_h p, \boldsymbol{\varphi}\cdot\boldsymbol{n})_{0,\Sigma} h\,, \end{aligned} \tag{2.3.31}$$

for all $(\boldsymbol{\tau}, \boldsymbol{s}, \boldsymbol{v}, \boldsymbol{\varphi}) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \mathcal{U}_h \times \Psi_h$.

Then, we have the following result.

Lemma 2.3.8. *There exists a constant $C > 0$, independent of h and λ, such that*

$$\begin{aligned} \|\Xi(\boldsymbol{\sigma}, p) - \Xi_h(\boldsymbol{\sigma}, p)\| \,\le\, C \Big\{ &\inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{\sigma}^* - \boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{Y}}} + \inf_{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{r}^* - \boldsymbol{s}_h\|_{0,\Omega_\mathrm{S}} \\ + &\inf_{\boldsymbol{v}_h \in \mathcal{U}_h} \|\boldsymbol{u}^* - \boldsymbol{v}_h\|_{0,\Omega_\mathrm{S}} + \inf_{\boldsymbol{\varphi}_h \in \Psi_h} \|\boldsymbol{\psi}^* - \boldsymbol{\varphi}_h\|_{0,\Sigma} + |p - \Pi_h p|_{1,\Omega_\mathrm{F}} \Big\}\,. \end{aligned} \tag{2.3.32}$$

Proof. A straightforward application of the first Strang lemma to the formulations (2.3.30) and (2.3.31) yields the existence of a constant $C_1 > 0$, independent of h and λ, such that

$$\begin{aligned} \|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*\|_{\boldsymbol{\mathcal{Y}}} + \|\boldsymbol{u}^* - \boldsymbol{u}_h^*\|_{0,\Omega_\mathrm{S}} + \|\boldsymbol{\psi}^* - \boldsymbol{\psi}_h^*\|_{0,\Sigma} \,\le\, C_1 \Bigg\{ &\inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{\sigma}^* - \boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{Y}}} \\ + &\inf_{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{r}^* - \boldsymbol{s}_h\|_{0,\Omega_\mathrm{S}} + \inf_{\boldsymbol{v}_h \in \mathcal{U}_h} \|\boldsymbol{u}^* - \boldsymbol{v}_h\|_{0,\Omega_\mathrm{S}} + \inf_{\boldsymbol{\varphi}_h \in \Psi_h} \|\boldsymbol{\psi}^* - \boldsymbol{\varphi}_h\|_{0,\Sigma} \\ + &\sup_{\boldsymbol{\varphi} \in \Psi_h} \frac{\int_\Sigma (\bar{p} - \Pi_h p) \boldsymbol{\varphi}\cdot\boldsymbol{n}}{\|\boldsymbol{\varphi}\|_{0,\Sigma}} + \sup_{\boldsymbol{\tau} \in \mathbf{K}_h} \frac{\int_\Sigma (\bar{p} - \Pi_h p)(\boldsymbol{n}, \boldsymbol{\tau n})_{0,\Sigma}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \Bigg\}\,. \end{aligned} \tag{2.3.33}$$

Then, employing the Cauchy-Schwarz inequality and the trace theorem, we find that

$$\sup_{\boldsymbol{\varphi}\in\Psi_h} \frac{\displaystyle\int_\Sigma (p-\Pi_h p)\boldsymbol{\varphi}\cdot\boldsymbol{n}}{\|\boldsymbol{\varphi}\|_{0,\Sigma}} \,\le\, C_2\,|p-\Pi_h p|_{1,\Omega_{\mathrm{F}}}$$

and

$$\sup_{\boldsymbol{\tau}\in\mathbf{K}_h} \frac{\displaystyle\int_\Sigma (\bar{p}-\Pi_h p)(\boldsymbol{n},\boldsymbol{\tau}\boldsymbol{n})_{0,\Sigma}}{\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{Y}}}} \,\le\, C_3\,|p-\Pi_h p|_{1,\Omega_{\mathrm{F}}}\,,$$

whereas the definitions of p^*, p_h^*, and $\|\cdot\|_{\boldsymbol{\mathcal{Y}}}$ (cf. (2.3.18), (2.3.28), and (2.3.5)) yield

$$\|p^*-p_h^*\|_{1,\Omega_{\mathrm{F}}} \,\le\, C_4\left\{|p-\Pi_h p|_{1,\Omega_{\mathrm{F}}} + \|\boldsymbol{\sigma}^*-\boldsymbol{\sigma}_h^*\|_{\boldsymbol{\mathcal{Y}}}\right\}.$$

In this way, bounding $\|\boldsymbol{\sigma}^*-\boldsymbol{\sigma}_h^*\|_{\boldsymbol{\mathcal{Y}}}$ according to (2.3.33), and combining this estimate with the last three inequalities, we arrive at (2.3.32) and finish the proof. □

We end this section with the following corollary of the a priori error estimate provided by Lemma 2.3.8.

Lemma 2.3.9. *Assume that $(\boldsymbol{\sigma},p)\in\mathbb{K}^\perp$ with $\boldsymbol{\sigma}\in[\mathrm{H}^\epsilon(\Omega_{\mathrm{S}})]^{n\times n}$ for some $\varepsilon>0$, and let $(\boldsymbol{u},\boldsymbol{r}):=\mathbf{D}(\boldsymbol{\sigma},p)$ and $\boldsymbol{\psi}:=\boldsymbol{u}|_\Sigma$. Then, there exists a constant $C>0$, independent of h and λ, such that*

$$\begin{aligned}\|(\boldsymbol{\sigma},p)-\Xi_h(\boldsymbol{\sigma},p)\| \,\le\, C\Big\{ &\|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})} + \|p\boldsymbol{n}-\boldsymbol{\pi}_h(p\boldsymbol{n})\|_{0,\Sigma} + \|\boldsymbol{r}-\mathbf{R}_h\boldsymbol{r}\|_{0,\Omega_{\mathrm{S}}}\\ + \quad &\|\boldsymbol{u}-U_h\boldsymbol{u}\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\psi}-\pi_h\boldsymbol{\varphi}\|_{0,\Sigma} + |p-\Pi_h p|_{1,\Omega_{\mathrm{F}}}\Big\}.\end{aligned}$$

Proof. We first recall from Lemma 2.3.4 that $\Xi(\boldsymbol{\sigma},p)=(\boldsymbol{\sigma},p)$ for all $(\boldsymbol{\sigma},p)\in\mathbb{K}^\perp$. Then, it follows from Lemma 2.3.5 that $(\boldsymbol{u}^*,\boldsymbol{r}^*):=\mathbf{D}(\boldsymbol{\sigma},p)=(\boldsymbol{u},\boldsymbol{r})$ and $\boldsymbol{\psi}^*=\boldsymbol{u}|_\Sigma=\boldsymbol{\psi}$, which combined with (2.3.32), gives

$$\begin{aligned}\|(\boldsymbol{\sigma},p)-\Xi_h(\boldsymbol{\sigma},p)\| \,\le\, C\Big\{&\inf_{\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{W}}_h}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{Y}}} + \inf_{\boldsymbol{s}_h\in\boldsymbol{\mathcal{Q}}_h}\|\boldsymbol{r}-\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}}\\ + \quad &\inf_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{U}}_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{0,\Omega_{\mathrm{S}}} + \inf_{\boldsymbol{\varphi}_h\in\Psi_h}\|\boldsymbol{\psi}-\boldsymbol{\varphi}_h\|_{0,\Sigma} + |p-\Pi_h p|_{1,\Omega_{\mathrm{F}}}\Big\}.\end{aligned} \tag{2.3.34}$$

Next, since $\boldsymbol{\sigma}\in\boldsymbol{\mathcal{W}}\cap[\mathrm{H}^\varepsilon(\Omega_{\mathrm{S}})]^{n\times n}$ with $\varepsilon>0$, we can employ the BDM-interpolation operator $\boldsymbol{\Pi}_h$ (cf. Section 2.2) to obtain

$$\begin{aligned}\inf_{\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{W}}_h}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{Y}}}^2 \,\le\, \|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{\boldsymbol{\mathcal{Y}}}^2 \,&=\, \|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})}^2 + \|\boldsymbol{\sigma}\boldsymbol{n}-\boldsymbol{\pi}_h(\boldsymbol{\sigma}\boldsymbol{n})\|_{0,\Sigma}^2\\ &\le\, \|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})}^2 + \|p\boldsymbol{n}-\boldsymbol{\pi}_h(p\boldsymbol{n})\|_{0,\Sigma}^2\,,\end{aligned} \tag{2.3.35}$$

which, together with (2.3.34) and the definitions of the projectors $\mathbf{R}_h$ and U_h (cf. Section 2.2), implies the required estimate and ends the proof. □

2.4 Analysis of the semi-discrete problem

From now on, we assume that the discrete Galerkin problem (2.2.11) is supplied with the initial data

$$(\boldsymbol{\sigma}_{0,h}, p_{0,h}) := \Xi_h(\boldsymbol{\sigma}_0, p_0) \quad \text{and} \quad (\boldsymbol{\sigma}_{1,h}, p_{1,h}) := \Xi_h(\boldsymbol{\sigma}_1, p_1).$$

Then, we introduce

$$\boldsymbol{e}_{\sigma,h}(t) = \boldsymbol{\sigma}_h^*(t) - \boldsymbol{\sigma}_h(t) \quad \text{and} \quad e_{p,h}(t) = p_h^*(t) - p_h(t),$$

where $(\boldsymbol{\sigma}_h^*(t), p_h^*(t)) = \Xi_h(\boldsymbol{\sigma}(t), p(t))$, and notice that $\boldsymbol{e}_{\sigma,h}(0) = \dot{\boldsymbol{e}}_{\sigma,h}(0) = \mathbf{0}$ and $e_{p,h}(0) = \dot{e}_{p,h}(0) = 0$.

The following lemma establishes an a priori estimate for the error between the solution $(\boldsymbol{\sigma}, p) \in \mathcal{C}^0((0,T); \mathbb{X}^{\text{sym}}) \cap \mathcal{C}^1((0,T); \mathbb{H}^{\text{sym}})$ of the continuous problem (2.1.12) and its semi-discrete approximation given by the solution $(\boldsymbol{\sigma}_h, p_h) \in \mathcal{C}^1((0,T); \mathbb{X}_h^{\text{sym}})$ of the Galerkin scheme (2.2.11).

Lemma 2.4.1. *Assume that $\boldsymbol{\sigma} \in \mathcal{C}^2((0,T); \mathbb{X} \cap \mathbb{H}^\epsilon(\Omega))$ for some $\epsilon > 0$, and that $p \in \mathcal{C}^2((0,T); \mathrm{H}^1(\Omega_{\mathrm{F}}))$. Then, there exists a constant $C > 0$, independent of λ, h and T, such that*

$$\begin{aligned}
&\max_{[0,T]} \|(\boldsymbol{\sigma}, p)(t) - (\boldsymbol{\sigma}_h, p_h)(t)\| + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}}, \dot{p})(t) - (\dot{\boldsymbol{\sigma}}_h, \dot{p}_h)(t)\|_{0,\mathcal{C}} \\
&\quad \leq C\,T \Big\{ \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathbf{H}(\mathbf{div},\Omega_{\mathrm{S}}))} + \|\boldsymbol{r} - \mathbf{R}_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}([\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{skew})} + \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Omega_{\mathrm{S}})^n)} \\
&\quad + \quad \|\nabla(p - \Pi_h p)\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Omega_{\mathrm{F}})^n)} + \|\boldsymbol{\psi} - \pi_h \boldsymbol{\psi}\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Sigma)^n)} + \|p\boldsymbol{n} - \boldsymbol{\pi}_h(p\boldsymbol{n})\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Sigma))} \Big\}.
\end{aligned} \tag{2.4.1}$$

where $(\boldsymbol{u}, \boldsymbol{r}) := \mathbf{D}(\boldsymbol{\sigma}, p)$ and $\boldsymbol{\psi} := \boldsymbol{u}|_\Sigma$.

Proof. Let us first notice that the fact that $(\boldsymbol{\sigma}(t), p(t)) \in \mathbb{K}^\perp$ for all $t \in [0,T]$ guarantees that

$$(\boldsymbol{\sigma}^*(t), p^*(t)) := \Xi(\boldsymbol{\sigma}(t), p(t)) = (\boldsymbol{\sigma}(t), p(t)) \qquad \forall\, t \in [0,T]. \tag{2.4.2}$$

Moreover, because of the regularity assumptions, it holds that

$$\Big(\frac{\mathrm{d}^i \boldsymbol{\sigma}^*}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i p^*}{\mathrm{d}t^i}(t)\Big) := \frac{\mathrm{d}^i \Xi(\boldsymbol{\sigma}(t), p(t))}{\mathrm{d}t^i} = \Xi(\frac{\mathrm{d}^i \boldsymbol{\sigma}}{\mathrm{d}t^i}(t), \frac{\mathrm{d}^i p}{\mathrm{d}t^i}(t)) \qquad \forall\, i \in \{1,2\}, \quad \forall\, t \in [0,T], \tag{2.4.3}$$

and hence, by virtue of Lemma 2.3.9 and (2.4.3), there exists $C_1 > 0$, independent of h, λ and T, such that

$$\begin{aligned}
&\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*, p - p_h^*)\|_{\mathrm{W}^{2,\infty}(\mathbb{X})} = \|(\boldsymbol{\sigma}, p) - \Xi_h(\boldsymbol{\sigma}, p)\|_{\mathrm{W}^{2,\infty}(\mathbb{X})} \\
&\quad \leq C_1 \Big\{ \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathbf{H}(\mathbf{div},\Omega_{\mathrm{S}}))} + \|p\boldsymbol{n} - \boldsymbol{\pi}_h(p\boldsymbol{n})\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Sigma))} + \|\boldsymbol{r} - \mathbf{R}_h \boldsymbol{r}\|_{\mathrm{W}^{2,\infty}([\mathrm{L}^2(\Omega_{\mathrm{S}})]^{n\times n}_{\mathrm{skew}})} \\
&\quad + \quad \|\boldsymbol{u} - U_h \boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Omega_{\mathrm{S}})^n)} + \|\boldsymbol{\psi} - \pi_h \boldsymbol{\psi}\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Sigma)^n)} + \|\nabla(p - \Pi_h p)\|_{\mathrm{W}^{2,\infty}(\mathrm{L}^2(\Omega_{\mathrm{F}})^n)} \Big\},
\end{aligned} \tag{2.4.4}$$

with $(\boldsymbol{u}(t), \boldsymbol{r}(t)) := \mathbf{D}(\boldsymbol{\sigma}(t), \boldsymbol{u}(t))$ and $\boldsymbol{\psi}(t) := \boldsymbol{u}(t)|_\Sigma$. Now, adding and substracting $(\ddot{\boldsymbol{\sigma}}, \ddot{p})$, and then using the identity (2.1.41) and the first equation of (2.2.11), we obtain the error equation,

$$\begin{aligned}
&\Big((\ddot{\boldsymbol{e}}_{\sigma,h}(t), \ddot{e}_{p,h}(t)), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{e}_{\sigma,h}(t), e_{p,h}(t)), (\boldsymbol{\tau}, q)\Big) \\
&\quad = \Big((\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}, \ddot{p}_h^* - \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} - (\ddot{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + A\Big((\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}, p_h^* - p)(t), (\boldsymbol{\tau}, q)\Big) \\
&\quad = \Big((\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}^*, \ddot{p}_h^* - \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} - (\ddot{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + A\Big((\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}, p_h^* - p)(t), (\boldsymbol{\tau}, q)\Big)
\end{aligned} \tag{2.4.5}$$

for all $(\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\rm sym}$, where the first expression of the last equality makes use of the fact that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^*$ (cf. (2.4.2)). In addition, by virtue of the inclusion $\mathbf{div}(\boldsymbol{\mathcal{W}}_h) \subset \mathcal{U}_h$, it turns out that

$$\big(\mathbf{div}(\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}^*)(t), \mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\rm S}} = 0 \qquad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\rm sym}. \tag{2.4.6}$$

Furthermore, according to the definitions of p^* and p_h^* (cf. (2.3.18) and (2.3.28)), there holds

$$(\nabla(p_h^* - p^*)(t), \nabla q)_{0,\Omega_{\rm F}} = (\nabla(p - \Pi_h p)(t), \nabla q)_{0,\Omega_{\rm F}} = 0 \qquad \forall q \in V_h, \tag{2.4.7}$$

and it is straightforward that

$$(\ddot{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_{\rm S}} = (\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h\ddot{\boldsymbol{r}}(t), \boldsymbol{\tau})_{0,\Omega_{\rm S}} \qquad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\rm sym}. \tag{2.4.8}$$

Next, rewriting (2.4.5) by taking into account (2.4.2) and (2.4.6)-(2.4.8), we deduce that

$$\begin{aligned}
&\Big((\ddot{\boldsymbol{e}}_{\sigma,h}(t), \ddot{e}_{p,h}(t)), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{e}_{\sigma,h}(t), e_{p,h}(t)), (\boldsymbol{\tau}, q)\Big) \\
&\quad = \Big((\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}, \ddot{p}_h^* - \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} - \big(\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h\ddot{\boldsymbol{r}}(t), \boldsymbol{\tau}\big)_{0,\Omega_{\rm S}} \qquad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\rm sym}.
\end{aligned}$$

Moreover, choosing $(\boldsymbol{\tau}, q) = (\dot{\boldsymbol{e}}_{\sigma,h}, \dot{e}_{p,h})(t)$ in the foregoing identity, recalling the definition of the energy functional $\mathcal{E}$ (cf. (2.1.19)), and applying the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}
\dot{\mathcal{E}}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)(t) \;&\le\; \|(\ddot{\boldsymbol{\sigma}} - \ddot{\boldsymbol{\sigma}}_h^*, \ddot{p} - \ddot{p}_h^*)(t)\|_{0,\mathcal{C}}\, \|(\dot{\boldsymbol{e}}_{\sigma,h}, \dot{e}_{p,h})(t)\|_{0,\mathcal{C}} \\
&+ \Big(\mathcal{C}\big(\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h\ddot{\boldsymbol{r}}(t)\big), \ddot{\boldsymbol{r}}(t) - \mathbf{R}_h\ddot{\boldsymbol{r}}(t)\Big)_{0,\Omega_{\rm S}}^{1/2} \Big(\mathcal{C}^{-1}\dot{\boldsymbol{e}}_{\sigma,h}, \dot{\boldsymbol{e}}_{\sigma,h}\Big)_{0,\Omega_{\rm S}}^{1/2} \\
&\le\; \Big(\|(\ddot{\boldsymbol{\sigma}} - \ddot{\boldsymbol{\sigma}}_h^*, \ddot{p} - \ddot{p}_h^*)(t)\|_{0,\mathcal{C}} + \sqrt{2\mu}\,\|\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h\ddot{\boldsymbol{r}}(t)\|_{0,\Omega_{\rm S}}\Big) \|(\dot{\boldsymbol{e}}_{\sigma,h}, \dot{e}_{p,h})(t)\|_{0,\mathcal{C}}
\end{aligned}$$

for all $(\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\rm sym}$, where we used that $\mathcal{C}\boldsymbol{s} = 2\mu\boldsymbol{s}$ for all $\boldsymbol{s} \in [\mathrm{L}^2(\Omega_{\rm S})]^{n\times n}_{\rm skew}$. Hence, by virtue of the estimate (2.1.14), there exists $C_2 > 0$, independent of h, λ and T, such that

$$\frac{\dot{\mathcal{E}}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)(t)}{2\sqrt{\mathcal{E}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)(t)}} \;\le\; C_2\left\{\|(\ddot{\boldsymbol{\sigma}} - \ddot{\boldsymbol{\sigma}}_h^*, \ddot{p} - \ddot{p}_h^*)(t)\|_0 + \|\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h\ddot{\boldsymbol{r}}(t)\|_{0,\Omega_{\rm S}}\right\},$$

and integrating in time yields

$$\max_{[0,T]} \mathcal{E}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)^{1/2}(t) \le C_2 \int_0^T \Big\{\|(\ddot{\boldsymbol{\sigma}}_h^* - \ddot{\boldsymbol{\sigma}}, \ddot{p}_h - \ddot{p})(t)\|_0 + \|\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h\ddot{\boldsymbol{r}}(t)\|_{0,\Omega_{\rm S}}\Big\}\,\mathrm{d}t. \tag{2.4.9}$$

On the other hand, according to the definition of $\mathcal{E}$ (cf. (2.1.19)), it holds

$$\max_{[0,T]} \mathcal{E}\big(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h)\big)^{1/2}(t) \;\le\; C_3\left\{\max_{[0,T]} \mathcal{E}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)^{1/2}(t) + \max_{[0,T]} \mathcal{E}\big(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*, p - p_h^*)\big)^{1/2}(t)\right\}, \tag{2.4.10}$$

and taking into account (2.1.15) and using fundamental theorem of calculus, we deduce that

$$\begin{aligned}
&\max_{[0,T]} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h)(t)\| + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_h, \dot{p} - \dot{p}_h)(t)\|_{0,\mathcal{C}} \\
&\quad \le C_4\, T\left\{\max_{[0,T]} \mathcal{E}\big((\boldsymbol{e}_{\sigma,h}, e_{p,h})\big)^{1/2}(t) + \max_{[0,T]} \mathcal{E}\big(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*, p - p_h^*)\big)^{1/2}(t)\right\}.
\end{aligned} \tag{2.4.11}$$

Then, combining (2.4.9) and (2.4.11) gives

$$\begin{aligned}
&\max_{[0,T]} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h)(t)\| + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_h, \dot{p} - \dot{p}_h)(t)\|_{0,\mathcal{C}} \\
&\quad \leq C_5\, T \left\{ \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*, p - p_h^*)\|_{\mathrm{W}^{2,\infty}(\mathbb{X})} + \max_{[0,T]} \|\ddot{\boldsymbol{r}}(t) - \mathbf{R}_h \ddot{\boldsymbol{r}}(t)\|_{0,\Omega_{\mathrm{S}}} \right\},
\end{aligned} \tag{2.4.12}$$

where C_3, C_4, and hence $C_5 > 0$, are all constants independent of h, λ and T. Finally, it is readily seen that the required estimate (2.4.1) follows from (2.4.4) and (2.4.12). □

As a straightforward consequence of Lemma 2.4.1 we have the following theorem establishing the rates of convergence of our semi-discrete scheme (2.2.11).

Theorem 2.4.1. *Assume that the solution* $(\boldsymbol{u}, p)$ *to* (2.1.30)-(2.1.35) *satisfies the regularity assumptions* $(\boldsymbol{u}, p) \in \mathcal{C}^2\big((0,T); \mathrm{H}^{1+s}(\Omega_{\mathrm{S}})^n \times \mathrm{H}^{1+s}(\Omega_{\mathrm{F}})\big)$ *and* $\mathbf{div}\, \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{C}^2((0,T); \mathrm{H}^s(\Omega_{\mathrm{S}})^n)$*, for some* $s > 1/2$*. Then, there exists a constant* $C > 0$*, independent of* h *and* λ*, such that*

$$\begin{aligned}
&\max_{[0,T]} \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h)(t)\| + \max_{[0,T]} \|(\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_h, \dot{p} - \dot{p}_h)(t)\|_{0,\mathcal{C}} \;\leq\; C\, h^{\min\{k,s\}} \Bigg\{ \|\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}([\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n})} \\
&\quad + \quad \|\mathbf{div}\, \boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}^s(\Omega_{\mathrm{S}})^n)} + \|\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}([\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n})} + \|\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}(\mathrm{H}^s(\Omega_{\mathrm{S}})^n)} \\
&\quad + \quad \|p\|_{\mathrm{W}^{2,\infty}(\mathrm{H}^{1+s}(\Omega_{\mathrm{F}}))} + \Big(\sum_{e\in\Sigma_h} \|\boldsymbol{\psi}\|^2_{\mathrm{W}^{2,\infty}(\mathrm{H}^s(e)^n)}\Big)^{1/2} + \Big(\sum_{e\in\Sigma_h} \|p\|^2_{\mathrm{W}^{2,\infty}(\mathrm{H}^s(e))}\Big)^{1/2} \Bigg\}.
\end{aligned}$$

Proof. It is clear from the hypotheses that $\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{C}^2((0,T); [\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n})$, moreover with $\mathbf{div}\, \boldsymbol{\sigma} \in \mathcal{C}^2((0,T); \mathrm{H}^s(\Omega_{\mathrm{S}})^n)$, $\boldsymbol{r} = (\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^{\mathrm{t}})/2 \in \mathcal{C}^2((0,T); [\mathrm{H}^s(\Omega_{\mathrm{S}})]^{n\times n})$, $\boldsymbol{\psi} = \boldsymbol{u}|_\Sigma \in \mathrm{H}^{s+1/2}(\Sigma)^n$, and $p|_\Sigma \in \mathrm{H}^{s+1/2}(\Sigma)$. Hence, the result follows directly from (2.4.1) by using the approximation properties given by (2.2.3), (2.2.4) and (2.2.6)-(2.2.9). □

In addition to the above, and inspired from (2.1.2), we are also able to postprocess the displacement $\boldsymbol{u}$ and the velocity $\dot{\boldsymbol{u}}$. In particular, we propose the following explicit expression for the semi-discrete displacement field:

$$\boldsymbol{u}_h(t) = \int_0^t \left\{ \int_0^s \rho_{\mathrm{S}}^{-1} \Big(\mathbf{div}\, \boldsymbol{\sigma}_h(z) + \mathbf{U}_h \boldsymbol{f}(z) \Big)\, \mathrm{d}z \right\} \mathrm{d}s + \boldsymbol{u}_{0,h} + t\boldsymbol{u}_{1,h}\,, \tag{2.4.13}$$

where $\boldsymbol{u}_{0,h}$ and $\boldsymbol{u}_{1,h}$ are obtained by solving (2.3.21) with data $(\mathbf{div}\, \boldsymbol{\sigma}_0, p_0)$ and $(\mathbf{div}\, \boldsymbol{\sigma}_1, p_1)$, respectively. It is then clear that, under the regularity conditions of Theorem 2.4.1, there holds

$$\max_{[0,T]} \|(\boldsymbol{u} - \boldsymbol{u}_h)(t)\|_{0,\Omega_{\mathrm{S}}} = O(h^{\min\{k,s\}})\,.$$

We end this paper by remarking that in a forthcoming chapter we show that the analysis given in [21, Section 6] for the dual-mixed formulation of the elastodynamic equations, can be adapted to deal with the time discretization, based on the Newmark trapezoidal rule, of our present problem (2.2.11).

Chapter 3

A fully discrete scheme for the pressure-stress formulation of the time-domain fluid-structure interaction problem

3.1 Stress-pressure variational formulation of the model problem

We consider a solid body represented by a connected polyhedral Lipschitz domain Ω_S whose boundary is given by two connected components Σ and Γ. The cavity Ω_F delimited by the inner boundary Σ is filled with an homogeneous, inviscid and compressible fluid (see Figure 3.1.1). Our objective is to compute the linear oscillations that take place in the fluid-solid domain $\Omega := \Omega_S \cup \Sigma \cup \Omega_F$ with $\Omega \subset \mathbb{R}^d (d = 2, 3)$, under the action of a given loading $\boldsymbol{f} : (0, T] \times \Omega_S \to \mathbb{R}^d$ prescribed in the solid domain. We assume that the solid is fixed at a nonempty part Γ_D of the external boundary $\Gamma := \partial\Omega$ and impose a traction-free boundary condition on its complement $\Gamma_N := \Gamma \setminus \Gamma_D$. We denote $\boldsymbol{n}$ the outward unit normal vector to $\Gamma \cup \Sigma$ and select on Σ the orientation that points outward to Ω_F. More precisely, the mathematical model associated to the physical phenomenon under interest is given by the set of equations

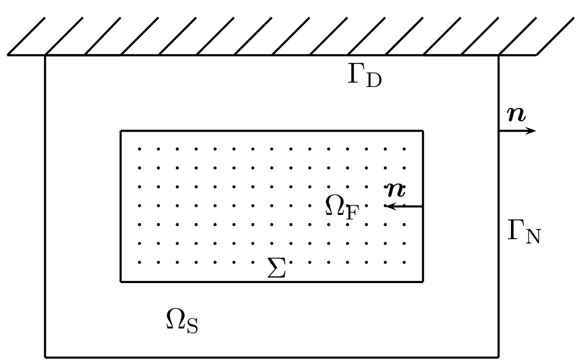

Figure 3.1.1: Fluid and solid domains

$$
\begin{aligned}
\rho_{\mathrm{S}}\ddot{\boldsymbol{u}} - \mathbf{div}\,\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) &= \boldsymbol{f} && \text{in } \Omega_{\mathrm{S}} \times (0,T], && (3.1.1)\\
c^{-2}\ddot{p} - \Delta p &= 0 && \text{in } \Omega_{\mathrm{F}} \times (0,T], && (3.1.2)\\
\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n} + p\boldsymbol{n} &= \mathbf{0} && \text{on } \Sigma \times (0,T], && (3.1.3)\\
\frac{\partial p}{\partial \boldsymbol{n}} + \rho_{\mathrm{F}}\ddot{\boldsymbol{u}}\cdot\boldsymbol{n} &= 0 && \text{on } \Sigma \times (0,T], && (3.1.4)\\
\boldsymbol{u} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{D}} \times (0,T], && (3.1.5)\\
\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{N}} \times (0,T], && (3.1.6)
\end{aligned}
$$

with the corresponding initial conditions. Here, p is the fluid pressure, $\mathcal{C}: \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is the Hooke operator given by

$$\mathcal{C}\boldsymbol{\tau} := \lambda\,(\mathrm{tr}\,\boldsymbol{\tau})\,\boldsymbol{I} + 2\mu\boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{R}^{d\times d},$$

$\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor, which is given, in terms of the solid displacement field $\boldsymbol{u}$, by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\Big\{\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\},$$

$\rho_{\mathrm{S}} > 0$ is the density of the solid, $\lambda > 0$ and $\mu > 0$ are its Lamé coefficients, $c > 0$ is the acoustic speed in the fluid, and $\rho_{\mathrm{F}} > 0$ is its density.

The stress tensor $\boldsymbol{\sigma} := \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u})$, which is imposed here as a primary unknown in the solid, is sought in the Sobolev space

$$\boldsymbol{\mathcal{W}} := \Big\{\boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}, \Omega_{\mathrm{S}}); \quad \boldsymbol{\tau}\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{N}}\Big\},$$

while the pressure p belongs to $\mathrm{H}^1(\Omega_{\mathrm{F}})$. These two variables are linked through equation (3.1.3), which can be interpreted as an implicitly prescribed normal stress on the contact boundary Σ. As we are dealing with a dual formulation in Ω_{S}, this transmission condition becomes essential, and hence we could impose it weakly through a suitable Lagrange multiplier (as we did in [24]), or alternatively, we could incorporate it into the continuous space. Here, we follow [25] and choose the second option by defining the global space

$$\mathbb{X} := \Big\{(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{W}} \times \mathrm{H}^1(\Omega_{\mathrm{F}}); \quad \boldsymbol{\tau}\boldsymbol{n} + q\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Sigma\Big\},$$

which is endowed with the Hilbertian norm $\|(\boldsymbol{\tau}, q)\|^2 := \|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})} + \|q\|^2_{1,\Omega_{\mathrm{F}}}$.

We still have to impose a further restriction in $\mathbb{X}$. Indeed, it is essential to take into account the conservation of the angular momentum, which is characterized by the symmetry of the stress tensor. This induces us to consider the closed subspace

$$\mathbb{X}^{\mathrm{sym}} := \big\{(\boldsymbol{\tau}, q) \in \mathbb{X}; \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}}\big\}.$$

We point out that, stable mixed finite elements for the linear elastostatic problem have been arduous to construct because of this symmetry restriction (cf. [1, 3, 4, 6, 9, 13]). One of the prevailing techniques [1, 4, 9, 13] used to deal with this difficulty consists in imposing weakly the symmetry through the introduction of a Lagrange multiplier, which turns out to be equal to the rotation $\boldsymbol{r} := \frac{1}{2}\big\{\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\big\}$. Recently, this mixed finite element strategy with reduced symmetry has been successfully applied to the elasticity eigenproblem [35], to the indefinite elasticity problem [33], to elastodynamics [5, 21], and to time-domain fluid-structure interaction problems [22]. It is important to bear in mind that, in what follows, there will be an underlying Lagrange multiplier (corresponding to the symmetry restriction) that we have chosen to hide for economy in notations. We refer to [21] (or its preliminary summarized version [20]) for a similar analysis for the elastodynamics in which the rotation variable is maintained as an active unknown.

We now notice that $\mathbb{X}^{\text{sym}}$ is dense in the space

$$\mathbb{H}^{\text{sym}} := [\mathrm{L}^2(\Omega_{\mathrm{S}})]^{d\times d}_{\text{sym}} \times \mathrm{L}^2(\Omega_{\mathrm{F}})$$

endowed with the norm $\|(\boldsymbol{\tau}, q)\|_0^2 := \|\boldsymbol{\tau}\|^2_{0,\Omega_{\mathrm{S}}} + \|q\|^2_{0,\Omega_{\mathrm{F}}}$. This allows us to pose the stress-pressure variational formulation of the fluid-solid interaction problem in the following terms (see [22, eq.(3.11)] for details):

$$\begin{aligned}
&\text{Find } (\boldsymbol{\sigma}, p) \in \mathrm{L}^\infty((0,T); \mathbb{X}^{\text{sym}}) \cap \mathrm{W}^{1,\infty}((0,T); \mathbb{H}^{\text{sym}}) \text{ such that}\\
&\Big((\ddot{\boldsymbol{\sigma}}, \ddot{p})(t), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} + A\Big((\boldsymbol{\sigma}, p)(t), (\boldsymbol{\tau}, q)\Big) = -\rho_{\mathrm{S}}^{-1}\big(\boldsymbol{f}(t), \mathbf{div}\,\boldsymbol{\tau}\big)_{0,\Omega_{\mathrm{S}}} \quad \forall (\boldsymbol{\tau}, q) \in \mathbb{X}^{\text{sym}},\\
&\big(\boldsymbol{\sigma}(0), p(0)\big) = (\boldsymbol{\sigma}_0, p_0), \qquad \big(\dot{\boldsymbol{\sigma}}(0), \dot{p}(0)\big) = (\boldsymbol{\sigma}_1, p_1),
\end{aligned} \tag{3.1.7}$$

where

$$\Big((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} := (\mathcal{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + \frac{1}{\rho_{\mathrm{F}} c^2}(p, q)_{0,\Omega_{\mathrm{F}}}$$

and

$$A\Big((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)\Big) := \rho_{\mathrm{S}}^{-1}(\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + \rho_{\mathrm{F}}^{-1}(\nabla p, \nabla q)_{0,\Omega_{\mathrm{F}}}.$$

Here, $\boldsymbol{f} \in \mathrm{L}^1((0,T); \mathrm{L}^2(\Omega_{\mathrm{S}})^d)$ is a given body force in Ω_{S} and $(\boldsymbol{\sigma}_0, p_0) \in \mathbb{X}^{\text{sym}}$ and $(\boldsymbol{\sigma}_1, p_1) \in \mathbb{H}^{\text{sym}}$ are prescribed initial data.

The well-posedness of problem (3.1.7) is established as follows (see 2.1.1).

Theorem 3.1.1. *Assume that $\boldsymbol{f} \in \mathrm{W}^{1,1}((0,T); \mathrm{L}^2(\Omega_{\mathrm{S}})^d)$. Then, problem (3.1.7) admits a unique solution $(\boldsymbol{\sigma}, p) \in \mathcal{C}^0((0,T); \mathbb{X}^{sym}) \cap \mathcal{C}^1((0,T); \mathbb{H}^{sym})$. Moreover, there exists a constant $C > 0$, independent of λ and T, such that*

$$\operatorname*{ess\,sup}_{[0,T]} \|(\boldsymbol{\sigma}, p)(t)\| + \operatorname*{ess\,sup}_{[0,T]} \|(\dot{\boldsymbol{\sigma}}, \dot{p})(t)\|_{0,\mathcal{C}} \le C\,T \Big\{ \|\boldsymbol{f}\|_{\mathrm{W}^{1,1}(\boldsymbol{L}^2(\Omega_{\mathrm{S}}))} + \|(\boldsymbol{\sigma}_0, p_0)\| + \|(\boldsymbol{\sigma}_1, p_1)\|_0 \Big\}.$$

Moreover, the projector Ξ and its discrete counterpart Ξ_h (introduced in 2.3)are the key tools in the convergence analysis that we will undertake in the following section. They are characterized by the following properties.

Lemma 3.1.1. *There exist a linear operators $\Xi : \mathbb{X} \to \mathbb{X}^{sym}$ and $\Xi_h : \mathbb{X} \to \mathbb{X}_h^{sym}$ such that*

$$\|\Xi(\boldsymbol{\tau}, q)\| + \|\Xi_h(\boldsymbol{\tau}, q)\| \le C\,\|(\boldsymbol{\tau}, q)\| \qquad \forall\,(\boldsymbol{\tau}, q) \in \mathbb{X},$$

with $C > 0$, independent of λ and h. Moreover, $\widetilde{\Xi} := \Xi|_{\mathbb{X}^{sym}}$ is the $(\cdot,\cdot)_{\mathcal{C}}$-orthogonal projection of $\mathbb{X}^{sym}$ onto $\mathbb{K}^\perp$.

Proof. See [22, Section 5]. □

Lemma 3.1.2. *Assume that* $(\boldsymbol{\tau}, q) \in \mathbb{K}^\perp$ *with* $\boldsymbol{\tau} \in [\mathrm{H}^s(\Omega_\mathrm{S})]^{d\times d}$ *for some* $s > 0$*, and let* $(\boldsymbol{v}, \boldsymbol{s}) := \mathbf{D}(\boldsymbol{\tau}, q)$ *and* $\boldsymbol{\psi} := \boldsymbol{v}|_\Sigma$*. Then, there exists a constant* $C > 0$*, independent of* h *and* λ*, such that*

$$\begin{aligned}\|(\boldsymbol{\tau}, q) - \Xi_h(\boldsymbol{\tau}, q)\| \,\leq\, C \Big\{ & \|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega_\mathrm{S})} + \|q\boldsymbol{n} - \boldsymbol{\pi}_h(q\boldsymbol{n})\|_{0,\Sigma} + \|\boldsymbol{s} - \boldsymbol{R}_h \boldsymbol{s}\|_{0,\Omega_\mathrm{S}} \\ + \quad & \|\boldsymbol{v} - U_h \boldsymbol{v}\|_{0,\Omega_\mathrm{S}} + \|\boldsymbol{\psi} - \boldsymbol{\pi}_h \boldsymbol{\psi}\|_{0,\Sigma} + |q - \Pi_h q|_{1,\Omega_\mathrm{F}} \Big\}.\end{aligned}$$

Proof. See [22, Lemma 5.8]. □

3.2 Time-space discretization

3.2.1 The fully discrete scheme

Given $L \in \mathbb{N}$, we consider a uniform partition of the time interval $[0, T]$ with step size $\Delta t := T/L$. Then, for any continuous function $\phi : [0, T] \to \mathbb{R}$ and for each $k \in \{0, 1, \ldots, L\}$ we denote $\phi^k := \phi(t_k)$, where $t_k := k\,\Delta t$. In addition, we adopt the same notation for vector/tensor valued functions and introduce the notations

$$t_{k+\frac{1}{2}} := \frac{t_{k+1} + t_k}{2}, \quad \phi^{k+\frac{1}{2}} := \frac{\phi^{k+1} + \phi^k}{2}, \quad \phi^{k-\frac{1}{2}} := \frac{\phi^k + \phi^{k-1}}{2},$$

and the discrete time derivatives

$$\partial_t \phi^k := \frac{\phi^{k+1} - \phi^k}{\Delta t} \quad \text{and} \quad \bar{\partial}_t \phi^k := \frac{\phi^k - \phi^{k-1}}{\Delta t},$$

from which we notice that

$$\partial_t \bar{\partial}_t \phi^k = \frac{\bar{\partial}_t \phi^{k+1} - \bar{\partial}_t \phi^k}{\Delta t} = \frac{\partial_t \phi^k - \partial_t \phi^{k-1}}{\Delta t} = \frac{\phi^{k+1} - 2\phi^k + \phi^{k-1}}{\Delta t^2}.$$

The Newmark trapezoidal rule applied to the Galerkin space-semidiscretization introduced in [22] for problem (3.1.7) reads as follows: For $k = 1, \ldots, L-1$, find $(\boldsymbol{\sigma}_h^{k+1}, p_h^{k+1}) \in \mathbb{X}_h^{\mathrm{sym}}$ such that

$$\begin{aligned}\Big(\partial_t \bar{\partial}_t (\boldsymbol{\sigma}_h^k, p_h^k), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} \,+\, A\Big(\big(\frac{\boldsymbol{\sigma}_h^{k+\frac{1}{2}} + \boldsymbol{\sigma}_h^{k-\frac{1}{2}}}{2}, \frac{p_h^{k+\frac{1}{2}} + p_h^{k-\frac{1}{2}}}{2}\big), (\boldsymbol{\tau}, q)\Big) \\ = \,-\, \rho_\mathrm{S}^{-1} \big(\boldsymbol{f}(t_k), \mathbf{div}\, \boldsymbol{\tau}\big)_{0,\Omega_\mathrm{S}} \qquad \forall\, (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\mathrm{sym}}.\end{aligned} \tag{3.2.1}$$

Moreover, for the sake of simplicity, we assume that the scheme (3.2.1) is started up with

$$(\boldsymbol{\sigma}_h^0, p_h^0) := \Xi_h(\boldsymbol{\sigma}_0, p_0) \quad \text{and} \quad (\boldsymbol{\sigma}_h^1, p_h^1) := \Xi_h(\boldsymbol{\sigma}(t_1), p(t_1)). \tag{3.2.2}$$

We insist here upon the fact that it is necessary to introduce a Lagrange multiplier in order to relax the weak symmetry constraint defining $\boldsymbol{\mathcal{W}}_h^{\mathrm{sym}}$. This permits one to deal with the well-known BDM-finite element basis functions of the space $\boldsymbol{\mathcal{W}}_h$ in order to obtain the linear systems of equations arising from (3.2.1) at each iteration step.

Now, recalling that $(\boldsymbol{\sigma}, p)$ stands for the solution of (3.1.7), we introduce the discrete errors

$$\boldsymbol{e}_{\sigma,h}^k := \boldsymbol{\sigma}_h^*(t_k) - \boldsymbol{\sigma}_h^k \in \boldsymbol{\mathcal{W}}_h^{\mathrm{sym}}, \quad \text{and} \qquad e_{p,h}^k := p_h^*(t_k) - p_h^k \in V_h,$$

where, as in [22], we define $(\boldsymbol{\sigma}_h^*, p_h^*) := \Xi_h(\boldsymbol{\sigma}, p)$, and observe that $(\boldsymbol{e}_{\sigma,h}^k, e_{p,h}^k) \in \mathbb{X}_h^{\rm sym}$. Then, thanks to (3.2.2), we have $\boldsymbol{e}_{\sigma,h}^0 = \boldsymbol{e}_{\sigma,h}^1 = \mathbf{0}$ and $e_{p,h}^0 = e_{p,h}^1 = 0$. In turn, the starting point of our convergence analysis is the following error equation

$$\begin{aligned}\Big(\partial_t\bar{\partial}_t(\boldsymbol{e}_{\sigma,h}^k, e_{p,h}^k), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} &+ A\Big(\big(\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} + \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{2}, \frac{e_{p,h}^{k+\frac{1}{2}} + e_{p,h}^{k-\frac{1}{2}}}{2}\big), (\boldsymbol{\tau}, q)\Big) \\ = \quad \Big((\boldsymbol{\chi}_{1,\sigma}^k, \boldsymbol{\chi}_{1,p}^k), (\boldsymbol{\tau}, q)\Big)_{\mathcal{C}} &+ A\Big((\boldsymbol{\chi}_{2,\sigma}^k, \boldsymbol{\chi}_{2,p}^k), (\boldsymbol{\tau}, q)\Big) \qquad \forall\, (\boldsymbol{\tau}, q) \in \mathbb{X}_h^{\rm sym}\,,\end{aligned} \tag{3.2.3}$$

where the consistency terms are, for $\xi \in \{\boldsymbol{\sigma}, p\}$,

$$\boldsymbol{\chi}_{1,\xi}^k := \partial_t\bar{\partial}_t\xi_h^*(t_k) - \ddot{\xi}(t_k) \quad \text{and} \quad \boldsymbol{\chi}_{2,\xi}^k := \frac{\xi_h^*(t_{k+1}) + 2\xi_h^*(t_k) + \xi_h^*(t_{k-1})}{4} - \xi(t_k)\,.$$

By definition of $(\boldsymbol{\sigma}_h^*, p_h^*)$, we have that

$$(\mathbf{div}(\boldsymbol{\sigma}_h^*(t_k) - \boldsymbol{\sigma}(t_k)), \mathbf{div}\,\boldsymbol{\tau})_{0,\Omega_{\rm S}} \;=\; 0 \qquad \forall\, (\boldsymbol{\tau}, q) \in \mathbb{X}_h\,,$$

and

$$(\nabla(p_h^*(t_k) - p(t_k)), \nabla q)_{0,\Omega_{\rm F}} \;=\; (\nabla(p(t_k) - \Pi_h p(t_k)), \nabla q)_{0,\Omega_{\rm F}} \;=\; 0 \qquad \forall\, (\boldsymbol{\tau}, q) \in \mathbb{X}_h\,.$$

Hence, we can substitute in the right hand side of (3.2.3) the functions $\boldsymbol{\chi}_{2,\sigma}^k$ and $\boldsymbol{\chi}_{2,p}^k$ by

$$\bar{\boldsymbol{\chi}}_{2,\xi}^k \;:=\; \boldsymbol{\chi}_{2,\xi}^k \;-\; (\xi_h^*(t_k) - \xi(t_k)) \;=\; \frac{\xi_h^*(t_{k+1}) - 2\xi_h^*(t_k) + \xi_h^*(t_{k-1})}{4} \qquad \forall\, \xi \in \{\boldsymbol{\sigma}, p\}$$

without altering the error equation.

3.2.2 Convergence analysis

We begin by establishing the following stability result.

Lemma 3.2.1. *There exists a constant $C > 0$, independent of λ, h and Δt, such that for each $n \in \mathbb{N}$ there holds*

$$\begin{aligned}\max_n \big\|(\partial_t\boldsymbol{e}_{\sigma,h}^n, \partial_t e_{p,h}^n)\big\|_{0,\mathcal{C}} &+ \max_n \Big\|\mathbf{div}\,\boldsymbol{e}_{\sigma,h}^{n+\frac{1}{2}}\Big\|_{0,\Omega_{\rm S}} + \max_n \Big\|\nabla e_{p,h}^{n+\frac{1}{2}}\Big\|_{0,\Omega_{\rm F}} \\ \leq\; C\Big\{ \max_n \big\|(\boldsymbol{\chi}_{1,\sigma}^n, \boldsymbol{\chi}_{1,p}^n)\big\|_{0,\mathcal{C}} &+ \max_n \big\|\mathbf{div}\,\partial_t\bar{\boldsymbol{\chi}}_{2,\sigma}^n\big\|_{0,\Omega_{\rm S}} + \max_n \big\|\nabla\partial_t\bar{\boldsymbol{\chi}}_{2,p}^n\big\|_{0,\Omega_{\rm F}} \\ + \max_n \big\|\mathbf{div}\,\bar{\boldsymbol{\chi}}_{2,\sigma}^n\big\|_{0,\Omega_{\rm S}} &+ \max_n \big\|\nabla\bar{\boldsymbol{\chi}}_{2,p}^n\big\|_{0,\Omega_{\rm F}} \Big\}\,.\end{aligned} \tag{3.2.4}$$

Proof. Taking $(\boldsymbol{\tau}, q) = (\frac{\boldsymbol{e}_{\sigma,h}^{k+1} - \boldsymbol{e}_{\sigma,h}^{k-1}}{2\Delta t}, \frac{e_{p,h}^{k+1} - e_{p,h}^{k-1}}{2\Delta t})$ in (3.2.3) and using

$$\frac{\boldsymbol{e}_{\sigma,h}^{k+1} - \boldsymbol{e}_{\sigma,h}^{k-1}}{2\Delta t} = \frac{\boldsymbol{e}_{\sigma,h}^{k+\frac{1}{2}} - \boldsymbol{e}_{\sigma,h}^{k-\frac{1}{2}}}{\Delta t} = \frac{\partial_t\boldsymbol{e}_{\sigma,h}^k + \partial_t\boldsymbol{e}_{\sigma,h}^{k-1}}{2}\,,$$

and the similar identity for $\dfrac{e_{p,h}^{k+1} - e_{p,h}^{k-1}}{2\Delta t}$, we find that

$$
\begin{aligned}
&\frac{1}{2\Delta t}\Big((\partial_t \boldsymbol{e}_{\sigma,h}^k - \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}, \partial_t e_{p,h}^k - \partial_t e_{p,h}^{k-1}), (\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}, \partial_t e_{p,h}^k + \partial_t e_{p,h}^{k-1})\Big)_{0,\mathcal{C}} \\
+ \quad & \frac{1}{2\Delta t} A\Big((\boldsymbol{e}_{\sigma,h}^{k+\frac12} + \boldsymbol{e}_{\sigma,h}^{k-\frac12}, e_{p,h}^{k+\frac12} + e_{p,h}^{k-\frac12}), (\boldsymbol{e}_{\sigma,h}^{k+\frac12} - \boldsymbol{e}_{\sigma,h}^{k-\frac12}, e_{p,h}^{k+\frac12} - e_{p,h}^{k-\frac12})\Big) \\
= \quad & \Big((\boldsymbol{\chi}_{1,\sigma}^k, \boldsymbol{\chi}_{1,p}^k), (\frac{\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}, \frac{\partial_t e_{p,h}^k + \partial_t e_{p,h}^{k-1}}{2})\Big)_{0,\mathcal{C}} \\
+ \quad & A\Big((\bar{\boldsymbol{\chi}}_{2,\sigma}^k, \bar{\boldsymbol{\chi}}_{2,p}^k), (\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac12} - \boldsymbol{e}_{\sigma,h}^{k-\frac12}}{\Delta t}, \frac{e_{p,h}^{k+\frac12} - e_{p,h}^{k-\frac12}}{\Delta t})\Big),
\end{aligned}
$$

which can also be written as

$$
\begin{aligned}
&\frac{1}{2\Delta t}\Big(\Big\|(\partial_t \boldsymbol{e}_{\sigma,h}^k, \partial_t e_{p,h}^k)\Big\|_{0,\mathcal{C}}^2 - \Big\|(\partial_t \boldsymbol{e}_{\sigma,h}^{k-1}, \partial_t e_{p,h}^{k-1})\Big\|_{0,\mathcal{C}}^2 \Big) \\
+ \quad & \frac{1}{2\Delta t}\Big\{ A\Big((\boldsymbol{e}_{\sigma,h}^{k+\frac12}, e_{p,h}^{k+\frac12}), (\boldsymbol{e}_{\sigma,h}^{k+\frac12}, e_{p,h}^{k+\frac12})\Big) - A\Big((\boldsymbol{e}_{\sigma,h}^{k-\frac12}, e_{p,h}^{k-\frac12}), (\boldsymbol{e}_{\sigma,h}^{k-\frac12}, e_{p,h}^{k-\frac12})\Big)\Big\} \\
= \quad & \Big((\boldsymbol{\chi}_{1,\sigma}^k, \boldsymbol{\chi}_{1,p}^k), (\frac{\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}, \frac{\partial_t e_{p,h}^k + \partial_t e_{p,h}^{k-1}}{2})\Big)_{0,\mathcal{C}} \\
+ \quad & A\Big((\bar{\boldsymbol{\chi}}_{2,\sigma}^k, \bar{\boldsymbol{\chi}}_{2,p}^k), (\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac12} - \boldsymbol{e}_{\sigma,h}^{k-\frac12}}{\Delta t}, \frac{e_{p,h}^{k+\frac12} - e_{p,h}^{k-\frac12}}{\Delta t})\Big).
\end{aligned}
$$

In this way, multiplying by $2\Delta t$ and summing up the foregoing identity over $k = 1, \ldots, n$, gives

$$
\begin{aligned}
&\big\|(\partial_t \boldsymbol{e}_{\sigma,h}^n, \partial_t e_{p,h}^n)\big\|_{0,\mathcal{C}}^2 + A\Big((\boldsymbol{e}_{\sigma,h}^{n+\frac12}, e_{p,h}^{n+\frac12}), (\boldsymbol{e}_{\sigma,h}^{n+\frac12}, e_{p,h}^{n+\frac12})\Big) \\
= \quad & 2\Delta t \sum_{k=1}^{n} \Big((\boldsymbol{\chi}_{1,\sigma}^k, \boldsymbol{\chi}_{1,p}^k), (\frac{\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}, \frac{\partial_t e_{p,h}^k + \partial_t e_{p,h}^{k-1}}{2})\Big)_{0,\mathcal{C}} \\
+ \quad & 2\Delta t \sum_{k=1}^{n} A\Big((\bar{\boldsymbol{\chi}}_{2,\sigma}^k, \bar{\boldsymbol{\chi}}_{2,p}^k), (\frac{\boldsymbol{e}_{\sigma,h}^{k+\frac12} - \boldsymbol{e}_{\sigma,h}^{k-\frac12}}{\Delta t}, \frac{e_{p,h}^{k+\frac12} - e_{p,h}^{k-\frac12}}{\Delta t})\Big) \\
= \quad & 2\Delta t \sum_{k=1}^{n} \Big((\boldsymbol{\chi}_{1,\sigma}^k, \boldsymbol{\chi}_{1,p}^k), (\frac{\partial_t \boldsymbol{e}_{\sigma,h}^k + \partial_t \boldsymbol{e}_{\sigma,h}^{k-1}}{2}, \frac{\partial_t e_{p,h}^k + \partial_t e_{p,h}^{k-1}}{2})\Big)_{0,\mathcal{C}} \\
- \quad & 2\,\Delta t \sum_{k=1}^{n-1} A\Big((\partial_t \bar{\boldsymbol{\chi}}_{2,\sigma}^k, \partial_t \bar{\boldsymbol{\chi}}_{2,p}^k), (\boldsymbol{e}_{\sigma,h}^{k+\frac12}, e_{p,h}^{k+\frac12})\Big) + 2A\Big((\bar{\boldsymbol{\chi}}_{2,\sigma}^n, \bar{\boldsymbol{\chi}}_{2,p}^n), (\boldsymbol{e}_{\sigma,h}^{n+\frac12}, e_{p,h}^{n+\frac12})\Big).
\end{aligned}
$$

It is now straightforward to deduce from the last identity and the Cauchy-Schwarz inequality, that there exists a constant $C_0 > 0$, independent of λ, h and Δt, such that

$$\max_n \big\|(\partial_t \boldsymbol{e}^n_{\sigma,h}, \partial_t e^n_{p,h}\big\|_{0,\mathcal{C}} \,+\, \max_n \sqrt{A\Big((\boldsymbol{e}^{n+\frac{1}{2}}_{\sigma,h}, e^{n+\frac{1}{2}}_{p,h}), (\boldsymbol{e}^{n+\frac{1}{2}}_{\sigma,h}, e^{n+\frac{1}{2}}_{p,h})\Big)}$$

$$\leq \quad C_0 \Big\{ \Delta t \sum_{k=1}^{L} \Big\|(\boldsymbol{\chi}^k_{1,\sigma}, \boldsymbol{\chi}^k_{1,p})\Big\|_{0,\mathcal{C}} \,+\, \Delta t \sum_{k=1}^{L-1} \sqrt{A\Big((\partial_t \bar{\boldsymbol{\chi}}^k_{2,\sigma}, \partial_t \bar{\boldsymbol{\chi}}^k_{2,p}), (\partial_t \bar{\boldsymbol{\chi}}^k_{2,\sigma}, \partial_t \bar{\boldsymbol{\chi}}^k_{2,p})\Big)}$$

$$+ \quad \max_n \sqrt{A\Big((\bar{\boldsymbol{\chi}}^n_{2,\sigma}, \bar{\boldsymbol{\chi}}^n_{2,p}), (\bar{\boldsymbol{\chi}}^n_{2,\sigma}, \bar{\boldsymbol{\chi}}^n_{2,p})\Big)} \Big\},$$

and the result follows from the lower bound of (2.1.1). □

We now aim to bound the expression

$$\mathcal{M}_h(\boldsymbol{\sigma}, p) \,:=\, \max_n \Big\|(\dot{\boldsymbol{\sigma}}, \dot{p})(t_{n+\frac{1}{2}}) - (\partial_t \boldsymbol{\sigma}^n_h, \partial_t p^n_h)\Big\|_{0,\mathcal{C}} \,+\, \max_n \Big\|\mathbf{div}\, \boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \mathbf{div}\, \boldsymbol{\sigma}^{n+\frac{1}{2}}_h\Big\|_{0,\Omega_{\mathrm{S}}}$$
$$+ \quad \max_n \Big\|\nabla\big(p(t_{n+\frac{1}{2}}) - p^{n+\frac{1}{2}}_h\big)\Big\|_{0,\Omega_{\mathrm{F}}}$$

To this end, we first observe thanks to the triangle inequality and the stability estimate (3.2.4) that

$$\mathcal{M}_h(\boldsymbol{\sigma}, p) \,\leq\, \widetilde{\mathcal{M}}_h(\boldsymbol{\sigma}, p) \,+\, C\, \widehat{\mathcal{M}}_h(\boldsymbol{\sigma}, p)\,, \tag{3.2.5}$$

where

$$\widetilde{\mathcal{M}}_h(\boldsymbol{\sigma}, p) \,:=\, \max_n \Big\|(\dot{\boldsymbol{\sigma}}, \dot{p})(t_{n+\frac{1}{2}}) - (\partial_t \boldsymbol{\sigma}^*_h(t_n), \partial_t p^*_h(t_n))\Big\|_{0,\mathcal{C}}$$
$$+ \quad \max_n \big\|\,\mathbf{div}\, \big(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - (\boldsymbol{\sigma}^*_h)^{n+\frac{1}{2}}\big)\big\|_{0,\Omega_{\mathrm{S}}} \,+\, \max_n \Big\|\nabla\big(p(t_{n+\frac{1}{2}}) - (p^*_h)^{n+\frac{1}{2}}\big)\Big\|_{0,\Omega_{\mathrm{F}}}$$

and

$$\widehat{\mathcal{M}}_h(\boldsymbol{\sigma}, p) \,:=\, \max_n \big\|(\boldsymbol{\chi}^n_{1,\sigma}, \boldsymbol{\chi}^n_{1,p})\big\|_{0,\mathcal{C}} \,+\, \max_n \big\|\mathbf{div}\, \partial_t \bar{\boldsymbol{\chi}}^n_{2,\sigma}\big\|_{0,\Omega_{\mathrm{S}}}$$
$$+ \quad \max_n \big\|\nabla \partial_t \bar{\boldsymbol{\chi}}^n_{2,p}\big\|_{0,\Omega_{\mathrm{F}}} \,+\, \max_n \big\|\mathbf{div}\, \bar{\boldsymbol{\chi}}^n_{2,\sigma}\big\|_{0,\Omega_{\mathrm{S}}} \,+\, \max_n \big\|\nabla \bar{\boldsymbol{\chi}}^n_{2,p}\big\|_{0,\Omega_{\mathrm{F}}}\,.$$

The following two lemmas apply Taylor expansions with integral remainder to derive upper bounds for the terms on the right hand side of (3.2.5).

Lemma 3.2.2. *Assume that the solution* $(\boldsymbol{\sigma}, p) \in \mathcal{C}^0((0,T); \mathbb{X}^{sym}) \cap \mathcal{C}^1((0,T); \mathbb{H}^{sym})$ *to problem* (3.1.7) *satisfies* $\boldsymbol{\sigma} \in \mathcal{C}^2((0,T); \mathbb{H}(\mathbf{div}, \Omega_{\mathrm{S}}) \cap H^s(\Omega_{\mathrm{S}})^{d\times d}) \cap \mathcal{C}^3((0,T); \mathbb{H}(\mathbf{div}, \Omega_{\mathrm{S}}))$ *for some* $s > 0$ *and* $p \in \mathcal{C}^3(H^1(\Omega_{\mathrm{F}}))$*. Then, there exists a constant* $C > 0$*, independent of* λ*,* h *and* Δt*, such that*

$$\begin{aligned}\widetilde{\mathcal{M}}_h(\boldsymbol{\sigma}, p) \,&\leq\, C \Big\{ \,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^*_h\|_{\mathrm{W}^{1,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p - p^*_h\|_{\mathrm{W}^{1,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))} \\ &+ \quad (\Delta t)^2 \Big(\, \|\boldsymbol{\sigma}^*_h\|_{\mathrm{W}^{3,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p^*_h\|_{\mathrm{W}^{3,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))} \Big) \Big\}.\end{aligned} \tag{3.2.6}$$

Proof. Using Taylor expansions centered at $t = t_{n+\frac{1}{2}}$ gives for each $\xi \in \{\boldsymbol{\sigma}, p\}$,

$$\xi(t_{n+\frac{1}{2}}) - (\xi^*_h)^{n+\frac{1}{2}} = \xi(t_{n+\frac{1}{2}}) - \xi^*_h(t_{n+\frac{1}{2}}) \,-\, \frac{1}{2} \int_{t_n}^{t_{n+1}} \ddot{\xi}^*_h(t) \Big(\frac{\Delta t}{2} - |t - t_{n+\frac{1}{2}}|\Big)\, \mathrm{d}t \tag{3.2.7}$$

and

$$\dot{\xi}(t_{n+\frac{1}{2}}) - \partial_t \xi_h^*(t_n) \,=\, \dot{\xi}(t_{n+\frac{1}{2}}) - \dot{\xi}_h^*(t_{n+\frac{1}{2}}) \,-\, \frac{1}{2\Delta t}\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \frac{\mathrm{d}^3\xi_h^*(t)}{\mathrm{d}t^3}(t_{n+1}-t)^2\,\mathrm{d}t \\ -\, \frac{1}{2\Delta t}\int_{t_n}^{t_{n+\frac{1}{2}}} \frac{\mathrm{d}^3\xi_h^*(t)}{\mathrm{d}t^3}(t_n-t)^2\,\mathrm{d}t\,. \tag{3.2.8}$$

Then, it is not difficult to see that using (3.2.8) with $\xi = \boldsymbol{\sigma}$ and $\xi = p$, and then applying the space differential operators $\mathbf{div}$ and ∇ to $\xi = \boldsymbol{\sigma}$ and $\xi = p$, respectively, in (3.2.7), we arrive at (3.2.6). □

Lemma 3.2.3. *Assume that the solution* $(\boldsymbol{\sigma}, p) \in \mathcal{C}^0((0,T);\mathbb{X}^{sym}) \cap \mathcal{C}^1((0,T);\mathbb{H}^{sym})$ *to problem* (3.1.7) *satisfies* $\boldsymbol{\sigma} \in \mathcal{C}^2((0,T);\mathbb{H}(\mathbf{div},\Omega_{\mathrm{S}}) \cap H^s(\Omega_{\mathrm{S}})^{d\times d}) \cap \mathcal{C}^4((0,T);\mathbb{H}(\mathbf{div},\Omega_{\mathrm{S}}))$ *for some* $s > 0$ *and* $p \in \mathcal{C}^4(H^1(\Omega_{\mathrm{F}}))$. *Then, there exists a constant* $C > 0$, *independent of* λ, h *and* Δt, *such that*

$$\widehat{\mathcal{M}}_h(\boldsymbol{\sigma}, p) \,\leq\, C\Big\{\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} +\, \|p - p_h^*\|_{\mathrm{W}^{2,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))} \\ +\quad (\Delta t)^2\Big(\, \|\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{4,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p_h^*\|_{\mathrm{W}^{4,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))}\Big)\Big\}\,. \tag{3.2.9}$$

Proof. Using now Taylor expansions centered at $t = t_n$ we have for each $\xi \in \{\boldsymbol{\sigma}, p\}$,

$$\boldsymbol{\chi}_{1,\xi}^n = \ddot{\xi}_h^*(t_n) - \ddot{\xi}(t_n) + \,\frac{1}{6(\Delta t)^2}\int_{t_{n-1}}^{t_{n+1}} \frac{\mathrm{d}^4\xi_h^*(t)}{\mathrm{d}t^4}\,(\Delta t - |t-t_n|)^3\,\mathrm{d}t\,, \tag{3.2.10}$$

$$\bar{\boldsymbol{\chi}}_{2,\xi}^n = \frac{1}{4}\int_{t_{n-1}}^{t_{n+1}} \ddot{\xi}_h^*(t)(\Delta t - |t-t_n|)\,\mathrm{d}t\,, \tag{3.2.11}$$

and

$$\partial_t \bar{\boldsymbol{\chi}}_{2,\xi}^n = \frac{\xi_h^*(t_{n+2}) - 3\xi_h^*(t_{n+1}) + 3\xi_h^*(t_n) - \xi_h^*(t_{n-1})}{4\Delta t} \,=\, \frac{1}{8\Delta t}\Big\{\int_{t_n}^{t_{n+2}} \frac{\mathrm{d}^3\xi_h^*(t)}{\mathrm{d}t^3}(t_{n+2}-t)^2\,\mathrm{d}t \\ -3\,\int_{t_n}^{t_{n+1}} \frac{\mathrm{d}^3\xi_h^*(t)}{\mathrm{d}t^3}(t_{n+1}-t)^2\,\mathrm{d}t \,+\, \int_{t_{n-1}}^{t_n} \frac{\mathrm{d}^3\xi_h^*(t)}{\mathrm{d}t^3}(t_{n-1}-t)^2\,\mathrm{d}t\Big\}\,. \tag{3.2.12}$$

In this way, proceeding similarly as for the previous lemma, that is by applying now (3.2.10), (3.2.11) and (3.2.12), we obtain (3.2.9) Further details are omitted. □

As a consequence of Lemmas 3.2.2, 3.2.3, and 3.1.2, we are able to establish next the required bound for $\mathcal{M}_h(\boldsymbol{\sigma}, p)$.

Lemma 3.2.4. *Assume that the solution* $(\boldsymbol{\sigma}, p) \in \mathcal{C}^0((0,T);\mathbb{X}^{sym}) \cap \mathcal{C}^1((0,T);\mathbb{H}^{sym})$ *to problem* (3.1.7) *satisfies* $\boldsymbol{\sigma} \in \mathcal{C}^2((0,T);\mathbb{H}(\mathbf{div},\Omega_{\mathrm{S}}) \cap H^s(\Omega_{\mathrm{S}})^{d\times d}) \cap \mathcal{C}^4((0,T);\mathbb{H}(\mathbf{div},\Omega_{\mathrm{S}}))$ *for some* $s > 0$ *and* $p \in \mathcal{C}^4(H^1(\Omega_{\mathrm{F}}))$. *Then, there exists a constant* $C > 0$, *independent of* λ, h *and* Δt, *such that*

$$\mathcal{M}_h(\boldsymbol{\sigma}, p) \,\leq\, C\Big\{\, \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}((0,T);\mathbf{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p\boldsymbol{n} - \boldsymbol{\pi}_h(p\boldsymbol{n})\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Sigma)^d)} \\ +\quad \|\boldsymbol{r} - \boldsymbol{R}_h\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}((0,T);[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{d\times d}_{skew})} \,+\, \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Omega_{\mathrm{S}})^d)} \\ +\quad \|\boldsymbol{\psi} - \boldsymbol{\pi}_h\boldsymbol{\psi}\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Sigma)^d)} \,+\, \|\nabla(p - \Pi_h p)\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Omega_{\mathrm{F}})^d)} \\ +\quad (\Delta t)^2\Big(\, \|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p\|_{\mathrm{W}^{4,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))}\Big)\Big\}\,, \tag{3.2.13}$$

where $(\boldsymbol{u}, \boldsymbol{r}) = \mathbf{D}(\boldsymbol{\sigma}, p)$ *and* $\boldsymbol{\psi} = \boldsymbol{u}|_{\Sigma}$.

Proof. It follows straightforwardly from the initial estimate (3.2.5) and Lemmas 3.2.2 and 3.2.3 that

$$\begin{aligned}\mathcal{M}_h(\boldsymbol{\sigma},p) \,\leq\, C\Big\{ \,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p-p_h^*\|_{\mathrm{W}^{2,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))} \\ +\,(\Delta t)^2\Big(\,\|\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{4,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p_h^*\|_{\mathrm{W}^{4,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))}\Big)\Big\},\end{aligned} \tag{3.2.14}$$

On the other hand, the uniform boundedness of $\Xi_h : \mathbb{X} \to \mathbb{X}_h^{\mathrm{sym}}$ with respect to h and λ, and our regularity assumptions, imply that there exists a constant $C > 0$, independent of h and λ, such that

$$\begin{aligned}&\|\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{4,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p_h^*\|_{\mathrm{W}^{4,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))} \\ &\quad\leq\, C\Big\{\,\|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p\|_{\mathrm{W}^{4,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))}\,\Big\}.\end{aligned} \tag{3.2.15}$$

Finally, combining (3.2.14) and (3.2.15) we conclude that

$$\begin{aligned}\mathcal{M}_h(\boldsymbol{\sigma},p) \leq\, C\Big\{\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^*\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p-p_h^*\|_{\mathrm{W}^{2,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))} \\ +\quad (\Delta t)^2\Big(\,\|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p\|_{\mathrm{W}^{4,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))}\Big)\Big\},\end{aligned}$$

and the result follows by applying Lemma 3.1.2 to $(\boldsymbol{\sigma},p) \in \mathbb{K}^\perp$. □

We notice here that while the constant $C > 0$ appearing in (3.2.13) is independent of λ, the first error term on the left-hand side, namely $(\dot{\boldsymbol{\sigma}},\dot{p})(t_{n+\frac{1}{2}}) - (\partial_t\boldsymbol{\sigma}_h^n, \partial_t p_h^n)$, is estimated in the λ-dependent norm $\|\cdot\|_{\mathcal{C}}$. Hence, Lemma 3.2.4 ensures that only the convergence of the semi-norms

$$\max_n \Big\|\mathbf{div}\left(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}\right)\Big\|_{0,\Omega_{\mathrm{S}}} \quad \text{and} \quad \max_n \Big\|\nabla\big(p(t_{n+\frac{1}{2}}) - p_h^{n+\frac{1}{2}}\big)\Big\|_{0,\Omega_{\mathrm{F}}}$$

remain unaltered when λ goes to infinity. We aim now to apply Lemma 2.1.1 to deduce the same stability behaviour in the full $\mathbb{X}$-norm. To this end, we first need the following intermediate result.

Lemma 3.2.5. *Under the hypotheses of Lemma 3.2.4 there exists a constant $C > 0$, independent of λ, h and Δt, such that*

$$\begin{aligned}&\max_n \Big\|(\boldsymbol{\sigma},p)(t_{n+\frac{1}{2}}) - (\boldsymbol{\sigma}_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})\Big\|_{0,\mathcal{C}} \\ &\leq \quad C\Big\{\,\|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}((0,T);\mathbf{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|\nabla(p-\Pi_h p)\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Omega_{\mathrm{F}})^d)} \\ &+ \quad \|p\boldsymbol{n} - \boldsymbol{\pi}_h(p\boldsymbol{n})\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Sigma)^d)} \,+\, \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Omega_{\mathrm{S}})^d)} \\ &+ \quad \|\boldsymbol{\psi} - \boldsymbol{\pi}_h\boldsymbol{\psi}\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Sigma)^d)} \,+\, \|\boldsymbol{r} - \boldsymbol{R}_h\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}((0,T);[\mathrm{L}^2(\Omega_{\mathrm{S}})]^{d\times d}_{skew})} \\ &+ \quad (\Delta t)^2\Big(\,\|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}}))} \,+\, \|p\|_{\mathrm{W}^{4,\infty}((0,T);H^1(\Omega_{\mathrm{F}}))}\Big)\Big\}.\end{aligned} \tag{3.2.16}$$

Proof. We first notice that for each $\xi \in \{\boldsymbol{\sigma}, p\}$ there holds

$$\begin{aligned}(\xi(t_{k+\frac{1}{2}}) - \xi_h^{k+\frac{1}{2}}) - (\xi(t_{k-\frac{1}{2}}) - \xi_h^{k-\frac{1}{2}}) = \xi(t_{k+\frac{1}{2}}) - \xi(t_{k-\frac{1}{2}}) - \frac{\Delta t}{2}(\dot{\xi}(t_{k+\frac{1}{2}}) + \dot{\xi}(t_{k-\frac{1}{2}})) \\ +\quad \frac{\Delta t}{2}(\dot{\xi}(t_{k+\frac{1}{2}}) - \partial_t\xi_h^k) \,+\, \frac{\Delta t}{2}\big(\dot{\xi}(t_{k-\frac{1}{2}}) - \partial_t\xi_h^{k-1}\big)\,.\end{aligned} \tag{3.2.17}$$

Then, using a Taylor expansion centered at $t = t_k$, we find that

$$\xi(t_{k+\frac{1}{2}}) - \xi(t_{k-\frac{1}{2}}) - \frac{\Delta t}{2}(\dot{\xi}(t_{k+\frac{1}{2}}) + \dot{\xi}(t_{k-\frac{1}{2}})) = \frac{1}{2}\int_{t_k}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\xi(t)}{\mathrm{d}t^3}(t_{k+\frac{1}{2}} - t)^2\,\mathrm{d}t$$
$$+ \frac{1}{2}\int_{t_{k-\frac{1}{2}}}^{t_k} \frac{\mathrm{d}^3\xi(t)}{\mathrm{d}t^3}(t_{k-\frac{1}{2}} - t)^2\,\mathrm{d}t - \frac{\Delta t}{2}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} \frac{\mathrm{d}^3\xi(t)}{\mathrm{d}t^3}(\frac{\Delta t}{2} - |t - t_k|)\,\mathrm{d}t \quad \forall\, \xi \in \{\boldsymbol{\sigma}, p\}\,. \tag{3.2.18}$$

Substituting (3.2.18) in (3.2.17), and summing the resulting identities over $k = 1, \ldots, n$, we deduce that there exists a constant $C_0 > 0$, independent of λ, h and Δt, such that

$$\max_n \left\|(\boldsymbol{\sigma}, p)(t_{n+\frac{1}{2}}) - (\boldsymbol{\sigma}_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})\right\|_{0,\mathcal{C}} \le C_0 \Big\{ (\Delta t)^2 \big(\|\boldsymbol{\sigma}\|_{\mathrm{W}^{3,\infty}((0,T);\mathrm{L}^2(\Omega_\mathrm{S})^{d\times d})}$$
$$+ \|p\|_{\mathrm{W}^{3,\infty}((0,T);\mathrm{L}^2(\Omega_\mathrm{F}))} \big) \qquad + \max_n \left\|(\dot{\boldsymbol{\sigma}}, \dot{p})(t_{n+\frac{1}{2}}) - (\partial_t \boldsymbol{\sigma}_h^n, \partial_t p_h^n)\right\|_{0,\mathcal{C}} \Big\}\,.$$

Finally, (3.2.16) is a direct consequence of the foregoing estimate and Lemma 3.2.4. □

We are now in a position to establish the following asymptotic error estimate.

Theorem 3.2.1. *Assume that the solutions* $(\boldsymbol{\sigma}, p)$ *to problem* (3.1.7) *satisfies the regularity assumptions* $(\boldsymbol{\sigma}, p) \in \mathcal{C}^4((0,T); \mathbb{X}^{sym})$ *and* $(\boldsymbol{u}, p) \in \mathcal{C}^2\Big((0,T); \mathrm{H}^{k+1}(\Omega_\mathrm{S})^d \times \mathrm{H}^{k+1}(\Omega_\mathrm{F})\Big)$, *for some* $k \ge 1$, *where* $\boldsymbol{u}$ *is the displacement associated to* $(\boldsymbol{\sigma}, p)$ *through operator* D. *Then, there exists a constant* $C > 0$, *independent of* λ, h *and* Δt, *such that*

$$\max_n \left\|\Big(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}, p(t_{n+\frac{1}{2}}) - p_h^{n+\frac{1}{2}}\Big)\right\| \le \; C \Big\{ h^k + (\Delta t)^2 \Big\}\,.$$

Proof. We deduce immediately from Lemmas 3.2.4 and 3.2.5 that there exists a constant $C_0 > 0$, independent of λ, h and Δt, such that

$$\max_n \left\|(\boldsymbol{\sigma}, p)(t_{n+\frac{1}{2}}) - (\boldsymbol{\sigma}_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})\right\|_{0,\mathcal{C}} + \max_n \left\|\mathbf{div}\,\big(\boldsymbol{\sigma}(t_{n+\frac{1}{2}}) - \boldsymbol{\sigma}_h^{n+\frac{1}{2}}\big)\right\|_{0,\Omega_\mathrm{S}}$$
$$+ \max_n \left\|\nabla\big(p(t_{n+\frac{1}{2}}) - p_h^{n+\frac{1}{2}}\big)\right\|_{0,\Omega_\mathrm{F}} \le C_0 \Big\{ \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{\mathrm{W}^{2,\infty}((0,T);\mathbf{H}(\mathbf{div},\Omega_\mathrm{S}))}$$
$$+ \quad \|p\boldsymbol{n} - \boldsymbol{\pi}_h(p\boldsymbol{n})\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Sigma)^d)} + \|\boldsymbol{r} - \boldsymbol{R}_h\boldsymbol{r}\|_{\mathrm{W}^{2,\infty}((0,T);[\mathrm{L}^2(\Omega_\mathrm{S})]^{d\times d}_{\mathrm{skew}})}$$
$$+ \quad \|\boldsymbol{u} - U_h\boldsymbol{u}\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Omega_\mathrm{S})^d)} + \|\boldsymbol{\psi} - \pi_h\boldsymbol{\psi}\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Sigma)^d)}$$
$$+ \quad \|\nabla(p - \Pi_h p)\|_{\mathrm{W}^{2,\infty}((0,T);\mathrm{L}^2(\Omega_\mathrm{F})^d)} + (\Delta t)^2\, \|\boldsymbol{\sigma}\|_{\mathrm{W}^{4,\infty}((0,T);\mathrm{H}(\mathbf{div},\Omega_\mathrm{S}))}$$
$$+ \quad (\Delta t)^2\, \|p\|_{\mathrm{W}^{4,\infty}((0,T);H^1(\Omega_\mathrm{F}))} \Big\},$$

and the result follows from the norm equivalency provided by Lemma 2.1.1 and the approximation properties given by (2.2.3), (2.2.4) and (2.2.6)-(2.2.9). □

$h=\Delta t$	N	$\mathtt{e}_h(\boldsymbol{\sigma})$	$\mathtt{r}_h(\boldsymbol{\sigma})$	$\mathtt{e}_h(\boldsymbol{u})$	$\mathtt{r}_h(\boldsymbol{u})$	$\mathtt{e}_h(p)$	$\mathtt{r}_h(p)$
1/16	7489	8.837e−03	–	8.041e−03	–	9.635e−02	–
1/32	29313	1.929e−03	2.195	1.901e−03	2.081	2.038e−02	2.241
1/64	115969	4.623e−04	2.061	4.688e−04	2.020	4.990e−03	2.030
1/128	461313	1.144e−04	2.014	1.166e−04	2.008	1.257e−03	1.990

Table 3.3.1: Convergence history in the case $\Gamma_{\mathrm{N}}=\emptyset$ and $\lambda=\mu=1.0$.

3.3 Numerical results

In this section we present several numerical experiments confirming the good performance of the fully discrete Galerkin scheme (3.2.1) as applied to a two-dimensional model problem. In all what follows, given the solution $(\boldsymbol{\sigma}_h^n, p^n)$ of (3.2.1) at a time level $n\Delta t$, we postprocess the corresponding displacement field $\boldsymbol{u}_h^n$ by solving the auxiliary saddle point problem:

$$\begin{aligned}
&\text{Find } \boldsymbol{\sigma}_h^* \in \boldsymbol{\mathcal{W}}_h \text{ with } \boldsymbol{\sigma}_h^*\boldsymbol{n} = -p_h^n\boldsymbol{n} \text{ on } \Sigma,\ \boldsymbol{r}_h^* \in \boldsymbol{\mathcal{Q}}_h \text{ and } \boldsymbol{u}_h^n \in \mathcal{U}_h \text{ such that}\\
&(\mathcal{C}^{-1}\boldsymbol{\sigma}_h^* + \boldsymbol{r}_h^*, \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} + (\boldsymbol{u}_h^n, \mathbf{div}\ \boldsymbol{\tau})_{0,\Omega_{\mathrm{S}}} = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h^{\Sigma},\\
&(\mathbf{div}\ \boldsymbol{\sigma}_h^*, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}} = (\mathbf{div}\ \boldsymbol{\sigma}_h^n, \boldsymbol{v})_{0,\Omega_{\mathrm{S}}} \quad \forall \boldsymbol{v} \in \mathcal{U}_h,\\
&(\boldsymbol{\sigma}_h^*, \boldsymbol{s})_{0,\Omega_{\mathrm{S}}} = 0 \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h,
\end{aligned} \tag{3.3.1}$$

where $\boldsymbol{\mathcal{W}}_h^{\Sigma} := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h;\quad \boldsymbol{\tau n} = \mathbf{0},\quad \text{on } \Sigma\}$.

For each mesh size h, the individual relative errors produced by the fully discrete Galerkin method (3.2.1) are measured at the final time step as follows:

$$\mathtt{e}_h(\boldsymbol{\sigma}) := \frac{\|\boldsymbol{\sigma}(t_{L-\frac{1}{2}}) - \boldsymbol{\sigma}_h^{L-\frac{1}{2}}\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})}}{\|\boldsymbol{\sigma}(t_{L-\frac{1}{2}})\|_{\mathrm{H}(\mathbf{div},\Omega_{\mathrm{S}})}}, \quad \mathtt{e}_h(p) := \frac{\|p(t_{L-\frac{1}{2}}) - p_h^{L-\frac{1}{2}}\|_{1,\Omega_{\mathrm{F}}}}{\|p(t_{L-\frac{1}{2}})\|_{1,\Omega_{\mathrm{F}}}},$$

$$\mathtt{e}_h(\boldsymbol{u}) := \frac{\|\boldsymbol{u}(t_{L-\frac{1}{2}}) - \boldsymbol{u}_h^{L-\frac{1}{2}}\|_{0,\Omega_{\mathrm{S}}}}{\|\boldsymbol{u}(t_{L-\frac{1}{2}})\|_{0,\Omega_{\mathrm{S}}}},$$

where $\{(\boldsymbol{\sigma}_h^n, p_h^n),\ n=0,\dots,L\}$ is the solution of (3.2.1) and $(\boldsymbol{\sigma}, p)$ is the solution of (3.1.7). In turn, we introduce the experimental rates of convergence

$$\mathtt{r}_h(\boldsymbol{\sigma}) := \frac{\log(\mathtt{e}_h(\boldsymbol{\sigma})/\mathtt{e}_{\hat{h}}(\boldsymbol{\sigma}))}{\log(h/\hat{h})}, \qquad \mathtt{r}_h(p) := \frac{\log(\mathtt{e}_h(p)/\mathtt{e}_{\hat{h}}(p))}{\log(h/\hat{h})},$$

$$\mathtt{r}_h(\boldsymbol{u}) := \frac{\log(\mathtt{e}_h(\boldsymbol{u})/\mathtt{e}_{\hat{h}}(\boldsymbol{u}))}{\log(h/\hat{h})},$$

where $\mathtt{e}_h$ and $\hat{\mathtt{e}}_{\hat{h}}$ are the errors corresponding to two consecutive triangulations with mesh sizes h and $\hat{h}$, respectively.

We now describe the main data of the three examples that will be reported in the following. For each one of them we consider $\Omega_{\mathrm{S}} = (0,1)^2 \backslash [0.25, 0.75]^2$, $\Omega_{\mathrm{F}} = (0.25, 0.75)^2$, $T=1$, $\rho_S = 1$, and $\rho_F = 1$.

$h = \Delta t$	N	$\mathtt{e}_h(\boldsymbol{\sigma})$	$\mathtt{r}_h(\boldsymbol{\sigma})$	$\mathtt{e}_h(\boldsymbol{u})$	$\mathtt{r}_h(\boldsymbol{u})$	$\mathtt{e}_h(p)$	$\mathtt{r}_h(p)$
1/16	7489	8.562e−03	–	6.453e−03	–	2.335e−01	–
1/32	29313	1.845e−03	2.214	1.450e−03	2.154	2.657e−02	3.136
1/64	115969	4.412e−04	2.064	3.572e−04	2.021	5.491e−03	2.274
1/128	461313	1.090e−04	2.017	8.905e−05	2.004	1.358e−03	2.016

Table 3.3.2: Convergence history in the case $\Gamma_{\mathrm{N}} \neq \emptyset$ and $\lambda = \mu = 1.0$.

$h = \Delta t$	N	$\mathtt{e}_h(\boldsymbol{\sigma})$	$\mathtt{r}_h(\boldsymbol{\sigma})$	$\mathtt{e}_h(\boldsymbol{u})$	$\mathtt{r}_h(\boldsymbol{u})$	$\mathtt{e}_h(p)$	$\mathtt{r}_h(p)$
1/16	7489	9.019e−03	–	3.362e−02	–	1.343e+00	–
1/32	29313	1.946e−03	2.212	2.755e−03	3.609	1.808e−01	2.893
1/64	115969	4.673e−04	2.058	8.749e−04	1.655	2.830e−02	2.675
1/128	461313	1.133e−04	2.044	2.404e−04	1.863	6.590e−03	2.103

Table 3.3.3: Convergence history in the case $E = 1.0$, $\nu = 0.49$.

In Example 1, we choose Lamé constants $\lambda = \mu = 1.0$, take $\Gamma_{\mathrm{D}} = \Gamma$ and select the datum $\boldsymbol{f}$ so that the exact solution for the displacement and pressure are given, respectively, by

$$\boldsymbol{u}(\mathbf{x}, t) := \sin(4\pi x_1)\sin(4\pi x_2)\begin{pmatrix}\sin t\\ \sin t\end{pmatrix} \qquad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{S}}, \forall\, t > 0\,,$$

and

$$p(\mathbf{x}, t) := \sin(4\pi x_1)\sin(4\pi x_2)\sin(4\sqrt{2}\pi t) \qquad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{F}},\, \forall\, t > 0\,.$$

In Example 2, we use again the same displacement and Lamé constants of the first example and choose $\boldsymbol{f}$ so that the exact solution for the pressure is given by

$$p(\mathbf{x}, t) := \sin(x_1 - 0.5)\sin(x_2 - 0.5)\sin(\sqrt{2}t) \qquad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{F}},\, \forall\, t > 0\,.$$

In addition, in this case we incorporate the traction boundary condition

$$\boldsymbol{\sigma}\mathbf{n} = \widehat{\boldsymbol{t}} \qquad \text{on} \qquad \Gamma_N\,,$$

with $\Gamma_N := \{x_2 = 0,\ 0 \leq x_1 \leq 1\}$.

Finally, in Example 3 we test the locking-free character of the method in the nearly incompressible case. For this purpose, we consider now Lamé constants corresponding to a Poisson ratio $\nu = 0.49$ and Young modulus $E = 1.0$, that is

$$\mu = \frac{E}{2(1+\nu)} = 0.336 \quad \text{and} \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = 16.443\,,$$

and maintain the displacement, pressure and traction condition of Example 2.

For all the above described examples we consider the AFW elements of order $k = 2$ for the spatial discretization in the solid, and the usual second order Lagrange element for the corresponding discretization in the acoustic medium. Tables 3.3.1 to 3.3.3 depict the convergence results obtained by taking equal time and space discretizations parameters Δt and h, respectively. The size of the linear systems solved at each iteration step is indicated by the parameter N. We report on the relative errors and the convergence orders for these three examples. As predicted by the theoretical results, we observe that in all cases the quadratic convergence rate of the error is attained in each variable. In addition, we remark from Example 3 that the method is also robust for nearly incompressible materials, thus confirming its locking-free character.

Concluding remarks and future work

Concluding remarks

In this thesis we have developed and analyzed mixed finite element methods for two models of wave propagation problems, namely elatodynamic and elastoacoustic, and have theoretically and experimentally demonstrated the convergence and stability of both methods. Further details are given in what follows.

1. Concerning the new mixed finite element method for the elastodynamic problem:

   (a) We have removed the displacement as unknown and formulated a time-dependent wave equation in $\mathbb{H}(\mathbf{div}, \Omega)$, whose unknowns are the stress tensor and the rotation, and have proved existence, uniqueness and regularity of solution for this equation. The rotation is introduced here as the Lagrange multiplier taking care of the weak symmetry of the stress.

   (b) We have considered a reduced wave equation in which the stress is the only unknown, and have proved that, under appropriate hypotheses on the initial conditions, the displacement can be recovered by integrating twice the momentum equation.

   (c) A new operator representing the mixed formulation of a stationary elasticity problem, together with its discrete version, is introduced. The properties of this operator and its discrete counterpart are essential to demonstrate the convergence of our schemes.

   (d) It is shown that the respective Galerkin scheme is well-posed for four families of finite elements, which satisfy the usual hypotheses, and a fully discrete scheme using the Newmark method is implemented, for which the corresponding stability and convergence are proved as well.

2. Concerning the new mixed finite element method for the elastoacoustic problem:

   (a) We have coupled, by means of appropriate transmission conditions, a mixed formulation in the structure with a primal formulation in the fluid. In adition, we eliminated the displacement, and therefore the only unknowns of the resulting formulation are the stress tensor in the structure and the pressure in the fluid, Also, we proved existence, uniqueness and regularity of solution for this formulation. Furthermore, under appropriate hypotheses on the initial conditions, the displacement in the structure can be recovered by integrating twice the momentum equation.

(b) We used the Arnold-Falk-Winther mixed elements and the classical Lagrange elements to approximate the stress tensor and the pressure, respectively, with the same polynomial order k, which allows us to couple the transmission conditions.

(c) A new operator representing the mixed formulation of a stationary elastoacoustic problem, together with its discrete version, is introduced. The properties of this operator and its discrete counterpart are essential to demonstrate the convergence of our schemes. Unlike the elastodynamic case, defining properly this operator and obtaining suitable properties become much more complex.

(d) It is shown that the respective Galerkin scheme is well-posed, and a fully discrete scheme using the Newmark method is implemented, for which the corresponding stability and convergence are proved.

(e) In order to simplify the analysis, and proceeding similarly as for elastodynamic, in this formulation we have hidden the rotation, which is originally incorporated in our formulations as the Lagrange multiplier imposing the weak symmetry required by the Arnold-Falk-Whinter spaces.

3. Finally, we note that the continuous and discrete analyses of the models share the following common features:

   (a) The use of energy arguments, Galerkin approximation, and compactness results, which are common tools in the continuous analysis of time-dependent problems. In addition, the use of classic results of ODE's theory to prove the well-posedness of the corresponding discrete formulations.

   (b) It is shown that it is possible to recover and approximate the displacement by post-processing formulae, which explains the fact that the displacement is included in the numerical results.

   (c) It is shown that the schemes employed are inmune to locking phenomenon in the nearly incompressible case. Also, numerical experiments in 2D illustrating that property and supporting the theoretical estimates obtained, are presented.

Future work

The methods employed and the results obtained in this thesis motivate several new projects, some of which are under development. Here are some of these new projects.

1. **A posteriori error analysis for a mixed formulation of the elastodynamic problem** As a natural continuation of the work done for the elastodynamic problem in [21], we are interested in developing an a posteriori error analysis for the mixed formulation used there. The time dependence of the model makes it necessary to adapt the techniques of the elliptical reconstruction, decomposing it into a spatial one first, and hence into a spatial and time reconstruction. For this purpose we plan to follow the techniques used in [27], where a posteriori error estimates are obtained for a primal formulation of the wave equation, in combination with what was done in [36], where a posteriori error estimates are obtained for the mixed formulation of a time-dependent linear parabolic problems

2. **A posteriori error analysis for a mixed formulation of the elastoacoustic problem** This is also the natural continuation of the work done in [22], for which we simply adapt the tools and logical sequence of the analysis to be employed in the previous item.

Conclusiones y trabajo futuro

Conclusiones

En esta tesis hemos desarrollado y analizado métodos de elementos finitos mixtos para dos modelos de problemas de propagación de ondas, más precisamente, los problemas de la elatodinámica y la elastoacústica, y hemos demostrado teórica y experimentalmente la convergencia y estabilidad de ambos métodos. Más detalles se dan en lo que sigue.

1. Con respecto al nuevo método de elementos finitos mixtos para el problema elastodinámico:

   (a) Hemos eliminado el desplazamiento como incógnita y hemos formulado una ecuación de la onda dependiente del tiempo en $\mathbb{H}(\mathbf{div}, \Omega)$ cuyas incógnitas son el tensor de esfuerzos y la rotación, y hemos probado existencia, unicidad y regularidad de solución para esta ecuación. La rotación es introducida aquí como el multiplicador de Lagrange, que incorpora la simetría débil del tensor de esfuerzos.

   (b) Hemos considerado una ecuación de la onda reducida en la que el tensor esfuerzo es la única incógnita, y hemos probado que, bajo hipótesis apropiadas sobre las condiciones iniciales, el desplazamiento puede ser recuperado integrando dos veces la ecuación de momentum.

   (c) Se introduce un nuevo operador que representa la formulación mixta de un problema de elasticidad estacionario, junto con la versión discreta del operador. Las propiedades de este operador y su contraparte discreta son esenciales para demostrar la convergencia de nuestros esquemas.

   (d) Se muestra que el respectivo esquema de Galerkin está bien planteado para cuatro familias de elementos finitos, que satisfacen las hipótesis usuales, y se implementa un esquema completamente discreto utilizando el método de Newmark, para el cual se demuestra también la estabilidad y convergencia correspondientes.

2. Con respecto al nuevo método de elementos finitos mixtos para el problema elastoacústico:

   (a) Hemos acoplado, mediante apropiadas condiciones de transmisión, una formulación mixta en la estructura con una formulación primal en el fluido. Además, eliminamos el desplazamiento, y por lo tanto las únicas incógnitas de la formulación resultante son el tensor de esfuerzos en la estructura y la presión en el fluido. Además, probamos la existencia, unicidad y regularidad de la solución para esta formulación. Finalmente, bajo hipótesis apropiadas sobre las condiciones iniciales, el desplazamiento en la estructura puede ser recuperado integrando dos veces la ecuación de momentum.

(b) Hemos usado los elementos de Arnold-Falk-Winther y los clásicos elementos de Lagrange para aproximar el tensor de esfuerzos y la presión, respectivamente, con el mismo orden polinomial k lo que nos permite acoplar las condiciones de transmisión.

(c) Se introduce un nuevo operador que representa la formulación mixta de un problema elastoacústico estacionario, junto con la versión discreta del operador. Las propiedades de este operador y su contraparte discreta son esenciales para demostrar la convergencia de nuestros esquemas. A diferencia del caso elastodinámico, definir adecuadamente este operador y obtener propiedades apropiadas se vuelve mucho más complejo.

(d) Se muestra que el respectivo esquema de Galerkin está bien planteado y se implementa un esquema completamente discreto utilizando el método de Newmark, para el cual se demuestra la estabilidad y convergencia correspondientes.

(e) Con el objeto de simplificar el análisis y proceder de forma similar al caso elastodinámico, en esta formulación hemos ocultado la rotación, que se incorpora originalmente en nuestras formulaciones como el multiplicador de Lagrange que impone la simetría débil requerida por los espacios de Arnold-Falk-Whinter.

3. Finalmente, observamos que los análisis continuos y discretos de los modelos comparten las siguientes características comunes:

   (a) El uso de argumentos de energía, aproximación de Galerkin y resultados de compacidad, que son herramientas comunes en el análisis continuo de problemas dependientes del tiempo. Además, el uso de resultados clásicos de la teoría de EDOs para probar el buen planteamiento de las formulaciones discretas correspondientes.

   (b) Se muestra que es posible recuperar y aproximar el desplazamiento mediante fórmulas de post-procesamiento, lo que explica el hecho de que el desplazamiento está incluido en los resultados numéricos.

   (c) Se demuestra que los esquemas empleados son inmunes al fenómeno del bloqueo en el caso casi incompresible. También se presentan experimentos numéricos en 2D que ilustran ésta propiedad y respaldan las estimaciones teóricas obtenidas.

Trabajo futuro

Los métodos empleados y los resultados obtenidos en esta tesis motivan varios proyectos nuevos, algunos de los cuales están en desarrollo. Estos son algunos de estos nuevos proyectos.

1. **Análisis de error a posteriori para una formulación mixta del problema elastodinámico.** Como una continuación natural del trabajo hecho para el problema elastodinámico estudiado en [21], estamos interesados en realizar un análisis de error a posteriori para la formulación mixta empleada en dicho trabajo. La dependencia del tiempo del modelo, hace necesario adaptar las técnicas de la reconstrucción elíptica, desglosando dicha reconstrucción en una reconstrucción espacial primeramente y luego una reconstrucción en espacial y temporal. Con este objetivo se pretende seguir las técnicas utilizadas en [27] en donde se obtienen estimaciones de error a posteriori para una formulación primal de la ecuación de la onda, en combinación con lo hecho en [36] en donde se obtienen estimaciones de error para una formulación mixta de problemas parabólicos lineales dependientes del tiempo.

2. **Análisis de error a posteriori para una formulación mixta del problema elastoacústico**
   Lo cual también constituye la continuación natural del trabajo hecho en [22], para lo cual simplemente adaptamos las herramientas y la secuencia lógica del análisis ya indicado en el ítem anterior.

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