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EXISTENCIA Y CONDICIONES DE OPTIMALIDAD EN OPTIMIZACIÓN VECTORIAL NO CONVEXA

Tesis para optar al título de
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Abstract

The main goal of this PhD thesis is to investigate weakly efficient vectorial minima under relaxed hypotheses of convexity for the involved functions.

The organization of the thesis is as follows. In the first part, our main concern is the existence of solutions for the compact case, without hypotheses of convexity and differentiability. Afterwards, by using notions of generalized convexity and recession analysis we treat the unbounded case for finite dimensional spaces. We study also problems for which the range of the involved vectorial function is contained in a finite or in an infinite dimensional space. For the latter case, we give several conditions for the nonemptiness and the boundedness of the set of weak solutions. In particular, we deal with the case of polyhedral and Lorentz cones. Moreover, we seek for weakly efficient solutions for the case when the domain of the vectorial function is a set of the real numbers. By the way, we obtain relationships for the nonemptiness of the set of weak minima and the set of minima of the component functions.

In the second part, we study theorems of alternative for vectorial optimization problems and we obtain optimal conditions for such theorems. With the aid of such results we characterize bi-dimensional spaces, we perform an scalarization by means of the positive polar cone for obtaining weakly efficient points, we characterize the zero duality gap and we obtain optimality conditions of Fritz-John type for vectorial optimization problems.

Finally, in connection with the first part, in the third part we seek for weak minima, when the domain of the vectorial function is a subset of the real numbers, the range is bi-dimensional and the component functions are quasiconvex without any hypothesis

of differentiability. This case is totally characterized and this allows us to develop a finite time algorithm for calculating weakly efficient solutions and the supremum of the set of weakly efficient minima.

Resumen

El propósito de esta Tesis es estudiar las propiedades de los mínimos vectoriales débilmente eficientes, bajo hipótesis de convexidad generalizada. En este trabajo se destacan tres partes.
En la primera parte, se resuelve el problema de existencia de soluciones para el caso compacto, sin hipótesis de convexidad y diferenciabilidad. Posteriormente, junto con una noción de convexidad generalizada y el análisis de recesión, se aborda el caso no acotado para espacios finito dimensional. Se estudia el problema existencia cuando el recorrido de la función vectorial, esta contenido en un espacio de dimension infinita y finita. Para el segundo caso, se muestra varias caracterizaciones para la no vacuidad y compacidad del conjunto solución de mínimos débiles, en particular aplicables a conos de tipo poliédrico y Lorentz. A continuación se estudia el problema de encontrar mínimos débiles eficientes, pero esta vez cuando el dominio de la función vectorial es un subconjuto de los números reales y cumple un tipo de convexidad generalizada, sin hipótesis de diferenciabilidad. De esta forma, se obtiene relaciones de no vacuidad del conjunto solución de mínimos débiles y el conjunto de mínimos de las funciones componentes.
En la segunda parte, se estudian los teoremas de alternativa para los problemas de optimización vectorial, encontrando condiciones optimales para dichos teoremas. Estos resultados nos permitiran caracterizar los espacios bi dimensionales , la escalarización por medio del cono polar positivo para la obtención de puntos débilmente eficientes, la nulidad del gap de dualidad y la obtención de condiciones de optimalidad de tipo Fritz-John en optimización vectorial.
Finalmente, en conexión con la primera parte se retorna nuevamente el problema de encontrar mínimos débiles, cuando el dominio de la función vectorial es un subconjunto de los números reales , de recorrido bidimensional y sus componentes son funciones casiconvexas, sin hipótesis de diferenciabilidad . En esta parte, se caracteriza totalmente este caso, lo que permite elaborar un algoritmo de tiempo finito, para calcular las soluciones débiles eficientes y el supremo del conjunto de mínimos débiles eficientes.

Glossary of Notations

Spaces	
$\mathbb{R}^n$	real $n-$dimensional space
$\mathbb{R}^n_+$	the nonnegative orthant of $\mathbb{R}^n$
$\mathbb{R}_{++}$	the interval $]0, +\infty[$
X	real Hausdorff topological space
Y	real locally convex topological vector space
Y^*	topological dual space
Sets	
E_w	the weak pareto minimum set
$\text{argmin} f_K$	$\bar{x} \in K \;:\; f(\bar{x}) \leq f(x)$ for all x in K
$\text{cone}(A)$	the conic hull of the set A
$\text{co}(A)$	the convex hull of the set A
$\text{ext}(A)$	the extreme points of the set A
$\text{int}\, A$	the topological interior of the set A
$\text{qint}\, A$	the quasi-interior of the set A
$\bar{A}$	the (topological) closure of the set A
A^*	the (positive) polar cone of the set A
A^∞	the asymptotic cone of the set A
$T(A; \bar{x})$	the contingent cone of A at $\bar{x}$
Mappings	
$T : X \rightrightarrows Y$	a multifunction from X to Y
$f : X \to \mathbb{R} \cup \{+\infty\}$	function from X to $\mathbb{R} \cup \{+\infty\}$

f^∞	the asymptotic function of f
$\langle \cdot, \cdot \rangle$	the duality pairing
l.s.c.	lower semicontinuity property

Introducción

En optimización vectorial uno de los problemas de interés es el siguiente:
Considere X un espacio topológico (Hausdorff) real, Y un espacio vectorial topológico real , $P \subseteq Y$ un cono convexo (no necesariamente puntiagudo o cerrado), tal que $P \neq Y$ y su interior topológico (int P), es no vacío y $K \subseteq X$. Se requiere

$$\text{encontrar } \bar{x} \in K \;:\; F(y) - F(\bar{x}) \notin -\text{int}P \quad \text{para todo } y \in K \qquad (\text{VP}),$$

donde $F: K \to Y$ es una función vectorial. El punto $\bar{x}$ se denomina mínimo débil eficiente, y se denota por E_w al conjunto de estos puntos.
En el caso particular, cuando $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ y $P = \mathbb{R}^m_+$, la función F puede escribirse como $F(x) = (f_1(x), \ldots, f_m(x))$. Si además tenemos que K es un conjunto convexo y cerrado, cada función componente es convexa y diferenciable, se prueba en [9], la siguiente caracterización

$$\bar{x} \in E_w \Leftrightarrow \nabla F(\bar{x})(x - \bar{x}) \notin -\text{int}\mathbb{R}^m_+$$

A continuación comentaremos algunos resultados existentes en la literatura, correspondiente a los ultimos 15 años del problema (VP).
Para la existencia de soluciones del problema (VP), cuando K es un conjunto compacto, algunos autores han usado alguna condición de convexidad para la función vectorial (ver [3, 6, 9, 13, 19, 21]). El motivo de esta hipótesis, es por el uso del lema de Knaster-Kuratowski- Mazurkiewicz ([4, 17]). Un resultado general de existencia para mínimos vectoriales aplicable en particular al problema (VP) se encuentra en [41].
Por otra parte, también ha sido un problema de interés, caracterizar la no vacuidad y

compacidad del conjunto E_w, en el artículo [13] (Teorema 2.1), se relaciona la compacidad y la no vacuidad de E_w con el conjunto de mínimo de las funciones componentes, con hipótesis de diferenciabilidad y convexidad. En [19], se obtienen estimaciones de $(E_w)^\infty$, para una clase de funciones no convexas. Podemos encontrar en las referencias [32, 33] más caracterizaciones de compacidad y la no vacuidad de E_w bajo hipótesis de convexidad y de diferenciabilidad generalizada.

Otro problema que ha tenido relevancia en el análisis convexo, son los teoremas del tipo alternativo, cuyas aplicaciones en el área de la optimización vectorial, son por ejemplo las escalarizaciones del problema (VP) por medio del cono polar positivo para la obtención de puntos débilmente eficientes, la existencia de multiplicadores de Lagrange, resultados de dualidad, etc.

Teóricamente un teorema alternativo en optimización vectorial, tiene la siguiente presentación:

Sean Y un espacio topológico localmente convexo real, $P \subseteq Y$ un cono convexo cerrado tal que $\text{int } P \neq \emptyset$, P^* cono polar positivo de P y $A \subseteq Y$. Se afirma la validez de exactamente una de las siguientes afirmaciones:

$$\exists\, a \in A \quad \text{tal que } a \in -\text{int } P;$$

$$\exists\, p^* \in P^*, \quad p^* \neq 0 \quad \text{tal que} \quad \langle p^*, a\rangle \geq 0 \quad \forall\, a \in A.$$

Notemos que para las aplicaciones de estos teoremas en optimización vectorial se considera $A = F(K)$.

En la literatura podemos encontrar, al menos dos caminos para probar teoremas de este tipo, el primero vía teoremas minmax, como en [36] y el segundo vía separación convexa, como en las referencias [34, 48, 50, 54, 55].

En esta tesis se estudia el problema (VP), sin hipótesis de diferenciabilidad. Primero se obtiene un teorema de existencia en espacios de dimension infinita, sin hipótesis de convexidad, por lo tanto se logra una versión vectorial del teorema de Weierstrass. Luego, usando nociones de convexidad generalizada para funciones vectoriales, se realizan estimaciones para $(E_w)^\infty$, lo que permite dar condiciones necesarias, suficientes

o ambas a la vez, dependiendo de los espacios y conos involucrados, logrando hacer una descripción muy detallada de E_w, para el caso de la recta real. Los resultados obtenidos en esta parte generalizan algunos de los resultados de las referencias antes mencionada.
El contenido de esta parte, corresponde a los trabajos:

- Flores-Bazán F. and Vera C., The vector Weierstrass theorem and generalized quasi-convex vector optimization, *pre-print 2003-04, Depto. Ingeniería Matemática Universidad de Concepción*, 2003.

- Flores-Bazán F. and Vera C., Characterization of the nonemptiness and compactness of solution sets in convex/nonconvex vector optimization, *J. of Optimization Theory and Applications*, Vol. 130, No 2, 2006.

- Flores-Bazán F. and Vera C., Weak eficiency in quasiconvex vector optimization without derivatives, sometido (2006).

En la segunda parte de esta Tesis, se estudia la siguiente implicación

$$A \cap (-\text{int } P) = \emptyset \Longrightarrow \text{co}(A) \cap (-\text{int } P) = \emptyset,$$

donde Y un espacio topológico localmente convexo real, $P \subseteq Y$ un cono convexo cerrado tal que $\text{int } P \neq \emptyset$. Se logra establecer varias equivalencias de la implicación anterior, lo que permitió, plantear un teorema alternativo óptimo, aplicable a problemas de optimizacion vectorial, además de caracterizar los espacios bi-dimensionales en términos de la validez de un teorema de tipo alternativo.
El contenido de esta parte, corresponde al trabajo:

- Flores-Bazán F., Vera C. and Hadjisavvas N., An optimal alternative theorem and applications to mathematical programming, *Journal of Global Optimization*, to appear, 2006.

En la parte Final, se estudia el problema (VP), cuando $X = \mathbb{R}$, $Y = \mathbb{R}^2$ y $P = \mathbb{R}^2_+$. Se logra caracterizar totalmente el conjunto E_w, con hipótesis de convexidad general-

izada, lo que permitió diseñar algunos algoritmos.
El contenido de esta parte, corresponde al trabajo:

- Flores-Bazán F. and Vera C., Weak eficiency in quasiconvex vector optimization without derivatives, sometido (2006).

A continuación damos una breve descripción de la tesis:

En el Capítulo 1 se introduce la notación y se revisan algunos resultados básicos del análisis convexo y asintótico, algunas definiciones de convexidad generalizada, como también nociones de mínimo vectorial también son revisadas.

El Capítulo 2 está dedicado a teoremas de existencia y caracterizaciones compactas de E_w. En la Sección 2.1, se dá una versión vectorial del teorema de Weierstrass. En la Sección 2.2 se estudia una noción generalizada de convexidad para funciones vectoriales introducidas en [21] y se dan resultados de existencia para un problema general de optimización. En la Sección 2.3 se estudia el caso cuando X es finito dimensional del problema (VP). En la Subsección 2.3.1 se estudian dos casos, para el primero se dan condiciones suficientes para que E_w sea compacto y no vacío, cuando Y es de dimensión infinita y F cumple una noción de convexidad generalizada . Para el segundo caso, se obtienen caracterizaciones compactas para el conjunto E_w, cuando F cumple una condicion de convexidad, Y es finito dimensional y tal que el conjunto puntos extremales de una base compacta del cono polar positivo de P, es cerrado. En la Sección 2.4 se estudia el caso $X = \mathbb{R}$ del problema (VP). En la Subsección 2.4.1, estudiamos los mismos casos que en la Subsección 2.3.1: para el primer caso se obtienen condiciones necesarias y suficientes para que E_w sea compacto y no vacío; para el segundo caso, se obtienen caracterizaciones compactas para el conjunto E_w, cuando F satisface una condicion de convexidad generalizada, Y es finito dimensional y tal que el conjunto puntos extremales de una base compacta del cono polar positivo de P, es cerrado. En la Subsección 2.4.2, se obtienen varias caracterizaciones para la no vacuidad del conjunto E_w, bajo hipótesis de convexidad relajada. En la Subsección 2.4.3, se estudia el problema (VP), cuando cada función componente de F es casi-convexa. En tal caso, se logra obtener varias relaciones entre el cunjunto E_w y el conjunto de mínimos de las funciones componentes.

El Capítulo 3 está dedicado a estudiar un problema de separación convexa. En la Sección 3.1, se obtiene un teorema óptimo para un problema de separación no convexa, en espacios de dimensión arbitraria, además se prueban algunas equivalencias entre las nociones de convexidad generalizada, que han utilizado algunos investigadores, ver [36, 47, 51, 54, 55]. En la Sección 3.2, se caracterizan los espacios bi-dimensionales por medio del los teoremas del tipo alternativo. En la Sección 3.3, se caracterizan las condiciones necesarias de optimalidad de tipo Fritz-John en optimizaciø'n vectorial. En la sección 3.4 se muestran dos aplicaciones: la primera permite caracterizar la nulidad del gap de dualidad, la segunda es para caracterizar las soluciones débilmente eficientes mediante la escalarización standard.

En el Capítulo 4, estudiamos el problema (VP), cuando $X = \mathbb{R}$, $Y = \mathbb{R}^2$ y $P = \mathbb{R}^2_+$, y se describe completamente el conjunto E_w, con hipótesis de casi convexidad. En las Secciones $4.1, \ldots, 4.5$, estudiamos los casos que surgen al asumir que $\mathrm{argmin}_K f_1$ y $\mathrm{argmin}_K f_2$, no se intersectan. En la Sección 4.1, se plantean algunos algoritmos, y además se muestran los resultados computacionales de las implementaciones de algunos de estos.

Contents

Chapter 1

Notation and Preliminary Facts

1.1 Convex analysis

In this section we introduce several notions and results from Convex Analysis.
Let X be a real vector space and Y be a real locally convex topological (Hausdorff) vector space. Throughout this work we shall use the following notation (x,y being elements of X):
$[x, y] := \{(1 - \lambda)x + \lambda y \ : \ \lambda \in [0, 1]\}$, $[x, y[:= \{(1 - \lambda)x + \lambda y \ : \ \lambda \in [0, 1[\}$,
$]x, y[:= \{(1 - \lambda)x + \lambda y \ : \ \lambda \in]0, 1[\}$.
Let $\emptyset \neq A \subseteq X$.

Definition 1.1.1 A is **convex** if $[x, y] \subseteq A$ for all $x, y \in A$.

Definition 1.1.2 A is **cone** if $\mathbb{R}_+ A = \{\lambda x \ : \ \lambda \geq 0, \quad x \in A\} \subseteq A$.

Definition 1.1.3

$$\begin{array}{lcl}
\mathrm{co}(A) & := & \bigcap\{C \subseteq X \;:\; A \subseteq C, \quad C \,\text{convex}\} \\
& = & \left\{ \sum_{i=1}^{n} \lambda_i x_i \;:\; n \in \mathbb{N}, (\lambda_i)_{1\le i \le n} \subseteq \mathbb{R}_+, \sum_{i=1}^{n} \lambda_i = 1 \right\}, \\
\mathrm{cone}(A) & := & \bigcap\{C \subseteq X \;:\; A \subseteq C, \quad C \,\text{cone}\} = \{\lambda x \;:\; \lambda \geq 0, \quad x \in A\} \\
& = & \bigcup_{t\geq 0} tA, \\
\mathrm{cone}_+(A) & := & \{\lambda x \;:\; \lambda > 0, \quad x \in A\} = \bigcup_{t>0} tA, \\
\overline{\mathrm{cone}}(A) & := & \bigcap\{C \subseteq X \;:\; A \subseteq C, \quad C \,\text{closed cone}\}.
\end{array}$$

Evidently, $\overline{\mathrm{cone}}(A) = \overline{\mathrm{cone}(A)}$, $\mathrm{cone}(A) = \mathrm{cone}_+(A) \cup \{0\}$, and therefore, $\overline{\mathrm{cone}(A)} = \overline{\mathrm{cone}_+(A)}$. In [47, 54, 55] the notation $\mathrm{cone}(A)$ instead of $\mathrm{cone}_+(A)$ is employed.

Given a convex subset K of Y, an element $x \in K$ is called a quasi-interior point if there is no closed hyperplane supporting K at x; i.e., if for all $x^* \in Y^*$ the following implication holds:

$$\langle x^*, y\rangle \geq \langle x^*, x\rangle \text{ for all } y \in K \;\Rightarrow\; x^* = 0.$$

Equivalently, x is a quasi-interior point if and only if $\overline{\mathrm{cone}}(K - x) = Y$ (see for instance [8] for details and references on quasi-interiors). We will denote by $\mathrm{qint}\ K$ the set of quasi-interior points of K. If $\mathrm{int}\ K \neq \emptyset$, then $\mathrm{int}\ K = \mathrm{qint}\ K$. For this reason, all results in this Tesis involving $\mathrm{qint}\ K$ are also true for $\mathrm{int}\ K$, provided the latter set is nonempty. On the other hand, for any $p \in [1, +\infty]$ the cone l^p_+ has nonempty quasi-interior, but its interior (and even the relative algebraic interior) is empty for all $p \in [1, +\infty)$. Quasi-interior points share some properties of the interior points; for instance, if $x \in \mathrm{qint}\ K$ and $y \in K$ then $[x, y[\ \subseteq \mathrm{qint}\ K$. In particular, $\mathrm{qint}\ K$ is convex and dense in K whenever it is not empty.

If P is a closed convex cone, then it is easy to check that $x \in \mathrm{qint}\ P$ if and only if $\langle x^*, x\rangle > 0$ for all $x^* \in P^*\backslash\{0\}$, or equivalently if the set $B = \{x^* \in P^* : \langle x^*, x\rangle = 1\}$ is a w*-closed base for P^* (we recall that a convex set B is called a base for P^* if 0 is not in the w*-closed hull of B and $P^* = \mathrm{cone}(B)$). If $P \neq Y$, then $0 \notin \mathrm{qint}\ P$. Note also that $\mathrm{qint}\ P = \mathrm{cone}_+(\mathrm{qint}\ P)$ and $P + \mathrm{qint}\ P = \mathrm{qint}\ P$.

In the rest of the Tesis, $P \subseteq Y$ will be a closed convex cone with $P \neq Y$ and $\operatorname{qint} P \neq \emptyset$.

Some elementary properties of sets to be used later are collected in the next proposition.

Proposition 1.1.1 Let $A,\ M \subseteq Y$ nonempty sets.

(a) $\alpha A+(1-\alpha)A \subseteq \operatorname{cone}(A)\ \forall\, \alpha \in\ ]0,1[\ \Longleftrightarrow \operatorname{cone}(A)$ is convex $\Longleftrightarrow \operatorname{co}(A) \subseteq \operatorname{cone}(A)$.

(b) $\alpha A + (1-\alpha)A \subseteq \operatorname{cone}_+(A)\ \forall\, \alpha \in\ ]0,1[\ \Longleftrightarrow \operatorname{cone}_+(A)$ is convex $\Longleftrightarrow \operatorname{co}(A) \subseteq \operatorname{cone}_+(A)$.

(c) $\operatorname{cone}_+(A+M) = \operatorname{cone}_+(A) + M$ provided that M is such that $tM \subseteq M\ \forall\, t > 0$.

(d) $\operatorname{cone}(A) + M \subseteq \overline{\operatorname{cone}}(A+M)$ *and* $\overline{\operatorname{cone}(A) + M} = \overline{\operatorname{cone}}(A+M)$, provided that M is a cone.

(e) $\overline{\operatorname{cone}}(A+\operatorname{qint} P) = \overline{\operatorname{cone}}(A+P)$, provided that P is a convex cone with $\operatorname{qint} P \neq \emptyset$.

(f) $\operatorname{cone}_+(A + \operatorname{int} P)$ is convex $\Longleftrightarrow \operatorname{cone}(A + \operatorname{int} P)$ is convex $\Longleftrightarrow \overline{\operatorname{cone}}(A+P)$ is convex, provided that P is a convex cone with $\operatorname{int} P \neq \emptyset$.

Proof. The proof of (a), (b) and (c) is straightforward.
(d): According to (c), $\operatorname{cone}_+(A) + M = \operatorname{cone}_+(A+M) \subseteq \overline{\operatorname{cone}}(A+M)$. On the other hand, for a fixed $a \in A$, every $p \in M$ can be obtained as the limit of $\frac{1}{n}(a+np)$. Hence $M \subseteq \overline{\operatorname{cone}}(A+M)$ and this shows the inclusion in (d). Since obviously $\operatorname{cone}(A+M) \subseteq \operatorname{cone}(A) + M$, the equality of closures also follows.
(e): Since $\operatorname{qint} P \subseteq P$, we have $\overline{\operatorname{cone}}(A + \operatorname{qint} P) \subseteq \overline{\operatorname{cone}}(A+P)$. Also, from $P \subseteq \overline{\operatorname{qint} P}$ it follows that $A + P \subseteq A + \overline{\operatorname{qint} P} \subseteq \overline{A + \operatorname{qint} P} \subseteq \overline{\operatorname{cone}}(A + \operatorname{qint} P)$, hence (e) follows.
(f): If $\operatorname{cone}_+(A + \operatorname{int} P)$ is convex, then it easily follows that $\operatorname{cone}(A + \operatorname{int} P)$ is convex. By using (e), we deduce that $\overline{\operatorname{cone}}(A+P)$ is convex. If $\overline{\operatorname{cone}}(A+P)$ is convex, then $\operatorname{cone}_+(A + \operatorname{int} P)$ is convex by Theorem 2.6 in [49]. □

Remark 1.1.1 Proposition 1.1.1(f) does not hold with $\mathrm{qint}\ P$ in the place of $\mathrm{int}\ P$. Indeed, let $Y = l^1$ and $P = l^1_+$. Then $\mathrm{qint}\ l^1_+ = \left\{(\alpha_i)_{i\in\mathbb{N}} : \alpha_i > 0\right\}$ while $\mathrm{int}\ l^1_+ = \emptyset$. Set

$$A = l^1 \backslash \left(-\mathrm{qint}\ l^1_+\right) = \left\{(\alpha_i)_{i\in\mathbb{N}} : \exists\, i \in \mathbb{N} \text{ with } \alpha_i \geq 0\right\}.$$

Each $(a_i)_{i\in\mathbb{N}} \in l^1$ can be written as a limit of a sequence of elements each of which has a finite number of nonzero coordinates. Thus $\overline{A} = l^1$ and $\overline{\mathrm{cone}}(A + l^1_+) = l^1$ is convex. However, one can readily check that $\mathrm{cone}_+(A + \mathrm{qint}\ P) = A + \mathrm{qint}\ P = \left\{(\alpha_i)_{i\in\mathbb{N}} : \exists\, i \in \mathbb{N} \text{ with } \alpha_i > 0\right\}$ is not convex.

Definition 1.1.4 Let x_0 be a point of a convex set A. x_0 is an **extreme point** of A if x, y $\in A$ and $tx + (1-t)y = x_0$ for some t $(0 \leq t \leq 1)$ entails $x = y = x_0$.

We denote for ext(A) to the set of extreme points of the set A.

Theorem 1.1.1 Let B be a subset of the compact convex set $C \subseteq Y$. The following conditions are equivalent:

(a) $\overline{\mathrm{co}}(B) = C$;

(b) $\mathrm{ext}(C) \subseteq \overline{B}$.

Proof. Can be found in Ref. [29]. □

Definition 1.1.5 A function $f : K \to \mathbb{R}$ with K being a convex set:

(a) is said to be **semistrictly quasiconvex**, if given any u, v in K,$f(u) \neq f(v)$, one has $f(z) < \max\{f(u), f(v)\}$ for all $z \in]u, v[$;

(b) is said to be **quasiconvex** if each of its level set is a convex set, or equivalently, if $f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$ for all x, y in K and all $t \in [0, 1]$.

Simple examples show that are functions that are semi-strictly quasiconvex but not quasiconvex. However, every l.s.c. and semistrictly quasiconvex function is quasiconvex [37].

1.2 Asymptotic analysis

We now assume that X is a reflexive Banach space. For any nonempty set C in X, we define the asymptotic cone of C as the set

$$C^\infty = \Big\{x \in X : \exists\, t_n \downarrow 0, \exists\, x_n \in C, \ \ t_n x_n \rightharpoonup x\Big\},$$

where "$\rightharpoonup$" means convergence in the weak topology. We set $\emptyset^\infty = \emptyset$.
In the case when C is weakly closed and starshaped at $x_0 \in C$, that is, if for every $x \in C$ one has $x_0 + t(x - x_0) \in C$ for all $t \in \ [0, 1[$, then

$$C^\infty = \Big\{v \in X : \ \ x_0 + tv \in C \;\forall\, t > 0\Big\} = \bigcap_{t>0} t(C - x_0).$$

If C is convex and closed this cone does not depend on $x_0 \in C$.

For any given function $h : X \to \mathbb{R} \cup \{+\infty\}$, the asymptotic function of h is defined as the function h^∞ such that

$$\text{epi } h^\infty = (\text{epi } h)^\infty.$$

Here, $\text{epi } h = \{(x, t) \in X \times \mathbb{R} : \ \ h(x) \leq t\}$ is the epigraph of h. Consequently, it is not difficult to prove that (Ref. [5])

$$h^\infty(v) = \inf \left[\liminf_{n\to+\infty} \ t_n h(\frac{x_n}{t_n}) : \ t_n \downarrow 0, x_n \rightharpoonup v\right].$$

When h is a convex and lower semicontinuous function, we have

$$h^\infty(v) = \lim_{\lambda\to+\infty} \frac{h(x_0 + \lambda v) - h(x_0)}{\lambda} = \sup_{\lambda>0} \frac{h(x_0 + \lambda v) - h(x_0)}{\lambda} \ \ \forall\, x_0 \in \text{dom } h,$$

where as usual, $\text{dom } h = \{x \in X : \ \ h(x) < +\infty\}$. We notice the independence of h^∞ on the choice of x_0. If $f : K \subseteq X \to \mathbb{R}$, f^∞ will denote the asymptotic function of f, where we extend f to the whole X by setting $f(x) = +\infty$ if $x \in X \setminus K$.

We collect some basic results on asymptotic cones in the next proposition (see Ref. [46] for instance) that will be useful in the sequel.

Proposition 1.2.1 The following assertions hold.

(a) $K_1 \subseteq K_2$ implies $K_1^\infty \subseteq K_2^\infty$;

(b) $(K + x)^\infty = K^\infty$ for all $x \in X$;

(c) let $K \subseteq \mathbb{R}^n$, then K is bounded if and only if $K^\infty = \{0\}$;

(d) Let (K_i), $i = 1, \ldots, m$, be any finite family of nonempty sets in X, then

$$\Big(\bigcup_{i=1}^{m} K_i\Big)^\infty = \bigcup_{i=1}^{m} (K_i)^\infty.$$

(e) Let (K_i), $i \in I$, be any family of nonempty sets in X, then

$$\Big(\bigcap_{i \in I} K_i\Big)^\infty \subseteq \bigcap_{i \in I} (K_i)^\infty.$$

If, in addition, $\cap_i K_i \neq \emptyset$ and each set K_i, $i \in I$, is closed and convex, then we obtain an equality in the previous inclusion.

1.3 Some basic definition in vector optimization

Let X be a real Hausdorff topological space , Y be a real locally convex topological vector space and P be a convex cone in Y, it will eventually be required to be closed with nonemtpty interior.
By Y^* we denote the topological dual space of Y, and the duality pairing between Y^* and Y is denoted by $\langle \cdot, \cdot \rangle$. The set $P^* \subseteq Y^*$ is the polar (positive) cone of P defined by

$$P^* = \Big\{ p^* \in Y^* : \ \langle p^*, p \rangle \geq 0 \ \ \forall \, p \in P \Big\}.$$

The closedness and convexity of the cone P is equivalent to (the bipolar theorem) $P = P^{**}$. In this case,

$$p \in P \iff \langle p^*, p \rangle \geq 0 \ \ \forall \, p^* \in P^*. \tag{1.3.1}$$

Moreover,

$$p \in \operatorname{int} P \iff \langle p^*, p \rangle > 0 \ \ \forall \, p^* \in P^* \setminus \{0\}. \tag{1.3.2}$$

In the following definitions we consider Y a real normed vector space.
Given a nonempty set $K \subseteq X$, we shall also need the notion of epigraph of $F : K \to Y$. It is, as usual, the set

$$\text{epi } F \doteq \Big\{(x, y) \in K \times Y : \; y \in F(x) + P\Big\}.$$

Definition 1.3.1 A mapping $F : K \to Y$ is said to be (Refs. [43, 40]) *P-lower semicontinuous* (lsc) at $x_0 \in K$ if for any open set $V \subseteq Y$ such that $F(x_0) \in V$ there exists an open neighborhood $U \subseteq X$ of x_0 such that $F(U \cap K) \subseteq V + P$. We shall say that F is P-lsc (on K) if it is at every point $x_0 \in K$.

We have the following proposition.

Proposition 1.3.1 Let $P \subseteq Y$ be a convex cone, $K \subseteq X$ and $S \subseteq Y$ be closed sets such that $S + P \subseteq S$ and $S \neq Y$. Further, we are given $F : K \to Y$. The following assertions hold.

(a) If F is a P-lsc function, then $\{x \in K : \; F(x) \in \lambda - S\}$ is closed for all $\lambda \in Y$;

(b) Assume $\text{int } P \neq \emptyset$ and P closed: F is P-lsc if and only if $\{x \in K : \; F(x) - \lambda \notin \text{int } P\}$ is closed for all $\lambda \in Y$;

(c) Assume $\text{int } P \neq \emptyset$ and P closed: $\text{epi } F$ is closed if and only if $\{x \in K : \; F(x) - \lambda \in -P\}$ is closed for all $\lambda \in Y$;

(d) Assume $\text{int } P \neq \emptyset$ and P closed: if F is P-lsc then $\text{epi } F$ is closed.

Proof. Part (a) is taken from Ref. [20], Part (b) can be found in Ref. [6], and (c) in Theorem 5.8 of Ref [40], Chapter 1. □

We now recall relaxed notions of convexity that are mostly used in vector optimization.

Definition 1.3.2 Assume $P \subseteq Y$ is a convex cone with $\text{int } P \neq \emptyset$. The function $F : K \to Y$, is said to be

(i) P-convex if for all $x, y \in K$,

$$\alpha F(x) + (1 - \alpha)F(y) \in F(\alpha x + (1 - \alpha)y) + P \quad \text{for all } \alpha \in \;]0, 1[;$$

(ii) properly P-quasiconvex ([15]) if for every $x, y \in K$, every $\alpha \in\ ]0,1[$,

$$F(\alpha x + (1-\alpha)y) \in F(x) - P \quad \text{or} \quad F(\alpha x + (1-\alpha)y) \in F(y) - P,$$

or equivalently, the set

$$\Big\{\xi \in K : \ F(\xi) - \lambda \notin P\Big\}$$

is convex for all $\lambda \in Y$;

(iii) P-quasiconvex ([15, 36]) if the set

$$\Big\{\xi \in K : \ F(\xi) \in \lambda - P\Big\}$$

is convex for all $\lambda \in Y$;

(iv) semi-strictly $(Y \setminus -\text{int } P)$-quasiconvex ([21]) if for every $x, y \in K$,

$$F(x) - F(y) \notin \text{int } P \implies F(\alpha x + (1-\alpha)y) - F(y) \notin \text{int } P \quad \forall\, \alpha \in\ ]0,1[.$$

As noted in [15] there is no relationship between the notions of P-convexity and properly P-quasiconvexity except in the situation described in the next theorem.

Certainly, properly P-quasiconvexity and P-quasiconvexity are the more common generalizations of the notion of quasiconvexity for real-valued functions to the vectorial setting. The semi-strict $(Y \setminus -\text{int } P)$-quasiconvexity is also another generalization.

The class of P-convex function has been studied extensively in [40]. The following theorem supplements the previous remarks.

Theorem 1.3.1 Assume $K \subseteq X$ is a convex set and $P \subseteq Y$ a convex cone such that $\text{int } P \neq \emptyset$. Concerning Definition 1.3.2 we have the following assertions:

(a) $(i) \Longrightarrow (iii)$;

(b) if in addition P is closed then $(iii) \Longrightarrow (iv)$;

(c) if additionally $P \cup (-P) = Y$ then, $(iii) \iff (ii)$;

(d) if additionally $P \cup (-P) = Y$ and P is closed, then $P = Y \setminus -\text{int}\, P$ and

$$(iv) \Longleftrightarrow (iii) \Longleftrightarrow (ii);$$

(e) if in addition P is closed then

$$P = Y \setminus -\text{int}\, P \iff P \cup (-P) = Y \iff \exists\, p^* \in P^* \setminus \{0\},\ P = \Big\{p \in Y :\ \langle p^*, p\rangle \geq 0\Big\}.$$

Proof. See [21]. □

Definition 1.3.3 *([21])*Let $K \subseteq X$ be a convex set and let $S \subseteq Y$ be any non-empty set. The function $H : K \to Y$ is said to be explicitly (S)-quasiconvex, or equivalently, explicitly $(Y \setminus -S)$-quasiconvex, if it is semi-strictly (S)-quasiconvex and semi-strictly $(Y \setminus -S)$-quasiconvex

Regarding others notions related to that of explicit $(\text{int}\, P)$-quasiconvexity , we have the following definition.

Definition 1.3.4 Let $K \subseteq X$ be a convex set and $P \subseteq Y$ be a convex cone such that $\text{int}\, P \neq \emptyset$. The function $F : K \to Y$ is said to be

(i) explicitly P-quasiconvex ([6]) if it is P-quasiconvex and semi-strictly $(\text{int}\, P)$-quasiconvex.

(ii) explicitly properly P-quasiconvex ([39]) if it is properly P-quasiconvex (see Definition 1.3.2) and semi-strictly P-quasiconvex;

(iii) (here $Y = \mathbb{R}^m$ and $P = \mathbb{R}^m_+$) explicitly quasiconvex componentwise ([45]) if each component of F is a quasiconvex and semi-strictly quasiconvex real-valued function.

Theorem 1.3.2 Let $K \subseteq X$ be a non-empty convex set, let $P \subseteq Y$ be a convex cone such that $\text{int } P \neq \emptyset$. Given any function $F : K \to Y$, the following assertions hold:

(a) if F is P-convex, it is explicitly $(\text{int } P)$-quasiconvex;

(b) if F is explicitly properly P-quasiconvex, it is explicitly $(\text{int } P)$-quasiconvex;

(c) (here $Y = \mathbb{R}^m$ and $P = \mathbb{R}^m_+$) if F is explicitly quasiconvex componentwise, it is explicitly $(\text{int } \mathbb{R}^m_+)$-quasiconvex.

Proof. See [21] . □

1.4 Notions of minimum in vector optimization

Let us consider a real topological vector space Y and X a Hausdorff topological space; a convex cone $P \subseteq Y$ such that $P \neq Y$ and (its topological interior) $\text{int } P \neq \emptyset$; a closed set $K \subseteq X$, and a vector-valued function $F : K \to Y$. A point $\bar{x} \in K$ is said to be: an *ideal* or *strong* minimizer of F on K, if

$$F(y) - F(\bar{x}) \in \; P \;\text{ for all } y \in K; \tag{1.4.1}$$

an *efficient* or *Pareto* minimizer of F on K, if ($l(P) \doteq P \cap (-P)$)

$$F(y) - F(\bar{x}) \notin -(P \setminus l(P)) \;\text{ for all } y \in K; \tag{1.4.2}$$

a *weakly efficient* or *weakly Pareto* minimizer of F on K, if

$$F(y) - F(\bar{x}) \notin -\text{int } P \;\text{ for all } y \in K. \tag{1.4.3}$$

Since

$$P \subseteq Y \setminus -(P \setminus l(P)) \subseteq Y \setminus -\text{int } P,$$

every solution to (1.4.1) is a solution to (1.4.2), and every solution to (1.4.2) is a solution to (1.4.3).

The solutions set to problem (1.4.3) is denoted by $E_w(K)$ or simply E_w. Let us notice that in the classic case, that is to said $P = \mathbb{R}^m_+$, where $\mathbb{R}^m_+$ is the non-negative orthant in $\mathbb{R}^n$. In this case $F(x) = (f_1(x), \cdots, f_m(x))$, with $f_i : K \longrightarrow \mathbb{R}$. Therefore a point $\bar{x} \in K$ that satisfies 1.4.3 it is equivalent to

$$\text{find } \bar{\mathbf{x}} \in K \quad \text{such that } \forall\, y \in K,\ \exists\, i:\ f_i(y) - f_i(\bar{x}) \geq 0. \tag{1.4.4}$$

Chapter 2

Existence results for weak efficiency

In this chapter we motivate ourselves for the following important theorem.

Given a compact (Hausdorff) topological space X; a function $f : X \to \mathbb{R} \cup \{+\infty\}$, the classical Weierstrass theorem asserts that f reaches its minimum value on X provided the sets $\{x \in X : \; f(x) \leq t\}$ are closed for every $t \in \mathbb{R}$, i.e., provided f is lower semicontinuous.

If we are interested in a vector version of this theorem, we need to specify the meaning of minimizer for f. Among the different notions of minimizer (all of them collapse in the real-valued case), see chapter 1, section 4.

To be more precise, in addition to X, let us consider a real topological vector space Y; a convex cone $P \subseteq Y$ such that $P \neq Y$ and (its topological interior) $\text{int } P \neq \emptyset$; a closed set $K \subseteq X$, and a vector-valued function $F : K \to Y$.

Most of the Weierstrass-type theorems for the vectorial case concern problem (1.4.3) (see Refs. [3, 6, 9, 13, 19, 21, 31] among others) and only a few problem (1.4.1) (see Refs. [19, 20]). The existence of solution to (1.4.2) is discussed in Ref. [15] under the assumption that

$$G^{\lambda} \doteq \{x \in K : F(x) - \lambda \in -P\}$$

is closed for all $\lambda \in Y$, which is equivalent to the closedness of $\text{epi } F \doteq \{(x, y) \in X \times Y : \; y \in F(x) + P\}$ provided P is closed, see Theorem 5.8 of Chapter 1 in Ref. [40].

The aim of the present chapter is twofold: the first one, is to provide existence results for the problem (1.4.3), without convexity on F, by assuming the closedness of

$$G(y) \doteq \{x \in K : \ F(x) - F(y) \notin \text{int } P\}$$

for all $y \in K$, and in the case when some additional structure on X is assumed, to get existence of solution under a weaker assumption than convexity (see Definition 2.2.1), secondly, to derive some equivalences for the nonemptiness and compactness of the weakly solution set under convexity/quasiconvexity conditions. We have to point out that the closedness of $G(y)$ for all $y \in K$ does not imply the closedness of G^λ, likewise, the closedness of G^λ for all $\lambda \in Y$ does not imply the closedness of $G(y)$. For the first assertion, simply take $P = \mathbb{R}^2_+$,$K = [0, 10] \subseteq \mathbb{R}$, $F(x) = (x, x)$, $x \neq 1$, $F(1) = (1, 5)$. Then, $G(y) = [0, y]$ if $y \in K \setminus \{1\}$ and $G(1) = [0, 5]$; whereas $G^\lambda = [0, 1[\ \cup \ ]1, 4]$ for $\lambda = (4, 4)$. For the second assertion, let us consider again $K = [0, 10]$, $P = \mathbb{R}^2_+$ and $F(x) = (x, \frac{1}{x})$ if $x \neq 0$, $F(0) = (1, 3)$. Then G^λ is closed for all $\lambda \in \mathbb{R}^2$, whereas $G(\frac{1}{2}) = \ ]0, \frac{1}{2}] \cup [\frac{1}{2}, 10] = \ ]0, 10]$.

2.1 An existence theorem without convexity

Let X be a real Hausdorff topological space, Y be a real normed vector space. Given a compact set K, a convex cone $P \subseteq Y$ such that $\text{int } P \neq \emptyset$, and any vector function $F : K \to Y$, we are interested in the problem (1.4.3), it is to say

$$\text{find } \bar{x} \in K : \ F(y) - F(\bar{x}) \notin -\text{int } P \ \text{ for all } y \in K. \tag{2.1.1}$$

We derive an existence result for the problem (2.1.1) as a consequence of a more general result proved in Ref. [41].

We first need three concepts taken from Refs. [40, 41]. We say that a convex cone $C \subseteq Y$ is *correct* if

$$\overline{C} + C \setminus l(C) \subseteq C,$$

or equivalently, $\overline{C} + C \setminus l(C) \subseteq C \setminus l(C)$. Here $l(C) \doteq C \cap (-C)$. A net $\{y_\alpha : \ \alpha \in J\}$ in Y for some directed set J, is said to be *decreasing* if $y_\alpha - y_\beta \in C$ and not $y_\beta - y_\alpha \in C$ for each $\alpha, \beta \in J$, $\beta > \alpha$. The set $A \subseteq Y$ is said to be C-*complete* if there are no covers of the form $\{(y_\alpha - \overline{C})^c\}$ where $\{y_\alpha\}$ is a decreasing net in A.

By $E(A|C)$ we mean the set of all the efficient points of A with respect to C, that is, the set of $\bar{y} \in A$ such that there is no $y \in A$ satisfying

$$\bar{y} - y \in C \text{ and not } y - \bar{y} \in C,$$

or equivalently, $y - \bar{y} \in Y \setminus -(C \setminus l(C))$ for all $y \in A$.

The next theorem was proved in Ref. [41].

Theorem 2.1.1 *(Ref. [41], Theorem 2.6])* Suppose that C is a correct convex cone in Y. If A is nonemtpy C-complete then $E(A|C)$ is nonempty.

From the previous theorem, we will derive an existence result to problem 2.1.1, which may be considered as a vectorial case of the Weierstrass theorem.

Theorem 2.1.2 Let X, Y be as above and P be a convex cone satisfying $P \neq Y$ and $\text{int } P \neq \emptyset$. Furthermore, let $K \subseteq X$ be a compact set. Assume that $F : K \to Y$ is such that $G(y) \doteq \{x \in K : \ F(x) - F(y) \notin \text{int } P\}$ is closed for all $y \in K$. Then E_w is nonempty and compact.

Proof. We first notice that $E_w = E(F(K)|C)$ for $C = (\text{int } P) \cup \{0\}$ (a correct cone). The closedness of E_w being evident it suffices to show that $E_w \neq \emptyset$. By assuming $E_w = \emptyset$ we arrive at a contradiction if we can show that $F(K)$ is C-complete because of Theorem 2.1.1 (actually the counter-reciprocal of Theorem 2.1.1). If it is not C-complete, let $\{F(x_\alpha)\}$ be a decreasing net with $\{(F(x_\alpha) - \overline{C})^c\}_\alpha$ forming a recovering of $F(K)$. By compactness one may assume that $x_\alpha \to x_0$ for some $x_0 \in K$. Since $E_w = \emptyset$, there is $y \in K$ such that $F(x_0) - F(y) \in \text{int } P$. For $F(y)$, there is some index α_0 such that $F(y) \notin F(x_\alpha) - \overline{C}$ for all $\alpha > \alpha_0$. This in particular implies $F(x_\alpha) - F(y) \notin \text{int } P$ for $\alpha > \alpha_0$. The closedness assumption on $G(y)$ yields $F(x_0) - F(y) \notin \text{int } P$, a contradiction. This proves E_w is nonempty. □

We now show another proof for Theorem 2.1.2 using a fixed-point theorem for discrete set-valued mapping.

Lemma 2.1.1 Let Y be a vector space, $Y_0 = \{v_1, \ldots, v_m\} \subset Y$, $m \geq 2$, and $Q \subseteq Y$ be a non-empty set such that $Q + Q \subseteq Q$. Define the set-valued map $T : Y_0 \rightrightarrows Y_0$ by

$$T(u) = \Big\{v \in Y_0 : u - v \in Q\Big\}.$$

Then, the following assertions are equivalent:

(a) $T(v) \neq \emptyset$ *for all* $v \in Y_0$*;*

(b) $0 \in Q$*;*

(c) $v_i \in T(v_i)$ *for all* $i = 1, \ldots, m$*.*

Proof. $(a) \Longrightarrow (b)$: Let us fix v_1. By using (a), for $i = 1, \ldots, m$, we may recursively find $j_i \in \{1, \ldots, m\}$ such that $v_{j_{i-1}} - v_{j_i} \in Q$ where $j_0 = 1$. It is not hard to check that

$$\text{there exist } i \leq m, \ s < i, \ \text{ such that } j_i = j_s, \tag{2.1.2}$$

for if not Y_0 must contain at least $m + 1$ elements, which is impossible. Hence,

$$0 = v_{j_i} - v_{j_s} = \sum_{p=s+1}^{i} (v_{j_p} - v_{j_{p-1}}) \in -Q,$$

proving (b).
The implications $(b) \Longrightarrow (c)$ and $(c) \Longrightarrow (a)$ are obvious. □

We are now in a position to establish the proof of the theorem 2.1.2.

Clearly,

$$E_w = \bigcap_{y \in K} \Big\{x \in K : \ F(x) - F(y) \notin \text{int } P\Big\}.$$

Since each of the sets involved in the intersection is closed (actually compact) by assumption, it suffices to prove that

$$\bigcap_{i=1}^{m} G(y_i) \neq \emptyset \tag{2.1.3}$$

for every finite set $\{y_1, \ldots, y_m\}$ in K, where

$$G(y) \doteq \Big\{x \in K : \ F(x) - F(y) \notin \text{int } P\Big\}. \tag{2.1.4}$$

Suppose on the contrary, that

$$\bigcap_{i=1}^{m} G(y_i) = \emptyset. \tag{2.1.5}$$

Set $Y_0 = \{v_1, \ldots, v_m\} \subset Y$, $v_i = F(y_i)$. From (2.1.5) it follows that the set-valued map $T : Y_0 \rightrightarrows Y_0$ defined by

$$T(v_i) = \Big\{v \in Y_0 : \ v_i - v \in \text{int } P\Big\},$$

has non-empty values, that is, $T(v_i) \neq \emptyset$ for all $i = 1, \ldots, m$. By the previous lemma $0 \in \text{int } P$, which is a contradiction. Thus (2.1.3) holds and hence E_w is non-empty. The closedness, and so compactness, follows from the assumption.

In the case when K is not compact, we need the following condition:
(C): there exists a nonempty compact set $D \subseteq K$ such that for all $x \in K \setminus D$ there exists $y \in D$ such that $F(y) - F(x) \in -\text{int } P$.

This condition requires that any element outside D cannot be a candidate for solution to problem (2.1.1). Therefore $E_w \subseteq D$, and consequently E_w must be compact, being it closed.

Corollary 2.1.1 Assume the assumptions on X, Y, P of Theorem 2.1.2 hold and that K is closed. If condition (C) is satisfied, then E_w is nonempty and compact, and $E_w(K) = E_w(D)$, where $E_w(D)$ denotes the set of solutions to $(2.1.1)$ on D instead of K.

Proof. By Theorem 2.1.2, there exists $\bar{x} \in D$ such that $F(y) - F(\bar{x}) \notin -\text{int } P$ for all $y \in D$. We claim that in fact $\bar{x} \in E_w$. If $y \in K \setminus D$, $y \in K$, condition (C) implies the existence of $y_0 \in D$ such that $F(y_0) - F(y) \in -\text{int } P$. Thus

$$F(y) - F(\bar{x}) = F(y) - F(y_0) + F(y_0) - F(\bar{x}) \in P + (Y \setminus -\text{int } P) \subseteq Y \setminus -\text{int } P,$$

proving the desired result. □

Example 2.1.1 Let us consider $K = \mathbb{R}^2$, $P = \mathbb{R}^2_+$, and the function $F(x_1, x_2) = (\sqrt{|x_1|} + \sqrt{|x_2|}, \sqrt{|x_1|} + 2\sqrt{|x_2|})$. Then, condition (C) holds by taking

$$D = \{(x_1, x_2) : \ \max\{|x_1|, |x_2|\} \leq 1\}$$

and $y = (0, 0)$.

A stronger assumption implying the closedness of $G(y)$ (as required in Theorem 2.1.2) for all $y \in K$, concerns the P-lower semicontinuity of F. In case P is additionally closed, it is proven in Ref. [6] that F is P-lsc on K if, and only if, the set $G_\lambda \doteq \{x \in K : \ F(x) - \lambda \notin \text{int } P\}$ is closed for all $\lambda \in Y$. We notice that the closedness of $G(y)$ for all $y \in K$ does not imply the closedness of G_λ for all $\lambda \in Y$, as the following example shows: take $P = \mathbb{R}^2_+$, $K = [0, 10] \subseteq \mathbb{R}$, $F(x) = (x, x)$, $x \neq 1$, $F(1) = (1, 5)$. Then, $G(y) = [0, y]$ if $y \in K \setminus \{1\}$ and $G(1) = [0, 5]$; whereas $G_\lambda = [0, 1[\, \cup \,]1, 4]$ for $\lambda = (1/2, 4)$.

Remark 2.1.1 As mentioned in the introduction, an existence theorem concerning problem (1.4.2) was established in Ref. [15] by imposing the closedness of

$$G^\lambda \doteq \{x \in K : F(x) - \lambda \in -P\}$$

for all $\lambda \in Y$. It is equivalent to requiring the closedness of $\text{epi } F \doteq \{(x, y) \in X \times Y : \ y \in F(x) + P\}$, whenever P is closed (see Theorem 5.8 of Chapter 1 in Ref. [40]). It was shown in the previous section that, in general, there is no relationship between the closedness of $G(y)$ for all $y \in K$ and the closedness of G^λ for all $\lambda \in Y$. However, in the case when $Y = \mathbb{R}^m$, $P = \mathbb{R}^m_+$ and every component of F is bounded from above on K, it is not hard to prove that

$$G^\lambda \ \text{ is closed for all } \lambda \in \mathbb{R}^m \ \text{ if, and only if, } \ G_\lambda \ \text{ is closed for all } \lambda \in \mathbb{R}^m.$$

2.2 A class of generalized quasiconvex vector mapping: existence results

Let Y be a real normed vector space and X be a reflexive Banach space. We are also given a nonempty set $S \subseteq Y$, a nonempty convex set $K \subseteq X$, and a mapping

$F : K \to Y$. It is requested to find

$$\bar{x} \in K \text{ such that } F(y) - F(\bar{x}) \in S \;\; \forall\, y \in K. \tag{2.2.1}$$

A point $\bar{x} \in K$ satisfying (2.2.1) is called a (global) S-*minimal* of F (on K); whereas a point $\bar{x}$ is a local S-*minimal* of F (on K) if there exists an open neighborhood U, $\bar{x} \in U$, such that

$$F(x) - F(\bar{x}) \in S \;\; \forall\, x \in U \cap K. \tag{2.2.2}$$

The set of S-minimal points of F is denoted by E_S.

In connection to problem (2.2.1) the following definition, introduced earlier in Ref. [35] and independently in Ref. [21] with different uses, will play an important role.

Definition 2.2.1 *(Refs. [10, 21])* Given S, K as above with K being convex, the mapping $F : K \to Y$, is said to be semistrictly (S)-quasiconvex at $y \in K$, if for every $x \in K$, $x \neq y$,

$$F(x) - F(y) \in -S \implies F(\xi) - F(y) \in -S \;\;\; \forall\, \xi \in \;]x, y[.$$

We say that F is semistrictly (S)-quasiconvex (on K) if it is at every $y \in K$.

One can easily check that the vector functions

$$F_1(x) = (e^{-x^2}, x^2), \;\; x \in \mathbb{R};\; F_2(x_1, x_2) = \Big(\frac{x_1^2}{1 + x_1^2}, x_2^3\Big), \;\; (x_1, x_2) \in \mathbb{R}^2, \;\; \text{and}$$

$$F_3(x) = (\frac{1}{1 + |x|^2}, |x|), \;\; x \in \mathbb{R},$$

are semistrictly $(\mathbb{R}^2 \setminus -\mathrm{int}\, \mathbb{R}^2_+)$-quasiconvex.
Before going further, we have to make some comments. Let us consider the case $Y = \mathbb{R}$. Semistrict $(\mathbb{R}_+)$-quasiconvexity amounts to saying quasiconvexity in the usual sense, that is, the set $\{x \in K : F(x) \leq t\}$ is convex for all $t \in \mathbb{R}$, or equivalently, given any $x, y \in K$,

$$F(x) \leq F(y) \Longrightarrow F(\xi) \leq F(y) \;\; \forall\, \xi \in \;]x, y[;$$

whereas semistrict $(\mathbb{R}_{++})$-quasiconvexity amounts to saying semistrict quasiconvexity as usually known in mathematical programming, that is, given any $x, y \in K$,

$$F(x) < F(y) \Longrightarrow F(\xi) < F(y) \;\; \forall \, \xi \in \;]x, y[.$$

The semistrict (P)-quasiconvexity was mentioned in Ref. [35]. Instead, we will use semistrictly $(Y \setminus -\text{int } P)$-quasiconvex functions. In case $Y = \mathbb{R}^2$ and $P \subseteq \mathbb{R}^2$ is a polyhedral, various equivalent conditions to semistrict (P)-quasiconvexity were derived in Ref. [10], where the term (P, P)-quasiconvexity is used. In particular, one is expressed in terms of the Jacobian matrix of the function involved. Moreover, it is also proved in Ref. [10] that semistrict (P)-quasiconvexity is equivalent to P-quasiconvexity whenever the function is continuous, and therefore such a function will be semistrictly $(\mathbb{R}^2 \setminus -\text{int } P)$-quasiconvex as Theorem 2.6 in Ref. [21] asserts.

The following theorem encompasses a well known result valid for real-valued functions.

Theorem 2.2.1 *(Ref. [35])* Let S, K be nonempty sets as above. Let $\bar{x} \in K$ be a local S-minimal for F on K. Then, $\bar{x} \in E_S$ if and only if F is semistrictly $(Y \setminus -S)$-quasiconvex at $\bar{x}$.

For a given $y \in K$, we set

$$\mathcal{S}_y \doteq \Big\{ x \in K : \; F(x) - F(y) \in -S \Big\}.$$

The proof of the next lemma follows immediately from Definition 2.2.1.

Lemma 2.2.1 Given S, K as above and $F : K \to Y$, the following two assertions are equivalent for fixed $y \in K$:

(a) F is semistrictly (S)-quasiconvex at y;

(b) $[x, y[\; \subseteq \mathcal{S}_y$ for all $x \in \mathcal{S}_y$.

In case $0 \in S$, $y \in \mathcal{S}_y$ for all $y \in K$, we also obtain

Theorem 2.2.2 Given S, K as above with $0 \in S$, and $F : K \to Y$.

(a) F is semistrictly (S)-quasiconvex at $y \in K$ if, and only if, $\mathcal{S}_y$ is starshaped at y;

(b) Assume $X = \mathbb{R}$. Then, F is semistrictly (S)-quasiconvex $\Longleftrightarrow$ $\mathcal{S}_y$ is convex for all $y \in K \Longleftrightarrow$ for every finite subset $\{y_1, \ldots, y_m\}$ of K and for every $j \in \{1, \ldots, m\}$,

$$F(y_i) - F(y_j) \in -S \;\; \forall\, i \neq j \Longrightarrow \; F(\xi) - F(y_j) \in -S \quad \forall\, \xi \in \mathrm{co}\,\{y_1, \ldots, y_m\},\; \xi \neq y_i \;\forall\, i;$$

(c) Assume $(Y \setminus -S) + (Y \setminus -S) \subseteq Y \setminus -S$. If F is semistrictly (S)-quasiconvex, then for every $x, y \in K$, $x \neq y$, every $\xi \in \;]x, y[$,

$$F(\xi) - F(x) \in -S \quad \text{or} \quad F(\xi) - F(y) \in -S,$$

that is, either $\xi \in \mathcal{S}_x$ or $\xi \in \mathcal{S}_y$;

(d) Assume that $X = \mathbb{R}$ and $(Y \setminus -S) + (Y \setminus -S) \subseteq Y \setminus -S$. If F is semistrictly (S)-quasiconvex, then for all $x < y$ in K, $\sup \mathcal{S}_x \geq \inf \mathcal{S}_y$.

Proof. (a): This is straightforward from the definition of $\mathcal{S}_y$ and Lemma 2.2.1.
(b): The "if" part of the first equivalence was proved in (a). The "only if" is as follows: take $x_1, x_2 \in \mathcal{S}_y$, $x_1 < x_2$, and $x \in \;]x_1, x_2[$. If $x > y$, then $x \in \;]y, x_2[$, and therefore, $F(x) - F(y) \in -S$, i.e., $x \in \mathcal{S}_y$. In case $x < y$ we proceed in the same manner, proving the convexity of $\mathcal{S}_y$. This completes the proof of the first equivalence. The "if" part of the second equivalence is obvious. For the "only if" part, let $\xi \in \mathrm{co}\{y_1, \ldots, y_n\}$, and assume that $F(y_i) - F(y_j) \in -S$ for all $i \neq j$. If $\xi < y_j$, then $\xi \in \;]y_p, y_j[$ for some $p \in \{1, \ldots, m\}$, and therefore $F(\xi) - F(y_j) \in -S$. The case $\xi > y_j$ is treated in the same manner.
(c): If $F(\xi) - F(x) \notin -S$ and $F(\xi) - F(y) \notin -S$ for some $\xi \in \;]x, y[$, then by the semistrict (S)-quasiconvexity of F, $F(x) - F(y) \notin -S$ and $F(y) - F(x) \notin -S$. Thus $0 \in Y \setminus -S$, which is absurd, proving the desired result.
(d): Suppose $\sup \mathcal{S}_x < \inf \mathcal{S}_y$ and take $\xi \in \mathbb{R}$ such that

$$x \leq \sup \mathcal{S}_x < \xi < \inf \mathcal{S}_y \leq y.$$

This implies $\xi \notin \mathcal{S}_x$ and $\xi \notin \mathcal{S}_y$, which contradicts (c). □

Example 2.2.1 This example shows that the reverse implication in (c) and (d) of the previous theorem may be false. Take $F(x) = (-x, 1-(x-1)^2)$, $x \in K = [0,2]$, $S = \mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+$. After some calculations, we obtain $F(0) - F(3/2) = (3/2, -3/4) \in -S$ but $F(4/5) - F(3/2) = (7/10, 21/100) \notin -S$, thus F is not semistrictly (S)-quasiconvex. On the other hand, it is not hard to check that for any $x, y \in K$, $x < y$, we get $F(\xi) - F(x) \in -S$ for all $\xi \in \ ]x, y[$. Moreover, for all $x \in K$, $\mathcal{S}_x = [0, 2-x] \cup [x, 2]$ which implies $\sup \mathcal{S}_x = 2$ and $\inf \mathcal{S}_x = 0$.

A simple criterion for a vector mapping to be semistrictly (S)-quasiconvex is given in the following proposition whose proof is straightforward.

Proposition 2.2.1 Let K, S be nonempty sets as above. If $F : K \to Y$ is such that $H(y) \doteq \{x \in K : \ F(x) - F(y) \notin S\}$ is empty for all $y \in K$, then F is semistrictly (S)-quasiconvex.

Example 2.2.2 By setting $S = \mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+$, the above mentioned function F_1 is such that $H(y) = \emptyset$ and $S_y = \mathbb{R}$ for all $y \in \mathbb{R}$, thus F_1 is semistrictly $(\mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+)$-quasiconvex. On the other hand, the function F_2 shows that one implication in (b) of Theorem 2.2.2 may be false if X has dimension greater or equal than two (for S as above), since in this case

$$\mathcal{S}_y = \Big([-|y_1|, |y_1|] \times \mathbb{R} \Big) \cup \Big(\mathbb{R} \times \ ]-\infty, y_2] \Big).$$

Moreover, even when $K \subseteq \mathbb{R}$, the reverse implication in Proposition 2.2.1 may be false as the function $F(x) = (\sqrt{x}, x)$, $x \geq 0$, shows, since $H(y) = [0, y[$, $y > 0$.

In the same lines of reasoning as in Refs. [19, 21], we introduce the following cones in order to deal with the case when K is an unbounded set,

$$R_P \doteq \bigcap_{y \in K} \Big\{ v \in K^\infty : \ F(y + \lambda v) - F(y) \in -P \ \ \forall \, \lambda > 0 \Big\}, \tag{2.2.3}$$

$$R_S \doteq \bigcap_{y \in K} \Big\{ v \in K^\infty : \ F(y + \lambda v) - F(y) \in -S \ \ \forall \, \lambda > 0 \Big\}. \tag{2.2.4}$$

We recall that E_S denotes the set of $\bar{x} \in K$ satisfying (2.2.1).

Theorem 2.2.3 Let K be a closed convex set; let $P \subseteq Y$ be a convex cone and $S \subseteq Y$ be nonempty such that $S + P \subseteq S$. If the function $F : K \to Y$ is semistrictly (S)-quasiconvex and the set $\mathcal{S}_y$ is weakly closed for all $y \in K$, then

$$E_S + R_P = E_S \ , \text{ and } (E_S)^\infty \subseteq R_S. \tag{2.2.5}$$

Moreover, if $E_S \neq \emptyset$ and either $X = \mathbb{R}$ or $Y = \mathbb{R}$ (here $P = [0, +\infty[$), then E_S is convex and

$$(E_S)^\infty = R_S. \tag{2.2.6}$$

Proof. Obviously $E_S \subseteq E_S + R_P$. Both inclusions in (2.2.5) trivially hold if $E_S = \emptyset$. Otherwise, take any $\bar{x} \in E_S$ and v in R_P. Then, for all $y \in K$ and all $\lambda > 0$,

$$F(y) - F(\bar{x} + \lambda v) = F(y) - F(\bar{x}) + F(\bar{x}) - F(\bar{x} + \lambda v) \in S + P \subseteq S.$$

This completes the proof of the equality in (2.2.5). The second inclusion follows from the equality

$$E_S = \bigcap_{y \in K} \mathcal{S}_y, \tag{2.2.7}$$

and Proposition 1.2.1. The equality in (2.2.6) is a consequence of the convexity of $\mathcal{S}_y$ (see (b) of Theorem 2.2.2). □

Remark 2.2.1 The second inclusion in (2.2.5) may be strict. Indeed, take $K = \mathbb{R}^2$, $P = \mathbb{R}^2_+$, $S = \mathbb{R}^2 \setminus -\text{int}\, \mathbb{R}^2_+$ and $F(x_1, x_2) = (x_1^2, e^{x_2})$. Then

$$E_S = \{0\} \times \mathbb{R}, \ \ R_S = \Big(\{0\} \times \mathbb{R}\Big) \cup \Big(\mathbb{R} \times \ ]-\infty, 0]\Big).$$

Notice that the components of F are convex. An additional instance in which we have $(E_S)^\infty = R_S$ will be presented in (b) of Corollary 2.3.1.

We shall need the following definition which has been introduced in Ref. [42].

Definition 2.2.2 A subset C of a normed vector space X is said to be recessively compact if for all sequence $\{x_n\}$ in K with $||x_n|| \to +\infty$ as $n \to +\infty$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{t_k\}$ of positive numbers converging to 0 such that $t_k x_{n_k}$ converges strongly to some non-zero element of X.

Proposition 2.2.2 Let $K \subseteq X$ be a closed convex set and $S \subseteq Y$ be a nonempty set. Assume that $F : K \to Y$ is semistrict (S)-quasiconvex, and that $\mathcal{S}_y$ is weakly closed for all $y \in K$. If K is recessively compact and $R_S = \{0\}$, then ($K_r = \{x \in K : \ ||x|| \leq r\} \neq \emptyset$)

$$\exists\, r > 0,\ \forall\, x \in K \setminus K_r,\ \exists\, y \in K_r :\ F(y) - F(x) \notin S. \tag{2.2.8}$$

Consequently condition (C), introduced in section 2.1, is satisfied for $S = Y \setminus -\mathrm{int}\, P$. Furthermore, if $X = \mathbb{R}$ then $R_S = \{0\}$ if and only if $(2.2.8)$ holds.

Proof. Suppose on the contrary that for all $n \in \mathbb{N}$ there exists $x_n \in K \setminus K_n$ such that

$$F(y) - F(x_n) \in S \ \ \forall\, y \in K_n.$$

Since $||x_n|| > n$ we have $||x_n|| \to +\infty$ as $n \to +\infty$, and therefore by the recessive compactness of K there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{t_k\}$ of positive numbers converging to 0 such that $t_k x_{n_k}$ converges strongly to some non-zero element v of X. Clearly $v \in K^{\infty}$. On the other hand, for any fixed $y \in K$, $F(y) - F(x_n) \in S$ for all $n \in \mathbb{N}$ sufficiently large (for instance $n > ||y||$). Hence, for every $\lambda > 0$ and all k sufficiently large, the semistrict (S)-quasiconvexity of F implies

$$F((1 - \lambda t_k)y + \lambda t_k x_{n_k}) - F(y) \in -S.$$

Thus, by the weak closedness assumption, $F(y+\lambda v) - F(y) \in -S$, showing that $v \in R_S$, which cannot happen if $R_S = \{0\}$ is assumed, proving that (2.2.8) has to hold.
We now consider $X = \mathbb{R}$. Take any $x \in K_r$ and assume that $0 \neq v \in R_S$. Then, due to the convexity of K and since $v \in K^{\infty}$, we have $x + \lambda_0 v \in K \setminus K_r$ for some $\lambda_0 > 0$ sufficiently large. Then, by (2.2.8), there exists $y \in K_r \subseteq K$ such that $F(y) - F(x + \lambda_0 v) \notin S$. By the convexity of K again, there is $\lambda_1 > 0$ such that $x + \lambda_0 v = y + \lambda_1 v$. Hence

$$F(y) - F(y + \lambda_1 v) \notin S,$$

which says $v \notin R_S$, a contradiction. □

2.3 The finite dimensional case on $\mathbb{R}^n$

2.3.1 Characterizing the nonemptiness and boundedness of the weakly efficient solution set

This section is devoted to characterizing the nonemptiness and compactness of the solution set to the problem 2.1.1 where $K \subseteq \mathbb{R}^n$ is possibly unbounded. In this part we study two cases: Y be a real normed vector space and $Y = \mathbb{R}^m$.
We consider the hypothesis
HYPOTHESIS (H0): The set $P \subseteq Y$ is a convex (not necessarily pointed) cone such that $P \neq Y$ and $\text{int } P \neq \emptyset$.

The first case

In this situation we use E_w (see (2.1.1)), R_w instead of E_S and R_S respectively (see 2.2.2 and 2.2.4). More precisely

$$R_w \doteq \bigcap_{y \in K} \Big\{ v \in K^\infty : F(y + \lambda v) - F(y) \notin \text{int } P \ \ \forall \lambda > 0 \Big\}. \tag{2.3.1}$$

This cone is always convex whenever $K \subseteq \mathbb{R}$. The next theorem was proved in Ref [21] under the convexity condition on each $H(y) \doteq \{x \in K : \ F(x) - F(y) \in -\text{int } P\}$, $y \in K$. This convexity assumption was needed because of the use of the Knaster-Kuratowski-Mazurkiewicz lemma Ref. [4]; instead we apply Theorem 2.1.2.

Theorem 2.3.1 Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $P \subseteq Y$ satisfies hypothesis (H0). Assume that $F : K \to Y$ is semistrictly $(Y \setminus -\text{int } P)$-quasiconvex and the set $G(y) \doteq \{x \in K : \ F(x) - F(y) \notin \text{int } P\}$ is closed for all $y \in K$. If $R_w = \{0\}$ then E_w is a nonempty compact set.

Proof. This a consequence of Corollary 2.1.1 and Proposition 2.2.2 with $S = Y \setminus -\text{int } P$. □

Remark 2.3.1 Unfortunately, we do not know whether the condition $R_w = \{0\}$ is also necessary for the nonemptiness and compactness of E_w in this general setting. However, when $P = \mathbb{R}^m_+$ and each component of F is lower semicontinuous and convex, $R_w = \{0\}$ becomes also a necessary condition as shown in Ref. [13] (see also Corollary 5.13 in Ref. [20]).

The second case

Since $\mathrm{int}\ P \neq \emptyset$, the polar cone $P^* \subseteq \mathbb{R}^m$, of P, can be written as $P^* = \mathrm{cone}\ (B)$ where B is a compact convex set such that $0 \notin B$. More precisely, the nonemptiness of $\mathrm{int}\ P$ allows us to take $B = \{\xi \in \mathbb{R}^m : \langle \xi, p_0 \rangle = 1\}$ for some $p_0 \in \mathrm{int}\ P$. Here, $\langle x, y \rangle$ denotes the inner product of x and y in $\mathbb{R}^m$. One can check that P^* is closed convex and pointed ($P^* \cap (-P)^* = \{0\}$). It is known that

$$\begin{array}{lcccc} p \in \mathrm{int}\ P & \Longleftrightarrow & \langle p^*, p \rangle > 0\ \ \forall\, p^* \in P^*,\ p^* \neq 0 & \Longleftrightarrow & \langle p^*, p \rangle > 0\ \ \forall\, p^* \in B; \\ p \in P & \Longleftrightarrow & \langle p^*, p \rangle \geq 0\ \ \forall\, p^* \in P^* & \Longleftrightarrow & \langle p^*, p \rangle \geq 0\ \ \forall\, p^* \in B. \end{array} \tag{2.3.2}$$

By the Krein Milman theorem (see Theorem 1.1.1) we have $B = \mathrm{co}\ B_0$ where B_0 is the set of extreme points of B. We actually need the following hypothesis on P^*.

HYPOTHESIS (H1): Let P be a closed convex cone such that $\mathrm{int}\ P \neq \emptyset$ and $P^* = \mathrm{cone}\ (\mathrm{co}(B_0))$ where B_0 is closed being the set of extreme points of B (hence B_0 is compact).

It is clear that the polhyedral and ice-cream (Lorentz) cones satisfy Hypothesis (H1).

Under hypothesis (H1), (2.3.2) reduces to

$$\begin{array}{lcl} p \in \mathrm{int}\ P & \Longleftrightarrow & \langle q, p \rangle > 0\ \ \forall\, q \in B_0; \\ p \in P & \Longleftrightarrow & \langle q, p \rangle \geq 0\ \ \forall\, q \in B_0. \end{array} \tag{2.3.3}$$

By recalling that E_w is the solution set to problem (2.1.1), the representation of P^* and (2.3.3) imply (for $S = \mathbb{R}^m \setminus -\mathrm{int}\ P$)

$$E_w \doteq E_S = \bigcap_{y \in K} \bigcup_{q \in B_0} \Big\{ x \in K :\ \langle q, F(x) - F(y) \rangle \leq 0 \Big\}, \tag{2.3.4}$$

$$R_P = \bigcap_{y\in K} \bigcap_{\lambda>0} \bigcap_{q\in B_0} \Big\{v \in K^\infty :\ \langle q, F(y+\lambda v) - F(y)\rangle \le 0\Big\}, \tag{2.3.5}$$

and (see (2.2.4))

$$R_w \doteq R_S = \bigcap_{y\in K} \bigcap_{\lambda>0} \bigcup_{q\in B_0} \Big\{v \in K^\infty :\ \langle q, F(y+\lambda v) - F(y)\rangle \le 0\Big\}. \tag{2.3.6}$$

Additionally, we also consider the cone

$$\tilde{R}_w = \bigcap_{y\in K} \bigcup_{q\in B_0} \Big\{v \in K^\infty :\ \langle q, F(y+\lambda v) - F(y)\rangle \le 0\ \ \forall\, \lambda > 0\Big\}, \tag{2.3.7}$$

and finally let us introduce the function

$$h_q(x) = \langle q, F(x)\rangle,\ q \in P^*,\ x \in K.$$

Clearly $\tilde{R}_w \subseteq R_w$. Conditions ensuring the equality are given in the next proposition.

Proposition 2.3.1 Let $K \subseteq \mathbb{R}^n$ be closed and convex. Assume that P satisfies hypothesis (H1) and $F : K \to \mathbb{R}^m$ is such that h_q is quasiconvex for all $q \in B_0$. Then, for any fixed $y \in K$,

$$\bigcap_{\lambda>0} \bigcup_{q\in B_0} \Big\{v \in K^\infty :\ h_q(y+\lambda v) \le h_q(y)\Big\} = \bigcup_{q\in B_0} \bigcap_{\lambda>0} \Big\{v \in K^\infty :\ h_q(y+\lambda v) \le h_q(y)\Big\}. \tag{2.3.8}$$

Consequently, $\tilde{R}_w = R_w$.

Proof. We only need to prove the inclusion "$\subseteq$". Let v be in the set of the left hand side in (2.3.8). Then, for all $k \in \mathbb{N}$ there is $q_k \in B_0$ such that $h_{q_k}(y+kv) \le h_{q_k}(y)$. We may assume that $q_k \to q_0 \in B_0$. Let us fix any $\lambda > 0$. The quasiconvexity of h_{q_k} implies that for all k sufficiently large,

$$h_{q_k}(y+\lambda v) \le \max\{h_{q_k}(y), h_{q_k}(y+kv)\} = h_{q_k}(y).$$

Letting $k \to +\infty$, we obtain $h_{q_0}(y+\lambda v) \le h_{q_0}(y)$. This proves v belongs to the set of the right hand side of (2.3.8). □

Proposition 2.3.2 Let $K \subseteq \mathbb{R}^n$ be closed and convex and P satisfies hypothesis (H1). Assume that $F : K \to \mathbb{R}^m$ is P-lsc and that $h_q : K \to \mathbb{R}$, is quasiconvex for all $q \in B_0$. If $E_w \neq \emptyset$, we have $R_P \subseteq (E_w)^\infty \subseteq \tilde{R}_w$.

Proof. This follows from Theorem 2.2.3, Proposition 1.3.1 and the previous proposition since quasiconvexity of h_q for all $q \in B_0$ implies the semistrict ($\mathbb{R}^m \setminus -\mathrm{int}\ P$)-quasiconvexity of F. □

In the case when the functions f_i are convex more precise estimates for $(E_w)^\infty$ are obtained.

Corollary 2.3.1 Assume that $h_q : K \to \mathbb{R}$ is convex for all $q \in B_0$, and $F : K \to \mathbb{R}^m$ is P-lsc. Then,

- (a) if $E_w \neq \emptyset$, we have

$$\bigcap_{q \in B_0} \Big\{ v \in K^\infty : \ h_q^\infty(v) \leq 0 \Big\} \subseteq (E_w)^\infty \subseteq \bigcup_{q \in B_0} \Big\{ v \in K^\infty : \ h_q^\infty(v) \leq 0 \Big\}; \qquad (2.3.9)$$

- (b) if $\mathrm{argmin_K}\ h_q \neq \emptyset$ for all $q \in B_0$, then

$$(E_w)^\infty = \bigcup_{q \in B_0} \Big\{ v \in K^\infty : \ h_q^\infty(v) \leq 0 \Big\} = R_w.$$

Proof. (a): This is a consequence of Propositions 2.3.1 and 2.3.2 since

$$R_P = \bigcap_{q \in B_0} \Big\{ v \in K^\infty : \ h_q(y + \lambda v) - h_q(y) \leq 0 \ \ \forall \, \lambda > 0, \ \forall \, y \in K \Big\},$$

$$\begin{aligned} \tilde{R}_w &= \bigcap_{y \in K} \bigcup_{q \in B_0} \Big\{ v \in K^\infty : \ \langle q, F(y + \lambda v) - F(y) \rangle \leq 0 \ \ \forall \, \lambda > 0 \Big\} \\ &= \bigcap_{y \in K} \bigcup_{q \in B_0} \Big\{ v \in K^\infty : \ h_q^\infty(v) \leq 0 \Big\} = \bigcup_{q \in B_0} \Big\{ v \in K^\infty : \ h_q^\infty(v) \leq 0 \Big\} \\ &= \bigcup_{q \in B_0} \bigcap_{y \in K} \Big\{ v \in K^\infty : \ h_q(y + \lambda v) - h_q(y) \leq 0 \ \ \forall \, \lambda > 0 \Big\} \end{aligned}$$

(b) Since

$$(\mathrm{argmin}_K h_q)^\infty = \Big\{ v \in K^\infty : \; h_q^\infty(v) \le 0 \Big\} \tag{2.3.10}$$

and $\mathrm{argmin}_K h_q \subseteq E_w$ for all $q \in B_0$, we obtain

$$\bigcup_{q \in B_0} \Big\{ v \in K^\infty : \; h_q^\infty(v) \le 0 \Big\} \subseteq (E_w)^\infty,$$

which together with (a) the desired result follows. □

Example 2.3.1

(i) An instance showing that inclusions in (a) of the previous corollary may be strict was exhibited in Remark 2.2.1.

(ii) Let $K = \mathbb{R}^2$ and $F(x_1, x_2) = (x_1^2, x_2^2)$. Then $f_i^\infty(v_1, v_2) = 0$ if $v_i = 0$, $f_i^\infty(v_1, v_2) = +\infty$ elsewhere. Thus,

$$R_P = \{(0,0)\}, \; R_w = \Big(\{0\} \times \mathbb{R}\Big) \cup \Big(\mathbb{R} \times \{0\}\Big) = (E_w)^\infty = E_w.$$

The next result is an extension of Theorem 2.1 in Ref. [13], where $P = \mathbb{R}^n_+$ is considered.

Theorem 2.3.2 Let $K \subseteq \mathbb{R}^n$ be a closed convex set and P be a cone satisfying hypothesis (H1). Assume that $F : K \to \mathbb{R}^m$ is P-lsc such that $\langle q, F(\cdot) \rangle : K \to \mathbb{R}$ is convex for all $q \in B_0$. The following assertions are equivalent:

(a) E_w is nonempty and compact;

(b) $\mathrm{argmin}_K \langle q, F(\cdot) \rangle$ is nonempty and compact for all $q \in B_0$;

(c) $\tilde{R}_w = \{0\}$;

(d) $R_w = \{0\}$.

Proof. $(b) \implies (a)$: This results from (b) of Corollary 2.3.1 and (2.3.10) since

$$(\mathrm{argmin}_K h_q)^\infty = \Big\{ v \in K^\infty : \ h_q^\infty(v) \leq 0 \Big\}.$$

$(a) \implies (b)$: Suppose that $\mathrm{argmin}_K h_{q_0} = \emptyset$ for some $q_0 \in B_0$. We take $x_0 \in \mathrm{argmin}_{E_w} h_{q_0}$ and $x_1 \in K$ such that $\langle q_0, F(x_1) \rangle < \langle q_0, F(x_0) \rangle$. We now consider the nonempty closed convex set

$$D \doteq \bigcap_{q \in B_0} \Big\{ x \in K : \ h_q(x) \leq h_q(x_1) \Big\}.$$

Then

$$D^\infty = \bigcap_{q \in B_0} \Big\{ v \in K^\infty : \ h_q^\infty(v) \leq 0 \Big\} = \{0\}$$

by (a) of Corollary 2.3.1. It means that D is bounded and thus compact. Therefore, by Theorem 2.1.2 there exists $z \in D$ such that

$$F(x) - F(z) \notin -\mathrm{int}\, P \ \ \forall\, x \in D.$$

It is not difficult to check that $z \in E_w$. We also have

$$\langle q_0, F(z) \rangle \leq \langle q_0, F(x_1) \rangle < \langle q_0, F(x_0) \rangle,$$

contradicting the choice of x_0. This proves the nonemptiness of $\mathrm{argmin}_K h_q$ for all $q \in B_0$. The boundedness follows from the inclusion $\mathrm{argmin}_K h_q \subseteq E_w$ for all $q \in B_0$.
$(c) \iff (d)$: This is a consequence of Proposition 2.3.1. □

When $P = \mathbb{R}^m_+$ the equivalence between (a) and (b) of Theorem 2.3.2 was proved in Theorem 2.1 of Ref. [13].

2.4 The finite dimensional case on the real-line

2.4.1 Characterizing the nonemptiness and boundedness of the weakly efficient solution set

We now proceed as in the previous section.
In this part we study two cases: Y be a real normed vector space and $Y = \mathbb{R}^m$.
The first case

The next theorem extends the result in Theorem 4.1 of Ref. [21].

Theorem 2.4.1 Let $K \subseteq \mathbb{R}$ be a closed convex set and assume that $P \subseteq Y$ satisfies hypothesis (H0). If $F : K \to Y$ is semistrictly $(Y \setminus -\text{int } P)$-quasiconvex such that $\mathcal{S}_y$ is closed for all $y \in K$, then E_w is a closed convex set, and the following assertions are equivalent:

(a) $R_w = \{0\}$;

(b) $\exists\, r > 0,\ \forall\, x \in K \setminus K_r,\ \exists\, y \in K_r :\ F(y) - F(x) \in -\text{int } P$, where $K_r = [-r, r] \cap K$;

(c) E_w is nonempty and bounded (it is already closed and convex).

Proof. $(a) \iff (c)$: One implication is a consequence of Theorem 2.3.1, and the other implication follows from Theorem 2.2.3.
$(a) \iff (b)$: This results from Proposition 2.2.2 with $S = Y \setminus -\text{int } P$. □

The second case

In this part we consider the same hypothese of the previous section for the cone P.

The following theorem proves the equivalence between (a) and (b) of Theorem 2.3.2 without the convexity assumption. We firstly have the following theorem.

Theorem 2.4.2 Let $K \subseteq \mathbb{R}$ be a closed convex set and P be a cone satisfying hypothesis (H1). Assume that

$$\text{argmin}_K \langle q, F(\cdot) \rangle \neq \emptyset \ \text{ for all } \ q \in B_0.$$

If for all $q \in B_0$, the function $\langle q, F(\cdot) \rangle$ is semistrictly quasiconvex and lsc, then

$$E_w = \overline{\text{co}}\Big(\bigcup_{q \in B_0} \text{argmin}_K \langle q, F(\cdot) \rangle \Big).$$

In case B_0 is finite (e.g. P is polhyedra) then we can delete the closure in the preceding equality.

Proof. We know that $\operatorname{argmin}_K\langle q, F(\cdot)\rangle$ is closed. Since every lsc and semistrictly quasiconvex real-valued function is quasiconvex, we thus obtain that F is semistrictly $(\mathbb{R}^m \setminus -\operatorname{int} P)$-quasiconvex. By Theorem 2.2.2 and (2.2.7), E_w is convex. Therefore

$$A_0 \doteq \overline{\operatorname{co}}\Big(\bigcup_{q\in B_0} \operatorname{argmin}_K\langle q, F(\cdot)\rangle\Big) \subseteq E_w. \tag{2.4.1}$$

Set $h_q(x) = \langle q, F(x)\rangle$. Thus, A_0 is of the form $]-\infty, +\infty[$, $[\alpha, +\infty[$, $]-\infty, \alpha]$, $[\alpha, \beta]$ for some $-\infty < \alpha \leq \beta < +\infty$. Obviously, in the first case there is nothing to prove. We only consider the case $A_0 = [\alpha, +\infty[$. If $x \in E_w \setminus A_0$, by (2.3.3) we may choose $q \in B_0$ such that $h_q(x) \leq h_q(\alpha)$. Take any $x_q \in \operatorname{argmin} h_q$; then $h_q(x_q) < h_q(x)$ since $x \notin A_0$, and therefore $h_q(\alpha) < h_q(x)$ reaching a contradiction. □

Remark 2.4.1

(i) The preceding theorem may be false if $\operatorname{argmin}_K\langle q_i, F(\cdot)\rangle = \emptyset$ for some i. Indeed, take $P = \mathbb{R}^2_+$, $K = \mathbb{R}$, $F(x) = (\sqrt{|x|}, \frac{x}{1+|x|})$, $x \in \mathbb{R}$. Here, $E_w = \ ]-\infty, 0]$, while $\operatorname{argmin}_{\mathbb{R}} f_1 = \{0\}$.

(ii) This instance shows the necessity of semistrict quasiconvexity of all components. In fact, simply take $K = [0, +\infty[$ and

$$f_1(x) = \begin{cases} 2 & \text{if} \quad x \notin [1,2], \\ 1 & \text{if} \quad x \in [1,2], \end{cases} \quad f_2(x) = \begin{cases} -e^{-x+5} & \text{if} \quad x \geq 5, \\ 4 - x & \text{if} \quad x \leq 5. \end{cases}$$

Here $E_w = [1, +\infty[$.

(iii) Clearly, one cannot expect Theorem 2.4.2 continues to be valid if $K \subseteq \mathbb{R}^n$, $n \geq 2$, since in general E_w is not convex. For instance take $F(x_1, x_2) = (x_1^2, x_2^2)$, $(x_1, x_2) \in \mathbb{R}^2$. Here $E_w = (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$.

The result of the next theorem is optimal in the sense described in Remark 2.3.1.

Theorem 2.4.3 Let $K \subseteq \mathbb{R}$ be a closed convex set and $P \subseteq \mathbb{R}^m$ be a cone satisfying hypothesis (H1). Assume that $\langle q, F(\cdot)\rangle : K \to \mathbb{R}$ is lsc and semistrictly quasiconvex for all $q \in B_0$. The following assertions are equivalent:

(a) E_w is a nonempty compact convex set;

(b) $\mathrm{argmin}_K \langle q, F(\cdot)\rangle$ is a nonempty compact convex set for all $q \in B_0$;

(c) $\exists\, r > 0,\ \forall\, x \in K \setminus K_r,\ \exists\, y \in K_r\ (K_r = [-r, r] \cap K)$:

$$\langle q, F(y) - F(x)\rangle < 0 \ \ \forall\, q \in B_0.$$

Proof. As in the first part of the proof of Theorem 2.4.2, F is semistrictly $(\mathbb{R}^m \setminus -\mathrm{int}\ P)$-quasiconvex. Therefore, E_w is convex by Theorem 2.2.2 and (2.2.7). The closedness of E_w follows again from (2.2.7) and Proposition 1.3.1. The implication $(b) \implies (a)$ is a consequence of Theorem 2.4.2 and the remark above. Let us prove $(a) \implies (b)$. Clearly

$$\bigcup_{q\in B_0} \bigcap_{y\in K} \bigcap_{\lambda>0} \Big\{ v \in K^\infty :\ \langle q, F(y+\lambda v) - F(y)\rangle \le 0 \Big\} \subseteq$$

$$\subseteq \bigcap_{y\in K} \bigcap_{\lambda>0} \bigcup_{q\in B_0} \Big\{ v \in K^\infty :\ \langle q, F(y+\lambda v) - F(y)\rangle \le 0 \Big\} = (E_W)^\infty = \{0\},$$

where the former equality was obtained by using Theorem 2.2.2 and (2.3.3) together with the remarks above. Hence, for all $q \in B_0$,

$$\bigcap_{y\in K} \bigcap_{\lambda>0} \Big\{ v \in K^\infty :\ \langle q, F(y+\lambda v) - F(y)\rangle \le 0 \Big\} = \{0\}.$$

We apply Theorem 2.3.1 (with $Y = \mathbb{R}$, $X = \mathbb{R}$ and $P = \mathbb{R}_+$) to the function $x \mapsto \langle q, F(x)\rangle$. This allows us to conclude that for all $q \in B_0$, $\mathrm{argmin}_K \langle q, F(\cdot)\rangle$ is nonempty and bounded, and therefore compact. The convexity easily follows. The equivalence between (a) and (c) is a consequence of Theorem 2.4.1 and (2.3.3). □

Remark 2.4.2 Although we were unable to prove the validity of Theorem 2.4.3 in higher dimension, we believed that it is true, at least in the polyhedral case.

2.4.2 Characterizations of the nonemptiness of the weakly efficient solution

We now will center to studying the case $P = \mathbb{R}^m_+$. Therefore we rewrite the cone 2.3.1 in this particular case

$$
\begin{aligned}
R_w &= \bigcap_{y\in K} \left\{ v \in K^\infty : F(y+\lambda v) - F(y) \in \mathbb{R}^m \setminus - \operatorname{int} \mathbb{R}^m_+ \quad \forall \lambda > 0 \right\} \\
&= \bigcap_{y\in K} \bigcap_{\lambda>0} \bigcup_{i=1}^{m} \{ v \in K^\infty : f_i(y+\lambda v) - f_i(y) \le 0 \} .
\end{aligned} \tag{2.4.2}
$$

When each f_i is quasiconvex , then it is easily seen that

$$
R_w = \bigcap_{y\in K} \bigcup_{i=1}^{m} \{ v \in K^\infty : f_i(y+\lambda v) - f_i(y) \le 0 \quad \forall\, \lambda > 0 \} . \tag{2.4.3}
$$

If, each f_i is convex and lower semicontinuous in K, then

$$
R_w = \bigcup_{i=1}^{m} \{ v \in K^\infty : f_i^\infty(v) \le 0 \} .
$$

We now present various alternative equivalent conditions for E_w to nonempty. In what follows $|x|$ stands for the absolute value of $x \in \mathbb{R}$.

(C_1) : for any sequence $\{x_k\}$ in K satisfying:

(i) $|x_k| \to +\infty$, $\frac{x_k}{|x_k|} \to v \in R_w$, and,

(ii) for all $y \in K$ there exists k_y such that $F(y) - F(x_k) \notin -\operatorname{int} \mathbb{R}^m_+$ for all $k \ge k_y$,

we assume the existence of $u \in K$ and $\bar{k}$, such that $|u| < |x_{\bar{k}}|$ and $F(u) - F(x_{\bar{k}}) \notin \mathbb{R}^m_+$.

(C_2) : there exists a nonempty compact set $D \subseteq K$ such that for all $x \in K \setminus D$ there exists $u \in D : \; F(u) - F(x) \notin \operatorname{int} \mathbb{R}^m_+$.

(C_3) : there exists $u \in K$ and $r > |u|$ such that $F(u) - F(x) \notin \operatorname{int} R^m_+$ for all $x \in K$, $|x| = r$.

(C_4) : there exists $r > 0$ such that $K_r = K \cap [-r, r] \neq \emptyset$ and for all $x \in K$, $|x| = r$ there exists $u \in K$, $|u| < r : F(u) - F(x) \notin \operatorname{int} \mathbb{R}^m_+$.

(C_5) : for every $|x_k| \to +\infty$ there exists $\bar{k}$, $u \in K$, such that $|u| < |x_{\bar{k}}|$ and $F(u) - F(x_{\bar{k}}) \notin \operatorname{int} \mathbb{R}^m_+$.

We point out that all of these conditions apply to situations in which the solution set may be unbounded. Notice that the cone R_w is not explicitly mentioned in C_i, $i = 2, 3, 4, 5$. Clearly $(C_2) \Longrightarrow (C_1)$, $(C_3) \Longrightarrow (C_4)$ and $(C_5) \Longrightarrow (C_1)$.

We are now in a position to establish the main existence theorem in case $K \subseteq \mathbb{R}$.

Theorem 2.4.4 Let $\emptyset \neq K \subseteq \mathbb{R}$ be closed and convex. Assume that $F : K \to \mathbb{R}^m$ is explicitly ($\operatorname{int} \mathbb{R}^m_+$)-quasiconvex and S_y is closed for all $y \in K$. Then E_w is a closed convex set, and the following assertions are equivalent:

(a) (C_1) is satisfied;

(b) E_w is non-empty;

(c) (C_2) is satisfied;

(d) (C_3) is satisfied;

(e) (C_4) is satisfied;

(f) (C_5) is satisfied.

Proof. The closednees and convexity of E_w follows Theorem 2.2.2.
$(c) \Longrightarrow (a)$: It is obvious.
$(a) \Longrightarrow (b)$: For every $k \in \mathbb{N}$, set $K_k = \{x \in K : |x| \leq k\}$. We may suppose,without loss of generality, that $K_k \neq \emptyset$ for all $k \in \mathbb{N}$. Let us consider the problem

$$\text{find} \quad \bar{x} \in K_k \quad \text{such} \quad \text{that} \quad F(y) - F(\bar{x}) \notin -\text{int}\,\mathbb{R}^m_+ \quad \forall\, y \in K_k \tag{2.4.4}$$

By Theorem 2.1.2 , problem (2.4.4) admits a solution, say $x_k \in K_k$ for all $k \in \mathbb{N}$. We have two possibilities: if $\{x_k\}$ is bounded, then it is no difficult to check that any limit point of $\{x_k\}$ is a solution to problem (2.1.1); if on the contrary $|x_k| \to +\infty$, we may assume, up to a subsequence, that $\frac{x_k}{|x_k|} \to v$, and therefore $v \in K^\infty$. We shall prove that $v \in R_w$. Indeed, let us fix any $y \in K$ and $\lambda > 0$. Then, for all k sufficiently large $(k > |y|)$, $F(y) - F(x_k) \notin -\text{int}\,\mathbb{R}^m_+$. The semistrict $(\mathbb{R}^m \setminus -\text{int}\,\mathbb{R}^m_+)$-quasiconvexity gives $F((1-\frac{\lambda}{|x_k|})y + \frac{\lambda}{|x_k|}x_k) - F(y) \notin \text{int}\,\mathbb{R}^m_+$ for all k sufficiently large. Hence $F(y+\lambda v) - F(y) \notin \text{int}\,\mathbb{R}^m_+$ because of the closedness of S_y. This show that $v \in R_w$, proving that (i) of assumption (C_1) is satisfied. For any fixed $y \in K$ one has, $F(y) - F(x_k) \notin -\text{int}\,\mathbb{R}^m_+$ for all $k \in \mathbb{N}$ sufficiently large $(k > |y|)$, showing that condition (ii) of C_1 is also satisfied. We are now in a position to use such an assumption. This implies the existence of $u \in K$ and $\bar{k}$ such that $|u| < |x_{\bar{k}}|$ and $F(u) - F(x_{\bar{k}}) \notin \text{int}\,\mathbb{R}^m_+$. Thus we also have $F(u) - F(x_{\bar{k}}) \notin -\text{int}\,\mathbb{R}^m_+$ and $F(u) - F(x_{\bar{k}}) \notin -\text{int}\,\mathbb{R}^m_+$ because of the choice of $x_{\bar{k}}$. We will prove that such $x_{\bar{k}}$ is a solution to $(2.1.1)$. It only remains to check that $F(y) - F(x_{\bar{k}}) \notin -\text{int}\,\mathbb{R}^m_+$ for all $y \in K$ with $|y| > \bar{k}$. Let us consider the case when $x_{\bar{k}} \in]u, y[$ or $x_{\bar{k}} \in]y, u[$. If $F(u) - F(y) \in \text{int}\,\mathbb{R}^m_+$, by using the semistrict $(\text{int}\,\mathbb{R}^m_+)$-quasiconvexity, we have in particular $F(x_{\bar{k}}) - F(u) \in -\text{int}\,\mathbb{R}^m_+$, giving a contradiction. Hence $F(u) - F(y) \notin \text{int}\,\mathbb{R}^m_+$. By the semistrict $(\mathbb{R}^m \setminus -\text{int}\,\mathbb{R}^m_+)$-quasiconvexity, we have obtain $F(x_{\bar{k}}) - F(y) \notin \text{int}\,\mathbb{R}^m_+$, which is the desired result. We now consider the case $u \in]y, x_{\bar{k}}[$ or $u \in]x_{\bar{k}}, y[$: if on the contrary $F(y) - F(x_{\bar{k}}) \in -\text{int}\,\mathbb{R}^m_+$, the semistrict $(\text{int}\,\mathbb{R}^m_+)$-quasiconvexity of F yields in particular $F(u) - F(x_{\bar{k}}) \in -\text{int}\,\mathbb{R}^m_+$, a contradiction. This completes the proof of $(a) \Longrightarrow (b)$.
$(b) \Longrightarrow (c)$: It follows by taking any fixed $u \in E_w$ and by setting $D = \{u\}$ in condition (C_2). Similarly $(b) \Longrightarrow (d)$ is proved by choosing $u \in E_w$ and $r > |u|$.

$(d) \Longrightarrow (e)$: It is straightforward.
$(e) \Longrightarrow (b)$: We consider the problem on K_r, see (2.1.2), which admits a solution, say x_r. If $|x_r| < r$, we claim that x_r is also solution to problem (2.1.1). Assume there is $y \in K$ with $|y| > r$ such that $F(y) - F(x_r) \in -\text{int } \mathbb{R}^m_+$. Since F is semiestrictly $(\text{int } \mathbb{R}^m_+)$-quasiconvex, we have $F(\xi) - F(x_r) \in -\text{int } \mathbb{R}^m_+$ for all $\xi \in]y, x_r[$ (or $\xi \in]x_r, y[$). We may choose $z \in]y, x_r[$ (or $z \in]x_r, y[$) with $|z| < r$ since $|x_r| < r < |y|$. Therefore $F(z) - F(x_r \in -\text{int } \mathbb{R}^m_+$, contradicting the choice of x_r. If $|x_r| = r$, by assumption there exists $u \in K$, $|u| < r$ such that $F(u) - F(x_r) \notin \text{int } \mathbb{R}^m_+$. Then we proceed as in the second part of the proof of $(a) \Longrightarrow (b)$, to conclude that x_r is also a solution to problem (2.1.1).
$(f) \Longrightarrow (a)$: It is straightforward.
$(b) \Longrightarrow (f)$: Take any $x_k \in K$ with $|x_k| \to +\infty$. Then (f) is satisfied if we consider u to be any element in E_w. □

Example 2.4.1

(i) Let $F(x) = (e^x, e^{-x})$, $x \in \mathbb{R}$. A direct computation verifies the validity of condition (C_1) or (C_2), showing that $E_w = \mathbb{R}$. Notice that no component of F admits a minimizer.

(ii) Let us consider any function $F : K \to \mathbb{R}^m$ such that at least one component has a minimum point on K. Then, it is not difficult to check that condition (C_1) or (C_2) hold. An instance is given by the function $F(x) = (\sqrt{|x|}, \frac{x}{1+|x|}),\ x \in \mathbb{R}$. Here $E_w =]-\infty, 0] = \mathbb{R}_w$.

(iii) Let $F(x) = (x, \min\{0, -x\})$, $x \in \mathbb{R}$. It is $\mathbb{R}^2_+$-quasiconvex and therefore semistrictly $(\mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+)$-quasiconvex. Moreover, it is semistrictly $(\text{int}\mathbb{R}^2_+)$-quasiconvex. All others asumptions of Theorem (2.4.4) are satisfied as well. Here $E_w = \mathbb{R}$.

(iv) Let $F(x) = (\frac{1}{1+|x|^2}, |x|)$, $x \in \mathbb{R}$. It is semistrictly $(\mathbb{R}^2 \setminus -\text{int } \mathbb{R}^2_+)$ (but not $\mathbb{R}^2_+$-quasiconvex). Moreover it is semistrictly $(\text{int } \mathbb{R}^2_+)$-quasiconvex. All others asumptions of Theorem (2.4.4) are satisfied as well. Here $E_w = \mathbb{R}$.

(v) Let $F(x) = (x, -x^2)$, $x \in]-\infty, 0]$. This function is semistrictly $(\mathbb{R}^2 \setminus -\text{ int } \mathbb{R}^2_+)$-quasiconvex and semistrictly $(\text{int } \mathbb{R}^2_+)$-quasiconvex. Such a function satisfies the assumptions of Theorem (2.4.4) except condition (C_1). It is easy to check that $E_w = \emptyset$.

2.4.3 Quasiconvex vector minimization

We start with a preliminary result which will be used subsequently.

Lemma 2.4.1 Let $\emptyset \neq K \subseteq \mathbb{R}$ be convex and $h : K \to \mathbb{R}$ be a quasiconvex and lsc function. Assume that $h(\alpha_2) < h(\alpha_1)$ for some $\alpha_1, \alpha_2 \in K$, $\alpha_2 > \alpha_1$ (resp. $\alpha_2 < \alpha_1$). The following assertions hold:

(a) $h(\alpha) \geq h(\alpha_1)\ \forall \alpha < \alpha_1$ (resp. $\forall \quad \alpha > \alpha_1$) and h is non-increasing in $]-\infty, \alpha_1[\cap K$ (resp. h is non-decreasing in $]\alpha_1, +\infty[\cap K$);

(b) if $\text{argmin}_K h = \emptyset$ then $h(\alpha) \leq h(\alpha_2)\ \forall \alpha > \alpha_2$ (resp. $\forall \alpha < \alpha_2$) and h is non-increasing in $]\alpha_2, +\infty[\cap K$ (resp. h is non-decreasing in $]-\infty, \alpha_2[\cap K$). Consequently, h is non-increasing (resp. non-decreasing) in K.

Proof. (a) The first part is obvious. Take α and β in K satisfying $\alpha < \beta < \alpha_1$, then $h(\beta) \leq \max\{h(\alpha), h(\alpha_1)\} = h(\alpha)$ by the first part.
(b) If there is $\alpha > \alpha_2$ such that $h(\alpha) > h(\alpha_2)$, then h is non-decreasing in $]\alpha, +\infty[\cap K$, and since h is also non-increasing in $]-\infty, \alpha_1[\cap K$, we obtain

$$\inf_K h = \inf_{[\alpha_1, \alpha]} h.$$

Thus $\text{argmin}_K h \neq \emptyset$, a contradiction. This proves the first part of (b). For the second part we reason as follows. If on the contrary, there exist $\alpha^{'} < \alpha^{''}$, $\alpha^{'} > \alpha_2$ such that $h(\alpha^{'}) < h(\alpha_{''})$, then h is non-decreasing in $]\alpha^{''}, +\infty[\cap K$, and since h is non-increasing in $]-\infty, \alpha_1[\cap K$, we conclude

$$\inf_K h = \inf_{[\alpha_1, \alpha^{''}]} h,$$

implying that $\text{argmin}_K h \neq \emptyset$, a contradiction. The remaining situation is treated in a similar way. □

In what follows the closednees of S_y for all $y \in K$ will be substituted by the stronger assumption of lower semicontinuity (lsc) of $f_i,\ i = 1, \ldots\ldots, m$.

Theorem 2.4.5 Let $K \subseteq \mathbb{R}$ be a convex closed set; $f_i : K \to \mathbb{R}$ be lsc and quasiconvex for all $i = 1, \ldots, m$. The following assertions hold:

(a) if $E_w \neq \emptyset$ and compact then $\text{argmin}_K f_i \neq \emptyset$ and compact for all $i = 1, \ldots, m$;

(b) if $\emptyset \neq E_w \neq \mathbb{R}$, then there exists j such that $\text{argmin}_K f_j \neq \emptyset$;

(c) if $K \neq \mathbb{R}$, then

$$E_w \neq \emptyset \Longleftrightarrow \quad \exists\, j, \quad \text{argmin}_K f_j \neq \emptyset.$$

Proof. (a): Assume that E_w is bounded. It is known from the previous section that E_w is closed convex and $(E_w)^\infty = R_w$. We have

$$\bigcup_{i=1}^{m} \bigcap_{y \in K} \bigcap_{\lambda > 0} \{v \in K^\infty \,:\, f_i(y + \lambda v) \leq f_i(y)\} \subseteq$$

$$\subseteq \bigcap_{y \in K} \bigcap_{\lambda > 0} \bigcup_{i=1}^{m} \{v \in K^\infty \,:\, f_i(y + \lambda v) \leq f_i(y)\} = (E_w)^\infty = \{0\},$$

Hence, for all $i = 1, \ldots, m$,

$$\bigcap_{y \in K} \bigcap_{\lambda > 0} \{v \in K^\infty \,:\, f_i(y + \lambda v) \leq f_i(y)\} = \{0\}.$$

We apply Theorem 2.4.1 to the function f_i. This allows us to conclude that for all $i = 1, \ldots, m$, $\text{argmin}_K f_i$ is nonempty and bounded, and therefore compact.
(b): we consider the case $E_w = [\alpha, +\infty[$ with $\alpha \in \mathbb{R}$ and $K = [a, +\infty[$ with $a \leq \alpha$ (the case $E_w =]-\infty, \alpha]$ and $K =]-\infty, a]$ is analyzed in a similar manner). Suppose that $\text{argmin}_K f_i = \emptyset$ for all $i = 1, \ldots, m$. We can choose for $i = 1, \ldots, m$, $a_i > a_{i-1}$ such that

$f_i(a_i) < f_i(a_{i-1})$ with $a_0 = \alpha$. By using the previous lemma, f_1 is non-increasing in $[a, +\infty[$. Thus $f_1(a_2) \leq f_1(a_1) < f_1(\alpha)$ and therefore $f_1(\alpha) > f_1(\alpha_m)$. One recursively may also deduce that $f_i(a_m) < f_i(\alpha)$ for $i = 1, \ldots, m$, which imply $\alpha \notin E_w$. This proves (b) when $K = [a, +\infty[$.

We now consider the case $E_w = [\alpha, +\infty[$ and $K = \mathbb{R}$ (again when $E_w =]-\infty, \alpha]$ and $K = \mathbb{R}$ the reasoning is similar). We can choose for $i = 1, \ldots, m$, $a_i \in \mathbb{R}$ such that $|a_i| > |a_{i-1}|$ and $f_i(a_i) < f_i(a_{i-1})$ with $a_0 = \alpha$.

We first prove that

$$a_1 < a_0 = \alpha \Longrightarrow a_m < a_{m-1} < \ldots < a_1 < a_0; \tag{2.4.5}$$

$$a_1 > a_0 = \alpha \Longrightarrow a_m > a_{m-1} > \ldots > a_1 > a_0. \tag{2.4.6}$$

By symmetry we only check (2.4.5). If $a_1 < a_0 = \alpha$, we apply the previous lemma to conclude that f_1 is non-decreasing in $\mathbb{R}$. If $a_1 < a_2$ and since $f_2(a_2) < f_2(a_1)$ then f_2 is non-increasing in K. Both assertions show that $E_w = \mathbb{R}$, which is impossible by assumption. Hence $a_2 < a_1$ and since $f_2(a_2) < f_2(a_1)$ again, f_2 is non-decreasing in $\mathbb{R}$. We proceed recursively to conclude that $a_m < a_{m-1} < \ldots < a_1 < a_0$ with f_i being non-decreasing in $\mathbb{R}$ for $i = 1, \ldots, m$. This fact allows us to deduce that, as before, $f_i(a_m) < f_i(\alpha)$ for $i = 1, \ldots, m$, implying that $\alpha \notin E_w$, which yields a contradiction. This completes the proof of (b). Part (c) is then straightforward. □

Example 2.4.2 We now exhibit four instances showing the results of the previous Theorem are, in some sense, optimal.

(i) We see, in general, the reverse implication in (a) of the previous theorem fails to hold (except when K is bounded). In fact, take $K = [0, +\infty[$ and

$$f_1(x) = \begin{cases} 2 & \text{if} \quad x \notin [1,2], \\ 1 & \text{if} \quad x \in [1,2], \end{cases} \qquad f_2(x) = \begin{cases} 2 & \text{if} \quad x \notin [3,4], \\ 1 & \text{if} \quad x \in [3,4]. \end{cases}$$

Here $E_w = [0, +\infty[$.

(ii) The quasiconvexity of all the functions f_i required in (a) cannot be removed as the following function shows: The necessity of the quasiconvexity of all components of (a) is showed by the function:

$$f_1(x) = e^{-x}, \quad f_2(x) = \begin{cases} e^{-x^2+1} & \text{if} \quad x \geq 1, \\ x & \text{if} \quad x \leq 1, \end{cases} \qquad K = [0, +\infty[.$$

Here $E_w = \{0\}$, while $\mathrm{argmin}_K f_1 = \emptyset \neq \mathrm{argmin}_K f_2$.
(iii) Part (b) (resp. (c)) of the previous theorem may be false if the assumption $E_w \neq \mathbb{R}$ (resp. $K = \mathbb{R}$) is deleted. Simply take $K = \mathbb{R}$, and $f_1(x) = x$, $f_2(x) = -x$. In this case $E_w = \mathbb{R}$.
(iv) part (b) cannot be extended to higher dimension. Take for instance $f_1(x_1, x_2) = e^{x_1}$, $f_2(x_1, x_2) = -x_1 + x_2^2,\ (x_1, x_2) \in \mathbb{R}^2$. Here $E_w = \mathbb{R} \times \{0\}$, while $\mathrm{argmin}_{\mathbb{R}^2} f_i = \emptyset,\ i = 1, 2$.

In what follows, we set

$$J = \{i \in \{1, \dots, m\} \,:\, \mathrm{argmin}_K f_i \neq \emptyset\}$$

Theorem 2.4.6 Let $\emptyset \neq K \subseteq \mathbb{R}$ be closed and convex and $f_i : K \to \mathbb{R}$ be lsc and semistrictly quasiconvex for all $i = 1, \dots, m$. Assume that $E_w \neq \emptyset$. Then

$$\mathrm{extr}(E_w) \subseteq \bigcup_{j \in J} \mathrm{argmin}_K f_j.$$

Proof. If $\mathrm{extr}(E_w) = \emptyset$ there is nothing to prove. Thus, suppose that $\mathrm{extr}(E_w) \neq \emptyset$. Then, $E_w \neq \mathbb{R}$ and by Theorem 2.4.5 $J \neq \emptyset$. Let $\alpha \in \mathrm{extr}(E_w)$. We choose, for $j \in J, x_j \in \mathrm{argmin}_K f_j$, and, for $i \notin J$, $x_i \in K$ such that $f_i(x_i) < f_i(\alpha)$. Since α is an extreme point of E_w, $x_j \geq \alpha$ for all $j \in J$ or $x_j \leq \alpha$ for all $j \in J$. Let us suppose $x_j > \alpha$ and $\alpha \notin \mathrm{argmin}_K f_i$ for all $j \in J$. Then $f_j(x_j) < f_j(\alpha)$ for all $j \in J$. If for all $i \notin J$, $\alpha < x_i$, we reach a contradiction since, by quasiconvexity, it is not difficult to prove that $f_i(\bar{x}) < f_i(\alpha)$ for all $i = 1, \dots, m$, with $\bar{x} = \min\{x_i \,:\, i = 1, \dots, m\}$. The latter implies that $\alpha \notin E_w$. If there exists $x_i \in K, \quad i \notin J$, such that $x_i < \alpha$. Since $f_i(x_i) < f_i(\alpha)$, we apply Lemma 2.4.1 to deduce that f_i is non-decreasing in K. It follows that $x_i \in E_w$. In fact: for $x > x_i$ one has $f_i(x) \geq f_i(x_i)$, and for $x < x_i$, one obtains $f_j(x) \geq f_j(x_i)$ since f_j is non-increasing in $]-\infty, x_j] \cap K$, proving that $x_i \in E_w$. The fact that $x_i < \alpha < x_j$ contradicts the extremality of α. Therefore the proof of the theorem is completed. □

Example 2.4.3 We really need that all the components be semistrictly quasiconvex in Theorem 2.4.6. In fact, take $K = [0, +\infty[$ and

$$f_1(x) = \begin{cases} 2 & \text{if} \quad x \notin [1, 2], \\ 1 & \text{if} \quad x \in [1, 2], \end{cases} \qquad f_2(x) = |x - 5|.$$

Here $E_w = [1, 8]$.

Theorem 2.4.7 Let $\emptyset \neq K \subseteq \mathbb{R}$ be closed and convex and $f_i : K \to \mathbb{R}$ be lsc and semistrictly quasiconvex for all $i = 1, \ldots, m$. Assume that $E_w \neq \emptyset$. Then, either

$$E_w = \mathbb{R} \quad \textit{or} \quad E_w = \operatorname{co}\left(\bigcup_{j \in J} \operatorname{argmin}_K f_j\right) + R_w. \tag{2.4.7}$$

Moreover, if $\operatorname{argmin}_K f_i \neq \emptyset \; \forall i = 1, \ldots, m$, then

$$E_w = \operatorname{co}\left(\bigcup_{i=1}^{m} \operatorname{argmin}_K f_j\right). \tag{2.4.8}$$

Proof. Assume that $E_w \neq \mathbb{R}$. Since E_w is convex, $\operatorname{extr}(E_w) \neq \emptyset$. By Theorem 2.4.5, $J \neq \emptyset$, and from Theorem 2.4.6 $\operatorname{extr}(E_w) \subseteq \bigcup_{j \in J} \operatorname{argmin}_K f_j$. Hence, the convexity and closedness of E_w imply

$$E_w = \operatorname{co}(\operatorname{extr}(E_w)) + (E_w)^\infty \subseteq \operatorname{co}\left(\bigcup_{j \in J} \operatorname{argmin}_K f_j\right) + R_w \subseteq E_w + R_w = E_w,$$

which is the desired result.

We now prove the last part. Assume first that $E_w = [\alpha, \beta]$ with $-\infty < \alpha < \beta < +\infty$. Thus $R_w = \{0\}$ and by Theorem 2.4.5 $\operatorname{argmin}_K f_i$ is nonempty and compact for all $i = 1, \ldots, m$, that is, $J = \{1, \ldots, m\}$, proving the result.

We consider the case $E_w = [\alpha, +\infty[$ (when $E_w =]-\infty, \alpha]$, a similar reasoning may be applied). Thus, $R_w = [0, +\infty[$ and $\alpha \in \operatorname{argmin}_K f_{i_1}$ for some i_1. Since, E_w is bounded if only if $R_w = \{0\}$, $\operatorname{argmin}_K f_{i_2} = [\beta, +\infty[$ for some i_2 and $\beta \geq \alpha$. If $\beta = \alpha$ there is nothing to prove. If $\beta > \alpha$, one may easily check that every $\alpha' \in E_w$ is in the set of the right hand side of (2.4.8). □

On combining Theorems 2.4.5 and 2.4.7 immediately obtains the following result which is a special case of the one established in [23].

Corollary 2.4.1 Let $\emptyset \neq K \subseteq \mathbb{R}$ be closed and convex and $f_i : K \to \mathbb{R}$ be lsc and semistrictly quasiconvex for all $i = 1, \ldots, m$. The following assertions are equivalent:

(a) E_w is nonempty and compact;

(b) $\text{argmin}_K f_i$ is nonempty and compact for all $i = 1, \ldots, m$.

Example 2.4.4 (i) We see the result in (2.4.7) may be false if at least one function is not semistrictly quasiconvex. In fact, let $K = [0, +\infty[$ and

$$f_1(x) = |x - 1|, \quad f_2(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 2, \\ -e^{-(x-2)} & \text{if} \quad x \geq 2. \end{cases}$$

Here $E_w = [0, +\infty[= R_w$.
(ii) This instance illustrates that (2.4.8) fails to hold if at least one function f_i is not semisrictly quasiconvex. Take $K = [0, +\infty[$ and

$$f_1(x) = \begin{cases} 2 & \text{if} \quad x \notin [1, 2], \\ 1 & \text{if} \quad x \in [1, 2], \end{cases} \qquad f_2(x) = \begin{cases} -e^{-x+5} & \text{if} \quad x \geq 5, \\ 4 - x & \text{if} \quad x \leq 5. \end{cases}$$

Here $E_w = [1, +\infty[$. An instance with the same purpose but with E_w bounded is exhibited in Example 2.4.3.
(iii) The preceding theorem may be false if $\text{argmin}_K f_i = \emptyset$ for some i. Indeed, take $K = \mathbb{R}$, $f_1(x) = \sqrt{|x|}$, $f_2(x) = \frac{x}{1+|x|}$, $\quad x \in \mathbb{R}$. Here, $E_w =]-\infty, 0]$, while $\text{argmin}_{\mathbb{R}} f_1 = \{0\}$.

Actually, corollary 2.4.1 remains valid if $\mathbb{R}^m_+$ is substituted by a more general closed convex cone (including the polyhedral and ice-cream cones) as proved in [23], Theorem 5.3. Although we were unable to find any counter-example in higher dimension, we conjecture that it is still true, at least for polyhedral cones. Certainly in the convex case, i.e., when each f_i is convex, corollary 2.4.1 holds when $K \subseteq \mathbb{R}^n$ [13], even for more general cones as described above [23], Theorem 5.1.

We now present a result which may be useful to understand the problem when $n > 1$.

Theorem 2.4.8 Let $\emptyset \neq K \subseteq \mathbb{R}^n$ be convex and closed; $f_i : K \to \mathbb{R}$ be lsc for all $i = 1, \dots, m$. Assume that E_w is nonempty bounded and that exists $i_0 \in \{1, \dots, m\}$ such that f_{i_0} is convex and $\mathrm{argmin}_K f_{i_0}$ is nonempty compact. Then $\mathrm{argmin}_K f_i$ is nonempty compact for all $i = 1, \dots, m$.

Proof. By assumption E_w is compact. Suppose that it exists j such that $\mathrm{argmin}_K f_j = \emptyset$. Take any $x_1 \in \mathrm{argmin}_{E_w} f_j$ and $z \in K$ such that $f_j(z) < f_j(x_1)$. Let $C = \{x \in K : f_i(x) \leq f_i(z) \, \forall \, i = 1, \dots, m\}$. Clearly C is a nonempty and closed set. We will prove that E_w is a bounded; and since

$$C^\infty \subseteq \{x \in K \, : \, f_{i_0}(x) \leq f_{i_0}(z)\}^\infty = \{v \in K^\infty \, : \, f_{i_0}^\infty(v) \leq 0\} = \{0\},$$

C is a bounded set. We apply the Theorem 3.2 of [23], to conclude that there exists $\bar{c} \in C$ such that $F(y) - F(\bar{c}) \notin -\mathrm{int}\; \mathbb{R}^m_+ \quad \forall \, y \in C$. We claim that $\bar{c} \in E_w$, indeed, if $y \in K \backslash C$ there is i_y such that $f_{i_y}(z) < f_{i_y}(y)$, which implies that $f_{i_y}(\bar{c}) < f_{i_y}(y)$, proving the claim. On the other hand $f_j(\bar{c}) \leq f_j(z) < f_j(x_1)$, which contradicts the choice of x_1. Hence $\mathrm{argmin}_K f_i$ is nonempty compact for all $i = 1, \dots, m$. □

We can relax the convexity assumption of f_{i_0} at the price of requiring K to be a subset of $\mathbb{R}$.

Theorem 2.4.9 Let $\emptyset \neq K \subseteq \mathbb{R}$ be convex and closed; $f_i : K \to \mathbb{R}$ be lsc for all $i = 1, \dots, m$. Assume that E_w is nonempty bounded and that exists $i_0 \in \{1, \dots, m\}$ such that f_{i_0} is quasiconvex and $\mathrm{argmin}_K f_{i_0}$ is nonempty compact. Then $\mathrm{argmin}_K f_i$ is nonempty compact for all $i = 1, \dots, m$.

Proof. By assumption $E_w \subseteq [a, b] \subseteq K$ for some $a, b \in R$. Suppose that $\mathrm{argmin}_K f_{i_1} = \emptyset$ for some i_1, we will show that $E_w \nsubseteq [a, b]$ yielding a contradiction. since $\mathrm{argmin}_K f_{i_1} =$

$\emptyset$, there exists $z \notin [a,b]$ such that $f_{i_1}(z) < f_{i_1}(x)$ for all $x \in [a,b]$. We only consider the case $z > b$ (when $z < a$ a similar reasoning can be applied). Take $\bar{z} \in \mathrm{argmin}_{[b,z]} f_{i_1}$. We distinguish two situations: (1) for all $y < a,\ y \in K$: either $f_{i_1}(\bar{z}) \leq f_{i_1}(y)$ or $f_{i_0}(\bar{z}) \leq f_{i_0}(y)$; (2) there exists $y_{i_1} < a$, $y_{i_1} \in K$ such that $f_{i_1}(y_{i_1}) < f_{i_1}(\bar{z})$ and $f_{i_0}(y_{i_1}) < f_{i_0}(\bar{z})$. We now check that in both situations a contradiction will be obtained. In case (1), we will prove that $z \in E_w$. In fact, if $x < a$ then either $f_{i_1}(\bar{z}) \leq f_{i_1}(x)$ or $f_{i_0}(\bar{z}) \leq f_{i_0}(x)$; if $a \leq x \leq b$ (resp. $b < x \leq z$) then $f_{i_1}(\bar{z}) \leq f_{i_1}(z) < f_{i_1}(x)$ by the choice of $\bar{z}$ and z (resp. $f_{i_1}(\bar{z}) \leq f_{i_1}(x)$ by the choice of $\bar{z}$); if $x > z$, we take $x_0 \in \mathrm{argmin}_K f_{i_0} \subseteq [a,b]$ and thus $f_{i_0}(\bar{z}) \leq \max\{f_{i_0}(x_0), f_{i_0}(x)\} = f_{i_0}(x)$, which completes the proof that $\bar{z} \in E_w$. This is impossible since $\bar{z} > b$ and $E_w \subseteq [a,b]$. If (2) occurs, let $\bar{y} \in \mathrm{argmin}_{[y_{i_1},a]} f_{i_1}$. We proceed as in the previous case to prove that $\bar{y} \in E_w$. This again is impossible if $E_w \subseteq [a,b]$ and $\bar{y} < a$. Thus $\mathrm{argmin}_K f_i$ it is nonempty compact for all $i = 1, \dots, m$. $\square$

Example 2.4.5 The preceding result may be false if we remove the quasiconvexity on the function whose argmin is compact. In fact, let $K = [0, +\infty[$ and $f_1(x) = e^{-x^2}$, $f_2(x) = e^{-x^2}$ if $x \neq 0$ and $f_2(0) = 0$. Here, $E_w = \{0\}$.

Chapter 3

An optimal alternative theorem and applications to vector optimization

Alternative theorems are very useful to derive many important results in convex and nonconvex optimization theory: the existence of Lagrange multipliers, duality results, scalarization of vector functions, etc. To be precise, let us consider a real locally convex topological vector space Y and a closed convex cone $P \subseteq Y$ such that $\mathrm{int}\, P \neq \emptyset$. We denote by Y^* the topological dual space of Y, and by P^* the (positive) polar cone of P. Given a nonempty set $A \subseteq Y$, alternative theorems assert the validity of exactly one of the following assertions:

$$\exists\, a \in A \,\text{such that}\, a \in -\mathrm{int}\, P; \tag{3.0.1}$$

$$\exists\, p^* \in P^*,\ p^* \neq 0,\ \text{such that}\ \langle p^*, a\rangle \geq 0\ \ \forall\, a \in A. \tag{3.0.2}$$

Here $\langle \cdot, \cdot \rangle$ stands for the duality pairing between Y and Y^* and $\mathrm{int}\, P$ denotes the topological interior of P. We recall that P^* is defined by

$$P^* = \{p^* \in Y^* :\ \langle p^*, p\rangle \geq 0\ \ \forall\, p \in P\}$$

and closedness and convexity of the cone P is equivalent to $P = P^{**}$ by the bipolar theorem. In this case,

$$p \in P \iff \langle p^*, p\rangle \geq 0\ \ \forall\, p^* \in P^*.$$

Moreover,

$$p \in \text{int } P \iff \langle p^*, p \rangle > 0 \ \forall\, p^* \in P^* \setminus \{0\}. \tag{3.0.3}$$

A separation theorem for convex sets and the above remarks allow us to write (3.0.1) and (3.0.2) in an equivalent way as, respectively,

$$A \cap (-\text{int } P) \neq \emptyset, \tag{3.0.4}$$

$$\text{co}(A) \cap (-\text{int } P) = \emptyset. \tag{3.0.5}$$

While the inconsistency of both assertions (3.0.4) and (3.0.5) is straightforward, the proof of the validity of (3.0.5) assuming (3.0.4) does not hold, in other words, the proof of the implication

$$A \cap (-\text{int } P) = \emptyset \Longrightarrow \text{co}(A) \cap (-\text{int } P) = \emptyset, \tag{3.0.6}$$

requires a careful analysis due to the lack of convexity of A. One of the goals of the present chapter is to characterize those sets A for which implication (3.0.6) is true. Most papers appearing in the literature (see for instance [1, 36, 47, 54, 55] and the references therein) were concerned with providing some (sufficient) conditions implying (3.0.6).

Throughout the chapter, X will be a vector space and Y a real locally convex vector space. We will denote by $\langle \cdot, \cdot \rangle$ the duality pairing between Y and Y^*.

3.1 An optimal theorem in spaces of arbitrary dimension

In search of conditions implying the validity of (3.0.6), several relaxed notions of convexity have appeared in the literature. Before reviewing and comparing some of them, we will first reformulate the conclusion of the alternative theorem in terms of the cone $\text{cone}\,(A + \text{qint } P)$. We recall the definition of pointedness for a cone that is not necessarily convex (see for instance [46]).

Definition 3.1.1 A cone $K \subseteq Y$ is called "pointed" if $x_1 + \ldots + x_k = 0$ is impossible for $x_1, x_2, \ldots, x_k$ in K unless $x_1 = x_2 = \ldots = x_k = 0$.

Our first result is the following:

Theorem 3.1.2 Let $A \subseteq Y$ be any nonempty set and $P \subseteq Y$, $P \neq Y$, be a convex and closed cone such that $\mathrm{qint}\ P \neq \emptyset$. The following assertions are equivalent:

(a) $\mathrm{cone}(A + \mathrm{qint}\ P)$ is pointed;

(b) $\mathrm{co}(A) \cap (-\mathrm{qint}\ P) = \emptyset$.

Proof. We first prove

$$\mathrm{cone}(A + \mathrm{qint}\ P) \text{ is pointed } \Longrightarrow A \cap (-\mathrm{qint}\ P) = \emptyset. \tag{3.1.1}$$

If there exists $x \in A \cap (-\mathrm{qint}\ P)$, then $x = 2(x - \frac{x}{2}) \in \mathrm{cone}(A + \mathrm{qint}\ P)$ and $-x = x + (-2x) \in A + \mathrm{qint}\ P \subseteq \mathrm{cone}(A + \mathrm{qint}\ P)$. By pointedness, $x = 0$, hence $0 \in \mathrm{qint}\ P$. As noted in Section 2, this implies $P = Y$, a contradiction.

Now assume that (a) holds. If (b) does not hold, then there exists $x \in -\mathrm{qint}\ P$ such that $x = \sum_{i=1}^{m} \lambda_i a_i$ with $\sum_{i=1}^{m} \lambda_i = 1$, $\lambda_i > 0$, $a_i \in A$. Thus, $0 = \sum_{i=1}^{m} \lambda_i (a_i - x)$. Using (a) we infer that $\lambda_i (a_i - x) = 0$ for all $i = 1, \ldots, m$. This contradicts (3.1.1).

Conversely, assume that (b) holds. If $\mathrm{cone}(A + \mathrm{qint}\ P)$ is not pointed, then there exist $x_i \in \mathrm{cone}(A + \mathrm{qint}\ P) \backslash \{0\}$, $i = 1, 2, \ldots n$, such that $\sum_{i=1}^{n} x_i = 0$. Each x_i can be written as $x_i = \lambda_i (y_i + u_i)$ with $\lambda_i > 0$, $y_i \in A$ and $u_i \in \mathrm{qint}\ P$. Hence $\sum_{i=1}^{n} \lambda_i y_i = -\sum_{i=1}^{n} \lambda_i u_i$. Setting $\mu_i = \lambda_i / \sum_{j=1}^{n} \lambda_j$ we get $\sum_{i=1}^{n} \mu_i y_i = -\sum_{i=1}^{n} \mu_i u_i \in \mathrm{co}(A) \cap (-\mathrm{qint}\ P)$, a contradiction. □

When $\mathrm{int}\ P \neq \emptyset$, then by the separation theorem $\mathrm{co}(A) \cap (-\mathrm{qint}\ P) = \emptyset$ is equivalent to the existence of $p^* \in P^* \backslash \{0\}$ such that $\langle p^*, y \rangle \geq 0$ for all $y \in A$. Thus, in case the set A is the image of some vector-valued mapping, the previous theorem implies the following

Corollary 3.1.3 Let $K \subseteq X$ be any nonempty set, $P \subseteq Y$ be a closed convex cone such that $\mathrm{int}\ P \neq \emptyset$, and $G : K \to Y$ be any mapping. Then the following assertions are equivalent:

(a) $\operatorname{cone}(G(K) + \operatorname{int} P)$ is pointed;

(b) $\exists\, p^* \in P^*,\ p^* \neq 0$, $\langle p^*, G(x)\rangle \geq 0\, \forall\, x \in K$.

We now recall the most general among the relaxed notions of convexity that were used in alternative theorems.

Definition 3.1.4 Let $P \subseteq Y$ be a closed convex cone with nonempty interior. A set $A \subseteq Y$ is called:

(a) generalized subconvexlike [55] if $\exists\, u \in \operatorname{int} P$, $\forall\, x_1,\ x_2 \in A,\ \forall\, \alpha \in\ ]0,1[,\ \forall\, \varepsilon > 0$, $\exists\, \rho > 0$ such that

$$\varepsilon u + \alpha x_1 + (1-\alpha)x_2 \in \rho A + P; \tag{3.1.2}$$

(b) presubconvexlike if $\exists\, u \in Y,\ \forall\, x_1,\ x_2 \in A,\ \forall\, \alpha \in\ ]0,1[,\ \forall\, \varepsilon > 0$, $\exists\, \rho > 0$ such that (3.1.2) holds;

(c) nearly subconvexlike [47, 54] if $\overline{\operatorname{cone}}(A + P)$ is convex.

Note that the definition of presubconvexlike sets is a transcription of an analogous definition for Y-valued functions given in [56]. Also, from Proposition 1.1.1(f) it follows that (c) above is equivalent to the convexity of $\operatorname{cone}_+(A + \operatorname{int} P)$ and also to the convexity of $\operatorname{cone}(A + \operatorname{int} P)$. In fact, we will show that all three notions of generalized convexity of sets given in Definition 3.1.4 are equivalent.

Proposition 3.1.5 In Definition 3.1.4, (a), (b) and (c) are equivalent.

Proof. $(a) \Leftrightarrow (b)$: It is obvious that (a) implies (b). If A is presubconvexlike, let $u \in Y$ be the element whose existence is required by (b). Since $\operatorname{int} P - \operatorname{int} P = Y$ (see, cg., [44]) we can write $u = v - w$ with $v, w \in \operatorname{int} P$. By assumption, for every $x_1,\ x_2 \in A,\ \alpha \in\ ]0,1[,\ \varepsilon > 0$ there exists $\rho > 0$ such that (3.1.2) holds. Then

$$\varepsilon v + \alpha x_1 + (1-\alpha)x_2 \in \rho A + P + \varepsilon w \subseteq \rho A + P.$$

Thus, A is generalized subconvexlike.

$(a) \Rightarrow (c)$: In Theorem 2.1 of [55], it is shown that a generalized subconvexlike set A is such that the set $\text{cone}_+(A) + \text{int } P$ is convex. By Proposition 1.1.1(c)(f), $\overline{\text{cone}}(A + P)$ is convex.

$(c) \Rightarrow (a)$: If $\overline{\text{cone}}(A + P)$ is convex then by Proposition 1.1.1(f), $\text{cone}_+(A + \text{int } P)$ is convex. From (a) of the same proposition applied to the set $A + \text{int } P$ it follows that

$$\alpha A + (1 - \alpha)A + \text{int } P \subseteq \text{cone}_+(A + \text{int } P) \;\; \forall \, \alpha \in \;]0, 1[.$$

This allows us to conclude that A is generalized subconvexlike. □

Thus, the two alternative theorems in [54] and [55] (with "int" instead of "qint") can be unified and extended as follows:

Theorem 3.1.6 Let $A \subseteq Y$ be any nonempty set. Assume that $A \cap (-\text{qint } P) = \emptyset$. Then

$$\text{cone}_+(A + \text{qint } P) \text{ is convex } \Longrightarrow \text{co}(A) \cap (-\text{qint } P) = \emptyset.$$

It is now clear that Theorem 3.1.6 is a consequence of Theorem 3.1.2 and the following easy proposition:

Proposition 3.1.7 If $\text{cone}_+(A + \text{qint } P)$ is convex and $A \cap (-\text{qint } P) = \emptyset$, then $\text{cone}(A + \text{qint } P)$ is pointed.

Proof. Since $\text{cone}(A + \text{qint } P)$ is also a convex cone, we have to show that whenever $x, -x \in \text{cone}(A + \text{qint } P)$, then $x = 0$. Indeed, assume that $x \neq 0$. Then $x, -x \in \text{cone}_+(A + \text{qint } P)$. This last set is convex, hence $0 = x + (-x) \in \text{cone}_+(A + \text{qint } P)$. Thus, there exist $\lambda > 0$, $y \in A$ and $u \in \text{qint } P$ such that $0 = \lambda(y + u)$. Then $y \in A \cap (-\text{qint } P)$, a contradiction. □

The converse of Proposition 3.1.7 (or Theorem 3.1.6) does not hold, as shown by the following example.

Example 3.1.8 Let us consider in $\mathbb{R}^3$ the polyhedral (closed convex) cone $P = \text{cone}(B)$, where

$$B = \Big\{(1, -x_2, x_3) : \; 0 \leq x_2, 0 \leq x_3, x_2 + x_3 \leq 1\Big\},$$

and the set

$$A = \Big\{(x_1, 1, \sqrt{1 - x_1^2}) :\ 0 \leq x_1 \leq 1\Big\}.$$

It is not difficult to check that $\mathrm{co}(A) \cap (-\mathrm{int}\ P) = \emptyset$ thus $\mathrm{cone}(A + \mathrm{int}\ P)$ is pointed. However, we will see that $\overline{\mathrm{cone}}(A + P)$ is nonconvex. To this end, it is enough to show that $z = (\frac{1}{2}, 1, \frac{1}{2}) \notin \overline{\mathrm{cone}}(A + P)$ since $z = \frac{1}{2}x + \frac{1}{2}y$ with $x = (0, 1, 1) \in A$ and $y = (1, 1, 0) \in A$. Assume on the contrary that there exist sequences $0 \leq x_1^k \leq 1$, $0 \leq x_2^k \leq 1$, $0 \leq x_3^k \leq 1$ and β_k, $\lambda_k \geq 0$ such that

$$\lambda_k(x_1^k + \beta_k) \to \frac{1}{2}, \tag{3.1.3}$$

$$\lambda_k(1 - \beta_k x_2^k) \to 1, \tag{3.1.4}$$

$$\lambda_k(\sqrt{1 - (x_1^k)^2} + \beta_k x_3^k) \to \frac{1}{2}. \tag{3.1.5}$$

If λ_k is bounded, we may assume that $\lambda_k \to \lambda$ for some $\lambda \geq 0$. From (3.1.4), we obtain $\lambda \geq 1$. On the other hand, up to a subsequence $x_1^k \to x_1$, thus (3.1.3) implies $x_1 \leq \frac{1}{2}$. By (3.1.5) we get $\sqrt{1 - x_1^2} \leq \frac{1}{2}$, which in turn gives $x_1 \geq \frac{\sqrt{3}}{2}$, contradicting a previous inequality. We now assume that $\lambda_k \to +\infty$. From (3.1.3) it follows $x_1^k \to 0$. Taking k sufficiently large, (3.1.5) yields a contradiction.

The preceding definitions of relaxed convexity for sets induce corresponding definitions for vector valued mappings: given a nonempty convex subset K of X, a multivalued mapping $G : K \rightrightarrows Y$ is called generalized subconvexlike [55] (respectively, nearly subconvexlike [47, 54], presubconvexlike [56]) if the set $G(K)$ is generalized subconvexlike (resp., nearly subconvexlike, presubconvexlike) . According to Proposition 3.1.5, these three notions are identical. Other definitions of generalized convexity for (single-valued) vector valued functions in view of using them to alternative theorems were given in [36] and [51]. A mapping $G : K \to Y$ is called $*$-quasiconvex [36] if $\langle x^*, G(\cdot)\rangle$ is quasiconvex for all $x^* \in P^*$. It is called naturally-P-quasiconvex [51] if for all $x, y \in K$, $G([x, y]) \subseteq [G(x), G(y)] - P$. We will first show that these notions are equivalent:

Proposition 3.1.9 Let $K \subseteq X$ be any nonempty convex set and $P \subseteq Y$ be a closed convex cone with nonempty interior. Then a mapping $G : K \to Y$ is $*$-quasiconvex if and only if it is naturally-P-quasiconvex.

Proof. Assume that G is naturally-P-quasiconvex. We need to check that given $t \in \mathbb{R}$ and $x^* \in P^*$, the set $K_t = \{z \in K : \langle x^*, G(z)\rangle \leq t\}$ is convex. Indeed, if $x, y \in K_t$ then by natural-P-quasiconvexity of G, for all $z \in [x, y]$ there exists $\lambda \in [0, 1]$ and $u \in P$ such that $G(z) = \lambda G(x) + (1 - \lambda)G(y) - u$. Hence,

$$\langle x^*, G(z)\rangle = \lambda \langle x^*, G(x)\rangle + (1 - \lambda) \langle x^*, G(y)\rangle - \langle x^*, u\rangle \leq t$$

thus $z \in K_t$, so K_t is convex.

Conversely, assume that G is not naturally-P-quasiconvex. Then there exist $x, y \in K$ and $z \in\ ]x, y[$ such that for all $\mu \in [0, 1]$, $G(z) \notin \mu G(x) + (1-\mu)G(y) - P$. Thus for every $\mu \in [0, 1]$ there exists $x^* \in Y^* \setminus \{0\}$ such that

$$\langle x^*, G(z)\rangle > \langle x^*, \mu G(x) + (1 - \mu)G(y) - u\rangle \ \ \forall\, u \in P.$$

Since P is a cone, we get $\langle x^*, u\rangle \geq 0$ for all $u \in P$, i.e., $x^* \in P^*$, and also $\langle x^*, G(z) - \mu G(x) - (1 - \mu)G(y)\rangle > 0$. Since by assumption $\operatorname{int} P \neq \emptyset$, there exists a w^*-compact base B of P^*. Setting $f(y^*, \mu) = \langle y^*, G(z) - \mu G(x) - (1 - \mu)G(y)\rangle$ we get

$$\max_{y^* \in B} \min_{\mu\in[0,1]} f(y^*, \mu) = \min_{\mu\in[0,1]} \max_{y^* \in B} f(y^*, \mu) > 0.$$

Hence there exists $x^* \in B$ such that

$$\langle x^*, G(z)\rangle > \mu\langle x^*, G(x)\rangle + (1 - \mu)\langle x^*, G(y)\rangle \ \ \forall\, \mu \in [0, 1].$$

In particular, we get $\langle x^*, G(z)\rangle > \langle x^*, G(x)\rangle$ and $\langle x^*, G(z)\rangle > \langle x^*, G(y)\rangle$. Thus G is not $*$-quasiconvex. □

In [36] it is proven that implication (3.0.6) holds for $A = G(K)$ under the $*$-quasiconvexity of G and the assumption

$$\forall p^* \in P^*, \text{ the restriction of } \langle p^*, G(\cdot)\rangle \text{ on any line segment of } K \text{ is lower semicontinuous.} \tag{3.1.6}$$

We will see that the $*$-quasiconvexity of G together with (3.1.6) imply the convexity of $\mathrm{cone}(G(K)+\mathrm{int}\ P)$ thus, in particular, that G is nearly subconvexlike. This follows from the next proposition which is of interest by itself. We refer the reader to [30] for the definition of upper semicontinuity and other properties of multivalued mappings that will be used in the proof.

Proposition 3.1.10 Let $K \subseteq X$ be any nonempty convex set, $P \subseteq Y$ be a closed convex cone and $G : K \to Y$ be naturally-P-quasiconvex and satisfying (3.1.6). Then

$$\forall\, x, y \in K,\ \ [G(x), G(y)] \subseteq G([x, y]) + P. \tag{3.1.7}$$

Proof. Given $x, y \in K$, define $H : [x, y] \rightrightarrows [G(x), G(y)]$ by $H(z) = (G(z) + P) \cap ([G(x), G(y)])$. We show first that H is closed. Let (z_n, w_n), $n \in \mathbb{N}$, be a sequence in the graph of H, converging to (z, w). Then $w_n \in H(z_n) \subseteq [G(x), G(y)]$. Obviously, $w \in [G(x), G(y)]$. Also, for every $n \in \mathbb{N}$ there exists $v_n \in P$ such that $w_n = G(z_n) + v_n$. For each $p^* \in P^*$ we get by assumption (3.1.6):

$$\begin{aligned} \langle p^*, w - G(z)\rangle &\geq \lim \langle p^*, w_n\rangle - \liminf \langle p^*, G(z_n)\rangle \\ &= \lim \langle p^*, w_n\rangle + \limsup \langle p^*, -G(z_n)\rangle \\ &= \limsup \langle p^*, v_n\rangle \geq 0. \end{aligned}$$

Since this is true for all $p^* \in P^*$, we deduce that $w - G(z) \in P$, i.e., $w \in H(z)$ and H is closed. Hence, H is upper semicontinuous.

Also, for every $z \in [x, y]$, $H(z) \neq \emptyset$ by the definition of natural-P-quasiconvexity. In addition, $H(z)$ is connected, being convex. Hence, the image of $[x, y]$ through H is connected (cf. Proposition 2.24, pg. 43 in [30]). This image is a subset of the line segment $[G(x), G(y)]$. Since $G(x) \in H(x)$ and $G(y) \in H(y)$, we deduce that $H([x, y]) = [G(x), G(y)]$. Thus, for every $w \in [G(x), G(y)]$ there exists $z \in [x, y]$ such that $w \in H(z)$, i.e., $w = G(z) + u$ for some $u \in P$. This shows inclusion (3.1.7). □

We deduce the following:

Corollary 3.1.11 Let X, Y, P, G be as in the previous proposition. Then $G(K) + P$ is convex.

Proof. It is sufficient to show that whenever $t \in [0,1]$, $x, y \in K$ and $u \in P$ then $tG(x) + (1-t)G(y) + u \in G(K) + P$. But this is obvious in view of the proposition. □

Thus, given a cone P with $\text{int } P \neq \emptyset$, if a mapping G is $*$-quasiconvex (or, equivalently, naturally-P-quasiconvex) and satisfies (3.1.6), then $G(K) + P$ is convex. This implies that G is nearly subconvexlike, so the alternative theorems of [36] and [51] are included in Theorem 3.1.6 and in particular in Theorem 3.1.2. The converse does not hold: the mapping $G(x) = (x, f(x))$, $x \in [-1, 1]$, where $f(x) = 1 - |x|$, is clearly nearly subconvexlike (with $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$), but the real-valued function $x \in [-1,1] \mapsto \langle (0,1), (x, f(x)) \rangle = f(x)$ is not quasiconvex, that is, G is not $*$-quasiconvex.

3.2 Characterizing the two-dimensionality through the alternative theorem

According to Theorem 3.1.6 (see also Proposition 1.1.1(f)), whenever $A \cap (-\text{int } P) = \emptyset$ holds, the convexity of $\text{cone}(A + \text{int } P)$ is a sufficient condition for $\text{co}(A) \cap (-\text{int } P) = \emptyset$ to hold. We will now see that in case $Y = \mathbb{R}^2$, it is also necessary.

Theorem 3.2.1 Let $P \subseteq \mathbb{R}^2$ be a convex closed cone such that $\text{int } P \neq \emptyset$, and $A \subseteq \mathbb{R}^2$ be any nonempty set satisfying $A \cap (-\text{int } P) = \emptyset$. Then the following assertions are equivalent:

(a) $\text{co}(A) \cap (-\text{int } P) = \emptyset$;

(b) $\text{cone}(A + P)$ is convex;

(c) $\text{cone}(A + \text{int } P)$ is convex;

(d) $\text{cone}(A) + P$ is convex;

(e) $\overline{\text{cone}}(A + P)$ is convex.

Proof. According to Proposition 1.1.1(f), $(c) \iff (e)$. Also

$$\mathrm{cone}(A + \mathrm{int}\ P) \subseteq \mathrm{cone}(A + P) \subseteq \mathrm{cone}(A) + P \subseteq \overline{\mathrm{cone}}(A + P) \tag{3.2.1}$$

where the last inclusion follows from Proposition 1.1.1(d). If $\mathrm{cone}(A + \mathrm{int}\ P)$ is convex, then its closure $\overline{\mathrm{cone}}(A + P)$ (see Proposition 1.1.1(e)) is convex. Due to the two-dimensionality of the space, we deduce that $\mathrm{cone}(A + P)$ and $\mathrm{cone}(A) + P$ are convex. Thus, (b), (c), (d) and (e) are equivalent.

That (e) implies (a) follows from Theorem 3.1.6 and Proposition 1.1.1(f).
$(a) \Rightarrow (b)$: There exists $x^* \in \mathbb{R}^2$ such that $\langle x^*, x\rangle \geq \langle x^*, u\rangle$ for all $x \in A$ and $u \in -\mathrm{int}\ P$. It follows that $x^* \in P^*$ and $\langle x^*, x\rangle \geq 0$ for all $x \in A$, thus also for all $x \in \mathrm{cone}(A + P)$.

Choose $u \in \mathrm{int}\ P$. Let $y, z \in A$. Then obviously

$$\mathrm{cone}(\{y\}) + \mathrm{cone}(\{u\}) = \{\lambda y + \mu u : \lambda, \mu \geq 0\}$$

is a closed convex cone containing y and u and contained in $\mathrm{cone}(A + P)$. The same is true for the cone $\mathrm{cone}(\{z\}) + \mathrm{cone}(\{u\})$. The two cones have the line $\mathrm{cone}(\{u\})$ in common and their union is contained in $\mathrm{cone}(A + P)$, thus it is contained in the halfspace $\{x \in \mathbb{R}^2 : \langle x^*, x\rangle \geq 0\}$. Hence, the set $B \doteq (\mathrm{cone}(\{y\}) + \mathrm{cone}(\{u\})) \cup (\mathrm{cone}(\{z\}) + \mathrm{cone}(\{u\}))$ is a convex cone. Since $y, z \in B$ we deduce that $[y, z] \subseteq B \subseteq \mathrm{cone}(A + P)$ thus $\mathrm{co}(A) \subseteq \mathrm{co}(B) = B \subseteq \mathrm{cone}(A + P)$. We deduce that $\mathrm{cone}(A + P)$ is convex. □

We now show that the equivalence between (a) and one of $(b), (c), (d), (e)$ in Theorem 3.2.1 is characteristic of 2–dimensional spaces. Since, say, $(b) \Rightarrow (a)$ is a consequence of Theorem 3.1.6, we only consider the implication $(a) \Rightarrow (b)$ etc.

Theorem 3.2.2 Let Y be a locally convex space and $P \subseteq Y$ be a closed, convex cone such that $\mathrm{int}\ P \neq \emptyset$ and $\mathrm{int}\ P^* \neq \emptyset$. The following assertions are equivalent:

(a) for all sets $A \subseteq Y$ one has

$$\mathrm{co}(A) \cap (-\mathrm{int}\ P) = \emptyset \Rightarrow \overline{\mathrm{cone}}(A + P) \text{ is convex;}$$

(b) for all sets $A \subseteq Y$ one has

$$\mathrm{co}(A) \cap (-\mathrm{int}\ P) = \emptyset \Rightarrow \mathrm{cone}(A) + P \ \text{ is convex;}$$

(c) for all sets $A \subseteq Y$ one has

$$\mathrm{co}(A) \cap (-\mathrm{int}\ P) = \emptyset \Rightarrow \mathrm{cone}(A + \mathrm{int}\ P) \ \text{ is convex;}$$

(d) Y is at most two-dimensional.

Proof. We show first that (a) implies (d). Assume that the dimension of Y is at least 3. Let $x^* \in \mathrm{int}\ P^*$. Then for all $x \in P\backslash\{0\}$, $\langle x^*, x\rangle > 0$. Fix $x \in \mathrm{int}\ P$, and choose linearly independent $y, z \in Y$ such that $\langle x^*, y\rangle = \langle x^*, z\rangle = 0$ (this is possible since the dimension of the kernel of x^* is at least 2). In particular, y and z are not zero. Let A be the set $[y + z, y + x] \cup [y + x, y - z]$. Every element w of A has the form: $w = t\,(y \pm z) + (1 - t)\,(y + x)$ with $t \in [0, 1]$. Hence $\langle x^*, w\rangle = (1 - t)\,\langle x^*, x\rangle \geq 0$. It follows that for every $w \in \mathrm{co}(A)$, $\langle x^*, w\rangle \geq 0$. Since for every $u \in -\mathrm{int}\ P$, $\langle x^*, u\rangle < 0$, it follows that $\mathrm{co}(A) \cap (-\mathrm{int}\ P) = \emptyset$.

We now show that $\overline{\mathrm{cone}}(A + P)$ is not convex. Since $y = \frac{y+z}{2} + \frac{y-z}{2} \in \mathrm{co}(A) \subseteq \mathrm{co}\,(\overline{\mathrm{cone}}\,(A + P))$, it is sufficient to show that $y \notin \overline{\mathrm{cone}}\,(A + P)$. Suppose to the contrary that $y \in \overline{\mathrm{cone}}\,(A + P)$. Then there exist $\lambda_i \geq 0$, $t_i \in [0, 1]$, $u_i \in P$ such that

$$\lambda_i(t_i(y \pm z) + (1 - t_i)(y + x)) + u_i \to y. \tag{3.2.2}$$

Then

$$\langle x^*, \lambda_i(t_i(y \pm z) + (1 - t_i)(y + x)) + u_i\rangle \to \langle x^*, y\rangle = 0 \Rightarrow$$

$$\lambda_i(1 - t_i)\,\langle x^*, x\rangle + \langle x^*, u_i\rangle \to 0 \Rightarrow$$

$$\lambda_i(1 - t_i) \to 0 \ \text{ and } \ \langle x^*, u_i\rangle \to 0.$$

If there is a subsequence of $\{\lambda_i\}$ converging to 0 then we get from (3.2.2) that $u_i \to y$ (since λ_i is multiplied with a bounded vector). This implies that $y \in \overline{P} = P$ which contradicts $\langle x^*, y\rangle = 0$.

If there is a subsequence of $\{\lambda_i\}$ converging to a number $\lambda \in \ ]0, +\infty[$ then $t_i \to 1$ and we get from (3.2.2) that $u_i \to y - \lambda(y \pm z)$. Since P is closed, this implies that $y - \lambda(y \pm z) \in P$. But $\langle x^*, y - \lambda(y \pm z)\rangle = 0$ while $\langle x^*, u\rangle > 0$ for all $u \in P \setminus \{0\}$. Hence $y - \lambda(y \pm z) = 0$. This is impossible, in view of the linear independence of y and z.

It follows that $\lambda_i \to +\infty$. Then $t_i \to 1$, and from $\lambda_i(1 - t_i) \to 0$ and (3.2.2) we obtain $\lambda_i t_i(y \pm z) + u_i \to y$. Thus, $y \pm z + \frac{u_i}{\lambda_i t_i} \to 0$ and $\frac{u_i}{\lambda_i t_i} \to -(y \pm z)$. However, $\frac{u_i}{\lambda_i t_i} \in P$ thus its limit should be in P. As before, this should imply that $y \pm z = 0$ which again contradicts the linear independence of y and z.

Thus, $y \notin \overline{\text{cone}}\,(A + P)$. Since $y \in \text{co}\,(\overline{\text{cone}}\,(A + P))$, we deduce that $\overline{\text{cone}}\,(A + P)$ is not convex. This contradicts (a).

To show that (b) implies (a), we simply remark that if $\text{cone}\,(A) + P$ is convex then its closure $\overline{\text{cone}\,(\text{A}) + \text{P}}$ is convex, and this is equal to $\overline{\text{cone}}\,(A + P)$ by Proposition 1.1.1(d). The same proposition shows that (c) implies (a). Finally, (d) implies (b) and (c) by Theorem 3.2.1. □

Remark 3.2.1 The assumption $\text{int}\ P^* \neq \emptyset$ (which corresponds to pointedness of P when Y is finite-dimensional) cannot be removed. Indeed, let $P = \{y \in Y : \langle p^*, y\rangle \geq 0\}$ where $p^* \in Y^*\backslash\{0\}$. Then $P^* = \text{cone}\,(\{p^*\})$, $\text{int}\ P^* = \emptyset$. For any nonempty $A \subseteq Y$, the set $A + \text{int}\ P$ is convex. Thus, (c) in Theorem 3.2.2 holds independently of the dimension of the space Y.

3.3 Characterizing a necessary optimality condition of the Fritz-John type

For simplicity we now consider X to be a real normed vector space. It is well known that if $\bar{x}$ is a local minimum point (in the usual sense) for the real-valued differentiable function F on K, then

$$\nabla F(\bar{x}) \in (T(K;\bar{x}))^*. \tag{3.3.1}$$

Here, $T(C;\bar{x})$ denotes the contingent cone of C at $\bar{x} \in C$, defined as the set of vectors v such that there exist $t_k \downarrow 0$, $v_k \in X$, $v_k \to v$ such that $\bar{x} + t_k v_k \in C$ for all k; C^* denotes the (positive) polar cone of C.

It is now our purpose to extend the previous optimality condition to the vector case without smoothness assumptions. More precisely, let $K \subseteq X$ be closed and consider a mapping $F : K \to \mathbb{R}^n$. Given a closed convex cone $P \subseteq \mathbb{R}^n$ with nonempty interior, a vector $\bar{x} \in K$ is a local weakly efficient solution for F on K, if there exists an open neighborhood V of $\bar{x}$ such that

$$(F(K \cap V) - F(\bar{x})) \cap (-\text{int } P) = \emptyset. \tag{3.3.2}$$

Following [50], we say that a function $h : X \to \mathbb{R}$ admits a Hadamard directional derivative at $\bar{x} \in X$ in the direction v if

$$\lim_{(t,u)\to(0^+,v)} \frac{h(\bar{x}+tu) - h(\bar{x})}{t} \in \mathbb{R}.$$

In this case, we denote such a limit by $dh(\bar{x};v)$.

If $F = (f_1, \ldots, f_n)$, we set

$$\mathcal{F}(v) \doteq ((df_1(\bar{x};v), \ldots, df_n(\bar{x};v)), \;\; \mathcal{F}(T(K;\bar{x})) = \{\mathcal{F}(v) \in \mathbb{R}^n : \; v \in T(K;\bar{x})\}.$$

It is known that if $df_i(\bar{x};\cdot)$, $i = 1, \ldots, n$ do exist in $T(K;\bar{x})$, and $\bar{x} \in K$ is a local weakly efficient solution for F on K, i.e., $\bar{x}$ satisfies (3.3.2), then (see for instance Lemma 3.2 of [50])

$$(df_1(\bar{x};v), \ldots, df_n(\bar{x};v)) \in \mathbb{R}^n \setminus -\text{int } P, \;\; \forall\, v \in T(K;\bar{x}),$$

or equivalently, $\mathcal{F}(T(K;\bar{x})) \cap (-\text{int } P) = \emptyset$. The following theorems provide complete characterizations for the validity of (a) as a necessary condition for $\bar{x}$ to be a local weakly efficient solution for F on K.

Theorem 3.3.1 Let $K \subseteq X$ be a closed set, $P \subseteq \mathbb{R}^n$ be a closed convex cone such that $\text{int } P \neq \emptyset$, and $F : K \to \mathbb{R}^n$ be a mapping. Set $F = (f_1, \ldots, f_n)$ and assume that

$\bar{x} \in K$ and $df_i(\bar{x};\cdot)$, $i = 1,\ldots,n$ do exist in $T(K;\bar{x})$. Then, the following assertions are equivalent:

(a) $\exists\,(\alpha_1^*,\ldots,\alpha_n^*) \in P^* \setminus \{0\},\ \ \alpha_1^* df_1(\bar{x},v) + \ldots + \alpha_n^* df_n(\bar{x},v) \geq 0\ \ \forall\, v \in T(K;\bar{x})$;

(b) $\operatorname{cone}(\mathcal{F}(T(K;\bar{x})) + \operatorname{int} P)$ is pointed.

Proof. We apply Corollary 3.1.3 to obtain the desired result. □

When $Y = \mathbb{R}^2$, more precise formulations can be obtained from Theorem 3.2.1.

Theorem 3.3.2 Let $K \subseteq X$ be a closed set, $P \subseteq \mathbb{R}^2$ be a closed convex cone such that $\operatorname{int} P \neq \emptyset$. Set $F = (f_1, f_2)$ and assume that $\bar{x} \in K$ and $df_i(\bar{x};\cdot)$, $i = 1,2$ do exist in $T(K;\bar{x})$. Then, the following assertions are equivalent:

(a) $\exists\,(\alpha_1^*,\alpha_2^*) \in P^* \setminus \{0\},\ \ \alpha_1^* df_1(\bar{x},v) + \alpha_2^* df_2(\bar{x},v) \geq 0\ \ \forall\, v \in T(K;\bar{x})$;

(b) $\mathcal{F}(T(K;\bar{x})) \cap (-\operatorname{int} P) = \emptyset$ and $\operatorname{cone}(\mathcal{F}(T(K;\bar{x})) + \operatorname{int} P)$ is convex.

Remark 3.3.1 When $P = \mathbb{R}^n_+$ and $f_1,\ldots,f_n$ are differentiable, Part (a) in Theorem 3.3.1 can be written as

$$\operatorname{co}(\{\nabla f_i(\bar{x}) :\ i = 1,\ldots,n\}) \cap (T(K;\bar{x}))^* \neq \emptyset, \tag{3.3.3}$$

which is the natural extension of (3.3.1). However, we have to point out that (3.3.3) is not in general a necessary optimality condition for $\bar{x}$ to be a local weakly efficient solution. This is shown in $\mathbb{R}^2$ by the example taken from [2], see also [12, 53] for additional discussion:

$$K = \{(x_1,x_2) :\ (x_1 + 2x_2)(2x_1 + x_2) \leq 0\},\ f_i(x_1,x_2) = x_i,\ \bar{x} = (0,0).$$

In this case $T(K;\bar{x}) = K$, which is nonconvex, thus $(T(K;\bar{x}))^* = \{(0,0)\}$, and therefore (3.3.3) does not hold since $\operatorname{co}(\{\nabla f_1(\bar{x}), \nabla f_2(\bar{x})\}) = \operatorname{co}\{(1,0),(0,1)\}$. Notice also that

$$\operatorname{cone}(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^2_+) = \bigcup_{t\geq 0} t(T(K;\bar{x}) + \mathbb{R}^2_+)$$

is nonconvex. On the other hand, due to the linearity of $\mathcal{F}$ (when f_1 and f_2 are differentiable), if $T(K;\bar{x})$ is convex then

$$\mathrm{cone}(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^2_+) = \bigcup_{t\geq 0} t(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^2_+)$$

is also convex. This fact was point out earlier in [52] (see also [12]). Therefore, (3.3.3) holds if $T(K;\bar{x})$ is convex. The following example shows that the necessary optimality condition (3.3.3) may be true without the convexity of $T(K;\bar{x})$. Take the same mapping F as before and

$$K = \{(x_1,x_2) \in \mathbb{R}^2_+ : x_1x_2 = 0\},\ \bar{x} = (0,0).$$

Then, (3.3.3) holds since in this case, $T(K;\bar{x}) = K$, $(T(K;\bar{x}))^* = \mathbb{R}^2_+$ *and*

$$\mathrm{co}\left(\{\nabla f_1(\bar{x}), \nabla f_2(\bar{x})\}\right) = \mathrm{co}\left\{(1,0),(0,1)\right\}.$$

3.4 Further applications

3.4.1 Characterizing the zero (Lagrangian) duality gap

We now obtain various equivalent conditions to the zero (Lagrangian) duality gap for a class of nonconvex minimization problems under a Slater-type condition.

Let us consider the following constrained minimization problem

$$\mu \doteq \inf_{x\in K} f(x), \tag{3.4.1}$$

where $K \doteq \{x \in C :\ g(x) \in -P\}$, C is a nonempty subset of a real locally convex topological vector space X, $f : C \to \mathbb{R}$, and $g : C \to Y$, with Y as before and $P \subseteq Y$ is a closed convex cone with nonempty interior. Let us introduce the Lagrangian

$$L(\lambda^*, x) = f(x) + \langle \lambda^*, g(x)\rangle.$$

Obviously,

$$\mu \geq \inf_{x\in C} L(\lambda^*, x)\ \ \forall\, \lambda^* \in P^*. \tag{3.4.2}$$

We set

$$A \doteq \Big\{(f(x)-\mu, g(x)) \in \mathbb{R}\times Y :\ x \in C\Big\}.$$

Theorem 3.4.1 Let us consider problem (3.4.1). If μ is finite and the Slater-type condition that for some $x_0 \in C$, $\langle y^*, g(x_0)\rangle < 0$ for all $y^* \in P^* \setminus \{0\}$ holds, then the following assertions are equivalent:

(a) there exists a Lagrange multiplier $\lambda^* \in P^*$ such that

$$\inf_{x\in K} f(x) = \inf_{x\in C} L(\lambda^*, x);$$

(b)

$$\inf_{x\in K} f(x) = \max_{\lambda^*\in P^*} \inf_{x\in C} L(\lambda^*, x);$$

(c) $\mathrm{cone}(A + \mathrm{int}(\mathbb{R}_+ \times P^*))$ is pointed.

Proof. $(a) \iff (b)$: One implication is obvious. From (a) it follows that

$$\mu \leq \max_{\lambda^*\in P^*} \inf_{x\in C} L(\lambda^*, x),$$

which together with (3.4.2) imply (b).
$(c) \implies (a)$: Applying Theorem 3.1.2 we infer that $\mathrm{co}(A) \cap (-\mathrm{int}(\mathbb{R}_+ \times P)) = \emptyset$. By the convex separation theorem, we obtain $\gamma^* \geq 0$ and $\lambda^* \in P^*$, not both zero, satisfying

$$\gamma^* f(x) + \langle \lambda^*, g(x)\rangle \geq \gamma^* \mu \;\; \forall\, x \in C. \tag{3.4.3}$$

If $\gamma^* = 0$, then $0 \neq \lambda^* \in P^*$ and $\langle \lambda^*, g(x)\rangle \geq 0$ for all $x \in C$, contradicting the Slater-type condition. Therefore, we may assume $\gamma^* = 1$ in (3.4.3). Hence,

$$f(x) + \langle \lambda^*, g(x)\rangle \geq \mu \;\; \forall\, x \in C, \tag{3.4.4}$$

which implies

$$\inf_{x\in C} L(\lambda^*, x) \geq \mu.$$

This together with (3.4.2) yield the desired result.
$(a) \implies (c)$: From (a), (3.4.4) holds, and this amounts to writing

$$\langle (1, \lambda^*), (f(x) - \mu, g(x))\rangle \geq 0 \;\; \forall\, x \in C.$$

We then apply Theorem 3.1.2 to get (c). □

3.4.2 Characterizing weakly efficient solutions

Let X be a real vector space $K \subseteq X$ a convex set and Y a real locally convex topological vector space. Given a vector mapping $F : K \to Y$, we consider the problem of finding

$$\bar{x} \in K : \; F(x) - F(\bar{x}) \notin -\text{int } P, \;\; \forall \, x \in K,$$

where $P \subseteq Y$ is a closed convex cone such that $\text{int } P \neq \emptyset$ (see Section 3.1). The set of such $\bar{x}$ is denoted by E_w, and its elements are termed weakly efficient solutions. Clearly

$$\bar{x} \in E_w \Longleftrightarrow (F(K) - F(\bar{x})) \cap (-\text{int } P) = \emptyset.$$

For a real-valued function h, by $\text{argmin}_K h$ we mean the set of minimum points of h on K.

The next theorem is a direct consequence of Corollary 3.1.3 with $G(x) = F(x) - F(\bar{x})$.

Theorem 3.4.2 Let $K \subseteq X$ be a convex set and F, P as above. The following assertions are equivalent:

(a)

$$\bar{x} \in \bigcup_{p^* \in P^*, p^* \neq 0} \text{argmin}_K \langle p^*, F(\cdot) \rangle;$$

(b) $\text{cone}(F(K) - F(\bar{x}) + \text{int } P)$ is pointed.

In case $Y = \mathbb{R}^2$, we get the following theorem whose proof follows from Theorem 3.2.1.

Theorem 3.4.3 Let $K \subseteq X$ be a convex set and F, P as above with $Y = \mathbb{R}^2$. Then the following assertions are equivalent:

(a)

$$\bar{x} \in \bigcup_{p^* \in P^*, p^* \neq 0} \text{argmin}_K \langle p^*, F(\cdot) \rangle;$$

(b) $\bar{x} \in E_w$ and $\text{cone}(F(K) - F(\bar{x}) + \text{int } P)$ is convex.

Notice that the cone appearing in (b) of the preceding theorem may be substituted by others cones by virtue of Theorem 3.2.1.

Remark 3.4.1 Some sufficient and in some situations also necessary conditions to get $E_w \neq \emptyset$ are established in [19, 23].

Chapter 4

The bicriteria case on the real-line and algorithms without derivatives

This chapter is motivated for the problem 1.4.3 in the finite dimensional case, this is given a nonempty set $K \subseteq \mathbb{R}^n$, $P = \mathbb{R}^m_+$ and a function $F : K \to \mathbb{R}^m$. We will be interested

$$\text{find} \quad \bar{x} \in K \quad \text{such that} \quad F(y) - F(\bar{x}) \in \mathbb{R}^m \setminus - \operatorname{int} \mathbb{R}^m_+ \quad \forall\, y \in K. \tag{4.0.1}$$

On the other hand, by algorithmic (to develop some gradient-type method as in [27]) and theoretical purposes (to develop a well-posedness theory Ref. [14], or optimality conditions) motivated by aplications, one needs the equivalence

$$\bar{x} \in E_w \Longleftrightarrow \bar{x} \in \bigcup_{p^* \in R^m_+,\, p^* \neq 0} \text{argmin}_K \langle p^*, F(\cdot) \rangle. \tag{4.0.2}$$

Actually, this union may be taken over the set $\Delta = \{\xi \in R^m_+ : \sum_{i=1}^m \xi_i = 1\}$. It is well known that the equivalence (4.0.2) is true whenever each component of F is convex. However, it still holds under weaker assumptions than convexity on F, as shown in Ref. [36]. The authors in [24] obtained a necessary and sufficient condition in order the equivalence in (4.0.2) holds: in case $m = 2$, it requires the convexity of $M = \text{cone}(F(K) - F(\bar{x})) + \mathbb{R}^2_+ = \text{cone}(F(K) - F(\bar{x}) + \mathbb{R}^2_+)$. The convexity of M holds if each component of F is convex, and it is independently obtained of $\bar{x}$, as expected.

It is now our purpose to discuss situations in which the equivalence in (4.0.2) is no longer true, that is, when the convexity of M fails in case $m = 2$. The implication " $\Longleftarrow$ " in (4.0.2), which always holds without any further assumption on F, gives rise to the so-called scalarization approach, or weighting method, for solving problem (4.0.1). It is known that there are many efficient algorithms for solving

$$\min_{x \in K} \langle p^*, F(x) \rangle, \quad p^* \in \Delta \tag{4.0.3}$$

at least, for smooth functions F. The main drawback lies on the choice of P^* since it is not known in advance for the decision-maker or modeler. In fact, take the example discussed in [28], $F(x) = (x, \sqrt{1+x^2})$, $x \in K = \mathbb{R}$. Here , each component of F is convex, but if $p_1^* > p_2^* > 0$, then the optimal value of (4.0.3) is $-\infty$. However $E_w =]-\infty, 0]$. Our Theorem 4.2.1 apply to this example.

On the other hand, when at least one component or F is quasiconvex but nonconvex (hence (4.0.2) does not necessarily hold), it may there be solutions to (4.0.1) that are not solutions to (4.0.3) for any $p^* \in \Delta$, as the following function shows: take $K =]0, +\infty]$ and $F = (f_1, f_2)$, where,

$$f_1(x) = \begin{cases} 2 & \text{if} \quad x \notin [1,2], \\ 1 & \text{if} \quad x \in [1,2], \end{cases} \qquad f_2(x) = |x - 5|.$$

Here $E_w = [1, 8]$ while the set of the right-hand side in (4.0.2) is $[1, 5]$. Our Theorem 4.1.3 applies in this case.

More precisely, in the present chapter, under the quasiconvexity (and lower semicontinuity) assumption on each component of F, we completely describe E_w without using any kind of derivative when $K \subseteq \mathbb{R}$ and $m = 2$. In this situation, it is proven in [23] that E_w is closed and convex.

The following proposition suggests us the case we will study in this chapter.

Proposition 4.0.1 Let $\emptyset \neq K \subseteq \mathbb{R}^n$ be convex and closed, $f_i : K \to \mathbb{R}$, $i = 1, \ldots, m$ be functions such that

$$\bigcap_{i=1}^{m} \operatorname{argmin}_K f_i \neq \emptyset.$$

Then

$$E_w = \bigcup_{i=1}^{m} \operatorname{argmin}_K f_i.$$

Proof. Let $\bar{x} \in \operatorname{argmin}_K f_i$ for all $i = 1, \ldots, m$, and $z \notin \bigcup_{i=1}^{m} \operatorname{argmin}_K f_i$. Then $f_i(\bar{x}) < f_i(z)$ for all $i = 1, \ldots, m$ implying $z \notin E_w$. This prove $E_w \subseteq \bigcup_{i=1}^{m} \operatorname{argmin}_K f_i$, and then the proof is completed. □

The inclusion

$$\bigcup_{i=1}^{m} \operatorname{argmin}_K f_i \subseteq E_w,$$

in general is true. However, as illustrated in Example 2.4.3, there are solutions which does not belong to the set on the left-hand side. This section is devoted to the full description of E_w, and it will be done in the case of two-objective functions under quasiconvexity conditions on each f_i.

By virtue of Proposition 4.0.1 we will consider

$$\operatorname{argmin}_K f_1 \cap \operatorname{argmin}_K f_2 = \emptyset.$$

For $i = 1, 2$ and $t \in \mathbb{R}$, we set

$$S_i[t] = \{x \in K \; : \; f_i(x) \leq t\}, \quad S_i^+[t] = \{x \in K \; : \; f_i(x) < t\}.$$

4.1 The first case: $\operatorname{argmin}_K f_1$ and $\operatorname{argmin}_K f_2$ are compact

$[\alpha_1, \beta_1] = \operatorname{argmin}_K f_1,\ [\alpha_2, \beta_2] = \operatorname{argmin}_K f_2.$

In this case we will assume:

$$-\infty < \alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < +\infty. \tag{4.1.1}$$

Set

$$A_+ = \{x \in [\beta_1, \alpha_2] :\ f_1(x) = f_1(\alpha_2)\}, \quad A_- = \{x \in [\beta_1, \alpha_2] :\ f_2(x) = f_2(\beta_1)\}.$$

Theorem 4.1.1 Let $\emptyset \neq K \subseteq \mathbb{R}$ be convex and closed, $f_i : K \to \mathbb{R}$ be lsc and quasiconvex for $i = 1, 2$. Assume that (4.1.1) holds. Then A_+ and A_- are convex and nonempty. Moreover, we also have:

(a) if $A_+ =]\alpha_0^+, \alpha_2]$, $\alpha_0^+ \geq \beta_1$, then

$$S_2[f_2(\alpha_0^+)] \cap \{x \in K : x > \beta_2,\ f_1(x) = f_1(\alpha_2)\} = E_w \cap \{x \in K : x > \beta_2\}; \tag{4.1.2}$$

(b) if $A_+ = [\alpha_0^+, \alpha_2]$, $\alpha_0^+ > \beta_1$, then

$$S_2[\lambda_+] \cap \{x \in K : x > \beta_2,\ f_1(x) = f_1(\alpha_2)\} = E_w \cap \{x \in K : x > \beta_2\}; \tag{4.1.3}$$

where $\lambda_+ = \lim_{t \downarrow 0} f_2(\alpha_0^+ - t) = \inf_{y < \alpha_0^+} f_2(y)$.

(c) if $A_- = [\beta_1, \alpha_0^-[$, $\alpha_0^- \leq \alpha_2$, then

$$S_1[f_1(\alpha_0^-)] \cap \{x \in K : x < \alpha_1,\ f_2(x) = f_2(\beta_1)\} = E_w \cap \{x \in K : x < \alpha_1\};$$

(d) if $A_- = [\beta_1, \alpha_0^-]$, $\alpha_0^- < \alpha_2$, then

$$S_1[\lambda_-] \cap \{x \in K : x < \alpha_1,\ f_2(x) = f_2(\beta_1)\} = E_w \cap \{x \in K : x < \alpha_1\},$$

where $\lambda_- = \lim_{t \downarrow 0} f_1(\alpha_0^- + t) = \inf_{y > \alpha_0^-} f_1(y)$.

Proof. We only prove the convexity of A_+. Take $y_i \in A_+$, $i = 1, 2$ with $y_1 < y_2$, and consider y such that $y_1 < y < y_2$. Then by quasiconvexity, $f_1(y) \leq f_1(\alpha_2)$. On the other hand, since $\beta_1 < y_1 < y$, we obtain $f_1(\alpha_2) = f_1(y_1) \leq f_1(y)$ by quasiconvexity. Thus $y \in A_+$. Hence either $A_+ =]\alpha_0^+, \alpha_2]$ or $A_+ = [\alpha_0^+, \alpha_2]$ for some α_0^+.
(a): Let $\bar{x}$ in the set of the left hand-side of (4.1.2). We will check that $\bar{x} \in E_w$. If $x > \bar{x}$ and $x \in K$, then $f_2(\bar{x}) \leq f_2(x)$ since $f_2(\alpha_2) < f_2(\bar{x})$; if $x \in]\alpha_0^+, \alpha_2] = A_+$ then obviously $f_1(x) = f_1(\alpha_2) = f_1(\bar{x})$; if $x \in]\alpha_2, \bar{x}[$ then, since $\beta_1 < \alpha_0^+ < \beta_2 < x < \bar{x}$, we have $f_1(x) \leq \max\{(f_1(\alpha_2), f_1(\bar{x})\} = f_1(\bar{x})$ and $f_1(\bar{x}) = f_1(\alpha_2) \leq \max\{f_1(\beta_1), f_1(x)\} = f_1(x)$, and then $f_1(x) = f_1(\bar{x})$; if $x < \alpha_0^+$, then $f_2(\alpha_0^+) \leq \max\{f_2(\alpha_2), f_2(x)\} = f_2(x)$ since $\alpha_0^+ < \alpha_2$. We also have $f_2(\bar{x}) \leq f_2(\alpha_0^+)$. Thus $f_2(\bar{x}) \leq f_2(x)$. This completes the proof that $\bar{x} \in E_w$, proving one inclusion in (4.1.2).
Let us prove the other inclusion. Take any $\bar{x} \in K$, $\bar{x} > \beta_2$.
If $\bar{x} \notin S_2[f_2(\alpha_0^+)]$, then $f_2(\bar{x}) > f_2(\alpha_0^+)$. On the other hand, $f_1(\bar{x}) > f_1(\beta_1)$ and therefore $f_1(\alpha_0^+) \leq f_1(\bar{x})$; in case $f_1(\alpha_0^+) = f_1(\bar{x})$ we had $\alpha_0^+ \in A_+$, giving a contradiction. Thus $f_1(\alpha_0^+) < f_1(\bar{x})$ implying that $\bar{x} \notin E_w$.
If $f_1(\bar{x}) \neq f_1(\alpha_2)$, then $f_1(\bar{x}) > f_1(\alpha_2)$ since $f_1(\beta_1) < f_1(\alpha_2)$ and f_1 is quasiconvex. We also have $f_2(\alpha_2) < f_2(\bar{x})$. Hence $\bar{x} \notin E_w$.
(b) : Let $\bar{x}$ in the set of the left hand-side of (4.1.3). We now check that $\bar{x} \in E_w$. If $x > \bar{x}$ and $x \in K$, then $f_2(\bar{x}) \leq f_2(x)$ since $f_2(\beta_2) < f_2(x)$; if $x \in]\alpha_0^+, \bar{x}[$ then obviously $f_1(\bar{x}) = f_1(\alpha_0^+) = f_1(\alpha_2)$ and therefore $f_1(x) \leq f_1(\alpha_0^+)$. Since $f_1(\beta_1) < f_1(x)$, $f_1(\alpha_0^+) \leq f_1(x)$, and hence $f_1(x) = f_1(\alpha_0^+)$; If $x < \alpha_0^+$ and $x \in K$, then $f_2(\bar{x}) \leq \lambda_+ \leq f_2(y)$ for all $y < \alpha_0^+$ because of monotonicity of f_2 in such an interval. This completes the proof that $\bar{x} \in E_w$ and thus one inclusion in (4.1.3) is proved.
For the other inclusion, take any $\bar{x} \in K$, $\bar{x} > \beta_2$.
If $\bar{x} \notin S_2[\lambda_+]$, $f_2(\bar{x}) > \lambda_+$. Since $\beta_1 < \alpha_0^+$, there is $t > 0$ such that $\beta_1 < \alpha_0^+ - t < \alpha_0^+$ and $f_2(\bar{x}) > f_2(\alpha_0^+ - t)$. On the other hand, $f_1(\alpha_0^+ - t) \leq f_1(\alpha_0^+) = f_1(\alpha_2)$. Obviously $f_1(\alpha_0^+ - t) < f_1(\alpha_2)$ since otherwise $\alpha_0^+ - t \in A_+$, which is impossible. By quasiconvexity $f_1(\alpha_2) \leq f_1(\bar{x})$, and hence $f_1(\alpha_0^+ - t) < f_1(\bar{x})$ implying $\bar{x} \notin E_w$.
If $f_1(\bar{x}) \neq f_1(\alpha_2)$, then $f_1(\bar{x}) > f_1(\alpha_2)$ since $f_1(\beta_1) < f_1(\alpha_2)$ and f_1 is quasiconvex. We also have $f_1(\alpha_2) < f_2(\bar{x})$. This shows $\bar{x} \notin E_w$.

$(c),(d)$: The proof of these parts is totally symmetrical to used in (a) and (b) . □

We use the following notation:

$$\gamma_+ = \begin{cases} f_2(\alpha_0^+) & \text{if} \quad A_+ =]\alpha_0^+, \alpha_2] \\ \lambda_+ & \text{if} \quad A_+ = [\alpha_0^+, \alpha_2] \end{cases}, \quad \gamma_- = \begin{cases} f_1(\alpha_0^-) & \text{if} \quad A_- =]\beta_1, \alpha_0^-] \\ \lambda_+ & \text{if} \quad A_- = [\beta_1, \alpha_0^-] \end{cases}$$

Let use consider the sets

$$M_1^+ = \{x \in K \ : \ x > \beta_2, \quad f_1(x) = f_1(\alpha_2)\}, \quad M_2^+ = \{x \in K \ : \ x > \beta_2, \quad f_2(x) = \gamma^+\},$$

and

$$M_1^- = \{x \in K \ : \ x < \alpha_1, \quad f_1(x) = \gamma_-\}, \quad M_2^- = \{x \in K \ : \ x < \alpha_1, \quad f_2(x) = f_2(\beta_1)\}.$$

Corollary 4.1.1 Assume the assumptions of Theorem 4.1.1 hold. Then M_i^+ and M_i^-, i=1,2, are convex.

(a) *If* $M_1^+ \cap M_2^+ \neq \emptyset$, then

$$\sup E_w = \min\{\sup M_1^+, \sup M_2^+\} = \sup M_1^+ \cap M_2^+,$$

(b) *If* $M_1^- \cap M_2^- \neq \emptyset$, then

$$\inf E_w = \max\{\inf M_1^-, \inf M_2^-\} = \inf M_1^- \cap M_2^-.$$

Proof. Due to the quasiconvexity of f_1 and f_2, it is not difficult to check that M_i^+ and M_i^-, $i = 1, 2$, are convex. We only check (a)(for the case (b) we proceed in a similar way). We analyze the different situations.

• $\sup M_1^+ < +\infty$ and $\sup M_2^+ < +\infty$: in this case we can prove that $\sup M_i^+ \in M_i^+$ for $i = 1, 2$. Set $\bar{x} = \min\{\sup M_1^+, \sup M_2^+\}$. If $\bar{x} \in M_1^+$ then $f_1(\bar{x}) = f_1(\alpha_2)$, and side $\bar{x} \in M_2^+$, we obtain $f_2(\bar{x}) = \gamma_+$, showing that $\bar{x} \in E_w$ by the previous theorem. We now prove that, $z > \bar{x},\ z \in K$ implies that $z \notin E_w$. In fact, $f_1(z) \neq f_1(\alpha_2)$ whenever $z > \bar{x},\ inK$. This implies tat $z \notin E_w$ again by the previous theorem. We now consider the case in which $\bar{x} = \max M_2^{+.}$ Then $f_2(\bar{x}) = \gamma_+$, and since $\bar{x} \in M_1^+$, we obtain $f_1(\bar{x}) = f_1(\alpha_2)$, showing that $\bar{x} \in E_w$ again by the previous theorem. We now prove that, $z > \bar{x},\ z \in K$ implies $z \notin E_w$. In fact, $f_2(z) \neq \gamma_+$ whenever $z > \bar{x},\ z \in K$. This automatically implies that $z \notin E_w$ by the previous theorem.

• $\max M_1^+ < +\infty$ and $\sup M_2^+ = +\infty$ (the case $\sup M_1^+ = +\infty$ and $\max M_2^+ < +\infty$ is treated similarly): set $\bar{x} = \max M_1^+ \in M_1^+$. The quasiconvexity of f_2 and the fact that $\sup M_2^+ = +\infty$ along with $M_1^+ \cap M_2^+ \neq \emptyset$ imply $f_2(\bar{x}) = f_2(\alpha_0^+)$. This shows that $\bar{x} \in E_w$. Obviously, if $z > \bar{x},\ \in K$, then $z \notin E_w$ by the quasiconvexity of f_1.

• $\sup M_1^+ = +\infty$ and $\sup M_2^+ = +\infty$: in this case $\sup E_w = +\infty$, since for $\bar{x} \in M_1^+ \cap M_2^+$, every $z > \bar{x}$ is in E_w by the previous theorem and the convexity of $M_1^+ \cap M_2^+$.

The remaining equalities in (a) and (b) are easily check. □

Corollary 4.1.2 Assume the assumptions of Theorem 4.1.1 hold.

(a) If $M_1^+ = \emptyset$ (resp. $M_2^- = \emptyset$), then $\max E_w = \beta_2$ (resp. $\min E_w = \alpha_1$).

(b) If $M_1^+ \cap M_2^+ = \emptyset$ and $M_1^+ \neq \emptyset$, then

$$\sup E_w = \min\{\sup M_1^+, \sup\{x \in K\ :\ x \geq \beta_2,\ f_2(x) \leq \gamma^+\}\}. \tag{4.1.4}$$

(c) If $M_1^- \cap M_2^- \emptyset$ and $M_1^- \neq \emptyset$, then

$$\inf E_w = \max\{\inf M_2^-, \inf\{x \in K\ :\ x \leq \alpha_1,\ f_1(x) \leq \gamma_-\}\}.$$

Proof. (a) Assume that $M_1^+ = \emptyset$ if ($M_2^- = \emptyset$ a similar reasoning may be applied). Then $f_1(\alpha_2) \neq f_1(x)$ for all $x > \beta_2,\ x \in K$. Since $\beta_1 < \alpha_2 \leq \beta_2$ we obtain that $f_1(\alpha_2) \leq f_1(x)$ for all $x > \beta_2$ by quasiconvexity, and therefore $f_1(\alpha_2) < f_2(x)$ for all $x > \beta_2$. Consequently $\max E_w = \beta_2$.

(b) We now consider $M_1^+ \neq \emptyset$.

• $\sup M_1^+ < +\infty$: in this case $\bar{x} = \sup M_1^+ \in M_1^+$. By assumption $\bar{x} \notin M_2^+$, thus $f_2(\bar{x}) \neq \gamma_+$. If $f_2(\bar{x}) < \gamma_+$ then $\bar{x} \in E_w$ by Theorem 4.1.1, and

$$\bar{x} \leq \sup\{x \in K\ :\ x \geq \beta_2,\ f_2(x) \leq \gamma_+\}.$$

Obviously, every $z \in K, z > \bar{x}$ is not in E_w, proving that $\sup E_w = \bar{x}$, and hence (4.1.4) is proved. If $f_2(\bar{x}) > \gamma_+$, we claim that

$$\bar{z} = \sup\{x \in K \ : \ x \geq \beta_2, \ f_2(x) \leq \gamma_+\} = \sup E_w. \tag{4.1.5}$$

First, since $f_2(\bar{x}) > \gamma_+$, we obtain $\bar{z} < \bar{x}$ because of the quasiconvexity of f_2, and since $f_1(x) = f_1(\alpha_2)$ for all $x \in]\beta_2, \bar{x}]$, we also get $\bar{z} \in E_w$ by Theorem 4.1.1, and if $z \in K$ with $z > \bar{z}$, then $z \notin E_w$, as claimed.

• $\sup M_1^+ = +\infty$: in this case $(4.1.5)$ is also verified as easily seen. □

We now present an algorithm which we approximated the value of γ_+. Let us remember that γ_+ is fundamental in the characterization presented in the Theorem 4.1.1. In this algorithm we consider f_2 be continuous and the tolerance like the length of the interval of search in every iterations. Let $\epsilon > 0$ be tolerance.

Algorithm 1.

0. Set $x_0 := \beta_1$, $y_0 := \alpha_2$ and $k := 0$.

1a. $z_k := \frac{x_k + y_k}{2}$.

1b. Evaluate $f_1(z_k)$. If $f_1(z_k) = f_1(\alpha_2)$ then
$y_{k+1} := z_k$, $x_{k+1} := x_k$, $k := k+1$ and GOTO **1d**;

else

1c. $x_{k+1} := z_k$, $y_{k+1} := y_k$, $k := k+1$ and GOTO **1a**;

1d. Evaluate $f_2(x_k)$ and $f_2(y_k)$. If $f_2(x_k) - f_2(y_k) \leq \epsilon$ STOP,
$\alpha_0^+(approx) := y_k$ and $\gamma_+(approx) := f_2(y_k)$.

else

1e. GOTO 1a.

Let us notice that if f_1 is lsc and semistrictly quasiconvex then $A_+ = \{\alpha_0^+\} = \{\alpha_2\}$. Therefore if f_2 be continuous then $\gamma_+ = f_2(\alpha_2)$.

On the other hand, $x_{k-1} \leq x_k < \alpha_0^+ \leq y_k \leq y_{k-1}$ and $f_2(y_{k-1}) \leq f_2(y_k) \leq f_2(\alpha_0^+) \leq f_2(x_k) \leq f_2(x_{k-1})$. Therefore $y_k - \alpha_0^+ \leq y_k - x_k = \frac{1}{2^k}(\alpha_2 - \beta_1)$, $\alpha_0^+ - x_k \leq y_k - x_k = \frac{1}{2^k}(\alpha_2 - \beta_1)$ and $f_2(\alpha_0^+) - f_2(y_k) \leq f_2(x_k) - f_2(y_k)$. Thus $f_2(x_k) - f_2(y_k) \to 0$ and $f_2(y_k) \to f_2(\alpha_0^+)$, $k \to +\infty$.

We now state our second algorithm based on Theorem 4.1.1 which determines a point $\bar{x} \in E_w$, if any, strictly greater than β_2.

Algorithm 2.

0. Take $x_0 \in K,\ x_0 > \beta_2$. Set $k = 0$.

1a. Evaluate $f_2(x_k)$. If $f_2(x_k) \leq \gamma_+$, GOTO **2a;**

else

1b. Set $x_{k+1} := \frac{x_k+\beta_2}{2}$, and GOTO **1a.**

2a. Evaluate $f_1(x_k)$. If $f_1(x_k) = f_1(\alpha_2)$ STOP, $x_k \in E_w$;

else

2b. Set $x_{k+1} := \frac{x_k+\beta_2}{2}$, $k := k+1$ and GOTO **2a.**

Let us notice that $x_k > x_{k+1} > \beta_2$, since $x_k - x_{k+1} = \frac{1}{2^k}(x_0 - \beta_2)$ and $x_{k+1} = \frac{x_k+\beta_2}{2}$. Thus the finiteness of this procedure is guaranteed if we know, a priori, that $E_w \cap \{x \in K :\ x > \beta_2\} \neq \emptyset$. Otherwise, the infinite sequence generated by the algorithm will converge to β_2 in which case $\max E_w = \beta_2$. This happens, for instance if $M_1^+ = \emptyset$ as Corollary 4.1.2 shows.

Observe that once we arrive at step 2b, the inequality in step la continues to be valid for the next iterate; and $f_1(x_k) \neq f_1(\alpha_2)$ amount to writing $f_1(x_k) > f_1(\alpha_2)$ since f_1 (and also f_2) is non-decreasing in $[\beta_2, +\infty[\cap K$.

The following procedure may used to compute $\bar{z}$ in case compact, precisely when $K = [a, b]$. In the same way as in the algorithm 1 we will consider the tolerance like the length of the interval of search in every iterations.

Algorithm 3.

0a. Evaluate $f_2(b)$ and $f_1(b)$. If $(f_2(b) \leq \gamma_+$ and $f_1(b) = f_1(\alpha_2))$, STOP $\sup E_w = b$.

else

0b. Set $x_0 := \beta_2$, $y_0 := b$. and $k := 0$.

1b. Set $z_k := \frac{x_k + y_k}{2}$

1c. Evaluate $f_1(z_k)$. If $f_1(z_k) = f_1(\alpha_2)$ then

$x_{k+1} := z_k$, $y_{k+1} := y_k$, $k := k + 1$ and GOTO **1e**

else

1d. $x_{k+1} := x_k$, $y_{k+1} := z_k$, $k := k + 1$ and GOTO **1b**;

1e. If $y_k - x_k > \epsilon$ then GOTO **1b**;

else GOTO **2a**.

2a. Evaluate $f_2(x_k)$. If $f_2(x_k) \leq \gamma_+$ then

STOP, $\sup E_w(approx) = x_k$

else

Set $x_0 := \beta_2$, $y_0 := x_k$ and $k := 0$.

2b. $z_k := \frac{x_k+y_k}{2}$.

2c. Evaluate $f_2(z_k)$. If $f_2(z_k) \leq \gamma_+$ then

$x_{k+1} := z_k$, $y_{k+1} := y_k$, $k := k+1$ and GOTO **2e**.

else

2d. $x_{k+1} := x_k$, $y_{k+1} := z_k$, $k := k+1$ and GOTO **2b**;

2e. If $y_k - x_k > \epsilon$ then GOTO **2b;**

else STOP, $\sup E_w(approx) = x_k$

A reasonning similar to the realized on in the algorithm 1, shows the convergence of this algorithm.

Based on the algorithms 1 and 3, let us show them proved obtained with implementation computational of these, applied to 2 examples.

Example 4.1.1 Let us consider the Example 2.4.3, take $y_0 = 10^8$. (see figure 1). We will determine the $\sup E_w$ *(see table 1).*

$$f_1(x) = \begin{cases} 2 & \textit{if} \quad x \notin [1,2], \\ 1 & \textit{if} \quad x \in [1,2], \end{cases} \qquad f_2(x) = |x-5|.$$

Here $E_w = [1, 8]$.

figure 1

TABLE 1

error	total cpu time	γ_+/iterations	$\sup E_w$/iterations
10^{-3}	0.0150000000	2.9992675781/12	7.9993314775/40
10^{-4}	0.0160000000	2.999908443/15	7.9999226491/42
10^{-5}	0.0160000000	2.9999942780/19	7.9999965455/45
10^{-6}	0.0160000000	2.9999992847/22	7.9999993877/49

Example 4.1.2 In this Example (see figure 2), take $y_0 = 10^8$, here $E_w = [0, 7]$. The approximations of the $\sup E_w$ can turn in the table 2

$$f_1(x) = \begin{cases} 2 & \textit{if}\ x < 1, \\ 1 & \textit{if} \quad x \in [1,2], \\ 2 & \textit{if} \quad x \in]2,7[, \\ \sqrt{x-7}+2 & \textit{if} \quad x > 7, \end{cases} \quad f_2(x) = \begin{cases} 6-x & \textit{if} \quad x < 4, \\ -e^{-(x-4)^2}+3 & \textit{if} \quad x \geq 4, \end{cases}$$

figura 2

TABLE 2

error	total cpu time	γ_+/iterations	$\sup E_w$/iterations
10^{-3}	0.014000	3.9990234375/11	6.9999681538/40
10^{-4}	0.015000	3.9999389648/15	6.9999908912/42
10^{-5}	0.015000	3.9999923706/18	6.9999997729/48
10^{-6}	0.016000	3.9999990463/21	6.9999997729/48

Theorem 4.1.2 Let $\emptyset \neq K \subseteq \mathbb{R}$ be convex and closed, $f_i : K \to \mathbb{R}$ be lsc for $i = 1, 2$, with f_1 being quasiconvex and f_2 semistrictly quasiconvex. Assume that (4.1.1) holds. Then $\min E_w = \alpha_1$, and if $\bar{x} \in K$, the following assertion are equivalent:

(a) $\bar{x} > \beta_2$, $\bar{x} \in E_w$;

(b) $\bar{x} > \beta_2$, $f_1(\alpha_2) = f_1(\bar{x})$,

$$S_1^+[f_1(\alpha_2)] \cap S_2^+[f_2(\bar{x})] = \emptyset;$$

(c) $\bar{x} > \beta_2$,

$$S_2^+[f_2(\bar{x})] \subseteq]\beta_1, \bar{x}[, \quad S_2^+[f_2(\bar{x})] \subseteq \{x \in K \,:\, f_1(x) = f_1(\bar{x})\}.$$

Proof. The semistrict quasiconvexity of f_2 implies $M_2^- = \emptyset$, the first part is a consequence of Corollary 4.1.2.
$(a) \Longrightarrow (b)$: From Theorem 4.1.1 it follows that $f_1(\alpha_2) = f_1(\bar{x})$. Let $x \in S_1^+[f_1(\alpha_2)]$: then $f_1(x) < f_1(\alpha_2) = f_1(\bar{x})$, and therefore $f_2(\bar{x}) \leq f_2(x)$ since $\bar{x} \in E_w$.
$(b) \Longrightarrow (c)$: Let $x \in S_2^+[f_2(\bar{x})]$ and suppose that $x > \bar{x}$. Since $f_2(\alpha_2) < f_2(x)$, the semistrict quasiconvexity of f_2 implies that $f_2(\bar{x}) < f_2(x)$, giving a contradiction, and $x < \bar{x}$. suppose now that $x \leq \beta_1$, then $f_2(\beta_1) < f_2(\bar{x})$ by the semistrict quasiconvexity of f_2 and the choice of x. By assumption and a previous inequality it follows that $f_1(\beta_1) \leq f_1(\alpha_2)$, yielding a contradiction. Finally, we observe that $\beta_1 \notin S_2^+[f_2(\bar{x})]$. This complete the proof of the first part in (c). We now prove de second inclusion. Take any $x \in S_2^+[f_2(\bar{x})]$. Then $f_1(x) \geq f_1(\alpha_2) = f_1(\bar{x})$. The quasiconvexity implies that $f_1(x) = f_1(\bar{x})$.
$(c) \Longrightarrow (a)$: We need to prove that $\bar{x} \in E_w$. Because of the second inclusion of (c), it suffices to check that $f_2(\bar{x}) \leq f_2(x)$ for all $x \notin S_2^+[f_2(\bar{x})]$, but it is straightforward. □

Theorem 4.1.3 Let $\emptyset \neq K \subseteq \mathbb{R}$ be convex and closed, $f_i : K \to \mathbb{R}$ be lsc and quasiconvex for $i = 1, 2$. Assume that (4.1.1) holds.

(a) If f_2 is semistrictly quasiconvex and $M_1^+ \cap M_2^+ \neq \emptyset$, then $E_w = [\alpha_1, \bar{x}]$, where $\bar{x} \in K$ solves the system

$$\bar{x} > \beta_2 \quad f_1(\bar{x}) = f_1(\alpha_2), \quad f_2(\bar{x}) = \gamma_+.$$

(b) If f_1 is semistrictly quasiconvex and $M_1^- \cap M_2^- \neq \emptyset$, then $E_w = [\bar{x}, \beta_2]$, where $\bar{x} \in K$ solves the system

$$\bar{x} < \alpha_1 \quad f_2(\bar{x}) = f_2(\beta_1), \quad f_1(\bar{x}) = \gamma_-.$$

Proof. (a) By of Corollary 4.1.2, $\min E_w = \alpha_1$, and by virtue of Corollary 4.1.1, it sufficies to prove that $M_1^+ \cap M_2^+ = \{\bar{x}\}$, which easily follows from the semistrict quasiconvexity of f_2. Part (b) follows from Corollaries 4.1.2 (b) and 4.1.1. □

4.2 The second case: $\mathrm{argmin}_K f_1$ compact and $\mathrm{argmin}_K f_2$ empty

In this case we will assume:

$$[\alpha, \beta] = \mathrm{argmin}_K f_1, \quad -\infty < \alpha \leq \beta < +\infty. \tag{4.2.1}$$

Set

$$B_+ = \{x \in K \ : \ x \leq \alpha, \ f_2(x) = f_2(\alpha)\}, \quad B_- = \{x \in K \ : \ x \geq \beta, \ f_2(x) = f_2(\beta)\}.$$

We recall from Lemma 2.4.1 that every lsc and quasiconvex function h on $K \subseteq \mathbb{R}$ such that $\mathrm{argmin}_K h = \emptyset$, is non-constant and monotone. This fact we will be used in the remaining of this chapter.

Theorem 4.2.1 Let $\emptyset \neq K \subseteq \mathbb{R}$ be convex and closed, $f_i : K \to \mathbb{R}$ be lsc and quasiconvex for $i = 1, 2$. Assume that (4.2.1) holds along with $\mathrm{argmin}_K f_2 = \emptyset$. Then, B_+, B_- are convex, nonempty. Moreover, the following assertions hold.

(a) If f_2 is non-decreasing then $]-\infty, \beta] \subseteq E_w$ and B_+ is bounded: in case $B_+ =]\alpha^+, \alpha]$ $(\alpha^+ < \alpha)$ we have

$$S_1[f_1(\alpha^+)] \cap \{x \in K \,:\, x > \beta,\ f_2(x) = f_2(\alpha)\} = E_w \cap \{x \in K \,:\, x > \beta\}; \quad (4.2.2)$$

in case $B_+ = [\alpha^+, \alpha]$ $(\alpha^+ \leq \alpha)$ we have

$$S_1[\mu_+] \cap \{x \in K \,:\, x > \beta,\ f_2(x) = f_2(\alpha)\} = E_w \cap \{x \in K \,:\, x > \beta\}; \quad (4.2.3)$$

where $\mu_+ = \lim_{t \downarrow 0} f_1(\alpha^+ - t) = \inf_{y < \alpha^+} f_1(y)$.

(b) If f_2 is non-increasing then $[\alpha, +\infty[\subseteq E_w$ and B_- is bounded : in case $B_- = [\beta, \beta^-[$ $(\beta < \beta^-)$ we have

$$S_1[f_1(\beta^-)] \cap \{x \in K \,:\, x < \alpha,\ f_2(x) = f_2(\beta)\} = E_w \cap \{x \in K \,:\, x < \alpha\};$$

in case $B_- = [\beta, \beta^-]$ $(\beta \leq \beta^-)$ we have

$$S_1[\mu_-] \cap \{x \in K \,:\, x < \alpha,\ f_2(x) = f_2(\beta)\} = E_w \cap \{x \in K \,:\, x < \alpha\},$$

where $\mu_- = \lim_{t \downarrow 0} f_1(\beta^- + t)$.

Proof. We only prove the convexity of B_+. Take $y_i \in B_+, i = 1, 2$, with $y_1 < y_2$, and consider y such that $y_1 < y < y_2$. By quasiconvexity, $f_2(y) \leq f_2(\alpha)$. If on the contrary $f_2(y) < f_2(\alpha)$ then f_2 is non-decreasing by Lemma 2.4.1. Hence, $f_2(\alpha) = f_2(y_1) \leq f_2(y)$, a contradiction, proving that B_+ is convex.
(a) : By assumption there exists $z < \alpha$ such that $f_2(z) < f_2(\alpha)$. We now check $\bar{x} < z$, imply $\bar{x} \in E_w$. If $\bar{x} < x$ then $f_2(\bar{x}) \leq f_2(x)$ by monotonicity; if $x < \bar{x}$ then $f_1(\bar{x}) \leq \max\{f_1(x), f_1(\alpha)\} = f_1(x)$. This prove that $\bar{x} \in E_w$, and since $\bar{x} < z$ was arbitrary we conclude that $] - \infty, \beta] \subseteq E_w$. The boundedness of B_+ is a consequence of $\operatorname{argmin}_K f_2 = \emptyset$. Let us prove (4.2.2). Take $\bar{x}$ in the set of the left hand-side of (4.2.2). We will check that $\bar{x} \in E_w$. If $x > \bar{x}$ then $f_2(\bar{x}) \leq f_2(x)$ for all $x \leq \bar{x}$ because of f_2 is non-decreasing. By the choice of α^+ and $\bar{x}$, we actually obtain that $f_2(\bar{x}) \leq f_2(x)$ for all $x > \alpha^+$. If $x \leq \alpha^+$, $x \in K$, then $f_1(\bar{x}) \leq f_1(\alpha^+) \leq f_1(x)$ by quasiconvexity and the choice of $\bar{x}$. This proves one inclusion in (4.2.2).

Let us prove the other inclusion. Take any $\bar{x} \in K$, $\bar{x} > \beta$. The monotonicity of f_2 along with the choice of B_+ imply $f_2(\alpha^+) < f_2(\alpha) \leq f_2(\bar{x})$. Therefore, if $\bar{x} \in E_w$ then $f_1(\bar{x}) \leq f_1(\alpha^+)$ and so $\bar{x} \in S_1[f_1(\alpha^+)]$. We already know that $f_2(\alpha) \leq f_2(\bar{x})$. We also have $f_1(\alpha) < f_1(\bar{x})$. Both inequalities imply $f_2(\alpha) = f_2(\bar{x})$, which completes the proof of (4.2.2).

Let $\bar{x}$ in the set of the left hand-side of (4.2.3). We now check that $\bar{x} \in E_w$. If $x > \bar{x}$ and $x \in K$, then, as in (a), we obtain $f_2(\bar{x}) \leq f_2(x)$ for all $x \geq \alpha^+$. If $x < \alpha^+$, and $x \in K$, then $f_1(\bar{x}) \leq \mu_+ \leq f_1(y)$ for all $y < \alpha^+$ because of the monotonicity of f_1 in such an interval (see Lemma 2.4.1). This completes the proof that $\bar{x} \in E_w$ and thus one inclusion in (4.2.3) is proved.

For the other inclusion, take any $\bar{x} \in K, \bar{x} > \beta$.

If $\bar{x} \notin S_1[\mu_+]$, $f_1(\bar{x}) > \mu_+$. Thus there exists $t > 0$ such that $f_1(\bar{x}) > f_1(\alpha^+ - t)$. Since f_2 is non-decreasing, $f_2(\alpha^+ - t) \leq f_2(\alpha)$. By the choice of B_+, $\quad f_2(\alpha^+ - t) < f_2(\alpha)$, and since $f_2(\alpha^+ - t) < f_2(\bar{x})$. Hence $\bar{x} \notin E_w$.

If $f_2(\bar{x}) \neq f_2(\alpha)$, then $f_2(\bar{x}) > f_2(\alpha)$ by monotonicity. We also have $f_1(\alpha) < f_1(\bar{x})$ since $\bar{x} > \beta$. This shows $\bar{x} \notin E_w$. □

We use the following notation:

$$\tilde{\gamma}_+ = \begin{cases} f_1(\alpha^+) & \text{if} \quad B_+ =]\alpha^+, \alpha] \\ \mu_+ & \text{if} \quad B_+ = [\alpha^+, \alpha] \end{cases}, \quad \tilde{\gamma}_- = \begin{cases} f_2(\beta^-) & \text{if} \quad B_- = [\beta, \beta^-[\\ \mu_- & \text{if} \quad B_- = [\beta, \beta^-] \end{cases}$$

Let use consider the sets

$$N_1^+ = \{x \in K \ : \ x > \beta, \quad f_1(x) = \tilde{\gamma}_+\}, \quad N_2^+ = \{x \in K \ : \ x > \beta, \quad f_2(x) = f_2(\alpha)\},$$

and

$$N_1^- = \{x \in K \ : \ x < \alpha_1, \quad f_1(x) = \tilde{\gamma}_-\}, \quad N_2^- = \{x \in K \ : \ x < \alpha, \quad f_2(x) = f_2(\beta)\}.$$

It is not difficult to check that N_i^+, N_i^-, $i = 1, 2$, are convex, and if $\sup N_i^+ \in \mathbb{R}$ then $\sup N_i^+ \in N_i^+$. A similar result is obtained for N_i^-.

Corollary 4.2.1 Let $\emptyset \neq K \subseteq \mathbb{R}$ be closed and convex $f_i : K \to \mathbb{R}$ be lsc and quasiconvex for $i = 1, 2$. Assume that (4.2.1) holds along with $\text{argmin}_K f_2 = \emptyset$, and $f_1$is semistrictly quasiconvex.

(a) If f_2 is non-decreasing and $N_1^+ \cap N_2^+ \neq \emptyset$, then $E_w =]-\infty, \bar{x}]$, were $\bar{x} \in K$ solves the system

$$\bar{x} > \beta \quad f_2(\bar{x}) = f_2(\alpha), \quad f_1(\bar{x}) = \tilde{\gamma}_+, \tag{4.2.4}$$

(b) If f_2 is non-increasing and $N_1^- \cap N_2^- \neq \emptyset$, then $E_w = [\bar{x}, +\infty[$, were $\bar{x} \in K$ solves the system

$$\bar{x} < \alpha \quad f_2(\bar{x}) = f_2(\beta), \quad f_1(\bar{x}) = \tilde{\gamma}_-. \tag{4.2.5}$$

Proof. We only check (4.2.4). It is easy to see that $N_1^+ \cap N_2^+$ is a singleton due to semistric quasiconvexity of f_1. Also, every $\bar{x} \in K$ satisfying (4.2.4) is in E_w. We now prove that every $z > \bar{x}$, $z \in K$, it does not belong to E_w. Indeed, if such a $z \in E_w$, then $f_1(z) \leq \tilde{\gamma}_+$ by Theorem 4.2.1. The case $f_1(z) = \tilde{\gamma}_+$, is excluded because of the semistrict quasiconvexity of f_1. Thus, we consider the case $f_1(z) < \tilde{\gamma_+}$, but it is impossible since $\tilde{\gamma}_+ = f_1(\bar{x}) \leq f_1(z)$ by quasiconvexity. This contradiction proves (4.2.4). Obviously $N_1^+ \cap N_2^+ = \{\bar{x}\}$. □

Corollary 4.2.2 Let $\emptyset \neq K \subseteq \mathbb{R}$ be closed and convex $f_i : K \to \mathbb{R}$ be lsc and quasiconvex for $i = 1, 2$. Assume that (4.2.1) holds along with $\text{argmin}_K f_2 = \emptyset$. If f_2 is semistrictly quasiconvex then either $E_w =]-\infty, \beta]$ (in case f_2 is non-decreasing) or $E_w = [\alpha, +\infty[$ (in case f_2 is non-increasing).

Proof. It is a consequence of Theorem 4.2.1. □

4.3 The third case: $\text{argmin}_K f_1$ unbounded and $\text{argmin}_K f_2$ empty

Theorem 4.3.1 Let $\emptyset \neq K \subseteq \mathbb{R}$ be closed and convex and $f_i : K \to \mathbb{R}$ be lsc and quasiconvex for $i = 1, 2$. Assume that $\text{argmin}_K f_2 = \emptyset$ and $\alpha \in \mathbb{R}$. The following assertions hold:

(a) If $\text{argmin}_K f_1 = [\alpha, +\infty[$ then either $E_w = \mathbb{R}$ (in case f_2 is non-decreasing) or $E_w = [\alpha, +\infty[$ (in the case f_2 is non-increasing);

(b) If $\text{argmin}_K f_1 =]-\infty, \alpha]$ then either $E_w = \mathbb{R}$ (in the case f_2 is non-increasing) or $E_w =]-\infty, \alpha]$ (in case f_2 is non-decreasing).

Proof. We only prove (a), being the other enterely similar. By assumption there exists $z \in K$ such that $f_2(z) < f_2(\alpha)$. (1) Assume first that $z < \alpha$. Then, Lemma 2.4.1 asserts that f_2 is non-decreasing in K. We now check $\bar{x} < z,\ \bar{x} \in K$, imply $\bar{x} \in E_w$. If $\bar{x} < x$ then $f_2(\bar{x}) \leq f_2(x)$ by monotonicity; if $x < \bar{x}$ then $f_1(\bar{x}) \leq \max\{f_1(x), f_1(\alpha)\} = f_1(x)$. This prove that $\bar{x} \in E_w$, and $\bar{x} < z$ since was arbitrary, we conclude that $E_w = \mathbb{R}$ because of the convexity of E_w. (2) Assume now that $z > \alpha$. By Lemma 2.4.1, f_2 is non-increasing in K. We will prove that $\bar{x} < \alpha,\ \bar{x} \in K$, imply $\bar{x} \notin E_w$. In fact, $f_2(z) < f_2(\alpha) \leq f_2(\bar{x})$, and the other hand, $f_1(z) < f_1(\bar{x})$ since $\bar{x} < \alpha$. This completes the proof that $\bar{x} \notin E_w$, and therefore $E_w = [\alpha, +\infty[$. □

4.4 The fourth case: $\text{argmin}_K f_1$ and $\text{argmin}_K f_2$ empty

In this case we obtain the following single theorem.

Theorem 4.4.1 Let $\emptyset \neq K \subseteq \mathbb{R}$ be closed and convex and $f_i : K \to \mathbb{R}$ be lsc and quasiconvex for $i = 1, 2$. (hence K is unbounded). The following assertions hold:

(a) if $K \neq \mathbb{R}$ then $E_w = \emptyset$;

(b) if $K = \mathbb{R}$ and $f_1,\ f_2$ are both non-decreasing or non-increasing, then $E_w = \emptyset$;

(c) if $K = \mathbb{R}$ and f_1 is non-decreasing (resp. non-increasing) and f_2 is non-increasing (resp. non-decreasing), then $E_w = \mathbb{R}$.

Proof. We see that if $K = [a, +\infty[$ (resp. $K =]-\infty, a]$) ($a > -\infty$) then $f_1,\ f_2$ are non-increasing (resp. non-decreasing) and therefore $E_w = \emptyset$. This proves (a). Parts (b) and (c) result follows a similar reasoning. □

The following case is present for the sake completennes.

4.5 The fifth case: $\operatorname{argmin}_K f_1$ and $\operatorname{argmin}_K f_2$ are unbounded

In this case $K = \mathbb{R}$. We will assume:

$$]-\infty, \alpha] = \operatorname{argmin}_K f_1,\ [\beta, +\infty[= \operatorname{argmin}_K f_2,\ -\infty < \alpha < \beta < +\infty. \qquad (4.5.1)$$

Theorem 4.5.1 Let $f_i : \mathbb{R} \to \mathbb{R}$ be lsc and quasiconvex for $i = 1, 2$. Assume that (4.5.1) holds. Then $E_w = \mathbb{R}$.

Proof. By assumption E_w is convex, therefore $E_w = \mathbb{R}$. □

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